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**Abstract:** Starting with a locally Lipschitz (patchy) Lyapunov function for a given switched system, we provide the construction of a continuously differentiable (smooth) Lyapunov function, obtained via a convolution-based approach. This smooth function approximates the patchy function when working with Clarke's generalized gradient. The convergence rate inherited by the smooth approximations, as a by-product of our construction, is useful in establishing the robustness with respect to additive inputs. With the help of an example, we address the limitations of our approach for other notions of directional derivatives, which generally provide less conservative conditions for stability of switched systems than the conditions based on Clarke's generalized gradient.

*Keywords:* Switched systems; patchy Lyapunov functions; generalized gradients.

## 1. INTRODUCTION

State-dependent switching systems can be used to model and analyze a large number of physical setups. These systems are defined by a family of vector field $\{f_1, \dots, f_M\} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and a switching signal $\sigma : \mathbb{R}^n \to \{1, \dots, M\}$, via an ordinary differential equation of the form

$$\dot{x} = f_{\sigma(x)}(x). \tag{1}$$

For studying generalized solutions of such systems, it turns out to be useful to introduce the *Filippov regularization* of the system (1), that leads us to the differential inclusion

$$\dot{x} \in F^{\rm sw}(x), \tag{2}$$

where $F^{\rm sw} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is upper semi-continuous with nonempty, compact and convex values for each $x \in \mathbb{R}^n$. To analyze the stability of an equilibrium of system (2) using Lyapunov-based methods, the search for almost everywhere differentiable Lyapunov functions fits well with the structure of the problem: the system (1) is an autonomous differential equation with discontinuous right-hand side, but that is continuous in some open sets whose closure covers the state space $\mathbb{R}^n$. For this reason, the choice of locally Lipschitz functions that are smooth in some open sets is quite intuitive and well-studied in literature. As a particular example, we refer the reader to (Della Rossa et al., 2018) to see the utility of functions obtained by max-min composition over a finite family of smooth functions in the context of stability of switched systems.

For the differential inclusion (2), the study of smooth Lyapunov functions has attracted some attention as well. A converse Lyapunov theorem has been proved for (2), see (Teel and Praly, 2000) or (Clarke et al., 1998) for the formal statement and (Kellett, 2015) for a thorough review. More precisely, we have the following result: If the origin of differential inclusion (2) is globally asymptotically stable (GAS), then there exists a *smooth* Lyapunov function. From a theoretical point of view, there are many advantages in having a *smooth* Lyapunov function: in particular the existence of such function also gives us information about the *robustness* of the stability for the considered dynamical system. Indeed, for general hybrid systems, it is shown in (Cai et al., 2007) that if a smooth Lyapunov function exists then the asymptotic stability is robust to small perturbations of the data.

In this paper, we want to fill the gap between these two sets of results concerning the construction of Lyapunov functions (almost everywhere differentiable, and the smooth ones) in the case of state dependent switching systems (1). In particular, we address the following question:

*Given a locally Lipschitz GAS-Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ for system* (1)*, is it possible to approximate $V$ with another smooth Lyapunov function?*

To answer this question, it will be crucial to define the concept of *"derivatives along the system's trajectories"* for locally Lipschitz functions (which are not $\mathcal{C}^1$ in general). Following the intuitions of (Ceragioli, 2000), we will focus on two different notions of "derivatives along trajectories" that we call *Clarke* and *Lie derivatives*, each of them being a set-valued map from the state space $\mathbb{R}^n$ to the real numbers. Results generalizing Lyapunov's direct method using these concepts can be found in (Baier et al., 2012), (Bacciotti and Ceragioli, 1999). We shall show that, among these two different concepts, Clarke notion leads to a positive answer to our question of obtaining "smooth approximations" of a given locally Lipschitz Lyapunov function. We shall also prove that, under certain conditions, such locally Lipschitz Lyapunov functions imply robustness with respect to perturbations in the dynamics, and this later property is formalized using the notion of input-to-state

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stability (ISS). Obtaining the same result for Lyapunov functions that satisfy conditions based on the concept of Lie derivative is not straightforward, and in this paper we will only scratch the surface of this problem.

The paper is organized as follows: In Section 2 we introduce some nonsmooth analysis prerequisites, while in Section 3, we give the definitions and the stability results for the state-dependent switching systems. In Section 4, we present our main result on smooth approximations, with some open-ended discussions for future research.

## 2. PRELIMINARIES

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be an upper semi-continuous mapping with nonempty, compact, convex values, and consider the differential inclusion (DI)

$$\dot{x}(t) \in F(x(t)). \tag{3}$$

In this section, we give sufficient conditions under which the origin of (3) is globally asymptotically stable, considering *non-smooth* (but locally Lipschitz) Lyapunov functions. Thus, in the following, we collect various notions of generalized derivatives and gradients. It can be seen that if $V$ is locally Lipschitz continuous (Clarke, 1990, Theorem 2.5.1, page 63) we have the following characterization of the Clarke's generalized gradient that, given the aim of this paper, can be seen as a definition:

$$\partial V(x) := \overline{\text{co}} \left\{ \lim_{k\to\infty} \nabla V(x_k) \,|\, x_k \to x,\, x_k \notin \mathcal{N}_V \right\} \tag{4}$$

is the *Clarke's generalized gradient* of $V$ at $x$, where $\mathcal{N}_V \subset \mathbb{R}^n$ is the set of zero measure where $\nabla V$ is not defined. We now introduce two different notions of generalized directional derivatives for locally Lipschitz functions with respect to differential inclusion (3), which appeared firstly in (Bacciotti and Ceragioli, 1999).

*Definition 1.* (Set-valued directional derivatives). Consider the differential inclusion (3); given a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$, the *Clarke generalized derivative* of $V$ along $F$, denoted $\dot{V}_F(x)$, is defined as

$$\dot{V}_F(x) := \{ \langle p, f\rangle \mid p \in \partial V(x),\, f \in F(x) \}. \tag{5}$$

Additionally, we define the *Lie generalized derivative* of $V$ with respect to $F$, denoted $\dot{\overline{V}}_F$, as

$$\dot{\overline{V}}_F(x) := \{ a \in \mathbb{R} \,|\, \exists f \in F(x) : \langle p, f\rangle = a,\, \forall p \in \partial V(x)\}. \tag{6}$$

Due to continuity of the scalar product, it can be proved that $\dot{V}_F(x)$ and $\dot{\overline{V}}_F(x)$ are closed and bounded intervals (possibly empty) of the real line, for each $x \in \mathbb{R}^n$. Adopting the convention $\max \emptyset = -\infty$, it follows that $\max \dot{V}_F(x)$ and $\max \dot{\overline{V}}_F(x)$ are well-defined, for each $x \in \mathbb{R}^m$. In the case where $V$ is continuously differentiable at $x$, one has $\dot{\overline{V}}_F(x) = \dot{V}_F(x) = \{\langle \nabla V(x), f\rangle \,|\, f \in F(x)\}$. It is clear that, in general

$$\dot{\overline{V}}_F(x) \subset \dot{V}_F(x). \tag{7}$$

We also consider a proper subclass of locally Lipschitz functions, introduced firstly in (Valadier, 1989): a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be *non-pathological* if, given any absolutely continuous function $\varphi \in AC(\mathbb{R}_+, \mathbb{R}^n)$, it holds that, for almost every $t \in \mathbb{R}_+$, there exists an $a_t \in \mathbb{R}$ such that

$$\langle v, \dot{\varphi}(t)\rangle = a_t, \;\; \forall v \in \partial V(\varphi(t)).$$

The usefulness of non-pathological functions in this context is mainly given by the following result:

*Proposition 2.* ((Ceragioli, 2000)). If $V : \mathbb{R}^n \to \mathbb{R}$ is a non-pathological function and $\varphi : \mathbb{R}_+ \to \mathbb{R}^n$ is a solution of the differential inclusion (3) then

$$\frac{d}{dt} V(\varphi(t)) \in \dot{\overline{V}}_F(\varphi(t))$$

for almost every $t \in \mathbb{R}_+$.

Now we are in the position to recall two main adaptations of the classical Lyapunov direct theorem in the context of differential inclusions and locally Lipschitz functions:

*Proposition 3.* (Clarke Sufficient Conditions). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function such that there exist [1] $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 \in \mathcal{PD}$, satisfying

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|),\; \forall x \in \mathbb{R}^n, \tag{8}$$

$$\max \dot{V}_F(x) \le -\alpha_3(|x|), \quad \forall x \in \mathbb{R}^n \tag{9}$$

then the origin of system (3) is GAS and we say that $V$ is *Clarke GAS-Lyapunov function* for the system (3).

The proof of this well-known result can be found for example in (Clarke, 1990) or in (Baier et al., 2012), but in the context of non-pathological functions, recalling inclusion (7), it can be deduced from the following proposition.

*Proposition 4.* (Lie Sufficient Conditions). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz and non-pathological function such that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 \in \mathcal{PD}$, satisfying

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|),\; \forall x \in \mathbb{R}^n, \tag{10}$$

$$\max \dot{\overline{V}}_F(x) \le -\alpha_3(|x|), \quad \forall x \in \mathbb{R}^n \tag{11}$$

then the origin of system (3) is GAS and we say that $V$ is a *Lie GAS-Lyapunov function* for the system (3).

Proposition 4 is proven, e.g., in (Ceragioli, 2000, Prop. 8).

## 3. SWITCHING SYSTEMS

In this section, we provide a formal description of the switched systems considered in this article and in particular, the partition of the state space that describes the switching rule. Then, we introduce the notion of *patchy Lyapunov* functions for such systems, and rewrite Propositions 3 and 4 in this setting.

*Definition 5.* A finite collection $\mathcal{X} := \{\mathcal{X}_1, \dots, \mathcal{X}_M\}$, with $\mathcal{X}_i \subset \mathbb{R}^n$, $i \in I := \{1, \dots, M\}$, is said to be a *proper partition* of the state space $\mathbb{R}^n$ if

(1) $\mathbb{R}^n = \bigcup_{i=1}^M \mathcal{X}_i$ ;
(2) $\overline{(\text{int}(\mathcal{X}_i))} = \mathcal{X}_i$ ;
(3) $\mathcal{X}_i \cap \mathcal{X}_j = \text{bd}(\mathcal{X}_i) \cap \text{bd}(\mathcal{X}_j), \quad \forall (i,j) \in I^2, i \ne j$ ;
(4) For each compact set $\mathcal{K} \subset \mathbb{R}^n \setminus \{0\}$, there exists a scalar $\kappa_0 > 0$ such that, for every $x \in \mathcal{K}$ and $i, j \in I$ $x \in \mathcal{X}_i$ and $\mathcal{X}_j \cap \mathbb{B}(x, \kappa_0) \ne \emptyset \Rightarrow \mathcal{X}_i \cap \mathcal{X}_j \cap \mathbb{B}(x, \kappa_0) \ne \emptyset$,

[1] A function $\alpha : \mathbb{R}^n \to \mathbb{R}$ is called *positive definite*, denoted by $\alpha \in \mathcal{PD}$, if it is continuous, $\mu(0) = 0$ and $\mu(x) > 0$ if $x \ne 0$. We say that $\alpha \in \mathcal{K}$ if $\alpha \in \mathcal{PD}$ and $\alpha$ is increasing. Also, $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and unbounded.

where $\mathrm{bd}(\mathcal{A})$ denotes the boundary of a set $\mathcal{A}$ and $\mathbb{B}(x,r) := \{y \in \mathbb{R}^n \mid |x-y| < r\}$. A similar definition can be found in (Ahmadi et al., 2017, Definition 1). To provide an interpretation, (2) is equivalent to saying that $\mathcal{X}_i$ are closures of some open sets $\mathcal{X}_i^\circ = \mathrm{int}(\mathcal{X}_i)$. Clearly (1) can be equivalently stated as $\bigcup_i \overline{\mathcal{X}_i^\circ} = \mathbb{R}^n$ and, perhaps not so obviously, (3) is equivalent to $\mathcal{X}_i^\circ \cap \mathcal{X}_j^\circ = \emptyset$, $\forall i \neq j$. Item (4) intuitively means that the regions of the partition are not "too close" to each other. Finally, let us define the set-valued map

$$I_{\mathcal{X}}(x) := \{i \in I \mid x \in \mathcal{X}_i\},$$

that represents the indices of the set containing $x$; by Definition 5, $I_{\mathcal{X}}$ is almost everywhere single valued.

Given a proper partition $\mathcal{X} = \{\mathcal{X}_1, \ldots, \mathcal{X}_M\}$ and a family $\mathcal{F} = \{f_1, \ldots, f_M\} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, we consider the system

$$\dot{x} = f_{\sigma(x)}(x), \tag{12}$$

where $\sigma : \mathbb{R}^n \to \{1, \ldots, M\}$ is the switching signal associated to the proper partition, that is

$$\sigma(x) = i, \ \ \forall\, x \in \mathrm{int}(\mathcal{X}_i),$$

and $\sigma(x)$ is arbitrarily defined on $\partial\mathcal{X} := \bigcup_{i=1}^M \mathrm{bd}(\mathcal{X}_i)$, that is the set where the function $f_{\sigma(\cdot)}(\cdot)$ is (possibly) not continuous, also called the *switching surface*. Due to this discontinuous behavior, we need to define an appropriate notion of solution of (12), conventionally arising from Filippov regularization, (Filippov, 1988).

*Definition 6.* Given a switching system defined by (12), we define its *Filippov regularization* as

$$F^{\mathrm{sw}}(x) := \overline{\mathrm{co}}\{f_i(x) \mid i \in I_{\mathcal{X}}(x)\}. \tag{13}$$

We say that $\varphi : \mathbb{R}_+ \to \mathbb{R}^n$ is a *Filippov solution* of system (12) starting at $x_0 \in \mathbb{R}^n$ if

- $\varphi$ is absolutely continuous, with $\varphi(0) = x_0$,
- $\dot{\varphi}(t) \in F^{\mathrm{sw}}(\varphi(t))$ for almost all $t > 0$.

It is well known that $F^{\mathrm{sw}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an upper semi-continuous map with nonempty, compact, convex values. We finally introduce a class of locally Lipschitz functions, obtained by "gluing" together a finite set of $\mathcal{C}^1$ functions that are well-defined on a proper partition of $\mathbb{R}^n$.

*Definition 7.* (Patchy Functions). A function $V : \mathbb{R}^n \to \mathbb{R}$ is a *Patchy Function (PF)* associated to the proper partition $\mathcal{X} = \{\mathcal{X}_1, \ldots, \mathcal{X}_M\}$ if

- $V$ is locally Lipschitz continuous;
- There exists a family $\{V_1, \ldots, V_M\}$, $V_i \in \mathcal{C}^1(\mathcal{X}_i, \mathbb{R})$ for all $i \in I$, such that

$$V(x) = V_i(x), \ \ \text{if } x \in \mathcal{X}_i.$$

A slightly different definition of "patchy functions" can be found in (Goebel et al., 2009) in the context of control Lyapunov functions and hybrid feedback. Even if the two definitions differ, the main idea behind them is the same: we consider a family of functions that are *smooth* in some regions of the space, with the additional requirement that they "glue" together.

*Remark 8.* We collect here some properties of the PF. First of all, let us underline that under the condition (2) of Definition 5 the boundary of each $\mathcal{X}_i$ has Lebesgue measure zero. Thus by Definition 7 a patchy function is $\mathcal{C}^1$ almost everywhere, in particular, for any i, for every $x \in \mathrm{int}(\mathcal{X}_i)$, we have $\nabla V(x) = \nabla V_i(x)$. Moreover, from (4), we have

$$\partial V(x) = \overline{\mathrm{co}}\,\{\nabla V_i(x) \mid i \in I_{\mathcal{X}}(x)\}\,, \tag{14}$$

and, given $F^{\mathrm{sw}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ in (13), the sets in (5) and (6) are

$$\begin{aligned}\dot{V}_{F^{\mathrm{sw}}}(x) &= \overline{\mathrm{co}}\{\langle \nabla V_i(x), f\rangle \mid i \in I_{\mathcal{X}}(x),\ f \in F^{\mathrm{sw}}(x)\},\\ \dot{\overline{V}}_{F^{\mathrm{sw}}}(x) &= \{a \in \mathbb{R} \mid \exists f \in F^{\mathrm{sw}}(x) : a = \langle \nabla V_i(x), f\rangle,\ \forall i \in I_{\mathcal{X}}(x)\}.\end{aligned} \tag{15}$$

The proofs of these equalities and of the fact that every patchy function is in particular non-pathological are quite technical, and thus they have been removed from this paper due to space constraints. It is also useful to introduce the following notation:

$$F_L^{\mathrm{sw}}(x) := \left\{ f \in F^{\mathrm{sw}}(x) \,\middle|\, \begin{array}{l} \langle \nabla V_i(x) - \nabla V_j(x), f\rangle = 0, \\ \forall (i,j) \in I_{\mathcal{X}}(x)^2 \end{array} \right\} \tag{16}$$

that represents the subset of $F^{\mathrm{sw}}(x)$ composed by vectors $f$ for which the scalar product with *all* the generalized gradients is constant, as introduced in the definition of $\dot{\overline{V}}_{F^{\mathrm{sw}}}(x)$. With (16), the set in (15) becomes

$$\dot{\overline{V}}_{F^{\mathrm{sw}}}(x) = \{\langle \nabla V_i(x), f\rangle \mid i \in I_{\mathcal{X}}(x),\ f \in F_L^{\mathrm{sw}}(x)\}.$$

We can now rewrite the stability conditions in Prop. 3 and Prop. 4 for switched system (13) using patchy functions.

*Corollary 9.* (Stability of Switched Systems). Consider the system (13) with proper partition $\mathcal{X} = \{\mathcal{X}_1, \ldots, \mathcal{X}_M\}$. Suppose that there exist a patchy function $V : \mathbb{R}^n \to \mathbb{R}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and $\alpha_3 \in \mathcal{PD}$ satisfying

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|),\ \forall x \in \mathbb{R}^n, \tag{17}$$

$$\langle \nabla V_i(x), f_i(x)\rangle \le -\alpha_3(|x|)\ \ \forall x \in \mathrm{int}(\mathcal{X}_i)\,\forall i \in I. \tag{18}$$

Let us also suppose that *at least one of the following* holds for each $x \in \partial\mathcal{X}$ :

$$\langle \nabla V_j(x), f_i(x)\rangle \le -\alpha_3(|x|)\ \ \forall (j,i) \in I_{\mathcal{X}}(x) \times I_{\mathcal{X}}(x) \tag{19}$$

$$\langle \nabla V_j(x), f\rangle \le -\alpha_3(|x|)\,\forall j \in I_{\mathcal{X}}(x), \forall f \in F_L^{\mathrm{sw}}(x), \tag{20}$$

then the origin of system (13) is GAS.

**Proof.** The result follows from the fact that $V$ is a non-pathological function, combined with the characterizations in (15), and statement of Prop. 3 and Prop. 4. In fact, conditions (18) combined with (19) (respectively, (18) combined with (20)) are the specifications of conditions (9) (resp. (11)) in the context of Filippov regularization (13) of a switching system. □

## 4. SMOOTH APPROXIMATION OF LYAPUNOV FUNCTIONS

In this section, we will address our main question: *Given a patchy Lyapunov function for system* (12) *is it possible to construct an approximating sequence of smooth functions for which the Lyapunov inequalities are (weakly) preserved?* Our response to this question is affirmative when working with Clarke derivatives, while for Lie derivatives we show with the help of an example that alternative methods need to be investigated. A primary motivation for constructing smooth approximations is that they provide more information about the robustness of the system. In Section 4.2, we show a connection between the existence of smooth Lyapunov functions and the ISS property with respect to additive inputs. It is emphasized that the interest in such approximation is not the prove of the existence of a *smooth* Lyapunov function for a globally asymptotically stable differential inclusion: such converse results are well-known, see (Teel and Praly, 2000), and (Clarke et al., 1998). Here, due to the specific structure of the problem, we carry out the construction of a smooth Lyapunov function that inherits the decay rate from an already-constructed locally Lipschitz (patchy) Lyapunov function.

### *4.1 Approximation of Clarke Lyapunov Functions*

In the following, we consider a proper partition $\mathcal{X} = \{\mathcal{X}_1, \ldots, \mathcal{X}_M\}$, a family of functions $\mathcal{F} = \{f_1, \ldots, f_M\} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and the ensuing switching system (13). We suppose that the system is GAS and that a Clarke patchy Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ is given, that is, $V$ satisfies (17), (18) and (19) for some functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 \in \mathcal{PD}$. We construct below a *smooth* function close to the given $V$. More precisely, we will prove the following

*Theorem 10.* (Smooth approximation). Consider a patchy function $V : \mathbb{R}^n \to \mathbb{R}$ associated to the proper partition $\mathcal{X}$, defined as in Definition 7. Given $F^{\rm sw} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined as in (13), let us suppose that there exists $\alpha_3 \in \mathcal{PD}$ such that

$$\dot{V}_{F^{\rm sw}}(x) \leq -\alpha_3(|x|), \ \forall x \in \mathbb{R}^n. \tag{21}$$

Then, for any given positive definite functions $\mu, \nu$, there exists $\widetilde{V} \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ such that

$$|\widetilde{V}(x) - V(x)| < \mu(x), \qquad \forall\, x \in \mathbb{R}^n,$$
$$\langle \nabla \widetilde{V}(x), f\rangle \leq -\alpha_3(|x|) + \nu(x), \ \ \forall x \in \mathbb{R}^n, \ \forall\, f \in F^{\rm sw}(x).$$

The proof of Theorem 10 builds on some tools that are provided in the sequel.

*Definition 11.* A *sequence of mollifiers* $\{\psi_\kappa\}_{\kappa>0}$ is any set of functions on $\mathbb{R}^n$ such that $\psi_\kappa \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, $\psi_\kappa(x) \geq 0$ for all $x \in \mathbb{R}^n$, $\text{supp}(\psi_\kappa) \subset \overline{\mathbb{B}(0, \kappa)}$ and $\int_{\mathbb{R}^n} \psi_\kappa(x)\, dx = 1$.

Given a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$, we define the *$\kappa$-regularization* of $V$ as

$$V^\kappa(x) := V \star \psi_\kappa(x) := \int_{\mathbb{R}^n} V(x - y)\psi_\kappa(y)\, dy. \tag{22}$$

A common way to construct an explicit sequence of mollifiers is to consider the following function

$$\psi(x) := \begin{cases} a e^{\frac{1}{|x|^2 - 1}}, & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases} \tag{23}$$

where $a = 1/\int_{\mathbb{R}^n} \psi(y)\, dy$; we can define the *$\kappa$-mollifier* as

$$\psi_\kappa(x) := \frac{1}{\kappa^n}\psi\left(\frac{x}{\kappa}\right), \tag{24}$$

where $\kappa \in \mathbb{R}_{>0}$. Throughout this section, we will use this family of mollifiers, but we underline that only the properties of Definition 11 are necessary to prove the following result.

*Lemma 12.* (Brezis, 2010, Proposition 4.21) Given a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$, the function in (22) satisfies $V^\kappa \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ for all $\kappa > 0$. Moreover, for any compact set $\mathcal{K} \subset \mathbb{R}^n$ and for any $\varepsilon > 0$, $\exists\, \widetilde{\kappa} > 0$ such that

$$|V^\kappa(x) - V(x)| < \varepsilon, \ \ \forall x \in \mathcal{K}, \ \forall \kappa \leq \widetilde{\kappa}. \tag{25}$$

Lemma 12 states that $V^\kappa \to V$ uniformly on compact subset of $\mathbb{R}^n$, as $\kappa \to 0$. Given the sequence of smooth functions $\{V^\kappa\}$ that converges to $V$ uniformly on compact subsets of $\mathbb{R}^n$, we establish that the directional derivatives of $V^\kappa$ along the vector fields of (13) also converge in some sense to the directional derivatives of $V$.

*Lemma 13.* Given a compact set $\mathcal{K} \subset \mathbb{R}^n \setminus \{0\}$ and an $\varepsilon > 0$, there exists $\widetilde{\kappa} > 0$ such that for all $\kappa \leq \widetilde{\kappa}$

$$\langle \nabla V^\kappa(x), f\rangle \leq -\alpha_3(|x|) + \varepsilon, \ \forall\, x \in \mathcal{K}, \ \forall f \in F^{\rm sw}(x).$$

**Proof.** Using the definition of $V^\kappa$ in (22) and the Lebesgue's Convergence Theorem, for a given $x, v \in \mathbb{R}^n$, we have

$$\begin{aligned} \langle \nabla V^\kappa(x), v\rangle &= \lim_{h \to 0^+} \frac{V^\kappa(x + hv) - V^\kappa(x)}{h} \\ &= \int_{\mathbb{B}(0,\kappa)} \langle \nabla V(x - y), v\rangle \psi_\kappa(y)\, dy, \end{aligned}$$

where we emphasize that $\nabla V(\cdot)$ is defined almost everywhere in $\mathbb{B}(x, \kappa)$. Let us fix $\mathcal{K} \subset \mathbb{R}^n \setminus \{0\}$ compact, $\varepsilon > 0$, $x \in \mathcal{K}$ and $f \in F^{\rm sw}(x)$. Given a $\kappa > 0$, let us define $I_{\mathcal{X}}(x; \kappa) := \bigcup_{z \in \mathbb{B}(x,\kappa)} I_{\mathcal{X}}(z)$. From Definition 5 we have

$$\begin{aligned} \langle \nabla V^\kappa(x), f\rangle &= \int_{\mathbb{B}(0,\kappa)} \langle \nabla V(x - y), f\rangle \psi_\kappa(y)\, dy \\ &= \sum_{i \in I_{\mathcal{X}}(x;\kappa)} \int_{\substack{y \in \mathbb{B}(0,\kappa) \\ x - y \in \mathcal{X}_i}} \langle \nabla V_i(x - y), f\rangle \psi_\kappa(y)\, dy. \end{aligned} \tag{26}$$

Let us fix $\kappa_0 > 0$ given by property (4) of Definition 5, then for all $i \in I_{\mathcal{X}}(x; \kappa_0)$ and for all $j \in I_{\mathcal{X}}(x)$ there exists $\widetilde{z}_{i,j} \in \mathbb{B}(x, \kappa_0) \cap \mathcal{X}_i \cap \mathcal{X}_j$. From (21), and from definition (13) this implies

$$\langle \nabla V_i(\widetilde{z}_{i,j}), f_j(\widetilde{z}_{i,j})\rangle \leq -\alpha_3(|\widetilde{z}_{i,j}|). \tag{27}$$

In fact if $i \in I_{\mathcal{X}}(x)$ then we can choose $\widetilde{z}_{i,j} = x$, otherwise we can take $\widetilde{z}_{i,j} \in \text{bd}(\mathcal{X}_j) \cap \text{bd}(\mathcal{X}_i) \cap \mathbb{B}(x, \kappa_0)$. By continuity of $f_1, \ldots, f_M$ and $\nabla V_1, \ldots \nabla V_M$ and by compactness of $\mathcal{K}$ there exists $\kappa_1 > 0$ small enough such that $|\langle \nabla V_i(x_2), f_j(x_2)\rangle - \langle \nabla V_i(x_1), f_j(x_1)\rangle| < \frac{\varepsilon}{2}$, if $x_1, x_2 \in \mathcal{K}$ and $|x_1 - x_2| < 2\kappa_1$, for all $(i, j) \in I^2$. Choosing $\kappa' := \min\{\kappa_0\, , \, \kappa_1\}$, (27) yields, for each $f \in F^{\rm sw}(x)$,

$$\langle \nabla V_i(x - y), f\rangle \leq -\alpha_3(|\widetilde{z}_{i,j}|) + \frac{\varepsilon}{2}, \ \forall\, y \in \mathbb{B}(0, \kappa), \ x - y \in \mathcal{X}_i \tag{28}$$

for all $\kappa \leq \kappa'$. Moreover by continuity of $\alpha_3$, there exists $\widetilde{\kappa} \leq \kappa'$ such that $\alpha_3(|x_1|) - \alpha_3(|x_2|) \leq \frac{\varepsilon}{2}$ if $|x_1 - x_2| \leq \widetilde{\kappa}$. This clearly implies

$$\alpha_3(|x|) - \alpha_3(|z|) \leq \frac{\varepsilon}{2}, \ \text{if } z \in \mathbb{B}(x, \widetilde{\kappa}). \tag{29}$$

To summarize, by combining (26), (28) and (29), we get

$$\begin{aligned} \langle \nabla V^\kappa(x), f\rangle \leq &- \sum_{i \in I_{\mathcal{X}}(x;\kappa)} \int_{\substack{y \in \mathbb{B}(0,\kappa) \\ x - y \in \mathcal{X}_i}} \alpha_3(|x|)\psi_\kappa(y)\, dy \\ &+ \int_{\mathbb{B}(0,\kappa)} \varepsilon\, \psi_\kappa(y)\, dy = -\alpha_3(|x|) + \varepsilon. \end{aligned}$$

for all $\kappa \leq \widetilde{\kappa}$. The choice of $\widetilde{\kappa} > 0$ neither depends on $x \in \mathcal{K}$, nor on $f \in F^{\rm sw}(x)$, but only on $\mathcal{K}$ and $\varepsilon > 0$. Hence, the assertion follows. □

Merging Lemmas 12 and 13, we obtain the following result:

*Theorem 14.* Consider any compact set $\mathcal{K} \subset \mathbb{R}^n \setminus \{0\}$. For any given $\varepsilon > 0$ there exists a $\kappa_0 > 0$ such that

$$|V(x) - V^\kappa(x)| < \varepsilon \ \text{ and } \ \langle \nabla V^\kappa(x), f\rangle \leq -\alpha_3(|x|) + \varepsilon,$$

for all $f \in F^{\rm sw}(x)$, $x \in \mathcal{K}$ and for all $\kappa \leq \kappa_0$.

**Proof of Theorem 10.** To complete the proof, we briefly recall the arguments used in (Lin et al., 1996, Theroem B.1), to which we refer for further details. We consider $\{\mathcal{U}_k\}$ a locally finite, countable cover of $\mathbb{R}^n \setminus \{0\}$, with $\overline{\mathcal{U}_k}$ compact and $\overline{\mathcal{U}_k} \subset \mathbb{R}^n \setminus \{0\}$. Let $\{\beta_k\}$ be a partition of unity on $\mathbb{R}^n \setminus \{0\}$ associated to the covering $\{\mathcal{U}_k\}$. Thus, given $\mu$ and $\psi$ of the statement one can properly define $\varepsilon_k > 0$ and apply Theorem 14 to $\mathcal{U}_k$ and $\varepsilon_k$, for each $k \in \mathbb{N}$, obtaining a smooth approximation $\widetilde{V}_k$ on $\mathcal{U}_k$. Defining $\widetilde{V}(x) = \sum_{k\in\mathbb{N}} \beta_k(x)\widetilde{V}_k(x)$, it can be seen that the statement holds. □

### *4.2 Application to ISS via Smooth functions*

As sample application of our smooth approximation of patchy functions, we discuss here some ISS results. Given a proper partition $\mathcal{X} = \{\mathcal{X}_1, \ldots, \mathcal{X}_M\}$, let us add to the system (13) a state dependent input map, that is,

$$\dot{x}(t) \in F_u^{\mathrm{sw}}(x,u) := \overline{\mathrm{co}}\,\{f_i(x) + g_i(u) \mid i \in I_{\mathcal{X}}(x)\}, \quad (30)$$

where $\{g_1, \ldots, g_M\} \subset \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^n)$. We have the following

*Proposition 15.* Let us consider the system (30) with input $u \equiv 0$. Suppose that there exist a $\widetilde{V} \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that, for all $x \in \mathbb{R}^n$

- $\alpha_1(|x|) \le \widetilde{V}(x) \le \alpha_2(|x|), \quad (31)$
- $\langle \nabla\widetilde{V}(x), f\rangle \le -\alpha_3(|x|), \quad \forall f \in F_u^{\mathrm{sw}}(x,0), \quad (32)$

i.e. $\widetilde{V}$ is a smooth GAS-Lyapunov function for the system (30) with input $u \equiv 0$. Moreover, let us suppose that there exist $\beta_1, \beta_2 \in \mathcal{K}_\infty$ and $\varepsilon > 0$ such that

(i) $g_i(u) \le \beta_1(|u|)$ for all $u \in \mathbb{R}^m$;
(ii) $|\nabla\widetilde{V}(x)| \le \beta_2(|x|) + \varepsilon$ for all $x \in \mathbb{R}^n$;
(iii) $\lim_{|x|\to\infty} \frac{\alpha_3(|x|)}{\beta_2(|x|)} = +\infty$.

Then there exists $\widetilde{\alpha}_3 \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}$ such that

$$\langle \nabla\widetilde{V}(x), \widetilde{f}\rangle \le -\widetilde{\alpha}_3(|x|), \quad \forall |x| \ge \gamma(|u|), \quad (33)$$

for every $\widetilde{f} \in F_u^{\mathrm{sw}}(x,u)$.

Property (33) implies that the system (30) is ISS, and $\widetilde{V}$ is an ISS-Lyapunov function (Sontag and Wang, 1995). The asymptotic ratio condition *(iii)* also appears in (Liberzon and Shim, 2015, Theorem 1). Let us note that quantifying the decay of $\widetilde{V}$ via $\alpha_3$ is the key to proving ISS, which motivates our result of Theorem 10, where the dissipation rate $\alpha_3$ is preserved by the smoothened construction. We want to underline again that if system (30) is GAS (in the case $u \equiv 0$), the existence of a smooth Lyapunov function is assured by the converse Lyapunov Theorem (Teel and Praly, 2000), but in this case we do not have informations on the decay function. Moreover, let us note that in items *(ii)* and *(iii)* of Prop. 15 we need to provide a $\mathcal{K}_\infty$-upper bound of the norm of the gradient of the smooth Lyapunov function. Under some assumptions, this bound can be deduced from the properties of the patchy function that $\widetilde{V}$ approximates, as we show next in Example 1.

**Proof.** We follow the idea of (Khalil, 2002, Lemma 4.6). First of all, note that

$$F_u^{\mathrm{sw}}(x,u) \subset \overline{\mathrm{co}}\{f_i(x) \mid x \in I_{\mathcal{X}}(x)\} + \overline{\mathrm{co}}\{g_i(u) \mid x \in I_{\mathcal{X}}(x)\}.$$

Thus, by *(i)* and *(ii)*, given $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $\widetilde{f} \in F_u^{\mathrm{sw}}(x,u)$, we have

$$\begin{aligned}\langle \nabla\widetilde{V}(x), \widetilde{f}\rangle \le& -\alpha_3(|x|) + \beta_1(|u|)(\beta_2(|x|) + \varepsilon)\\ =& -(1-\delta)\alpha_3(|x|)\\ &- \delta\alpha_3(|x|) + \beta_1(|u|)(\beta_2(|x|) + \varepsilon),\end{aligned}$$

for all $0 < \delta < 1$. Thus if $\beta_1(|u|)(\beta_2(|x|)+\varepsilon) - \delta\alpha_3(|x|) \le 0$ then $\langle \nabla\widetilde{V}(x), \widetilde{f}\rangle \le -(1-\delta)\alpha_3(|x|)$. Computing we have

$$\beta_1(|u|)(\beta_2(|x|)+\varepsilon) - \delta\alpha_3(|x|) \le 0 \Leftrightarrow \beta_1(|u|) \le \frac{\delta\alpha_3(|x|)}{\beta_2(|x|)+\varepsilon}.$$

By *(iii)*, there exists $\widetilde{\gamma} \in \mathcal{K}_\infty$ such that $\widetilde{\gamma}(s) \le \delta\frac{\alpha_3(s)}{\beta_2(s)+\varepsilon}$ for all $s \in \mathbb{R}_{\ge 0}$, and thus, by letting $\widetilde{\alpha}_3 = (1-\delta)\alpha_3$ and $\gamma = \widetilde{\gamma}^{-1} \circ \beta_1$, we can deduce (33). □

**Example 1.** (Linear conewise switching). Let us consider a partition $\mathcal{X} = \{\mathcal{X}_1, \ldots, \mathcal{X}_M\}$ such that all the $\mathcal{X}_i$ are cones, that is, if $x \in \mathcal{X}_i$ then $\lambda x \in \mathcal{X}_i$, for all $\lambda \in \mathbb{R}_+$, for all $i \in I$. Consider the linear switched system

$$\dot{x}(t) \in \overline{\mathrm{co}}\,\{A_i x + B_i u \mid i \in I_{\mathcal{X}}(x)\} =: F_{\mathrm{lin}}^{\mathrm{sw}}(x,u) \quad (34)$$

where $\{A_1, \ldots, A_M\} \subset \mathbb{R}^{n\times n}$ and $\{B_1, \ldots, B_M\} \subset \mathbb{R}^{n\times m}$. Considering system (34) with $u \equiv 0$, let us suppose that a GAS-patchy Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$ is given. In particular, without loss of generality, we can suppose that $V$ is homogeneous of degree 2. If it is not, by linearity of (34) and the fact that every $\mathcal{X}_i$ is a cone, we can apply the idea presented in (Rosier, 1992, Prop 2) to each $V_i$, obtaining a patchy Lyapunov function $\widetilde{V}$ homogeneous of degree 2. For that reason, there exist $a_1, a_2, a_3 > 0$ such that

$$\begin{aligned}&a_1|x|^2 \le V(x) \le a_2|x|^2, \ \forall x \in \mathbb{R}^n,\\ &\langle \nabla V_i(x), A_i x\rangle \le -a_3|x|^2 \ \ \forall x \in \mathrm{int}(\mathcal{X}_i)\, \forall i \in I,\\ &\langle \nabla V_j(x), A_i x\rangle \le -a_3|x|^2 \ \forall x \in \partial\mathcal{X}, \forall (j,i) \in I_{\mathcal{X}}(x)^2.\end{aligned}$$

Moreover, following Rosier (1992), if $V$ is homogeneous of degree 2, then the generalized gradient $\partial V(\cdot)$ is homogeneous of degree 1 in the sense that $\partial V(\lambda x) = \lambda \partial V(x)$, $\forall x \in \mathbb{R}^n$, $\forall \lambda \in \mathbb{R}$. Thus we can find a $b \in \mathbb{R}$ such that

$$\max_{v\in\partial V(x)} |v| \le b|x|, \ \forall x \in \mathbb{R}^n. \quad (35)$$

Applying Theorem 10, for any $\varepsilon > 0$ we construct a smooth function $\widetilde{V} \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ such that

$$\begin{aligned}&(a_1-\varepsilon)|x|^2 \le \widetilde{V}(x) \le (a_2+\varepsilon)|x|^2, \ \forall x \in \mathbb{R}^n,\\ &\langle \nabla\widetilde{V}(x), f\rangle \le -(a_3+\varepsilon)|x|^2 \ \ \forall x \in \mathbb{R}^n\, \forall f \in F_{\mathrm{lin}}^{\mathrm{sw}}(x,0).\end{aligned}$$

Moreover, the gradient of $\widetilde{V}$ inherits the property (35) in the sense that it can be bounded by a linear function, that is there exists a $\widetilde{b} \in \mathbb{R}$ such that $|\nabla\widetilde{V}(x)| \le \widetilde{b}|x| + \varepsilon$. Thus, all the hypotheses of Prop. 15 are satisfied by the function $\widetilde{V}$, and we can conclude that the system (34) is ISS, and function $\widetilde{V}$ is a smooth ISS-Lyapunov function.

We want to underline that, without the smoothness assumption on the function $\widetilde{V}$, Prop. 15 does not hold; this provides a motivation to study the approximation technique for the family of patchy Lyapunov functions.

### *4.3 Lie Lyapunov Functions: A counterexample*

Let us consider a patchy Lyapunov function for system (13) in the sense of Lie, i.e. a patchy function $V : \mathbb{R}^n \to \mathbb{R}$ that satisfies (17), (18) and (20). The smoothing technique used in Section 4.1 is not useful in this context, as shown in the following counterexample:

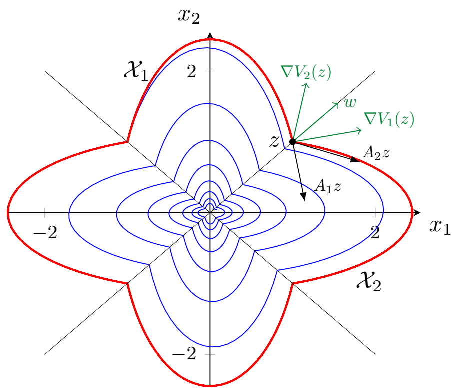


Fig. 1. Illustration of Example 2. The blue curve is a solution starting at $z_0 = (-1, 1)^\top$; the red curve shows a level set of the patchy function $V$; the black and green vectors represent, respectively, the fields and the gradients at $z = (1, 1)^\top$.

**Example 2.** We consider the dynamical system from (Johansson and Rantzer, 1998, Example 1). Given the proper partition $\mathcal{X} = \{\mathcal{X}_1, \mathcal{X}_2\}$ of $\mathbb{R}^2$ defined by

$$\mathcal{X}_i := \{x \in \mathbb{R}^2 \mid x^\top Q_i x \geq 0\},$$

with $Q_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and $Q_2 = -Q_1$, we consider the switching system

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x \in \mathcal{X}_1, \\ A_2 x, & \text{if } x \in \mathcal{X}_2, \end{cases}$$

with $A_1 = \begin{pmatrix} -0.1 & 1 \\ -5 & -0.1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} -0.1 & 5 \\ -1 & -0.1 \end{pmatrix}$. It is easy to see that the patchy function defined by

$$V(x) = \begin{cases} V_1(x) = x^\top P_1 x & \text{if } x \in \mathcal{X}_1, \\ V_2(x) = x^\top P_2 x & \text{if } x \in \mathcal{X}_2, \end{cases}$$

with $P_1 = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$, $P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ is a Lie Lyapunov function: outside the switching surfaces (i.e. if $x \in \operatorname{int}(\mathcal{X}_1) \cup \operatorname{int}(\mathcal{X}_2)$), we have $\langle \nabla V(x), \dot{x} \rangle < 0$ and hence (18) holds. On the switching lines (i.e if $x \in \partial\mathcal{X} = \operatorname{bd}(\mathcal{X}_1) \cap \operatorname{bd}(\mathcal{X}_2)$), we have $\dot{\overline{V}}_{F^{\mathrm{sw}}}(x) = \emptyset$, which means that nothing has to be checked in (20). See Figure 1 for a graphical illustration of a trajectory of the system and a particular level set of the function $V$. It is clear that $V$ is *not* a Clarke Lyapunov function, for example at the point $z = (1, 1)^\top \in \partial\mathcal{X}$ we have $\langle \nabla V_1(z), A_2 z \rangle = 23.4 > 0$, which contradicts (19). If we try to apply the smoothing technique of Section 4.1 to the function $V$, we would have

$$\nabla V^\kappa(z) \to w := \frac{1}{2}\nabla V_1(z) + \frac{1}{2}\nabla V_2(z) \in \partial V(z),$$

and for this vector $w$, it holds that $\langle w, A_2 z \rangle = 11.4 > 0$. The sequence of smooth functions $V_\kappa$ obtained via convolution with $\psi_\kappa$ defined as in (24), even if it converges uniformly on the compact sets to $V$, does not preserve the property of being monotonically decreasing along the solutions, which holds for $V$ by assumption.

This example shows that the convolution-based approximation, when working with the sequence of mollifiers (24), does not permit us to approximate a patchy Lie Lyapunov function by a smooth one. The problem of the existence and construction of such smooth approximate functions remains open for further research.

## 5. CONCLUSION

Having proposed certain patchy Lyapunov functions for certifying the stability of autonomous switched systems, we addressed the question of constructing a smooth approximation of such functions. Such constructions are particularly relevant for studying stability in the presence of bounded disturbances in the dynamics. Some questions on generalizing this construction with Lie notion of derivative are still being investigated.

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