Assume-guarantee contracts for discrete and continuous-time systems

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Abstract

Many techniques for verifying properties for discrete or continuous-time systems are limited to systems of moderate size. In this paper, we propose an approach based on assume-guarantee contracts and compositional reasoning for verifying properties of a broad class of discrete-time and continuous-time systems consisting of interconnected components. The notion of assume-guarantee contracts makes it possible to divide responsibilities among the system components: a contract specifies the property that a component must fulfill under some assumptions on the behavior of its environment (i.e. of the other components). We define weak and strong semantics of assume-guarantee contracts for both discrete-time and continuous-time systems. We then establish a certain number of results for compositional reasoning, which allow us to show that a global assume-guarantee contract of the whole system is satisfied when all components satisfy their own contracts. We show that the weak satisfaction of the contract is sufficient to deal with interconnections described by a directed acyclic graph, while strong satisfaction is needed to reason about general interconnections containing cycles. Specific results for systems described by differential inclusions and invariance assume-guarantee contracts are then developed. Finally, we show how the proposed assume-guarantee framework can recast different versions of the small-gain theorem as a particular case. Throughout the paper, the main results are illustrated using simple examples.

Key words: component-based design, assume-guarantee contracts, prefix-closed properties, small-gain theorem.

1 Introduction

Cyber-physical systems (CPS) result from integrations of computational devices with physical processes and are to become ubiquitous in modern societies (autonomous vehicles, smart buildings, robots, etc.) (see [22] and the references therein). Despite considerable progress in the field, current techniques apply to system of moderate complexity (the complexity is quantified by the number of interacting components). Thus, the design of complex CPS requires to divide large design problems in smaller sub-problems that can be solved using existing tools.

Compositional approaches for the analysis and the design of continuous or discrete-time dynamical systems have been long known in the field of control theory, where the celebrated small-gain theorem [16,18,11] makes it possible to prove stability of a system from the stability of its components. Other compositional approaches for the analysis and design of cyber-physical systems have been mainly initiated in the field of computer science [15,1,13].

The study of properties of dynamical systems using decentralized approaches has been an ongoing research area in recent years [27,7,26,32,8,4,12]. Other composi-

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tional approaches, using formal methods and symbolic techniques, are presented in [25,9,19,23,31,21,28,36,33,24]. All these works develop efficient computational techniques by making specific assumptions on the classes of dynamical systems and of properties to which they can be applied.

In the current work, we aim at proposing a general theoretical framework and thus we make weak assumptions on systems and properties. We initiate a high-level framework for verifying properties of complex systems, consisting of interconnected components, using a contract-based approach [6]. Each component is assigned an assume-guarantee contract, which specifies the property that the component must fulfill under assumption about its environment (i.e. the other components). We introduce contracts and define weak and strong semantics for both discrete-time and continuous-time systems. We then establish results that allow us to reason compositionally using assume-guarantee contracts: i.e. if all components satisfy their own contract then a global contract of the whole system is satisfied. We show that the weak satisfaction of the contract is sufficient to deal with interconnections described by a directed acyclic graph, while strong satisfaction is needed to reason about general interconnections containing cycles. We then investigate two important questions: how one can go from weak to strong satisfaction of a contract and how to measure the robustness of assume-guarantee contracts against imperfect state measurements. We then show that for systems described by differential inclusions and invariance assume-guarantee contracts, weak satisfaction of contracts is sufficient to reason on general interconnections. Finally, we show how the proposed assume-guarantee framework can recast different versions of the small-gain theorem as a particular case.

There are several advantages in using contract based design for CPS. Firstly, by dividing a complex design problem into several smaller sub-problems, one is able to address design challenges that would be out of reach of current state-of-the-art design methods. Secondly, contract-based design makes it possible to replace a component without jeopardizing the behavior of the overall system: one just has to make sure that the new component satisfies the assigned contract. Thirdly, components are reusable when similar contracts appear in the decomposition of a global contract.

Assume-guarantee reasoning has been previously used in control theory. The authors in [17] presented a compositionality result for linear dynamical systems based on the notion of simulation introduced in [35]. In spirit, our work is closer to the framework presented in [20] for verifying general properties using parametric assume-guarantee contracts and compositional reasoning by means of small-gain theorems for discrete-time systems. However, the main compositionality result in that work requires to assume that at least one component satisfies a contract (for some parameter value), independently of the behavior of other components. This breaks the circularity of implications of the assume-guarantee contracts, which is arguably the main difficulty in contract-based design, and the reason why we introduce weak and strong semantics for assume-guarantee contracts.

The present paper focuses on the theoretical development of a general framework for contract-based reasoning. Applications of this framework to the design of symbolic controllers are reported in [28,38,29].

The paper is organized as follows. In Section 2, we introduce the class of prefix-closed properties. In Section 3, we introduce the class of systems and interconnections considered through the paper. In Section 4, we introduce assume-guarantee contracts, their weak and strong semantics and we establish compositionality results for reasoning about interconnected systems. In Section 5, we develop specific results for systems described by differential inclusions and invariance assume-guarantee contracts. Finally, in Section 6, we show that different versions of the classical small-gain theorem can be recast as particular applications of our framework. Throughout the paper, simple examples are used as illustrations of the main results.

A preliminary version of this work has been presented in the conference paper [30]. The current paper extends the approach in different directions: we generalize the approach from cascade and feedback compositions to any composition structure and from invariance to more general properties. We also provide a robustness analysis of assume-guarantee contracts against imperfect state measurement. In [30], we have shown that invariance relative to an assume-guarantee contracts implies the weak satisfaction of such contract. In the current work, we show the converse results, which means that for systems described by Lipschitz differential inclusions and invariance assume-guarantee contracts, weak satisfaction of contracts is sufficient to reason about general interconnections containing cycles. Finally, in the current work, we show that different versions of the classical small-gain theorem can be recovered using our framework.

Notations $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{R}^+$ and $\mathbb{N}$ denote the set or reals, nonnegative reals, positive reals and positive integers respectively. For $p \in \mathbb{N}$, $[0,p]_{\mathbb{N}} = [0,p] \cap \mathbb{N}$ is an interval of integers. The set of discrete-time domains is $\mathbb{E}(\mathbb{N}) = \{[0,a]_{\mathbb{N}},\ a \in \mathbb{N}\} \cup \{\mathbb{N}\}$. Similarly, the set of continuous-time domains is $\mathbb{E}(\mathbb{R}_0^+) = \{[0,a],\ a \in \mathbb{R}_0^+\} \cup \{[0,a),\ a \in \mathbb{R}^+\} \cup \{\mathbb{R}_0^+\}$. For $Z \subseteq \mathbb{R}^n$, we denote by $M_d(Z)$ the set of discrete-time maps $z : E \to Z$, where $E \in \mathbb{E}(\mathbb{N})$. Similarly, $M_c(Z)$ is the set of continuous-time maps $z : E \to Z$, where $E \in \mathbb{E}(\mathbb{R}_0^+)$. $M(Z)$ is used to denote both continuous and discrete-time cases. For $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm. Given a space X equipped with a metric $d : X \times X \to \mathbb{R}_0^+$, the closure of the set $A \subseteq X$ is denoted $\mathrm{cl}(A)$ and its complement is denoted $\overline{A}$. For $\varepsilon > 0$ and $x \in X$, the ball with center in x and a radius ε is $\mathcal{R}_\varepsilon(x) = \{y \in X \mid d(x, y) \le \varepsilon\}$. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if γ is $\mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(\cdot, s)$ belongs to class $\mathcal{K}$, and for each fixed nonzero r, the map $\beta(r, \cdot)$ is strictly decreasing and $\beta(r, s) \to 0$ as $s \to \infty$.

2 Preliminaries on prefix-closed sets

Given a set $Z \subseteq \mathbb{R}^n$, prefix closed sets are subsets $P \subseteq M(Z)$ that can be defined as follows: if a trajectory $z : E \to Z$ belongs to the prefix-closed set P, then any prefix of z belongs to P. In this part, we first give a formal definition of a prefix-closed set, we then give a necessary and sufficient condition for a set to be prefix-closed, finally we give some examples of such sets.

Definition 1 *Let $Z \subseteq \mathbb{R}^n$. Let $z : E \to Z$ and $z' : E' \to Z$ in $M(Z)$. z is said to be a prefix of z' and denoted $z \in pref(z')$ if $E \subseteq E'$ and for all $t \in E$, $z(t) = z'(t)$. In this case, z can be seen as a restriction of z' and is also denoted $z = z'_{|E}$.*

This notion is generalized toward sets of continuous or discrete-time maps in the usual way: for $A \subseteq M(Z)$, $\mathrm{pref}(A) = \bigcup_{z \in A} \mathrm{pref}(z)$.

Definition 2 *Let $Z \subseteq \mathbb{R}^n$ and $P \subseteq M(Z)$. P is said to be prefix-closed if the following logical implication is satisfied:*

$$z \in P \text{ and } \hat{z} \in pref(z) \Rightarrow \hat{z} \in P.$$

In the following we will give a characterization of prefix-closed sets.

Proposition 1 *Let $Z \subseteq \mathbb{R}^n$ and $P \subseteq M(Z)$. P is prefix-closed if and only if $pref(P) = P$.*

PROOF. Suppose that $\mathrm{pref}(P) = P$ and let us prove that P is prefix-closed. Let $z \in P$ and $\hat{z} \in \mathrm{pref}(z)$. Since $z \in P$, we have $\hat{z} \in \mathrm{pref}(z) \subseteq \mathrm{pref}(P) = P$. Then, $\hat{z} \in P$ and P is prefix-closed. Now suppose that P is prefix-closed and let us prove that $\mathrm{pref}(P) = P$. The inclusion $P \subseteq \mathrm{pref}(P)$ is verified by definition of the prefix. Let $\hat{z} \in \mathrm{pref}(P)$, then there exists $z \in P$ such that $\hat{z} \in \mathrm{pref}(z)$. Since P is prefix-closed we get $\hat{z} \in P$. Then, $\mathrm{pref}(P) \subseteq P$ which ends the proof. □

In the following we give some examples of prefix-closed sets. This notion allows us to represent different type of properties such as invariance or systems described by differential or difference inclusions.

Example 1 *(Invariance) Let the set $S \subseteq \mathbb{R}^n$ such that $S \neq \emptyset$ and let us define:*

$$A = \{z : E \to \mathbb{R}^n \in M(\mathbb{R}^n) \mid \forall t \in E, z(t) \in S\}.$$

Example 2 *Let $S_1, S_2 \ldots, S_q \subseteq \mathbb{R}^n$ such that for all $i \in [1, q]_{\mathbb{N}}$, $S_i \neq \emptyset$ and let us define:*

$$A = \{z : E \to \mathbb{R}^n \in M_c(\mathbb{R}^n) \cap \mathcal{D}^q \mid \forall i \in [1, q]_{\mathbb{N}}, \\ \forall t \in E,\ z^{(i)}(t) \in S_i\}$$

Where $\mathcal{D}^q$ denotes the set of continuous-time maps q times differentiable and $z^{(i)}$ denotes the ith derivative of z.

Example 3 *(Differential inclusion) Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map we define:*

$$A = \{z : E \to \mathbb{R}^n \in M_c(\mathbb{R}^n) \cap \mathcal{D}^1 \mid \forall t \in E, \dot{z}(t) \in F(z(t))\}$$

An example of a non prefix-closed property is the reachability property described as follows:

Example 4 *Let the set $K \subseteq \mathbb{R}^n$ such that $K \neq \emptyset$ and let us define:*

$$A = \{z : E \to \mathbb{R}^n \in M(\mathbb{R}^n) \mid \exists t \in E,\ z(t) \in K\}.$$

3 Systems and interconnections

3.1 Systems

In this section, we introduce the classes of systems and interconnections considered throughout this paper, it is important to note that the classes of systems used in the paper are quite general, and includes deterministic and nondeterministic systems, in discrete-time or in continuous-time, described by difference or differential equations and inclusions and allows us to deal with phenomena such as sampling, time delays...

Definition 3 *A discrete-time system is a tuple $\Sigma = (W_1, W_2, X, Y, \mathcal{T})$ where*

- *$W_1 \subseteq \mathbb{R}^{m_1}$, $W_2 \subseteq \mathbb{R}^{m_2}$, $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^p$, are the sets of external and internal inputs, states, and outputs;*

- $\mathcal{T} \subseteq M_d(W_1 \times W_2 \times X \times Y)$ *is a set of discrete-time trajectories.*

Definition 4 *A continuous-time system is a tuple* $\Sigma = (W_1, W_2, X, Y, \mathcal{T})$ *where*

- $W_1 \subseteq \mathbb{R}^{m_1}$, $W_2 \subseteq \mathbb{R}^{m_2}$, $X \subseteq \mathbb{R}^n$ *and* $Y \subseteq \mathbb{R}^p$, *are the sets of external and internal inputs, states, and outputs;*
- $\mathcal{T} \subseteq M_c(W_1 \times W_2 \times X \times Y)$ *is a set of continuous-time trajectories.*

3.2 Interconnections

Let us first introduce some notations for interconnected systems. A network of systems consists of a collection of $N \in \mathbb{N}_{>0}$ systems $\{\Sigma^1, \ldots, \Sigma^N\}$, a set of vertices $I = \{1, \ldots, N\}$ and a binary connectivity relation $\mathcal{I} \subseteq I \times I$ where each vertex $i \in I$ is labelled with the system Σ^i. For $i \in I$, we define $\mathcal{N}(i) = \{j \in I \mid (j, i) \in \mathcal{I}\}$ as the set of neighbouring components from which the incoming edges originate. We define $I_{\text{init}} = \{i \in I \mid \mathcal{N}(i) = \emptyset\}$ as the set of components for which there exist no incoming edge.

Given a directed graph $\mathcal{G} = (I, \mathcal{I})$ over the set of vertices $I = \{1, \ldots, N\}$ and binary connectivity relation $\mathcal{I}$. A walk is a sequence $\sigma = a_1 a_2 \ldots a_m$ such that for all $i \in \{1, \ldots, m-1\}$, $(a_i, a_{i+1}) \in \mathcal{I}$, in such case we say that a_i is an element of σ. For a walk σ, if the vertices $a_1, \ldots, a_{m-1}$ are distinct and $a_1 = a_m$, then σ is a cycle. The set of all cycles is denoted $\zeta = \{\zeta_1, \ldots, \zeta_p\}$, $p \in \mathbb{N}$.

Remark 1 *We recall that for a directed graph, if we drop one edge of every cycle in the graph, a directed acyclic graph (DAG) denoted* $\mathcal{G}^{DAG}$ *is obtained. In such case, the set of initial vertices for the new directed acyclic graph is characterized by* $I_{init}^{DAG} \subseteq I_{init} \cup A$, *where* $A \subseteq I$ *is the set of vertices to which we dropped an edge. An illustration of this approach is given in Figure 1.*

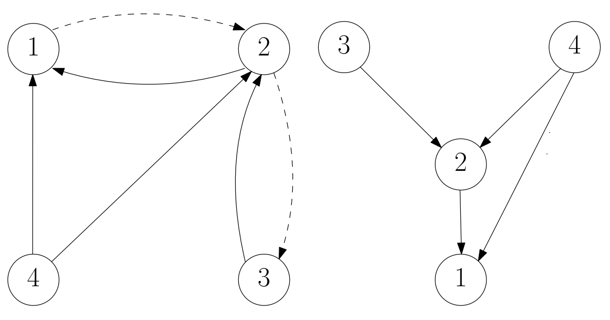

Fig. 1. Left: A graph $\mathcal{G}$ of four vertices, containing two cycles. The set of initial vertices is $I_{\text{init}} = \{4\}$. Right: A new DAG graph is constructed by removing dashed edges $\mathcal{G}^{\text{DAG}}$. The set of initial vertices is $I_{\text{init}}^{\text{DAG}} = \{4, 3\} \subseteq I_{\text{init}} \cup A = \{4\} \cup \{2, 3\}$.

In the following, we consider interconnections of systems of the same temporal nature (discrete or continuous-time) and defined as follows:

Definition 5 *Given a collection of systems* $\{\Sigma^i\}_{i \in I}$ *and a binary connectivity relation* $\mathcal{I} \subseteq I \times I$. *We say that* $\{\Sigma^i\}_{i \in I}$ *is compatible for composition w.r.t.* $\mathcal{I}$, *if for each* $i \in I$, *we have* $\prod_{j \in \mathcal{N}(i)} Y^j = W_2^i$, *i.e., the internal input space of* Σ^i *is the same as the Cartesian product of the output spaces of all the neighbors in* $\mathcal{N}(i)$. *The composed system* Γ *denoted* $\langle (\Sigma^i)_{i \in I}, \mathcal{I} \rangle$, *is given by a tuple* $\Gamma = (W_1, \{0\}, X, Y, \mathcal{T})$ *where:* $W_1 = \prod_{i \in I} W_1^i$, $X = \prod_{i \in I} X^i$, $Y = \prod_{i \in I} Y^i$ *and* $(w_1, 0, x, y) : E \to W_1 \times \{0\} \times X \times Y \in \mathcal{T}$ *is a trajectory of* Γ *if and only if for all* $i \in I$, *there exists a trajectory* $(w_1^i, w_2^i, x^i, y^i) : E \to W_1^i \times W_2^i \times X^i \times Y^i \in \mathcal{T}^i$ *of* Σ^i *such that the internal inputs are constrained by the relation* $w_2^i(t) = (y^{j_1}(t), \ldots, y^{j_p}(t))$ *where* $\mathcal{N}(i) = \{j_1, \ldots, j_p\}$, *for all* $i \in I$ *and for all* $t \in E$. *By abuse of notation, the constraint on the internal inputs will be written as* $w_2^i = \prod_{j \in \mathcal{N}(i)} \{y^j\}$.

Remark 2 *Let us remark that in the proposed interconnection structure, all the internal inputs of a system are outputs of other systems. Then, the composed system* $\Gamma = \langle (\Sigma^i)_{i \in I}, \mathcal{I} \rangle$ *has trivial null internal inputs. Hence, with an abuse of notation, we will denote* $\Gamma = (W_1, X, Y, \mathcal{T})$ *and* $(w_1, x, y) \in \mathcal{T}$, *with* $(w_1, x, y) : E \to W_1 \times X \times Y$ *as a trajectory of* Γ. *Similarly, all initial elements* Σ_i, *where* $i \in I_{init}$ *have trivial null internal inputs and we use the same notation for their trajectories.*

We should emphasize that trajectories of systems need not be defined on the whole time domains $\mathbb{N}$ or $\mathbb{R}_0^+$. This makes it possible to avoid forward-completeness issues related to systems composition as shown in the following example.

Example 5 *Let us consider the system* $\Sigma^1 = (W_1, W_2, X, Y, \mathcal{T})$ *where* $W_1 = \{0\}, W_2 = X = Y = \mathbb{R}$. *A trajectory of* Σ^1 *is a quadruple* $(0, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ *in* $\mathcal{T}$ *where* $E \in \mathbb{E}(\mathbb{R}_0^+)$, w_2 *is continuous,* x *and* y *are differentiable and such that* $x(0) = 1$ *and for all* $t \in E$,

$$\begin{cases} \dot{x}(t) = w_2(t) \\ y(t) = (x(t))^2. \end{cases}$$

Let $I = \{1\}$ *and the interconnection relation* $\mathcal{I} = \{(1, 1)\}$. *It is clear that* $\{\Sigma^i\}_{i \in I}$ *is compatible for composition w.r.t.* $\mathcal{I}$. *It can be seen that* Σ^1, *has trajectories defined on the whole time domain* $\mathbb{R}_0^+$. *However, if we only consider those trajectories, the set of trajectories* $\mathcal{T}_\Gamma$ *of the composed system* $\Gamma = \langle (\Sigma^i)_{i \in I}, \mathcal{I} \rangle$ *would be empty since the trajectories of* $\mathcal{T}_\Gamma$ *are of the form* $(0, x, y) : E \to W_1 \times X \times Y$ *where* $E \subseteq [0, 1)$, *and for all* $t \in E$, $x(t) = \frac{1}{1-t}$ *and* $y(t) = \frac{1}{(1-t)^2}$.

4 Assume-guarantee reasoning

4.1 Assume-guarantee contracts

An assume-guarantee contract is a compositional tool that specifies how a system behaves under assumptions about its inputs [6]. The use of assume-guarantee contracts makes it possible to reason on a global system based on properties of its components. In this section, we introduce assume-guarantee contracts to reason on properties for discrete or continuous-time systems. These contracts are equipped with a weak and a strong semantics, which will allow us to establish compositionality results. Let us first define contracts for discrete-time systems:

Definition 6 *Let $\Sigma = (W_1, W_2, X, Y, \mathcal{T})$ be a discrete-time system, an assume-guarantee contract for Σ is a tuple $\mathcal{C} = (A_{W_1}, A_{W_2}, G_X, G_Y)$ where*

- *$A_{W_1} \subseteq M_d(W_1)$ and $A_{W_2} \subseteq M_d(W_2)$ are sets of assumptions on the external and internal inputs;*
- *$G_X \subseteq M_d(X)$ and $G_Y \subseteq M_d(Y)$ are sets of guarantees on the states and outputs.*

We say that Σ (weakly) satisfies $\mathcal{C}$, denoted $\Sigma \models \mathcal{C}$, if for all trajectories $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ in $\mathcal{T}$:

- *for all $l \in E$, if $w_{1|[0,l]_{\mathbb{N}}} \in A_{W_1}$ and $w_{2|[0,l]_{\mathbb{N}}} \in A_{W_2}$, then:*
  - $x_{|[0,l]_{\mathbb{N}}} \in G_X$;
  - $y_{|[0,l]_{\mathbb{N}}} \in G_Y$.

We say that Σ strongly satisfies $\mathcal{C}$, denoted $\Sigma \models_s \mathcal{C}$, if for all trajectories $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ in $\mathcal{T}$:

- *if $w_{1|[0,0]_{\mathbb{N}}} \in A_{W_1}$ then $y_{|[0,0]_{\mathbb{N}}} \in G_Y$*
- *for all $l \in E$, if $w_{1|[0,l]_{\mathbb{N}}} \in A_{W_1}$ and $w_{2|[0,l]_{\mathbb{N}}} \in A_{W_2}$, then:*
  - $x_{|[0,l]_{\mathbb{N}}} \in G_X$;
  - $y_{|[0,l]_{\mathbb{N}}} \in G_Y$ *and* $y_{|[0,l+1]_{\mathbb{N}} \cap E} \in G_Y$.

Let us remark that $\Sigma \models_s \mathcal{C}$ obviously implies $\Sigma \models \mathcal{C}$. Intuitively, an assume-guarantee contract for a discrete-time system states that if the restrictions of the external and internal inputs of the system up to a time $l \in \mathbb{N}$ belongs to A_{W_1} and A_{W_2}, respectively , then the restriction of the state of the system up to a time l belongs to G_X, and the restriction of the output of the system up to a time l (or up to a time $l + \delta$, where $\delta \in \{0, 1\}$, in the case of strong satisfaction) belongs to G_Y. One may remark that if the set of guarantees on the outputs G_Y is prefix closed, the notion of strong satisfaction of contract can be defined by: Σ strongly satisfies $\mathcal{C}$, if for all trajectories $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ in $\mathcal{T}$:

- if $w_{1|[0,0]_{\mathbb{N}}} \in A_{W_1}$ then $y_{|[0,0]_{\mathbb{N}}} \in G_Y$
- for all $l \in E$, if $w_{1|[0,l]_{\mathbb{N}}} \in A_{W_1}$ and $w_{2|[0,l]_{\mathbb{N}}} \in A_{W_2}$, then:
  - $x_{|[0,l]_{\mathbb{N}}} \in G_X$;
  - $y_{|[0,l+1]_{\mathbb{N}} \cap E} \in G_Y$.

We now introduce contracts for continuous-time systems:

Definition 7 *Let $\Sigma = (W_1, W_2, X, Y, \mathcal{T})$ be a continuous-time system, an assume-guarantee contract for Σ is a tuple $\mathcal{C} = (A_{W_1}, A_{W_2}, G_X, G_Y)$ where*

- *$A_{W_1} \subseteq M_c(W_1)$ and $A_{W_2} \subseteq M_c(W_2)$ are sets of assumptions on the external and internal inputs;*
- *$G_X \subseteq M_c(X)$ and $G_Y \subseteq M_c(Y)$ are sets of guarantees on the states and outputs.*

We say that Σ (weakly) satisfies $\mathcal{C}$, denoted $\Sigma \models \mathcal{C}$, if for all trajectories $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ in $\mathcal{T}$:

- *for all $t \in E$, if $w_{1|[0,t]} \in A_{W_1}$ and $w_{2|[0,t]} \in A_{W_2}$, then:*
  - $x_{|[0,t]} \in G_X$;
  - $y_{|[0,t]} \in G_Y$.

We say that Σ strongly satisfies $\mathcal{C}$, denoted $\Sigma \models_s \mathcal{C}$, if for all trajectories $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ in $\mathcal{T}$:

- *if $w_{1|[0,0]_{\mathbb{N}}} \in A_{W_1}$ then $y_{|[0,0]} \in G_Y$*
- *for all $t \in E$, if $w_{1|[0,t]} \in A_{W_1}$ and $w_{2|[0,t]} \in A_{W_2}$, then:*
  - $x_{|[0,t]} \in G_X$;
  - *there exists $\delta > 0$ such that for all $s \in [0, \delta]$ $y_{|[0,t+s] \cap E} \in G_Y$.*

Again, $\Sigma \models_s \mathcal{C}$ obviously implies $\Sigma \models \mathcal{C}$. An assume-guarantee contract for a continuous-time system states that if the restriction of the external and internal inputs to the system up to a time $t \in \mathbb{R}_0^+$ belongs to A_{W_1} and A_{W_2}, respectively, then the restriction of the state of the system up to time t belongs to G_X, and the restriction of the output of the system up to time t (or up to a time $t + s$ with $s \in [0, \delta]$ and $\delta > 0$, in the case of strong satisfaction) belongs to G_Y. Let us remark that the value of δ may depend on the trajectory $(w_1, w_2, x, y) \in \mathcal{T}$ and on the value of the time instant $t \in E$, which makes a noticeable difference with the discrete-time case. One may remark that if the set of guarantees on the outputs G_Y is prefix closed, the notion of strong satisfaction of a contract can be defined by: Σ strongly satisfies $\mathcal{C}$, if for all trajectories $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ in $\mathcal{T}$:

- if $w_{1|[0,0]_{\mathbb{N}}} \in A_{W_1}$ then $y_{|[0,0]} \in G_Y$
- for all $t \in E$, if $w_{1|[0,t]} \in A_{W_1}$ and $w_{2|[0,t]} \in A_{W_2}$, then:
  - $x_{|[0,t]} \in G_X$;
  - there exists $\delta > 0$ such that $y_{|[0,t+\delta] \cap E} \in G_Y$.

Remark 3 *Similarly to Remark 2, a contract for the composed system $\Gamma = \langle (\Sigma^i)_{i \in I}, \mathcal{I} \rangle$ has trivial null assumptions on internal inputs. Hence, with an abuse of notation, a contract for the composed system Γ will be denoted $\mathcal{C} = (A_{W_1}, G_X, G_Y)$*

4.2 Compositional reasoning

We now provide results allowing us to reason about interconnected systems based on contracts satisfied by the components.

4.2.1 Acyclic interconnections

Firstly, we provide the following result on the composition of assume-guarantee contracts, where the interconnection graph $\mathcal{G}$ between the components is a DAG. This result applies equally to discrete or continuous-time systems.

Theorem 1 *Let a network of components $\{\Sigma^i\}_{i \in I}$ compatible for composition w.r.t. $\mathcal{I}$. Let the composed system $\Gamma = \langle (\Sigma^i)_{i \in I}, \mathcal{I} \rangle$ and assume that $\mathcal{G} = (I, \mathcal{I})$ is a DAG. To each component Σ^i we associate a contract $\mathcal{C}^i = (A_{W_1^i}, A_{W_2^i}, G_{X^i}, G_{Y^i})$, and let $\mathcal{C} = (\prod_{i \in I} A_{W_1^i}, \prod_{i \in I} G_{X^i}, \prod_{i \in I} G_{Y^i})$ be a contract for Γ. If for all $i \in I$, $\Sigma^i \models \mathcal{C}^i$ and $\prod_{j \in \mathcal{N}(i)} G_{Y^j} \subseteq A_{W_2^i}$ then $\Gamma \models \mathcal{C}$.*

PROOF. We provide the proof for continuous-time systems only, but the proof for discrete-time systems can be derived similarly. Let $(w, x, y) : E \to W \times X \times Y$ in $\mathcal{T}$ be a trajectory of the system Γ. Then, for all $i \in I$, there exists a trajectory $(w_1^i, w_2^i, x^i, y^i) : E \to W_1^i \times W_2^i \times X^i \times Y^i \in \mathcal{T}^i$ of Σ^i such that $w_2^i = \prod_{j \in \mathcal{N}(i)} \{y^j\}$. Let $t \in E$ such that for all $i \in I$, $w^i_{1|[0,t]} \in A_1^i$. Then, since initial components $\{\Sigma^i\}_{i \in I_{\text{init}}}$ do not have internal inputs, and from the satisfaction of contracts for all components, we have:

$$\forall i \in I_{\text{init}}, \;\; x^i_{|[0,t]} \in G_{X^i} \text{ and } y^i_{|[0,t]} \in G_{Y^i}. \tag{1}$$

Let us assume the existence of $i \in I \setminus I_{\text{init}}$, such that $x^i_{|[0,t]} \notin G_{X^i}$ or $y^i_{|[0,t]} \notin G_{Y^i}$. Since $\Sigma^i \models \mathcal{C}^i$ and $w^i_{1|[0,t]} \in A_1^i$, we have that $w^i_{2|[0,t]} \notin A_2^i$, then using the fact that $w_2^i = \prod_{j \in \mathcal{N}(i)} \{y^j\}$ and $\prod_{j \in \mathcal{N}(i)} G_{Y^j} \subseteq A_{W_2^i}$, we have the existence of $j \in \mathcal{N}(i)$ such that $y^j_{|[0,t]} \notin G_{Y^j}$. Hence, using the structure of a DAG, we have by iterating this procedure, the existence of $k \in I_{\text{init}}$ such that $y^k_{|[0,t]} \notin G_{Y^k}$ which contradicts (1). Hence, we have for all $i \in I$, $x^i_{|[0,t]} \in G_{X^i}$ and $y^i_{|[0,t]} \in G_{Y^i}$. Then, $\Gamma \models \mathcal{C}$. $\square$

Let us remark that the previous result is a generalization of Theorem 1 in [30] for cascade composition.

4.2.2 Cyclic interconnections of discrete-time systems

We now provide a result on general interconnections, without any restriction on the interconnection graph. We first present a result for the discrete-time case.

Theorem 2 *Let a network of discrete-time components $\{\Sigma^i\}_{i \in I}$ compatible for composition w.r.t. $\mathcal{I}$. Let the composed system $\Gamma = \langle (\Sigma^i)_{i \in I}, \mathcal{I} \rangle$. To each component Σ^i we associate a contract $\mathcal{C}^i = (A_{W_1^i}, A_{W_2^i}, G_{X^i}, G_{Y^i})$, and let $\mathcal{C} = (\prod_{i \in I} A_{W_1^i}, \prod_{i \in I} G_{X^i}, \prod_{i \in I} G_{Y^i})$ a contract for Γ. Let us assume the following:*

(i) for all $i \in I$, $\Sigma^i \models \mathcal{C}^i$;
(ii) for all $i \in I$, $\prod_{j \in \mathcal{N}(i)} G_{Y^j} \subseteq A_{W_2^i}$;
(iii) for any cycle ζ_q in $\mathcal{G}$, there exists an element $k \in \zeta_q$ such that $\Sigma^k \models_s \mathcal{C}^k$;
(iv) for all $i \in I$, $A_{W_1^i}$ is a prefix-closed set.

then $\Gamma \models \mathcal{C}$.

PROOF. Let $(w, x, y) : E \to W \times X \times Y$ in $\mathcal{T}$ be a trajectory of the system Γ. Then, for all $i \in I$, there exists a trajectory $(w_1^i, w_2^i, x^i, y^i) : E \to W_1^i \times W_2^i \times X^i \times Y^i \in \mathcal{T}^i$ of Σ^i such that $w_2^i = \prod_{j \in \mathcal{N}(i)} \{y^j\}$. Let $l \in E$ such that for all $i \in I$, $w^i_{1|[0,l]} \in A_{W_1^i}$. All initial components $\{\Sigma^i\}_{i \in I_{\text{init}}}$ do not have internal inputs, then from the satisfaction of contracts for all components and since $A_{W_1^i}$ is prefix-closed for all $i \in I$, we have:

$$\forall i \in I_{\text{init}}, \; \forall m \in [0, l]_{\mathbb{N}}, \; x^i_{|[0,m]} \in G_{X^i} \text{ and } y^i_{|[0,m]} \in G_{Y^i}. \tag{2}$$

To prove that $\Gamma \models \mathcal{C}$, we proceed by induction. First, let us prove that for all $i \in I$, $y^i_{|[0,0]} \in G_{Y^i}$. We have the existence of an element k in any cycle ζ_q such that $\Sigma^k \models_s \mathcal{C}^k$, which implies from prefix closedeness of $A_{W_1^k}$ that $y^k_{|[0,0]} \in G_{Y^k}$. To prove that this initial condition is satisfied by all the components Σ^i, $i \in I$, we proceed as follows: for any component Σ^k that strongly satisfies its contract, we drop the incoming edge into the vertex k in the cycle ζ_q. Then, in view of remark 1, a new DAG, $\mathcal{G}^{\text{DAG}}$ is obtained. Then from (2) we have:

$$\forall i \in I^{\text{DAG}}_{\text{init}} \subseteq I_{\text{init}} \cup A, \;\; y^i_{|[0,0]} \in G_{Y^i}. \tag{3}$$

with A is the set of vertices to which we dropped an edge (vertices corresponding to components that stronlgy satisfy their contracts). Now let an element $i \in I \setminus I^{\text{DAG}}_{\text{init}}$ and let us assume that $y^i_{|[0,0]} \notin G_{Y^i}$. From prefix-closedness of $A_{W_1^i}$ it follows that $w^i_{1|[0,0]} \in A_{W_1^i}$, moreover $\Sigma^i \models \mathcal{C}^i$, then we have that $w^i_{2|[0,0]} \notin A_{W_2^i}$, and using the fact that $w_2^i = \prod_{j \in \mathcal{N}(i)} \{y^j\}$ and $\prod_{j \in \mathcal{N}(i)} G_{Y^j} \subseteq A_{W_2^i}$, we have the existence of $j \in \mathcal{N}(i)$ such that $y^j_{|[0,0]} \notin G_{Y^j}$.

Hence, using the structure of a DAG, we have by iterating this procedure, the existence of $h \in I_{\text{init}}^{\text{DAG}}$ such that $y^h_{|[0,0]} \notin G_{Y^h}$ which contradicts (3). Hence, we have for all $i \in I$, $y^i_{|[0,0]} \in G_{Y^i}$.

Now let $m \in [0,l]_{\mathbb{N}}$, let us assume that for all $i \in I$, $y^i_{|[0,m-1]} \in G_{Y^i}$ and let us prove that for all $i \in I$, $y^i_{|[0,m]} \in G_{Y^i}$. We have the existence of an element k in any cycle ζ_q such that $\Sigma^k \models_s \mathcal{C}^k$. From prefix-closedness of $A_{W_1^i}$ it follows that $w^k_{1|[0,m-1]} \in A_{W_1^i}$, moreover we have that $w^k_{2|[0,m-1]} = \prod_{j\in\mathcal{N}(k)}\{y^k_{|[0,m-1]}\} \in \prod_{j\in\mathcal{N}(k)} G_{Y^j} \subseteq A_{W_2^k}$, then since $\Sigma^k \models_s \mathcal{C}^k$ we have that $y^k_{|[0,m]} \in G_{Y^k}$. Hence, by using the same procedure as before (dropping the incoming edge into the vertex k in the cycle ζ_q), we have from (2) that:

$$\forall i \in I_{\text{init}}^{\text{DAG}} \subseteq I_{\text{init}} \cup A, \;\; y^i_{|[0,m]} \in G_{Y^i}. \qquad (4)$$

Now let an element $i \in I \setminus I_{\text{init}}^{\text{DAG}}$ and let us assume that $y^i_{|[0,m]} \notin G_{Y^i}$. From prefix-closedeness of $A_{W_1^i}$ we have that $w^i_{1|[0,m]} \in A_{W_1^i}$, then since $\Sigma^i \models \mathcal{C}^i$, we have that $w^i_{2|[0,m]} \notin A_{W_2^i}$, then using the fact that $w_2^i = \prod_{j\in\mathcal{N}(i)}\{y^j\}$ and $\prod_{j\in\mathcal{N}(i)} G_{Y^j} \subseteq A_{W_2^i}$, we have the existence of $j \in \mathcal{N}(i)$ such that $y^j_{|[0,m]} \notin G_{Y^j}$. Hence, using the structure of a DAG, we have by iterating this procedure, the existence of $h \in I_{\text{init}}^{\text{DAG}}$ such that $y^h_{|[0,m]} \notin G_{Y^h}$ which contradicts (4). Hence, we have for all $i \in I$, $y^i_{|[0,m]} \in G_{Y^i}$.

Let $i \in I$, we have $w^i_{1|[0,l]} \in A_{W_1^i}$ and $w^i_{2|[0,l]} = \prod_{j\in\mathcal{N}(i)}\{y^i_{|[0,l]}\} \in \prod_{j\in\mathcal{N}(i)} G_{Y^j} \subseteq A_{W_2^i}$, then we have from (i) that for all $i \in I$, $x^i_{|[0,l]} \in G_{X^i}$. Hence, $\Gamma \models \mathcal{C}$. □

4.2.3 Cyclic interconnections of continuous-time systems

In order to deal with continuous-time systems, we need the following assumption on the set of guarantees on the output G_Y. This assumption will be explained later on different examples.

Assumption 1 *Let $\Sigma = (W_1, W_2, X, Y, \mathcal{T})$ be a continuous-time system, and $\mathcal{C} = (A_{W_1}, A_{W_2}, G_X, G_Y)$ an assume-guarantee contracts for Σ. For any trajectory $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ of the system Σ, the following logical implication is satisfied for all $t \in E$:*

$$\forall s \in [0,t), \;\; y_{|[0,s]} \in G_Y \Rightarrow y_{|[0,t]} \in G_Y.$$

First, we explain on the following example the necessity of Assumption 1.

Example 6 *Let us consider the system $\Sigma_1 = (W_1, W_2, X, Y, \mathcal{T})$ where $W_1 = W_2 = X = Y$. A trajectory of Σ_1 is a quadruple $(w_1, w_2, x, y) : \mathbb{R}_0^+ \to W_1 \times W_2 \times X \times Y$ in $\mathcal{T}$. Let $I = \{1\}$ and let the interconnection relation $\mathcal{I} = \{(1,1)\}$. It is clear that $\{\Sigma_i\}_{i\in I}$ is compatible for composition w.r.t $\mathcal{I}$. Let us consider the assume-guarantee contract $\mathcal{C} = (A_{W_1}, A_{W_2}, G_X, G_Y)$ for Σ_1 with $G_Y \subseteq A_{W_2}$. Let the contract $\mathcal{C}_\Gamma$ for the composed system $\Gamma = \langle(\Sigma^i)_{i\in I}, \mathcal{I}\rangle$ defined as in Theorem 2. Let us assume that $\Sigma_1 \models_s \mathcal{C}$ and that for all $t \in \mathbb{R}_0^+$, $w_{1|[0,t]} \in A_{W_1}$. From strong satisfaction of the contract we have that $y_{|[0,0]} \in G_Y$. Hence, $w_{2|[0,0]} = y_{|[0,0]} \in G_Y \subseteq A_{W_2}$. Since $w_{1|[0,0]} \in A_{W_1}$ and $\Sigma_1 \models_s \mathcal{C}$ we have the existence of $\delta_1 > 0$ such that for all $s \in [0, \delta_1]$, $y_{|[0,s]} \in G_Y$. Particularly, we have that $y_{|[0,\delta_1]} \in G_Y$. Hence, $w_{2|[0,\delta_1]} = y_{|[0,\delta_1]} \in G_Y \subseteq A_{W_2}$. Then, using the fact that $w_{1|[0,\delta_1]} \in A_{W_1}$ and from the strong satisfaction of contract, we have the existence of $\delta_2 > 0$ such that $s \in [0, \delta_2]$, $y_{|[0,\delta_1+s]} \in G_Y$. By iterating, we have the existence of a sequence of strictly positive elements $(\delta_i)_{i\in\mathbb{N}_{>0}}$ such that for all $s \in [0, \delta)$, $y_{|[0,s]} \in G_Y$ with $\delta = \sum_{i=1}^{+\infty} \delta_i$. However, if δ is finite, the conclusion of the previous theorem does not hold. Indeed, we have that $w_{1|[0,t]} \in A_{W_1}$ for all $t \in \mathbb{R}_0^+$ and we only have that $y_{|[0,s]} \in G_Y$ for all $s \in [0, \delta)$. Hence, the contract $\mathcal{C}_\Gamma$ of the system Γ is not satisfied.*

Hence, even if the strong satisfaction allows to evolve within the time, Assumption 1 is crucial for ruling out Zeno phenomena.

Now we give some sufficient conditions on systems and contracts in order to satisfy Assumption 1 for different examples. Let $\Sigma = (W_1, W_2, X, Y, \mathcal{T})$ be a system, and $\mathcal{C} = (A_{W_1}, A_{W_2}, G_X, G_Y)$ an assume-guarantee contract for Σ, where the set of guarantees G_Y is described in the corresponding examples introduced in Section 2.

- Example 1: If for any trajectory $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ for the system Σ, $y : E \to Y$ is left continuous and the set of guarantees G_Y is closed then Assumption 1 is satisfied.
- Example 2: Similarly to the previous example, it can be shown that if for any trajectory $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ for the system Σ, $y : E \to Y$ is q times differentiable, the qth derivative $y^{(q)}$ is left continuous and the sets S_i, $i = 1, \ldots, q$, are closed then Assumption 1 is satisfied.
- Example 3: It can be shown that if the set valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous [1] and for any trajectory $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ for the system Σ, $y : I \to Y$ is differentiable and

[1] A set-valued mapping $M : \mathbb{R}^m \rightrightarrows R^n$ outer semicontinuous at $x \in \mathbb{R}^m$ if for every sequence of points x_i convergent to x and any convergent sequence of points $y_i \in M(x_i)$, one has $y \in M(x)$, where $\lim\limits_{y_i \to +\infty} = y$. The mapping M is outer semicontinuous if it is outer semicontinuous at each $x \in \mathbb{R}^m$.

its derivative is left continuous, then Assumption 1 is satisfied.

The following result relates the satisfaction of Assumption 1 for a global system to its satisfaction for the components. The result is straightforward and is stated without proof.

Claim 1 *Given a collection of components $\{\Sigma^i\}_{i\in I}$, such that each component Σ^i satisfies Assumption 1 w.r.t the contract $\mathcal{C}^i = (A_{W_1^i}, A_{W_2^i}, G_{X^i}, G_{Y^i})$. Then the composed system $\Gamma = \langle (\Sigma^i)_{i\in I}, \mathcal{I}\rangle$ satisfies Assumption 1 w.r.t the contract $\mathcal{C} = (\prod_{i\in I} A_{W_1^i}, \prod_{i\in I} G_{X^i}, \prod_{i\in I} G_{Y^i})$.*

Theorem 3 *Let a network of continuous-time components $\{\Sigma^i\}_{i\in I}$ compatible for composition w.r.t. $\mathcal{I}$. Let the system $\Gamma = \langle (\Sigma^i)_{i\in I}, \mathcal{I}\rangle$ be the composed system. To each component Σ^i, we associate a contract $\mathcal{C}^i = (A_{W_1^i}, A_{W_2^i}, G_{X^i}, G_{Y^i})$ and let $\mathcal{C} = (\prod_{i\in I} A_{W_1^i}, \prod_{i\in I} G_{X^i}, \prod_{i\in I} G_{Y^i})$ a contract for Γ. Let us assume the following:*

(i) for all $i \in I$, $\Sigma^i \models \mathcal{C}^i$;
(ii) for all $i \in I$, $\prod_{j\in\mathcal{N}(i)} G_{Y^j} \subseteq A_{W_2^i}$;
(iii) for all $i \in I$, Σ^i satisfies Assumption 1;
(iv) for any cycle ζ_q in $\mathcal{G}$, there exists an element $k \in \zeta_q$ such that $\Sigma^k \models_s \mathcal{C}^k$;
(v) for all $i \in I$, $A_{W_1^i}$ is a prefix-closed set.

then $\Gamma \models \mathcal{C}$.

PROOF. Let $(w, x, y) : E \to W \times X \times Y$ in $\mathcal{T}$ be a trajectory of the system Γ. Then, for all $i \in I$, there exists a trajectory $(w_1^i, w_2^i, x^i, y^i) : E \to W_1^i \times W_2^i \times X^i \times Y^i \in \mathcal{T}^i$ of Σ^i such that $w_2^i = \prod_{j\in\mathcal{N}(i)}\{y^j\}$. Let $t \in E$ such that for all $i \in I$, $w^i_{1|[0,t]} \in A_{W_1^i}$. All initial components $\{\Sigma^i\}_{i\in I_{\text{init}}}$ do not have internal inputs, then from the satisfaction of contracts for all components and since $A_{W_1^i}$ is-prefix closed for all $i \in I$, we have:

$$\forall i \in I_{\text{init}},\ \ \forall s \in [0, t],\ \ x^i_{|[0,s]} \in G_{X^i} \text{ and } y^i_{|[0,s]} \in G_{Y^i}. \tag{5}$$

Using the same proof as for the discrete-time case, in Theorem 2, we can show that:

$$\forall i \in I,\ \ y^i_{|[0,0]} \in G_{Y^i}. \tag{6}$$

Let us define

$$\begin{aligned} T &= \sup\{s \in [0, t] \mid \forall s' \in [0, s],\ y_{|[0,s']} \in G_Y\}; \qquad (7)\\ &= \sup\{s \in [0, t] \mid \forall i \in I, \forall s' \in [0, s], y^i_{|[0,s']} \in G_{Y^i}\}. \end{aligned}$$

From (6) we have $y_{|[0,0]} \in G_Y$, it then follows that $T \in [0, t]$. Let us remark that by (7), we have that $y_{|[0,s]} \in G_Y$ for all $s \in [0, T)$. Let us suppose that $y_{|[0,t]} \notin G_Y$. Hence, $T < t$.

We have $y_{|[0,s]} \in G_Y$ for all $s \in [0, T)$. Then, from (iii) and using Claim 1, we have that $y_{|[0,T]} \in G_Y$. We have the existence of an element k in any cycle ζ_q such that $\Sigma^k \models_s \mathcal{C}^k$. We have from prefix-closedeness of the set $A_{W_1^k}$ that $w^k_{1|[0,T]} \in A_{W_1^k}$. Then, since $w^k_{2|[0,T]} = \prod_{j\in\mathcal{N}(k)}\{y^j_{|[0,T]}\} \in \prod_{j\in\mathcal{N}(k)} G_Y^j \subseteq A_{W_2^k}$, we have from (iv) the existence of $\delta_k > 0$ such that for all $s_k \in [0, \delta_k]$, $y^k_{|[0,T+s_k]\cap E} \in G_{Y^k}$. Let $\delta = \min_{k\in A} \delta_k$, where A is the set of vertices corresponding to components that strongly satisfy their contracts, we have that for all $s \in [0, \delta]$, $y^k_{|[0,T+s]\cap E} \in G_{Y^k}$ by using the same procedure as for the discrete-time case (dropping the incoming edges into the vertex k in the cycle ζ_q), we have from (5):

$$\forall i \in I^{\text{DAG}}_{\text{init}} \subseteq I_{\text{init}} \cup A,\ \ y^i_{|[0,T+\delta]\cap[0,t]} \in G_{Y^i}. \tag{8}$$

Now let an element $i \in I \setminus I^{\text{DAG}}_{\text{init}}$ and let us assume that $y^i_{|[0,T+\delta]\cap[0,t]} \notin G_{Y^i}$. From prefix-closedeness of $A_{W_1^i}$ we have that $w^i_{1|[0,T+\delta]\cap[0,t]} \in A_{W_1^i}$. Then, since $\Sigma^i \models \mathcal{C}^i$ we have that $w^i_{2|[0,T+\delta]\cap[0,t]} \notin A_{W_2^i}$, then using the fact that $w_2^i = \prod_{j\in\mathcal{N}(i)}\{y^j\}$ and $\prod_{j\in\mathcal{N}(i)} G_{Y^j} \subseteq A_{W_2^i}$, we have the existence of $j \in \mathcal{N}(i)$ such that $y^j_{|[0,T+\delta]\cap[0,t]} \notin G_{Y^j}$. Hence, using the structure of a DAG, we have by iterating this procedure, the existence of $h \in I^{\text{DAG}}_{\text{init}}$ such that $y^h_{|[0,T+\delta]\cap[0,t]} \notin G_{Y^h}$ which contradicts (8). Hence, we have for all $i \in I$, $y^i_{|[0,T+\delta]\cap[0,t]} \in G_{Y^i}$. Which contradicts our assumption.

Then, we have that $y_{|[0,t]} \in G_Y$ which is equivalent to $y^i_{|[0,t]} \in G_{Y^i}$ for all $i \in I$. Now let $i \in I$, we have $w^i_{1|[0,t]} \in A_{W_1^i}$ and $w^i_{2|[0,t]} = \prod_{j\in\mathcal{N}(i)}\{y^j_{|[0,t]}\} \in \prod_{j\in\mathcal{N}(i)} G_{Y^j} \subseteq A_{W_2^i}$, then we have from (i) that for all $i \in I$, $x^i_{|[0,t]} \in G_{X^i}$. Hence, $\Gamma \models \mathcal{C}$. $\square$

It can be seen that the previous results represent generalization of Theorem 2 in [30] for feedback composition.

Remark 4 *It was shown in Theorems 2 and 3 that prefix-closedeness of the set of assumptions A_{W_1} is critical for the compositionality result for general interconnections containing cycles. Given a non-prefix-closed set of assumptions A_{W_1}, the set pref(A_{W_1}) is prefix-closed. Hence, the results of Theorems 2 and 3 remain correct if we assign to each component Σ_i the contract $\mathcal{C}^i = (\mathit{pref}(A_{W_1^i}), A_{W_2^i}, G_{X^i}, G_{Y^i})$. This approach allows to overcome the prefix-closedeness assumption, at the cost of an additional conservatism.*

Remark 5 *Proposition 1 in [26] can be recovered by this approach, where our prefix-closed sets and general sets* *corresponds to invariants and LTL specifications, respectively, in that work.*

Let us point out that weak semantics is generally insufficient to reason on general compositions containing cycles, as shown by the following counter-example:

Example 7 *Let us consider the system* $\Sigma^1 = (W_1, W_2, X, Y, \mathcal{T})$ *where ,* $W_1 = W_2 = X = Y = \mathbb{R}_0^+$*. A trajectory of* Σ^1 *is a quadruple* $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ *in* $\mathcal{T}$ *where* $E \in \mathbb{E}(\mathbb{R}_0^+)$*,* w_1 *and* w_2 *are continuous,* x *and* y *are differentiable and such that* $x(0) = 0$*, and for all* $t \in \mathbb{R}_0^+$*,*

$$\begin{cases} \dot{x}(t) = \sqrt{w_2(t)} + w_1(t) \\ y(t) = x(t). \end{cases}$$

Let $I = \{1\}$ *and let the interconnection relation* $\mathcal{I} = \{(1,1)\}$*. It is clear that* $\{\Sigma^i\}_{i\in I}$ *is compatible for composition w.r.t.* $\mathcal{I}$*. Let us consider the assume-guarantee contract* $\mathcal{C} = (A_{W_1}, A_{W_2}, G_X, G_Y)$ *for* Σ^1*, given by:*

$$\begin{aligned} A_{W_1} &= \{w_1 : E \to W_1 \in M_c(W_1) |\ \forall t \in E, w_1(t) = 0\} \\ A_{W_2} &= \{w_2 : E \to W_2 \in M_c(W_2) |\ \forall t \in E, w_2(t) = 0\} \\ G_X &= \{x : E \to X \in M_c(X) |\ \forall t \in E, x(t) = 0\} \\ G_Y &= \{y : E \to Y \in M_c(Y) |\ \forall t \in E, y(t) = 0\} \end{aligned}$$

Let the contract $\mathcal{C}_\Gamma$ *for the composed system* $\Gamma = \langle(\Sigma^i)_{i\in I}, \mathcal{I}\rangle$ *defined as in Theorem 3. We can easily check that* $\Sigma^1 \models \mathcal{C}$*. However, the conclusion of the previous theorem does not hold. Indeed, the map* $(w_1, x, y) : \mathbb{R}_0^+ \to W_1 \times X \times Y$ *defined by* $w_1(t) = 0$ *and* $x(t) = y(t) = t^2/4$ *for all* $t \in \mathbb{R}_0^+$ *is a trajectory of* Γ *and the contract* $\mathcal{C}_\Gamma$ *of the system* Γ *is not satisfied.*

It is clear from the previous example that strong satisfaction is needed to reason about general interconnections containing cycles. We show two modifications of the previous example, based on sampling or time-delays, which lead to strong satisfaction of the contract.

Example 8 *Let the system* $\Sigma^1 = (W_1, W_2, X, Y, \mathcal{T})$ *where* $W_1 = W_2 = X = Y = \mathbb{R}_0^+$*. A trajectory of* Σ^1 *is a quadruple* $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ *in* $\mathcal{T}$ *where* $E \in \mathbb{E}(\mathbb{R}_0^+)$*,* w_1 *and* w_2 *are continuous,* x *and* y *are differentiable and such that* $x(0) = 0$*, and for all* $t \in \mathbb{R}_0^+$*,*

$$\begin{cases} \dot{x}(t) = \sqrt{w_2(t)} + w_1(t) & \\ y(t) = 0 & 0 \le t \le t_0 \\ y(t) = x(t_k) & t_k < t \le t_{k+1},\ k \in \mathbb{N}. \end{cases}$$

where $(t_k)_{k\in\mathbb{N}}$ *a strictly increasing sequence of sampling instants with* $t_0 \ge 0$ *and* $t_k \to +\infty$ *when* $k \to +\infty$*. We consider the same assume-guarantee contract as in the previous example. Let us remark that* y *is left-continuous and Assumption 1 is satisfied. We can easily check that* $\Sigma^1 \models_s \mathcal{C}$*, where the value of* δ *as in Definition 7 is given by* $\delta = t_{k+1} - t$ *if* $t_k \le t < t_{k+1}$*. Let* $I = \{1\}$ *and let the interconnection relation* $\mathcal{I} = \{(1,1)\}$*. Let the contract* $\mathcal{C}_\Gamma$ *for the composed system* $\Gamma = \langle(\Sigma^i)_{i\in I}, \mathcal{I}\rangle$ *defined as in Theorem 3. Now we can check that the conclusion of the previous theorem holds since the only trajectory* $(w_1, x, y) : \mathbb{R}_0^+ \to W_1 \times X \times Y$ *of the composed system* Γ *is given by* $w_1(t) = x(t) = y(t) = 0$*, for all* $t \in \mathbb{R}_0^+$*.*

Example 9 *Let the system* $\Sigma^1 = (W_1, W_2, X, Y, \mathcal{T})$ *where* $W_1 = W_2 = X = Y = \mathbb{R}_0^+$*. A trajectory of* Σ^1 *is a quadruple* $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ *in* $\mathcal{T}$ *where* $E \in \mathbb{E}(\mathbb{R}_0^+)$*,* w_1 *and* w_2 *are continuous,* x *and* y *are differentiable and such that* $x(0) = 0$*, and for all* $t \in \mathbb{R}_0^+$*,*

$$\begin{cases} \dot{x}(t) = \sqrt{w_2(t)} + w_1(t) & \\ y(t) = 0 & 0 \le t \le T \\ y(t) = x(t - T) & T < t. \end{cases}$$

where $T > 0$ *is a time delay. We consider the same assume-guarantee contract as in Example 7. Let us remark that* y *is left-continuous and Assumption 1 is satisfied. We can easily check that* $\Sigma \models_s \mathcal{C}$*, where the value of* δ *as in Definition 7 is given by* $\delta = T$*. Let* $I = \{1\}$ *and let the interconnection relation* $\mathcal{I} = \{(1,1)\}$*. Let the contract* $\mathcal{C}_\Gamma$ *for the composed system* $\Gamma = \langle(\Sigma^i)_{i\in I}, \mathcal{I}\rangle$ *defined as in Theorem 3. Then, we can check that the conclusion of the previous theorem holds since the only trajectory* $(w_1, x, y) : \mathbb{R}_0^+ \to W_1 \times X \times Y$ *of the composed system* Γ *is given by* $w_1(t) = x(t) = y(t) = 0$*, for all* $t \in \mathbb{R}_0^+$*.*

It can be seen from the Examples 8 and 9 that our framework is suitable to reason on systems that includes some sampled or delayed behaviors. Moreover, these examples suggest that by sampling or delaying the output of a component, strong satisfaction of a contract can be obtained. These examples also show how one can go from weak to strong satisfaction by slightly modifying the system, in the next section we show that this is also possible by slightly modifying the contract.

Remark 6 *Theorems 1, 2 and 3 apply to a very general class of systems. When considering more specific classes, one can sometimes reason on general interconnections without strong contract satisfaction. Such a case will be shown in Section 5, where we consider systems modeled by Lipschitz differential inclusions and invariance assume-guarantee contracts.*

4.3 From weak to strong contract satisfaction

In this section, we show that under some additional assumptions, it is possible to reason about general compositions using the weak semantics of assume guarantee contracts. The results of this section only apply to continuous-time systems.

In order to measure the distance between two continuous-time trajectories, which might not have the same time domain. We use the notion of ε-closeness of trajectories [14], which is related to the Hausdorff distance between the graphs of the trajectories.

Definition 8 *(ε-closeness of trajectories) Let $Z \subseteq \mathbb{R}^n$. Given $\varepsilon > 0$ and two continuous-time trajectories $z_1 : E_1 \to Z$ and $z_2 : E_2 \to Z$ in $M_c(Z)$. z_2 is said to be ε-close to z_1, if for all $t_1 \in E_1$, there exists $t_2 \in E_2$ such that $\mid t_1 - t_2 \mid \leq \varepsilon$ and $\|z_1(t_1) - z_2(t_2)\| \leq \varepsilon$. We define the ε-expansion of z_1 by: $\mathcal{B}_\varepsilon(z_1) = \{z' : E' \to Z \mid z'$ is ε-close to $z\}$.*

This notion is generalized toward sets of continuous-time maps in the usual way: For $A \subseteq M_c(Z)$, $\mathcal{B}_\varepsilon(A) = \bigcup\limits_{z \in A} \mathcal{B}_\varepsilon(z)$.

Proposition 2 *Let $\Sigma = (W_1, W_2, X, Y, \mathcal{T})$ be a continuous-time system and let $\mathcal{C} = (A_{W_1}, A_{W_2}, G_X, G_Y)$ be an assume-guarantee contract for Σ. Let us assume that for all trajectories $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y \in \mathcal{T}$, $y : E \to Y$ is continuous and $y_{|[0,0]} \in G_Y$. If $\Sigma \models \mathcal{C}$, then for all $\varepsilon > 0$, $\Sigma \models_s \mathcal{C}_\varepsilon$ where $\mathcal{C}_\varepsilon = (A_{W_1}, A_{W_2}, G_X, \mathcal{B}_\varepsilon(G_Y) \cap M_c(Y))$.*

PROOF. Let $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y \in \mathcal{T}$, then $y_{|[0,0]} \in G_Y \subseteq \mathcal{B}_\varepsilon(G_Y) \cap M_c(Y)$. Let $t \in E$, such that $w_{1|[0,t]} \in A_{W_1}$ and $w_{2|[0,t]} \in A_{W_2}$. Then, satisfaction of $\mathcal{C}$ gives that $x_{|[0,t]} \in G_X$ and $y_{|[0,t]} \in G_Y$. By continuity of y, there exists $\delta > 0$ such that for all $s \in [0, \delta]$, $y_{|[0,t+s] \cap E} \in \mathcal{B}_\varepsilon(G_Y)$. Also by definition, $y_{|[0,t+s] \cap E} \in M_c(Y)$ for all $s \in [0, \delta]$. Hence, $y_{|[0,t+s] \cap E} \in \mathcal{B}_\varepsilon(G_Y) \cap M_c(Y)$, for all $s \in [0, \delta]$, which ends the proof. $\Box$

The following example shows an application of the previous corollary:

Example 10 *Let the system $\Sigma^1 = (W_1, W_2, X, Y, \mathcal{T})$ where $W_1 = W_2 = X = Y = \mathbb{R}_0^+$. A trajectory of Σ^1 is a triple $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ in $\mathcal{T}$ where $E = \mathbb{R}_0^+$, w_1 and w_2 are continuous, x and y are differentiable and such that $x(0) = 0$, and for all $t \in \mathbb{R}_0^+$,*

$$\begin{cases} \dot{x}(t) = \sqrt{w_2(t)} - x(t) + w_1(t) \\ y(t) = x(t). \end{cases}$$

Let $I = \{1\}$ and let the interconnection relation $\mathcal{I} = \{(1,1)\}$. It is clear that $\{\Sigma^i\}_{i \in I}$ is compatible for composition w.r.t. $\mathcal{I}$. Let $a > 1$ and let us consider the assume-guarantee contract $\mathcal{C} = (A_{W_1}, A_{W_2}, G_X, G_Y)$ for Σ^1, given by:

$$\begin{aligned} A_{W_1} &= \{w_1 : E \to W_1 \in M_c(W_1) \mid \forall t \in E, w_1(t) = 0\} \\ A_{W_2} &= \{w_2 : E \to W_2 \in M_c(W_2) \mid \forall t \in E, w_2(t) \in [0, a^2]\} \\ G_X &= \{x : E \to X \in M_c(X) \mid \forall t \in E, x(t) \in [0, a]\} \\ G_Y &= \{y : E \to Y \in M_c(Y) \mid \forall t \in E, y(t) \in [0, a]\} \end{aligned}$$

We can easily check that $\Sigma^1 \models \mathcal{C}$ and for all trajectories $(w_1, w_2, x, y) \in \mathcal{T}$, $y : E \to Y$ is continuous and $y_{|[0,0]} \in G_Y$. Then, from Proposition 2, we have that $\Sigma \models_s \mathcal{C}_\varepsilon$ for any $\varepsilon > 0$, where $\mathcal{C}_\varepsilon = (A_{W_1}, A_{W_2}, G_X, \mathcal{B}_\varepsilon(G_Y) \cap M_c(Y))$. Now let $\varepsilon > 0$, such that $\mathcal{B}_\varepsilon(G_Y) \cap M_c(Y) = \{y : E \to Y \in M_c(Y) \mid \forall t \in E, y(t) \in [0, a + \varepsilon]\} \subseteq \{y : E \to Y \in M_c(Y) \mid \forall t \in E, y(t) \in [0, a^2]\} = A_{W_2}$. Then, since the system Σ^1 satisfies Assumption 1 (the output trajectory $y : E \to Y$ is continuous and the set $[0, a]$ is closed), we have from Theorem 3 that the composed system $\Gamma = \langle (\Sigma^i)_{i \in I}, \mathcal{I} \rangle$ satisfies the composed contract $\mathcal{C}_\Gamma = (A_{W_1}, G_X, \mathcal{B}_\varepsilon(G_Y) \cap M_c(Y))$. Let us remark that there exists trajectories of the composed system Γ given by: $(w_1, x, y) : \mathbb{R}_0^+ \to W_1 \times X \times Y$, where $w_1(t) = 0$ and $x(t) = y(t) = (1 - e^{-t/2})^2$, for all $t \in \mathbb{R}_0^+$.

We have shown how one can go from weak to strong satisfaction of a contract, by relaxing the guarantees on the output. In the next result, we show that it is also possible to do so by relaxing the assumptions.

Proposition 3 *Let $\Sigma = (W_1, W_2, X, Y, \mathcal{T})$ be a continuous-time system and let $\mathcal{C} = (A_{W_1}, A_{W_2}, G_X, G_Y)$ be an assume-guarantee contract for Σ. Let us assume that for all trajectories $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y \in \mathcal{T}$, $w_1 : E \to W_1$ and $w_2 : E \to W_2$ are continuous and $y_{|[0,0]} \in G_Y$. For an $\varepsilon > 0$, if $\Sigma \models \mathcal{C}^\varepsilon$, with $\mathcal{C}^\varepsilon = (\mathcal{B}_\varepsilon(A_{W_1}) \cap M_c(W_1), \mathcal{B}_\varepsilon(A_{W_2}) \cap M_c(W_2), G_X, G_Y)$. Then, $\Sigma \models_s \mathcal{C}$.*

PROOF. Let $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y \in \mathcal{T}$, then $y_{|[0,0]} \in G_Y$. Let $t \in E$, such that $w_{1|[0,t]} \in A_{W_1}$ and $w_{2|[0,t]} \in A_{W_2}$. By continuity of w_1 and w_2, there exists $\delta > 0$ such that for all $s \in [0, \delta]$, $w_{1|[0,t+s] \cap E} \in \mathcal{B}_\varepsilon(A_{W_1})$ and $w_{2|[0,t+s] \cap E} \in \mathcal{B}_\varepsilon(A_{W_2})$. Also by definition, $w_{1|[0,t+s] \cap E} \in M_c(W_1)$ and $w_{2|[0,t+s] \cap E} \in M_c(W_2)$ for all $s \in [0, \delta]$. Then, satisfaction of $\mathcal{C}^\varepsilon$ gives that $x_{|[0,t+s]} \in G_X$ and $y_{|[0,t+s]} \in G_Y$ for all $s \in [0, \delta]$. Which ends the proof. $\Box$

Remark 7 *We recall that this approach to ensure strong satisfaction of contract has been used in [28] to construct symbolic controllers for sampled-data systems. Interestingly, this technique is useful in practice, since it allows to ensure strong satisfaction of contracts without reasoning in terms of δ which may depend on time and trajectory.*

4.4 Robustness of assume-guarantee contracts

In real applications of control theory, state measurements are not perfect, they are generally subject to measurement errors. The objective of this section is to show that, the concept of assume guarantee contracts is robust against imperfect state measurements. The use of such type of measurement errors is just for the sake of illustration, the robustness of the assume-guarantee framework can be extended to deal with different types of errors introduced for example by time-delays or unmodeled dynamics.

Let a continuous-time system $\Sigma = (W_1, W_2, X, Y, \mathcal{T})$ where $X = Y$ and for any trajectory $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y \in \mathcal{T}$, we have $x(t) = y(t)$ for all $t \in E$. Let the measured system $\hat{\Sigma} = (W_1, W_2, X, Y, \hat{\mathcal{T}})$, where $(\hat{w}_1, \hat{w}_2, \hat{x}, \hat{y}) : E \to W_1 \times W_2 \times X \times Y \in \hat{\mathcal{T}}$ if and only if there exists $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y \in \mathcal{T}$ such that for all $t \in E$, $\hat{w}_1(t) = w_1(t)$, $\hat{w}_2(t) = w_2(t)$, $\hat{x}(t) = x(t) + e(t)$ and $\hat{y}(t) = y(t) + e(t)$. Where $\hat{x}(t)$ is the state measurement received at t and $\mid e(t) \mid \leq \eta$ is a time varying measurement error bounded by η.

Given a system Σ and a contract $\mathcal{C}$, the following result provides the nature of the contract that needs to be satisfied by $\hat{\Sigma}$ in order to enforce the satisfaction of $\mathcal{C}$ by Σ. First, we introduce some notations.

Let $Z \subseteq \mathbb{R}^n$ and $A \subseteq M_c(Z)$. We define the $-\varepsilon$-expansion of A by: $\mathcal{B}_{-\varepsilon}(A) = \{z : E \to Z \mid \mathcal{B}_\varepsilon(z) \subseteq A\}$.

Proposition 4 *Let the systems Σ and $\hat{\Sigma}$ described above. Let $\mathcal{C} = (A_{W_1}, A_{W_2}, G_X, G_Y)$ be a contract for Σ and $\hat{\mathcal{C}} = (A_{W_1}, A_{W_2}, \mathcal{B}_{-\eta}(G_X), \mathcal{B}_{-\eta}(G_Y))$ a contract for $\hat{\Sigma}$. If $\hat{\Sigma} \models \hat{\mathcal{C}}$, then $\Sigma \models \mathcal{C}$. Similarly, if $\hat{\Sigma} \models_s \hat{\mathcal{C}}$ then $\Sigma \models_s \mathcal{C}$.*

PROOF. We provide the proof for the weak satisfaction only, but the proof for strong satisfaction can be derived similarly. Let us assume that $\hat{\Sigma} \models \hat{\mathcal{C}}$. Let $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y \in \mathcal{T}$ and assume that for all $t \in E$, $w_{1|[0,t]} \in A_{W_1}$ and $w_{2|[0,t]} \in A_{W_1}$. Hence, $\hat{w}_{1|[0,t]} \in A_{W_1}$ and $\hat{w}_{2|[0,t]} \in A_{W_1}$. Since $\hat{\Sigma} \models \hat{\mathcal{C}}$, we have that $\hat{x}_{|[0,t]} \in \mathcal{B}_{-\eta}(G_X)$ and $\hat{y}_{|[0,t]} \in \mathcal{B}_{-\eta}(G_Y)$. Then, $\mathcal{B}_\eta(\hat{x}_{|[0,t]}) \subseteq G_X$ and $\mathcal{B}_\eta(\hat{y}_{|[0,t]}) \subseteq G_Y$. Hence, $x_{|[0,t]} \in \mathcal{B}_\eta(\hat{x}_{|[0,t]}) \subseteq G_X$ and $y_{|[0,t]} \in \mathcal{B}_\eta(\hat{y}_{|[0,t]}) \subseteq G_Y$. Which ends the proof. □

Given a system Σ and an assume-guarantee contract $\mathcal{C}$, if the objective is to synthesize a controller for Σ enforcing the satisfaction (or strong satisfaction) of the contract [2] $\mathcal{C}$, and if the state measurements are not perfect, one can synthesize a controller for the measured system $\hat{\Sigma}$ enforcing the satisfaction of the contract $\hat{\mathcal{C}}$. Then, in view of Proposition 4, the real system Σ will satisfy the contract $\mathcal{C}$.

5 Compositional invariants for differential inclusions

In this section, we focus on continuous-time systems $\Sigma = (W_1, W_2, X, Y, \mathcal{T})$ defined by differential inclusions, and invariance assume-guarantee contracts, where assumptions and guarantees are defined as in Example 1. We use the classical characterization of invariant sets for differential inclusions developed using the concept of contingent cone (see [5] and the references therein) to derive necessary and sufficient conditions for weak satisfaction of assume-guarantee contracts. We also show that under some technical assumptions (Lipschitzness of the vector field and the output map), weak satisfaction makes it possible to reason on general interconnections containing cycles.

A trajectory of Σ is a triple $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ in $\mathcal{T}$ where $E \in \mathbb{E}(\mathbb{R}_0^+)$, w_1 and w_2 are locally measurable, x and y are absolutely continuous and continuous, respectively, and satisfy for almost all $t \in E$:

$$\begin{cases} \dot{x}(t) \in F(x(t), w_1(t), w_2(t)), \; x(0) \in X_0 \\ y(t) = h(x(t)) \end{cases} \tag{9}$$

where $F : \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \rightrightarrows \mathbb{R}^n$ is a set-valued map, $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuous and X_0 is the set of initial conditions. Let us introduce the following assumption on the system Σ:

Assumption 2 *The set-valued map [3] $F : \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \rightrightarrows \mathbb{R}^n$ is Lipschitz, has compact values and $X \times W_1 \times W_2 \subseteq Int(dom(F))$. The map [4] $h : \mathbb{R}^n \to \mathbb{R}^p$ satisfies $X \subseteq Int(dom(h))$ and $h(X) \subseteq Y$.*

[2] See [28] for an illustration to the synthesis of a controller enforcing the satisfaction of an invariance assume-guarantee contract.

[3] Given a set-valued map $F : \mathbb{R}^q \rightrightarrows \mathbb{R}^n$, the domain of F is $dom(F) = \{z \in \mathbb{R}^q | \; F(z) \neq \emptyset\}$. F is said to be locally Lipschitz if for all $z \in Int(dom(F))$, there exists a neighborhood U of z and a constant $L \geq 0$ (the Lipschitz constant) such that for every $z_1, z_2 \in U \cap dom(F)$, $F(z_1) \subseteq F(z_2) + L||z_1 - z_2||\mathbb{B}$. F is said to be Lipschitz if the constant L is independent of $z \in Int(dom(F))$. It has compact values if for all $z \in dom(F)$, $F(z)$ is compact.

[4] Given a map $h : \mathbb{R}^n \to \mathbb{R}^p$, the domain of h is denoted $dom(h)$ and consists of elements $x \in \mathbb{R}^n$ such that $h(x)$ is defined.

Assumption 3 *A contract $\mathcal{C} = (A_{W_1}, A_{W_2}, G_X, G_Y)$ is an invariance contract, where the sets of assumptions and guarantees are described as follows:*

- *For $S_{W_i} \subseteq \mathbb{R}^{m_i}$, $A_{W_i} = \{w_i : E \to \mathbb{R}^{m_i} \in M_c(\mathbb{R}^{m_i}) \mid \forall t \in E,\ w_i(t) \in S_{W_i}\}$, $i \in \{1,2\}$;*
- *For $S_X \subseteq \mathbb{R}^n$, $G_X = \{x : E \to \mathbb{R}^n \in M_c(\mathbb{R}^n) \mid \forall t \in E,\ x(t) \in S_X\}$;*
- *For $S_Y \subseteq \mathbb{R}^p$, $G_Y = \{y : E \to \mathbb{R}^p \in M_c(\mathbb{R}^p) \mid \forall t \in E,\ y(t) \in S_Y\}$;*

Let a network of components $\{\Sigma^i\}_{i\in I}$, compatible for composition w.r.t. $\mathcal{I}$, where each component have the form of (9). Each component Σ^i have maps and initial sets F^i, h^i, X_0^i, $i \in I$, the composed system $\Gamma = \langle(\Sigma^i)_{i\in I}, \mathcal{I}\rangle$ can be written under the same form with maps F, h and initial set X_0 given by:

$$\begin{aligned} F(x,w^1) &= \textstyle\prod_{i\in I} F^i(x^i,w_1^i,w_2^i), \quad w_2^i = \prod_{j\in\mathcal{N}(i)}\{h^j(x^j)\} \\ h(x) &= (h^1(x^1),\dots,h^N(x^N)), \\ X_0 &= \textstyle\prod_{i\in I} X_0^i. \end{aligned}$$

Note that this representation is consistent with the one given in Definition 5.

The following technical result is straightforward and is stated without proof:

Claim 2 *If h_i is Lipschitz and Assumption 2 holds for all Σ_i, $i \in I$, then it holds for $\Gamma = \langle(\Sigma^i)_{i\in I}, \mathcal{I}\rangle$;*

5.1 Invariants relative to assume-guarantee contracts

We give necessary and sufficient conditions for weak satisfaction of assume-guarantee contracts based on the classical characterization of invariant sets for differential inclusions (see e.g. Theorem 5.3.4 in [5]).

Definition 9 *Let $K \subseteq \mathbb{R}^n$ and $x \in K$, the contingent cone to set K at point x, denoted $T_K(x)$, is given by:*

$$T_K(x) = \left\{ z \in \mathbb{R}^n \mid \liminf_{h\to 0^+} \frac{d_K(x+hz)}{h} = 0 \right\}$$

where $d_K(y)$ denotes the distance of y to K, defined by $d_K(y) = \inf_{y'\in K} ||y - y'||$.

Definition 10 *Let $\Sigma = (W_1, W_2, X, Y, \mathcal{T})$ be a continuous-time system described by (9). Let $\mathcal{C} = (A_{W_1}, A_{W_2}, G_X, G_Y)$ be an invariance assume-guarantee contract for Σ satisfying Assumption 3, where the sets S_{W_1}, S_{W_2} are compact. A closed set $K \subseteq X$ is said to be an invariant of Σ relative to the contract $\mathcal{C}$ if the following conditions hold:*

(i) $X_0 \subseteq K \subseteq S_X \cap h^{-1}(S_Y)$;

(ii) for all $x \in K$, $F(x, S_{W_1}, S_{W_2}) \subseteq T_K(x)$.

where the set-valued map is given by: $F(., S_{W_1}, S_{W_2}) = \bigcup_{w_1\in S_{W_1}} \bigcup_{w_2\in S_{W_2}} F(., w_1, w_2)$.

We prove that the existence of an invariant of Σ relative to a contract $\mathcal{C}$ is equivalent to the weak satisfaction of this contract.

Proposition 5 *Let $\Sigma = (W_1, W_2, X, Y, \mathcal{T})$ be a continuous-time system described by (9) such that Assumption 2 holds. Let $\mathcal{C} = (A_{W_1}, A_{W_2}, G_X, G_Y)$ be an invariance assume-guarantee contract for Σ satisfying Assumption 3, where the sets S_{W_1}, S_{W_2} are compact. Then, $\Sigma \models \mathcal{C}$, if and only if there exists a closed set $K \subseteq X$ invariant of Σ relative to the contract $\mathcal{C}$*

PROOF. First let us prove that the existence of an invariant of Σ relative to a contract $\mathcal{C}$ implies the weak satisfaction of this contract. Let $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ in $\mathcal{T}$. Let $t \in E$ and suppose that $w_{i|[0,t]} \in A_{W_i}$, $i \in \{1,2\}$. Then, we have for all $s \in [0,t]$, $w_i(s) \in S_{W_i}$, then for almost all $s \in [0,t]$, $\dot{x}(s) \in F(x(s), S_{W_1}, S_{W_2})$. From Assumption 2, we have $X \subseteq Int(dom(F(., S_{W_1}, S_{W_2})))$ and then $K \subseteq Int(dom(F(., S_{W_1}, S_{W_2})))$. Moreover, from the compactness of S_{W_i}, $i \in \{1,2\}$, it follows that the set-valued map $F(., S_{W_1}, S_{W_2})$ is Lipschitz and has compact values. Then, since for all $x \in K$, $F(x, S_{W_1}, S_{W_2}) \subseteq T_K(x)$, we have by Theorem 5.3.4 in [5] that for all $s \in [0,t]$, $x(s) \in K \subseteq S_X$ and then for all $s \in [0,t]$, $y(s) = h(x(s)) \in h(K) \subseteq S_Y$. Then, $x_{|[0,t]} \in G_X$ and $y_{|[0,t]} \in G_Y$. Hence, $\Sigma \models \mathcal{C}$.

We now deal with the second implication. Let us assume that $\Sigma \models \mathcal{C}$. Then for any trajectory $(w_1, w_2, x, y) : E \to W_1 \times W_2 \times X \times Y$ of the system Σ. We have for all $t \in E$, if for all $s \in [0,t]$, $w_1(s) \in S_{W_1}$ and $w_2(s) \in S_{W_1}$, then for all $s \in [0,t]$, $x(s) \in S_X$ and $y(s) \in S_Y$. Let us prove the existence of a non empty set $K \subseteq X$ satisfying the conditions of Definition 10. Let us define

$$\begin{aligned} \tilde{K} = \{p \in X \mid \exists (w_1, w_2, x, y) : E \to S_{W_1} \times S_{W_2} \times X \times Y \in \mathcal{T} \\ \text{with } x(0) \in X_0 \text{ and } \exists t \in E \text{ with } x(t) = p\}. \end{aligned} \quad (10)$$

The set $\tilde{K}$ is the set of reachable states for the differential inclusion (9) initialized in X_0, where the external and internal inputs belongs to S_{W_1} and S_{W_2}, respectively. From the satisfaction of the contract, we have that $X_0 \subseteq \tilde{K} \subseteq S_X \cap h^{-1}(S_Y)$. Let $(w_1', w_2') \in S_{W_1} \times S_{W_2}$ and let us prove that $\tilde{K}$ is an invariant for the differential inclusion

$$\dot{x}(t) \in F(x(t), w_1', w_2'). \quad (11)$$

Let $z^0 \in \tilde{K}$, and let $z : E' \to X$ be a solution of (11) with $z(0) = z^0$. Since $z^0 \in \tilde{K}$, we have the existence of a trajectory $\sigma = (w_1, w_2, x, y) : [0, s] \to S_{W_1} \times S_{W_2} \times X \times Y$ of the system Σ described in (9) such that $x(0) \in X_0$ and $x(s) = z^0$ and for which the external and internal inputs belong to S_{W_1} and S_{W_2}, respectively. Let the time domain E_c defined as follows:

$$\begin{cases} E^c &= [0, a+s] \text{ if } E' = [0, a] \\ &= [0, a+s) \text{ if } E' = [0, a) \\ &= \mathbb{R}_0^+ \text{ if } E' = \mathbb{R}_0^+ \end{cases}$$

and let the trajectory $\sigma^c = (w_1^c, w_2^c, x^c, y^c) : E^c \to S_{W_1} \times S_{W_2} \times X \times Y$ of the system Σ defined as follows: for all $t \in [0, s]$, $\sigma(t) = \sigma^c(t)$ and for all $t \in E^c \setminus [0, s]$ we have, $x^c(t) = z(t-s)$, $y^c(t) = h(x^c(t))$, $w_1^c(t) = w_1'$ and $w_2^c(t) = w_2'$. From construction of $\tilde{K}$, we have that $x(t) \in \tilde{K}$ for all $t \in E^c$. Hence, for all $t \in E'$, $z(t') = x(t'+s) \in \tilde{K}$, where $t'+s \in E^c$. Hence, $\tilde{K}$ is an invariant for the differential inclusion (11). Let us now prove that $K = \mathrm{cl}(\tilde{K})$ is also an invariant for (11). Let $v^0 \in K$, and let assume the existence of $v : E \to X$ solution to (11) with $v(0) = v^0$ and $s \in E$ such that $v(s) \in \overline{K}$. Since, $\overline{K}$ is an open, we have the existence of $\varepsilon > 0$ such that $\mathcal{R}_\varepsilon(v(s)) \subseteq \overline{K}$. Then, using the continuity of solutions of (11) in initial conditions (see Corollary 5.3.3 in [5]), we have the existence of $\eta > 0$ and $x^0 \in \tilde{K}$ such that $x^0 = x(0) \in \mathcal{R}_\eta(v^0)$ and $x(s) \in \mathcal{R}_\varepsilon(v(s)) \in \overline{K}$, which contradicts the invariance of $\tilde{K}$. Hence, $K = \mathrm{cl}(\tilde{K})$ is an invariant for the differential inclusion (11).

For $(w_1', w_2') \in S_{W_1} \times S_{W_2}$, we have that K is closed. Moreover by Claim 2, F and thus $F(., w_1', w_2')$ is Lipschitz and has compact values. Moreover, $X \times W_1 \times W_2 \subseteq Int(dom(F))$ and thus $X \subseteq Int(F(., w_1', w_2'))$, which in turn implies that $K \subseteq Int(F(., w_1', w_2'))$. Then, from Theorem 5.3.4 in [5], we have

$$\forall x \in K,\ F(x, w_1', w_2') \subseteq T_K(x). \tag{12}$$

Since equation (12) is verified for all $(w_1', w_2') \in S_{W_1} \times S_{W_2}$, we have for all $x \in K$, $F(., S_{W_1}, S_{W_2}) = \bigcup_{w_1' \in S_{W_1}} \bigcup_{w_2' \in S_{W_2}} F(., w_1', w_2') \subseteq T_K(x)$. Then, K is an invariant of the system Σ relative to the contract $\mathcal{C}$. $\square$

Remark 8 *Let us remark that in view of Proposition 5, the Lipschitzness property of the system Σ is needed only on a neighbourhood of the set of interest given by $G_X \times S_{W_1} \times S_{W_2}$.*

5.2 Composition of invariants

We now provide results allowing us to reason about interconnected systems based on invariants of their components.

Theorem 4 (Invariants under composition) *Let a network of components $\{\Sigma^i\}_{i \in I}$ compatible for composition w.r.t. $\mathcal{I}$, where each component have the form of (9) and satisfies Assumption 2. Each component Σ^i have maps and initial sets F^i, h^i, X_0^i, $i \in I$. Let $\mathcal{C}^i = (A_{W_1^i}, A_{W_2^i}, G_{X^i}, G_{Y^i})$ be an invariance assume-guarantee contract for Σ^i satisfying Assumption 3, where the sets $S_{W_1^i}, S_{W_2^i}$ are compact. Let $\mathcal{C} = (\prod_{i \in I} A_{W_1^i}, \prod_{i \in I} G_{X^i}, \prod_{i \in I} G_{Y^i})$ be a contract for the composed system $\Gamma = \langle (\Sigma^i)_{i \in I}, \mathcal{I} \rangle$. Let us assume the following:*

(i) for all $i \in I$, there exists a closed set $K^i \subseteq X^i$ invariant of Σ^i relative to the contract $\mathcal{C}^i$;
(ii) for all $i \in I$, $\prod_{j \in \mathcal{N}(i)} S_{Y^j} \subseteq S_{W_2^i}$.

then $K = \prod_{i \in I} K^i$ is an invariant of Γ relative to the contract $\mathcal{C}$.

PROOF. We first prove that the closed set K is an invariant for the differential inclusion:

$$\dot{x}(t) \in F(x(t), S_{W_1}). \tag{13}$$

Where $x(t) = (x_1(t), \dots, x_N(t))$ and $S_{W_1} = \prod_{i \in I} S_{W_1^i}$. Let $x \in K$, then for all $i \in I$, we have that

$$\begin{aligned} F^i(x^i, S_{W_1^i}, w_2^i) &= F^i(x^i, S_{W_1^i}, \prod_{j \in \mathcal{N}(i)} \{h^j(x^j)\}) \\ &\subseteq F^i(x^i, S_{W_1^i}, \prod_{j \in \mathcal{N}(i)} \{h^j(K^j)\}) \\ &\subseteq F^i(x^i, S_{W_1^i}, \prod_{j \in \mathcal{N}(i)} \{S_{Y^j}\}) \\ &\subseteq F^i(x^i, S_{W_1^i}, S_{W_2^i}). \end{aligned}$$

Where the first equality comes from the definition of an interconnection relation, the second inclusion comes from (i) and the last inclusion comes from (ii). Following the same line as in the proof of Proposition 5, we can show that $x_i(t) \in K_i$ for all $t \in E$. Hence, for all $t \in E$, $x(t) \in K$, which is therefore an invariant of the differential inclusion (13).

Since K_i, $i \in I$ is closed, so is K. Moreover, by Claim 2 and compactness of S_{W_1}, F and thus $F(., S_{W_1})$ is Lipchitz and has compact values. Moreover, $X \times W_1 \subseteq Int(dom(F))$ and thus $X \subseteq Int(F(., S_{W_1}))$, which in turn implies that $K \subseteq Int(F(., S_{W_1}))$. Then, from Theorem 5.3.4 in [5], we have

$$\forall x \in K,\ F(x, S_{W_1}) \subseteq T_K(x).$$

Finally, we have $X_0 = \prod_{i \in I} X_0^i \subseteq \prod_{i \in I} K^i = K$. Moreover, $K = \prod_{i \in I} K^i \subseteq \prod_{i \in I} G_{X^i}$, and $K = \prod_{i \in I} K^i \subseteq$

$\prod_{i\in I}(h^i)^{-1}(G_{Y^i}) = h^{-1}(G_Y)$. Hence, K is an invariant of Γ relative to the contract $\mathcal{C}$. $\square$

Let us remark that the previous result can also be stated in terms of weak satisfaction of contracts, as shown in the next corollary. The proof follows immediately from the equivalence between the invariance relative to contracts and the weak satisfaction of contracts (see Proposition 5).

Corollary 1 *Let a network of components $\{\Sigma^i\}_{i\in I}$ compatible for composition w.r.t. $\mathcal{I}$, where each component have the form of (9) and satisfies Assumption 2. Each component Σ^i have maps and initial sets F^i, h^i, X_0^i, $i \in I$. Let $\mathcal{C}^i = (A_{W_1^i}, A_{W_2^i}, G_{X^i}, G_{Y^i})$ be an invariance assume-guarantee contract for Σ^i satisfying Assumption 3, where the sets $S_{W_1^i}, S_{W_2^i}$ are compact. Let $\mathcal{C} = (\prod_{i\in I} A_{W_1^i}, \prod_{i\in I} G_{X^i}, \prod_{i\in I} G_{Y^i})$ be a contract for the composed system $\Gamma = \langle(\Sigma^i)_{i\in I}, \mathcal{I}\rangle$. Let us assume the following:*

(i) for all $i \in I$, $\Sigma^i \models \mathcal{C}^i$;
(ii) for all $i \in I$, $\prod_{j\in\mathcal{N}(i)} S_{Y^j} \subseteq S_{W_2^i}$.

then $\Gamma \models \mathcal{C}$.

We show an example to illustrate the application of the previous theorem.

Example 11 *Consider systems $\Sigma^i = (W_1^i, W_2^i, X_i, Y_i, \mathcal{T}_i)$, $i = 1, 2$ where $W_1^i = W_2^i = X^i = Y^i = \mathbb{R}$. A trajectory of Σ^i is a triple $(w_1^i, w_2^i, x^i, y^i) : E \to W_1^i \times W_2^i \times X^i \times Y^i$ in $\mathcal{T}_i$ where $E = \mathbb{R}_0^+$, w_1^i and w_2^i are locally measurable, x^i and y^i are absolutely continuous and continuous, respectively, and satisfy for almost all $t \in E$:*

$$\begin{cases} \dot{x}^i(t) = f^i(x^i(t), w_1^i(t), w_2^i(t)) \\ \qquad = -a_i x^i(t) + a_i w_2^i(t) + w_1^i(t), \\ y^i(t) = h^i(x(t)) = x^i(t). \end{cases}$$

where $x_i(0) \in [0, b_i]$ with $a_i, b_i \in \mathbb{R}_0^+$, let $b = \max(b_1, b_2)$. Let us remark that h_i is Lipschitz and that Assumption 2 holds for Σ_i. Let the interconnection relation $\mathcal{I} = \{(1,2),(2,1)\}$. It is clear that $\{\Sigma^i\}_{i\in I}$ is compatible for composition w.r.t. $\mathcal{I}$. Let the contract $\mathcal{C}^i = (A_{W_1^i}, A_{W_2^i}, G_{X^i}, G_{Y^i})$ for the system Σ^i satisfying Assumption 3 with $S_{W_1^i} = \{0\}$, $S_{W_2^i} = S_{X^i} = S_{Y^i} = [0,b]$. We can easily check that for all $x^i \in [0,b]$, $f_i(x^i, [0,b], \{0\}) \subseteq T_{[0,b]}(x^i)$, since

$$T_{[0,b]}(x^i) = \begin{cases} \mathbb{R}^+ & \text{if } x^i = 0, \\ \mathbb{R}^- & \text{if } x^i = b, \\ \mathbb{R} & \text{if } x^i \in (0,b) \end{cases}$$

Then $[0,b]$ is an invariant of the system Σ_i, relative to the contract $\mathcal{C}^i$. By Theorem 4, $[0,b]^2$ is an invariant of the composed system $\Gamma = \langle(\Sigma^i)_{i\in I}, \mathcal{I}\rangle$ relative to the composed contract $\mathcal{C}$.

6 Small gain results

In this part, we show how the proposed framework can recover different versions of the classical small gain theorem as a particular case. Indeed, we show how the framework allows to recover the classical BIBO stability result [11]. Moreover, we construct a new small-gain result for the concept of growth bound [3]. To the best of our knowledge, this result is new and have not been investigated before in the literature. We suppose for the sake of simplicity that for each system $\Sigma = (W_1, W_2, X, Y, \mathcal{T})$, we have $W_1 = \{0\}$, $X = Y$ and for all $(w_1, w_2, x, y) : \mathbb{R}_0^+ \to W_1 \times W_2 \times W \times X \times Y$ in $\mathcal{T}$, $x(t) = y(t)$, for all $t \in \mathbb{R}_0^+$.

6.1 BIBO stability

Given a system Σ satisfying a BIBO stability condition [11], we show that if the gain of the system is lower than 1 then the feedback [5] composed system is bounded for all the time domain.

Theorem 5 *Let a system $\Sigma^1 = (\{0\}, W_2, X, Y, \mathcal{T})$, $I = \{1\}$ and let the interconnection relation $\mathcal{I} = \{(1,1)\}$ such that $\{\Sigma^i\}_{i\in I}$ is compatible for composition w.r.t. $\mathcal{I}$. Let $\gamma < 1$ and $\beta \in \mathbb{R}_0^+$ such that for any trajectory $(0, w_2, x, y) : \mathbb{R}_0^+ \to W_1 \times W_2 \times X \times Y$ in $\mathcal{T}$, $x : \mathbb{R}_0^+ \to X$ is continuous, $|x(0)| \le \frac{\beta}{1-\gamma}$ and for all $t \in \mathbb{R}_0^+$ we have:*

$$\|x_{|[0,t]}\|_\infty \le \gamma \|w_{2|[0,t]}\|_\infty + \beta. \tag{14}$$

Then for any trajectory $(0, x, y) : \mathbb{R}_0^+ \to \{0\} \times X \times Y$ of the composed system $\Gamma = \langle(\Sigma^i)_{i\in I}, \mathcal{I}\rangle$, we have for all $t \in \mathbb{R}_0^+$: $\|x_{|[0,t]}\|_\infty \le \frac{\beta}{1-\gamma}$.

PROOF. We first start by constructing a suitable contract for the system Σ^1. Let the map $a : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, a parameter $\varepsilon > 0$ and a parametrized contract $\mathcal{C}(\varepsilon) = (A_{W_1}^\varepsilon, A_{W_2}^\varepsilon, G_X^\varepsilon, G_Y^\varepsilon)$ for Σ, where:

- $A_{W_1}^\varepsilon = \{w_1 : \mathbb{R}_0^+ \to W_1 \in M_c(W_1) |\ \forall t \in \mathbb{R}_0^+, w_1(t) = 0\}$;
- $A_{W_2}^\varepsilon = \{w_2 : \mathbb{R}_0^+ \to W_2 \in M_c(W_2) |\ \forall t \in \mathbb{R}_0^+, \|w_{2|[0,t]}\|_\infty \le a(\varepsilon)\}$;

[5] Given a system Σ^1 and a set of vertices $I = \{1\}$, the feedback composition of the system Σ^1 is the composition with an interconnection relation $\mathcal{I} = \{(1,1)\}$.

- $G_X^\varepsilon = G_Y^\varepsilon = \{x : \mathbb{R}_0^+ \to X \in M_c(X) |\ \forall t \in \mathbb{R}_0^+, \|x_{|[0,t]}\|_\infty \leq \gamma a(\varepsilon) + \beta\}$.

Let us choose $a(\varepsilon) = \frac{\beta+\varepsilon}{1-\gamma}$, where $\varepsilon > 0$. We have that $|x(0)| \leq \frac{\beta}{1-\gamma} \leq \gamma a(\varepsilon) + \beta = \frac{\beta+\gamma\varepsilon}{1-\gamma}$, for any $\varepsilon > 0$. Hence, $x_{|[0,0]} \in G_X^\varepsilon$. We also have from (14) that $\Sigma^1 \models \mathcal{C}(\varepsilon)$, and for all trajectories $(w_1, w_2, x, y) \in \mathcal{T}$, $x : \mathbb{R}_0^+ \to X$ is continuous. Then, from Proposition 2, we have that $\Sigma^1 \models_s \mathcal{C}'(\varepsilon)$ for any $\varepsilon > 0$, where $\mathcal{C}'(\varepsilon) = (A_{W_1}^\varepsilon, A_{W_2}^\varepsilon, G_X^\varepsilon, \mathcal{B}_\varepsilon(G_Y^\varepsilon) \cap M_c(Y))$. Now, using the fact that $\gamma a(\varepsilon) + \beta - a(\varepsilon) = -\varepsilon < 0$, we have that $\mathcal{B}_\varepsilon(G_Y^\varepsilon) \cap M_c(Y) \subseteq A_{W_2}^\varepsilon$. Moreover, from continuity of $x : \mathbb{R}_0^+ \to X$ Assumption 1 is satisfied. Then from Theorem 3, the composed system $\Gamma = \langle(\Sigma^i)_{i\in I}, \mathcal{I}\rangle$ satisfies the composed contract $\mathcal{C}_\Gamma^\varepsilon = (A_{W_1}^\varepsilon, G_X^\varepsilon, \mathcal{B}_\varepsilon(G_Y^\varepsilon) \cap M_c(Y))$. Then, we have for all $t \in \mathbb{R}_0^+$: $\|x_{|[0,t]}\|_\infty \leq \gamma a(\varepsilon) + \beta = \frac{\beta+\gamma\varepsilon}{1-\gamma}$.

Since the last inequality is verified for all $\varepsilon > 0$ we have for all $t \in \mathbb{R}_0^+$, $\|x_{|[0,t]}\|_\infty \leq \frac{\beta}{1-\gamma}$. □

6.2 Growth bound

The notion of growth bound allows to analyse the growth or contraction properties of a system, particularly, this notion coincide with forward completeness (see Corollary 2.3 in [3]) for finite-dimensional systems described by nonlinear differential equations $\dot{x}(t) = F(x(t), w_1(t), w_2(t))$ and with a locally Lipschitz map F. Given a continuous-time system with a given growth bound, in the following we show how to characterize the growth bound of the feedback composed system.

Theorem 6 *Let a system $\Sigma^1 = (\{0\}, W_2, X, Y, \mathcal{T})$, $I = \{1\}$ and let the interconnection relation $\mathcal{I} = \{(1,1)\}$ such that $\{\Sigma^i\}_{i\in I}$ is compatible for composition w.r.t. $\mathcal{I}$. Let $\gamma_1, \gamma_2, \gamma_3$ be class $\mathcal{K}$ maps and a constant $c \in \mathbb{R}$, where $\gamma_3 < Id$ and such that for any trajectory $(0, w_2, x, y) : \mathbb{R}_0^+ \to W_1 \times W_2 \times X \times Y$ in $\mathcal{T}$, $x : \mathbb{R}_0^+ \to X$ is continuous, $|x(0)| \leq \gamma_2(|x(0)|) + c$ and for all $t \in \mathbb{R}_0^+$ we have:*

$$|x(t)| \leq \gamma_1(t) + \gamma_2(|x(0)|) + \gamma_3(\|w_{2|[0,t]}\|_\infty) + c. \quad (15)$$

Then there exist $\mathcal{K}$ functions α_1, α_2 and $c' \in \mathbb{R}$ such that for any trajectory $(0, x, y) : \mathbb{R}_0^+ \to \{0\} \times X \times Y$ of the composed system $\Gamma = \langle(\Sigma^i)_{i\in I}, \mathcal{I}\rangle$ we have for all $t \in \mathbb{R}_0^+$, $\|x_{|[0,t]}\|_\infty \leq \alpha_1(t) + \alpha_2(|x(0)|) + c'$.

PROOF. We first define the system $\Sigma_{x(0)}^1 = (\{0\}, W_2, X, Y, \mathcal{T}_{x(0)})$ where $(0, w_2, x, y) : \mathbb{R}_0^+ \to \{0\} \times W_2 \times X \times Y \in \mathcal{T}_{x(0)}$ is a trajectory of the system $\Sigma_{x(0)}^1$ is and only if it is a trajectory of the system Σ^1 initialized in $x(0) \in X$.

We start by constructing a suitable contract for the system $\Sigma_{x(0)}^1$. Let the map $a : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$. A parameter $\varepsilon > 0$ and a parametrized contract $\mathcal{C}(\varepsilon) = (A_{W_1}^\varepsilon, A_{W_2}^\varepsilon, G_X^\varepsilon, G_Y^\varepsilon)$ for Σ^1, where:

- $A_{W_1}^\varepsilon = \{w_1 : \mathbb{R}_0^+ \to W_1 \in M_c(W_1) |\ \forall t \in \mathbb{R}_0^+, w_1(t) = 0\}$;
- $A_{W_2}^\varepsilon = \{w_2 : \mathbb{R}_0^+ \to W_2 \in M_c(W_2) |\ \forall t \in \mathbb{R}_0^+, |w_2(t)| \leq a(t, \varepsilon)\}$;
- $G_X^\varepsilon = G_Y^\varepsilon = \{x : \mathbb{R}_0^+ \to X \in M_c(X) |\ \forall t \in \mathbb{R}_0^+, |x(t)| \leq \gamma_1(t) + \gamma_2(|x(0)|) + \gamma_3(|a(t, \varepsilon)|) + c\}$.

Let us choose the map $a : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying $a(t, \varepsilon) = (Id - \gamma_3)^{-1}(\gamma_1(t) + \gamma_2(|x(0)|) + c + \varepsilon)$, where $\varepsilon > 0$. Since γ_1 and $(Id - \gamma_3)^{-1}$ are class $\mathcal{K}$ maps (see [18]), we have for all $t \in \mathbb{R}_0^+$,

$$\|a_{|[0,t]}(., \varepsilon)\|_\infty = |a(t, \varepsilon)| \quad (16)$$

Let us now prove that $\Sigma^1 \models \mathcal{C}(\varepsilon)$. Let $t \in \mathbb{R}_0^+$ and assume that $|w_2(s)| \leq a(s, \varepsilon)$ for all $s \in [0, t]$. We have from (15) that for all $s \in [0, t]$

$$\begin{aligned} |x(s)| &\leq \gamma_1(s) + \gamma_2(|x(0)|) + \gamma_3(\|w_{2|[0,s]}\|_\infty) + c \\ &\leq \gamma_1(s) + \gamma_2(|x(0)|) + \gamma_3(\|a_{|[0,s]}(., \varepsilon)\|_\infty) + c \\ &\leq \gamma_1(s) + \gamma_2(|x(0)|) + \gamma_3(|a(s, \varepsilon)|) + c \end{aligned}$$

where the last inequality comes from (16). Hence, $\Sigma^1 \models \mathcal{C}(\varepsilon)$. Moreover we have that for all trajectories $(w_1, w_2, x, y) \in \mathcal{T}$, $x : \mathbb{R}_0^+ \to X$ is continuous and using the fact that $|x(0)| \leq \gamma_2(|x(0)|) + c$, we have that $x_{|[0,0]} \in G_X^\varepsilon$, for all $\varepsilon > 0$. Then, from Proposition 2, we have that $\Sigma^1 \models_s \mathcal{C}'(\varepsilon)$ for any $\varepsilon > 0$, where $\mathcal{C}'(\varepsilon) = (A_{W_1}^\varepsilon, A_{W_2}^\varepsilon, G_X^\varepsilon, \mathcal{B}_\varepsilon(G_Y^\varepsilon) \cap M_c(Y))$. Now, using the fact that $\gamma_1(t) + \gamma_2(|x(0)|) + \gamma_3(a(t, \varepsilon)) + c - a(t, \varepsilon) = -\varepsilon < 0$ we have that $\mathcal{B}_\varepsilon(G_Y^\varepsilon) \cap M_c(Y) \subseteq A_{W_2}^\varepsilon$. Moreover, from continuity of $x : \mathbb{R}_0^+ \to X$, Assumption 1 is satisfied. Then from Theorem 3, the composed system $\Gamma = \langle(\Sigma^i)_{i\in I}, \mathcal{I}\rangle$ satisfies the composed contract $\mathcal{C}_\Gamma^\varepsilon = (A_{W_1}^\varepsilon, G_X^\varepsilon, \mathcal{B}_\varepsilon(G_Y^\varepsilon) \cap M_c(Y))$. Then, we have for all $t \in \mathbb{R}_0^+$: $|x(t)| \leq \gamma_1(t) + \gamma_2(|x(0)|) + \gamma_3(a(t, \varepsilon)) + c \leq a(t, \varepsilon) = (Id - \gamma_3)^{-1}(\gamma_1(t) + \gamma_2(|x(0)|) + c + \varepsilon)$.

The last inequality is verified for all $\varepsilon > 0$, which implies from the continuity of $(Id - \gamma_3)^{-1}$ that for all $t \in \mathbb{R}_0^+$, $|x(t)| \leq (Id - \gamma_3)^{-1}(\gamma_1(t) + \gamma_2(|x(0)|) + c) \leq (Id - \gamma_3)^{-1}(2\gamma_1(t)) + (Id - \gamma_3)^{-1}(2\gamma_2(|x(0)|)) + (Id - \gamma_3)(2c)$, where the last inequality comes the fact that $Id - \gamma_3$ is a class $\mathcal{K}$ map (see the weak triangular inequality in [16]). By choosing $\alpha_1 = (Id - \gamma_3)^{-1} \circ (2\gamma_1)$ and $\alpha_2 = (Id - \gamma_3)^{-1} \circ (2\gamma_2)$, and $c' = (Id - \gamma_3)(2c)$ where α_1 and α_2 are class $\mathcal{K}$ (see Lemma 4.2 in [18]), we have for all $t \in \mathbb{R}_0^+$, $\|x_{|[0,t]}\|_\infty \leq \alpha_1(t) + \alpha_2(|x(0)|) + c'$. □

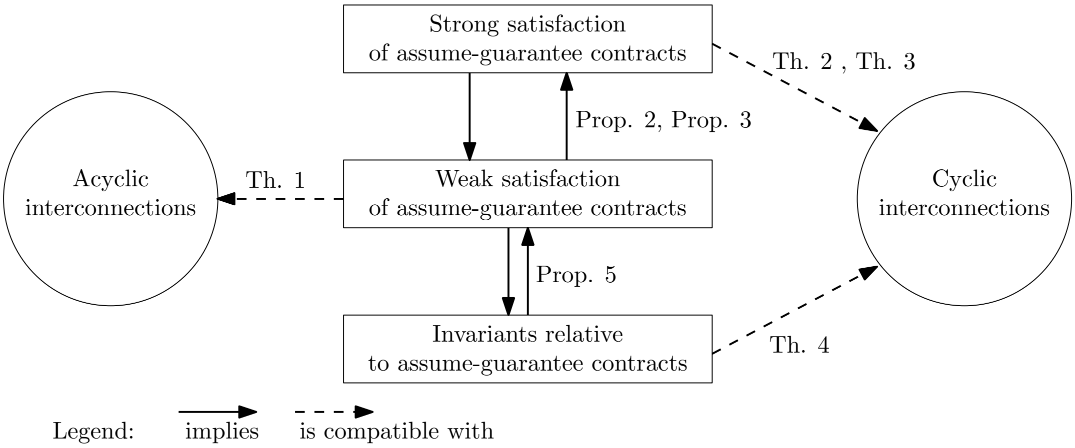

Fig. 2. Summary of main results in the paper

Remark 9 *Let us remark that for finite-dimensional systems, described by nonlinear differential equations and with a locally Lipschitz map F, the previous result states that if a system is forward complete with a gain γ_3 lower than identity, then the feedback composed system is forward complete.*

Remark 10 *Let us remark that for the particular case when $\gamma_1 = 0$, $\gamma_2 = 0$ and γ_3 is a linear map, the result of Theorem 6 can be seen as a generalisation of the BIBO small-gain result presented in Theorem 5.*

Remark 11 *The results presented for BIBO stability and forward completeness can be generalized using similar proofs to the cases of BIBO incremental stability [11] and incremental forward completeness [37].*

Remark 12 *Let us emphasize that using the same approach, and similar to the work of [10], one can generalize different small-gain results to different interconnection structures.*

7 Conclusion

In this paper, we proposed a contract based approach for verifying compositionally properties of discrete-time and continuous-time interconnected systems. The main notions considered in the paper and their relationships are sketched in Figure 2. The main contributions are summarized below. We introduced a notion of assume-guarantee contracts equipped with a weak and a strong semantics. We showed that weak semantics are sufficient to deal with acyclic interconnections (Theorem 1), strong semantics are required to reason on cyclic interconnections (Theorems 2,3 and Example 7) and that strong semantics of a contract can sometimes be obtained from weak ones (Propositions 2 and 3).

We then developed specific results for systems described by differential inclusions and invariance assume-guarantee contracts. We showed that sufficient and necessary conditions for weak satisfaction of contracts can be given using invariant sets (Proposition 5) and that invariants are compatible with cylic interconnections (Theorem 4). Finally, we have shown how the proposed assume-guarantee framework can recast different versions of the small-gain theorem as a particular case (Theorems 5 and 6).

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