Forwarding stabilization in discrete time

Mattia Mattioni, Salvatore Monaco, Dorothée Normand-Cyrot

Forwarding stabilization in discrete time ⋆

Mattia Mattioni [a], Salvatore Monaco [a] and Dorothée Normand-Cyrot [b]

[a]*Dipartimento di Ingegneria Informatica, Automatica e Gestionale 'Antonio Ruberti', Università di Roma "La Sapienza", via Ariosto 25, 00185 Roma, Italy.*

[b] *Laboratoire des Signaux et Systèmes, L2S-CNRS, 3 rue Joliot Curie, 91190 Gif-sur-Yvette, France*

Abstract

The paper deals with stabilization of discrete-time cascade dynamics. The notion of average passivity introduced by the authors is used to achieve stability through an iterative design procedure for feedforward cascaded connections. Academic simulated examples illustrate the performances and compare the proposed solution with the one available in the literature.

Key words: asymptotic stabilization; discrete-time systems; application of nonlinear analysis and design.

1 Introduction

Starting from the 80s a systematic body of nonlinear control methods has been developed (e.g., Isidori (1995), Sepulchre et al. (1997), Khalil (2002)). Despite the numerous important results achieved throughout the years (e.g., Wei and Byrnes (1994); Mazenc and Nijmeijer (1998); Nešić et al. (1999); Kazakos and Tsinias (1994); Jiang and Wang (2001); Navarro-LóPez and Fossas-Colet (2004); Kazantzis (2004)) the theory is less developed in discrete time where several problems remain unsolved. This is mainly due to the loss of the differential structure in the evolution equations and the complex algebraic nonlinearities which must be handled when dealing with compositions of functions.
Among the issues which got in the way the development of the theory, one can include the primitive concept of passivity which is ambiguous in discrete time so directly impacting passivity-based design when addressing stabilization of cascaded dynamics (see Lin and Gong (2003), Chiang et al. (2010), Lin and Pongvuthithum (2002), Jankovic (2006)). Several works by the authors are aimed at bridging this gap. In this sense, a different way of representing discrete-time dynamics was introduced in Monaco and Normand-Cyrot (1997) as an alternative to the usual one, employing difference equations, for providing a differential geometric flow interpretation to the input to state evolutions. It was then profitably exploited to define a notion of u-average passivity Monaco and Normand-Cyrot (2011).
This paper formulates in this framework stabilization of discrete-time cascade systems exhibiting an upper-triangular (or feedforward) mathematical model. In the literature, a few studies and design methodologies are concerned with these cascade forms, with particular emphasis on a class of strict-feedforward structures. In Aranda-Bricaire and Moog (2004), the authors investigate equivalence to feedforward dynamics, up to coordinates change and preliminary feedback. In Mazenc and Nijmeijer (1998), the design is carried out through bounded control and then extended to the presence of disturbances in Ahmed-Ali et al. (1999). In Monaco and Normand-Cyrot (2013), a stabilizing procedure for dynamics in strict feedforward-form is developed through the computation of successive coordinates change making each successive sub-dynamics driftless and passive. In Monaco et al. (2016), forwarding is revisited via Immersion and Invariance so relaxing the a-priori knowledge of a Lyapunov function for initializing the design.
Stabilizing discrete-time systems in feedforward form remains challenging, because these structures are recovered in the formulation of many control problems.
The design approach proposed in this work represents the discrete-time counterpart of the so-called continuous-time forwarding design (see Sepulchre et al. (1997)). Stabilization is achieved through an iterative procedure by ensuring at each step global asymptotic stability (GAS) of the feedforward interconnection of two dynamics via u-average passivity based feedback. Under suitable growth assumptions on the coupling nonlinearities, the design is extended to multiple cascades according to an iterative procedure which, at each step, makes use of the Lyapunov function and cross-

⋆ Supported by the *Université Franco-Italienne/Italo-Francese* through the program *Vinci*.

Email addresses: mattia.mattioni@uniroma1.it (Mattia Mattioni), salvatore.monaco@uniroma1.it (Salvatore Monaco), cyrot@l2s.centralesupelec.fr (Dorothée Normand-Cyrot).

term arguments proposed in (see Sepulchre et al. (1997)). Finally, the stabilizing feedback is obtained through output-damping.

This work extends the results in Mattioni et al. (2017) by weakening the corresponding assumptions via the concept of Average Output-Feedback-Passivity. Finally, the discrete-time forwarding technique has an immediate application into the sampled-data context, since the feedforward structure is preserved through sampling. However, taking advantage of the continuous-time original system, one might deduce a less conservative sampled-data forwarding strategy which stays in-between the continuous and discrete-time scenarios. In this sense, the work by Mattioni et al. (2019) is devoted to the sampling of continuous-time feedforward design where the difference among the two approaches are discussed as well.

The paper is organized as follows. Preliminaries on discrete-time dynamics and average passivity are in Section 2. Section 3 is devoted to the computation of a Lyapunov function for uncontrolled feedforward dynamics with constructive aspects for classes of systems. Section 4 states the main results. Section 5 develops some computations over simulated examples while conclusions are in Section 6.

Notations and basic assumptions: All mappings and vector fields are assumed smooth in their arguments. Given a mapping $H : \mathbb{R}^n \to \mathbb{R}^m$ with $H(x_1,\dots,x_n)$ we define $\nabla_{x_i} H = \frac{\partial H}{\partial x_i}$ and $\nabla H = (\nabla_{x_1} H \dots \nabla_{x_n} H)$. Accordingly, $\nabla_{x_i} H(\bar{x}) = \nabla_{x_i} H(x)\big|_{x=\bar{x}}$ and, equivalently, $\nabla_x H(\bar{x}) = \nabla_x H(x)\big|_{x=\bar{x}}$.
Given a vector field G over $\mathbb{R}^n$ and a scalar function $V : \mathbb{R}^n \to \mathbb{R}$, we define the Lie derivative of V along G as $\mathrm{L}_{G(\cdot)} V(x) = \nabla V(x) G(x)$. A function $\rho : [0,\infty) \to [0,\infty)$ is said of class $\mathscr{K}$ if it is continuous, strictly increasing and $\rho(0) = 0$. It is said of class $\mathscr{K}_\infty$ if it is $\mathscr{K}$ and it is unbounded. Given a mapping $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $F^{-1}(\cdot,u)$ denotes the inverse function verifying $F(F^{-1}(x,u),u) = x$. The symbol "$\circ$" denotes the composition of functions.

2 Preliminaries on discrete-time systems

Consider a nonlinear discrete-time single-input dynamics described as usual in the form of a map

$$\Sigma_D : \; x_{k+1} = F(x_k, u_k) \tag{1}$$

where $F(\cdot,u) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is smoothly parametrized by the control variable u. It is assumed that $F_0(x) := F(x,0)$ is invertible in the first argument. As proposed in Monaco and Normand-Cyrot (1997), Σ_D can be rewritten in the form of two coupled differential and difference equations ((F_0,G)-representation)

$$x^+ = F_0(x) \tag{2a}$$

$$\frac{\mathrm{d}x^+(u)}{\mathrm{d}u} = G(x^+(u),u) \tag{2b}$$

with $x^+ = x^+(0) = F_0(x)$ and $G(\cdot,u) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ defined according to the equality

$$G(F(x,u),u) := \nabla_u F(x,u). \tag{3}$$

In equations (2), $x^+(u)$ denotes any curve over $\mathbb{R}^n$, parametrized by u. For any given pair (x_k,u_k) for which a solution to (2b) with initial condition fixed by (2a) exists, the integration of (2b) over $u \in [0,u_k[$ gives

$$x_k^+(u_k) = x_k^+(0) + \int_0^{u_k} G(x_k^+(v),v)\mathrm{d}v$$

so recovering $x_k^+(u_k) = F(x_k,u_k)$. This is straightforward from (3) when computing the Taylor expansion of the map $F(x,u)$ around $u = 0$ so obtaining

$$F(x,u) = F_0(x) + \int_0^u \nabla_v F(x,v)\mathrm{d}v.$$

Conversely, a given $\mathbb{R}^n$-valued smooth map $F(\cdot,u)$ can be split into the form (2) whenever there exists a vector field $G(\cdot,u)$ over $\mathbb{R}^n$, parametrized by u satisfying (3). The existence, uniqueness and completeness of $G(\cdot,u)$ are ensured by invertibility of the mapping $F(x,u)$ in (1) with respect to x so uniquely defining $G(x,u)$ as $G(x,u) := \nabla_u F(F^{-1}(x,u),u)$. Such an assumption can be relaxed by requiring $F(\cdot,0)$ invertible so implying existence of $G(\cdot,u)$ for u small enough. The (F_0,G) representation (2) can be extended along the same lines to the multi-input case (see Monaco and Normand-Cyrot (2011) for further details).

A useful consequence of working in the (F_0,G) context, is that given any function $\Lambda(\cdot) : \mathbb{R}^n \to \mathbb{R}$ its variation with respect to u, under the one step ahead evolution (1), can be expressed in the integral form as

$$\Lambda(F(x,u)) - \Lambda(F_0(x)) = \int_0^u \mathrm{L}_{G(\cdot,v)} \Lambda(x^+(v))\mathrm{d}v. \tag{4}$$

In the sequel, $\Sigma_D(H)$ will denote either the dynamics (1) with invertible drift term $F_0(\cdot)$ or its (F_0,G) representation with output mapping $H(\cdot) : \mathbb{R}^n \to \mathbb{R}$. Without loss of generality, it will be assumed that $\Sigma_D(H)$ possesses an equilibrium at $x = 0$ that is $F_0(0) = 0$ with moreover $H(0) = 0$.

2.1 u-average passivity

The notion of *u-average passivity* has been introduced in discrete time by Monaco and Normand-Cyrot (2011) to overpass the necessity of a direct input-output link when referring to a more usual passivity notion.

Definition 2.1 (u-average passivity) *$\Sigma_D(H)$ is said to be* u-average passive *(or* average passive*) if it is passive in the* *usual sense with respect to the u*-average output

$$H^{av}(x,u) := \frac{1}{u}\int_0^u H(x^+(v))\mathrm{d}v \tag{5}$$

$H^{av}(x,0) = H(x^+(0)) = H(F_0(x))$; i.e, there exists a positive semi-definite storage function $S:\mathbb{R}^n \to \mathbb{R}$ such that, for $k \in \mathbb{N}$

$$S(x_{k+1}) - S(x_k) \le H^{av}(x_k,u_k)u_k. \tag{6}$$

Remark 2.1 *A necessary condition for u-average passivity of $\Sigma_D(H)$ is*

$$\mathrm{L}_{G(\cdot,0)}H(F_0(x)) = \nabla_u(H(F(x,u))\big|_{u=0} > 0 \tag{7}$$

in a neighborhood of $x=0$. This corresponds to require that $\Sigma_D(H)$ has relative degree equal to 1 at $x=0$.

Exploiting the (F_0,G) representation, the passivity inequality (6) rewrites as

$$S(F_0(x)) - S(x) + \int_0^u \mathrm{L}_{G(\cdot,v)}S(x^+(v))\mathrm{d}v \le \int_0^u H(x^+(v))\mathrm{d}v \tag{8}$$

where, by definition $\int_0^u H(x^+(v))\mathrm{d}v = uH^{\mathrm{av}}(x,u) = u\int_0^1 H(x^+(su))\mathrm{d}s$. In addition, when the output map depends on u, one has $H^{\mathrm{av}}(x,u) := \frac{1}{u}\int_0^u H(x^+(v),v)\mathrm{d}v$. More in general, one can define u-average passivity from some nominal control value $\bar{u}$ as follows.

Definition 2.2 (u-average passivity from $\bar{u}$) *Given $\bar{u} \in \mathbb{R}$, $\Sigma_D(H)$ is u-average passive from $\bar{u}$ if there exists a positive semi-definite storage function $S:\mathbb{R}^n \to \mathbb{R}$ such that, for any $k \in \mathbb{N}$*

$$S(x_{k+1}) - S(x_k) \le (u_k - \bar{u})H^{av}_{\bar{u}}(x_k,u_k) \tag{9}$$

with

$$H^{av}_{\bar{u}}(x,u) = \frac{1}{u-\bar{u}}\int_{\bar{u}}^u H(x^+(v),v)\mathrm{d}v. \tag{10}$$

u-average passivity from $\bar{u}$ provides the increment of average passivity with respect to a fixed $\bar{u}$ as

$$\int_{\bar{u}}^u H(x^+(v),v)\mathrm{d}v = \int_0^{u-\bar{u}} H(x^+(\bar{u}+v),\bar{u}+v)\mathrm{d}v$$
$$= (u-\bar{u})\int_0^1 H(x^+((1-s)\bar{u}+su),(1-s)\bar{u}+su)\mathrm{d}s.$$

When $\bar{u}=0$ one recovers u-average passivity.

Remark 2.2 *u-average passivity from $\bar{u}$ is strictly reminiscent of the notion of incremental passivity (see Pavlov and Marconi (2008)). It defines incremental-like passivity of the overall system with respect to trajectories that are parametrized by different inputs u rather than time. Moreover, contrarily to incremental passivity, u-average passivity from $\bar{u}$ is referred to the influence of the incremental-like input $\Delta u = u - \bar{u}$ over the same output trajectories.*

The following definition is useful to specify an excess of average passivity

Definition 2.3 (Average OFP(ρ)) *Let a a storage function $S:\mathbb{R}^n \to \mathbb{R}_{\ge 0}$ be positive semi-definite. $\Sigma_D(H)$ is said to be*

- u-average output feedback passive *with $\rho \in \mathbb{R}$ (u-OFP(ρ)), if it is output-feedback passive in the classical sense with respect to the u-average output (5); i.e. for all $k \ge 0$*

$$S(x_{k+1}) - S(x_k) \le H^{av}(x_k,u_k)u_k - \rho(H^{av}(x_k,u_k))^2;$$

- $(u-\bar{u})$-average output feedback passive *with $\rho \in \mathbb{R}$ (u-OFP(ρ)), if it is output-feedback passive in the classical sense with respect to the $(u-\bar{u})$-average output (10); i.e., for all $k \ge 0$*

$$S(x_{k+1}) - S(x_k) \le H^{av}_{\bar{u}}(x_k,u_k)u_k - \rho(H^{av}_{\bar{u}}(x_k,u_k))^2.$$

2.2 u-average passivity-based controller

On these bases, stabilizing u-average passivity based controller (u-AvPBC) can be deduced. For, the notion of zero state detectability is instrumental.

Definition 2.4 *Consider the discrete-time system $\Sigma_D(H)$ and let for $u=0$, $\mathscr{Z} \subset \mathbb{R}^n$ be the largest positively invariant set contained in $\{x \in \mathbb{R}^n \mid y = H(x) = 0\}$. $\Sigma_D(H)$ is said Zero-State-Detectable (ZSD) if $x=0$ is asymptotically stable conditionally to $\mathscr{Z}$.*

The following result extends the celebrated negative output feedback to the discrete-time context via the notion of u-average passivity.

Theorem 2.1 (Monaco and Normand-Cyrot (2011)) *Let $\Sigma_D(H)$ be u-average passive with positive storage function $S:\mathbb{R}^n \to \mathbb{R}$ and be ZSD. Then, any feedback $u = \gamma(x)$ solving the algebraic equation*

$$u + KH^{av}(x,u) = 0, \quad K > 0 \tag{11}$$

achieves global asymptotic stability of the origin of $\Sigma_D(H)$.

The existence of a solution to (11) is guaranteed by the condition

$$1 + \frac{K}{2}\mathrm{L}_{G(\cdot,0)}H(F_0(x)) > 0 \tag{12}$$

which is locally ensured around $x = 0$ by (7). However, computing a closed-form solution requires the inversion of the corresponding series expansion in u deduced from (11). In practice, only approximate solutions can be computed by solving such algebraic equality up to a certain degree of approximation in u so yielding local properties for the closed loop (see Monaco and Normand-Cyrot (1997, 2011)).

2.3 A computable bounded solution

Solving the equality (11) in $O(u^2)$, one easily computes

$$u^{\mathrm{ap}}(x) = -L(x)H(F_0(x)) \tag{13}$$
$$L(x) = \frac{K}{1+\frac{K}{2}\mathrm{L}_{G(\cdot,0)}H(F_0(x))}$$

with $K > 0$. The approximate solution (13) defines a negative feedback on the output, computed one step ahead over free evolution (i.e., $H(F_0(x))$), which only guarantees local asymptotic stability of the closed-loop system.
Inspired by Lemma B.1 in Mazenc and Praly (1996), where an approximate and bounded solution to an implicit equation is proposed, it is possible by setting a suitable gain, to derive a stabilizing feedback that is also bounded (Monaco et al. (2016)).

Theorem 2.2 *Let $\Sigma_D(H)$ be u-average passive with positive storage function $S(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and be ZSD. Then, for any real $\mu > 0$, the feedback $u^b(x) = -\lambda(x)H(F_0(x))$ with*

$$0 < \lambda(x) \leq \frac{\mu \min\{1, C\}}{(2\mu+1)(1+|H(F_0(x))|)} \tag{14}$$

with

$$C = \min_{|u|\leq\frac{1}{2}} \Big\{ \frac{|u|}{|\int_0^1 H(x^+(su))\mathrm{d}s - H(F_0(x))|} \Big\} \tag{15}$$

is bounded (i.e., $|u^b(x_k)| < \mu$ for any $x_k \in \mathbb{R}^n$) and ensures global asymptotic stability of the origin of $\Sigma_D(H)$.

Remark 2.3 *We underline that the bound over the feedback $u^b(x) = -\lambda(x)H(F_0(x))$ can be further developed and generalized to get that $|u^b(x_k)| \leq \frac{\mu}{2\mu+1} \leq 1$ for all $k \in \mathbb{N}$.*

3 Lyapunov cross term for cascade dynamics

Consider the elementary feedforward uncontrolled dynamics

$$\Sigma_0 : \quad \begin{cases} z_{k+1} = & f(z_k) + \varphi(z_k, \xi_k) \\ \xi_{k+1} = & a(\xi_k) \end{cases}$$

where $\xi \in \mathbb{R}^{n_\xi}$, $z \in \mathbb{R}^{n_z}$ and $u \in \mathbb{R}$; f, φ and a are assumed smooth functions in their arguments with $\varphi(z, 0) = 0$. Let Σ_0 possess an equilibrium at the origin. In the sequel, for the sake of brevity, we might refer to the properties of the equilibria of the system as properties of the corresponding dynamics. The following standing assumptions are set.

A.1 $z_{k+1} = f(z_k)$ is Globally Stable - GS - with $\mathscr{K}_\infty$ Lyapunov function $W(z)$;
A.2 $\xi_{k+1} = a(\xi_k)$ is Globally Asymptotically Stable - GAS - and Locally Exponentially Stable - LES - with a C^2 and $\mathscr{K}_\infty$ Lyapunov function $U(\xi)$;
A.3 $\varphi(z, \xi)$ satisfies the *linear growth assumption*; i.e. there exist two class $\mathscr{K}$-functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ such that

$$||\varphi(z,\xi)|| \leq \gamma_1(||\xi||)||z|| + \gamma_2(||\xi||);$$

A.4 $W(z)$ is C^2 and verifies what follows:
- given any $s(\cdot) : \mathbb{R}^{n_z} \to: \mathbb{R}^{n_z}$ and $d(\cdot,\cdot) : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_z}$

$$|W(s(z)+d(z,\xi)) - W(s(z))| \leq \Big|\nabla W(s(z))d(z,\xi)\Big|;$$

- there exist $c, M \in \mathbb{R}_{>0}$ such that for $\|z\| > M$, $\|\nabla W(f(z))\|\|z\| \leq cW(f(z))$.

For concluding GS of the origin of Σ_0, Assumptions A.1 and A.2 are not enough because of the coupling term $\varphi(z,\xi)$ which might grow unboundedly albeit ξ converges to zero exponentially fast. To this end, we show how assumptions A.3 and A.4 enable us to deduce GS of Σ_0 and, furthermore, to build a Lyapunov function $V_0 : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$ for Σ_0.
One starts by assuming V_0 of the form

$$V_0(z,\xi) = W(z) + U(\xi) + \Psi(z,\xi) \tag{16}$$

where the additional *cross-term*, $\Psi : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$, is properly chosen to ensure the semi-negativity of the increment $\Delta_k V_0(z,\xi) = V(z_{k+1}, \xi_{k+1}) - V(z_k, \xi_k)$ along Σ_0. More precisely $\Psi(z,\xi)$ is chosen so to get rid of all the coupling terms with indefinite sign in $\Delta_k V_0(z,\xi)$; i.e., it has to satisfy the equality

$$\Delta_k \Psi(z,\xi) = -W(f(z_k) + \varphi(z_k,\xi_k)) + W(f(z_k)). \tag{17}$$

A solution to (17) is provided by the infinite sum

$$\Psi(z,\xi) = \sum_{k=0}^{\infty} \big[W(f(z_k) + \varphi(z_k,\xi_k)) - W(f(z_k))\big] \tag{18}$$

computed along the trajectories $(z_k, \xi_k) = (z_k(z,\xi), \xi_k(\xi))$ of Σ_0 starting at $(z_0, \xi_0) = (z, \xi)$. With such a choice, one gets that the Lyapunov function is not increasing along the trajectories of Σ_0; i.e., $\Delta_k V_0(z,\xi) \leq \Delta_k U(\xi) \leq 0$. The existence of a solution is guaranteed by the Theorem below; its proof can be found appendix and, in a preliminary version, in Mattioni et al. (2017).

Theorem 3.1 *Consider Σ_0 under **A.1** to **A.4**, then:*

(i) *there exists a continuous function* $\Psi : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$ *solution of (17);*
(ii) *the function* $V_0 : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$ *in (16) is positive-definite and radially unbounded.*

Remark 3.1 *An alternative approach for constructing a Lyapunov function for the system* Σ_0 *consists in defining the so-called composite Lyapunov function as developed by Mazenc and Praly (1996) in continuous time. The extension of this methodology to the discrete-time scenario is not straightforward as notable difficulties arise from the composition (rather than differentiation) of nonlinear functions defining the increment of a given Lyapunov over* Σ_0.

A particular situation arises when Σ_0 exhibits the so-called strict-feedforward structure as discussed in the next section with emphasis on the cross-term interpretation.

3.1 The case of strict-feedforward dynamics

Let the *strict-feedforward dynamics*

$$\Sigma_{20} : \begin{cases} z_{k+1} = & F z_k + \varphi(\xi_k) \\ \xi_{k+1} = & a(\xi_k) \end{cases}$$

where $\varphi(0) = 0$ and the matrix F satisfies $F^\top F = I$ (all the eigenvalues are on the unit circle and with unitary geometric multiplicity). In this case, Assumption **A.1** is satisfied with $W(z) = z^\top z$ and **A.4** follows.
Specifying (17) for Σ_{20} one gets that $\Psi(\cdot)$ must satisfy the equality

$$\Delta_k \Psi(z,\xi) = -2 z_k^\top F^\top \varphi(\xi_k) - \varphi^\top(\xi_k)\varphi(\xi_k). \qquad (19)$$

Because in this case $\Delta_k \Psi(z,\xi) = -\Delta_k W(z)$, a solution to (19) is given by

$$\begin{aligned}\Psi(z,\xi) = &\sum_{k=0}^{\infty} \left[z_{k+1}^\top(z,\xi) z_{k+1}(z,\xi) - z_k^\top(z,\xi) z_k(z,\xi) \right] \\ =&(z_k^\top(z,\xi) z_k(z,\xi))_\infty - z^\top z\end{aligned}$$

with $(z_k^\top(z,\xi) z_k(z,\xi))_\infty = \lim_{k\to\infty} z_k^\top(z,\xi) z_k(z,\xi)$ so getting, according to (16), the Lyapunov function for Σ_{20}

$$V_0(z,\xi) = U(\xi) + (z_k^\top(z,\xi) z_k(z,\xi))_\infty. \qquad (20)$$

Other than studying the stability properties of Σ_{20} through Lyapunov functions and the cross-term, one might notice that Σ_{20} possesses a stable set $\mathscr{S}$ over which the trajectories are described by $\xi_{k+1} = a(\xi_k)$. $\mathscr{S}$ is implicitly defined by

$$\mathscr{S} = \{(z,\xi) \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \text{ s.t. } \phi(\xi) = 0\} \qquad (21)$$

where $\phi : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_z}$ is the smooth mapping

$$\phi(\xi) = -\sum_{\ell=k_0}^{\infty} F^{k_0-1-\ell}\varphi(\xi_\ell) \qquad (22)$$

with $\xi = \xi_{k_0}$ verifying

$$\phi(a(\xi)) = F\phi(\xi) + \varphi(\xi). \qquad (23)$$

Introducing now the coordinates transformation

$$\zeta = z - \phi(\xi) = z + \sum_{\ell=k_0}^{\infty} F^{k_0-1-\ell}\varphi(\xi_\ell) \qquad (24)$$

one gets that Σ_{20} rewrites as the decoupled dynamics

$$\zeta_{k+1} = F\zeta_k \qquad (25a)$$
$$\xi_{k+1} = a(\xi_k) \qquad (25b)$$

possessing a globally stable equilibrium at the origin. A Lyapunov function for the decoupled dynamics (25) is then

$$\tilde{V}_0(\zeta,\xi) = U(\xi) + \zeta^\top \zeta. \qquad (26)$$

Such a Lyapunov function comes to coincide, up to a coordinates change, with the one computed through cross-term in (20). This fact provides an interesting interpretation to the cross-term (18) as stated in the following result.

Proposition 3.1 *Let the strict-feedforward dynamics satisfy* ***A.1****. Then, the Lyapunov function (20) deduced from (18) and (26) computed through (24) coincide, up to a coordinates transformation; namely,* $V_0(z,\xi) = \tilde{V}_0(z-\phi(\xi),\xi)$. *As a consequence, the cross-term takes the form*

$$\Psi(z,\xi) = (z-\phi(\xi))^\top (z-\phi(\xi)) - z^\top z. \qquad (27)$$

Proof: First, rewrite $\zeta^\top\zeta$ for $k_0 = 0$ as

$$\begin{aligned}&(z + \sum_{\ell=0}^{\infty} F^{-1-\ell}\varphi(\xi_\ell))^\top (F^k)^\top F^k (z + \sum_{\ell=0}^{\infty} F^{-1-\ell}\varphi(\xi_\ell)) \\ &= \| z_k(z,\xi) + \sum_{\ell=0}^{\infty} F^{k-\ell-1}\varphi(\xi_\ell) - \sum_{\ell=0}^{k-1} F^{k-\ell-1}\varphi(\xi_\ell)\|^2\end{aligned}$$

because $(F^k)^\top F^k = I$. Letting $k \to \infty$, one gets $\zeta^\top\zeta = (z_k^\top(z,\xi))(z_k(z,\xi))_\infty$. Accordingly, setting $\Psi(z,\xi) = (z-\phi(\xi))^\top(z-\phi(\xi)) - z^\top z$ one easily recovers that the cross term verifies (19) due to the invariance equality (23). ◁

It is important to note that the cross-term in (20) depends on $\lim_{k\to\infty}\|z_k(z,\xi)\|^2$ which always exists, for strict-feedforward structures, albeit $\lim_{k\to\infty} z_k(z,\xi)$ does not (but for the particular case of $F = 1$ and $n_z = 1$).
In this section, the existence of a cross-term of the form (18) is linked to the one of the invariant set $\mathscr{S}$ in (21) for the strict-feedforward dynamics Σ_{20} as well as a coordinates transformation (24) decoupling the subsystems dynamics. In this special case, this is a consequence of the non-resonance condition among the eigenvalues of both F and $\nabla a(0)$ representing Σ_{20}.

3.2 Some further particular cases

Some particular cases are examined below. Let Σ_0 verify **A.1** with Lyapunov function $W(z)$ such that $W(f(z)) - W(z) = 0, \forall z \in \mathbb{R}^{n_z}$. Then, (17) specializes as

$$\Delta_k \Psi(z,\xi) = -W(f(z_k) + \varphi(z_k,\xi_k)) + W(z_k) = -\Delta_k W(z)$$

and the cross-term takes the form

$$\Psi(z,\xi) = \sum_{k=0}^{\infty} \left[W(z_{k+1}) - W(z_k)\right] = W_\infty(z,\xi) - W(z)$$

with $W_\infty(z,\xi) := \lim_{k\to\infty} W(z_k(z,\xi))$ so getting $V_0(z,\xi) = U(\xi) + W_\infty(z,\xi)$.
If in addition $f(z) = z$ in Σ_0, one computes $z_\infty(z,\xi) = z + \sum_{\ell=0}^{\infty} \varphi(z_\ell,\xi_\ell)$ and thus $W_\infty(z,\xi) = W(z_\infty(z,\xi))$. Accordingly, the mapping $(z,\xi) \mapsto (z_\infty,\xi)$ defines a *local* coordinates change since $\nabla_z z_\infty(z,\xi) = I + \sum_{\ell=0}^{\infty} \nabla_z \varphi(z_\ell,\xi_\ell)$ and the sum vanishes at $\xi = 0$. When the connection term $\varphi(\xi,z)$ does not depend on z, the above coordinates change is globally defined as one recovers a strict-feedforward form.

4 Forwarding stabilization

The previous arguments are used in the sequel to achieve stabilization of controlled feedforward dynamics of the form

$$\Sigma_e : \begin{cases} z^n_{k+1} = & f_n(z^n) + \varphi_n(z^1,\dots,z^n,\xi) + g_n(z^1,\dots,z^n,\xi,u) \\ & \vdots \\ z^1_{k+1} = & f_1(z^1) + \varphi_1(z^1,\xi) + g_1(z^1,\xi,u) \\ \xi_{k+1} = & a(\xi) + b(\xi,u) \end{cases}$$

with $\xi \in \mathbb{R}^{n_\xi}$ and $z^i \in \mathbb{R}^{n_{z^i}}$ $(i = 1,\dots,n)$, $u \in \mathbb{R}$; moreover, $g_i(0,\dots,0,z^i,0,0) = g_i(z^1,\dots,z^i,\xi,0) = 0$, $\varphi_i(0,\dots,0,z^i,0,0) = \varphi_i(z^1,\dots,z^i,\xi,0) = 0$ and $b(\xi,0) = 0$. It is assumed that each $f_i(\cdot) + \varphi_i(\cdot)$ is invertible with respect to the corresponding z^i $(i = 1,\dots,n)$ and $a(\cdot)$ invertible with respect to ξ.
The results is first discussed with reference to the two block cascade and then generalized to Σ_e. Basically, it is shown that, whenever Σ_e verifies, for $u = 0$, Assumptions **A.1**, **A.3**, **A.4** and a relaxed version of **A.2**, one can deduce an average passivity-based feedback achieving stabilization in closed loop. When specified to the lower two block cascade, these arguments are then iteratively applied to the augmented cascade embedding at each step a new upper block.

4.1 The two block controlled cascade

Let the augmented two-blocks feedforward cascade

$$\Sigma_1 : \begin{cases} z_{k+1} = f(z_k) + \varphi(z_k,\xi_k) + g(z_k,\xi_k,u_k) & \text{(28a)} \\ \xi_{k+1} = a(\xi_k) + b(\xi_k,u_k) & \text{(28b)} \end{cases}$$

defined on $\mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi}$ with $u \in \mathbb{R}$, $g(z,\xi,0) = b(\xi,0) = 0$, which recovers Σ_0 when setting $u = 0$ and let the origin be an equilibrium. The following assumption is set.

A.5 The mapping $g(z,\xi,u)$ satisfies the *linear growth assumption* in z for all (ξ,u).

According to Section 2, the existence of vector fields $G(\cdot,\cdot,u) : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_z}, B(\cdot,u) : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_\xi}$ satisfying

$$\nabla_u g(z,\xi,u) = G(z^+(u),\xi^+(u),u); \ \nabla_u b(\xi,u) = B(\xi^+(u),u)$$

or, equivalently,

$$g(z,\xi,u) = \int_0^u G(z^+(v),\xi^+(v),v)\mathrm{d}v$$
$$b(\xi,u) = \int_0^u B(\xi^+(v),v)\mathrm{d}v$$

is guaranteed by requiring that the mappings $f(z) + \varphi(z,\xi)$ and $a(\xi)$ are invertible. When necessary, we denote $\bar{F}_0(z,\xi) = \mathrm{col}(f(z) + \varphi(z,\xi),a(\xi))$ and $\bar{G}(z,\xi,u) = \mathrm{col}(G(z,\xi,u),B(\xi,u))$ and $\bar{F}(z,\xi,u) = \mathrm{col}(f(z) + \varphi(z,\xi) + g(z,\xi,u),a(\xi) + b(\xi,u))$.

4.1.1 uOFP(ρ) and uPBC

The design of the feedback under Assumption **A.2** has been studied in Mattioni et al. (2017). Here, such a requirement is weakened as follows.

AR.2 The ξ-dynamics (28b) with output $Y_0(\xi,u) = \mathrm{L}_{B(\cdot,\xi)}U(\xi)$ is uOFP(-$\frac{1}{2}$) with radially unbounded storage function $U(\xi)$; i.e. for all $k \geq 0$

$$\Delta_k U(\xi) \leq Y_0^{\mathrm{av}}(\xi_k,u_k)u_k + \frac{1}{2}(Y_0^{\mathrm{av}}(\xi_k,u_k))^2 \qquad (29)$$

with by definition

$$Y_0^{\mathrm{av}}(\xi,u) := \frac{1}{u}\int_0^u \mathrm{L}_{B(\cdot,v)}U(\xi^+(v))\mathrm{d}v = \frac{1}{u}\int_0^u \nabla_v U(\xi^+(v))\mathrm{d}v.$$

According to Theorem 2.1, the following Lemma is straightforward.

Lemma 4.1 *Let the subdynamics (28b) verify* ***AR.2*** *and be ZSD with output* $Y_0(\xi,0) = \mathrm{L}_{B(\cdot,0)}U(\xi)$. *Then the control* $u_0 = u_0(\xi)$ *solution to*

$$u_0 = -Y_0^{av}(\xi,u_0) \qquad (30)$$

makes the closed-loop equilibrium of the ξ*-dynamics GAS. Moreover, if the linearization of (28b) at the origin is stabilizable, then* u_0 *achieves LES of the closed-loop equilibrium.*

The above result follows from Theorem 3.1.

Lemma 4.2 *Let Σ_1 verify **A.1**, **AR.2**, **A.3**, **A.4** and **A.5** and let the linearization of (28b) at $\xi = 0$ be stabilizable. Then the cross-term $\Psi(\cdot,\cdot) : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$ provided by*

$$\Psi(z,\xi) = \sum_{k=0}^{\infty} (W(z_{k+1}) - W(f(z_k))$$

computed along the trajectories of

$$\tilde{\Sigma}_1 : \begin{cases} z_{k+1} = & f(z_k) + \varphi(z_k,\xi_k) + g(z_k,z_k,u_0(\xi_k)) \\ \xi_{k+1} = & a(\xi_k) + b(\xi_k,u_0(\xi_k)) \end{cases}$$

exists and $V_0(z,\xi) = U(\xi) + \Psi(z,\xi) + W(z)$ is a Lyapunov function for $\tilde{\Sigma}_1$.

The result below is deduced from Theorem 2.1 and Lemmas 4.1 and 4.2. It shows that the partial state feedback $u_0(\xi)$ enables to conclude $(u - u_0(\xi))$-OFP($-\frac{1}{2}$) of Σ_1 with storage function V_0 so recovering assumption **AR.2** stated on Σ_1.

Theorem 4.1 *Let Σ_1 verify **A.1**, **A.3**, **A.4** and **A.5** and let (28b) verify **AR.2** and be ZSD with output $Y_0(\xi,0) = \mathrm{L}_{B(\cdot,0)}U(\xi)$. Let $u_0(\xi)$ be the solution to (30). Then, the following holds:*

(i) *Σ_1 is $(u - u_0(\xi))$-OFP(-$\frac{1}{2}$) with respect to the output*

$$Y_1(z,\xi,u) = \mathrm{L}_{\bar{G}(\cdot,u)}V_0(z,\xi) \quad (31)$$

and radially unbounded storage function $V_0(z,\xi)$ in (16);

(ii) *the feedback $u_1(z,\xi)$ solution of*

$$\begin{aligned} u_1 &= -Y^{av}_{u_0(\xi)}(z,\xi,u_1) \quad (32) \\ &= -\frac{1}{(u_1 - u_0(\xi))} \int_{u_0(\xi)}^{u_1} \mathrm{L}_{\bar{G}(\cdot,v)}V_0(z^+(v),\xi^+(v),v)\mathrm{d}v \end{aligned}$$

achieves GAS of the origin of Σ_1 in closed loop;

(iii) *if the linearization of Σ_1 at the origin is stabilizable, then (32) yields LES of the closed-loop equilibrium.*

Proof: When $u = u_0(\xi)$, Lemmas 4.1 and 4.2 imply

$$\Delta_k V_0(z,\xi)\big|_{u=u_0(\xi_k)} \leq -\frac{1}{2}(Y_0^{\mathrm{av}}(\xi_k,u_0(\xi_k)))^2.$$

Thus, one gets, along Σ_1,

$$\begin{aligned} &\Delta_k V_0(z,\xi) = U(a(\xi)) - U(\xi) + \int_0^u \mathrm{L}_{B(\cdot,v)}U(\xi^+(v))\mathrm{d}v \\ &+ W(f(z) + \varphi(z,\xi)) - W(z) + \int_0^u \mathrm{L}_{G(\cdot,\xi^+(v),v)}W(z^+(v))\mathrm{d}v \\ &+ \Psi(F(z,\xi)) - \Psi(z,\xi) + \int_0^u \mathrm{L}_{\bar{G}(\cdot,v)}\Psi(z^+(v),\xi^+(v))\mathrm{d}v. \end{aligned}$$

Exploiting now the properties of the cross-term $\Psi(\cdot)$ computed for $u = u_0(\xi)$, one verifies that

$$\begin{aligned} &\Delta_k V_0(z,\xi) = \Delta_k V_0(z,\xi)\big|_{u=u_0(\xi)} + \int_{u_0(\xi)}^u \mathrm{L}_{\bar{G}(\cdot,v)}V_0(z^+(v),\xi^+(v))\mathrm{d}v \\ &\leq -\frac{1}{2}|Y_0^{\mathrm{av}}(\xi,u_0(\xi))|^2 + \int_{u_0(\xi)}^u \mathrm{L}_{\bar{G}(\cdot,v)}V_0(z^+(v),\xi^+(v))\mathrm{d}v \\ &= -\frac{1}{2}|Y_0^{\mathrm{av}}(\xi,u_0(\xi))|^2 + (u - u_0(\xi))Y^{\mathrm{av}}_{u_0(\xi)}(z,\xi,u) \\ &= -\frac{1}{2}(Y^{\mathrm{av}}_{u_0(\xi)}(z,\xi,u) - Y_0^{\mathrm{av}}(z,\xi,u_0(\xi)))^2 \\ &\quad + \frac{1}{2}(Y^{\mathrm{av}}_{u_0(\xi)}(z,\xi,u))^2 + uY^{\mathrm{av}}_{u_0(\xi)}(z,\xi,u) \\ &\leq \frac{1}{2}(Y^{\mathrm{av}}_{u_0(\xi)}(z,\xi,u))^2 + uY^{\mathrm{av}}_{u_0(\xi)}(z,\xi,u) \end{aligned}$$

so implying $(u - u_0(\xi))$-OFP(-$\frac{1}{2}$) with respect to the dummy output $Y_1(z,\xi,u) = \mathrm{L}_{\bar{G}(\cdot,u)}V_0(z,\xi)$. Consequently, the feedback solution to the damping implicit equality (32) ensures

$$\begin{aligned} \Delta_k V_0(z,\xi) \leq &-\frac{1}{2}(Y^{\mathrm{av}}_{u_0(\xi)}(z,\xi,u) - Y_0^{\mathrm{av}}(z,\xi,u_0(\xi)))^2 \\ &-\frac{1}{2}(Y^{\mathrm{av}}_{u_0(\xi)}(z,\xi,u))^2 \leq 0. \end{aligned}$$

Accordingly, GAS under $u = u_1(z,\xi)$ as in (32) follows if the equilibrium of Σ_2 is GAS conditionally to the largest invariant set contained into

$$\begin{aligned} &\{(z,\xi) \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \text{ s.t. } (Y^{\mathrm{av}}_{u_0(\xi)}(z,\xi,0) - Y_0^{\mathrm{av}}(z,\xi,u_0(\xi)))^2 \\ &\quad + (Y^{\mathrm{av}}_{u_0(\xi)}(z,\xi,0))^2 = 0\} \equiv \\ &\{(z,\xi) \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \text{ s.t. } Y^{\mathrm{av}}_{u_0(\xi)}(z,\xi,0) = Y_0^{\mathrm{av}}(z,\xi,u_0(\xi)), \\ &\quad Y^{\mathrm{av}}_{u_0(\xi)}(z,\xi,0) = 0\} \equiv \\ &\{(z,\xi) \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \text{ s.t. } Y_0^{\mathrm{av}}(z,\xi,0) = 0\}. \end{aligned}$$

Thus, ZSD of (28b) with respect to $Y_0(\xi,0) = \mathrm{L}_{B(\cdot,0)}U(\xi)$ ensures the result. LES follows when the dynamics is stabilizable in first approximation. ◁

Along the lines of Section 2.3, the equality (32) admits a local solution in $O(u^2)$ of the form

$$u_1^{\mathrm{ap}}(z,\xi) = -L_1(z,\xi,u_0^{\mathrm{ap}}(\xi))Y_1(F(z,\xi,u_0^{\mathrm{ap}}(\xi)),u_0^{\mathrm{ap}}(\xi))$$

with $u_0^{\mathrm{ap}}(\xi) = -L_0(\xi)Y_0(a(\xi),0)$ and

$$L_1(z,\xi,u_0^{\mathrm{ap}}(\xi)) = \frac{1}{1 + \frac{1}{2}\mathrm{L}_{\bar{G}(\cdot,u_0^{\mathrm{ap}}(\xi))}Y_1(\bar{F}(z,\xi,u_0^{\mathrm{ap}}(\xi)),u_0^{\mathrm{ap}}(\xi))}$$

$$L_0(\xi) = \frac{1}{1 + \frac{1}{2}\mathrm{L}_{B(\cdot,0)}Y_0(a(\xi),0)}$$

with $L_0(\xi)$ and $L_1(z,\xi,u_0^{\mathrm{ap}}(\xi))$ being well-defined because of u-average passivity as recalled in Section 2.2. Such a feedback will ensure local asymptotic stability of the origin in closed loop. Along the lines of Theorem (2.2) the corollary below follows.

Corollary 4.1 *Let Σ_1 verify **A.1**, **A.3**, **A.4** and **A.5** and let (28b) verify **AR.2** and be ZSD with output $Y_0(\xi,0) = \mathrm{L}_{B(\cdot,0)}U(\xi)$. Let $u_0(\xi)$ be the solution to (30) and let*

$$u_0^b(\xi) = -\lambda_0(\xi)Y_0(a(\xi),0)$$

be defined as in Theorem 4.1. Then, for any real $\mu_1 > 0$, the feedback

$$u_1^b(z,\xi) = -\lambda_1(z,\xi)Y_1(F(z,\xi,u_0^b(\xi)),u_0^b(\xi))$$

with $\lambda_1(\cdot) > 0$ satisfying

$$\lambda_1(z,\xi) \leq \frac{\mu_1 \min\{1, C_1\}}{(2\mu_1+1)(1+|Y_1(F(z,\xi,u_0^b(\xi)),u_0^b(\xi))|)}$$

with $C_1 = \min_{|u|\leq\frac{1}{2}}\{\frac{|u|}{|Y^{av}_{u_0^b(\xi)}(z,\xi,u)-Y_1(F(z,\xi,u_0^b(\xi)),u_0^b(\xi))|}\}$ is bounded (i.e., $|u_1^b(z_k,\xi_k)| < \mu_1$ for all $k \geq 0$) and ensures global asymptotic stability of the origin of Σ_1.

Remark 4.1 *For classes of feedforward systems of the form*

$$\begin{aligned} z_{k+1} &= Fz_k + \alpha(z_k,\xi_k)\xi_k + \beta(z_k,\xi_k,u_k)u_k \\ \xi_{k+1} &= a(\xi_k) + b(z_k,\xi_k,u_k) \end{aligned}$$

an alternative and different solution has been proposed by Mazenc and Nijmeijer (1998). Such an approach does not involve passivity and passivitation of the dynamics and is rather focused on the definition of composite Lyapunov functions so requiring stronger assumptions on the coupling mappings. Moreover, $\xi_{k+1} = a(\xi_k)$ is assumed to possess a globally asymptotically stable equilibrium whereas F is demanded to be weakly Schur not necessarily verifying $F^\top F = I$. Summarizing, when $W(z) = z^\top Qz$ and κ is suitably defined, the idea is to compute a bounded feedback $u = u_m(z,\xi)$ so to make the composite Lyapunov function $\mathrm{V}(z,\xi) = \kappa(U(\xi)) + \ln(1+z^\top Qz)$ negative (semi)definite by involving nested upper bounds so getting, in the end, $u_m(z,\xi) = -L_m(z,\xi)N(z,\xi,0)$ and mapping $N(z,\xi,u)$ (that is not the the average passive output) deduced from the upper bounds over $\Delta_k\mathrm{V}(z,\xi)$.

4.2 Extended feedforward structures

The proposed procedure extends to the n-blocks feedforward dynamics Σ_e under the same assumptions **A.1**, **A.3**, **A.4**, **A.5** reformulated for each sub-dynamics $j = 1,\dots,n$ in a straightforward manner. Moreover, the ξ-dynamics is required to verify Assumption **AR.2**.

Basically, If the linearization of Σ_e at the origin is stabilizable, then GAS and LES of the closed-loop equilibrium can be achieved by extending the here presented strategy in a bottom-up way. The consequent procedure is aimed at exploiting OFP-like properties that are implicitly ensured, at each step i, with respect to the corresponding output Y_i. For the sake of compactness, we introduce the following notations. $\mathbf{z} = \mathrm{col}(z^n,\dots,z^1)$, $G(\mathbf{z}^+(u),u) = \mathrm{col}(G_n(\cdot),\dots,G_1(\cdot),B(\cdot))$ with any $G_i(\cdot,u)$ being such that $\nabla_u g_i(z^1,\dots,z^i,\xi,u) = G_i(z^{1+}(u),\dots,z^{i+}(u),\xi^+(u),u)$.

Initialization: Set $Y_0(\xi,u) = \mathrm{L}_{B(\cdot,u)}U(\xi^+(u))$, $u_0 = -\frac{1}{u_0}\int_0^{u_0}\mathrm{L}_{B(\cdot,v)}U(\xi^+(v))\mathrm{d}v$ ensuring GAS and LES of the ξ-dynamics.

Step 1: Set

$$\begin{aligned} &V_0(z^1,\xi) = W_0(z^1) + \Psi_0(z^1,\xi) + U(\xi) \\ &\Psi_0(z^1,\xi) = \sum_{k=0}^{\infty}\left[W_0(f_1(z_k^1) + \tilde{\varphi}_1(z_k^1,\xi_k)) - W_0(f_1(z_k^1))\right] \\ &\tilde{\varphi}_1(z_1,\xi) = \varphi_1(z_1,\xi) + \int_0^{u_0}\mathrm{L}_{G(\cdot,v)}V_0(\mathbf{z}^+(v),\xi^+(v))\mathrm{d}v \\ &Y_1(z^1,\xi,u) = \mathrm{L}_{G(\cdot,u)}V_0(\mathbf{z},\xi) \\ &u_1 = -\frac{1}{u_1-u_0}\int_{u_0}^{u_1}Y_1(z^{1+}(v),\xi^+(v),v)\mathrm{d}v. \end{aligned}$$

Now the design can be reported to the case $n = 2$. For, one sets at each step i, $\bar{\xi}_i = col(z^{i-1},\dots,z,\xi)$ that clearly verifies **AR.2** by construction as detailed here below.

Step i: Define

$$\begin{aligned} &V_{i-1}(\cdot) = W_{i-1}(z^i) + \Psi_{i-1}(z^1,\dots,z^i,\xi) + V_{i-2}(z^1,\dots,z^{i-1},\xi) \\ &\Psi_{i-1}(\cdot) = \sum_{k=0}^{\infty}\left[W_{i-1}(f_i(z_k^1) + \tilde{\varphi}_i(z_k^1,\dots,z_k^i,\xi_k)) - W_{i-1}(f_1(z_k^1))\right] \\ &\tilde{\varphi}_i(\cdot) = \varphi_i(z^1,\dots,z^i,\xi) + \int_0^{u_{i-1}}\mathrm{L}_{G(\cdot,v)}V_{i-1}(\mathbf{z}^+(v),\xi^+(v))\mathrm{d}v \\ &Y_i(z^1,\dots,z^i,\xi,u) = \mathrm{L}_{G(\cdot,u)}V_{i-1}(\mathbf{z},\xi) \\ &u_i = -\frac{1}{u_i-u_{i-1}}\int_{u_{i-1}}^{u_i}Y_i(z^{1+}(v),\dots,z^{i+}(v),\xi^+(v),v)\mathrm{d}v \end{aligned}$$

where the sum is evaluated along the trajectories of Σ_e from the initial state $(z^i,\dots,z^1,\xi)$ and under the feedback u_{i-1}. Applying this procedure n times one gets the result below.

Theorem 4.2 *Let all sub-dynamics z^j of Σ_e verify **A.1**, **A.3**, **A.4**, **A.5** and the ξ-dynamics verify **AR.2** being ZSD with respect to the output $Y_0(\xi,u) = \mathrm{L}_{B(\cdot,u)}U(\xi)$. Then, if Σ_e is stabilizable in first approximation, the control control $u = u_n(\mathbf{z},\xi)$ computed as the implicit solution of*

$$\begin{aligned} &u_n = -\frac{1}{u_n-u_{n-1}}\int_{u_{n-1}}^{u_n}\mathrm{L}_{G(\cdot,v)}V_{n-1}(\mathbf{z}^+(v),\xi^+(v))\mathrm{d}v \\ &V_{n-1}(\mathbf{z},\xi) = U(\xi) + \sum_{i=1}^{n}\left[W_{i-1}(z^i) + \Psi_{i-1}(z^1,\dots,z^i,\xi)\right] \end{aligned}$$

makes the origin of Σ_e GAS and LES.

4.3 The case of strict-feedforward dynamics

Consider now the augmented strict-feedforward dynamics

$$\Sigma_2 : \begin{cases} z_{k+1} = & Fz_k + \varphi(\xi_k) + g(\xi_k, u_k) \\ \xi_{k+1} = & a(\xi_k) + b(\xi_k, u_k) \end{cases}$$

with F satisfying $F^\top F = I$ and the dynamics $\xi_{k+1} = a(\xi_k)$ invertible. Moreover, one verifies by definition that

$$\nabla_u g(\xi, u) = G(a(\xi) + b(\xi, u), u)$$
$$\nabla_u b(\xi, u) = B(a(\xi) + b(\xi, u), u).$$

As previously noted for Σ_0, **A.1** and **A.4** are verified by setting $W(z) = z^\top z$, while **A.3** and **A.5** relax to requiring that $\|\varphi(\xi)\| \leq \gamma_1(\|\xi\|)$ and $\|g(\xi,u)\| \leq \gamma_2(\|(\xi,u)\|)$ for some $\mathscr{K}$ functions $\gamma_i(\cdot)$ $(i = 1,2)$. Assuming now **AR.2** and stabilizability of the ξ-system at the origin, the control $u = u_0(\xi)$ can be constructed so to make the equilibrium of the ξ-dynamics GAS and LES. Consequently, Lemma 4.2 applies and one can find a cross-term $\Psi(z,\xi)$ solution to $\Delta_k \Psi(z,\xi) = -\Delta_k W$ along the closed-loop trajectories of

$$\tilde{\Sigma}_2 \begin{cases} z_{k+1} = & Fz_k + \varphi(\xi_k) + g(\xi_k, u_0(\xi_k)) \\ \xi_{k+1} = & a(\xi_k) + b(\xi_k, u_0(\xi_k)). \end{cases}$$

As discussed before, one computes the coordinates change $\zeta = z - \tilde{\phi}(\xi)$ as $\tilde{\phi}(\xi) = -\sum_{\ell=k_0}^{\infty} F^{k_0-1-\ell}\tilde{\varphi}(\xi_\ell)$ with $\tilde{\varphi}(\xi) = \varphi(\xi) + g(\xi, u_0(\xi))$ so getting the decoupled dynamics

$$\zeta_{k+1} = F\zeta_k, \quad \xi_{k+1} = \tilde{a}(\xi_k).$$

The Lyapunov function $\tilde{V}_0(\zeta,\xi) = U(\xi) + \zeta^\top\zeta$ coincides with $V_0(z,\xi)$. and the problem of stabilizing Σ_2 via cross-term can be lead to the one of stabilizing the equivalent

$$\zeta_{k+1} = F\zeta_k + \int_{u_0(\xi_k)}^{u_k} G_\zeta(\xi^+(v), v)\mathrm{d}v$$
$$\xi_{k+1} = a(\xi_k) + \int_0^{u_0(\xi_k)} B(\xi^+(v), v)\mathrm{d}v + \int_{u_0(\xi_k)}^{u_k} B(\xi^+(v), v)\mathrm{d}v$$

with $G_\zeta(\xi^+(u), u) = G(\xi^+(u), u) - \mathrm{L}_{B(\cdot,u)}\phi(\xi^+(u))$. Hence, Theorem 4.1 holds with output $Y_1(\zeta,\xi,u) = \mathrm{L}_{\bar{G}_\zeta(\cdot,u)}\tilde{V}_0(\zeta,\xi)$ and stabilizing feedback $u = u_1(z,\xi)$ solution of

$$u = -\frac{1}{(u - u_0(\xi))}\int_{u_0(\xi)}^{u} \mathrm{L}_{G_\zeta(\cdot,v)}\tilde{V}_0(\zeta^+(v), \xi^+(v), v)\mathrm{d}v.$$

Remark 4.2 *When $\varphi(\xi) = \hat{\varphi}(\xi)\xi$ with $|\varphi(\xi)| \leq \Gamma(\xi)|\xi|$ for some positive function $\Gamma(\cdot)$ and the feedback is implemented as in Corollary 4.1, one recovers the solution by Mazenc and Nijmeijer (1998).*

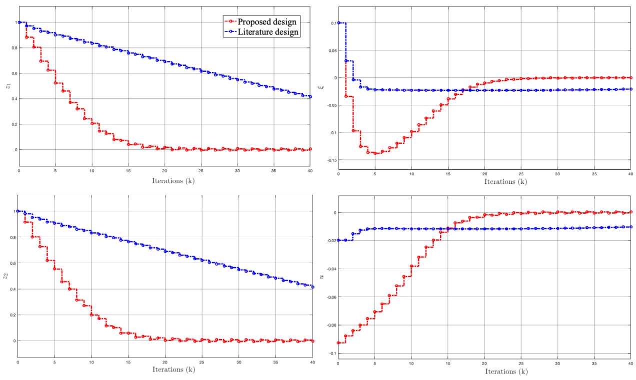

Fig. 1. The proposed bounded u-avPBC feedback vs standard bounded feedforwarding

Remark 4.3 *When $F = I$ and $n_z = 1$, the coordinates change $\zeta = z - \tilde{\phi}(\xi)$ makes the ζ-dynamics driftless once the preliminary control $u_0(\xi)$ has been applied. Accordingly, one recovers the result in Monaco and Normand-Cyrot (2013) proposed when assuming directly in Σ_2, $\xi_{k+1} = u_k$.*

Remark 4.4 *In Monaco et al. (2016), the stabilization problem of strict-feedforward systems is set in the framework of Immersion and Invariance (I&I, Astolfi and Ortega (2003)) when $n_z = 1$. Assuming **AR.2**, a stable set over which the closed loop ξ-dynamics evolves is exhibited. The design aims at stretching the off-stable set components ζ to zero while ensuring boundedness of the full state trajectories. Moreover, I&I is less demanding since the knowledge of a Lyapunov function $U(\xi)$ for the ξ-system is not necessary.*

Remark 4.5 *The procedure in Section 4.2 specifies to multi-block strict feedforward dynamics along the same lines. At each step, one looks for a coordinates change $\zeta^i = z^i - \phi_i(\zeta^1, \ldots, \zeta^{i-1}, \xi)$ that decouples the corresponding dynamics in the new coordinates when $u = u_{i-1}(z^1, \ldots, z^{i-1}, \xi)$. As a matter of fact, at each step, one makes the set $\zeta_i = 0$ globally asymptotically stable for the augmented cascade. Furthermore, such a set is made invariant by the control $u_i(z^1, \ldots, z^i, \xi)$ which also makes it attractive and achieves GAS of the augmented cascade.*

5 Examples

5.1 A strict-feedforward dynamics

Let us compare the proposed control with the one available in Mazenc and Nijmeijer (1998) through the *almost* strict-feedforward dynamics described by

$$z_{1,k+1} = z_{2,k} + \frac{3}{2}\xi_k^2 + \frac{3}{2}u_k + (\theta_1(z_k,\xi_k) + \theta_2(\xi_k))u_k^2 \quad (33a)$$
$$z_{2,k+1} = z_{1,k} + \xi_k^2 + u_k + \theta_2(\xi_k)u_k^2 \quad (33b)$$
$$\xi_{k+1} = \frac{1}{2}\xi_k + u_k + \theta_2(\xi_k)u_k^2. \quad (33c)$$

The dynamics satisfies the Assumptions in Theorem 4.1 with (33c) possessing a globally exponentially stable equilibrium at the origin with $U(\xi) = \frac{1}{2}U(\xi)$. Accordingly, setting $z = \text{col}\{z_1, z_2\}$, along Section 3.1, the cross-term can be computed through the decoupling change of coordinates $\zeta = z - \varphi(\xi)$ with $\varphi(\xi) = -\frac{1}{25}\begin{pmatrix}22 & 28\end{pmatrix}^\top \xi^2$. Thus, when $u = 0$, the origin of (33) is GS with Lyapunov function $V_1(z,\xi) = U(\xi) + \frac{1}{2}\|z - \varphi(\xi)\|^2$ so that u-average passivity follows and the bounded stabilizing damping feedback can be deduced via Corollary 2.2.
As far as the control laws are concerned, the comments in Remark 4.1 hold to this case and the solution proposed in Mazenc and Nijmeijer (1998) through composite Lyapunov functions does not invoke or imply u-average passivation in closed loop. For completeness, a simple simulation is reported in Figure 1 by applying both (bounded) feedbacks with the same bounding constant $\mu_1 = 1$. The result shows that both feedbacks are comparable and provide satisfactory performances.

5.2 A general feedforward dynamics

Let us apply the results in Section 4 to the dynamics in feedforward form

$$z_{k+1} = e^{\xi_k + \frac{u_k}{2}} z_k; \quad \xi_{k+1} = \xi_k + u_k$$

also described as

$$z^+ = e^{\xi} z; \quad \frac{\partial z^+(u)}{\partial u} = \frac{1}{2} z^+(u)$$
$$\xi^+ = \xi; \quad \frac{\partial \xi^+(u)}{\partial u} = 1.$$

The standing assumptions are verified with $U(\xi) = \frac{1}{2}\xi^2$, $W(z) = \frac{1}{2}z^2$. According to the initial step of the forwarding procedure, one computes over the ξ-dynamics $Y_0(\xi) = \xi$ and $u_0(\xi) = -\frac{2}{3}\xi$. As a consequence, because $f(z) = z$, one has that the cross term can be directly computed as in Section 3.2 so getting $\Psi(z,\xi) = \frac{1}{2}(e^{2\xi} - 1)z^2$. The corresponding Lyapunov function is $V_0(z,\xi) = \frac{1}{2}\xi^2 + \frac{1}{2}e^{2\xi}z^2$. Thus, the overall system is u average passive from u_0 with respect to the output $Y_1(z,\xi) = \xi + \frac{3}{2}e^{2\xi}z^2$ verifying $Y_1(0,\xi) = Y_0(\xi)$. Thus, the stabilizing feedback $u = u_1(z,\xi)$ is the solution to (32) which takes the form $u = -\xi - \frac{1}{2}(u + u_0) - \frac{1}{2}e^{4\xi}\frac{e^{3u} - e^{3u_0}}{u - u_0}z^2$. Since the above equation is hard to be solved, one can compute the approximation proposed in Corollary 2.2. Starting from the exact solution $u_0(\xi) = -\frac{2}{3}\xi$, one computes $u_1^b(z,\xi) = -\lambda_1(z,\xi)Y_1(F(z,\xi,u_0),u_0)$ with $Y_1(F(z,\xi,u_0),u_0) = \xi + \frac{1}{2}u_0 + \frac{1}{2}e^{4\xi}\frac{1 - e^{3u_0}}{1 - u_0}z^2$.

The behavior of the closed-loop dynamics under the approximate feedback $u_1^b(z,\xi)$ is tested through simulations fixing $(\xi_0 \; z_0)^\top = (1 \; 1)^\top$ and for different values of μ_1. The results are in Figure 2 where the trajectories showing that asymptotically stability is guaranteed by the bounded u-average passivity based feedback. Moreover, we underline that as μ_1 increases, the trajectories of the closed-loop dynamics are basically identical as $|u_b^1| \leq \frac{\mu_1}{2\mu_1 + 1} \approx 0.5$ as $\mu \geq 25$. As expected, as μ decreases, the convergence rate to zero does.

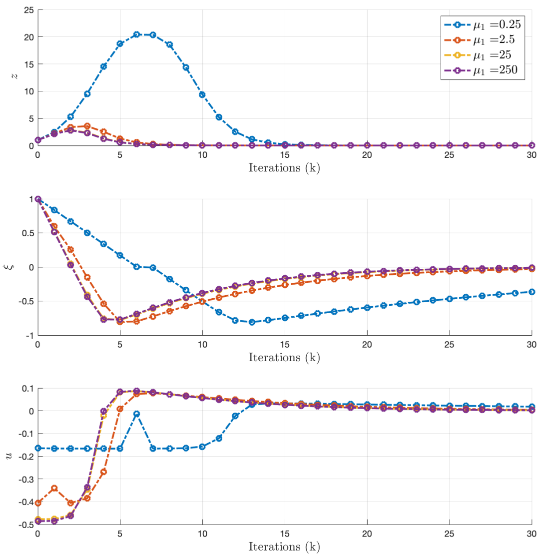

Fig. 2. The dynamics in Example 5.2 for $(\xi_0 \; z_0)^\top = (1 \; 1)^\top$ and several values of the bounding constant $\mu_1 > 0$.

6 Conclusions

This paper describes a constructive forwarding design for discrete-time cascade dynamics. The design is iterative and involves, at each step, average-passivation and the construction of a Lyapunov function. In case of strict-feedforward dynamics, the proposed strategy recovers the one developed in the literature through successive coordinate transformation and average passivation.

Acknowledgement

The authors are grateful to the anonymous reviewers for the relevant comments and suggestions that were helpful to improve the quality of the work.

References

Ahmed-Ali, T., Mazenc, F., Lamnabhi-Lagarrigue, F., 1999. Disturbance attenuation for discrete-time feedforward nonlinear systems. In: Stability and Stabilization of Nonlinear Systems. Springer, pp. 1–17.

Aranda-Bricaire, E., Moog, C. H., 2004. Invariant codistributions and the feedforward form for discrete-time nonlinear systems. Systems & control letters 52 (2), 113–122.

Astolfi, A., Ortega, R., 2003. Immersion and invariance: a new tool for stabilization and adaptive control of nonlinear systems. IEEE Transactions on Automatic Control 48 (4), 590–606.

Chiang, C. J., Huang, C. C., Jankovic, M., June 2010. Discrete-time cross-term forwarding design of robust controllers for hcci engines. In: Proceedings of the 2010 American Control Conference. pp. 2218–2223.

Isidori, A., 1995. Nonlinear Control Systems. Springer.

Jankovic, M., 2006. Control of cascade systems with time delay–the integral cross-term approach. In: 45th IEEE Conference on Decision and Control. pp. 2547–2552.

Jiang, Z.-P., Wang, Y., 2001. Input-to-state stability for discrete-time nonlinear systems. Automatica 37 (6), 857–869.

Kazakos, D., Tsinias, J., 1994. The input-to-state stability condition and global stabilization of discrete-time systems. Automatic Control, IEEE Transactions on 39 (10), 2111–2113.

Kazantzis, N., 2004. A new approach to the zero-dynamics assignment problem for nonlinear discrete-time systems using functional equations. Systems & Control Letters 51 (3), 311–324.

Khalil, H., 2002. Nonlinear Systems. Prentice-Hall.

Lin, W., Gong, Q., 2003. A remark on partial-state feedback stabilization of cascade systems using small gain theorem. IEEE transactions on automatic control 48 (3), 497–499.

Lin, W., Pongvuthithum, R., 2002. Global stabilization of cascade systems by c/sup 0/partial-state feedback. IEEE transactions on automatic control 47 (8), 1356–1362.

Mattioni, M., Monaco, S., Normand-Cyrot, D., 2019. Feedforwarding under sampling. IEEE Transactions on Automatic Control.

Mattioni, M., Monaco, S., Normand-Cyrot, S. D., Dec 2017. Lyapunov stabilization of discrete-time feedforward dynamics. In: 2017 IEEE 56th Annual Conference on Decision and Control (CDC). pp. 4272–4277.

Mazenc, F., Nijmeijer, H., 1998. Forwarding in discrete-time nonlinear systems. International Journal of Control 71 (5), 823–835.

Mazenc, F., Praly, L., 1996. Adding an integration and global asymptotic stabilization of feedforward systems. IEEE Transactions on Automatic Control 41 (11), 1559–1578.

Monaco, S., Normand-Cyrot, D., 1997. A unified representation for nonlinear discrete-time and sampled dynamics. Journal of Mathematical Systems Estimation and Control 7, 477–503.

Monaco, S., Normand-Cyrot, D., 2011. Nonlinear average passivity and stabilizing controllers in discrete-time. Systems & Control Letters 60, 431–439.

Monaco, S., Normand-Cyrot, D., 2013. Stabilization of nonlinear discrete-time dynamics in strict-feedforward form. In: European Control Conference 13. pp. 2186–2191.

Monaco, S., Normand-Cyrot, D., Mattioni, M., 2016. Stabilization of feedforward discrete-time dynamics through immersion and invariance. In: 2016 American Control Conference (ACC). pp. 264–269.

Navarro-LóPez, E. M., Fossas-Colet, E., 2004. Feedback passivity of nonlinear discrete-time systems with direct input–output link. Automatica 40 (8), 1423–1428.

Nešić, D., Teel, A. R., Sontag, E. D., 1999. Formulas relating kl stability estimates of discrete-time and sampled-data nonlinear systems. Systems & Control Letters 38 (1), 49–60.

Pavlov, A., Marconi, L., 2008. Incremental passivity and output regulation. Systems & Control Letters 57 (5), 400–409.

Sepulchre, R., Janković, M., Kokotović, P., 1997. Constructive nonlinear control. Springer New York.

Wei, L., Byrnes, C., 1994. Design of discrete-time nonlinear control systems via smooth feedback. IEEE Transactions on Automatic Control 39 (11), 2340–2346.

A Proof of Theorem 3.1

The proof starts by showing **(i)**. To this purpose, since the equilibrium of $\xi_{k+1} = a(\xi_k)$ is LES, we can write that for a real constant $\alpha \in (0,1)$ and function $\gamma(\cdot) \in \mathscr{K}$, then $\gamma(\|\xi_k\|) \le \alpha^k \gamma(\|\xi\|)$ for any $k \ge 0$. To prove the result we need to prove that $W(f(z_k) + \varphi(z_k, \xi_k)) - W(f(z))$ is summable. To this end, we first prove that z_k is bounded for any $k \ge 0$. For, by exploiting **A.4**, we compute

$$\begin{aligned} W(z_{k+1}) =& W(f(z_k) + \varphi(z_k, \xi_k)) \\ \le& W(z_k) + \|\nabla W(f(z_k))\| \|\varphi(z_k, \xi_k)\|. \end{aligned}$$

Using now **A.3** we have

$$\begin{aligned} W(z_{k+1}) \le& W(z_k) + \|\nabla W(f(z_k))\| \big(\gamma_1(\|\xi_k\|)\|z_k\| + \gamma_2(\|\xi_k\|)\big) \\ \le& W(z_k) + \|\nabla W(f(z_k))\| \alpha^k \big(\gamma_1(\|\xi\|)\|z_k\| + \gamma_2(\|\xi\|)\big) \end{aligned}$$

where $\xi = \xi_0$ and the latter bound exploits the LES property of $\xi_{k+1} = a(\xi_k)$. Accordingly, now, because $\gamma_i(\|\xi\|)$ for $i = 1,2$ are constant, one can find $\gamma(\cdot) \in \mathscr{K}$ such that

$$\begin{aligned} W(z_{k+1}) \le& W(z_k) + \|\nabla W(f(z_k))\| \gamma(\|\xi\|) \alpha^k \big(1 + \|z_k\|\big) \\ \le& W(z_k) + 2\gamma(\|\xi\|)\alpha^k \|\nabla W(f(z_k))\| \|z_k\|. \end{aligned}$$

Applying now **A.1** and **A.4** and assuming $\|z_k\| > \max\{1, M\}$

$$W(z_{k+1}) \le (1 + \mathbf{c}_1(\|\xi\|)\alpha^k) W(z_k)$$

with constant $\mathbf{c}_1(\|\xi\|) = 2c\gamma(\|\xi\|)$ implying that, as $k \to \infty$, $W(z_{k+1}) = W(z_k)$ and, thus, boundedness of $W(z_k)$ for any $k \ge 0$. Because $W(\cdot)$ is assumed radially unbounded, boundedness of $W(z_k)$ implies the one of $\|z_k\|$.

Accordingly, considering now $W(f(z_k) + \varphi(z_k, \xi_k)) - W(f(z_k))$ and exploiting the above bound, one gets

$$W(f(z_k) + \varphi(z_k, \xi_k)) - W(f(z_k)) \le \mathbf{c}_1(\|\xi\|)\alpha^k W(z_k) \tag{A.1}$$

Because $W(z_k)$ and $\|z_k\|$ are bounded for any time $k \ge 0$, one gets that there exists a constant $\mathbf{c_2}(\|(z, \xi)\|)$ depending on the initial state (z,ξ) such that

$$W(f(z_k)+\varphi(z_k,\xi_k))-W(f(z_k))\le \mathbf{c}_2(\|(z,\xi)\|)\alpha^k \quad \text{(A.2)}$$

so getting that $W(f(z_k)+\varphi(z_k,\xi_k))-W(f(z_k))$ is summable over $[0,\infty)$ and (18) exists and is bounded for all (z,ξ). Continuity of (18) comes from the fact that it is the composition and the sum of continuous-functions on $[0,\infty)$.
As far as **(ii)** is concerned, positive definiteness of V_0 is obtained by exploiting the radial unboundedness of $W(z)$.

$$\begin{aligned}&W(z_k)=W(z)+\sum_{t=0}^{k-1}\big[W(f(z_t)+\varphi(z_t,\xi_t))-W(z_t)\big]=\\&W(z)+\sum_{t=0}^{k-1}\big[W(f(z_t)+\varphi(z_t,\xi_t))-W(f(z_t))\big]+\\&\sum_{t=0}^{k-1}\big[W(f(z_t))-W(z_t)\big]\end{aligned}$$

where the term $W(f(z_t))-W(z_t)$ is non-increasing for any $t\ge 0$. By substracting both sides of the last equality by $W(f(z_t))-W(z_t)$ and taking the limit for $k\to\infty$ one gets

$$\begin{aligned}&W_\infty(z)-\sum_{t=0}^{\infty}\big[W(f(z_t))-W(z_t)\big]=\\&W(z)+\sum_{t=0}^{\infty}\big[W(f(z_t)+\varphi(z_t,\xi_t))-W(f(z_t))\big]\end{aligned}$$

where $W_\infty(z)=\lim_{k\to\infty}W(z_k)$ and $\Psi(z,\xi)=\sum_{t=0}^{\infty}\big[W(f(z_t)+\varphi(z_t,\xi_t))-W(f(z_t))\big]$. Hence, one gets that $V_0(z,\xi)$ rewrites

$$V_0(z,\xi)=W_\infty(z)-\sum_{t=0}^{\infty}\big[W(f(z_t))-W(z_t)\big]+U(\xi)\ge 0. \quad \text{(A.3)}$$

From the radially unboundedness of W and U one has that if $V_0(z,\xi)=0$ then $\xi=0$. By construction, $V_0(z,0)=W(z)$ so concluding that $V_0(z,\xi)=0$ implies $(z,\xi)=(0,0)$. According to the last inequality this proves that V_0 is positive-definite.
To prove its radial unboundedness we first point out that from (A.3) it follows that $V_0(z,\xi)\to\infty$ as $\|\xi\|\to\infty$ for any z. Hence, one has to show that

$$\lim_{\|z\|\to+\infty}\Big[W_\infty(z)-\sum_{t=0}^{\infty}\big(W(f(z_t))-W(z_t)\big)\Big]=+\infty. \quad \text{(A.4)}$$

This will be achieved by lowerbounding (A.4) by means of a radially unbounded function deduced from $W(z)$. For, consider $C=\mathbf{c}(\|\xi\|)$ in (A.1). Accordingly, for any $k\ge 0$ we write

$$\begin{aligned}&|W(f(z_k)+\varphi(z_k,\xi_k))-W(f(z_k))|\le\\&\|\frac{\partial W}{\partial z}\|(C|\alpha|^k+C|\alpha|^k\|z_k\|).\end{aligned}$$

It follows that

$$\begin{aligned}&W(f(z_k)+\varphi(z_k,\xi_k))-W(f(z_k))\ge\\&-|W(f(z_k)+\varphi(z_k,\xi_k))-W(f(z_k))|\ge\\&\ge -2\|\frac{\partial W}{\partial z}\|C|\alpha|^k\|z_k\|-C(1-\|z_k\|)\|\frac{\partial W}{\partial z}\||\alpha|^k.\end{aligned}$$

When $1-\|z_k\|>0$ the term $-C(1-\|z_k\|)\frac{\partial W}{\partial z}\||\alpha|^k$ can be discarded without affecting the inequality. On the other hand, when $1-\|z_k\|\le 0$, it is bounded by $K_2|\alpha|^k$ so that

$$\begin{aligned}&W(f(z_k)+\varphi(z_k,\xi_k))-W(f(z_k))\ge\\&-2\|\frac{\partial W}{\partial z}\|C|\alpha|^k\|z_k\|-K_2|\alpha|^k.\end{aligned}$$

Using **A.4** we obtain

$$W(f(z_k)+\varphi(z_k,\xi_k))-W(z_k)\ge \quad \text{(A.5)}$$
$$\begin{cases}-K|\alpha|^kW(z_k)-K_2|\alpha|^k+W(f(z_k))-W(z_k), & \|z\|>r\\ -K_1|\alpha|^kW(z_k)-K_2|\alpha|^k+W(f(z_k))-W(z_k), & \|z\|\le r\end{cases}$$

with $r\ge 1$ and real K,K_1,K_2.

$\|z\|>r$ and $k\in[0,t)$

$$\begin{aligned}W(z_k)\ge\ &\phi(k,0)W(z)+\sum_{t=0}^{k-1}\phi(k-1,t)\big[-K_2|\alpha_1|^t+\\&W(f(z_t))-W(z_t)\big]\end{aligned}$$

$\|z\|\le r$ and $k\in[0,t)$

$$W(z_k)\ge W(z)+\sum_{t=0}^{k-1}\big[-K_1|\alpha|^t-K_2|\alpha|^t+W(f(z_t))-W(z_t)\big]$$

with $\phi(k,t)=\prod_{j=t}^{k}(1-K|\alpha|^j)$. Accordingly, by mixing both the bounds, one gets

$$\begin{aligned}W(z_k)\ge&\phi(k,0)W(z)+\\&\sum_{t=0}^{k-1}(-K_1|\alpha|^t-K_2|\alpha|^t+W(f(z_t))-W(z_t))\end{aligned}$$

so that for all $k\ge 0$, $\phi(k,0)$ admits a lower bound K_3 and

$$W(z_k)\ge K_3W(z)+\sum_{t=0}^{k-1}\big[W(f(z_t))-W(z_t)\big]+r_k$$

with $r_k:=\sum_{t=0}^{k-1}\big[-K_1|\alpha|^t-K_2|\alpha|^t\big]$ which converges to a bounded solution r^* over $[0,\infty)$. So, taking the limit when $k\to\infty$ one obtains

$$W_\infty(z,\xi)-\sum_{t=0}^{k-1}\big[W(f(z_t))-W(z_t)\big]\ge K_3W(z)+r^*.$$

It is clear that r^* and K_3 may depend on ξ but are independent of z so that (A.4) holds.

Accordingly, by construction $V_0(z_{k+1}, \xi_{k+1}) - V_0(z_k, \xi_k)) = W(f(z_k)) - W(z_k) + U(a(\xi_k)) - U(\xi_k) \leq 0$ so concluding the proof.