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New axiomatisations of discrete quantitative and qualitative possibilistic integrals [☆]

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Abstract

Necessity (resp. possibility) measures are very simple min-decomposable (resp. max-decomposable) representations of epistemic uncertainty due to incomplete knowledge. They can be used in both quantitative and qualitative settings. In the present work, we revisit Choquet and Sugeno integrals as criteria for decision under uncertainty and propose new axioms when uncertainty is representable in possibility theory. First, a characterization of Choquet integral with respect to a possibility or a necessity measure is proposed. We respectively add an optimism or a pessimism axiom to the axioms of the Choquet integral with respect to a general capacity. This new axiom enforces the maxitivity or the minitivity of the capacity without requiring the same property for the functional. It essentially assumes that the decision-maker preferences only reflect the plausibility ordering between states of nature. The obtained pessimistic (resp. optimistic) criterion is an average maximin (resp. maximax) criterion of Wald across cuts of a possibility distribution on the state space. The additional axiom can be also used in the axiomatic approach to Sugeno integral and generalized forms thereof to justify possibility and necessity measures. The axiomatization of these criteria for decision under uncertainty in the setting of preference relations among acts is also discussed. We show that the new axiom justifying possibilistic Choquet integrals can be expressed in this setting. In the case of Sugeno integral, we correct a characterization proof for an existing set of axioms on acts, and study an alternative set of axioms based on the idea of non-compensation.

Keywords: Choquet integral; Sugeno integral; Possibility theory

1. Introduction

In multiple-criteria decision making, discrete fuzzy integrals are commonly used as aggregation functions. They calculate a global evaluation for objects or alternatives evaluated according to some criteria, taking into account dependencies between them. When the evaluation scale is quantitative, Choquet integrals are often used, while in the case of a qualitative scale Sugeno integrals are more naturally considered [26]. The definition of the discrete fuzzy

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integrals is based on a monotonic set function named capacity after Choquet (see [35,44] and a complete bibliography in [25]), or fuzzy measure after Sugeno [43], that attaches importance or uncertainty weights to groups of criteria or states. Capacities are used in many areas such as uncertainty modeling [7], multicriteria aggregation [11,27,26] or in game theory [40,25]. The similarity between Sugeno and Choquet integrals was noticed quite early [5,4].

The characterization of Choquet integral on quantitative scales is based on a general capacity [41,4,31], which covers the case of a lower or an upper probability defined from a family of probability functions. There are no results concerning the characterisation of the Choquet integral with respect to a possibility or a necessity measure. In contrast, for the qualitative setting, characterizations of Sugeno integrals with respect to possibility measures exist [23,6]. This does not come as a surprise since Sugeno integrals with respect to necessity (resp. possibility) measures are still minitive (resp. maxitive) functionals, while this is not the case for the corresponding Choquet integrals.

This paper proposes a new property called *levelwise-pessimism* (resp. *levelwise-optimism*) to be added to axioms characterizing Choquet integrals that may justify the use of a necessity (resp. possibility) measure for representing uncertainty or weights of groups of criteria. This additional axiom enforces the minitivity (resp. the maxitivity) of the capacity without forcing the functional to have these properties. It essentially assumes that the decision-maker preferences only reflect the plausibility ordering between criteria or states of nature. Such specific Choquet integrals, which generalize maximin and maximax criteria of Wald, are currently used in signal processing based on maxitive kernels [24] or in sequential decision [1].

We also show that the same levelwise-pessimism and levelwise-optimism properties can be added to characterisations of Sugeno integrals, to obtain possibilistic qualitative integrals (weighted min and max). Finally we show that these properties can be expressed in the Savage setting of preference between acts, and discuss the possibility of act-based characterization of possibilistic Choquet integrals. In this paper, we also contribute to the decision-theoretic foundations of Sugeno integrals by revisiting and improving existing proofs, and proposing an alternative non-compensation axiom.

This program contrasts with the work of Rebillé [37], which is a qualitative counterpart to Von Neumann and Morgenstern approach to expected utility, where loteries involving frequentist probabilities are replaced by possibility distributions, and Choquet integrals with respect to necessity measures are obtained. Here, we do not assume that uncertainty is represented by possibility theory, but the latter is derived from the axioms.

The paper is organized in two parts. The next section is devoted to Choquet integral. We recall possibilistic Choquet integrals and introduce the new properties (one expressing optimism, one expressing pessimism) that, added to existing characteristic properties, justify possibility theory as the underlying uncertainty representation. Then we point out that adapting the framework to the setting of Savage acts and decision under uncertainty is somewhat problematic due to the difficulty of retrieving infinitary versions of maxitivity and minitivity. The rest of the paper applies the same program to Sugeno integral in the finite, totally ordered setting. In Section 3, it is first shown that characteristic properties of Sugeno integrals can be augmented with the same levelwise-pessimism and levelwise-optimism properties as in the case of Choquet integral so as to characterize the weighted min or max for multi-criteria decision making (MCDM). Then we reconsider the preference-based axiomatisations of Sugeno integrals, pointing out a gap in existing proofs, and proposing an alternative proof, as well as one approach based on a non-compensation axiom, and finally an alternative axiomatisation of qualitative possibilistic criteria based on the levelwise-pessimism and levelwise-optimism axioms.

2. Characterization of possibilistic Choquet integrals

We adopt the notations used in MCDM, where some objects or alternatives are evaluated according to a common finite set $\mathcal{C} = \{1, \dots, n\}$ of criteria. In the case of decision under uncertainty (DMU), $\mathcal{C}$ is the set of the possible states of the world. A common, totally ordered, evaluation scale V is assumed to provide ratings according to the criteria. Each object is identified with a function $f = (f_1, \dots, f_n) \in V^n$, called a *profile*, where f_i is the evaluation of f according to criterion i . The set of all these objects (or acts in the setting of DMU) is denoted by $\mathcal{V}$.

A capacity or fuzzy measure is a non-decreasing set function $\mu : 2^{\mathcal{C}} \rightarrow L$, where L is a totally ordered scale with top 1 and bottom 0 such that $\mu(\emptyset) = 0$ and $\mu(\mathcal{C}) = 1$, with $L \subseteq V$. The two scales are commensurate. The reason for choosing two scales L and V instead of a single one, is first that V is a utility scale while L is an uncertainty scale. In the numerical setting, one may take $L = [0, 1]$ while $V = [0, \infty)$. In the qualitative setting, L and V are finite, and we assume they have the same top and bottom, but one may have that the image $\mu(2^{\mathcal{C}})$ of μ be a proper subset of V .

When L is equipped with a negation denoted by $1 - \cdot$, the conjugate of a capacity μ is defined by $\mu^c(A) = 1 - \mu(\bar{A})$, where $\bar{A}$ is the complement of A .

A possibility measure Π is a capacity obeying the *maxitivity* property: $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$ [45]. If $\pi = (\pi_1, \dots, \pi_n)$ is the possibility distribution associated with Π (i.e., $\pi_i = \Pi(\{i\})$), we have $\Pi(A) = \max_{i \in A} \pi_i$, which makes it clear that $\pi_i = 1$ for some i . In multi-criteria decision making, π_i is the importance of the criterion i . In the case of decision under uncertainty, π_i represents the plausibility of the state i . A necessity measure is a capacity N obeying the *minitivity* property $N(A \cap B) = \min(N(A), N(B))$ [14]; then we have $N(A) = \min_{i \notin A} 1 - \pi_i$ since functions Π and N are conjugate capacities. See [14] for an introduction to possibility theory, and [16] for a recent bibliography.

2.1. Discrete possibilistic Choquet integrals

In this part, L is supposed to be the unit interval. The Möbius transform associated with a numerical capacity μ is the set function m_μ attaching to each set $T \subseteq \mathcal{C}$ a value

$$m_\mu(T) = \sum_{K \subseteq T} (-1)^{|T \setminus K|} \mu(K),$$

where $\sum_{T \subseteq \mathcal{C}} m_\mu(T) = 1$ (see for instance [25]). The sets T such that $m_\mu(T) \neq 0$ are called the focal sets of μ . Note that subsets other than singletons may have negative Möbius values. Using m_μ , the discrete Choquet integral of a function $f : \mathcal{C} \rightarrow \mathbb{R}$ with respect to a capacity μ can be simply expressed as a generalized weighted mean [9]:

$$C_\mu(f) = \sum_{T \subseteq \mathcal{C}} m_\mu(T) \min_{i \in T} f_i. \quad (1)$$

Suppose μ is a necessity measure N and let σ be the permutation of the criteria such that $1 = \pi_{\sigma(1)} \geq \dots \geq \pi_{\sigma(n)} \geq \pi_{\sigma(n+1)} = 0$. The Choquet integral of f with respect to N boils down to:

$$C_N(f) = \sum_{i=1}^n (\pi_{\sigma(i)} - \pi_{\sigma(i+1)}) \min_{j: \pi_j \geq \pi_{\sigma(i)}} f_j = \sum_{i=1}^n (\pi_{\sigma(i)} - \pi_{\sigma(i+1)}) \min_{j=1}^i f_{\sigma(j)} \quad (2)$$

since the focal sets of N are among the sets $\{\sigma(1), \dots, \sigma(i)\}_{i=1, \dots, n}$ and their value for the Möbius transform is $\pi_{\sigma(i)} - \pi_{\sigma(i+1)}$ respectively. Using the identity $C_\Pi(f) = 1 - C_N(1 - f)$ one obtains the Choquet integral of f with respect to the conjugate possibility measure in the form:

$$C_\Pi(f) = \sum_{i=1}^n (\pi_{\sigma(i)} - \pi_{\sigma(i+1)}) \max_{j: \pi_j \geq \pi_{\sigma(i)}} f_j = \sum_{i=1}^n (\pi_{\sigma(i)} - \pi_{\sigma(i+1)}) \max_{j=1}^i f_{\sigma(j)} \quad (3)$$

Note that if $\pi_1 = \dots = \pi_n = 1$ then $C_N(f) = \min_{i=1}^n f_i$ and $C_\Pi(f) = \max_{i=1}^n f_i$ are Wald maximin and maximax criteria, respectively. Moreover if many criteria have the same importance π_i , then the expression of C_N (resp. C_Π) proves that we take into account the worst (resp. best) value of f_j according to these criteria.

It is well known and worth noticing that the functional C_N is not minitive and C_Π is not maxitive [8] as shown by the following example.

Example 1. We consider $\mathcal{C} = \{1, 2\}$, the possibility distribution π and the following profiles f and g as in the table below:

	{1}	{2}
π	1	0.5
f	0.2	0.3
g	0.4	0.1

	{1}	{2}
$\min(f, g)$	0.2	0.1
$\max(f, g)$	0.4	0.3

Then, $C_N(f) = 0.5 \cdot 0.2 + 0.5 \cdot 0.2 = 0.2$ and $C_N(g) = 0.5 \cdot 0.4 + 0.5 \cdot 0.1 = 0.25$, but $C_N(\min(f, g)) = 0.5 \cdot 0.2 + 0.5 \cdot 0.1 = 0.15 \neq \min(C_N(f), C_N(g))$ and $C_N(\max(f, g)) = 0.5 \cdot 0.4 + 0.5 \cdot 0.3 = 0.35 \neq \max(C_N(f), C_N(g))$.

2.2. Pessimistic and optimistic substitute profiles

Using the permutation σ on the criteria associated with π , a pessimistic profile $f^{\sigma,-}$ and an optimistic profile $f^{\sigma,+}$ can be associated with each profile f [15]: $\forall i = 1, \dots, n$,

$$f_i^{\sigma,-} = \min_{j:\pi_j \geq \pi_{\sigma(i)}} f_j = \min_{j=1}^i f_{\sigma(j)}; \quad f_i^{\sigma,+} = \max_{j:\pi_j \geq \pi_{\sigma(i)}} f_j = \max_{j=1}^i f_{\sigma(j)}. \quad (4)$$

Clearly, $f^{\sigma,-}$ and $f^{\sigma,+}$ are respectively obtained by taking the worst and the best evaluation at each level of π . Observe that only the ordering of elements i induced by π on $\mathcal{C}$ is useful in the definition of the pessimistic and optimistic profiles associated with f . These profiles correspond to the values of f appearing in the weighted mean expressions (2) and (3). Substituting pessimistic and optimistic profiles associated with f in these expressions, possibilistic Choquet integrals take the form of usual discrete expectations E_p w.r.t. a probability distribution p on $\mathcal{C}$, with $p_{\sigma(i)} = m_{\sigma(i)} = \pi_{\sigma(i)} - \pi_{\sigma(i+1)}$:

$$C_N(f) = \sum_{i=1}^n m_{\sigma(i)} f_i^{\sigma,-} = E_p(f^{\sigma,-}) = C_N(f^{\sigma,-});$$

$$C_{\Pi}(f) = \sum_{i=1}^n m_{\sigma(i)} f_i^{\sigma,+} = E_p(f^{\sigma,+}) = C_{\Pi}(f^{\sigma,+}).$$

Two profiles f and g are said to be comonotonic if and only if for all $i, j \in \mathcal{C}$, $f_i > f_j$ implies $g_i \geq g_j$. So f and g are comonotonic if and only if there exists a permutation τ on $\mathcal{C}$ such that $f_{\tau(1)} \leq \dots \leq f_{\tau(n)}$ and $g_{\tau(1)} \leq \dots \leq g_{\tau(n)}$. For any profile f , we have $f_1^{\sigma,-} \geq \dots \geq f_n^{\sigma,-}$ and $f_1^{\sigma,+} \leq \dots \leq f_n^{\sigma,+}$. So for any pair of profiles f and g , $f^{\sigma,-}$ and $g^{\sigma,-}$ (resp. $f^{\sigma,+}$ and $g^{\sigma,+}$) are comonotonic.

There are several profiles that are evaluated the same as a given one by a possibilistic Choquet integral. Especially we can define a sequence of progressively changing profiles $(\phi_k)_{1 \leq k \leq n}$, ranging from f to $f^{\sigma,-}$, that are equivalently evaluated by C_N . Namely, $\phi_1 = f$, $\phi_{k+1} \leq \phi_k$, $\phi_n = f^{\sigma,-}$ where $\phi_{k+1} = \phi_k$ except for one coordinate. The profiles ϕ_k are defined by

$$\phi_k(i) = \begin{cases} \min_{l:\pi_l \geq \pi_{\sigma(i)}} f_l & \text{if } i \leq k \\ f_i & \text{otherwise.} \end{cases}$$

Similarly we can define a sequence of profiles, ranging from f to $f^{\sigma,+}$, that are equivalently evaluated by C_{Π} . Namely, $(\phi^k)_{1 \leq k \leq n}$ such that $\phi^1 = f$, $\phi^{k+1} \geq \phi^k$, $\phi^n = f^{\sigma,+}$ where $\phi^{k+1} = \phi^k$ except for one coordinate. The profiles ϕ^k are defined by

$$\phi^k(i) = \begin{cases} \max_{l:\pi_l \geq \pi_{\sigma(i)}} f_l & \text{if } i \leq k \\ f_i & \text{otherwise.} \end{cases}$$

We observe that $C_N(f) = C_N(\phi_k)$, $C_{\Pi}(f) = C_{\Pi}(\phi^k)$, for all $1 \leq k \leq n$.

Example 2. We consider $\mathcal{C} = \{1, 2, 3\}$, the permutation associated with Π such that $\pi_1 \geq \pi_2 \geq \pi_3$ and the profile $f = (2, 1, 3)$. We have $\phi_2 = (2, 1, 3)$ and $\phi_3 = f^{\sigma,-} = (2, 1, 1)$; and $\phi^2 = (2, 2, 3)$ and $\phi^3 = f^{\sigma,+} = (2, 2, 3)$. We can check that $C_N(f) = C_N(\phi_2) = C_N(\phi_3) = 2(1 - \pi_2) + \pi_2$, while $C_{\Pi}(f) = C_{\Pi}(\phi^2) = C_{\Pi}(\phi^3) = 2(1 - \pi_3) + 3\pi_3$.

2.3. Representation theorem

Consider the case of Boolean functions, corresponding to subsets A, B of $\mathcal{C}$. Their profiles are just characteristic functions $\mathbf{1}_A, \mathbf{1}_B$. Given a permutation σ induced by π , let us find their corresponding optimistic and pessimistic Boolean profiles.

Lemma 1. For all $A \subseteq \mathcal{C}$ non-empty, $\mathbf{1}_A^{\sigma,-} = \mathbf{1}_B$ for a subset $B = A^{\sigma,-} \subseteq A$ and $\mathbf{1}_A^{\sigma,+} = \mathbf{1}_B$ for a superset $B = A^{\sigma,+} \supseteq A$.

Proof. $\mathbf{1}_A^{\sigma,-}(i) = \min_{k=1}^i \mathbf{1}_A(\sigma(k)) = 1$ if $\forall k \leq i, \sigma(k) \in A$, and 0 otherwise. So $\mathbf{1}_A^{\sigma,-} = \mathbf{1}_B \leq \mathbf{1}_A$, hence $B \subseteq A$.
 $\mathbf{1}_A^{\sigma,+}(i) = \max_{k=1}^i \mathbf{1}_A(\sigma(k)) = 1$ if $\exists k \leq i, \sigma(k) \in A$, and 0 otherwise. So $\mathbf{1}_A^{\sigma,+} = \mathbf{1}_B \geq \mathbf{1}_A$, hence $A \subseteq B$. $\square$

It is easy to realize that the set $A^{\sigma,-}$ exactly contains the largest sequence of consecutive criteria $(\sigma(1), \dots, \sigma(k^-))$ in A , while the set $A^{\sigma,+}$ exactly contains the smallest sequence of consecutive criteria $(\sigma(k^+), \dots, \sigma(n))$ that includes A .

Lemma 2. *A capacity μ is a necessity measure if and only if there exists a permutation σ on $\mathcal{C}$ such that for all A we have $\mu(A) = \mu(A^{\sigma,-})$.*

A capacity μ is a possibility measure if and only if there exists a permutation σ on $\mathcal{C}$ such that for all A we have $\mu(A) = \mu(A^{\sigma,+})$.

Proof. Let σ be such that $\mu(A) = \mu(A^{\sigma,-}), \forall A \subseteq \mathcal{C}$. Let us prove that for all $A, B \subseteq \mathcal{C}$, we have $\mu(A \cap B) = \min(\mu(A), \mu(B))$.

By assumption, $\mu(A \cap B) = \mu((A \cap B)^{\sigma,-})$ with $(A \cap B)^{\sigma,-} = \{\sigma(1), \dots, \sigma(k^-)\}$. From Lemma 1, $(A \cap B)^{\sigma,-} \subseteq A \cap B$, but $\sigma(k^- + 1) \notin A \cap B$. Hence, $\sigma(k^- + 1) \notin A$ or $\sigma(k^- + 1) \notin B$. Suppose without loss of generality that $\sigma(k^- + 1) \notin A$. Then $A^{\sigma,-} = (A \cap B)^{\sigma,-}$ hence $\mu(A \cap B) = \mu(A) \leq \mu(B)$ so $\mu(A \cap B) = \min(\mu(A), \mu(B))$. Consequently μ is a necessity measure.

Conversely we consider a necessity measure N and the permutation σ such that $\pi_{\sigma(1)} \geq \dots \geq \pi_{\sigma(n)}$. Clearly, $N(A) = 1 - \pi_{\sigma(i_0)}$ with $i_0 = \min\{j : \sigma(j) \notin A\}$. So the set $A^{\sigma,-}$ is $\{1, \dots, i_0 - 1\}$ and $N(A^{\sigma,-}) = 1 - \pi_{i_0} = N(A)$.

A similar proof can be developed for the case of possibility measures. Let σ be such that $\mu(A) = \mu(A^{\sigma,+})$. Let us prove that for all $A, B \subseteq \mathcal{C}$, we have $\mu(A \cup B) = \max(\mu(A), \mu(B))$.

By assumption, $\mu(A \cup B) = \mu((A \cup B)^{\sigma,+})$, where $\{\sigma(k^+), \dots, \sigma(n)\}$. From Lemma 1, $(A \cup B)^{\sigma,+} \supseteq A \cup B, \forall A \subseteq \mathcal{C}$. As $A \cup B \not\subseteq \{\sigma(k^+ + 1), \dots, \sigma(n)\}$, then $\sigma(k^+) \in A$ or $\sigma(k^+) \in B$. Suppose without loss of generality that $\sigma(k^+) \in A$. Then $A^{\sigma,+} = (A \cup B)^{\sigma,+}$ hence $\mu(A \cup B) = \mu(A) \geq \mu(B)$, so $\mu(A \cup B) = \max(\mu(A), \mu(B))$. Consequently μ is a possibility measure.

Conversely we consider a possibility measure Π and the permutation σ such that $\pi_{\sigma(1)} \geq \dots \geq \pi_{\sigma(n)}$. Then, $\Pi(A) = \pi_{\sigma(i_0)}$ with $i_0 = \min\{j : \sigma(j) \in A\}$. $A^{\sigma,+}$ is $\{i_0, \dots, n\}$ so $\Pi(A^{\sigma,+}) = \pi_{i_0}$. $\square$

Now we add suitable axioms to a known representation theorem for Choquet integral [41,4], and obtain a characterisation theorem for the case when the capacity is a possibility or a necessity measure. When $f_i \geq g_i, \forall i \in \mathcal{C}$, we write $f \geq g$, for short.

Theorem 1. *A function $I : \mathcal{V} \rightarrow \mathbb{R}$ satisfies the following properties:*

C1 $I(1, \dots, 1) = 1$,

C2 *Comonotonic additivity: $I(f + g) = I(f) + I(g)$ for f and g comonotonic,*

C3 *Pareto-dominance: $f \geq g$ implies $I(f) \geq I(g)$,*

Π4 *Levelwise-optimism property: There exists a permutation σ on $\mathcal{C}$ such that $\forall A, I(\mathbf{1}_A) = I(\mathbf{1}_A^{\sigma,+})$,*

if and only if $I = C_\Pi$, where $\Pi(A) = I(\mathbf{1}_A)$ is a possibility measure.

Proof. It is easy to check that the Choquet integral with respect to Π satisfies the properties C1-C3 and Π4 according to the permutation associated with π . Conversely, if I satisfies the properties C1-C3, then according to the results presented in [41,4], I is a Choquet integral with respect to the fuzzy measure μ defined by $\mu(A) = I(\mathbf{1}_A)$. The property Π4 implies $\mu(A) = I(\mathbf{1}_A^{\sigma,+}) = \mu(A^{\sigma,+})$, and, using Lemma 2, this equality is equivalent to have a possibility measure. $\square$

Note that, in the previous theorem, Axiom Π4 can be replaced by : There exists a permutation σ on $\mathcal{C}$ such that $\forall f, I(f) = I(f^{\sigma,+})$.

We have a similar characterisation theorem for necessity measures:

Theorem 2. A function $I : \mathcal{V} \rightarrow \mathbb{R}$ satisfies the following properties:

C1 $I(1, \dots, 1) = 1$,

C2 Comonotonic additivity: $I(f + g) = I(f) + I(g)$ for f and g comonotonic,

C3 Pareto-dominance: $f \geq g$ implies $I(f) \geq I(g)$,

N4 Level-pessimism property: There exists a permutation σ on $\mathcal{C}$ such that $\forall A, I(\mathbf{1}_A) = I(\mathbf{1}_A^{\sigma, -})$

if and only if $I = C_N$, where $N(A) = I(\mathbf{1}_A)$ is a necessity measure.

Note that in the previous Theorem 2, Axiom N4 can be replaced by : There exists a permutation σ on $\mathcal{C}$ such that $\forall f, I(f) = I(f^{\sigma, -})$.

Axioms $\Pi 4$ and N4 also make sense in decision under uncertainty, where the set $\mathcal{C}$ is a set of states of the world, and f_i is the consequence of an act in state i . An optimistic decision maker is represented using a possibility measure since under $\Pi 4$, the attractiveness of the profile in state i ($f_i^{\sigma, +}$) is never less than the greatest value f_j among the states more plausible than i . Particularly, if the state i is the most plausible state and $f_i = 1, f_j = 0, \forall j \neq i$ then we have $C_\Pi(f) = C_\Pi(g)$, where $g_i = 1, \forall i \in \mathcal{C}$. The expected profit in a very plausible state is not affected by the expected losses in less plausible states. The Choquet integral calculates the average of the best consequences for each plausibility level, hence the name “levelwise-optimism”. For a pessimistic decision maker, the attractiveness of the act in state i ($f_i^{\sigma, -}$) is never greater than the smallest value f_j among the states that are more plausible than j . Particularly, if the state i is the most plausible one, and $f_i = 0$ while $f_j = 1, \forall j \neq i$, then we have $C_N(f) = C_N(g)$, with $g_i = 0, \forall i \in \mathcal{C}$. The expected profits in the least plausible states cannot compensate the expected losses in more plausible states. In this case, the Choquet integral calculates the average of the worst consequences for each plausibility level.

The proposal of $\Pi 4$ and N4 as axioms can be questioned by pointing out their difficulty to be tested in practice, as there are $n!$ possible ordering of elements in $\mathcal{C}$. However, one may directly ask a decision-maker for the importance ordering between criteria (or the plausibility ordering of states), as this information is less demanding than, e.g., asking for genuine numerical weights. Then we can verify if the properties expressed by the axioms $\Pi 4$ and N4 correspond to the attitude of the decision-maker or not, through pairwise comparison of well-chosen profiles.

These results indicate that Choquet integrals w.r.t possibility and necessity measures are additive for a larger class of pairs of functions than usual, as first pointed out in [15], for instance $C_N(f + g) = C_N(f) + C_N(g)$ as soon as $(f + g)^{\sigma, -} = f^{\sigma, -} + g^{\sigma, -}$, which does not imply that f and g are comonotonic.

Example 3. We consider $\mathcal{C} = \{1, 2, 3\}$, the permutation associated with Π such that $\pi_1 \geq \pi_2 \geq \pi_3$ and the profiles $f = (1, 2, 3), g = (1, 3, 2)$ which are not comonotonic. We have $f^{\sigma, -} = (1, 1, 1), g^{\sigma, -} = (1, 1, 1)$ and $(f + g)^{\sigma, -} = (2, 2, 2)$ so $(f + g)^{\sigma, -} = f^{\sigma, -} + g^{\sigma, -}$.

The above result should be analyzed in the light of a claim by Mesiar and Šipoš [34] stating that if the capacity μ is additive on the set of cuts $\{\{i : f_i \geq \alpha\} : \alpha > 0\} \cup \{\{i : g_i \geq \alpha\} : \alpha > 0\}$ of f and g , then $C_\mu(f + g) = C_\mu(f) + C_\mu(g)$. For a general capacity, additivity holds only if f and g are comonotonic. For more particular capacities, the set of pairs of acts for which additivity holds on cuts can be larger. This is what seems to happen with possibility and necessity measures.

2.4. Possibilistic Choquet integrals: the infinite setting

In the context of decision under uncertainty (DMU), the set of criteria is replaced by a set S of more or less plausible states of affairs and profiles are called acts. Then, in the classical DMU framework, S is an infinite set, typically the real line or an interval of the real line; f is then a real function. Possibilistic uncertainty can be modelled by fuzzy intervals [10], that is, possibility distributions π on S such that $\{r : \pi(r) \geq \alpha\}$ is a closed interval, $\forall \alpha \in (0, 1]$. The pessimistic Choquet integral of a positive value function f is of the form [15]:

$$C_N(f) = \int_0^{+\infty} N(f \geq r) dr = \int_0^1 f^{\pi^-}(\alpha) d\alpha$$

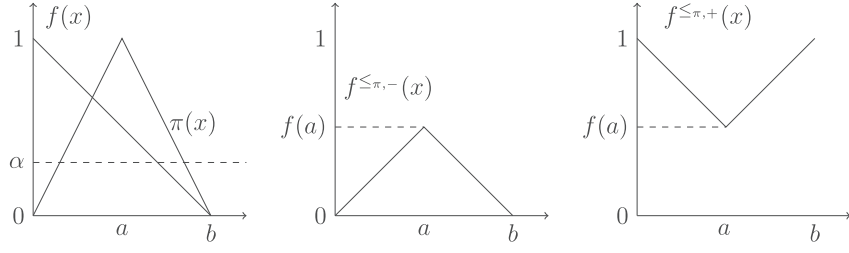


Fig. 1. Continuous case.

where $f^{\pi-}(\alpha) = \inf\{f(r) : \pi(r) \geq \alpha\}$. Likewise for the optimistic Choquet integral:

$$C_{\Pi}(f) = \int_0^{+\infty} \Pi(f \geq r) dr = \int_0^1 f^{\pi+}(\alpha) d\alpha$$

where $f^{\pi+}(\alpha) = \sup\{f(r) : \pi(r) \geq \alpha\}$. They are continuous counterparts of Equations (2) and (3) noticing that a possibility distribution can be viewed as a nested random set $\alpha \in [0; 1] \mapsto \{s : \pi(s) \geq \alpha\}$ based on Lebesgue measure on $[0, 1]$ [13].

If π is the membership function of a fuzzy interval M , under the random set interpretation, upper and lower mean values of M can be defined, i.e., $E_*(M)$ and $E^*(M)$, respectively, such that [13,29]:

$$E_*(M) = \int_0^1 \inf M_{\lambda} d\lambda; \quad E^*(M) = \int_0^1 \sup M_{\lambda} d\lambda, \quad (5)$$

where M_{λ} is the λ -cut of M . Note that these expressions are Choquet integrals of the identity function with respect to the possibility and the necessity measures induced by M . The mean interval of a fuzzy interval M is defined as $E(M) = [E_*(M), E^*(M)]$. It is thus the interval containing the mean values of all random variables compatible with M (i.e., such that $P(A) \leq \Pi(A)$, $\forall A$ measurable). Then it is clear that the following result holds:

Proposition 1. *If f is a continuous function, $C_N(f) = E_*(f(M))$ and $C_{\Pi}(f) = E^*(f(M))$, where $f(M)$ is the image of π by f via the extension principle: $\mu_{f(M)}(s) = \sup_{x:s=f(x)} \pi(x)$.*

Proof. It suffices to recall that the α cut of $f(M)$ is precisely of the form $[f^{\pi-}(\alpha), f^{\pi+}(\alpha)]$. $\square$

The notion of pessimistic and optimistic profiles can be generalised when $\mathcal{C}$ is changed into a continuous set S . A possibility distribution π on the set of possible states defines a complete plausibility ordering $\leq_{\pi}$ on S , and given an act f , its pessimistic counterpart is $f^{\leq\pi,-}(s) = \inf_{s' \leq_{\pi} s'} f(s')$ and its optimistic counterpart is $f^{\leq\pi,+}(s) = \sup_{s' \leq_{\pi} s'} f(s')$.

Let us present an example of $C_N(f)$ and $C_{\Pi}(f)$ when the set of the states of the world is represented with a real interval (see Fig. 1).

Example 4. *We consider $\mathcal{C} = [0, b]$, and let $\pi(x)$ be the triangular fuzzy number with support $[0, b]$ and core $a < b$. More precisely $\pi(x) = x/a$ if $0 \leq x \leq a$ and $\frac{b-x}{b-a}$ otherwise. We consider the value function $f(x) = (b-x)/b$ which is decreasing on $[0, b]$.*

If $\pi(x) \geq \alpha$, $[a\alpha, b - (b-a)\alpha]$ is the associated α -cut, then the pessimistic value of the utility function on this interval is $f^{\pi-}(\alpha) = f(b - (b-a)\alpha) = \frac{(b-a)\alpha}{b}$ and the optimistic one is $f^{\pi+}(\alpha) = f(a\alpha) = 1 - \frac{a\alpha}{b}$.

If $b = 1$ and $a = 1/2$: $C_N(f) = \int_0^1 f(b - (b-a)\alpha) d\alpha = \frac{(b-a)}{2b} = 1/4$; $C_{\Pi}(f) = \int_0^1 f(a\alpha) d\alpha = 1 - \frac{a}{2b} = 3/4$. Finally, the pessimistic and optimistic substitutes of the value function f are (see Fig. 1):

$$f^{\leq\pi,-}(x) = \min\left(\frac{(b-a)x}{ab}, f(x)\right); \quad f^{\leq\pi,+}(x) = \max\left(\frac{(b-a)^2 + a(x-a)}{b(b-a)}, f(x)\right).$$

$C_N(f)$ is also the expectation of $f^{\leq \pi, -}$ with respect to the uniform probability on $[a, b]$ (pessimistic area), and $C_\Pi(f)$ is the expectation of $f^{\leq \pi, +}$ with respect to the uniform probability on $[0, a]$ (optimistic area).

The use of fuzzy intervals for computing possibilistic Choquet integrals is of practical interest [24], and Example 4 can be extended to more general (non-linear) fuzzy intervals representing utility functions and possibility distributions. The difficulty lies in finding intersection points between membership functions, that could be facilitated by suitable parametrization of fuzzy intervals. The result in Proposition 1 may facilitate the computation of the possibilistic Choquet integrals, if it is easy to compute $f(M)$ for the fuzzy interval M representing the uncertainty. For instance, in the example, the triangular possibility distribution π can be represented by the 3-tuple $M = (0, a, b)$ that encodes the core and the support of π . Then it is obvious that $f(M) = (0, 1 - a/b, 1)$ is triangular too, and $C_N(f)$ is the midpoint of the interval $[0, 1 - a/b]$ while $C_\Pi(f)$ is the midpoint of the interval $[1 - a/b, 1]$.

2.5. Preference relations between profiles

In axiomatic approaches to decision under uncertainty, the starting point is a preference relation between profiles that satisfies a number of prescribed axioms. Under some conditions, such a preference relation between profiles can be represented by Choquet integrals. There have been attempts to justify continuous Choquet integrals [9] by means of axioms bearing on the preference relation.

Let $\succeq$ be a preference relation on a set of infinite profiles $\mathcal{V}$ representing acts $f : S \rightarrow [0, +\infty)$, and given by the decision maker on such infinite profiles. In [7], the following axioms are proposed, in the infinite setting, i.e., the set of possible states is a dense (continuous) set of states S .

- A1** *Totality and Non-triviality*: $\succeq$ is a non-trivial total preorder on $\mathcal{V}$ (transitive and complete, and a strict preference $f \succ g$ for at least two acts f, g).
- A2** *Continuity according to uniform monotone convergence*:
 - A2.1** $[f_n, f, g \in \mathcal{V}, f_n \succeq g, f_n \downarrow^u f] \Rightarrow f \succeq g$;
 - A2.2** $[f_n, f, g \in \mathcal{V}, g \succeq f_n, f_n \uparrow^u f] \Rightarrow g \succeq f$;
- A3** *Monotonicity*: If $f \succeq g + \epsilon$, where ϵ is any positive constant function, then $f \succ g$
- A4** *Comonotonic independence*: Let f, g, h be profiles, where both f and h , and g and h are comonotonic; then $f \succeq g \Leftrightarrow f + h \succeq g + h$

We recall the following result [7]:

Theorem 3. A preference relation $\succeq$ satisfies axioms A1–A4 if and only if there exists a capacity μ such that C_μ represents the preference relation. This capacity is unique.

Since only the ordering on states matters for possibilistic Choquet integrals, the levelwise-pessimism and levelwise-optimism axioms can be expressed in terms of comparison between acts:

- N4** There exists a complete preordering $\succeq_\pi$ on S such that $f \sim f^{\succeq_\pi, -}$, where $f^{\succeq_\pi, -}(s) = \inf_{s' \succeq_\pi s} f(s')$.
- Π4** There exists a complete preordering $\succeq_\pi$ on S such that $f \sim f^{\succeq_\pi, +}$, where $f^{\succeq_\pi, +}(s) = \sup_{s' \succeq_\pi s} f(s')$.

We can conjecture the following result in the pessimistic and optimistic cases: A preference relation $\succeq$ satisfies axioms A1–A4 and **N4** (resp. **Π4**) if and only if there exists a necessity (resp. possibility) measure N (resp. Π) such that C_N (resp. C_Π) represents the preference relation. This necessity (resp. possibility) measure is unique.

The proof requires showing that the unique capacity obtained from axioms A1 – A4 is a necessity measure. However, a necessity measure must then satisfy the infinite minitivity axiom, $N(\cap_{i \in I} A_i) = \inf_{i \in I} N(A_i)$, for any index set I , in order to ensure the existence of a possibility distribution underlying the capacity. But it is not clear how to extend Lemma 2 to infinite families of sets. As it stands, Lemma 2 only justifies finite minitivity. In a finite setting, the permutation σ , indicating the relative plausibility of states can be extracted from the preference relation on profiles, by observing special ones. The possibility ordering $\succeq_\pi$ underlying the preference relation $\succeq$ can be extracted by

considering binary profiles of the form f_s such that $f_s(s) = 0$ and $f_s(t) = 1$ if $t \neq s$. More precisely, $s \succeq_\pi t$ whenever $f_s \succeq f_t$. In the infinite setting, it may be that $\succeq$ is non-trivial while $s \sim_\pi t$ for all $s, t \in S$. For instance, on a denumerable set, such as the positive integers, assigning necessity degree 1 to all complements of finite subsets and 0 otherwise yields a finitely minitive set function that corresponds to assigning zero possibility degrees to all integers.

The same difficulty arises for the optimistic counterpart of the above tentative result. This is left for further research.

3. Preference-based axiomatisations of Sugeno integral: a review with new results

In this section, we stick to a qualitative setting using a finite value scale. To distinguish from the numerical case, we denote by $\wedge$ and $\vee$ the minimum and the maximum on L . The Sugeno integral [42,43] of a profile f can be defined by means of several expressions, among which the two following normal forms [32]¹:

$$\int f \circ \mu = \bigvee_{A \subseteq C} \mu(A) \wedge \bigwedge_{i \in A} f_i = \bigwedge_{A \subseteq C} (1 - \mu^c(A)) \vee \bigvee_{i \in A} f_i \quad (6)$$

A first statement of this equality was proved quite early by Kandel and Byatt [30] in 1978. The aim of this section is first to show that the new axioms for possibilistic Choquet integrals can be also used to characterize possibilistic Sugeno integrals. Moreover we review the existing axiomatizations of Sugeno integrals in the style of Savage, based on preference relations between acts expressing the choices of a decision-maker. In the axiomatization proposed in [21,22], the proof of the representation theorem turns out to be too sketchy. A careful analysis shows that some gaps remain. The aim of this section is also to fix these defects, and propose an alternative axiomatic framework based on a non-compensation axiom, all of which, augmented with the preference-based version of the levelwise pessimism and optimism axioms, provide a new preference-based justification of possibility theory.

3.1. A new characterisation of possibilistic Sugeno integrals

Sugeno integral can be characterized as follows [38,27]:

Theorem 4. *Let $I : L^C \rightarrow L$. There is a capacity μ such that $I(f) = \int f \circ \mu$ for every $f \in L^C$ if and only if the following properties are satisfied*

1. $I(f \vee g) = I(f) \vee I(g)$, for any comonotonic $f, g \in L^C$.
2. $I(a \wedge f) = a \wedge I(f)$, for every $a \in L$ and $f \in L^C$.
3. $I(1_C) = 1$.

Equivalently, conditions (1–3) can be replaced by conditions (1'–3') below:

- 1'. $I(f \wedge g) = I(f) \wedge I(g)$, for any comonotonic $f, g \in L^C$.
- 2'. $I(a \vee f) = a \vee I(f)$, for every $a \in L$ and $f \in L^C$.
- 3'. $I(0_C) = 0$.

Most older formulations of this theorem [4,36] add an assumption of increasing monotonicity of the functional I (if $f \geq g$ then $I(f) \geq I(g)$) to the three conditions (1–3). But these papers do not point out the equivalent conditions (1'–3'). The existence of these two equivalent characterisations is due to the possibility of equivalently writing Sugeno integral in conjunctive and disjunctive form – Eq. (6), so that $\int f \circ \mu = 1 - \int (1 - f) \circ \mu^c$ (an equality proved in [28]) and the fact that a functional $I(f)$ satisfies conditions (1–3) if and only if $1 - I(1 - f)$ satisfies conditions (1'–3'). Marichal [32] provides several similar characterizations, especially one assuming maxitive and minitive comonotonicity (conditions 1 and 1') along with idempotence, one assuming homogeneity conditions 2 and 2' plus

¹ We use Sugeno's original notation, so as to distinguish from the usual integral.

increasing monotonicity. However, the proof that conditions (1–3) are necessary and sufficient seems to first appear only in the thesis [38] (and then used in [27,25]). We put a short version of it in [Appendix A](#) for the sake of completeness. This proof shows that we only need to request conditions 2 and 2' for characteristic functions of sets ($f = \mathbf{1}_A$), as done in [25], p. 244.

Moreover, for a necessity measure N , $\int f \circ N = \bigwedge_{i=1}^n (1 - \pi_i) \vee f_i$ (weighted min); and for a possibility measure Π , $\int f \circ \Pi = \bigvee_{i=1}^n \pi_i \wedge f_i$ (weighted max); see for instance [12] pp. 138–139, for the latter and [28] for the former. The Sugeno integral with respect to a possibility (resp. necessity) measure is maxitive (resp. minitive), hence the following characterization results for them:

Theorem 5. *Let $I : L^{\mathcal{C}} \rightarrow L$. There is a possibility measure Π such that $I(f) = \int f \circ \Pi$ for every $f \in L^{\mathcal{C}}$ if and only if the following properties are satisfied*

1. $I(f \vee g) = I(f) \vee I(g)$, for any $f, g \in L^{\mathcal{C}}$.
2. $I(a \wedge f) = a \wedge I(f)$, for every $a \in L$ and $f \in L^{\mathcal{C}}$.
3. $I(\mathbf{1}_{\mathcal{C}}) = 1$.

Theorem 6. *There is a necessity measure N such that $I(f) = \int f \circ N$ for every $f \in L^{\mathcal{C}}$ if and only if the following properties are satisfied*

- 1'. $I(f \wedge g) = I(f) \wedge I(g)$, for any $f, g \in L^{\mathcal{C}}$.
- 2'. $I(a \vee f) = a \vee I(f)$, for every $a \in L$ and $f \in L^{\mathcal{C}}$.
- 3'. $I(\mathbf{0}_{\mathcal{C}}) = 0$.

This is easy to check as the comonotonic maxitivity (resp. minitivity) of I enforces the maxitivity (resp. minitivity) of the underlying capacity. However, we can alternatively characterise those simplified Sugeno integrals in the same style as we did for possibilistic Choquet integrals, using optimistic and pessimistic substitutes, due to the following

Lemma 3. $\int f \circ N = \int f^{\sigma, -} \circ N, \quad \int f \circ \Pi = \int f^{\sigma, +} \circ \Pi.$

Proof. Assume $\pi_1 \geq \dots \geq \pi_n$ for simplicity, i.e., $\sigma(i) = i$. By definition $f^{\sigma, -} \leq f$ so $\int f^{\sigma, -} \circ N \leq \int f \circ N$ since the Sugeno integral is an increasing function. Let i_0 and i_1 be the indices such that $\int f^{\sigma, -} \circ N = (1 - \pi_{i_0}) \vee \bigwedge_{j \leq i_0} f_{i_0} = (1 - \pi_{i_0}) \vee f_{i_1}$ where $i_1 \leq i_0$. Hence $\pi_{i_1} \geq \pi_{i_0}$, i.e., $1 - \pi_{i_1} \leq 1 - \pi_{i_0}$ and $\int f^{\sigma, -} \circ N \geq (1 - \pi_{i_1}) \vee f_{i_1} \geq \int f \circ N$.

By definition $f \leq f^{\sigma, +}$ so $\int f \circ \Pi \leq \int f^{\sigma, +} \circ \Pi$. Let i_0 and i_1 be the indices such that $\int f^{\sigma, +} \circ \Pi = \pi_{i_0} \wedge \bigvee_{j \leq i_0} f_j = \pi_{i_0} \wedge f_{i_1}$ where $i_1 \leq i_0$. Hence $\pi_{i_0} \leq \pi_{i_1}$ and $\int f^{\sigma, +} \circ \Pi \leq \pi_{i_1} \wedge f_{i_1} \leq \int f \circ \Pi$. $\square$

In particular, using notations of Subsection 2.2, $\int f \circ N = \int \phi_k \circ N, \int f \circ \Pi = \int \phi^k \circ \Pi$, for all $1 \leq k \leq n$, as for Choquet integral.

Now we can state qualitative counterparts of [Theorems 1 and 2](#), that are proved in the same way:

Theorem 7. *There is a possibility measure Π such that $I(f) = \int f \circ \Pi$ for every $f \in L^{\mathcal{C}}$ if and only if the following properties are satisfied*

1. $I(f \vee g) = I(f) \vee I(g)$, for any comonotonic $f, g \in L^{\mathcal{C}}$.
2. $I(a \wedge f) = a \wedge I(f)$, for every $a \in L$ and $f \in L^{\mathcal{C}}$.
3. $I(\mathbf{1}_{\mathcal{C}}) = 1$.

$\Pi 4$ There exists a permutation σ on $\mathcal{C}$ such that $\forall A, I(\mathbf{1}_A) = I(\mathbf{1}_A^{\sigma,+})$

Theorem 8. There is a necessity measure N such that $I(f) = \int f \circ N$ for every $f \in L^{\mathcal{C}}$ if and only if the following properties are satisfied

1. $I(f \wedge g) = I(f) \wedge I(g)$, for any comonotonic $f, g \in L^{\mathcal{C}}$.
2. $I(a \vee f) = a \vee I(f)$, for every $a \in L$ and $f \in L^{\mathcal{C}}$.
3. $I(0_{\mathcal{C}}) = 1$.

$N4$ There exists a permutation σ on $\mathcal{C}$ such that $\forall A, I(\mathbf{1}_A) = I(\mathbf{1}_A^{\sigma,-})$

Axiom $\Pi 4$ (resp., $N4$) can be replaced by: there exists a permutation σ on $\mathcal{C}$ such that for all f , $I(f) = I(f^{\sigma,+})$ (resp. $I(f) = I(f^{\sigma,-})$).

Remark 1. The same weakened condition $I(a \wedge \mathbf{1}_A) = a \wedge \mathbf{1}_A$ as in [25] (Theorem 4.58, p. 244), could be used in place of the second assumption in the above Theorems 5, 7, and $I(a \vee f) = a \vee I(f)$ in Theorems 6, 8.

These results make it clear that the levelwise-pessimism and optimism axioms are generic assumptions in the sense that they apply equally to Sugeno and Choquet discrete integrals. In the case of Sugeno integral, for the MCDM context, there are two equivalent axiomatisations of the possibility-based and necessity-based versions, because we can take advantage of the respective maxitivity and minitivity properties of weighted max and min, that carry over to non-Boolean profiles, while this is impossible in the case of Choquet integral. The levelwise-pessimism and optimism axioms are properties that enforce the possibilistic setting regardless of the considered characterisation of the integral.

3.2. Act-based axiomatisation of Sugeno integral

In this section, the set of criteria is replaced by a *finite* set S of states and acts are just functions f from S to a set of consequences X . We consider again a finite totally ordered scale $(L, \leq)$ with bottom 0 and top 1. A mapping $u : X \rightarrow L$ is named a utility function. We assume that X contains an ideal consequence x^* with $u(x^*) = 1$ and a worst consequence x_* with $u(x_*) = 0$.

The decision-maker is supposed to supply a preference relation $\succeq$ on the set X^S of acts, that is, a non-trivial preorder: $\succeq$ is transitive and complete.

We introduce new notations that will be useful in the following:

- A constant act x is such that $\exists x \in X, \forall s \in S, x(s) = x$. In particular, the acts x^* and x_* are such that $x^*(s) = x^*, \forall s \in S$ and $x_*(s) = x_*, \forall s \in S$.
- Given $x, y \in X$, a binary act $f = xAy$ is an act such that $f(s) = x$ if $s \in A$ and $f(s) = y$ if $s \in \bar{A}$.
- For acts f, g , fAg is the act defined by $fAg(s) = f(s)$ for all s in A and $fAg(s) = g(s)$ for all s in $\bar{A}$.

Note that the preference relation $\succeq$ induces a complete preordering $\geq_P$ on consequences : $x \geq_P y$ if and only if $x \succeq y$; this ordering can be extended to acts as follows: $f \geq_P g$ if and only if $f(s) \geq_P g(s), \forall s \in S$. This is the Pareto-ordering. Then, one can define an act $f \vee g$ making the best of f and g , such that $\forall s \in S, f \vee g(s) = f(s)$ if $f(s) \geq_P g(s)$ and $g(s)$ otherwise; and an act $f \wedge g$ making the worst of f and g , such that $\forall s \in S, f \wedge g(s) = f(s)$ if $g(s) \geq_P f(s)$ and $g(s)$ otherwise. Acts are thus combined like fuzzy sets.

The axioms proposed in [21] are as follows:

A1 Totality: $\succeq$ is a non-trivial total preorder, i.e., it is transitive and complete, and $f \succ g$ for some acts.

WP3 Weak compatibility with constant acts: $\forall A \subseteq S, \forall x, y \in X, \forall f, x \geq y$ implies $xAf \geq yAf$.

RCD Restricted conjunctive dominance: For any acts g, h and any constant act $x, x \succ h$ and $g \succ h$ imply $x \wedge g \succ h$.

Upper-bounding the utility of an act g better than another act h by a constant value that is better than the utility of act h still yields an act better than h .

RDD *Restricted max-dominance*: For any acts g, h and any constant act x , $h \succ x$ and $h \succ g$ imply $h \succ x \vee g$.

If an act h is preferred to an act g and also to the constant act x then if the bad consequences of g are upgraded to x , h is still preferred to g .

Note that axioms **A1** and **WP3** entail Pareto-dominance: if $f \geq_P g$ then $f \succeq g$ (see Lemma 4 in [22]). Moreover, **RCD** and **RDD** make sense for one-shot decisions, i.e., without repetition, making the compensation of bad results by good ones impossible.

We recall here the main result about this axiomatization for decision under uncertainty.

Theorem 9 ([21]). *Let $(X^S, \succeq)$ be a preference structure. The following propositions are equivalent:*

- $(X^S, \succeq)$ satisfies **A1**, plus **WP3**, **RCD**, **RDD**.
- There exists a finite chain L of preference levels, an L -valued monotonic set-function μ , and an L -valued utility function u on X , such that $f \succeq g$ if and only if $\int u(f) \circ \mu \geq \int u(g) \circ \mu$.

This result is the counterpart, using preference between acts, of Theorem 4.2 in [32]. However the existing proof of this result as given in [21], and stated again in [22], is incomplete. It develops in 4 steps recalled here:

Step 1: The scale Since the preference relation $\succeq$ is complete (**A1**), the finite qualitative scale L is the set of acts quotiented by the associated indifference relation $\sim$. We evaluate f by $I(f) \in L$, the element in L associated to the equivalence class of f . The utility function u is constructed on X by restricting $\leq$ to constant acts (equivalence classes of $\geq_P$), namely $u(x) = I(x)$.

Step 2: The capacity We define a set function μ by means of extreme binary acts as $\mu(A) = I(x^*Ax_*)$. If $A \subseteq B$, then $x^*Ax_* \leq_P x^*Bx_*$ i.e. $\mu(B) \geq \mu(A)$ from Pareto-dominance. In particular, we have $\mu(S) = I(x^*) = I(x^*, \dots, x^*) = 1$, $\mu(\emptyset) = I(x_*) = I(x_*, \dots, x_*) = 0$. So μ is a fuzzy measure.

Step 3: The internal operator We extend the functional I to binary acts of the form xAx_* . By Pareto-dominance, since $xAx_* \leq_P x^*Ax_*$ and $xAx_* \leq_P x$, it follows that $xAx_* \leq x^*Ax_*$ and $xAx_* \leq x$. So we have $I(xAx_*) \leq I(x^*Ax_*) = \mu(A)$ and $I(xAx_*) \leq I(x) = u(x)$ hence $I(xAx_*) \leq \mu(A) \wedge u(x)$.

Note that $xAx_* = x \wedge (x^*Ax_*)$. The contrapositive form of **RCD** implies $x^*Ax_* \leq xAx_*$ or $x \leq xAx_*$. So either $I(xAx_*) \geq \mu(A)$ or $I(xAx_*) \geq u(x)$. We conclude that $I(xAx_*) = \mu(A) \wedge u(x)$.

Step 4: The external operator In [21] it is claimed that for any two xAx_* and yBx_* , where $B \subseteq A$, axiom **RDD** implies $I((xAx_*) \vee (yBx_*)) = I(xAx_*) \vee I(yBx_*)$. Finally, let $x^* = x_m > \dots > x_1 = x_*$, that is, $m = \#X$. Any act f can be expressed as $\bigvee_{i=1}^m x_i F_i x_*$, where $F_i = \{s : f(s) \geq_P x_i\}$, and $m = \#X$ is the number of consequences, so that $I(f) = \bigvee_{i=1}^m I(x_i F_i x_*) = \bigvee_{i=1}^m (\mu(F_i) \wedge u(x_i))$.

However in Step 4, it is clear that we cannot directly apply axiom **RDD** to acts xAx_* , yBx_* and their disjunction, like it is done in Step 3 for conjunction and **RDD**. This is because to apply axiom **RDD**, one of the acts must be constant. We could directly get the result of Step 4 if we replace **RDD** by a comonotonicity axiom, namely [6]:

CDD: if g and h are comonotonic, then $f \succ g$ and $f \succ h$ imply $f \succ g \vee h$.

Then, since $\bigvee_{i=1}^j x_i F_i x_*$ and $x_{j+1} F_{j+1} x_*$ are comonotonic, **CDD** and Pareto-dominance imply $I(\bigvee_{i=1}^{j+1} x_i F_i x_*) = I(\bigvee_{i=1}^j x_i F_i x_*) \vee I(x_{j+1} F_{j+1} x_*)$, and $I(f) = \bigvee_{i=1}^m I(x_i F_i x_*)$ follows by induction.

In [Appendix B](#), we propose a new version of Step 4 of the proof, showing that axiom **RDD** is indeed enough to get the representation theorem at this point. In the next section we provide an alternative proof of [Theorem 9](#) based on the idea of non-compensation.

3.3. A non-compensation axiom

Sugeno integral is non-compensatory by nature: it returns a result that is always in the set consisting of the $u(f_i)$'s and the values $\mu(A)$.

This is natural for one-shot decisions. For instance, if it is only known that $s \in E \subseteq S$ then the actual utility of an act is the utility of the (only) outcome, that is, it should be a value in $\{u(x), x \in E\}$. For instance, flipping a coin once, suppose A stands for *head* and x means that you get 100 euros. Its complement $\bar{A}$ stands for *tail* and y means that you lose 100 euros. Then, the worth of this lottery xAy is either $u(100)$ (if you think you win), $u(-100)$ if you think you lose, or it reflects your confidence in A , which in case of ignorance is evaluated as 0 (pessimistic case) or 1 (optimistic case).

This remark leads to a simpler axiomatization based on this non-compensation property:

$$\textbf{Axiom NC} : \begin{cases} x^*Ay \sim y \text{ or } x^*Ay \sim x^*Ax_* \text{ (DNC)} \\ \text{and} \\ xAx_* \sim x \text{ or } xAx_* \sim x^*Ax_* \text{ (CNC)} \end{cases}$$

For a binary act, this axiom reflects the fact that $\int u(xAy) \circ \mu$ equals either $u(x)$, $u(y)$, or the likelihood of A , $\mu(A)$, which is $u(x^*Ax_*)$, or the likelihood of $\bar{A}$, $\mu(\bar{A})$.

Axiom **NC** also reflects the fact that Sugeno integral is decomposable in terms of medians [33]. In particular, $\int u(xAy) \circ \mu = \text{median}(u(x), u(y), \mu(A))$, if $x >_P y$. Axiom **NC** indeed accounts for the following features:

- If $u(x) > u(y)$ and $\mu(A)$ is too small ($\mu(A) \leq u(y)$), then we get $\int u(xAy) \circ \mu = u(y)$;
- If $u(x) > u(y)$ and $\mu(A)$ is high enough ($\mu(A) \geq u(x)$), then we get $\int u(xAy) \circ \mu = u(x)$;
- $u(x) \geq \int u(xAy) \circ \mu \geq u(y)$ (no way of expecting results better than x or worse than y);
- if $u(x) = 1$ then $\int u(xAy) \circ \mu = \int u(x^*Ay) \circ \mu \geq \mu(A) \vee u(y)$, and in fact, $\int u(x^*Ay) \circ \mu = \mu(A) \vee u(y)$;
- if $u(y) = 0$ then $\int u(xAy) \circ \mu = \int u(xAx_*) \circ \mu \leq \mu(A) \wedge u(x)$, and in fact, $\int u(xAx_*) \circ \mu = \mu(A) \wedge u(x)$.

Then we can prove the following result (original, to the best of our knowledge):

Theorem 10. *Theorem 9 still holds if RCD and RDD are replaced by the non-compensation axiom NC.*

Proof. Like before, the utility scale L is the set of acts quotiented by the indifference relation $\sim$, and $I(f)$ denotes the equivalence class of f . Again construct the ordinal utility function u on X , by restricting $\succeq$ which is complete, to constant acts. We again derive the capacity μ as $\mu(A) = I(x^*Ax_*)$ and assume $x_1 = x^* > x_2 > \dots > x_m = x_*$. For any act f , and any consequence x_i , let again $F_i = \{s : f(s) \succeq_P x_i\}$. Consider acts $x^*F_{i+1}x_i$ and $x_iF_i x_*$.

- $x^*F_{i+1}x_i \succeq f \succeq x_iF_i x_*$ (Pareto-dominance);
- by **DNC**, $I(x^*F_{i+1}x_i) \in \{\mu(F_{i+1}), u(x_i)\}$, hence $I(f) \leq I(x^*F_{i+1}x_i) \leq \mu(F_{i+1}) \vee u(x_i)$;
- by **CNC**, $I(x_iF_i x_*) \in \{\mu(F_i), u(x_i)\}$, hence $I(f) \geq I(x_iF_i x_*) \geq \mu(F_i) \wedge u(x_i)$.

Then for all states x_i , $\mu(F_{i+1}) \vee u(x_i) \geq I(f) \geq \mu(F_i) \wedge u(x_i)$.

Hence, $\bigwedge_{i=1}^m \mu(F_{i+1}) \vee u(x_i) \geq I(f) \geq \bigvee_{i=1}^m \mu(F_i) \wedge u(x_i)$.

But the two bounds are equal and are Sugeno integral. $\square$

The reader may have noticed that in fact axioms **RCD** and **RDD** are completely related to the **NC** axiom since:

Proposition 2. *Under the Pareto dominance property, RCD implies CNC and RDD implies DNC.*

Proof. RCD implies CNC is established in [21,22] at Step 3 of the proof of Theorem 9, as recalled above. Now let us prove that Axiom RDD and Pareto Dominance imply $\forall A \subseteq S, \forall x, x^*Ax \sim x$ or $x^*Ax \sim x^*Ax_*$.

Pareto Dominance implies $x^*Ax \succeq x$ and $x^*Ax \succeq x^*Ax_*$.

We decompose x^*Ax as $(x^*Ax_*) \vee x$ and apply axiom RDD. Since $x^*Ax \sim (x^*Ax_*) \vee x$ we conclude $x^*Ax \preceq x$ or $x^*Ax \preceq x^*Ax_*$. Hence $x^*Ax \sim x$ or $x^*Ax \sim x^*Ax_*$. $\square$

So proving Theorem 10 is a simpler way to prove the representation Theorem 9, avoiding Step 4 of the proof proposed by Dubois et al. [21].

3.4. A new characterisation for qualitative possibilistic integrals

In the case of a Sugeno integral with respect to a possibility measure, RDD is replaced by the stronger axiom of disjunctive dominance:

Axiom DD (disjunctive dominance) : $\forall f, g, h, h \succ f$ and $h \succ g$ imply $h \succ f \vee g$.

And we get a similar result as the above Theorem 9, whereby $f \succeq g$ if and only if $\int u(f) \circ \Pi \geq \int u(g) \circ \Pi$ for a possibility measure Π [23].

In the case of a Sugeno integral with respect to a necessity measure, RCD is replaced by the stronger axiom of conjunctive dominance:

Axiom CD (conjunctive dominance) : $\forall f, g, h, f \succ h$ and $g \succ h$ imply $f \wedge g \succ h$.

And we get a similar result as the above Theorem 9, whereby $f \succeq g$ if and only if $\int u(f) \circ \mu \geq \int u(g) \circ \mu$ for a necessity measure μ [23].

We can then replace axioms DD and CD in these representation results by the levelwise-optimism and levelwise-pessimism axioms, proposed in this paper, expressed in terms of preference equivalence between binary acts. We just add them to the characteristic axioms for Sugeno integrals in terms of a preference relation between acts

Theorem 11. Let $(X^S, \succeq)$ be a preference structure. The following propositions are equivalent:

- $(X^S, \succeq)$ satisfies A1, plus WP3, RCD, RDD and $\Pi 4'$
- There exists a permutation σ on S such that for all $A \subseteq S, x^*Ax_* \sim (x^*Ax_*)^{\sigma,+}$.
- There exists a finite chain of preference levels L , an L -valued monotonic possibility measure Π , and an L -valued utility function u on X , such that $f \succeq g$ if and only if $\int u(f) \circ \Pi \geq \int u(g) \circ \Pi$.

Theorem 12. Let $(X^S, \succeq)$ be a preference structure. The following propositions are equivalent:

- $(X^S, \succeq)$ satisfies A1, plus WP3, RCD, RDD and $N 4'$
- There exists a permutation σ on S such that for all $A \subseteq S, x^*Ax_* \sim (x^*Ax_*)^{\sigma,-}$.
- There exists a finite chain of preference levels L , an L -valued necessity measure N , and an L -valued utility function u on X , such that $f \succeq g$ if and only if $\int u(f) \circ N \geq \int u(g) \circ N$.

The reason for the validity of those theorems in the case of Sugeno integral is exactly the same as the reason for the validity of Theorems 1, 2, 7, 8: adding $\Pi 4$ (resp. $N 4$) to the representation theorem of Sugeno integral forces the capacity to be a possibility (resp. necessity) measure. Note that in $\Pi 4'$ above, we have that $(x^*Ax_*)^{\sigma,+} = x^*A^{\sigma,+}x_*$ and in $N 4'$ above, $(x^*Ax_*)^{\sigma,-} = x^*A^{\sigma,-}x_*$, so that the above results follow from Lemma 1.

This axiomatisation of possibilistic integrals in the qualitative setting is one among other possible ones as seen in this section. However, the level optimism and pessimism axioms seem to be unavoidable to axiomatize possibilistic Choquet integrals as they are no longer maxitive nor minitive functionals. It is interesting to notice that the same axioms are instrumental to specialize Sugeno and Choquet integrals to possibility and necessity measures. In the case of qualitative possibilistic integrals, the maxitivity or minitivity property of the preference functional makes it possible to propose more choices of axioms.

4. Conclusion

This paper proposes an original axiomatization of discrete Choquet integrals with respect to possibility and necessity measures, and shows that it is enough to add, to existing axiomatisations of general instances of Choquet integrals, a property of indifference between some profiles or acts, that singles out possibility or necessity measures. Remarkably, this property, which says that the decision-maker only considers relevant the relative importance of single criteria, is qualitative in nature and can thus be added as well to axiom systems for Sugeno integrals, to yield qualitative weighted min and max aggregation operations. It can be expressed as well in the ordinal act-based preference setting à la Savage. We conjecture that results on the qualitative side go beyond Sugeno integrals and apply to more general qualitative functionals, replacing the inner minimum and maximum connectives by more general conjunctions and disjunctions or implications (t-norms and conorms [5], semi-copulas [2], residuated implications and their semi-duals [18], more generally non-commutative conjunctions and their semi-duals [19,20]). This is a matter of further research.

Possibilistic Choquet integrals are the simplest ones (apart from the standard integral), and are instrumental in representing a numerical capacity in terms of several convex sets of probabilities, each generating upper and lower probabilities that are possibility and necessity measures as studied by Brüning and Denneberg [3]. These authors observe that any numerical capacity is an upper necessity measure, each corresponding to a convex set of probabilities. This property seems to carry over to Choquet integrals; this is the so-called maximin representation of capacities. Interestingly similar findings exist for qualitative capacities and Sugeno integral [17]. This additional similarity between qualitative and numerical possibilistic integrals confirms the important role of possibility theory in the mathematics of uncertainty modeling.

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Appendix A. Characterization of Sugeno integral without monotonicity assumption [38]

The fact that Sugeno integral verifies conditions (1–3) of Theorem 4 is obvious. Now, for the sufficiency. Rico [38] proceeds in 3 steps (conditions 1, 2, 3 refer to those in the theorem).

1. A profile f can be written as $\bigvee_{i=1}^m \lambda_i \wedge \mathbf{1}_{F_i}$ with $F_i = \{j : f_j \geq \lambda_i\}$ and $\lambda_i \in L$. Profiles $\lambda_i \wedge \mathbf{1}_{F_i}$ are comonotonic, and take value λ_i on F_i and 0 otherwise. By condition 1, $I(f) = \bigvee_{i=1}^m I(\lambda_i \wedge \mathbf{1}_{F_i})$. By condition 2, $I(\lambda_i \wedge \mathbf{1}_{F_i}) = \lambda_i \wedge I(\mathbf{1}_{F_i})$.
2. The set-function defined by $\mu(A) = I(\mathbf{1}_A)$ is a capacity. Indeed, condition 2 implies that $I(a \wedge \mathbf{1}_A) = a \wedge I(\mathbf{1}_A)$, so that $I(\mathbf{1}_\emptyset) = I(0 \wedge \mathbf{1}_A) = 0$. Condition 3 means $I(\mathbf{1}_C) = 1$. Now consider $A \subseteq B$. $\mathbf{1}_A$ and $\mathbf{1}_B$ are comonotonic and $\mathbf{1}_A \vee \mathbf{1}_B = \mathbf{1}_B$. Condition 1 implies $I(\mathbf{1}_A \vee \mathbf{1}_B) = I(\mathbf{1}_B) = I(\mathbf{1}_A) \vee I(\mathbf{1}_B)$. So $I(\mathbf{1}_B) \leq I(\mathbf{1}_A)$.

So $I(f) = \bigvee_{i=1}^m \lambda_i \wedge \mu(F_i)$, which is the original form of Sugeno integral [42,43].

Appendix B. A new proof for Step 4 of Theorem 9

This is inspired by the proof published in Sabbadin's Ph.D thesis [39], written in French, which however contains a small mistake as well. Let $x^* = x_m > \dots > x_1 = x_*$ and act f is expressed as $\bigvee_{i=1}^n x_i F_i x_*$ as in Step 4 of Theorem 9.

Proposition 3. $I(f) = \bigvee_{i=1}^m I(x_i F_i x_*)$.

Proof. Clearly $f \succeq x_i F_i x_*$, $\forall i = 1, \dots, m$ by Pareto-dominance. So $I(f) \geq \bigvee_{i=1}^m I(x_i F_i x_*)$. Now we need two lemmas:

Lemma 4. $f \preceq x_i$ and $f \preceq f F_i x_*$ imply $f \sim x_i F_i x_*$.

Proof. Assume $f \preceq x_i$ and $f \preceq f F_i x_*$. By Pareto-Dominance, $f F_i x_* \preceq x^* F_i x_*$. So we have $f \preceq x_i$ and $f \preceq x^* F_i x_*$. But using Step 3 of the proof of [Theorem 9](#), we know, by **RCD** and Pareto-dominance, that $x_i \wedge (x^* F_i x_*) = x_i F_i x_* \sim x_i$ or $\sim x^* F_i x_*$. So $f \preceq x_i F_i x_*$, but due to Pareto-dominance, $f \sim x_i F_i x_*$. $\square$

Lemma 5. $f \preceq x_i$ implies either $f \preceq f F_i x_*$ or $f \preceq x_{i-1}$.

Proof. Assume $f \preceq x_i$, $f \succ f F_i x_*$ and $f \succ x_{i-1}$. Hence $f \succ f F_i x_{i-1}$ by **RDD**, which is in contradiction with Pareto-Dominance, as $f \leq_P f F_i x_{i-1}$. $\square$

More precisely, since $f \geq_P f F_i x_*$, the conclusion of [Lemma 5](#) reads: either $f \sim f F_i x_*$ or $f \preceq x_{i-1}$. If $x_* \prec f \leq x^*$, there is an index i such that $x_{i-1} \prec f \preceq x_i$, so $f \sim f F_i x_*$, that is $I(f) = I(f F_i x_*) > u(x_{i-1})$ and using the claim in [Lemma 4](#) (via **RCD**), we conclude that $\exists i : f \sim x_i F_i x_*$. So $\exists i : I(f) = I(x_i F_i x_*)$. Hence $I(f) \leq \bigvee_{i=1}^m I(x_i F_i x_*)$ which together with $I(f) \geq \bigvee_{i=1}^m I(x_i F_i x_*)$ yields $I(f) = \bigvee_{i=1}^m I(x_i F_i x_*)$. In case $f \sim x_*$, then as $\forall i > 1$, $f \geq x_i F_i x_*$ by Pareto-dominance, it implies that $\forall i > 1$, $x_i F_i x_* \sim x_*$ as well, hence $I(x_i F_i x_*) = 0$ and $I(f) = \bigvee_{i=1}^m I(x_i F_i x_*)$ trivially. $\square$

Now using Step 3 of the proof of [Theorem 9](#) along with [Proposition 3](#) that replaces Step 4, it yields $I(f) = \bigvee_{i=1}^m (\mu(F_i) \wedge u(x_i))$.

Remark. In his Ph.D. dissertation [\[39\]](#), Sabbadin claims to prove that $f \preceq x_i$ implies either $f \preceq x_i F_i x_*$ or $f \preceq x_{i-1}$ directly. To prove it, Sabbadin assumes $f \preceq x_i$, $f \succ f F_i x_*$ and $f \succ x_{i-1}$. But the right assumption should include $f \succ x_i F_i x_*$ in place of $f \succ f F_i x_*$. And then we cannot directly conclude that $f \preceq x_i F_i x_*$ or $f \preceq x_{i-1}$; indeed $f \succ f F_i x_*$ and $f \succ x_{i-1}$ just implies $f \succ f F_i x_{i-1}$ via **RDD**, which is not in contradiction with $f \preceq x_i$. The above proof provides a correction to what seems to have been an oversight.

References

- [1] N. Ben Amor, H. Fargier, W. Guezguez, Possibilistic sequential decision making, *Int. J. Approx. Reason.* 55 (5) (2014) 1269–1300.
- [2] J. Borzová-Molnárová, L. Halčinová, O. Hutník, The smallest semicopula-based universal integrals, part I, *Fuzzy Sets Syst.* 271 (2015) 1–17.
- [3] M. Brüning, D. Denneberg, Max–min (σ)-additive representation of monotone measures, *Stat. Pap.* 43 (2002) 23–35.
- [4] L.M. de Campos, M.J. Bolaños, Characterization and comparison of Sugeno and Choquet integrals, *Fuzzy Sets Syst.* 52 (1992) 61–67.
- [5] L.M. de Campos, M.T. Lamata, S. Moral, A unified approach to define fuzzy integrals, *Fuzzy Sets Syst.* 39 (1) (1991) 75–90.
- [6] A. Chateauneuf, M. Grabisch, A. Rico, Modeling attitudes toward uncertainty through the use of the Sugeno integral, *J. Math. Econ.* 44 (11) (2008) 1084–1099.
- [7] A. Chateauneuf, Modeling attitudes towards uncertainty and risk through the use of Choquet integral, *Ann. Oper. Res.* 52 (1994) 3–20.
- [8] G. De Cooman, Integration and conditioning in numerical possibility theory, *Ann. Math. Artif. Intell.* 32 (1–4) (2001) 87–123.
- [9] D. Denneberg, *Non-Additive Measure and Integral*, Kluwer Academic Publishers, Dordrecht, 1994.
- [10] D. Dubois, E. Kerre, R. Mesiar, H. Prade, Fuzzy interval analysis, in: D. Dubois, H. Prade (Eds.), *Fundamentals of Fuzzy Sets*, in: The Handbooks of Fuzzy Sets Series, Kluwer, Dordrecht, 2000, pp. 483–581.
- [11] D. Dubois, J.-L. Marichal, H. Prade, M. Roubens, R. Sabbadin, The use of the discrete Sugeno integral in decision making: a survey, *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.* 9 (2001) 539–561.
- [12] D. Dubois, H. Prade, *Fuzzy Sets and Systems, Theory and Applications*, Academic Press, 1980.
- [13] D. Dubois, H. Prade, The mean value of a fuzzy number, *Fuzzy Sets Syst.* 24 (1987) 279–300.
- [14] D. Dubois, H. Prade, *Possibility Theory: An Approach to Computerized Processing of Uncertainty*, Plenum Press, New York, 1988.
- [15] D. Dubois, H. Prade, Quantitative possibility theory and its probabilistic connections, in: P. Grzegorzewski, O. Hryniewicz, M.-A. Gil (Eds.), *Soft Methods in Probability, Statistics and Data Analysis*, in: *Adv. Soft Comput.*, vol. 3, Physica Verlag, Heidelberg, 2002.
- [16] D. Dubois, H. Prade, Possibility theory and its applications: where do we stand ?, in: J. Kacprzyk, W. Pedrycz (Eds.), *Springer Handbook of Computational Intelligence*, Springer, 2015, pp. 31–60.
- [17] D. Dubois, H. Prade, A. Rico, Representing qualitative capacities as families of possibility measures, *Int. J. Approx. Reason.* 58 (2015) 3–24.
- [18] D. Dubois, H. Prade, A. Rico, Residuated variants of Sugeno integrals, *Inf. Sci.* 329 (2016) 765–781.
- [19] D. Dubois, H. Prade, A. Rico, B. Teheux, Generalized Sugeno integrals, in: *Proc. 16th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU 2016*, in: *Communications in Computer and Information Science* 610, vol. 1, Springer, 2016, pp. 363–374.
- [20] D. Dubois, H. Prade, A. Rico, B. Teheux, Generalized qualitative Sugeno integrals, in press *Inf. Sci.* (2017).⁶

- [21] D. Dubois, H. Prade, R. Sabbadin, Qualitative decision theory with Sugeno integrals, in: Proc. of 14th Conference on Uncertainty in AI, 1998, pp. 121–128.
- [22] D. Dubois, H. Prade, R. Sabbadin, Qualitative decision theory with Sugeno integrals, in: M. Grabisch, T. Murofushi, M. Sugeno (Eds.), *Fuzzy Measures and Integrals. Theory and Applications, Studies in Fuzziness and Soft Computing*, Physica-Verlag, 2000, pp. 314–322.
- [23] D. Dubois, H. Prade, R. Sabbadin, Decision theoretic foundations of qualitative possibility theory, *Eur. J. Oper. Res.* 128 (2001) 459–478.
- [24] F. Graba, O. Strauss, An interval-valued inversion of the non-additive interval-valued F-transform: use for upsampling a signal, *Fuzzy Sets Syst.* 288 (2016) 26–45.
- [25] M. Grabisch, *Set Functions, Games and Capacities in Decision-Making*, Springer-Verlag, 2016.
- [26] M. Grabisch, C. Labreuche, A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid, *Ann. Oper. Res.* 175 (2010) 247–286.
- [27] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, *Aggregation Functions*, Cambridge University Press, 2009.
- [28] M. Grabisch, T. Murofushi, M. Sugeno, Fuzzy measure of fuzzy events defined by fuzzy integrals, *Fuzzy Sets Syst.* 50 (3) (1992) 293–313.
- [29] S. Heilpern, The expected value of a fuzzy number, *Fuzzy Sets Syst.* 47 (1992) 81–87.
- [30] A. Kandel, W.J. Byatt, Fuzzy sets, fuzzy algebra and fuzzy statistics, *Proc. IEEE* 68 (1978) 1619–1639.
- [31] J.-L. Marichal, An axiomatic approach of the discrete Choquet integral as a tool to aggregate interacting criteria, *IEEE Trans. Fuzzy Syst.* 8 (2000) 800–807.
- [32] J.-L. Marichal, On Sugeno integral as an aggregation function, *Fuzzy Sets Syst.* 114 (2000) 347–365.
- [33] J.-L. Marichal, Weighted lattice polynomials, *Discrete Math.* 309 (4) (2009) 814–820.
- [34] R. Mesiar, J. Šipoš, A theory of fuzzy measures: integration and its additivity, *Int. J. Gen. Syst.* 23 (1) (1993) 49–57.
- [35] E. Pap, *Null-Additive Set Functions*, Kluwer, 1995.
- [36] D. Ralescu, M. Sugeno, Fuzzy integral representation, *Fuzzy Sets Syst.* 84 (1996) 127–133.
- [37] Y. Rébillé, Decision-making over necessity measures through the Choquet integral criterion, *Fuzzy Sets Syst.* 157 (2006) 3025–3039.
- [38] A. Rico, *Modélisation des Préférences pour l’Aide à la Décision par l’Intégrale de Sugeno*, Ph.D. Thesis, Université Paris I Sorbonne, 2002.
- [39] R. Sabbadin, *Une Approche Ordinale de la Décision dans L’Incertain*, Ph.D. Thesis, Université Paul Sabatier, 1998.
- [40] D. Schmeidler, Cores of exact games, *J. Math. Anal. Appl.* 40 (1) (1972) 214–225.
- [41] D. Schmeidler, Integral representations without additivity, *Proc. Am. Math. Soc.* 97 (1986) 255–261.
- [42] M. Sugeno, *Theory of Fuzzy Integrals and Its Applications*, PhD Thesis, Tokyo Institute of Technology, 1974.
- [43] M. Sugeno, Fuzzy measures and fuzzy integrals: a survey, in: M.M. Gupta, et al. (Eds.), *Fuzzy Automata and Decision Processes*, North-Holland, 1977, pp. 89–102.
- [44] Z. Wang, G.J. Klir, *Generalized Measure Theory*, Springer, 2009.
- [45] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets Syst.* 1 (1978) 3–28. [é](#)