CENTRAL-LIMIT THEOREM FOR CONSERVATIVE FRAGMENTATION CHAINS

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Abstract. We are interested in a fragmentation process. We observe fragments frozen when their sizes are less than ε ($\varepsilon > 0$). Is is known ([BM05]) that the empirical measure of these fragments converges in law, under some renormalization. In [HK11], the authors show a bound for the rate of convergence. Here, we show a central-limit theorem, under some assumptions.

1. Introduction

1.1. **Scientific and economic context.** One of the main goals in the mining industry is to extract blocks of metallic ore and then separate the metal from the valueless material. To do so, rock is fragmented into smaller and smaller rocks. This is carried out in a series of steps, the first one being blasting, after which the material goes through a sequence of crushers. At each step, the particles are screened, and if they are smaller than the diameter of the mesh of a classifying grid, they go to the next crusher. The process stops when the material has a sufficiently small size (more precisely, small enough to enable physicochemical processing).

This fragmentation process is energetically costly (each crusher consumes a certain quantity of energy to crush the material it is fed). One of the problems that faces the mining industry is that of minimizing the energy used. The optimisation parameters are the number of crushers and the technical specifications of these crushers.

In [BM05], the authors propose a mathematical model of what happens in a crusher. In this model, the rock pieces/fragments are fragmented independently of each other, in a random and auto-similar manner. This is consistent with what is observed in the industry, and this is supported by the following publications: [PB02, DM98, Wei85, Tur86]. Each fragment has a size s (in $\mathbb{R}^+$) and is then fragmented into smaller fragments of sizes s_1, s_2, … such that the sequence $(s_1/s, s_2/s, \dots)$ has a law ν which does not depend on s (which is why the fragmentation is said to be auto-similar). This law ν is called the *dislocation measure* (each crusher has its own dislocation measure). The dynamic of the fragmentation process is thus modelized in a stochastic way.

In each crusher, the rock pieces are fragmented repetitively until they are small enough to slide through a mesh whose holes have a fixed diameter. So the fragmentation process stops for each fragment when its size is smaller than the diameter of the mesh, which we denote by ε ($\varepsilon > 0$). We are interested in the *statistical distribution* of the fragments coming out of a crusher. If we renormalize the sizes of these fragments by dividing them by ε, we obtain a measure $\gamma_{-\log(\varepsilon)}$, which we call the *empirical measure* (the reason for the index $-\log(\varepsilon)$ instead of ε will be made clear later). In [BM05], the authors show that the energy consumed by the crusher to reduce the rock pieces to fragments whose diameters are smaller than ε can be computed as an integral of a bounded function against the measure $\gamma_{-\log(\varepsilon)}$ (they cite [Bon52, Cha57, WLMG67] on this particular subject). For each crusher, the empirical measure $\gamma_{-\log(\varepsilon)}$ is one of the two only observable variables (the other one being the size of the pieces pushed into the grinder). The specifications of a crusher are summarized in ε and ν.

1.2. **State of the art.** In [BM05], the authors show that the energy consumed by a crusher to reduce rock pieces of a fixed size into fragments whose diameter are smaller than ε behaves asymptotically like a power of ε when ε goes to zero. More precisely, this energy multiplied by a power of ε converges towards a constant of the form $\kappa = \nu(\varphi)$ (the integral of ν, the

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dislocation measure, against a bounded function φ). In [BM05], the authors also show a law of large numbers for the empirical measure $\gamma_{-\log(\varepsilon)}$. More precisely, if f is bounded continuous, $\gamma_{-\log(\varepsilon)}(f)$ converges in law, when ε goes to zero, towards an integral of f against a measure related to ν (this result also appears in [HK11], p. 399). We set $\gamma_\infty(f)$ to be this limit (check Equations (5.1), (2.5), (2.2) to get an exact formula). The empirical measure $\gamma_{-\log(\varepsilon)}$ thus contains information relative to ν and one could extract from it an estimation of κ or of an integral of any function against ν.

It is worth noting that by studying what happens in various crushers, we could study a family $(\nu_i(f_j))_{i\in I,j\in J}$ (with an index i for the number of the crusher and the index j for the j-th test function in a well-chosen basis). Using statistical learning methods, one could from there make a prediction for $\nu(f_j)$ for a new crusher for which we know only the mechanical specifications (shape, power, frequencies of the rotating parts ...). It would evidently be interesting to know ν before even building the crusher.

In [HKK10], the authors prove a convergence result for the empirical measure similar to the one in [BM05], the convergence in law being replaced by an almost sure convergence. In [HK11], the authors give a bound on the rate of this convergence, in a L^2 sense, under the assumption that the fragmentation is conservative. This assumption means there is no loss of mass due to the formation of dust during the fragmentation process.

$$\begin{array}{ccc} \gamma_{-\log(\varepsilon)} & \underset{\varepsilon\to 0}{\overset{\text{(bound on rate)}}{\longrightarrow}} & \gamma_\infty \\ & & \updownarrow \text{relation} \\ \text{energy} \times (\text{power of } \varepsilon) & \underset{\varepsilon\to 0}{\sim} & \kappa = \nu(\varphi) \end{array}$$

Figure 1.1. State of the art.

So we have convergence results ([BM05, HKK10]) of an empirical quantity towards constants of interest (a different constant for each test function f). Using some transformations, these constants could be used to estimate the constant κ. Thus it is natural to ask what is the exact rate of convergence in this estimation, if only to be able to build confidence intervals. In [HK11], we only have a bound on the rate.

When a sequence of empirical measures converges to some measure, it is natural to study the fluctuations, which often turn out to be Gaussian. For such results in the case of empirical measures related to the mollified Boltzmann equation, one can cite [Mel98, Uch88, DZ91]. When interested in the limit of a n-tuple as in Equation (1.1) below, we say we are looking at the convergence of a U-statistics. Textbooks deal with the case where the points defining the empirical measure are independent or with a known correlation (see [dlPG99, DM83, Lee90]). The problem is more complex when the points defining the empirical measure are in interaction with each other like it is the case here.

1.3. **Goal of the paper.** As explained above, we want to obtain the rate of convergence in the convergence of $\gamma_{-\log(\varepsilon)}$ when ε goes to zero. We want to produce a central-limit theorem of the kind: for a bounded continuous f, $\varepsilon^\beta(\gamma_{-\log(\varepsilon)}(f)-\gamma_\infty(f))$ converges towards a non-trivial measure when ε goes to zero (the limiting measure will in fact be Gaussian), for some exponent β. The technics used will allow us to prove the convergence towards a multivariate Gaussian of a vector of the kind

$$\varepsilon^\beta(\gamma_{-\log(\varepsilon)}(f_1)-\gamma_\infty(f_1),\dots,\gamma_{-\log(\varepsilon)}(f_n)-\gamma_\infty(f_n)) \tag{1.1}$$

for functions $f_1, \dots, f_n$.

More precisely, if by $Z_1, Z_2, \dots, Z_N$ we denote the fragments sizes that go out from a crusher (with mesh diameter equal to ε). We would like to show that for a bounded continuous f,

$$\gamma_{-\log(\varepsilon)}(f) := \sum_{i=1}^N Z_i f(Z_i) \longrightarrow \gamma_\infty(f), \text{ almost surely, when } \varepsilon\to 0\,,$$

and that for all n, and $f_1, \dots, f_n$ bounded continuous function such that $\gamma_\infty(f_i) = 0$,

$$\varepsilon^\beta(\gamma_{-\log(\varepsilon)}(f_1), \dots, \gamma_{-\log(\varepsilon)}(f_n))$$

converges in law towards a multivariate Gaussian when ε goes to zero.

The exact results are stated in Proposition 5.1 and Theorem 5.2.

1.4. **Outline of the paper.** We will state our assumptions along the way (Assumptions A, B, C, D). Assumption D can be found at the beginning of Section 3. We define our model in Section 2. The main idea is that we want to follow tags during the fragmentation process. Let us imagine the fragmentation is the process of breaking a stick (modeled by $[0, 1]$) into smaller sticks. We suppose that the original stick has painted dots and that during the fragmentation process, we take note of the sizes of the sticks supporting the painted dots (we call them the painted sticks). When the sizes of the painted sticks get smaller than ε ($\varepsilon > 0$), the fragmentation is stopped for these sticks. In Section 3, we make use of classical results on renewal processes and of [Sgi02] to show that the size of one painted stick has an asymptotic behavior when ε goes to zero and that we have a bound on the rate with which it reaches this behavior. Section 4 is the most technical. There we study the asymptotics of symmetric functionals of the sizes of the painted sticks (always when ε goes to zero). In Section 5, we precisely define the measure we are interested in (γ_T with $T = -\log(\varepsilon)$). Using the results of Section 4, it is then easy to show a law of large numbers for γ_T (Proposition 5.1) and a central-limit Theorem (Theorem 5.2). Proposition 5.1 and Theorem 5.2 are our two main results. The proof of Theorem 5.2 is based on a simple computation involving characteristic functions (the same technique was already used in [DPR09, DPR11a, DPR11b, Rub16]).

1.5. **Notations.** For x in $\mathbb{R}$, we set $\lceil x \rceil = \inf\{n \in \mathbb{Z} : n \geq x\}$, $\lfloor x \rfloor = \sup\{n \in \mathbb{Z} : n \leq x\}$. The symbol $\sqcup$ means "disjoint union". For n in $\mathbb{N}^*$, we set $[n] = \{1, 2, \dots, n\}$. For f an application from a set E to a set F, we write $f : E \hookrightarrow F$ if f is injective and, for k in $\mathbb{N}^*$, if $F = E$, we set

$$f^{\circ k} = \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}$$

2. Statistical model

2.1. **Fragmentation chains.** Let $\varepsilon > 0$. Like in [HK11], we start with the space

$$\mathcal{S}^\downarrow = \left\{ \mathbf{s} = (s_1, s_2, \dots),\ s_1 \geq s_2 \geq \cdots \geq 0,\ \sum_{i=1}^{+\infty} s_i \leq 1 \right\}.$$

A fragmentation chain is a process in $\mathcal{S}^\downarrow$ characterized by

- a dislocation measure ν which is a finite measure on $\mathcal{S}^\downarrow$,
- a description of the law of the times between fragmentations.

A fragmentation chain with dislocation measure ν is a Markov process $X = (X(t), t \geq 0)$ with values in $\mathcal{S}^\downarrow$. Its evolution can be described as follows: a fragment with size x lives for some time (which may or may not be random) then splits and gives rise to a family of smaller fragments distributed as $x\xi$, where ξ is distributed according to $\nu(.)/\nu(\mathcal{S}^\downarrow)$. We suppose the life-time of a fragment of size x is an exponential time of parameter $x^\alpha \nu(\mathcal{S}^\downarrow)$, for some α. We could here make different assumptions on the life-time of fragments, but this would not change our results.

We denote by $\mathbb{P}_m$ the law of X started from the initial configuration $(m, 0, 0, \dots)$ with m in $(0, 1]$. The law of X is entirely determined by α and $\nu(.)$ (Theorem 3 of [Ber02]).

We make the same assumption as in [HK11] and we will call it Assumption A.

Assumption A. *We have* $\nu(\mathcal{S}^\downarrow) = 1$ *and* $\nu(s_1 \in]0; 1[) = 1$.

Let

$$\mathcal{U} := \{0\} \cup \bigcup_{n=1}^{+\infty} (\mathbb{N}^*)^n$$

denote the infinite genealogical tree. For $u = (u_1, \dots, u_n) \in \mathcal{U}$ and $i \in \mathbb{N}^*$, we say that u is in the n-th generation and we write $|u| = n$, and we write $ui = (u_1, \dots, u_n, i)$, $u(k) = (u_1, \dots, u_k)$ for all

$k \in [n]$. For any $u = (u_1, \dots, u_n)$ and $v = ui$ ($i \in \mathbb{N}^*$), we say that u is the ancestor of v. For any u in $\mathcal{U}\backslash\{0\}$ ($\mathcal{U}$ deprived of its root), u has exactly one ancestor and we denote it by $\mathbf{a}(u)$. The set $\mathcal{U}$ is ordered alphanumerically :

- If u and v are in $\mathcal{U}$ and $|u| < |v|$ then $u < v$.
- If u and v are in $\mathcal{U}$ and $|u| = |v| = n$ and $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$ with $u_1 = v_1$, $\dots$, $u_k = v_k$, $u_{k+1} < v_{k+1}$ then $u < v$.

A mark is an application from $\mathcal{U}$ to some other set. We associate a mark on the tree $\mathcal{U}$ to each path of the process X. The mark at node u is ξ_u, where ξ_u is the size of the fragment indexed by u. The distribution of this random mark can be described recursively as follows.

Proposition 2.1. *(Consequence of Proposition 1.3, p. 25,* [Ber06]*) There exists a family of i.i.d. variables indexed by the nodes of the genealogical tree, $((\widetilde{\xi}_{ui})_{i\in\mathbb{N}^*}, u \in \mathcal{U})$, where each $(\widetilde{\xi}_{ui})_{i\in\mathbb{N}^*}$ is distributed according to the law $\nu(.)/\nu(\mathcal{S}^{\downarrow})$, and such that the following holds:*
Given the marks $(\xi_v, |v| \le n)$ of the first n generations, the marks at generation $n + 1$ are given by

$$\xi_{ui} = \widetilde{\xi}_{ui}\xi_u \,,$$

where $u = (u_1, \dots, u_n)$ and $ui = (u_1, \dots, u_n, i)$ is the $i-th$ child of u.

2.2. **Tagged fragments.** From now on, we suppose that we start with a block of size $m = 1$. We assume that the total mass of the fragments remains constant through time, as follows.

Assumption B. ***(Conservative property).***
We have $\nu(\sum_{i=1}^{+\infty} s_i = 1) = 1$.

2.2.1. *First definition.* We can now define tagged fragments. We use the representation of fragmentation chains as random infinite marked tree to define a fragmentation chain with q tagged fragments. Suppose we have a fragmentation process X. On each node $u \in \mathcal{U}$, we set a mark

$$(\xi_u, A_u) \,,$$

with ξ_u defined as above and $A_u \subset [q]$, denoting the tags present on the fragment labeled by u. The random variables $(A_u)_{u\in\mathcal{U}}$ are defined as follows.

- We set $A_{\{0\}} = [q]$.
- We suppose we have i.i.d. random variables $((U_{u,j})_{j\in[q]}, u \in \mathcal{U})$ of law $\mathcal{U}([0, 1])$. For all $n \in \mathbb{N}$, given the marks of the first n generations, the marks at generation $n+1$ are given by Proposition 2.1 (concerning $\xi_.$) and

$$A_{ui} = \{j \in A_u \,:\, \widetilde{\xi}_{u1} + \dots + \widetilde{\xi}_{u(i-1)} \le U_{u,j} < \widetilde{\xi}_{u1} + \dots + \widetilde{\xi}_{u(i-1)} + \widetilde{\xi}_{ui}\} \,,\, \forall u \,:\, |u| = n \,,\, \forall i \in \mathbb{N}^* \,.$$

We observe that, for all $j \in [q]$, $u \in \mathcal{U}$, $i \in \mathbb{N}^*$,

$$\mathbb{P}(j \in A_{ui} | j \in A_u, \widetilde{\xi}_{ui}) = \widetilde{\xi}_{ui} \,. \tag{2.1}$$

In the case $q = 1$, the branch $\{u \in \mathcal{U} \,:\, A_u \ne \emptyset\}$ has the same law as the randomly tagged branch of Section 1.2.3 of [Ber06]. The presentation is simpler in our case because the Malthusian exponent is 1 under Assumption B.

2.2.2. *Second definition.* There is a different way to define the law of the random mark (ξ_u, A_u), which we will present now. This definition is strictly equivalent to the first definition above. We take $(Y_1, Y_2, \dots, Y_q)$ to be q i.i.d. variables of law $\mathcal{U}([0, 1])$. We set, for all u in $\mathcal{U}$,

$$(\xi_u, l_u, A_u)$$

with ξ_u defined as above. The random variables A_u take values in the subsets of $[q]$. The random variables l_u take values in $[0, 1]$. These variables are defined as follows.

- We set $A_{\{0\}} = [q]$, $l_{\{0\}} = 0$.
- For all $n \in \mathbb{N}$, given the marks of the first n generations, the marks at generation $n + 1$ are given by Proposition 2.1 (concerning $\xi_.$) and

$$l_{ui} = l_u + \xi_u(\widetilde{\xi}_{u1} + \widetilde{\xi}_{u2} + \dots + \widetilde{\xi}_{u(i-1)}) \,,\, \forall u \,:\, |u| = n \,,\, \forall i \in \mathbb{N}^* \,,$$

$$k \in A_{ui} \text{ if and only if } Y_k \in [l_{ui}, l_{ui} + \xi_{ui}) \,,\, \forall u \,:\, |u| = n \,,\, \forall i \in \mathbb{N}^* \,.$$

We obtain $(\xi_u, A_u)_{u\in\mathcal{U}}$ having the same law as in Section 2.2.1. So the two definitions are equivalent.

2.3. **Observation scheme.** We ofreeze the process when the fragments become smaller than a given threshold $\varepsilon > 0$. That is, we have the following data

$$(\xi_u)_{u\in\mathcal{U}_\varepsilon}\,,$$

where

$$\mathcal{U}_\varepsilon = \{u \in \mathcal{U},\, \xi_{\mathbf{a}(u)} \geq \varepsilon,\, \xi_u < \varepsilon\}\,.$$

We now look at q tagged fragments ($q \in \mathbb{N}^*$). For each i in $[q]$, we call $L_0^{(i)} = 1$, $L_1^{(i)}$, $L_2^{(i)}\dots$ the successive sizes of the fragment having the tag i. More precisely, for each $n \in \mathbb{N}^*$, there is almost surely exactly one $u \in \mathcal{U}$ such that $|u| = n$, $i \in A_u$; and so, $L_n^{(i)} = \xi_u$. For each i, the process $S_0^{(i)} = -\log(L_0^{(i)}) = 0 \leq S_1^{(i)} = -\log(L_1^{(i)}) \leq \dots$ is a renewal process without delay, with waiting-time following a law π (see [Asm03], Chapter V for an introduction to renewal processes). This law π is defined by the following.

(2.2)

$$\text{For all bounded measurable } f : [0,1] \to [0,+\infty)\,,\ \int_{\mathcal{S}^\downarrow} \sum_{i=1}^{+\infty} s_i f(s_i)\nu(d\boldsymbol{s}) = \int_0^{+\infty} f(e^{-x})\pi(dx)\,,$$

(see Proposition 1.6, p. 34 of [Ber06], or Equations (3), (4), p. 398 of [HK11]).

We make the following assumption on π.

Assumption C. *There exist $a, b > 0$ ($a < b$) such that the support of π is $[a, b]$. We set $\delta = e^{-b}$.*

We set

$$T = -\log(\varepsilon)\,.$$

We set, for all $i \in [q]$, $t \geq 0$,

$$B_t^{(i)} = \inf\{S_j^{(i)} \,:\, S_j^{(i)} > t\} - t\,. \tag{2.3}$$

The process $B^{(i)}$ is a homogeneous Markov process (Proposition 1.5 p. 141 of [Asm03]). We call it the residual lifetime of the fragment tagged by i. In the following, we will treat t as a time parameter. This has nothing to do with the time in which the fragmentation process X evolves.

We observe that, for all t, $(B_t^{(1)}, \dots, B_t^{(q)})$ is exchangeable (meaning that for all σ in the symmetric group of order q, $(B_t^{(\sigma(1))}, \dots, B_t^{(\sigma(q))})$ has the same law as $(B_t^{(1)}, \dots, B_t^{(q)})$).

2.4. **Stationary age process.** We define $\widetilde{X}$ to be an independent copy of X. We suppose it has q tagged fragments. Therefore it has a mark $(\widetilde{\xi}, \widetilde{A})$ and renewal processes $(\widetilde{S}_k^{(i)})_{k\geq 0}$ (for all i in $[q]$) defined in the same way as for X. We let $(\widetilde{B}^{(1)}, \widetilde{B}^{(2)})$ be the residual lifetimes of the fragments tagged by 1 and 2.

Let

$$\mu = \int_0^{+\infty} x\pi(dx)$$

and let π_1 be the distribution with density $x \mapsto x/\mu$ with respect to π. We set $\overline{C}$ to be a random variable of law π_1. We set U to be independent of $\overline{C}$ and uniform on $(0,1)$. We set $\widetilde{S}_{-1} = \overline{C}(1-U)$. The process $\overline{S}_0 = \widetilde{S}_{-1}$, $\overline{S}_1 = \widetilde{S}_{-1} + \widetilde{S}_0^{(1)}$, $\overline{S}_2 = \widetilde{S}_{-1} + \widetilde{S}_1^{(1)}$, $\overline{S}_2 = \widetilde{S}_{-1} + \widetilde{S}_2^{(1)}$, $\dots$ is a renewal process with delay π_1. We set $(\overline{B}_t^{(1)})_{t\geq 0}$ to be its residual lifetime process :

$$\overline{B}_t^{(1)} = \begin{cases} \overline{C}(1-U) - t & \text{if } t < \overline{S}_0\,, \\ \inf_{n\geq 0}\{\overline{S}_n \,:\, \overline{S}_n > t\} - t & \text{if } t \geq \overline{S}_0\,. \end{cases} \tag{2.4}$$

Theorem 3.3 p.151 of [Asm03] tells us that $(\overline{B}_t^{(1)})_{t\geq 0}$ has the same transition as $(B_t^{(1)})_{t\geq 0}$ defined above and that $(\overline{B}_t^{(1)})_{t\geq 0}$ is stationary.

We define a measure η on $\mathbb{R}^+$ by its action on bounded measurable functions:
(2.5)
For all bounded measurable $f : \mathbb{R}^+ \to \mathbb{R}$, $\eta(f) = \frac{1}{\mu}\int_{\mathbb{R}^+} \mathbb{E}(f(Y-s)\mathbb{1}_{\{Y-s\geq 0\}})ds$, $(Y \sim \pi)$.

Lemma 2.2. *The measure η is the law of $\overline{B}_t^{(1)}$ (for any t).*

Proof. Let $\xi \geq 0$. We set $f(y) = \mathbb{1}_{y\geq\xi}$, for all y in $\mathbb{R}$. We have (with Y of law π)

$$\begin{aligned}
\frac{1}{\mu}\int_{\mathbb{R}^+} \mathbb{E}(f(Y-s)\mathbb{1}_{Y-s\geq 0})ds &= \frac{1}{\mu}\int_{\mathbb{R}^+}\left(\int_0^y \mathbb{1}_{y-s\geq\xi}ds\right)\pi(dy) \\
&= \frac{1}{\mu}\int_{\mathbb{R}^+}(y-\xi)_+\pi(dy) \\
&= \int_\xi^{+\infty}\left(1-\frac{\xi}{y}\right)\frac{y}{\mu}\pi(dy) \\
&= \mathbb{P}(\overline{C}(1-U)\geq\xi)\,.
\end{aligned}$$

□

For v in $\mathbb{R}$, we now want to define a process $(\overline{B}_t^{(1),v})_{t\geq v}$ having the same transition as $B_t^{(1)}$ and being stationary. We set $\overline{B}_v^{(1),v}$ such that it has the law η. As we have given its transition, the process $(\overline{B}_t^{(1),v})_{t\geq v}$ is well defined in law. In addition, we suppose that it is independent of all the other processes.

For v in $[0,T]$, we define a process $(\widehat{B}^{(1),v}, \widehat{B}^{(2),v})$ such that $\widehat{B}^{(1),v} = B^{(1)}$ and $(\widehat{B}^{(1),v}, \widehat{B}^{(2),v})$ has the law of $(B^{(1)}, B^{(2)})$ conditioned on

$$\forall u \in \mathcal{U}\,,\; 1 \in A_u \Rightarrow [2 \in A_u \Leftrightarrow -\log(\xi_u) \leq v]\,,$$

which reads as follows : the tag 2 remains on the fragment bearing the tag 1 until the size of the fragment is smaller than e^{-v}. We observe that, conditionally on $\widehat{B}_v^{(1),v}$, $\widehat{B}_v^{(2),v}$: $(\widehat{B}^{(1),v}_{v+\widehat{B}_v^{(1),v}+t})_{t\geq 0}$ and $(\widehat{B}^{(2),v}_{v+\widehat{B}_v^{(2),v}+t})_{t\geq 0}$ are independent.

Let k in $\mathbb{N}^*$ be such that

$$(k-1)\times(b-a) \geq a\,. \tag{2.6}$$

Now we state a small Lemma that will be useful below.

Lemma 2.3. *Let v be in $\mathbb{R}$. The variables $\overline{B}_v^{(1),v}$ and $\widehat{B}_{kb}^{(1),kb}$ have the same support (and it is $[0,-\log(\delta)]$).*

Proof. By Equation (2.4), the support of η is $[0,b]$; and so the support of $\overline{B}_v^{(1),v}$ is $[0,b]$. By Assumption C, the support of $S_{k-1}^{(1)}$ is $[(k-1)a,(k-1)b]$ and the support of $S_k^{(1)} - S_{k-1}^{(1)}$ is $[a,b]$. If $S_k^{(1)} > (k-1)b$ then $B_{kb}^{(1)} = S_k^{(1)} - S_{k-1}^{(1)} - ((k-1)b - S_{k-1}^{(1)})$. As $S_{k-1}^{(1)}$ and $S_k^{(1)} - S_{k-1}^{(1)}$ are independent, we get that the support of $B_{kb}^{(1)}$ includes $[0,b]$ (because of Equation (2.6)). As this support is included in $[0,b]$, we have proved the desired result. □

For v in $\mathbb{R}$, we define a process $(\overline{B}_t^{(2),v})_{t\geq v}$ by: $(\overline{B}_t^{(1),v}, \overline{B}_t^{(2),v})_{t\geq v}$ has the law of

$$(\widehat{B}^{(1),kb}_{t-v+kb}, \widehat{B}^{(2),kb}_{t-v+kb})_{t\geq v}$$

conditioned on $(\widehat{B}^{(1),kb}_{t-v+kb})_{t\geq v} = (\overline{B}_t^{(1),v})_{t\geq v}$. This conditioning is correct because $\overline{B}_v^{(1),v}$ and $\widehat{B}_{kb}^{(1),kb}$ have the same support.

3. Rate of convergence in the Key Renewal Theorem

We need the following regularity assumption.

Assumption D. *The probability $\pi(dx)$ is absolutely continuous with respect to the Lebesgue measure (we will write $\pi(dx) = \pi(x)dx$). The density function $x \mapsto \pi(x)$ is continuous on $(0;+\infty)$.*

Fact 3.1. *Let $\theta > 1$ (θ is fixed in the rest of the paper). The density π satisfies*

$$\limsup_{x\to+\infty} \exp(\theta x)\pi(x) < +\infty\,.$$

For φ a nonnegative Borel-measurable function on $\mathbb{R}$, we set $S(\varphi)$ to be the set of complex-valued measures κ (on the Borelian sets) such that $\int_{\mathbb{R}} \varphi(x)|\kappa|(dx) < \infty$, where $|\kappa|$ stands for the total variation norm. If κ is a finite complex-valued measure on the Borelian sets of $\mathbb{R}$, we define $T\kappa$ to be the σ-finite measure with the density

$$v(x) = \begin{cases} \kappa((x,+\infty)) & \text{if } x \geq 0\,, \\ -\kappa((-\infty,x]) & \text{if } x < 0\,. \end{cases}$$

Let F be the cumulative distribution function of π.

We set $B_t = B_t^{(1)}$ (see Equation (2.3) for the definition of $B^{(1)}$, $B^{(2)}$, ...). By Theorem 3.3 p.151 and Theorem 4.3 p. 156 of [Asm03], we know that B_t converges in law to a random variable B_∞(of law η). The following Theorem is a consequence of [Sgi02], Theorem 5.1, p. 2429. It shows there is actually a rate of convergence for this convergence in law.

Theorem 3.2. *Let $\varepsilon' \in (0,\theta)$. Let*

$$\varphi(x) = \begin{cases} e^{(\theta-\varepsilon')x} & \textit{if } x \geq 0\,, \\ 1 & \textit{if } x < 0\,. \end{cases}$$

If Y is a random variable of law π then

$$\sup_{\alpha\,:\,|\alpha|\leq M} \left| \mathbb{E}(\alpha(B_t)) - \frac{1}{\mu}\int_{\mathbb{R}^+} \mathbb{E}(\alpha(Y-s)\mathbb{1}_{\{Y-s\geq 0\}})ds \right| = o\left(\frac{1}{\varphi(t)}\right)$$

as t approaches $+\infty$ outside a set of Lebesgue measure zero (the supremum is taken on α in the set of Borel-measurable functions on $\mathbb{R}$).

Proof. Let $*$ stands for the convolution product. We define the renewal measure $U(dx) = \sum_{n=0}^{+\infty} \pi^{*n}(dx)$ (notations: $\pi^{*0}(dx) = \delta_0$, the Dirac mass at 0, $\pi^{*n} = \pi * \pi * \dots * \pi$ (n times)). We take i.i.d. variables $X, X_1, X_2 \dots$ of law π. We set $f(x) = M$, for all x in $\mathbb{R}$. We have, for all $t \geq 0$,

$$\begin{aligned} \mathbb{E}(f(B_t)) &= \mathbb{E}\left(\sum_{n=0}^{+\infty} f(X_1 + X_2 + \dots + X_{n+1} - t)\mathbb{1}_{\{X_1+\dots+X_n < t \leq X_1+\dots+X_{n+1}\}}\right) \\ &= \int_0^t \mathbb{E}(f(s+X-t)\mathbb{1}_{\{s+X-t\geq 0\}})U(ds)\,. \end{aligned}$$

We set

$$g(t) = \begin{cases} \mathbb{E}(f(X-t)\mathbb{1}_{\{X-t\geq 0\}}) & \text{if } t \geq 0\,, \\ 0 & \text{if } t < 0\,. \end{cases}$$

We have, for all $t \geq 0$,

$$\begin{aligned} \left|\mathbb{E}(f(X-t)\mathbb{1}_{\{X-t\geq 0\}})\right| &\leq \|f\|_\infty \mathbb{P}(X \geq t) \\ &\leq \|f\|_\infty e^{-(\theta-\frac{\varepsilon'}{2})t}\mathbb{E}(e^{(\theta-\frac{\varepsilon'}{2})X})\,. \end{aligned}$$

We have: $\mathbb{E}(e^{(\theta-\frac{\varepsilon'}{2})X}) < \infty$. The function φ is submultiplicative and it is such that

$$\lim_{x\to-\infty} \frac{\log(\varphi(x))}{x} = 0 \leq \lim_{x\to+\infty} \frac{\log(\varphi(x))}{x} = \theta - \varepsilon'\,.$$

The function g is in $L^1(\mathbb{R})$. The function $g.\varphi$ is in $L^\infty(\mathbb{R})$. We have $g(x)\varphi(x) \to 0$ as $|x| \to \infty$. We have

$$\varphi(t)\int_t^{+\infty} |g(x)|dx \underset{t\to+\infty}{\longrightarrow} 0\,,\ \varphi(t)\int_{-\infty}^{t} |g(x)|dx \underset{t\to-\infty}{\longrightarrow} 0\,.$$

We have $T^2(\pi) \in S(\varphi)$.

Let us now take a function α such that $|\alpha| \le M$. We set

$$\widehat{\alpha}(t) = \begin{cases} \mathbb{E}(\alpha(X-t)\mathbb{1}_{\{X-t\ge 0\}}) & \text{if } t \ge 0\,,\\ 0 & \text{if } t < 0\,. \end{cases}$$

Then we have $|\widehat{\alpha}| \le |g|$ and (computing as above for f)

$$\mathbb{E}(\alpha(B_t)) \quad = \quad \widehat{\alpha} * U(t)$$

So, by [Sgi02], Theorem 5.1, we have proved the desired result. □

Corollary 3.3. *There exists a constant Γ_1 bigger than 1 such that: for any bounded measurable function F on $\mathbb{R}$ such that $\eta(F) = 0$,*

$$|\mathbb{E}(F(B_t))| \le \|F\|_\infty \times \frac{\Gamma_1}{\varphi(t)}$$

for t outside a set of Lebesgue measure zero.

Proof. We take $M = 1$ in the above Theorem. Keep in mind that η is defined in Equation (2.5). There exists a constant Γ_1 such that: for all measurable function α such that $\|\alpha\|_\infty \le 1$,

$$|\mathbb{E}(\alpha(B_t)) - \eta(\alpha)| \le \frac{\Gamma_1}{\varphi(t)} \text{ (for } t \text{ outside a set of Lebesgue measure zero).} \tag{3.1}$$

Let us now take a bounded measurable F such that $\eta(F) = 0$. By Equation (3.1), we have (for t outside a set of Lebesgue measure zero)

$$\begin{aligned} \left|\mathbb{E}\left(\frac{F(B_t)}{\|F\|_\infty}\right) - \eta\left(\frac{F}{\|F\|_\infty}\right)\right| &\le \frac{\Gamma_1}{\varphi(t)}\\ |\mathbb{E}(F(B_t))| &\le \|F\|_\infty \times \frac{\Gamma_1}{\varphi(t)}\,. \end{aligned}$$

□

4. Limits of symmetric functionals

4.1. **Notations.** We fix $q \in \mathbb{N}^*$. We set $\mathcal{S}_q$ to be the symmetric group of order q. A function $F : \mathbb{R}^q \to \mathbb{R}$ is symmetric if

$$\forall \sigma \in \mathcal{S}_q\,,\ \forall (x_1, \dots, x_q) \in \mathbb{R}^q\,,\ F(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(q)}) = F(x_1, x_2, \dots, x_q)\,.$$

For $F : \mathbb{R}^q \to \mathbb{R}$, we define a symmetric version of F by

$$F_{\text{sym}}(x_1, \dots, x_q) = \frac{1}{q!}\sum_{\sigma\in\mathcal{S}_q} F(x_{\sigma(1)}, \dots, x_{\sigma(q)})\,,\text{ for all } (x_1, \dots, x_q) \in \mathbb{R}^q\,.$$

We set $\mathcal{B}_{\text{sym}}(q)$ to be the set of bounded, measurable, symmetric functions F on $\mathbb{R}^q$, and we set $\mathcal{B}^0_{\text{sym}}(q)$ to be the F of $\mathcal{B}_{\text{sym}}(q)$ such that

$$\int_{x_1} F(x_1, x_2, \dots, x_q)\eta(dx_1) = 0\,,\ \forall (x_2, \dots, x_q) \in \mathbb{R}^{q-1}\,.$$

We set

$$L_T = \sum_{u\in\mathcal{U}_\varepsilon : A_u \neq \emptyset} (\#A_u - 1)\,.$$

Suppose that k is in $[q]$ and $l \ge 1$. For t in $[0, T]$, we consider the following collections of nodes of $\mathcal{U}$:

$$\mathcal{T}_1 = \{u \in \mathcal{U}\backslash\{0\} \,:\, A_u \neq \emptyset\,,\ \xi_{\mathbf{a}(u)} \ge \varepsilon\} \cup \{0\}\,,$$
$$S(t) = \{u \in \mathcal{T}_1 \,:\, -\log(\xi_{\mathbf{a}(u)}) \le t\,,\ -\log(\xi_u) > t\}\,.$$

We set $\mathcal{L}_1$ to be the set of leaves in the tree $\mathcal{T}_1$. For t in $[0,T]$ and i in $[q]$, there exists one and only one u in $S(t)$ such that $i \in A_u$. We call it $u\{t,i\}$. Under Assumption C, there exists a constant bounding the numbers vertices of $\mathcal{T}_1$ almost surely. Let us look at an example in Figure 4.1. Here,

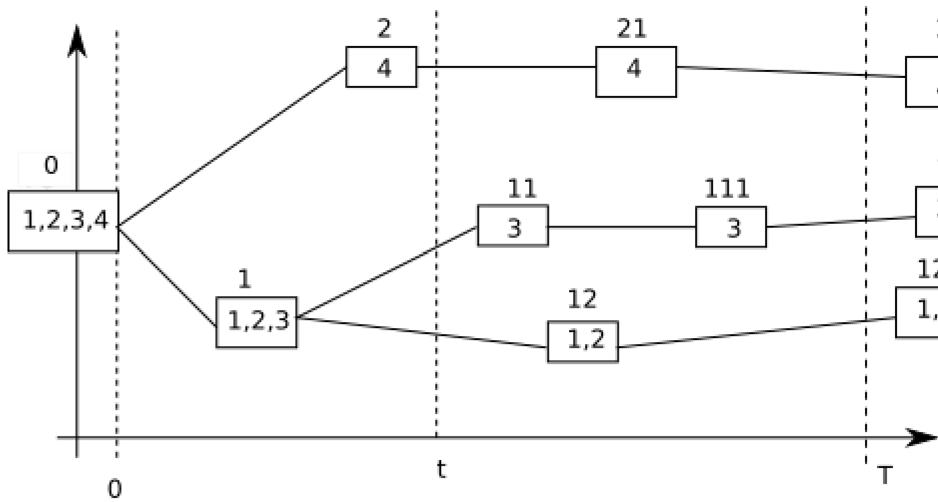

FIGURE 4.1. Tree and marks

we have a graphic representation of a realization of $\mathcal{T}_1$. Each node u of $\mathcal{T}_1$ is written above a rectangular box in which we read A_u; the right side of the box has the coordinate $-\log(\xi_u)$ on the X-axis. For simplicity, the node $(1,1)$ is designated by 11, the node $(1,2)$ is designated by 12, and so on. In this example: $\mathcal{T}_1 = \{0,(1),(2),(1,1),(2,1),(1,2),(1,1,1),(2,2),(1,1,2),(1,2,1)\}$, $\mathcal{L}_1 = \{(2,2),(1,1,2),(1,2,1)\}$, $A_{(1)} = \{1,2,3\}$, $A_{(1,2)} = \{1,2\}$, $\dots$, $S(t) = \{(1,2),(1,1),(2,1)\}$, $u\{t,1\} = (1,2)$, $u\{t,2\} = (1,2)$, $u\{t,3\} = (1,1)$, $u\{t,4\} = (2,1)$.

For k, l in $\mathbb{N}$, we define the event

$$C_{k,l}(t) = \{ \sum_{u \in S(t)} \mathbb{1}_{\#A_u = 1} = k \, , \sum_{u \in S(t)} (\#A_u - 1) = l \} \, .$$

For example, in Figure 4.1, we are in the event $C_{2,1}(t)$.

We define

$$\mathcal{T}_2 = \{u \in \mathcal{T}_1 \backslash \{0\} \, : \, \#A_{\mathbf{a}(u)} \geq 2\} \cup \{0\} \, ,$$

$$m_2 \, : \, u \in \mathcal{T}_2 \mapsto (\xi_u, \inf\{i, i \in A_u\}) \, .$$

For example, in Figure 4.1, $\mathcal{T}_2 = \{(0),(1),(2),(1,1),(1,2),(1,2,1)\}$. Let α be in $(0,1)$. We observe that $C_{k,l}(\alpha T)$ is measurable with respect to $(\mathcal{T}_2, m_2)$ if $T - \alpha T > b$ (we suppose that this is the case in the following). We set, for all u in $\mathcal{T}_2$, $T_u = -\log(\xi_u)$. Let $\mathcal{L}_2$ be the set of leaves u in the tree $\mathcal{T}_2$ such that the set A_u has a single element n_u. For example, in Figure 4.1, $\mathcal{L}$ =$\{(2),(1,1)\}$.

For q even ($q = 2p$) and for all t in $[0,T]$, we define the events

$$P_t = \{\forall i \in [p] \, , \exists u_i \in \mathcal{U} \, : \, \xi_{u_i} < e^{-t} \, , \xi_{\mathbf{a}(u_i)} \geq e^{-t} \, , A_{u_i} = \{2i-1, 2i\}\} \, ,$$

$$\forall i \in [p] \, , P_{i,i+1}(t) = \{\exists u \in S(t) \, : \, \{2i-1, 2i\} \subset A_u\} \, .$$

We set, for all t in $[0,T]$,

$$\mathcal{F}_{S(t)} = \sigma(S(t), (\xi_u, A_u)_{u \in S(t)}) \, .$$

4.2. **Intermediate results.**

Lemma 4.1. *We suppose that F is in $\mathcal{B}^0_{sym}(q)$ and that F is of the form $F = (f_1 \otimes f_2 \otimes \cdots \otimes f_q)_{sym}$. Let A be in $\sigma(\mathcal{L}_2)$. For any α in $]0,1[$, k in $[q]$ and l in $\{0,1,\dots,(q-k-1)_+\}$, we have*

$$|\mathbb{E}(\mathbb{1}_{C_{k,l}(\alpha T)}\mathbb{1}_A F(B_T^{(1)}, B_T^{(2)}, \dots, B_T^{(q)}))| \leq \|F\|_\infty \Gamma_1^q C_{tree}(q) \left(\frac{1}{\delta}\right)^q \varepsilon^{q/2},$$

(for a constant $C_{tree}(q)$ defined below in the proof) and

$$\varepsilon^{-q/2}\mathbb{E}(\mathbb{1}_{C_{k,l}(\alpha T)}\mathbb{1}_A F(B_T^{(1)}, B_T^{(2)}, \dots, B_T^{(q)})) \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Proof. We have $\{\#\mathcal{L}_1 = q\} \in \sigma(\mathcal{L}_2)$. Let A be in $\sigma(\mathcal{L}_2)$. Since the event $C_{k,l}(\alpha T)$ is in $\sigma(\mathcal{L}_2) \vee \sigma(\mathcal{T}_2) \vee \sigma(m_2)$, we have

$$\begin{aligned}
&|\mathbb{E}(\mathbb{1}_{C_{k,l}(\alpha T)}\mathbb{1}_A F(B_T^{(1)}, B_T^{(2)}, \dots, B_T^{(q)}))| \\
&\qquad = |\mathbb{E}(\mathbb{1}_{C_{k,l}(\alpha T)}\mathbb{1}_A \mathbb{E}(F(B_T^{(1)}, B_T^{(2)}, \dots, B_T^{(q)})|\mathcal{L}_2, \mathcal{T}_2, m_2)) \\
&\qquad = |\mathbb{E}(\sum_{f:\mathcal{T}_2 \to \mathcal{P}([q])} \mathbb{1}_{C_{k,l}(\alpha T)}\mathbb{1}_A \mathbb{E}(F(B_T^{(1)}, B_T^{(2)}, \dots, B_T^{(q)})\mathbb{1}_{A_u = f(u), \forall u \in \mathcal{T}_2}|\mathcal{L}_2, \mathcal{T}_2, m_2)).
\end{aligned}$$

If u in $\mathcal{L}_2$ and if $T_u < T$, then, conditionally on $\mathcal{T}_2$, m_2, $B_T^{(n_u)}$ is independent of all the other variables and has the same law as $B_{T-T_u}^{(1)}$. Thus, using Theorem 3.2 and Corollary 3.3, we get, for any $\varepsilon' \in (0, \theta - 1)$, $u \in \mathcal{L}_2$, $i \in A_u$,

$$\mathbb{E}(f_i(B_T^{(i)})|\mathcal{L}_2, \mathcal{T}_2, m_2) \leq e^{-(\theta - \varepsilon')(T-T_u)_+},$$

for $T - T_u \notin Z_0$ where Z_0 is of Lebesgue measure zero.

Thus we get

$$|\mathbb{E}(\mathbb{1}_{C_{k,l}(\alpha T)}\mathbb{1}_A F(B_T^{(1)}, B_T^{(2)}, \dots, B_T^{(q)}))|$$

(since F is of the form $F = (f_1 \otimes \cdots \otimes f_q)_{sym}$,
since, conditionally on $u \in \mathcal{L}_2$, the distribution of T_u is absolutely continuous
with respect to the Lebesgue measure)

$$\leq \|F\|_\infty \Gamma_1^q \mathbb{E}(\sum_{f:\mathcal{T}_2 \to \mathcal{P}([q])} \left[\mathbb{1}_{C_{k,l}(\alpha T)} \prod_{u \in \mathcal{L}_2} e^{-(\theta - \varepsilon')(T-T_u)_+} \times \mathbb{1}_A \mathbb{E}(\mathbb{1}_{A_u = f(u), \forall u \in \mathcal{T}_2}|\mathcal{L}_2, \mathcal{T}_2, m_2)\right])$$

(because of Assumption C and because $\theta - \varepsilon' > 1$)

$$\leq \|F\|_\infty \Gamma_1^q \mathbb{E}(\sum_{f:\mathcal{T}_2 \to \mathcal{P}([q])} \left[\mathbb{1}_{C_{k,l}(\alpha T)} \prod_{u \in \mathcal{L}_2} e^{-(T-T_{\mathbf{a}(u)})-\log(\delta)} \times \mathbb{1}_A \mathbb{E}(\mathbb{1}_{A_u = f(u), \forall u \in \mathcal{T}_2}|\mathcal{L}_2, \mathcal{T}_2, m_2)\right])$$

(because of Equation (2.1))

$$\leq \|F\|_\infty \Gamma_1^q \mathbb{E}(\sum_{f:\mathcal{T}_2 \to \mathcal{P}([q])} \mathbb{1}_{C_{k,l}(\alpha T)}\mathbb{1}_A \left[\prod_{u \in \mathcal{L}_2} e^{-(T-T_{\mathbf{a}(u)})-\log(\delta)} \times \prod_{u \in \mathcal{T}_2 \backslash \{0\}} e^{-(A_u - 1)(T_u - T_{\mathbf{a}(u)})}\right]).$$

For a fixed ω, we have

$$\prod_{u \in \mathcal{L}_2} e^{-(T-T_{\mathbf{a}(u)})-\log(\delta)} \times \prod_{u \in \mathcal{T}_2 \backslash \{0\}} e^{-(A_u - 1)(T_u - T_{\mathbf{a}(u)})} = \left(\frac{1}{\delta}\right)^{\#\mathcal{L}_2} \exp\left(-\int_0^T a(s)ds\right),$$

where, for all s,

$$\begin{aligned} a(s) &= \sum_{u\in\mathcal{T}_2\backslash\{0\}\,:\,T_{a(u)}\le s<T} \mathbb{1}_{A_u=1} + \sum_{u\in\mathcal{T}_2\backslash\{0\}\,:\,T_{\mathbf{a}(u)}\le s\le T_u} (\#A_u-1) \\ &= \sum_{u\in S(s)} \mathbb{1}_{\#A_u=1} + \sum_{u\in S(s)} (\#A_u-1)\,. \end{aligned}$$

We observe that, for all ω:

$$a(t)\ge \left\lceil \frac{q}{2}\right\rceil\,,\ \forall t\,,$$
$$a(\alpha T)=k+l\ ,\ \text{if } \omega\in C_{k,l}(\alpha T)\,,$$

and if t is such that

$$\sum_{u\in S(t)}(\#A_u-1)=l'\,,\ \sum_{u\in S(t)}\mathbb{1}_{\#A_u=1}=k'$$

for some integers l', k', then for all $s\ge t$,

$$a(s)\ge k'+\left\lceil\frac{q-k'}{2}\right\rceil\,.$$

We observe that, under Assumption C, there exists a constant which bounds $\#\mathcal{T}_1$ almost surely and so there exists a constant $C_{tree}(q)$ which bounds $\#\{f:\mathcal{T}_1\to\mathcal{P}([q])\}$ almost surely. So, we have

$$\begin{aligned} &|\mathbb{E}(\mathbb{1}_{C_{k,l}(\alpha T)}\mathbb{1}_A F(B_T^{(1)},B_T^{(2)},\dots,B_T^{(q)}))| \\ &\le \|F\|_\infty\Gamma_1^q\mathbb{E}\big(\sum_{f:\mathcal{T}_2\to\mathcal{P}([q])}\mathbb{1}_A\mathbb{1}_{C_{k,l}(\alpha T)}\left(\frac{1}{\delta}\right)^{\#\mathcal{L}_2} e^{-\lceil q/2\rceil\alpha T}e^{-(k+\lceil\frac{q-k}{2}\rceil)(T-\alpha T)}\big) \\ &\le \|F\|_\infty\Gamma_1^q C_{tree}(q)\left(\frac{1}{\delta}\right)^{q} e^{-\lceil q/2\rceil\alpha T}e^{-(k+\lceil\frac{q-k}{2}\rceil)(1-\alpha)T}\,. \end{aligned}$$

As $k\ge 1$, then $k+\left\lceil\frac{q-k}{2}\right\rceil>\frac{q}{2}$, and so we have proved the desired result. □

Lemma 4.2. *Let k be an integer $\ge q/2$. Let $\alpha\in[q/(2k),1]$. We have*

$$\mathbb{P}(L_{\alpha T}\ge k)\le K_1(q)\varepsilon^{q/2}\,,$$

where $K_1(q)=\sum_{i\in[q]}\frac{q!}{(q-i)!}$.

Let k be an integer $>q/2$. Let $\alpha\in(q/(2k),1)$. We have

$$\varepsilon^{-q/2}\mathbb{P}(L_{\alpha T}\ge k)\underset{\varepsilon\to 0}{\longrightarrow}0\,.$$

Proof. Let k be an integer $\ge q/2$ and let $\alpha\in[q/(2k),1]$. We decompose

$$\{L_{\alpha T}\ge k\}=\cup_{i\in[q]}\cup_{m:[i]\hookrightarrow[q]}\left(F(i,m)\cap\{L_{\alpha T}\ge k)\cap\{\#S(\alpha T)=i\}\right),$$

where

$$F(i,m)=\{i_1,i_2\in[i]\text{ with } i_1\ne i_2\Rightarrow\exists u_1,u_2\in S(\alpha T),u_1\ne u_2,m(i_1)\in A_{u_1}\,,\,m(i_2)\in A_{u_2}\}\,.$$

Suppose we are in the event $F(i,m)$. For $u\in S(\alpha T)$ and for all j in $[i]$ such that $m(j)\in A_u$, we define

$$T_{|u|}^{(j)}=-\log(\xi_u)\,,\,T_{|u|-1}^{(j)}=-\log(\xi_{\mathbf{a}(u)})\,,\,\dots\,,\,T_1^{(j)}=-\log(\xi_{\mathbf{a}^{\circ(|u|-1)}(u)})\,,T_0^{(j)}=0\,,$$
$$l(j)=|u|\,,\,v(j)=u\,.$$

We have

$$\begin{aligned} &\mathbb{P}(L_{\alpha T}\ge k)\le\sum_{i\in[q]}\sum_{m:[i]\hookrightarrow[q]}\mathbb{P}(F(i,m)\cap\{L_{\alpha T}\ge k\}\cap\{\#S(\alpha T)=i\}) \\ &=\sum_{i\in[q]}\sum_{m:[i]\hookrightarrow[q]}\mathbb{E}(\mathbb{1}_{L_{\alpha T}\ge k}\mathbb{1}_{F(i,m)}\mathbb{E}(\mathbb{1}_{\#S(\alpha T)=i}|F(i,m),L_{\alpha T},(T_p^{(j)})_{j\in[i],p\in[l(j)]},\,(v(j))_{j\in[i]},\,(A_{v(j)})_{j\in[i]})) \end{aligned}$$

(because of Equation (2.1))

$$= \sum_{i\in[q]} \sum_{m:[i]\hookrightarrow[q]} \mathbb{E}\left(\mathbb{1}_{L_{\alpha T}\geq k} \mathbb{1}_{F(i,m)} \prod_{j\in[i]} \prod_{r\in A_{v(j)}\setminus m(j)} \prod_{k=1}^{l(j)} \exp((-T_k^{(j)}+T_{k-1}^{(j)})) \right)$$

$$\leq \sum_{i\in[q]} \sum_{m:[i]\hookrightarrow[q]} \mathbb{E}\left(\mathbb{1}_{L_{\alpha T}\geq k} \mathbb{1}_{F(i,m)} \prod_{j\in[i]} (e^{-\alpha T})^{\# A_{v(j)}-1} \right)$$

$$\leq \sum_{i\in[q]} \sum_{m:[i]\hookrightarrow[q]} \mathbb{E}(\mathbb{1}_{L_{\alpha T}\geq k} e^{-k\alpha T}) \leq e^{-k\alpha T} \times \sum_{i\in[q]} \frac{q!}{(q-i)!} .$$

If we suppose that $k > q/2$ and $\alpha \in (q/(2k), 1)$, then

$$\exp\left(\frac{qT}{2}\right) \exp(-k\alpha T) \underset{T\to+\infty}{\longrightarrow} 0 .$$

□

Immediate consequences of the two above lemmas are the following Corollaries.

Corollary 4.3. *If q is odd and if $F \in \mathcal{B}^0_{sym}(q)$ is of the form $F = (f_1 \otimes \cdots \otimes f_q)_{sym}$, then*

$$\varepsilon^{-q/2}\mathbb{E}(F(B_T^{(1)},\dots,B_T^{(q)})\mathbb{1}_{\#\mathcal{L}_1=q}) \underset{\varepsilon\to 0}{\longrightarrow} 0 .$$

Proof. We take $\alpha \in \left(\frac{q}{2}\left\lceil\frac{q}{2}\right\rceil^{-1}, 1\right)$. We can decompose

$$\begin{aligned}
&\varepsilon^{-q/2}\left|\mathbb{E}(F(B_T^{(1)},\dots,B_T^{(q)})\mathbb{1}_{\#\mathcal{L}_1=q})\right| \\
&\quad = |\varepsilon^{-q/2} \sum_{k\in[q]} \sum_{l\in\{0,1,\dots,(q-k-1)_+\}} \mathbb{E}(\mathbb{1}_{C_{k,l}(\alpha T)} \mathbb{1}_{\#\mathcal{L}_1=q} F(B_T^{(1)},\dots,B_T^{(q)})) \\
&\qquad + \varepsilon^{-q/2}\mathbb{E}(\mathbb{1}_{L_{\alpha T}\geq\lceil q/2\rceil}\mathbb{1}_{\#\mathcal{L}_1=q}F(B_T^{(1)},\dots,B_T^{(q)}))| \\
&\qquad\qquad \text{(by Lemmas 4.1, 4.2)} \underset{\varepsilon\to 0}{\longrightarrow} 0 .
\end{aligned}$$

□

Corollary 4.4. *Suppose $F \in \mathcal{B}^0_{sym}(q)$ is of the form $F = (f_1 \otimes \cdots \otimes f_q)_{sym}$. Let A in $\sigma(\mathcal{L}_2)$. Then*

$$|\mathbb{E}(F(B_T^{(1)},\dots,B_T^{(q)})\mathbb{1}_A)| \leq \|F\|_\infty \varepsilon^{q/2}\left\{K_1(q) + \Gamma_1^q C_{tree}(q)\left(\frac{1}{\delta}\right)^q q^2\right\}$$

Proof. From Lemmas 4.1, 4.2, we get

$$\begin{aligned}
|\mathbb{E}(F(B_T^{(1)},\dots,B_T^{(q)})\mathbb{1}_A)| &= |\mathbb{E}(F(B_T^{(1)},\dots,B_T^{(q)})\mathbb{1}_A(\mathbb{1}_{L_{\alpha T}\geq q/2} + \sum_{k'\in[q]} \sum_{0\leq l\leq (q-k'-1)_+} \mathbb{1}_{C_{k',l}(\alpha T)}))| \\
&\leq \|F\|_\infty \varepsilon^{q/2}\left\{K_1(q) + \Gamma_1^q C_{\mathrm{tree}}(q)\left(\frac{1}{\delta}\right)^q \sum_{k'\in[q]} (q-k'-1)_+\right\} .
\end{aligned}$$

□

We now want to find the limit of $\varepsilon^{-q/2}\mathbb{E}(\mathbb{1}_{L_T\leq q/2}\mathbb{1}_{\#\mathcal{L}_1=q}F(B_T^{(1)},\dots,B_T^{(q)}))$ when ε goes to 0, for q even. First we need a technical lemma.

For any i, the process $(B_t^{(i)})$ has a stationary law (see Theorem 3.3 p. 151 of [Asm03]). Let B_∞ be a random variable having this stationary law η (it has already appeared in Section 3). We can always suppose that it is independent of all the other variables.

Lemma 4.5. *Let f_1 , f_2 be in $\mathcal{B}^0_{sym}(1)$. Let α belong to $(0,1)$. We have*

$$\int_{-\infty}^{-\log(\delta)} e^{-v}|\mathbb{E}(f_1(\overline{B}_0^{(1),v})f_2(\overline{B}_0^{(2),v}))|dv<\infty$$

and

$$\Big|e^{T-\alpha T-B^{(1)}_{\alpha T}}\mathbb{E}(f_1\otimes f_2(B^{(1)}_{T,}B^{(2)}_T)\mathbb{1}_{P_{1,2}(T)^{\complement}}|\mathcal{F}_{S(\alpha T)},P_{1,2}(\alpha T)) - \int_{-\infty}^{-\log(\delta)} e^{-v}\mathbb{E}(\mathbb{1}_{v\leq \overline{B}_0^{(1),v}}f_1(\overline{B}_0^{(1),v})f_2(\overline{B}_0^{(2),v}))dv\Big| \leq \Gamma_2\|f_1\|_\infty\|f_2\|_\infty\exp\left(-(T-\alpha T)\left(\frac{\theta-\varepsilon'-1}{2}\right)\right),$$

where

$$\Gamma_2=\frac{\Gamma_1^2}{\delta^{2+2(\theta-\varepsilon')}(2(\theta-\varepsilon')-1)}+\frac{\Gamma_1}{\delta^{\theta-\varepsilon'}}+\frac{\Gamma_1^2}{\delta^{2(\theta-\varepsilon')}(2(\theta-\varepsilon')-1)}.$$

Proof. From now on, we suppose that $\alpha T-\log(\delta)<(T+\alpha T)/2$, $(T+\alpha T)/2-\log(\delta)<T$ (this is true if T is large enough). We have, for all s in $[\alpha T+B^{(1)}_{\alpha T},T]$,

$$\mathbb{P}(u\{s,2\}=u\{s,1\}|\mathcal{F}_{S(\alpha T)},P_{1,2}(\alpha T),(\xi_{u\{t,1\}})_{0\leq t\leq T})=\exp(-(s+B^{(1)}_s-(\alpha T+B^{(1)}_{\alpha T}))\,.$$

And so,

$$\begin{aligned}&\mathbb{E}(f_1\otimes f_2(B^{(1)}_{T,}B^{(2)}_T)\mathbb{1}_{P_{1,2}(T)^{\complement}}|\mathcal{F}_{S(\alpha T)},P_{1,2}(\alpha T))\\ &=\mathbb{E}\Big(\mathbb{E}(f_1\otimes f_2(B^{(1)}_{T,}B^{(2)}_T)\mathbb{1}_{P_{1,2}(T)^{\complement}}|\mathcal{F}_{S(\alpha T)},P_{1,2}(\alpha T),(\xi_{u\{t,1\}})_{0\leq t\leq T})|\mathcal{F}_{S(\alpha T)},P_{1,2}(\alpha T)\Big)\\ &\qquad\text{(keep in mind that } \widehat{B}^{(1),v}=B^{(1)} \text{ for all } v)\\ &=\mathbb{E}(\mathbb{E}(\int_{\alpha T+B^{(1)}_{\alpha T}}^{T+B^{(1)}_T} e^{-(v-\alpha T-\widehat{B}^{(1),v}_{\alpha T})}f_1(\widehat{B}^{(1),v}_T)f_2(\widehat{B}^{(2),v}_T)dv|\mathcal{F}_{S(\alpha T)},P_{1,2}(\alpha T),(\xi_{u\{t,1\}})_{0\leq t\leq T})\\ &\qquad|\mathcal{F}_{S(\alpha T)},P_{1,2}(\alpha T))\\ &=\mathbb{E}(\int_{\alpha T+B^{(1)}_{\alpha T}}^{T+B^{(1)}_T} e^{-(v-\alpha T-\widehat{B}^{(1)v}_{\alpha T})}f_1(\widehat{B}^{(1),v}_T)f_2(\widehat{B}^{(2),v}_T)dv|\mathcal{F}_{S(\alpha T)},P_{1,2}(\alpha T))\end{aligned}$$

We have

$$\begin{aligned}(4.1)\quad&\left|e^{T-\alpha T-B^{(1)}_{\alpha T}}\mathbb{E}\left(\int_{\alpha T+B^{(1)}_{\alpha T}}^{(T+\alpha T)/2} e^{-(v-\alpha T-\widehat{B}^{(1),v}_{\alpha T})}f_1(\widehat{B}^{(1),v}_T)f_2(\widehat{B}^{(2),v}_T)dv|\mathcal{F}_{S(\alpha T)},P_{1,2}(\alpha T)\right)\right|\\ &=e^{T-\alpha T-B^{(1)}_{\alpha T}}|\mathbb{E}(\int_{\alpha T+B^{(1)}_{\alpha T}}^{(T+\alpha T)/2} e^{-(v-\alpha T-\widehat{B}^{(1),v}_{\alpha T})}\\ &\times\mathbb{E}(f_1(\widehat{B}^{(1),v}_T)f_2(\widehat{B}^{(2),v}_T)|\widehat{B}^{(1),v}_v,\widehat{B}^{(2),v}_v,\mathcal{F}_{S(\alpha T)},P_{1,2}(\alpha T))dv\\ &|\mathcal{F}_{S(\alpha T)},P_{1,2}(\alpha T))|\\ &\text{(using the fact that } \widehat{B}^{(1),v}_T \text{ and } \widehat{B}^{(2),v}_T \text{ are independant}\\ &\text{conditionally to } \widehat{B}^{(1),v}_v,\widehat{B}^{(2),v}_v,\mathcal{F}_{S(\alpha T)},P_{1,2}(\alpha T)\text{, if } T\geq v-\log(\delta)\text{,}\\ &\text{we get, by Theorem 3.2 and Corollary 3.3)}\\ &\leq e^{T-\alpha T-B^{(1)}_{\alpha T}}\\ &\times\mathbb{E}(\int_{\alpha T+B^{(1)}_{\alpha T}}^{(T+\alpha T)/2} e^{-(v-\alpha T-\widehat{B}^{(1)}_{\alpha T})}(\Gamma_1\|f_1\|_\infty e^{-(\theta-\varepsilon')(T-v-\widehat{B}^{(1),v}_v)_+}\times\Gamma_1\|f_2\|_\infty e^{-(\theta-\varepsilon')(T-v-\widehat{B}^{(2),v}_v)_+})dv\\ &|\mathcal{F}_{S(\alpha T)},P_{1,2}(\alpha T))\end{aligned}$$

$$\leq \Gamma_1^2 \|f_1\|_\infty \|f_2\|_\infty e^{T-\alpha T-\log(\delta)} \int_{\alpha T}^{(T+\alpha T)/2} e^{-(v-\alpha T+\log(\delta))} e^{-2(\theta-\varepsilon')(T-v+\log(\delta))} dv$$
$$= \frac{\Gamma_1^2 \|f_1\|_\infty \|f_2\|_\infty}{\delta^{2+2(\theta-\varepsilon')}} e^{T-2(\theta-\varepsilon')T} \left[\frac{e^{(2(\theta-\varepsilon')-1)v}}{2(\theta-\varepsilon')-1} \right]_{\alpha T}^{(T+\alpha T)/2}$$
$$\leq \frac{\Gamma_1^2 \|f_1\|_\infty \|f_2\|_\infty}{\delta^{2+2(\theta-\varepsilon')}} \frac{\exp\left(-(2(\theta-\varepsilon')-1)T + (2(\theta-\varepsilon')-1)\frac{(T+\alpha T)}{2}\right)}{2(\theta-\varepsilon')-1}$$
$$= \frac{\Gamma_1^2 \|f_1\|_\infty \|f_2\|_\infty}{\delta^{2+2(\theta-\varepsilon')}} \frac{\exp\left(-(2(\theta-\varepsilon')-1)\left(\frac{T-\alpha T}{2}\right)\right)}{2(\theta-\varepsilon')-1} .$$

We have

$$(4.2) \quad \left| e^{T-\alpha T - B^{(1)}_{\alpha T}} \mathbb{E}\left(\int_{(T+\alpha T)/2}^{T+B_T^{(1)}} e^{-(v-\alpha T - B^{(1)}_{\alpha T})} f_1(\widehat{B}_T^{(1),v}) f_2(\widehat{B}_T^{(2),v}) dv | \mathcal{F}_{S(\alpha T)}, P_{1,2}(\alpha T) \right) \right.$$
$$\left. - \int_{(T+\alpha T)/2}^{T-\log(\delta)} e^{-(v-T)} \mathbb{E}(\mathbb{1}_{v \leq T + \overline{B}_T^{(1),v}} f_1(\overline{B}_T^{(1),v}) f_2(\overline{B}_T^{(2),v})) dv \right|$$
$$= \left| e^{T-\alpha T - B^{(1)}_{\alpha T}} \mathbb{E}(\int_{(T+\alpha T)/2}^{T-\log(\delta)} e^{-(v-\alpha T - B^{(1)}_{\alpha T})} \mathbb{1}_{v \leq T + B_T^{(1)}} f_1(\widehat{B}_T^{(1),v}) f_2(\widehat{B}_T^{(2),v}) dv | \mathcal{F}_{S(\alpha T)}, P_{1,2}(\alpha T)) \right.$$
$$\left. - e^{T-\alpha T - B^{(1)}_{\alpha T}} \mathbb{E}(\int_{(T+\alpha T)/2}^{T-\log(\delta)} e^{-(v-\alpha T - B^{(1)}_{\alpha T})} \mathbb{1}_{v \leq T - \overline{B}_T^{(1),v}} f_1(\overline{B}_T^{(1),v}) f_2(\overline{B}_T^{(2),v}) dv | \mathcal{F}_{S(\alpha T)}, P_{1,2}(\alpha T)) \right|$$
$$= e^{T-\alpha T - B^{(1)}_{\alpha T}} \left| \int_{(T+\alpha T)/2}^{T-\log(\delta)} e^{-(v-\alpha T - B^{(1)}_{\alpha T})} \mathbb{E}(\mathbb{E}(\mathbb{1}_{v \leq T + B_T^{(1)}} f_1(\widehat{B}_T^{(1),v}) f_2(\widehat{B}_T^{(2),v}) \right.$$
$$| \widehat{B}_v^{(1),v}, \mathcal{F}_{S(\alpha T)}, P_{1,2}(\alpha T)) | \mathcal{F}_{S(\alpha T)}, P_{1,2}(\alpha T)) dv$$
$$\left. - \int_{(T+\alpha T)/2}^{T-\log(\delta)} e^{-(v-\alpha T - B^{(1)}_{\alpha T})} \mathbb{E}(\mathbb{E}(\mathbb{1}_{v \leq T + \overline{B}_T^{(1),v}} f_1(\overline{B}_T^{(1),v}) f_2(\overline{B}_T^{(2),v})) dv | \overline{B}_v^{(1),v})) dv \right|$$

We observe that, for all v in $[(T+\alpha T)/2, T-\log(\delta)]$,

$$\mathbb{E}(\mathbb{1}_{v \leq T + B_T^{(1)}} f_1(\widehat{B}_T^{(1),v}) f_2(\widehat{B}_T^{(2),v}) | \widehat{B}_v^{(1),v}, \mathcal{F}_{S(\alpha T)}, P_{1,2}(\alpha T)) = \Psi(\widehat{B}_v^{(1),v}) ,$$
$$\mathbb{E}(\mathbb{1}_{v \leq T + \overline{B}_T^{(1),v}} f_1(\overline{B}_T^{(1),v}) f_2(\overline{B}_T^{(2),v}) | \overline{B}_v^{(1),v}) = \Psi(\overline{B}_v^{(1),v}) \overset{\text{law}}{=} \Psi(B_\infty) ,$$

for some function Ψ (the same on both lines) such that $\|\Psi\|_\infty \leq \|f_1\|_\infty \|f_2\|_\infty$. So, by Theorem 3.2 and Corollary 3.3, the quantity in Equation (4.2) can be bounded by

$$e^{T-\alpha T - B^{(1)}_{\alpha T}} \int_{(T+\alpha T)/2}^{T-\log(\delta)} e^{-(v-\alpha T - B^{(1)}_{\alpha T})} \Gamma_1 \|f_1\|_\infty \|f_2\|_\infty e^{-(\theta-\varepsilon')(v-\alpha T - B^{(1)}_{\alpha T})} dv$$

(coming from Corollary 3.3 there is an integral over a set of Lebesgue measure zero in the above bound, but this term vanishes). The above bound can in turn be bounded by:

$$(4.3) \quad \frac{\Gamma_1 \|f_1\|_\infty \|f_2\|_\infty}{\delta^{(\theta-\varepsilon')}} e^T \int_{(T+\alpha T)/2}^{T-\log(\delta)} e^{(\theta-\varepsilon')\alpha T} e^{-(\theta-\varepsilon'+1)v} dv$$
$$\leq \frac{\Gamma_1 \|f_1\|_\infty \|f_2\|_\infty}{\delta^{\theta-\varepsilon'}} e^{T+\alpha T(\theta-\varepsilon')} \exp(-(\theta-\varepsilon'+1)(\frac{T+\alpha T}{2}))$$
$$= \frac{\Gamma_1 \|f_1\|_\infty \|f_2\|_\infty}{\delta^{\theta-\varepsilon'}} \exp\left(-(\theta-\varepsilon'-1)\left(\frac{T-\alpha T}{2}\right)\right) .$$

We have

$$(4.4) \quad \int_{\frac{T+\alpha T}{2}}^{T-\log(\delta)} e^{-(v-T)} \mathbb{E}(\mathbb{1}_{v \leq T + \overline{B}_T^{(1),v}} f_1(\overline{B}_T^{(1),v}) f_2(\overline{B}_T^{(2),v})) dv$$

$$= \mathbb{E}\left(\int_{-\left(\frac{T-\alpha T}{2}\right)}^{-\log(\delta)} e^{-v} \mathbb{1}_{v \leq \overline{B}_0^{(1),v}} f_1(\overline{B}_0^{(1),v}) f_2(\overline{B}_0^{(1),v}) dv\right)$$

and

$$(4.5) \quad \int_{-\infty}^{-\frac{(T-\alpha T)}{2}} e^{-v} |\mathbb{E}(f_1(\overline{B}_0^{(1),v}) f_2(\overline{B}_0^{(2),v}))| dv$$

(since $\overline{B}_0^{(1),v}$ and $\overline{B}_0^{(2),v}$ are independant conditionnaly on $\overline{B}_v^{(1),v}$, $\overline{B}_v^{(2),v}$ if $v - \log(\delta) \leq 0$)

(using Theorem 3.2 and Corollary 3.3)

$$\leq \int_{-\infty}^{-\frac{(T-\alpha T)}{2}} e^{-v} \Gamma_1^2 \|f_1\|_\infty \|f_2\|_\infty \mathbb{E}(e^{-(\theta-\varepsilon')(-v-\overline{B}_v^{(1),v})_+} e^{-(\theta-\varepsilon')(-v-\overline{B}_v^{(2),v})_+}) dv$$

(again, coming from Corollary 3.3 there is an integral over a set of Lebesgue measure zero in the above bound, but this term vanishes)

$$\leq \int_{-\infty}^{-\frac{(T-\alpha T)}{2}} e^{-v} \Gamma_1^2 \|f_1\|_\infty \|f_2\|_\infty e^{-2(\theta-\varepsilon')(-v+\log(\delta))} dv$$

$$= \frac{\Gamma_1^2 \|f_1\|_\infty \|f_2\|_\infty}{\delta^{2(\theta-\varepsilon')}} \frac{\exp\left(-(2(\theta-\varepsilon')-1)\frac{(T-\alpha T)}{2}\right)}{2(\theta-\varepsilon')-1} .$$

Equations (4.1), (4.3), (4.4) and (4.5) give us the desired result. □

Lemma 4.6. *Let k in $\{0,1,2,\dots,p\}$. We suppose q is even and $q = 2p$. Let $\alpha \in (q/(q+2), 1)$. We suppose $F = f_1 \otimes f_2 \otimes \dots \otimes f_q$, with $f_1, \dots, f_q$ in $\mathcal{B}^0_{sym}(1)$. We then have :*

$$(4.6) \quad \varepsilon^{-q/2} \mathbb{E}(F(B_T^{(1)}, \dots, B_T^{(q)}) \mathbb{1}_{P_{\alpha T}} \mathbb{1}_{\#\mathcal{L}_1 = q}) \underset{\varepsilon \to 0}{\longrightarrow} \prod_{i=1}^{p} \int_{-\infty}^{-\log(\delta)} e^{-v} \mathbb{E}(\mathbb{1}_{v \leq \overline{B}_0^{(1),v}} f_{2i-1}(\overline{B}_0^{(1),v}) f_{2i}(\overline{B}_0^{(2),v})) dv .$$

Proof. We have

$$P_{\alpha T} \cap \{\#\mathcal{L}_1 = q\} = P_{\alpha T} \cap \bigcap_{1 \leq i \leq p} P_{2i-1,2i}(T)^{\complement} .$$

By Lemma 4.5, we have, for some constant C,

$$(4.7) \quad \left| e^{pT} \mathbb{E}\left(\mathbb{1}_{P_{\alpha T}} \prod_{i=1}^{p} \mathbb{E}(f_{2i-1} \otimes f_{2i}(B_T^{(2i-1)}, B_T^{(2i)})) \mathbb{1}_{P_{2i-1,2i}(T)^{\complement}} \middle| \mathcal{F}_{S(\alpha T)} \right) - \mathbb{E}\left(\mathbb{1}_{P_{\alpha T}} \prod_{i=1}^{p} e^{B_{\alpha T}^{(2i-1)} + \alpha T} \int_{-\infty}^{-\log(\delta)} e^{-v} \mathbb{E}(\mathbb{1}_{v \leq \overline{B}_0^{(1),v}} f_{2i-1}(\overline{B}_0^{(1),v}) f_{2i}(\overline{B}_0^{(2),v})) dv \right) \right|$$
$$\leq \prod_{i=1}^{p} (\Gamma_2 \|f_{2i-1}\|_\infty \|f_{2i}\|_\infty) \times \mathbb{E}\left(\mathbb{1}_{P_{\alpha T}} \prod_{i=1}^{p} [e^{B_{\alpha T}^{(2i-1)} + \alpha T} e^{-(T-\alpha T)\frac{(\theta-\varepsilon'-1)}{2}}] \right) .$$

We introduce the events (for $t \in [0,T]$)

$$O_t = \{\#\{u\{t, 2i-1\}, 1 \leq i \leq p\} = p\} ,$$

and the tribes (for i in $[q]$, $t \in [0,T]$)

$$\mathcal{F}_{t,i} = \sigma(u\{t,i\}, \xi_{u\{t,i\}}) .$$

We have :

(4.8)
$$\mathbb{E}(\mathbb{1}_{P_{\alpha T}}\prod_{i=1}^{p}e^{B_{\alpha T}^{(2i-1)}+\alpha T})=\mathbb{E}(\mathbb{1}_{O_{\alpha T}}\prod_{i=1}^{p}e^{B_{\alpha T}^{(2i-1)}+\alpha T}\mathbb{E}(\prod_{i=1}^{p}\mathbb{1}_{u\{\alpha T,2i-1\}=u\{\alpha T,2i\}}|\vee_{1\leq i\leq p}\mathcal{F}_{\alpha T,2i-1}))$$
$$=\mathbb{E}(\mathbb{1}_{O_{\alpha T}})\,.$$

We then observe that
$$O_{\alpha T}^{\complement}=\cup_{i\in[p]}\cup_{j\in[p],j\neq i}\{u\{\alpha T,2i-1\}=u\{\alpha T,2j-1\}\}\,,$$
and, for $i\neq j$,
$$\begin{aligned}\mathbb{P}(u\{\alpha T,2i-1\}=u\{\alpha T,2j-1\}) &= \mathbb{E}(\mathbb{E}(\mathbb{1}_{u\{\alpha T,2i-1\}=u\{\alpha T,2j-1\}}|\mathcal{F}_{\alpha T,2i-1}))\\ &= \mathbb{E}(e^{-\alpha T-B_{\alpha T}^{(2i-1)}})\\ \text{(because of Assumption (C))} &\leq \mathbb{E}(e^{-\alpha T-\log(\delta)})\,.\end{aligned}$$
So
$$\mathbb{P}(O_{\alpha T})\underset{\varepsilon\to 0}{\longrightarrow}1\,.$$
This finishes the proof of Equation (4.6).

□

4.3. **Convergence result.** For f and g bounded measurable functions, we set

(4.9)
$$V(f,g)=\int_{-\infty}^{-\log(\delta)}e^{-v}\mathbb{E}(\mathbb{1}_{v\leq\overline{B}_0^{(1),v}}f(\overline{B}_0^{(1),v})g(\overline{B}_0^{(2),v}))dv\,.$$

For q even, we set $\mathcal{I}_q$ to be the set of partitions of $[q]$ into subsets of cardinality 2. For I in $\mathcal{I}_q$ and t in $[0,T]$, we introduce
$$P_{t,I_q}=\{\forall\{i,j\}\in I\,,\,\exists u\in\mathcal{U}\text{ such that }\xi_u<e^{-t}\,,\,\xi_{\boldsymbol{a}(u)}\geq e^{-t}\,,\,A_u=\{i,j\}\}\,.$$
For t in $[0,T]$, we define
$$\mathcal{P}_t=\cup_{I\in\mathcal{I}_q}P_{t,I}\,.$$

Proposition 4.7. *Let q be in $\mathbb{N}^*$. Let $F=(f_1\otimes\dots\otimes f_q)_{sym}$ with $f_1,\dots,f_q$ in $\mathcal{B}^0_{sym}(1)$. If q is even ($q=2p$) then*

(4.10)
$$\varepsilon^{q/2}\mathbb{E}(F(B_T^{(1)},\dots,B_T^{(q)})\mathbb{1}_{\#\mathcal{L}_1=q})\underset{\varepsilon\to 0}{\longrightarrow}\sum_{I\in\mathcal{I}_q}\prod_{\{a,b\}\in I}V(f_a,f_b)\,.$$

Proof. Let α be in $(q/(q+2),1)$. We have
$$\varepsilon^{-q/2}\mathbb{E}(F(B_T^{(1)},\dots,B_T^{(q)})\mathbb{1}_{\#\mathcal{L}_1=q}) = \varepsilon^{-q/2}\mathbb{E}(F(B_T^{(1)},\dots,B_T^{(q)})\mathbb{1}_{\#\mathcal{L}_1=q}(\mathbb{1}_{\mathcal{P}_{\alpha T}}+\mathbb{1}_{\mathcal{P}_{\alpha T}^{\complement}}))\,.$$
By Lemma 4.1 and Lemma 4.2, we have that
$$\lim_{\varepsilon\to 0}\varepsilon^{-q/2}\mathbb{E}(F(B_T^{(1)},\dots,B_T^{(q)})\mathbb{1}_{\#\mathcal{L}_1=q}\mathbb{1}_{\mathcal{P}_{\alpha T}^{\complement}})=0$$
(because $(B_T^{(1)},\dots,B_T^{(q)})$ is exchangeable). We compute :
$$\varepsilon^{-q/2}\mathbb{E}(F(B_T^{(1)},\dots,B_T^{(q)})\mathbb{1}_{\#\mathcal{L}_1=q}\mathbb{1}_{\mathcal{P}_{\alpha T}})=\varepsilon^{-q/2}\mathbb{E}(F(B_T^{(1)},\dots,B_T^{(q)})\mathbb{1}_{\#\mathcal{L}_1=q}\sum_{I_q\in\mathcal{I}_q}\mathbb{1}_{P_{\alpha T,I_q}})$$
(as F is symmetric and $(B_T^{(1)},\dots,B_T^{(q)})$ is exchangeable)
$$=\frac{q!}{2^{q/2}\left(\frac{q}{2}\right)!}\varepsilon^{-q/2}\mathbb{E}(F(B_T^{(1)},\dots,B_T^{(q)})\mathbb{1}_{\#\mathcal{L}_1=q}\mathbb{1}_{P_{\alpha T}})$$
$$=\frac{q!\varepsilon^{-q/2}}{2^{q/2}\left(\frac{q}{2}\right)!}\frac{1}{q!}\sum_{\sigma\in\mathcal{S}_q}\mathbb{E}((f_{\sigma(1)}\otimes\dots\otimes f_{\sigma(q)})(B_T^{(1)},\dots,B_T^{(q)})\mathbb{1}_{\#\mathcal{L}_1=q}\mathbb{1}_{P_{\alpha T}})$$
(by Lemma 4.6)

$$\underset{\varepsilon\to 0}{\longrightarrow} \frac{1}{2^{q/2}\left(\frac{q}{2}\right)!}\sum_{\sigma\in\mathcal{S}_q}\prod_{i=1}^{p} V(f_{\sigma(2i-1)}, f_{\sigma(2i)}) = \sum_{I\in\mathcal{I}_q}\prod_{\{a,b\}\in I} V(f_a, f_b)\,.$$

□

5. Results

We are interested in the probability measure γ_T defined by its action on bounded measurable functions $F : [0,1] \to \mathbb{R}$ by

$$\gamma_T(F) = \sum_{u\in\mathcal{U}_\varepsilon} X_u F\left(\frac{X_u}{\varepsilon}\right)\,.$$

We define, for all q in $\mathbb{N}^*$, F from $[0,1]^q$ to $\mathbb{R}$,

$$\gamma_T^{\otimes q}(F) = \sum_{a\,:\,[q]\to\mathcal{U}_\varepsilon} X_{a(1)}\dots X_{a(q)} F\left(\frac{X_{a(1)}}{\varepsilon},\dots,\frac{X_{a(q)}}{\varepsilon}\right)\,,$$

$$\gamma_T^{\odot q}(F) = \sum_{a\,:\,[q]\hookrightarrow\mathcal{U}_\varepsilon} X_{a(1)}\dots X_{a(q)} F\left(\frac{X_{a(1)}}{\varepsilon},\dots,\frac{X_{a(q)}}{\varepsilon}\right)\,,$$

where the last sum is taken over all the injective applications a from $[q]$ to $\mathcal{U}_\varepsilon$. If we set

$$\Phi(F)\,:\,(y_1,\dots,y_q)\in\mathbb{R}^+\mapsto F(e^{-y_1},\dots,e^{-y_q})\,,$$

then

$$\mathbb{E}(\gamma_T^{\otimes q}(F)) = \mathbb{E}(\Phi(F)(B_T^{(1)},\dots,B_T^{(q)}))\,,$$

$$\mathbb{E}(\gamma_T^{\odot q}(F)) = \mathbb{E}(\Phi(F)(B_T^{(1)},\dots,B_T^{(q)})\mathbb{1}_{\#\mathcal{L}_1} = q)\,.$$

We define, for all bounded continuous $f : \mathbb{R}^+ \to \mathbb{R}$,

$$\gamma_\infty(f) = \eta(\Phi(f))\,. \tag{5.1}$$

Proposition 5.1 (Law of large numbers)**.** *Let f be a continuous function from $[0,1]$ to $\mathbb{R}$. We have:*

$$\gamma_T(f) \underset{T\to+\infty}{\overset{a.s.}{\longrightarrow}} \gamma_\infty(f)\,.$$

Proof. We take a bounded measurable function $f : [0,1] \to \mathbb{R}$. We define $\overline{f} = f - \eta(\Phi(f))$. We take an integer $q \geq 2$. We introduce the notation :

$$\forall g\,:\,\mathbb{R}^+\to\mathbb{R}\,,\,\forall(x_1,\dots,x_q)\in\mathbb{R}^q\,,\; g^{\otimes q}(x_1,\dots,x_q) = g(x_1)g(x_2)\dots g(x_q)\,.$$

We have

$$\begin{aligned}
\mathbb{E}((\gamma_T(f)-\eta(\Phi(f)))^q) &= \mathbb{E}((\gamma_T(\overline{f}))^q)\\
&= \mathbb{E}(\gamma_t^{\otimes q}(\overline{f}^{\otimes q}))\\
&= \mathbb{E}(\gamma_t^{\otimes q}((\overline{f}^{\otimes q})_{\text{sym}}))\\
\text{(by Corollary 4.4)}\quad &\leq \|\overline{f}\|_\infty^q \varepsilon^{q/2}\left\{K_1(q) + \Gamma_1^q C_{\text{tree}}(q)\left(\frac{1}{\delta}\right)^q q^2\right\}\,.
\end{aligned}$$

We now take sequences $(T_n = -\log(n))_{n\geq 1}$, $(\varepsilon_n = 1/n)_{n\geq 1}$. We then have, for all n and for all $\iota > 0$,

$$\mathbb{P}([\gamma_{T_n}(f)-\eta(\Phi(f))]^4 \geq \iota) \leq \frac{\|\overline{f}\|_\infty^4}{\iota n^2}\left\{K_1(4) + \Gamma_1^4 C_{\text{tree}}(4)\left(\frac{1}{\delta}\right)^4 \times 16\right\}\,.$$

So, by Borell-Cantelli's Lemma,

$$\gamma_{T_n}(f) \underset{n\to+\infty}{\overset{\text{a.s.}}{\longrightarrow}} \eta(\Phi(f))\,. \tag{5.2}$$

Let n be in $\mathbb{N}^*$. We can decompose

$$\mathcal{U}_{\varepsilon_n} = \mathcal{U}_{\varepsilon_n}^{(1)} \sqcup \mathcal{U}_{\varepsilon_n}^{(2)} \text{ where } \mathcal{U}_{\varepsilon_n}^{(1)} = \mathcal{U}_{\varepsilon_n} \cap \mathcal{U}_{\varepsilon_{n+1}} \,,\, \mathcal{U}_{\varepsilon_n}^{(2)} = \mathcal{U}_{\varepsilon_n} \backslash \mathcal{U}_{\varepsilon_{n+1}} \,.$$

For u in $\mathcal{U}_{\varepsilon_n} \backslash \mathcal{U}_{\varepsilon_{n+1}}$, we set $\boldsymbol{d}(u) = \{v \in \mathcal{U}_{\varepsilon_{n+1}} \,:\, \boldsymbol{a}(v) = u\}$. We can then write

$$\begin{aligned}
\sum_{u\in\mathcal{U}_{\varepsilon_n}} X_u f\left(\frac{X_u}{\varepsilon_n}\right) &= \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(1)}} X_u f(nX_u) + \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(2)}} X_u f(nX_u)\,, \\
\sum_{u\in\mathcal{U}_{\varepsilon_{n+1}}} X_u f\left(\frac{X_u}{\varepsilon_{n+1}}\right) &= \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(1)}} X_u f((n+1)X_u) + \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(2)}} \sum_{v\in\boldsymbol{d}(u)} X_v f((n+1)X_v)\,.
\end{aligned}$$

So we have, for all n,

$$\begin{aligned}
&\left| \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(2)}} \sum_{v\in\boldsymbol{d}(u)} X_v f((n+1)X_v) - \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(2)}} X_u f(nX_u) \right| \\
&\qquad \le |\gamma_{T_{n+1}}(f) - \gamma_{T_n}(f)| + \left| \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(1)}} X_u f((n+1)X_u) - \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(1)}} X_u f(nX_u) \right| .
\end{aligned}$$

If we take $f = \mathrm{Id}$, the two terms in the equation above can be transformed:

$$\begin{aligned}
&\left| \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(2)}} \sum_{v\in\boldsymbol{d}(u)} X_v f((n+1)X_v) - \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(2)}} X_u f(nX_u) \right| \\
&\ge \left| \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(2)}} \left(X_u f(nX_u) - \sum_{v\in\boldsymbol{d}(u)} X_v f(nX_v) \right) \right| - \left| \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(2)}} \sum_{v\in\boldsymbol{d}(u)} (X_v f(nX_v) - X_v f((n+1)X_v)) \right| \\
&\qquad \text{(by Assumption C)} \\
&\ge \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(2)}} \left(X_u f(nX_u) - \sum_{v\in\boldsymbol{d}(u)} X_v f(nX_u) e^{-a} \right) - \left| \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(2)}} \sum_{v\in\boldsymbol{d}(u)} (X_v f(nX_v) - X_v f((n+1)X_v)) \right| \\
&\qquad \ge \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(2)}} X_u (1-e^{-a}) \frac{n}{n+1} - \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(2)}} \sum_{v\in\boldsymbol{d}(u)} X_v \frac{1}{n+1}\,,
\end{aligned}$$

$$\begin{aligned}
&|\gamma_{T_{n+1}}(f) - \gamma_{T_n}(f)| + \left| \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(1)}} X_u f((n+1)X_u) - \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(1)}} X_u f(nX_u) \right| \\
&\qquad \le |\gamma_{T_{n+1}}(f) - \gamma_{T_n}(f)| + \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(1)}} X_u \frac{1}{n}\,.
\end{aligned}$$

Let $\iota > 0$. We fix ω in Ω. Almost surely, there exists n_0 such that, for $n \ge n_0$, $|\gamma_{T_{n+1}}(f) - \gamma_{T_n}(f)| < \iota$. For $n \ge n_0$, we can then write (still with $f = \mathrm{Id}$):
(5.3)

$$\sum_{u\in\mathcal{U}_{\varepsilon_n}^{(2)}} X_u \le \frac{n+1}{n(1-e^{-a})} \left(\iota + \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(2)}} \sum_{v\in\boldsymbol{d}(u)} X_v \frac{1}{n+1} + \sum_{u\in\mathcal{U}_{\varepsilon_n}^{(1)}} X_u \frac{1}{n} \right) \le \frac{n+1}{n(1-e^{-a})} \left(\iota + \frac{1}{n} \right) .$$

Let $n \ge n_0$ and t in (T_n, T_{n+1}). We can decompose

$$\mathcal{U}_{\varepsilon_n} = \mathcal{U}_{\varepsilon_n}^{(1)}(t) \sqcup \mathcal{U}_{\varepsilon_n}^{(2)}(t) \text{ where } \mathcal{U}_{\varepsilon_n}^{(1)}(t) = \mathcal{U}_{\varepsilon_n} \cap \mathcal{U}_{e^{-t}} \,,\, \mathcal{U}_{\varepsilon_n}^{(2)}(t) = \mathcal{U}_{\varepsilon_n} \backslash \mathcal{U}_{e^{-t}} \,.$$

For u in $\mathcal{U}_{\varepsilon_n}\backslash\mathcal{U}^{(1)}_{\varepsilon_n}(t)$, we set $\boldsymbol{d}(u,t)=\{v\in\mathcal{U}_{e^{-t}}\,:\,\boldsymbol{a}(v)=u\}$. For any continuous f from $[0,1]$ to $\mathbb{R}$, there exists $n_1\in\mathbb{N}^*$ such that, for all $x,y\in[0,1]$, $|x-y|<1/n_1\Rightarrow|f(x)-f(y)|<\iota$. Suppose that $n\geq n_0\vee n_1$. Then we have (for all $t\in[T_n,T_{n+1}]$),

$$
\begin{aligned}
(5.4)\quad |\gamma_t(f)-\gamma_{T_n}(f)| &\leq \left|\sum_{u\in\mathcal{U}^{(1)}_{\varepsilon_n}(t)} X_u f(e^t X_u) - \sum_{u\in\mathcal{U}^{(1)}_{\varepsilon_n}(t)} X_u f(nX_u)\right| \\
&\quad + \left|\sum_{u\in\mathcal{U}^{(2)}_{\varepsilon_n}(t)} X_u f(e^t X_u) - \sum_{u\in\mathcal{U}^{(2)}_{\varepsilon_n}(t)} X_u f(nX_u)\right| \\
&\leq \sum_{u\in\mathcal{U}^{(1)}_{\varepsilon_n}(t)} X_u\iota + 2\sum_{u\in\mathcal{U}^{(2)}_{\varepsilon_n}(t)} X_u\|f\|_\infty \\
\text{(using Equation 5.3)} &\leq \iota + 2\|f\|_\infty\frac{n+1}{n(1-e^{-a})}\left(\iota+\frac{1}{n}\right).
\end{aligned}
$$

Equations (5.2) and (5.4) prove the desired result. □

Theorem 5.2 (Central-limit Theorem)**.** *Let q be in $\mathbb{N}^*$. For functions f_1, ..., f_q which are continuous and in $\mathcal{B}^0_{sym}(1)$, we have*

$$
\varepsilon^{-q/2}(\gamma_T(f_1),\dots,\gamma_T(f_q)) \xrightarrow[T\to+\infty]{law} \mathcal{N}(0,(K(f_i,f_j))_{1\leq i,j\leq q})\ (\varepsilon=e^{-T})
$$

(K is given in Equation (5.5)).

Proof. Let f_1, ..., f_q $\mathcal{B}^0_{\mathrm{sym}}(1)$ and $v_1,\dots,v_q\in\mathbb{R}$.
First, we develop the product below

$$
\begin{aligned}
&\prod_{u\in\mathcal{U}_\varepsilon}\left(1+\sqrt{\varepsilon}\frac{X_u}{\varepsilon}(iv_1f_1+\dots+iv_qf_q)\left(\frac{X_u}{\varepsilon}\right)\right) = \\
&\exp\left(\sum_{u\in\mathcal{U}_\varepsilon}\log\left[1+\sqrt{\varepsilon}\,\mathrm{Id}\times(iv_1f_1+\dots+iv_qf_q)\left(\frac{X_u}{\varepsilon}\right)\right]\right) = \\
&\text{(for } \varepsilon \text{ small enough)} \\
&\exp\left(\sum_{u\in\mathcal{U}_\varepsilon}\sum_{k\geq1}\frac{(-1)^{k+1}}{k}\varepsilon^{k/2}(\mathrm{Id}\times(iv_1f_1+\dots+iv_qf_q))^k\left(\frac{X_u}{\varepsilon}\right)\right) = \\
&\text{(because, for } u\in\mathcal{U}_\varepsilon,\ X_u/\varepsilon\leq 1 \text{ a.s.)} \\
&\exp\left(\frac{1}{\sqrt{\varepsilon}}\gamma_T(iv_1f_1+\dots+iv_qf_q)+\frac{1}{2}\gamma_T(\mathrm{Id}\times(v_1f_1+\dots+v_qf_q)^2)+R_\varepsilon\right),
\end{aligned}
$$

where

$$
\begin{aligned}
R_\varepsilon &= \sum_{k\geq3}\sum_{u\in\mathcal{U}_\varepsilon}\frac{(-1)^{k+1}}{k}\varepsilon^{k/2-1}X_u\left(\frac{X_u}{\varepsilon}\right)^{k-1}(iv_1f_1+\dots+iv_qf_q)^k\left(\frac{X_u}{\varepsilon}\right) \\
&= \sum_{k\geq3}\frac{(-1)^{k+1}}{k}\varepsilon^{k/2-1}\gamma_T((\mathrm{Id})^{k-1}(iv_1f_1+\dots+iv_qf_q)^k),
\end{aligned}
$$

$$
|R_\varepsilon|\leq\sum_{k\geq3}\frac{\varepsilon^{k/2-1}}{k}(|v_1|\|f_1\|_\infty+\dots+|v_q|\|f_q\|_\infty)=O(\sqrt{\varepsilon}).
$$

We have, for some constant C,

$$
\mathbb{E}\left(\left|\exp\left(\frac{1}{\sqrt{\varepsilon}}\gamma_T(iv_1f_1+\dots+iv_qf_q)+\frac{1}{2}\gamma_T(\mathrm{Id}\times(v_1f_1+\dots+v_qf_q)^2)+R_\varepsilon\right)\right.\right.
$$

$$- \exp\left(\frac{1}{\sqrt{\varepsilon}}\gamma_T(iv_1f_1+\dots+iv_qf_q)+\frac{1}{2}\eta(\Phi(\mathrm{Id}\times(v_1f_1+\dots+v_qf_q)^2)\right)\Bigg|\Bigg)$$
$$\leq \mathbb{E}\left(C\left|\frac{1}{2}\gamma_T(\mathrm{Id}\times(v_1f_1+\dots+v_qf_q)^2)-\frac{1}{2}\eta(\Phi(\mathrm{Id}\times(v_1f_1+\dots+v_qf_q)^2)+R_\varepsilon\right|\right)$$
$$\text{(by Proposition 5.1)} \underset{\varepsilon\to 0}{\longrightarrow} 0\,.$$

Second, we develop the same product in a different manner. We have

$$\prod_{u\in\mathcal{U}_\varepsilon}\left(1+\sqrt{\varepsilon}\frac{X_u}{\varepsilon}(iv_1f_1+\dots+iv_qf_q)\left(\frac{X_u}{\varepsilon}\right)\right)=$$
$$\sum_{k\geq 0}\varepsilon^{-k/2}i^k\sum_{1\leq j_1,\dots,j_k\leq q}v_{j_1}\dots v_{j_k}\sum_{\substack{u_1,\dots,u_k\in\mathcal{U}_\varepsilon\\ u_1<\dots<u_k}}X_{u_1\dots}X_{u_k}f_{j_1}\left(\frac{X_{u_1}}{\varepsilon}\right)\dots f_{j_k}\left(\frac{X_{u_k}}{\varepsilon}\right)=$$
$$\sum_{k\geq 0}\varepsilon^{-k/2}i^k\sum_{1\leq j_1,\dots,j_k\leq q}v_{j_1}\dots v_{j_k}\frac{1}{k!}\gamma_T^{\odot k}(f_{j_1}\otimes\dots\otimes f_{j_k})\,.$$

By Corollary 4.4, we have, for all k,

$$\left|\varepsilon^{-k/2}\sum_{1\leq j_1,\dots,j_k\leq q}v_{j_1}\dots v_{j_k}\frac{1}{k!}\mathbb{E}(\gamma_T^{\odot k}(f_{j_1}\otimes\dots\otimes f_{j_k}))\right|$$
$$\leq\frac{q^k\sup(|v_1|,\dots,|v_q|)^k\sup(\|f_1\|_\infty,\dots,\|f_q\|_\infty)^k}{k!}\left\{K_1(q)+\Gamma_1^qC_{\mathrm{tree}}(q)\left(\frac{1}{\delta}\right)^q q^2\right\}\,.$$

So, by Corollary 4.3 and Proposition 4.7, we get that

$$\mathbb{E}\left(\prod_{u\in\mathcal{U}_\varepsilon}\left(1+\sqrt{\varepsilon}\frac{X_u}{\varepsilon}(iv_1f_1+\dots+iv_qf_q)\left(\frac{X_u}{\varepsilon}\right)\right)\right)$$
$$\underset{\varepsilon\to 0}{\longrightarrow}\sum_{\substack{k\geq 0\\ k\text{ even}}}(-1)^{k/2}\sum_{1\leq j_1,\dots,j_k\leq q}\frac{1}{k!}\sum_{I\in I_k}\prod_{\{a,b\}\in I}V(v_{j_a}f_{j_a},v_{j_b}f_{j_b})$$
$$=\sum_{\substack{k\geq 0\\ k\text{ even}}}\frac{(-1)^{k/2}}{2^{k/2}(k/2)!}\sum_{1\leq j_1,\dots,j_k\leq q}V(v_{j_1}f_{j_1},v_{j_2}f_{j_2})\dots V(f_{j_{k-1}},f_{j_k})$$
$$=\sum_{\substack{k\geq 0\\ k\text{ even}}}\frac{(-1)^{k/2}}{2^{k/2}(k/2)!}\left(\sum_{1\leq j_1,j_2\leq q}v_{j_1}v_{j_2}V(f_{j_1},f_{j_2})\right)^{k/2}$$
$$=\exp\left(-\frac{1}{2}\sum_{1\leq j_1,j_2\leq q}v_{j_1}v_{j_2}V(f_{j_1},f_{j_2})\right)\,.$$

In conclusion, we have

$$\mathbb{E}\left(\exp\left(\frac{1}{\sqrt{\varepsilon}}\gamma_T(iv_1f_1+\dots+iv_qf_q)\right)\right)$$
$$\underset{\varepsilon\to 0}{\longrightarrow}\exp\left(-\frac{1}{2}\eta(\Phi(\mathrm{Id}\times(v_1f_1+\dots+v_qf_q)^2))-\frac{1}{2}\sum_{1\leq j_1,j_2\leq q}v_{j_1}v_{j_2}V(f_{j_1},f_{j_2})\right)\,.$$

So we get the desired result with, for all f, g,

$$K(f,g) = \eta(\Phi(\mathrm{Id} \times fg) + V(f,g)) \tag{5.5}$$

(V is defined in Equation (4.9)). □

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