



European Research in Mathematics Education II Proceedings

Jarmila Novotná

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European Research in Mathematics Education II

Mariánské Lázně, Czech Republic
Charles University, Faculty of Education

February 24 - 27, 2001

Proceedings

Edited by Jarmila Novotná



Prague 2002

**European Research
in Mathematics Education II**

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CERME 2 is closely linked to the Research Project *Cultivation of mathematical thinking and education in European culture*.

Since all papers and other presentations here are presented in English, which is not usually the first language of the presenters, the responsibility for spelling and grammar lies with the authors of the papers themselves.

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A YOUNG EUROPEAN SOCIETY ESTABLISHING A TRADITION FOR SCIENTIFIC CONFERENCES

CERME 2 was the second conference of the new society ERME, i.e. the European Society for Research in Mathematics Education. CERME 1 was held in Haus Ohrbeck (near Osnabrück, Germany) in August 1998. The Programme Committee under the chair of Barbara Jaworski established a new culture for scientific conferences in the field of mathematics education in Europe. Work was mainly done in working groups. The spirit of communication, co-operation and collaboration was meant to be a characteristic trademark for this new series of European conferences on research in mathematics education.

CERME 2 was held in Mariánské Lázně from February, 24th to 27th 2001 in the Czech Republic. The Programme Committee decided to follow in the footsteps of CERME 1 and to plan CERME 2 again in a style of collaborative group work. The intention was that each group would engage in scientific debate with the purpose of deepening mutual knowledge about topics, problems and methods of research in this field. The scientific programme consisted mainly of this group work. This time even plenary lectures were completely omitted.

CERME 2 was organised by the following Programme Committee:

- Elmar Cohors-Fresenborg (chairman, Germany)
- Christer Bergsten (Sweden)
- Tommy Dreyfus (Israel)
- Barbara Jaworski (United Kingdom)
- Maria Alessandra Mariotti (Italy)
- Jarmila Novotná (Czech Republic)
- Julianna Szendrei (Hungary)

Setting up the Groups

The Programme Committee first had to consider the topics for the groups. The final list was a result of a process in which the continuation of working groups from CERME 1 and new proposals from PC members or members of the ERME-Board were involved. Eventually, themes were agreed upon and group leaders were sought for the 7 groups. First, the PC invited group co-ordinators, acknowledged experts, each having research interest and expertise in the topic of the group. In several cases, the colleagues asked could not accept the invitation, therefore, other decisions had to be made. In a second step, the PC, together with the group co-ordinators, looked for group leaders with the aim that – depending on the estimated size of the group – 2 to 4 group leaders from different countries should form the leading team. A balance of nations was sought in the group

leadership. Of course, not every person invited was able to accept. So some compromises on this balance had to be made.

The groups chosen and the group coordinator and group leaders who finally participated were as follows:

	Theme	Coordinator	Further group leaders
1	Building Structures in Mathematical Knowledge	Milan Hejný (CZ)	Graham Littler (UK) Pessia Tsamir (IL)
2	Tools and Technologies in Mathematical Didactics	Keith Jones (UK)	J-Bapt. Lagrange (F)
3	Theory and Practice of Teaching -from Pre-service to In-service Teacher Education	Fulvia Furinghetti (I)	Barbro Grevholm (S) Konrad Krainer (AT)
4	Social Interactions in Mathematical Learning Situations	Götz Krummheuer (D)	Gérard Sensevy (F)
5	Mathematical Thinking and Learning as Cognitive Processes	Inge Schwank (D)	Pearla Nesher (IL)
6	Assessment and Curriculum	Ole Björkvist (FIN)	Klára Tompa (H)
7	The Role of Metaphors and Images in the Learning and Understanding of Mathematics	Bernard Parzysz (F)	Nuria Gorgorio (E)

The team of group leaders organised a process of reviewing the delivered research papers. In some groups, the reviewing process was done among the group leaders, in others, group leaders identified other suitable and competent reviewers. As an outcome of this reviewing process, papers were accepted or rejected or a proposal was made to transfer the paper into another group where it would fit the topic better. The reviewers were asked to give supportive comments on the papers; even in the case of acceptance, the writers of the papers should receive support to improve the paper. The intention was that these accepted papers should form the first step of the scientific debate of CERME 2.

The decision in CERME 1 that there should be no “oral delivery” of a paper within the group at the conference was identified as a good decision by most of the participants. Due to the fact that it takes some time to become acquainted with this new style of scientific conference, these goals were not fully achieved in the first conference. Therefore a new attempt was made with CERME 2 to promote this new idea of preparing and executing a scientific conference as a forum of scientific debate. The change of scientific traditions and the establishment of new styles need a lot of staying power. It has to be understood more as a process than as a breaking point. Considering this, CERME 2 has made a big step forward.

Publication of Conference Proceedings

During a meeting at the end of the conference, the members of the old and the newly elected ERME-Board discussed ways of how to improve the scientific level of European research in mathematics education and what role the publication of the outcome of the conferences could play. Publication was regarded as a third step in the scientific debate: The first step consists of writing, reviewing and re-writing of papers. The second step consists of debating the scientific themes within the working groups; here, the accepted papers are merged into the debate because they have been read by the members of the groups before the meeting. The third step consists of a new process of rewriting and reviewing the papers, as a result of the scientific debate, for publication in the proceedings. It was decided that only those papers which had finally been accepted by the team of the group leaders for each group should be published in the conference proceedings.

We are glad that we can now present the results of this intensive scientific debate to the public. We take the opportunity to thank, very warmly, Jarmila Novotná as head of the local organising committee for the marvellous work which she and her team did to draw nearer to the common goal of promoting European research in mathematics education. We also thank the many unknown helpers, not only before and during the conference, but on the occasion of writing the preface also in terms of all the technical support. As a result, an interesting and stimulating research publication can now act as a catalyst in strengthening and enriching the research in our growing discipline.

Elmar Cohors-Fresenborg

on behalf of the CERME 2 Programme Committee, December 2001

WORKING GROUP 1

Creating experience for structural thinking

Group leaders:

Milan Hejný
Graham Littler
Pessia Tsamir

INTRODUCTION TO WG1

BUILDING STRUCTURES IN MATHEMATICAL KNOWLEDGE

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Our brain does not store individual pieces of knowledge separately. Some of the pieces are joined together to form a linkage and there are many such linkages in our brain. Some linkages are inter-linked by common piece(s) of knowledge. We understand the structure of knowledge to be the web of all these linkages.

The cumulative understanding of the structure of knowledge considers all mathematical knowledge to be divided into large linkages such as geometry, algebra, logic.... A learning process in this view is understood to be adding a new piece of knowledge onto one of the already existing linkages.

The generic approach to the structure of knowledge considers the structure to be dynamic, in which each new piece of knowledge causes a re-organisation of the existing structure. A form of stock-taking process takes place. Does the new piece of knowledge join one or more of the already established linkages? Does it cause two or more linkages to join? Is it a completely new piece of knowledge which cannot join any existing knowledge? This process means the restructuring of the already existing mental structure.

The majority of the contributions to WG1 were based on the generic approach to the building of mathematical structures and in all of these, the research interest was focussed on those phenomena which play an important role in this process. A broad variety of topics were presented in the papers – algebra, arithmetic, 2D and 3D geometry, set theory, logic, argumentation, concept creation and research methods.

CREATING MATHEMATICAL STRUCTURE

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Abstract: *Structuring mathematics belongs to the most complex of long-term cognitive processes. Some results which we gained whilst studying this phenomenon will be presented in this contribution which is aimed at general ideas (research methods, the anatomy of the process, different ways of structuring), and two other contributions aimed at non-standard structures: the finite algebraic structure of “restricted arithmetic” and the infinite arithmetical structure of “triads”.*

1. The aim of the paper

In this paper we would like to give a general view of our understanding of what creating a mathematical structure means and how it can be studied. Our study is influenced by ideas of L. Kvasz (1998), and P. Vopěnka (2000), based on long-term experimental education and the research of D. Jírotková, J. Perný, J. Perenčaj, B. Rozek, E. Swoboda, M. Tichá and is closely connected to the research of N. Stehlíková, J. Kratochvílová presented in our WG1.

2. Internal mathematical structure (IMS)

The idea of a mathematical structure was profoundly explained in the Bourbaki’s famous Architecture of Mathematics. Our aim is to investigate how a mathematical structure is created in an individual’s mind. To avoid the danger of confusing the two different readings of the term *mathematical structure*, we will add the adjectives *external* and *internal* to distinguish the two. The differentiation corresponds to Bolzano’s idea which was elaborated by Karl Popper into the idea of three worlds (Popper, Lorenz, 1994). For the world of individual minds – the second of Popper’s worlds – we use the word *internal*, and for his third world – the world of culture – we use the word *external*.

¹ The research has been supported by the grant VZ J13/98/114100004, Kultivace matematického myšlení a vzdělanosti v evropské kultuře.

There are several possibilities of how the term *internal mathematical structure* (hereinafter IMS) can be understood. In his study, van Hiele (1986) discusses two different ways of how structure can be characterized:

“For Piaget there are three characteristics:

1. Structure has a totality,
2. Structure is achieved by transformations,
3. Structure is autoregulating. ...

In structural psychology (Gestalt psychology) there are four important properties that govern structure:

1. It is possible to extend a structure. Whoever knows a part of the structure also knows the extension of it.
2. A structure may be seen as a part of a finer structure. ...
3. A structure may be seen as a part of a more-inclusive structure....
4. A given structure may be isomorphic with another structure...”

Piaget’s characteristics are too general; we have used them as an inspiration. The four properties of Gestalt psychology helped us to understand the problem, but they are only focused onto a part of the building process of IMS. Moreover, the second sentence in property 1 is, according to our experience, not true. In our opinion, the most important phenomenon of IMS is its connectedness.

Bell (1993), when discussing psychological principles that underlie designing teaching, starts with *connectedness* which he characterizes by the statement - *a fundamental fact about learned material is that richly connected bodies of knowledge are well retained; isolated elements are quickly lost.*

The same idea with the stress on a constructivistic approach is expressed by Hiebert & Carpenter (1992, p. 66, 67): *We propose that when relationships between internal representations of ideas are constructed, they produce networks of knowledge....*

These two ideas, construction and connectedness, give the core of our approach to IMS. In our understanding, IMS is a dynamic set of networks with different pieces of knowledge like ideas, concepts, facts, relations, examples, solving strategies, arguments, algorithms, procedures, hypotheses, ... as centroids of these networks. IMS binds all these networks together and equipped this set with an organization. *Networks may be structured like vertical hierarchies, or, they may be structured like webs. ... A mathematical idea or procedure or fact is understood if it is part of an internal network. ... The degree*

of understanding is determined by the number and the strength of the connections Hiebert & Carpenter (1992, p. 67).

In our long term experimental teaching, we observed that a student's ability to built IMS is a deep characteristic of his/her cognitive style and profoundly depends on the *autonomy* of his/her approach to mathematics and on his/her *structuring appetite* i.e. the desire to create different graphs, tables, lists and overviews in order to get understanding of how 'all these facts are connected together'.

Our comparative analyses of how IMS is built in the traditional transmissive vs. the constructivistic teaching showed that the crucial role in building IMS is played by concepts. *Conceptual knowledge is knowledge that is rich in relationships. A unit of conceptual knowledge is not stored as an isolated piece of information; it is conceptual knowledge only if it is part of a network.* Hiebert & Carpenter (1992, p. 78).

In the transmissive teaching the stress is put on how to deal with it; in the constructivistic teaching the stress is put on what it is (see example 1 below).

3. How can IMS be studied?

Five of the research methods which we have elaborated for the investigation of concept creation (Hejný et al, 1990, 28-34) are listed below. The methods were focused to networks of particular ideas, concepts and procedures, hence they could be, and were used in analyzing IMS.

Explain some idea (prime, area, fraction, triangle, Pythagorean theorem) to your a) classmate, b) younger friend.

We found out that if the explanation retained the formal school approach, the quality of the student's understanding of this idea is low, and, as a rule, is more procedural than conceptual and the network of this idea, as a part of IMS, is usually poor.

Example 1. An eight-year-old boy asked his older sister Alice what 'per cent' was. She showed him formulas: $\text{percentage} = 100 \text{ rate/base}$, $b = 100r/p$, $r = pb/100$. He was not satisfied and said: "I did not ask you how you can calculate it, I would like to know what it is". The girl was not able to answer this question which was focused to the concept 'per cent'. Alice's knowledge was rather procedural and formal. Her three-parts knowledge of 'per cent' is isolated.

Describe a concept in a non-standard way. A student is asked to define a ‘circle’ without the concept of ‘distance’, or a ‘median of a triangle’ without the concept of the ‘midpoint of a segment’ or a ‘L.C.M’ without the concept of ‘multiple’.

Example 2. University students, future teachers, were asked to define a ‘prime’ without the concept of divisor. Their answer was “It is not possible, the definition of a ‘prime’ requires the concept of division or multiplication”. The same task was perfectly solved by seventh graders: “A number n is not prime, if you can arrange n pebbles into the rectangular shape. If you can not do that and the only shape is the line of n pebbles, n is prime”. Students used the linkage between concepts ‘area’ and ‘multiplication’ to find a new definition of a prime.

Use a non-standard notation in a counting procedure. E.g. a student is asked to use ‘ \uparrow ’ for addition and ‘ \downarrow ’ for subtraction; e.g. $5\uparrow 2 = 7$, $4\uparrow(-3) = 1$, $6\downarrow 1 = 5$, $2\downarrow(-1) = 3$. We observed that fifth graders deal with this notation better than eighth graders who exhibit a strong tendency to rewrite the arrow notation into the usual one. When asked to explain the symbol $5\downarrow\downarrow 2$, the majority of eighth graders take it as $5 - (-2) = 7$, for majority of fifth graders it was just the confirmation of the subtraction ($5 - 2 = 3$) and for some of them it was the doubled subtraction ($5 - 2 - 2 = 1$). One these students gave the following argument: he wrote ‘ $5\downarrow\uparrow 2$ ’ and said “I had 5 crowns, I lost 2 crowns and I found 2 crowns, so I have 5 crowns; in your case I lost 2 crowns twice, so I have just 1 crown”.

A more demanding task is counting in Roman numerals (LXI times $CIL = ?$).

Solve a problem with restricted instruments. E.g. find the midpoint of the side of a given rectangle if you only have one straight-edge (not a ruler), or use a calculator with a 12-digit display to multiply two 10-digit numbers.

Derive one idea from an other. E.g. derive the idea of subtraction from the idea of addition via ‘count on’ calculation. Derive the formula for the area of a trapezoid from the formula for the area of a triangle. Derive the formula for $\sin(\alpha - \beta)$ from the formula for $\sin(\alpha + \beta)$.

4. The research on the building of IMS

A researcher who wants to study IMS and its creation faces at least one serious difficulty. The object of the study is a long-term process and if studied ‘in vitro’ it gives only particular results. To get a complex understanding of how an IMS is built we have to study it ‘in vivo’.

In this respect, a great help for our research was our ten years of experimental teaching in a primary school (years 1975-1980 - fifth to eighth grades, years 1984-1989 third to eighth grades) during which we not only gained valuable experience, but also kept detailed pedagogical diaries which profoundly facilitated the understanding the problems.

Generally speaking, each new piece of mathematical knowledge can be regarded as a part of building an IMS. However, we are going to restrict our attention to those mental activities, which bring new linkage between already existing and/or new pieces of knowledge. We saw such a linkage in example 2.

In our experimental teaching, we identified many different kinds of such building IMS-steps. Probably the most useful were:

1. the appearance of a question which concerns mathematical structure (e.g. What it is? Why does it work? How are these two ideas linked together?)
2. the appearance of a strategic mathematical problem² in an individual's mind
3. finding the linkage between two or more already existing pieces of knowledge
4. finding the connection between new and existing knowledge
5. introducing some organization into already existing knowledge
6. the extension (generalization) of a piece of knowledge
7. looking for new non-standard solving strategies
8. the need to give an explanation of disharmony if it appears
9. the reorganization of an already existing structure
10. finding the linkage between two or more structures
11. using abstraction to create a new, more abstract structure
12. organizing the whole structure in a clear, simple written form

To elucidate some of these building IMS-steps, we will illustrate them. The illustrations come from the author's experimental teaching. We have to add that there was no difficulty in observing these IMS-steps, since each student who was involved in such an activity had a strong need to discuss his or her problem and investigation with the teacher. This will be clearly seen in example 4.

More detailed illustrations of the presented points will be given in the above contributions of the author's colleagues.

² As an (internal) strategic we mean such a problem which survives in an individual's mind for a long time. In example 3 we will see an illustration: finding strategy for the two-pile REMOVE game with the characteristic 4 was the strategic problem for Ben for the period of one year. In the history of mathematics (external) strategic problems were e.g. the trisection of an angle, the Fermat problem, etc.

5. Linkage between pieces of knowledge

Example 3. A die was on the table and three four graders, Adam, Ben and Cid, were asked what number of spots were on the bottom face of the die. All three used the strategy of the missing number – they found out that sides 1, 2, 4, 5 and 6 could be seen, hence the missing number had to be 3. We repeated the experiment three more times and then I rolled two dice. This time, Ben immediately gave the correct answer for both dice. He explained his strategy: “Here, the top number is 1, so the bottom number must be 6. The top and the bottom numbers create a pair. It is 2-5, or 1-6, or 4-3.” Adam and Cid grasped this idea. Neither of boys found the pattern “the sum of two opposite numbers is 7”, or even the rule “bottom number = $7 - \text{top number}$ ”.

Next day, we did the same experiment with an octahedron. First, all boys applied the missing number strategy. Then Adam took the octahedron and looked at it carefully. He tried to remember all four pairs of opposite numbers. When the die was rolled for the second time, Adam immediately said the bottom number. His answer was correct. Adam said: “You know it is Ben’s trick; pairs are (he pointed to the corresponding sides of the die) 3 and 6, 1 and 8, 7 and 2, and 4 and 5; you know, it is now easy”. Ben said that Adam had a good memory. Adam answered that the task was not so difficult, since in each pair “you have one small and one large number”.

A couple of days later, I played the same game with Adam and Ben. Cid was not in school. This time, the die was a dodecahedron, a solid both boys were familiar with. Before we started the game, Adam asked me if he could look at the die. Ben said that it was not fair but then he changed his mind and started to write a list of opposite numbers on this die. The list was not completed when Ben exclaimed: “Thirteen, it is thirteen, it must be thirteen, the sum of each pair is thirteen.” He was very happy and proud of this investigation. After a while, Adam agreed with this idea. He added that it was seven for the hexahedron and nine for the octahedron. So I rolled the dodecahedron and both boys used the strategy “count on to 13” ($\text{top number} + ? = 13$).

After that, I prepared an icosahedron die. This time I put the numbers on the faces randomly. The rule “the sum of opposite numbers is 21” was not true. Some half an hour later, I asked the boys to play the game with the icosahedron. They agreed and Ben asked me, how many numbers there were on the solid. I answered twenty and rolled the die. The top number was 12. Ben said: “So the bottom number must be eight.” I hesitated: “Are you sure?” Adam said “Sure”, but he stopped for a moment and corrected Ben’s answer: “It’s nine, you have one more.” I repeated my hesitation: “Are you sure?” “Oh yes, nine for certain, sure,” said Adam and Ben agreed. So I carefully took the solid and showed the bottom number to boys. It was 3. The boys were surprised and Adam took the

die in his hands. He observed it for a while and said: “You know, this is a false die, we cannot play that game with it.” So I asked Ben and Adam to correct the numbers on the false icosahedron which they did successfully. They knew that for the correctly numbered (point symmetrical convex) polyhedron the sum of numbers on parallel faces must be $n + 1$, where n is the number of faces. They also knew that this rule holds for all “suitable” solids and that a pyramid is not a suitable solid, since it has no parallel faces.

In the above experiment, we identify the boys’ six discoveries:

- a) the opposite-side numbers create pairs; you have to remember these pairs (hexahedron, octahedron and dodecahedron)
- b) in each pair, there is one small and one large number – this helps the memory (octahedron)
- c) the rule: the sum of each pair of the opposite numbers in the dodecahedron is thirteen (it is an invariant)
- d) the rule holds for the hexahedron and the octahedron and therefore should be true for each shape, particularly for the icosahedron
- e) if the rule fails in the icosahedron, the die is false
- f) generalisation of the rule to all suitable polyhedrons.

Three of these ideas can be classified as steps of the building of IMS.

The observations b) and c) paves the way for the discovery d) - the rule which links six pieces of knowledge, six pairs 1-12, 2-11, 3-10, 4-9, 5-8 and 6-7. At the same time, the rule can be regarded as the principle which introduces an organisation into the set of six pairs.

The idea e) transforms the octahedron rule for the hexahedron rule and the octahedron rule. The decision f) proves that boys generalised these rules to “all suitable polyhedrons” and took it as the criterion of the correctness of a die.

6. Linkage between structures

Two examples illustrate students discovering of the linkage between structures.

Example 4. In the fifth grade, we used to play several REMOVE games. Students discovered the strategy for some of them. However, the two-pile REMOVE³ with the characteristic 4 was too difficult for them. They found the strategy for small m, n , but the general strategy was not found.

³ Two piles of m and n stones are given. Two players remove in turn either any number of stones from one pile or k stones from each pile, where $1 \leq k \leq 4$. The player removing the last stone wins.

A year later, in the sixth grade, we introduced the set of REACH THE CORNER⁴ games. The strategy of such a game can be easily visualised: each critical⁵ square of the board will be marked. To do this we start with marking the terminal square (1,1) which is critical. Then we cross all squares from which the critical square (1,1) is accessible by one move: $\{(1,y), 1 < y \leq v\} \cup \{(x,1), 1 < x \leq u\} \cup \{(x,x), 1 < x \leq 5\}$. The further critical squares are (2,3) and (3,2); those uncrossed squares that are closest to the (1,1) square. We mark them as critical squares and continue this process.

When Diana completed such a rectangle for $u = v = 20$, she exclaimed that there was a nice pattern and showed the pattern to the class. Ben said that now he was able to win the REMOVE game with the characteristic 4 for any numbers of stones in the piles. The boy discovered the isomorphism between both games and after some corrections, he was able to describe this isomorphism clearly.

In the above example, the class teacher guided the work of the students to create the possibility of them finding the linkage between structures. The next example shows how the linkage of structures was found by the student himself.

Example 5. From the fourth grade on, some students were fond of magic squares. During four years, they discovered about two dozens magic squares and each such discovery was displayed on the class wall newspaper for several months. The majority of these discoveries belonged to Finn. He was very fond of all computations and his dream was to find a general procedure how to create $n \times n$ magic square for any n . As a seventh grader, Finn posed the following problem: to find the magic square in which numbers in rows, columns and diagonals are not added but multiplied. However, there were no other restrictions on the numbers in the square except that there were no two equal numbers among them.

Finn was so excited about the problem and so disappointed by the lack of interest from his classmates, that I felt it my duty to be his discussion partner, and to be involved. We had two decisive discussions. In the first one, I asked him to give me an example. After some hesitation, he refused, saying that it

⁴ The board is a rectangular grid with u columns and v rows. Unit squares of this board will be denoted by the co-ordinate system: the left-bottom square is denoted by (1,1), the right-bottom square by (u,1), the left-top square by (1,v) and the right-top square by (u,v). At the beginning, a counter is placed in the square (u,v). Players move the counter in turn either left, down, or in the diagonal left-down direction. Moves to the left and down are not restricted, but the diagonal move is limited by k squares (in our case $k = 4$). The player who reaches the (1,1) square wins.

⁵ A square is critical if the player making the next move from this square cannot win provided his partner will play the best possible way.

would be a great hint since any example is, in fact, the explanation of the discovery. However, during our communication he told me so much information that I was almost sure about his finding: if in any additive square we replace each its number n by its power 2^n we obtain a multiplicative square. After some days I brought a horrible example of the 3x3 magic square with rows 39 366, 256, 46656 and 9216, 7776, 6561 and 1296, 236196, 1536. Finn took my solution of his problem and started checking it. After a while he came back and asked me for my solving strategy. “Tell me how you found these strange numbers?” I answered “No! It’s my secret. You have yours, I have mine.” He was surprised and likely also disappointed and said: “It is not a very nice solution. I have a solution where all my numbers are less than one thousand. Try to find such a solution.” Next day, I showed Finn my new solution which was also Finn’s solution the boy informed me that he understood my solution and knew that I knew his strategy.

This example shows four kinds of IMS-steps listed in section 4:

1. The existence of a strategical problem in a student’s mind. In Finn’s mind, it was a problem of finding a general strategy for the creation of magic squares.
2. Finding the linkage between two structures, namely $(\mathbb{N}, +)$ and $(\mathbb{N}, *)$. Finn applied this procedural knowledge in action ($2^m * 2^n = 2^{m+n}$) to his strategical problem and created an interesting problem.
3. The need to give the explanation of disharmony if it appears. It was my horrible solution which surprised the boy and he did a clever analysis of it to discover my strategy.
4. The extension (generalization) of a piece of knowledge. The boy’s previous investigation only concerned powers of number 2, or, possibly any other number. My example was constructed as an amalgam of powers of 2 and 3.

7. Ways of creating structure

Within the research, four ways of building structure have been identified so far.

- a) **Spontaneously.** Dealing with some mathematical field, a student step-by-step gets a progressively better insight into it and from time to time links isolated pieces of knowledge in his/her mind. This process is long-term. In our experiments, we used several such fields: grid, additive triangles, cubes, arrow diagrams, and triads (see the Kratochvílová’s contribution).
- b) **Via generalisation.** Two or more particular structures are given to a student and he/she has to describe the general structure. Both particular

structures should be examples of this general structure. In our experiments, the role of particular structures was played by the arithmetic structure (arithmetic mean of any pairs of real numbers) and the geometrical structure (midpoint of any pair of points).

- c) **Via analogy.** Transmission of an already known structure to a new context. In the research, we used some non-standard metric spaces, and non-standard arithmetic. The most deeply elaborated research will be shown in the contribution of Stehlíková-Jírotková.
- d) **Problem-oriented way.** E. Galois found the structure of group when analysing the problem of solvability of the equations of the fifth degree.

8. Conclusions

The results of our research presented here concern the processes of building IMS in arithmetic and algebra on the basis of a spontaneous approach and via analogy and methods which can be used for such a study. Our next goal is to analyse

- (1) Building IMS in geometry and combinatorics (some particular results were obtained by J. Perný (1999) and B. Rozek (1997.)).
- (2) Building IMS using problem oriented way (promising experiments and analyses are in papers of E. Swoboda (1997) and A. Zeromska (2000) and in on going research of D. Jírotková, J. Kratochvílová, N. Stehlíková).
- (3) Two types of cognitive styles: problem-solvers and structure-creators.
- (4) The dynamism of the process of building IMS. The main question in this direction is “Is it possible to explain the process of building IMS as a sequence of stages?”

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BUILDING THE INFINITE ARITHMETIC STRUCTURE¹

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Abstract: *The set of all triads $(a,b,a+b) \in \mathbb{N}^3$, $(0 \notin \mathbb{N})$, $a \leq b$ with the mappings L_t (the left mapping): $(a,b,a+b) \rightarrow (a,a+b,2a+b)$, and R_t (the right mapping): $(a,b,a+b) \rightarrow (b,a+b,a+2b)$ creates the structure. This was used as a tool in the research aimed at investigating the building of an infinite arithmetic structure. Pupils were given problems in this context and their solutions were analysed. Two of them are presented.*

1. Rationale

Structural thinking does not take place at the level of discovering relationships in an existing structure but at the level of a gradual building of a new structure through solutions of adequately demanding problems. While carrying out experiments aimed at this area, the author was creating her own ideas of the process of structuring and of a structure. If a pupil solves a problem of a certain type ($3+?=5$) for the first time, he/she uses the trial-error method. If he/she solves other similar problems ($2+?=5$, $5+?=6$, ...), his/her work becomes quicker, he/she gains an insight into the situation and new experiences. After some time, he/she discovers that the required number can be found e.g. by counting on or later on by subtraction. This knowledge changes the original strategy of trial-error into the direct counting method and it becomes the basis of the building of structure (in this case, an arithmetic one). Next, in other series of problems, a pupil discovers other connections, not only among objects, but among pieces of new knowledge too. A set of individual pieces of knowledge becomes interconnected, more consistent and this process is considered as a process of structuring and its result is a structure (M. Hejný in his paper in these proceedings uses the term: IMS - Internal Mathematical Structure).

The richest mathematical structure, which is well known by a student of lower secondary school, is the structure of integers and possibly rational numbers. The building of this structure is a long-term process that goes on gradually and intermittently. Therefore it is very difficult to investigate this process experimentally. However the researcher's efforts were focussed on

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looking for some area of mathematics where the experimental investigation of structuring was realisable. For example you could use the product of isometries (creating of the group of isometries) for your experiment. But the difficulty of this area, especially the key concept – product of isometries – does not allow the use of this tool for investigation of structural thinking of students aged less than 15. One solution of this problem is to use the structure of triads.

2. The research tool

The structure of triads is an appropriate tool for investigating the first stages of a structural process for pupils aged already 10-11. It needs minimal mathematical knowledge but it offers various, sometimes even surprising structural situations. The important characteristic of the structure is based on its unfamiliarity, the pupil works with a new mathematical object and new mappings and he/she cannot use nor rely on memory/skills used in the classroom.

The structure was given to the pupils in three phases. The first phase concerned understanding a new object, the triad. The second one concerned understanding the mappings L_t and R_t given to extend the set of triads. In both of them pupils were given concrete numerical problems without any theory or algebra. The third phase concerned “movements” in the structure with the help of a graphical structure – paper with lines 1-10. (See Fig. 1, 2 above). This phase also enabled the researcher to observe whether the pupils could follow the mappings and whether they did this in a logical or random way.

3. Overview of experiments

The experiments were carried out with pupils whose ages were between 10 and 11 years, thirty in the United Kingdom and eighteen in the Czech Republic. All the experiments were done in quiet situations, the researcher working with three pupils who could discuss amongst each other. Each meeting in the United Kingdom took about three hours. However each meeting in the Czech Republic took about one hour because the researcher met the same group of pupils three times on a weekly basis.

4. Scenario of experiment

The scenario involved the thirteen prototypes of problems and only six of them, because of time reasons, were used in the experiments. In future research starting, in March and expected to last one-year, all prototypes will be used.

The first phase with pupils was focused on the concept of triad. After a short explanation, the following problems² were given to the pupils:

P1. Create a triad with the numbers: 3,7,4.

P2. Choose three of the numbers 2, 3, 4, 5 and create a triad.

P3³. Circle the triplets that are triads: (1,5,6); (10,10,20); (3,2,1); (6,4,10); (7,5,17); (0,0,2).

P4. Fill in the missing numbers in the triplets to form triads: (7,9,_); (_,9,10); (14,78,_); (7,_,12)⁴; (75,_,74)⁴.

P5. Which number from the following four is necessary to cross out to create a triad: (5,6,9,11)?

P6. Fill in the missing numbers (_,_,8) to create a triad. Find all possibilities.

P7. Fill in the missing numbers (_,6,_) to create a triad. Find several possibilities.

P8. Similarly for (3,_,_).

P9 Try to find several triads such that a) all the numbers are even; b) all the numbers are odd.

P10. Try to find a triad with the first two numbers divisible by 7 and the third number indivisible by 7.

The second phase of experiment was focussed on the mappings Lt and Rt. The language used in this phase was modified. The phrases “left mapping” and “right mapping” were replaced by “first triad (of the given triad)” and “second triad (of the given triad)”. Words in the brackets were often omitted. It was introduced to the pupils through the following actions:

Construction of the first⁵ triad from a given⁵ triad was a five-steps procedure:

- Take the first number of the given triad.
- This number is the first number of the first triad.
- Take the third number of the given triad.
- This number is the second number of the first triad.
- Complete the first triad by adding its two numbers and putting the result in the third position.

² In the pilot experiments the pupils were given the mappings immediately after the introduction of concept of triad and it showed as a mistake. Therefore the new set of problems was prepared.

³ The problems marked in bold were used in the experiments.

⁴ The last two non-existing “triads” serve as a test if pupils understand the idea of triad properly.

⁵ The usage of adjectives “first” (“second”) and “given” which seem to be confusing was for all pupils quite clear. There was no misunderstanding caused by this terminology.

Construction of the second triad from a given triad was described by an analogous five-steps procedure. In the notation an arrow was used, e.g. $(1,3,4) \rightarrow (1,4,5)$ for the first triad; $(1,3,4) \rightarrow (3,4,7)$ for the second triad.

We can look at the process of creating the structure through the APOS theory - action-process-object-scheme (Czarnocha, B., Dubinsky, E., Prabhu, V., Vidakovic, D., 1999). In this case by applying the actions (construction of the first and second triad from a given triad and construction of the first and second triad from the first “new” triad and so on) the scheme creates part of the structure.

After introducing the constructions the following problems were given to the pupils:

P11. Find the first and second triads of the triad: $(1,5,6)$.

P12. Fill in the missing numbers in the triads to form the triad and its first or second triad. $(1, _, _) \rightarrow (1, _, _)$; $(_, 6, _) \rightarrow (_, 10, _)$; $(_, _, 15) \rightarrow (15, _, _)$

P13. $(_, _, _) \rightarrow (_, _, _)$ Choose six of the following eight numbers (you can use them twice) and put them into the given arrow scheme: 2, 3, 5, 6, 8, 11, 14, 18.

The third phase was focussed on the movement in the structure. The pupils were given the paper with lines numbered 1-10 and the following problems were given:

P14. Find further triads on lines 3, 4 and 5.

P15. What is the smallest triad on line 10? The⁶ smallest triad is defined as the triad whose elements give the smallest sum.

P16. What is the largest triad on line 10? The⁶ largest triad is defined as the triad whose elements give the largest sum.

5. Methods applied

The first two phases used in the research resembled class work rather than research observation. However even these phases gave valuable information, especially observing the pupils' mistakes. The pupils solving the problems in the third phase needed to use other abilities and knowledge that were not part of previous class work. This was shown by the increasing number of pupils' solutions available to help us get an insight into the solving processes. The problems and questions (in the third phase) used in the experiments with pupils

⁶ In the future experiments this sentence will be omitted and we will observe how the terms the smallest/largest triad will be used by pupils.

were inspired by some of the research methods described in M. Hejný's paper. For instance the method – 'Use a non-standard notation' was present in solving the problem P14 as a need to reduce the notation. In some of the experiments in the third phase the pupils were reminded of the construction of the first and second triad and they were asked for inverse construction (re-construction) in the sense of M. Hejný's method – Derive one idea from the other.

6. Experiments

Two of ten experiments analysed are presented in this paper. In the case of "Rebecca", we will see an interesting attempt to grasp and describe a large structure using an "economical" notation. In "Simon's" case, there are two interesting phenomena:

1. The structure helped the boy to find and correct his mistake.
2. One formal similarity shows a profound misunderstanding.

From the methodological point of view, the analysis of the first part of "Simon" work was done using Atomic Analysis (Stehlíková, 1995) which was elaborated by the Bratislava and Prague research groups. Pupil's written work is broken into the smallest meaningful fragments – the static atoms. The whole set of static atoms is ordered in a time sequence. Then the smallest mental fragments are looked for – the dynamic atoms that were used for writing the static atoms. Finally we try to link all these dynamic atoms into a consistent solving process. From our previous experiences Atomic Analysis is one way in which it is possible to get a deep understanding of pupil's written work.

6.1. Case 1 - Rebecca (10-year-old girl)

After the implementing phase, Rebecca was given the problem P11 and P14. Rebecca created the triads without difficulty. (See Fig. 1) She found all the triads on lines 1, 2, 3 and 4. Then she wrote the eight pairs on line 5: (10,17), (22,23), (25,32), (27,33), (26,37), (38,41), (29,40), (39,45). The researcher interrupted her work with the question:

Res. 44: Are you making new triads?

Rebecca 19: Yes.

Res. 45: How many numbers do you have in a triad?

Rebecca 20: (Pause 2 seconds.) It has only 2 numbers. The "triad" (10,17) is not a real triad.

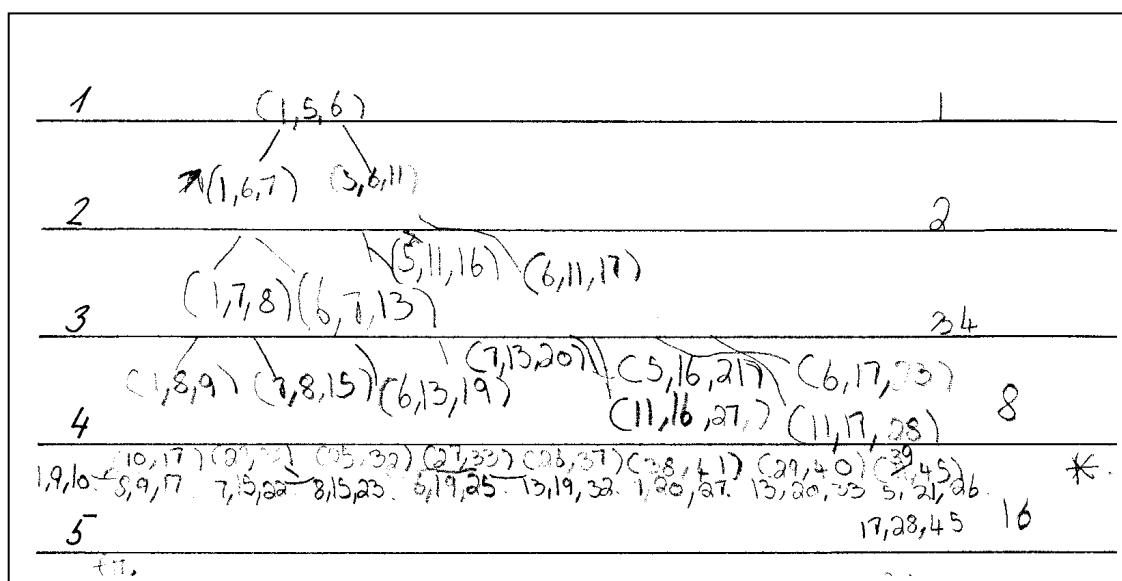
Analysis

Two interesting phenomena of Rebecca's solution deal with the graphic organisation of triads:⁷

1. The triads on lines 1, 2 and 3 are on the left side of Fig. 1.

This is a consequence of the position of the triad on line 1 and caused no problem to Rebecca's solving process.

Fig. 1



2. The great number of triads on line 5 forces Rebecca to think about a more economical notation.

It is not difficult to understand Rebecca's idea of her "new" notation. The pairs of triads were written down as the pairs of numbers, e.g. there are two triads made from (1,8,9), these triads (1,9,10) and (8,9,17) are represented by the pair (10,17). Similarly the triads (7,15,22) and (8,15,23) are represented by the pair (22,23).

From the density of line 4 Rebecca saw that the situation on line 5 would be graphically very complicated. She tried to solve the problem in two ways:

1. She decided to write small numbers.
2. She introduced her own abbreviation by means of coding a pair of triads into an ordered-pair of digits.

⁷ From now on, the text in italic means an analysis.

Both these ideas – small numbers and the ordered-pair - are applied systematically: the size of all numbers is same, all eight representatives are found without any mistakes.

Rebecca did not find the ordered-pairs insufficient. The researcher expected that she would be in a conflict situation when problems P15 or P16 were given. By using the same shape of brackets as for the triads Rebecca caused communicative misunderstanding which could have been eliminated by using different brackets.

Rebecca was then given the problem P15. She pointed to the smallest triads on lines 1, 2, 3, 4 without any difficulties and stopped on line 5. It took her two minutes to find how to continue the sequence. She had to overcome the obstacle caused by her usage of the ordered-pairs and reconstituted them into the two triads for which they stood. The smallest triad on line 5 was found and Rebecca continued on line 6 with her notation, writing the ordered-pair (11,19). This time, she immediately recognised that her form of coding triads did not help with continuing the sequence of the smallest triads. She crossed it out and put (1,10,11). Once more on line 7, she tried to use the ordered-pair (writing the number 12) and rejected the idea by crossing out this number. From then on, the sequence of the smallest triads (1,11,12); (1,12,13); (1,13,14); (1,14,15) was written spontaneously.

Analysis

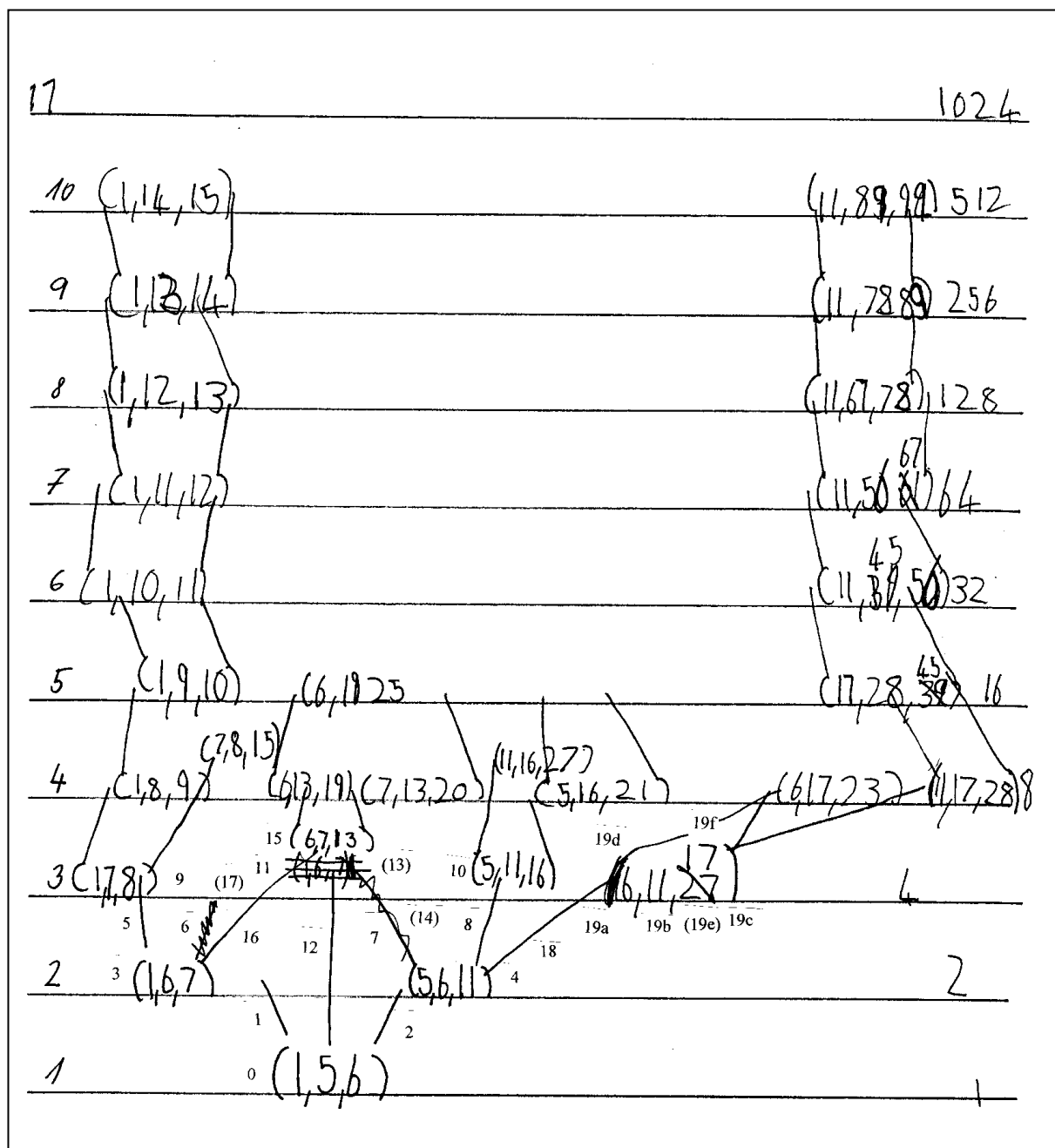
It seems that Rebecca did not realise that her idea of coding was very weak and could not be used in solving the presented triad problems. Rebecca knew what she was doing and could translate her coding to her own satisfaction but she realised that it was too 'condensed' for the problem P15 and so had to revert to triad coding. Rebecca's discovery of the notation was very valuable for her interior evaluation as we can see from the fact that it had been rejected and after that it appeared twice again.

Rebecca's solving process shows her intellectual culture and autonomy, her ability to create and test hypotheses. However she did not seem to realise that the pair (10,17) could represent more than two triads.

6.2. Case 2 - Simon (10-year-old boy)

The introductory part of the experiment with Simon was the same as with other pupils. Then, the work up to line 5 – solving the problem P14 – was done by Simon without any help from the researcher. (See Fig. 2)

Fig. 2



Analysis

When analysing the first part of Simon's solution to the problems P11 and P14, we first looked for the order in which he wrote the individual symbols on the paper. Our solution of this didactic problem is denoted in Figure 2 by small numbers which represent this order. There are four kinds of symbols: 1. triads or their parts (0, 3, 4, 9, 10, 11, 15, 19a, 19b, 19c, 19f); 2. line segments (1, 2, 5, 6, 7, 8, 12, 16, 18); 3. crossings-out ((13), (14), (17), (19e)); 4. rewritings (19d).

The sequence of symbols in Fig. 2 shows that Simon adhered to the following procedure: as soon as he finished writing triads on the appropriate line, he immediately drew the two line segments for two successors. His procedure is evidence of the importance of the tree bifurcated structure for him. It is therefore a structure-creating element.

Solving problem P15 he wrote the sequence of the smallest triads up to line 10 without any problems. The rewriting on line 9 is caused by his lack of concentration. The regularity of this sequence is still in his mind: the first number is 1, the third number equals the second number plus 1, both the second and third numbers increase by 1 gradually. When creating the sequence of the biggest elements (problem P16), the above regularities played the key role and caused the conceptual misconception in the second sequence. In this process, the transfer of the “left” structure to the “right” structure appears, while the latter does not correspond to the rules of the creation of the right successor.

The illustration shows that a pupil’s strong structural thinking can help him/her (it happened when Simon discovered the wrong triad 11 and replaced it by triad 15), however, it can also be misleading as we have seen on the example of the right sequence. Let us say that in the context of the teaching and not research we would have shown Simon that his triad (11,28,39) was created in a wrong way and let this cognitive conflict take its course. Simon would have discovered his mistake and what caused it and made the necessary correction. In this research, we only began the whole process.

7. Conclusions

The set of triads that is equipped with left and right mappings serves as a good tool for research, diagnosing pupils’ abilities to build a structure and is an educational field within recreational mathematics. The structure of triads as a research tool enables us to observe the whole process of creating an arithmetical structure because the graphical model of the structure evidences many thinking structural processes of the pupils. The research has illustrated the following four facts:

1. Creating a global structure presumes necessarily previous insight into local structures (involving from two to five elements – the triads joined by the mappings).
2. The ability to create a concept of the structure of triads is profoundly individual. Pupils showing the same level of understanding of the structure of natural numbers get insights into the structure of triads at different rates.

3. A powerful tool for the pupil is the ability to grasp the structure graphically and leads to the understanding of structure which does not depend whether 'a tree', representing the structure, is orientated up or down. (See Fig. 1 and Fig. 2)
4. As the pupils solved the problems step by step they created other elements of the structure. During these activities the pupils had misconceptions. By realising this and by understanding the reasons for them, contributed to getting an insight into the structure.

The following four cases of misconceptions were found:

- A. The notation of triads can be reduced, e.g. triads coded as dyads or monads.
- B. The triads on a line higher than line 1 can be written down automatically without thinking about mappings.
- C. The generated pattern from the left 'branch' can be applied to the right 'branch'.
- D. The number of line can be used as an operator for generating an appropriate triad on the line (the triad on line 10 was made by doubling the numbers in the triad on line 5).

The short time spent on the first two phases focussing on the concept of a triad and the left and right mappings was a fault of the research. Future research will give much more time to these both phases and will include them as a part of the research and not only as preparation.

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FROM FRACTIONS TO RATIONAL NUMBERS IN THEIR STRUCTURE: OUTLINES OF AN INNOVATIVE DIDACTICAL STRATEGY AND THE QUESTION OF DENSITY

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Abstract: *We trace the guidelines of an in-progress research of didactic innovation on the approach to rational numbers from the structural point of view. A key element in the implementation of experimental activities is how to develop a view of rationals as a numerical extension of natural numbers by setting students the problem of defining the operations and order in the new set, using their knowledge of natural numbers as a reference point. Here we focus on the part of the research devoted to the comparison between the order of natural and rational numbers, with reference to the question of density. The results obtained show the effectiveness of this approach as far as active participation on behalf of the students in the solving of problems is concerned, as well of tackling the question of motivation. However, the results also show the need for more time to be given to the study of certain issues and their organisation in a more general frame*

1. Introduction

Building the concept of rational numbers is a very long process starting right from primary school. Here pupils start with the study of problem situations, which, little by little, become more complicated and involve a great number of fields of experience and branch out even into fractions by various mathematical routes (part/whole, quotient, measure, ratio and operator, according Behr et al. 1992).

The importance and breadth of this theme has been documented by numerous studies, most of which look at the approach to fractions in the period from primary to middle school. Great attention is paid here to the psychological and/or didactic issues which the approach to one of these aspects of fractions entails: i) the importance of the context of reference; ii) the iconic and graphic representations adopted and the flexibility used to exploit them; iii) the incidence of the model of naturals; to name but a few. But there are not many studies on fractions from a strictly numerical point of view (Barash & Klein

1996, Tirosch 1997) and those on rational numbers from an algebraic-structural viewpoint are almost absent.

Our research concerns a didactically innovative strategy for middle schools ranging from fractions to rational numbers, and it is aimed at highlighting structural aspects of rational numbers for facilitating the continuity between lower and upper secondary schools. This research forms part of a wider project aimed at renovating the teaching and learning of algebra. In this project, an early use of letters is promoted for expressing relationships, solving algebraic problems (even demonstrative ones) and for highlighting relational and structural aspects in numerical scope (Malara & Iaderosa 1999, 2001).

We have to consider that there is a certain amount of epistemological divergence between the modern, structural vision of the curriculum (which concentrates on the introduction of the various numerical concepts in the sixth and seventh years, particularly rational numbers), and the old, consolidated teaching tradition (which deals with fractions from a merely operative point of view). The classical tradition does not even get to the concept of rational numbers as a class of equivalent fractions, while nevertheless studying proportions or the function of proportionality in a more general sense. In this tradition, the approach to operations is aimed at determining their result for particular pairs of fractions though almost never explaining its laws of correspondence in general – a process which leads students to carry out calculation processes blindly and at times even distortedly (Vinner et al. 1987). Moreover, the relationship between rationals and decimals is not well clarified. In the praxis, decimals are introduced in primary school as “numbers” for expressing measures of magnitude (Brousseau 1981) and, even though they are later revised as representations of rationals, issues about coherence or compatibility of the operations in the two different systems are usually not faced with the students. The comparison between fractions is then usually carried out using the decimal representation of the quotient between numerator and denominator (of course, usually in very approximate terms), though students are not required to reflect on how a fraction varies when its numerator and/or denominator varies (Lopez-Real 1998). This approach allows neither for the comparison of fractions in general terms, nor for the conceptualisation of how a generic fraction changes when its terms of reference do.

Alongside these aspects, which are typical of current mathematics teaching, we must also consider the scarce knowledge many teachers have of the subject and the convictions held with regard to teaching practices (Carpenter & Fennema 1991). In our research, we therefore tried to operate for and with the teachers involved, taking their initial beliefs into account, and encouraging them towards new attitudes and concepts through their critical analysis of various studies on the topic.

2. The Theoretical Framework

We used several studies in the field as our main points of reference. These were analysed and discussed in the awareness-building seminar activities held in order for teachers to look at the issues involved.

One important study was Pitkethly & Hunting's survey (1996), which gathered together the results of research carried out in the first half of the '90s. In this survey, the constructive mechanisms for rational number knowledge-building are taken into consideration. These consist of: a) whole number schemes (the use of counting for the identification of individual units as a means of division); b) partitioning schemes (halving, dealing, folding or splitting); measuring schemes (fractional units seen as chunks or measures); equivalencing schemes (the reconfiguration of a certain unit in terms of other ones); relational schemes (relational understanding between parts and wholes). The research highlights several discrepancies in the findings. a) Discrete vs continuous: certain scholars maintain that the continuous approach, associated with measuring, is the most suitable to introduce initial fraction concepts, while for others, the discrete approach, associated with counting, is fundamental. b) The role of natural numbers: for some (like Kieren (1993)), the knowledge of natural numbers is productive, while for others (like Streefland (1991)), it is inhibiting. A new key result, which has still to be put to the test, concerns the relationship between ratio and early fraction learning. It seems that ratio is integral to children's earliest conceptions of fractions. It has also become clear that progress in concept development is not linear and that genuine growth in understanding involves 'folding back' to reconstruct one's concept of fractions.

Another study we found particularly interesting was that carried out by Kieren (1993), his model of the implicit and generative order of rational number thinking (p. 65), fits with our idea of the development of the knowledge of the ordered field of rational numbers¹. Kieren takes into consideration the relationship between rational number knowledge as it is formally characterised in mathematics books (ordered, and coherent with formal logic), and conceptual knowledge (that is, the interweaving of the intuitive and formal knowledge on a personal basis; the ongoing doing constitutes knowing and the enfolding and unfolding of such actions). He conceives this relationship in the sense of the

¹ The model is structured on four levels: the lowest level contains knowledge based on intuitive tools, level II contains sub-construct knowledge, which is subsumed by more formal multiplicative thinking, which comes under Level III. Lastly, level IV represents the structural knowledge of rationals, that is, their mathematical meaning.

implicative and generative orders construct as advocated by Bohm and Peat (p. 51)².

In other studies the approach to rational numbers highlights the role of active participation on behalf of the students and the importance of entrusting them with the building of their own knowledge. Confrey's research (1995) may be considered part of this clearly constructivist approach. Confrey states that splitting and counting are two independent primitive roots of number operations; splitting gives rise to multiplication division and ratio through actions such as sharing, mixing or observing similarity, whereas counting leads to addition and subsequently multiplication (division) as repeated addition (subtraction). We approve of working in parallel, distinguishing between additive and multiplicative models and structures. Another aspect common of our study is the great importance given to the teacher, to his/her resources, attitudes, experiences, conversations and beliefs about mathematics. Also to his/her ability to listen and to draw out different impressions from the students so as to support rather than to suppress diversity (an approach which he calls "voice-perspective dialectic"). This approach is also to be found in the studies carried out by Murray et al. (1996) and Newstead & Murray (1998) in which, alongside the constructive aspects, space is also given to meta-cognitive ones. Other studies that we have looked at concern more particular aspects, such as the construction of classes of equivalence and the question of density (Sanchez & Llinares 1992, Gimenez 1990) or the use of the number line (Baturro & Cooper 1999).

3. The Research Hypothesis

A large part of the wide variety of studies examined are closely bound up with the various fields of experience, as well as being based on graphic or iconic representation as a strong support to the various learning processes without paying any attention to the numeric aspects per se. However, we feel that given the very diversity and the reliance on intuitiveness of the classroom experience, the students often acquire fragments of ideas and concepts which need to be revised and developed. The students need to be led progressively towards the

² This is conceived to contrast with the explicitly sequential observable patterns that seem invariable in a world with dynamic orders that underlie the explicit order. It is from these underlying orders that the explicit world draws its meaning. The implied order has the dynamic feature of new thought growing out of but enfolding or embedding previous thinking; the generative orders observe the unfolding of a central generative intuitive theme into a kind of hierarchy of successively more elaborated and sophisticated forms. Bohm & Peat suggest that "the really creative act of a mathematician is to perceive the germ of this vast structure ... that is constantly tested for coherence"... Kieren maintains that knowledge-building for the child and the creation of a related curriculum by the teacher is a creative act of the same order.

purely numerical aspect so as to build bridges for the further development of abstract thought. In this sense, we agree with Bonotto (quoted in Pitkethly & Hunting 1996), who states: “The child must quickly move away from particular concrete situations. The visual representation must not become the unique point of reference for the concept. Rather, the context is just one contribution to the concept’s formation.”

In order to help students to develop highly flexible, effective and transparent conceptual models for rational numbers, their order and the operations between them in general terms, we believe we must lead the students: i) to think of fractions as an unifying concept arising from various different situations and to co-ordinate the associated symbolic representations; ii) to think of a rational number as a class of equivalent fractions and to move flexibly across the class; iii) to define the operations between rationals by themselves, consciously choosing the most convenient representations given the situation in question; iv) to use letters early on to express operations and relationships between rational numbers in general terms.

4. The Methodology

The study, with six middle-school teachers one of whom is also a researcher started in 1998/99 and is still in progress. It consists of alternating periods of study and the critical analysis of articles from the field, and other periods dedicated to the trial, experimentation and analysis of various interventions made in the classroom. Thus it is necessary to distinguish between the research methodology and the didactical strategy adopted in class.

The research methodology approach is developed through the following steps carried out with the teachers involved: i) a joint study of research literature on the chosen theme and the creation of research hypotheses; ii) planning the experimental activity needed to verify the hypotheses in their essential points and a prior analysis of the potential difficulties for the pupils; iii) joint analysis of the protocols produced during experimentation (pupils' output, reports on construction discussions and evaluation discussions, etc.); iv) selection of documents considered to be meaningful in order to exemplify thinking patterns, behaviour and difficulties for the pupils; v) elaboration of the results obtained and reflection on the processes that determined them.

In the classroom, the teachers always work constructively, stimulating and orchestrating the pupils' intentions, promoting group reflection on what is gradually being carried out until the acquired knowledge is eventually institutionalised. Usually they: i) use verbalisation as a tool (the pupils get used

to always writing down their ideas, conjectures, reasons for their procedural choices, etc.); ii) offer open problem situations, which can be read from many points of view so that pupils are provoked to think hypothetically; iii) let the pupils analyse other pupils' reasoning and procedures (either individually or in small groups); iv) give time for collective discussion so that socially agreed conclusions - that each pupil feels as his/her own - arise from comparison and a plurality of ideas.

5. The Didactic Development

The didactic development which we put into practice in the classes may be placed between levels II and IV of Kieren's model (1993). This, as requested by the teachers anxious to follow the classic order, is based on the construction of a suitable context for absolute rationals and, after the introduction of relative whole numbers, that of all rationals (these last aspects are treated in year eight). The approaches to directed numbers and rationals are somewhat interweaving, giving positive reciprocal outcomes alternately in both fields. A fundamental element of the entire strategy appears to be the work done on multiple representations of natural numbers, the co-ordination of " a/b " and " $a:b$ ", the reference to decimal numbers, and division quotients as another representation of these (despite being in many cases approximate).

The experimentation covered all the points reported in table 1. These were developed mainly in year seven, but preparatory and then more investigative work was done in year six and year eight respectively. It must be stressed that the experiments were carried out during normal school lessons over a period covering the entire school year, together with the other topics of the maths curriculum. Generally speaking, the activity was carried out through the following stages: a) the class was posed the problem which was then solved collectively through group discussion; b) the students were given problems to solve individually at home or in class, and then asked to discuss the resulting strategies and points of view as a class; c) individual evaluation tests at the end of each key stage of the course; d) final evaluation then carried out at a later stage to test acquired knowledge.

For reasons of space, we have decided to concentrate here on the part of the research which deals with the question of density of rational numbers, for its novelty. The activities reported here took place in the teacher-researcher's class.

Table 1**The main steps of the experiment**

1. The equivalence of fractions and the construction of rational numbers as classes of equivalent fractions.
2. The construction of the directed and inverse operations of rational numbers in general terms, starting from particular cases and considering the questions of preservation and coherence as what is already known about natural and decimal numbers.
3. The concepts of opposite and inverse for a rational number different from zero and their analogy.
4. The comparison of rationals in general terms and the character of density of the related order structure.

Table 2**Summary of the discussion of the first problem**

The teacher starts off by citing particular cases and getting the students to say what happens in N. S/he then goes on to rationals and suggests students think about whether there are numbers between the $\frac{1}{5}$ and $\frac{1}{2}$. The kids then go on to use finite decimal representation, which induces them to consider the decimals 0.3 and 0.4 as rationals included within them. The teacher then asks if they can continue the ‘filling in’ exercise between 0.4 and 0.5. One student converts 0.4 into $\frac{40}{100}$ and 0.5 into $\frac{50}{100}$ and offers up $\frac{45}{100}$ as an intermediate rational, while another says that there is also $\frac{41}{100}$; $\frac{42}{100}$; ... ; $\frac{49}{100}$. The teacher resets the problem with the $\frac{41}{100}$ and $\frac{42}{100}$ pair and the kids get as far as thousandths following the previous procedure. It becomes clear from the discussion that it is impossible to determine the number of rationals between two given numbers as the issue of density comes out. As regards the hypothetical number successive to a rational, yet another student (starting from $\frac{7}{5}$, that is, 1.4, writes 0.40001, 0.400001 and says that he has seen with other numbers that you can’t establish what comes next because you can put as many zeros on a 4 as you want and then put on a 1 to make the difference, so every time you get a smaller number because you can add an infinite number of zeros making each figure slip one place to the right. It’s interesting to note that students also refer back to numeric representation on the number line, but soon realise the limits inherent in such a system. As one says, “Only space limits the infinity of fractions that there are”, and another adds, “Using the number line for natural numbers, you go by jumps, while when you use it for rationals, you go by cuts.” This vision of rationals between two given points becoming ever denser, a vision which is independent of the limits of graphic representation, leads them to exclude all talk of successive numbers when dealing with rationals.

As far as the research already carried out on the building of the concept of rational numbers, operations and how to compare rationals is concerned, see Malara (2000).

The question of density

We focused on three particular problems proposed. The first problem deals with the differences in ordering between natural and rational numbers.

Can you establish how many and what numbers come a) between two natural numbers, b) between two rational numbers? 2. Can we talk about a successive number for rationals like we can with natural numbers? The problem is looked at in a group discussion. In order to solve it, the students take part in the discussion with different points of view: some turn to decimal representation, the ‘navigation’ within the different classes of equivalence, number line representation (see table 2).

A second problem was about ordering rationals. (This question was the object of an individual test). *The teacher writes: I have a doubt. Seeing that between two fractions there are an infinity of others, is the set of rationals an ordered or disordered set?* Table 3 shows several significant example of pupils’ elaboration. P1 shows that this student, has more than a good conceptualisation of the problem and has a great awareness of the differences between the two orderings. P2 shows that the student is fixed on the idea of natural numbers which, given the diversity the problems at hand, leads him to exclude that rational numbers may be ordered. P3 is a very interesting case.

Table 3

Initial elaboration of the second problem
<p>Despite being at a different level, the following examples exemplify the type of activity undertaken and the (mis)concepts nurtured:</p> <p>P1: “I don’t think you can just write like we usually do, 1,2, 3, with fractions but I think that if we have $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{11}$, I can put them into order. I mean, (in various ways like decimal numbers and equivalent fractions) I can establish which of two fractions is bigger or smaller than the other. I’m thinking that if there are an infinity of fractions between any two other fractions, I could still put a certain number of fractions in the right order but I will never manage to line up <u>all</u> the numbers without leaving any gaps between one and the other”.</p> <p>P2: I’ve reached a solution: in a set of rationals, there is never the successor number to any given number. For example, after $\frac{9}{7}$ you think there is $\frac{10}{7}$, but there might be $\frac{19}{14}$ or $\frac{91}{70}$ instead. As you go on and on, you find that it is endless and so the successor to $\frac{9}{7}$ does not exist. Therefore it is a disordered collection.</p> <p>P3: I reckon that fractions are ordered because, both with the rational ones and the equivalent ones you can put them in order (especially if you look at the denominators) both in order from biggest to smallest and smallest to biggest. However, it’s a bit of a disordered order, I mean, there are infinite types of</p>

fractions and so there can be different types of order. You can take any fraction you want and put it in order, but you can't really do it with all those infinitive ones. For example, for $5/6$, you can write: $1/6, 2/6, 3/6, 4/6, 5/6, 6/6, 7/6, 8/6, \dots$, but if you take $5/6$ and $7/14$, these can't go together in the same order, the same class, but you can see which is bigger and which is smaller by turning the fractions into decimal numbers or by transforming the fraction with the handiest denominator and comparing them. For example, $5/6 \rightarrow 10/12$; $10/12 \dots$ and $\dots 7/14$ now you can see which is bigger. $10/12$ bigger, $7/14$ smaller. $4/6 \rightarrow 5 \cdot 7/6 \cdot 7 = 35/42$; $7/14 \rightarrow 7 \cdot 3/14 \cdot 3 = 21/42$, now it's OK.

The student takes into consideration the sequence of multiples of $5/6$ in its “natural” order, and at the same time the “ordered” class of equivalence generated by $5/6$. However, all this is clear to the student, who does not confuse the “local” orderings of these sets with those of the classes.

The third question (see table 4), posed as a final check, highlighted several issues which were overlooked in the prior analyses. All three of the protocols reported underline erroneous conceptual aspects or incomplete mental images. For example, P4's comment, “The gap between fractions gets bigger the further on you go,” shows the student's understanding of the fact that substituting one pair of fractions with another pair of fractions which are equivalent but with greater denominators than the second pair lets you ‘see’ a greater number of fractions between the two rationals. The term ‘gap’, however, stresses the student's partial mental image, disconnected from the number line, or rather an enlargement of it giving the zoom effect (Maher *et al.* 1994) which stops him from grasping the notion of the ‘getting denser’ between the two numbers in the ‘given’ fractions. P5 proves even more problematic, showing an erroneous view of the general situation. The student sees the diminishing progression of successive powers start from $1/2$, but makes absolutely no comparison of these powers with $1/5$. Elaboration P6 seems initially disconnected, but a certain co-ordination between measure and decimal numbers and between rational and decimal numbers can be observed.

Table 4

A question for the final evaluation

(The situation and the language used denote the style and philosophy of the work in class.)

Andrea and Matteo sum up the differences and similarities between the organisation of natural numbers and that of rationals. Andrea maintains that, all in all, the organisation is the same, only that in general, calculations are more complicated in rationals. There are hidden traps – for example, you can't just do $3/4 + 5/8$, while you can do $11 + 10$ – but anyway you can always check

out of two numbers which comes first and which comes second. For example, 12 is before 14 just as $3/5$ is before $4/5$, and then you can say how many numbers there are between two, for example: between 7 and 12 (8, 9, 10, 11) just as between $1/2$ and $1/5$ there is $1/3$ and $1/4$. Matteo takes a deep breath, 'puts his thinking cap on', and starts to contradict Andrea. What do you think Matteo will say?

Some significant protocols

P4: Matteo takes a deep breath and maintains that $3/5$ is not followed immediately by $4/5$, but that between these numbers there is an infinity of fractions or decimal numbers and that the divide between the two fractions gets bigger the further you go. It's true that the calculations are more difficult (in Q) and there are some hidden traps, but what Andrea implies when he says that between $1/2$ and $1/5$ there's $1/3$ and $1/4$ is not true because there are a lot more. Therefore, the organisation of rational and natural numbers is not the same, but similar.

P5: If I was Matteo, I would answer by denying the fact that between $1/2$ and $1/5$ there's $1/3$ and $1/4$ because also $1/8$ is between $1/2$ and $1/5$. Also $1/16$ and also $1/32$ and if we carry on multiplying the denominator by 2, you get all the numbers that can fit in the middle; in fact, there are infinite fractions between $1/2$ and $1/5$, while in natural numbers, there aren't.

P6: I think that between natural numbers like 7 and 12, there aren't just 8-9-10-11, but also infinite decimal numbers (infinite between each number) which add various bits of another whole number to each whole number. For example, between 7 and 8, 7.1; 7.11; 7.2; But I agree with Andrea that between 7 and 12 there are only 8-9-10-11 as natural numbers. On the other hand, as for rational numbers, it gets more complicated because: if we take a fraction and multiply **or** divide its divisor and its numerator by the same number, you can find other numbers that represent the same value as the initial number (*examples given*). I remember that rational numbers can be represented in decimal numbers.

Final Considerations

The experimental studies carried out so far show a good level of participation from the students – in their ability to handle the problems set, to approach them from a variety of directions, and to solve them. These aspects aside, the issue of the stability of knowledge acquired and the insertion of this knowledge in a wider frame remains a difficult one. This aspect is shown in the example given above. In the group discussion, the question posed about the comparison of orders in N and Q+ entailed a whole-hearted response from the class, a general sharing of reasoning and the assumption of the discreteness of natural numbers and the density of rationals. The individual tests focused on somewhat hazy or

contradictory concepts, while the final evaluation – apart from the matter of the clear difficulties in explaining their own ideas – demonstrated the following: a) a lack of ability to co-ordinate the individual pieces of knowledge; b) that many ideas are still not well formed and in certain cases wrong, even in capable students who have made (and usually make) a constructive and important contribution to the discussion. From a general point of view, since many students find themselves in the zone of proximal development, the matter of the timing needed to reiterate and reflect on certain concepts is left open (we must not forget that these activities are undertaken during normal school lessons). This factor would require a didactic continuity between years eight and nine in Italy which is still yet to come. From a more general pedagogical point of view, this would also entail greater reflection on the dilemma of the “inter-cognitive processes and intra cognitive processes of learning” (Hershkowitz 1999).

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INSTRUMENTS TO DETECT VARIABLES IN PRIMARY SCHOOL ¹

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Abstract: *Variables are one of the most important mathematical instruments and their learning is one of the goals of teaching. Variables are presented expressly in secondary school in connection with algebra and functions. It is possible to detect a very precocious use of variables by analysing some standard classroom activities for the primary school. The substitutions are the best tool to single out these uses.*

1. The theoretical frame

First order formal languages in Mathematical Logic are presented by giving arbitrary (infinite) sets whose elements are called variables (or indeterminates or unknowns), cf. Bell & Machover 1977. Variables are used to build terms and formulae. In formal language interpretations, variables can have different meanings, sometimes quantitative ones, depending on the interpretations.

In the first school approach with algebra or pre-algebra in the sense of Linchevski 1995, see also Arzarello 1989, and Boero 1994, variables have quantitative origins and meanings. A critical point in algebraic thinking is the contemporary presence in variables of extensional and intensional aspects, cf. Arzarello & Bazzini & Chiappini 1993 and 1994.

Here variables, or better indeterminates, are studied in their intensional aspects.

2. The hypotheses of the research

The research starts with the assumption that variables, in logical meaning, are used in standard didactic activities, from the first year in school. If pupils are requested to manage variables without a specific learning itinerary on that, then

¹ The contribution is executed in the sphere of Local Research Unity in Research on Mathematics Education of University of Parma - Italy.

the hypothesis is that there are capacities or a sort of mental structure that can be innate, ready to learn some more abstract topics of Mathematics.

In order to prove the previous statement, it is necessary to build suitable instruments. I propose the use of substitutions, in the sense of Logic morphologic operations, for detecting these capacities. In my opinion substitutions can also clarify inner processes of some pupils in the solution of problems.

3. Variables and natural language

This first assertion, i.e. that variables are used in standard didactic activities, can be proved by analysing textbooks and teachers activities in language learning.

The learning of natural language, reading and writing is one of the first topics for pupils of primary school. Teachers are requested to establish a correspondence between phonetics and writing. In order to obtain this correspondence they use many different techniques. One of these is the alphabet with drawings, representing objects of everyday use and (capital) letters. The pupil must know the name of each object represented and at the same time she/he must listen to the sound and keep it in memory. The first two years of primary school present many activities of this kind. I think that different natural languages have similar problems. My experience is with the Italian language and some examples I show later are taken from it. I hope the reader can "translate" what I show in her/his own natural language. I want to emphasise here those aspects that can be treated and explained in Mathematical Logic, reckoning that Logic is one of the contents indicated in Italian compulsory school standards by National Education Board, even if variables and substitutions aren't present explicitly in Italian national standards for primary school.

The learning of natural language is very complex and abstract. It can be said that it is more difficult than the learning of many mathematical topics. With a simple example taken from linguistic domain I can explain some aspects of logical education. A lot of Italian words are written with two consecutive equal consonants; pupils write only one of them, or write two consonants when only one is requested, changing in this way the meaning of the word. In order to overcome this sort of problem, the teacher presents (semantic) exercises of this kind: complete the word pa...a in which the dots are spaces that must be filled² and suitable drawings help the pupil. If dots are replaced with **pp** the word is **pappa** (pap), using only one letter **p**, the word obtained is **papa** (Pope). If dots in

² The same dots appear in the application form for CERME 2!

pa...a are replaced with *ll*, the word is *palla* (ball), with only one *l*, the word is *pa^{la}* (shovel).³ These exercises are very close to equations in finite domain (the alphabet or the "Cartesian square" of the alphabet) and the solutions can be obtained by replacing dots with letters in order to get the name of the objects represented in each drawing. I will show later a similar mathematical example (the square perimeter). In primary school we can find "equations"⁴ or open phrases similar to above linguistic exercises. Find the number such that $3 + \dots = 8$, or $2 \times \dots = 6$, and so on, "equations" that can be solved by trials, or by considering the previous examples as verbal problems, see Moser 1985a, Moser 1985b, and more recent and pertinent, Malara 1999.

The main differences between linguistic variables and mathematical ones is that for arithmetical or algebraic "equations" there are algorithms of solution, on the contrary linguistic "equation" can be solved only by trials or by knowledge of an everyday context. But in the first years of primary school these algorithms aren't available, and, even if the algorithms are available pupils can refuse to use them, as Basso & Bonotto 1996 evidences. In Italy algebraic equations are introduced in Mathematics at 8-th school degree, far from these first uses of them. I'm not interested here in the "arithmetical" algorithms, I want to stress that in these exercises the dots assume the role of variable and that "solutions" are obtained by trials and by substitution, using the semantic context given by the pupils' personal experience. Another more subtle difference is that in the common knowledge of first degree pupils, quantitative aspects of variables are present with a sort of *intended* semantics of the numbers (e.g. "two" is an adjective, i.e. a determination of something else); "*pp*" in itself has no meaning at all.

In my opinion the same mental structures are required to solve the aforesaid linguistic exercises and to learn the first steps in algebra.

Another phonetic problem can well prove the pupils' awareness of use of variables. In Italian language there some "blocks" of letters *qu* or *cu* or *cqu*, that have nearly the same pronunciation, similar to one used in the English word *cuirass*. The teacher's strategy is the same: some incomplete words are given and she/he asks for their completion: ...adro, a...a, ...oco and from these ones the pupil obtains *quadro* (picture), *acqua* (water), *cuoco* (cook). Drawings help to determine these words, but what kind of representation can help the students in the case of the word *quiddità* (quiddity)? This philosophical term isn't used in primary school, but in the present multiethnic society, also words of everyday

³ In Northern Italy local dialects are poor in double liquid consonants. On the contrary, in the regional language of Sardegna, all consonants are in general too strong and become double even if it is not the case. Hence this kind of exercise is frequent in Italian schools.

⁴ The term "equation" is used here in a rough meaning.

use for mother-tongue speaking pupils, are unknown to other people, even if the unknown words can be represented by drawings. In this case the semantic by-trial technique isn't suitable for solving these "equations", since there is too little phonetic difference among the proposed "blocks". The solution of the problems posed by the word ...iddità or by equation $3 : \dots = \sqrt{7}$, in primary school have at least the same difficulty degree; it may be that the linguistic question is more complex than algebraic equations, for which a solution procedure will be presented in the following school years.

The previous simple examples can be checked in textbooks for primary school, some with a slightly different presentation: instead of dots, some other different graphic notations are used. Hence the exercises can appear in the form: complete the word $pa\square a$ or $\square oco$. I asked a primary teacher the reason why the handbooks preferred this presentation, less befitting from a typographic point of view. She told me that pupils sometimes used three letters when there were three dots! These wrong answers are produced by two concurrent reasons:

1. the pupils are unaware that "..." in this context must be regarded as a unique thing that must be substituted with a unique linguistic entity,
2. they use each dot as a name for a variable disregarding a mathematical implicit *dictum* that each occurrence of the variable in an expression must be substituted with the same value.

The sign \square seems to avoid the first problem. Mistakes such as *cquoco*, *cquadro*, can be motivated for example by the parlour game "hangman" or something like that, but in every case these answers reveal an interpretation of each dot as a variable that can assume values on the alphabet.

These examples, and many others, show that pupils face variables in an early stage of their development, at primary school. The teacher, on the contrary, doesn't explain the "tricks" that must be used in manipulation of variables and the pupil is left alone. The consequences of this have long term effects on the whole student's curriculum. In each Italian primary school, in the same classroom, there are different teachers for linguistic education and mathematical education. They can be unaware of the fact that they teach with the same logical instruments, cf. Marchini, 1990a. Moreover the teacher for linguistic education should not take care of the (mathematical) difficulties shown by pupils in the treatment of variables at this stage. Timely measures can avoid major difficulties in the learning.

Unusual symbols (different from x , y , ...) for variables don't prevent us from considering them according to mathematical standards. The following

problem has been presented to 200 pupils in 4-th year classes⁵: replace each symbol with the same letter in order to obtain meaningful words

#ie◊e pa◊◊a #a##a ga#ba.

The results gave 58% correct and 23% incorrect but in the answers the symbols were replaced with letters taking into account only one word at a time⁶ in order to obtain meaningful words. It can be said that 81% of pupils can solve these "equations" and 71,6% of them read the problem as a "system of equations". But the solutions of equations or systems of this kind of equations require treatment of variables.

I want to underline these linguistic aspects because they are used very early in teaching and they offer good opportunities to begin important mathematical learning "outside" mathematics, see Bonotto 1996, or can give essential and timely information on pupils' competency.

4. Substitutions

The topic "substitutions" originated in Mathematical Logic as a morphologic operation used in transformations of formulae and terms of a formal language, cf. Rasiowa & Sikorski 1963. In Halmos 1962, substitutions are the core of algebraic logic. In Mathematics Education researches, the argument was partially presented in Freudenthal 1975, Adda 1982, Byers & Erlwanger 1984, and Lowenthal 1989. Malara 1997 describes substitutions as metacognitive abilities, see also Linchevski & Vinner 1990. Pesci 1990 uses some examples of them in a test identifying weak pupils. From a different point of view this argument is treated in Laborde's thesis, in Caron-Pargue 1981 and in Janvier 1987. But there isn't a specific entry for substitutions (and variables) in ZDM classification. This research domain seems scarcely explored. The semantic concept of *interpretation* of a first order formal language is another example of an application of substitutions, in a sense more general than morphologic one. A deeper analysis of substitution can help in the determination of assessment criteria cf. Marchini, 1990b.

⁵ This problem is presented by Pezzi, 1997. The problem is one of a diagnostic set of questions used in assessing background knowledge on substitutions and variables before a teaching experiment on these arguments. The symbols used in the item are a circle, here replaced with #, and a rhomb similar to the one used here. The good results in substitution management of 4-th degree pupils involved in Pezzi's teaching experiment, cannot be produced by previous experiences with geometry problems, since one of the goals of the experiment was to facilitate the learning of perimeters and areas of polygons.

⁶ The correct answer is *miele* (honey), *palla* (ball), *mamma* (mummy), *gamba* (leg); the last is the key-word for the solution. In this group of 46 pupils the more frequent answers are *piede* (foot), *pappa* (pap), *bacca* (berry), and *ganba* (an incorrect writing for leg).

There are three base points for substitutions: the starting point (initial configuration or IC) the destination point (final configuration or FC) and the code (CD). In *direct* substitutions the given CD is the set of instructions to apply to the given IC in order to obtain the unknown FC. In *inverse* substitutions the unknown CD must be determined in order to justify the given FC starting from the given IC. This taxonomy is consistent with Krutetskii as quoted in Pesci 1991. In the previous example of □oco, pupil has the IC given by the text and the FC is suggested by a drawing. She/he selects CD by trials among the proposed **qu**, **cu**, **cqu**; then she/he conjectures the correct FC from her/his linguistic experience. Hence this is an example of a sort of inverse substitution: the FC isn't completely given, since pupil knows the phonetic of the required word, and must translate the sound in a written FC.

In general, direct substitutions are simpler than inverse ones, since a decidable procedure of terms transformation can be applied. Inverse substitutions are difficult: in fact if the search of the CD isn't delimited to a given context, the procedure can be undecidable. In a lot of linguistic exercises the type of substitution required isn't simple to determine: often only the CD, or IC, or FC is given. An example of an exercise with only the CD given is: write words with **ll**; in the exercise: write words derived from *acqua* (water), only the FC is given; and so on. In Mathematics the same happens.

I will give an example of a geometric problem analysed by the means of substitutions: find the side of the square whose perimeter is 36. The text of the problem gives explicitly only the CD: $p = 36$. The IC is implicitly present in the problem by the information that the figure under examination is a square, hence the IC is the formula $p = 4 \times s$, where s is the unknown side. Applying the CD to the IC we get a (first) FC: $36 = 4 \times s$. This can be treated as a simple algebraic equation that can be solved by standard algebraic techniques, i.e. dividing both members of the equality by 4.

The solution of this problem can be obtained in a different way, by substitutions, without arithmetic algorithms. In this case neither IC nor CD, nor FC are clearly given, but pupil can assume $36 = 4 \times s$ as a new IC and searches a new FC in multiplication table, by inspection of the cells (of the table with entries from 1 to 12) in which appears 36: $36 = 3 \times 12$, $36 = 4 \times 9$, $36 = 6 \times 6$, $36 = 9 \times 4$, $36 = 12 \times 3$. The only one that is suggested by the IC $36 = 4 \times s$ is (FC) $36 = 4 \times 9$. At the end pupil performs an inverse substitution determining the new CD (and the problem solution) $s = 9$.


















This problem about a square, can be proposed in Italian primary school from the 3-rd to the 5-th year; its solution requires managing a complex situation in which geometric intuition has little or nothing to do with it. Note also the change of point of view: at the first step $4 \times s$ is regarded as a whole with symbol

awareness, cf. MacGregor & Price, 1999; afterwards the same writing is "decomposed" in order to obtain the CD comparing $36 = 4 \times s$ with $36 = 4 \times 9$. Similar remarks can be made for many other inverse arithmetical problems. I hope that this mathematical example clarifies that some arithmetical or geometrical problems can be solved, using substitutions, disregarding the numerical values involved, but with an intensional use of numerals viewed only as symbols or icons, cf. Malara & Navarra, 2000.

5. Instruments for detecting capacities of managing variables






In § 2, I pose the problem of building instruments suitable for evidencing the pupils' management of variables in the sense which I explained above. My answer to this problem is a proposal: I produced⁷ a set of questions for primary school presented only by graphical means, icons; the choice of using only graphical means is motivated by the lack of ability to write in first year classrooms.

I give a version of the test below, typographically reduced through lack of space. The original test has one page for each of the eight questions. The three lines in each item are spaced and the answer must be written in the empty central line. This choice can be unusual but it was necessary to avoid the pupil who interprets the drawing as a unique thing (this phenomenon was present in the "α-release" of the test given to other pupils). The test was accompanied with a drawing (HFT human figure test inspired by Polacek) and other kinds of questions addressed to teachers (a sort of profile for each pupil).⁸



IC	    	IC	    
FC	_____	CD	_____
₁ CD	 = 	₂ FC	    






⁷ With the help of Dott. Franco Priore.

⁸ The elaboration of these informations connected with the use of substitutions and variables will form the content of other papers.






IC     






FC _____

₃ CD  = 





IC     






CD _____

₄ FC     



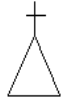

IC     






FC _____

₅ CD  =  ;  = 





IC     






FC _____

₆ CD  =  ;  = 






IC     

FC _____

₇ CD  =  ;  = 


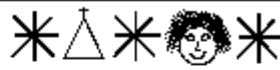


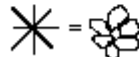
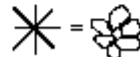
IC     

CD _____

₈ FC     

The sample was of 82 first year pupils (from 5 classes chosen in 3 different schools). In the 33% of the 656 items variables are correctly used; in 44% of the 656 items substitutions are correctly performed. The test was given in classroom without any preceding activity on the work. The teacher explained the task with a few examples presented immediately before the execution of test.

With the aim to clarify how the test could identify the correct management of variables, I reproduce here protocols of two pupils of the same class. In my opinion is evident that both apply correctly substitutions, only the first pupil succeeds in the management of variables.

IC		IC	
FC		FC	
CD		CD	

6. Open problems and didactic perspectives

The instrument I produced opens up some interesting problems. For instance are substitutions and variables connected or are they different kinds of mental structures? Until now I have not been able to prepare questions on variables, not involving substitutions for pupils aged 6,. But can variables in themselves be introduced avoiding (implicit or explicit) substitutions?

Another difficult question: are the correct answers to questions about substitutions and variables determined by learning or by innate talent?

The number of good solutions to the tests in primary school isn't negligible and it suggests that there is at least one specific mental structure since pupils had not learnt about variables and substitutions before in school.

It seems to me that previous arguments show that if the teacher can detect at an early stage of schooling what is the pupils' level on this field, substitutions are available as simple instruments. Some didactic action can be performed in order to improve the results on substitutions and variables. Pezzi 1997 writes about it.

Moreover substitution applies in other mathematical fields and further. The mathematical minded reader has present systems of algebraic equations, integrals calculus and so on. Substitutions are applied in fields other than Mathematics: cf. Marchini 1989, Chirenti & Marchini 1988, Margiotta 1990, Margiotta 1991a, Margiotta 1991b, Margiotta 1992. Therefore substitutions are instruments for the learning of many subjects presented in compulsory schooling and they are involved in many activities adapted for pupils, cf. Alberti 1998.

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PROCEPTS IN GEOMETRY

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Abstract: *We discuss the theory of encapsulating a process (Davis, Sfard, Tall & Gray et al, Dubinsky et al). All these papers concentrate on “procepts” in arithmetic / algebra / calculus, though the description of the theory is more general. (The authors exclude the discussion of procepts in geometry implicitly, or explicitly in Tall et al, 2000b). In this paper we briefly discuss processes of learning geometry (like a natural science, Struve 1987) and - on the base of case studies in class room interactions (Meissner & Pinkernell, 2000) - we argue that there also are procepts in geometry and we will give some examples.*

1. Procept Formation

Which are the cognitive processes when children develop their individual mathematical concepts? Vinner (in Tall 1991, p. 65ff) especially discussed the role of definitions in learning mathematics and how children may overcome the "conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition". Mathematical definitions mainly describe objects or a static view while the process of acquiring new insight often runs in parallel with activities or procedures or mental processes in time. Thus there are divergent roots to develop individual mathematical concepts and Tall & Vinner (1981) use the term "evoked concept image" to describe the part of the memory evoked in a given context.

How can these inconsistent views, an object on the one hand and procedures on the other hand, grow together to form an appropriate mathematical concept (or rich and powerful "concept image" with the words of Vinner)? Piaget (1985, p. 49) already has pointed out that "actions and operations become thematized objects of thought or assimilation". This idea has become very powerful today to understand the development of certain concept images in mathematics education as a process of "interiorization" or "reification" or "encapsulation".

We may quote Davis (1984, p.29f): "When a procedure is first being learned, one experiences it almost one step at time; the overall patterns and continuity and flow of the entire activity are not perceived. But as the procedure is practiced, the procedure itself becomes an entity - it becomes a *thing*. It, itself,

is an input or object of scrutiny. All of the full range of perception, analysis, pattern recognition and other information processing capabilities that can be used on any input data can be brought to bear on this particular procedure. Its similarities to some other procedure can be noted, and also its key points of difference. The procedure, formerly only a thing to be done - a verb - has now become an object of scrutiny and analysis; it is now, in this sense, a noun."

Sfard (1987) distinguishes also two kinds of mathematical definitions, referring to abstract concepts as if they were real objects or speaking about processes and actions. "The structural descriptions seem to be more abstract. ... To speak about mathematical objects, we must be able to deal with products of some processes without bothering about the processes THEMSELVES." She claims (1987, p. 168) that the operational conceptions develop at an early stage of learning even if they are not deliberately fostered at school. In Sfard (1992, p. 64f) she identified a constant three-step pattern in the successive transitions from operational to structural conceptions: "First there must be a process performed on the already familiar objects, then the idea of turning this process into a more compact, self-contained whole should emerge, and finally an ability to view this new entity as a permanent object in its own right must be acquired."

These three components of concept development Sfard calls "interiorization", "condensation", and "reification", respectively. "Condensation means a rather technical change of approach, which expresses itself in an ability to deal with a given process in terms of input/output without necessarily considering its component steps. Reification is the next step: in the mind of the learner, it converts the already condensed process into an object-like entity."

Dubinsky (in Tall 1991) and his colleagues (Cottrill et al, 1996) also studied the ENCAPSULATION phenomenon and they developed the APOS theory. They see three steps (**Action** → **Process** → **Object**) to get mental objects which then become part of a **Schema S**.

Gray & Tall analyzed the duality between process and concept and came to a similar view. They consider (1991, p. 72ff) "the duality between process and concept in MATHEMATICS, in particular using the same symbolism to present both a process (such as the addition of two numbers $3+2$) and the product of that process (the sum $3+2$). The ambiguity of notation allows the successful thinker the flexibility in thought to move between the process to carry out a mathematical task and the concept to be mentally manipulated as part of a wider mental schema." They hypothesized that the successful mathematical thinker uses a mental structure "which is an amalgam of process and concept". Tall (1991, p. 251ff), reflecting the dual roles of several symbols and notations: "Given the importance of a concept which is both process and product, I find it

somewhat amazing that it has no name. So I coined the portmanteau term "procept". In 1994 Tall & Gray proposed the following definitions:

"An *elementary procept* is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object. A *procept* consists of a collection of elementary procepts which have the same object." In Tall et al (2000a) we find examples for symbols as process and concept.

Especially when discussing advanced mathematical thinking we can discover a lot of "procepts". Dubinsky (2000, p. 43) lists such concepts: "functions, binary operations, groups, subgroups, cosets, normality, quotient groups, **induction**, permutations, symmetries, existential and universal quantifiers, limits, chain rule, derivatives, infinite sequences, mean, standard deviation, central limit theorem, place value, base conversion and fractions". For more details see <http://www.cs.gsu.edu/~rumeec/index.htm>.

2. Procepts in Geometry?

Studying the above lists we miss geometry, at least "concrete", visual geometry. Are there no procepts in geometry? Is the process of learning geometry that much different from the process of learning arithmetic and algebra and calculus? Are there no procedures or processes in geometry to become objects on a procept level? Most of the work on the "encapsulation of a process to an object" concentrates on examples in arithmetic, algebra, and calculus. We do not know papers on examples in geometry.

One of the reasons might be that in many countries geometry is not in the center of teaching mathematics and therefore there is not much research on how children learn geometry. In German primary school books for example we have only about 5% of the pages with geometry topics. (And even less than 5% of the time spent for mathematics education in German primary school classes then really is used to teach geometry).

Another reason might be that there still is a method of teaching and learning geometry which is similar to an axiomatic approach: We start with "definitions" and properties (line, point, circle, square, ...), discover relations and prove statements. Of course, it will be difficult then to discover (like in arithmetic) "processes which may produce a mental mathematical object". Then there also is no necessity in geometry for getting symbols which are used to represent either a process or an object.

Tall et al (2000b) formulate the hypothesis that there are three types of mathematics (space & shape, symbolic mathematics, axiomatic mathematics) and that each of them is accompanied by a different type of cognitive development. They consider - before focusing on the growth of symbolic thinking - "briefly ... the very different cognitive development in geometry". There are perceptions of real objects initially recognized as whole gestalts and classifications of prototypes. Reconstructions are necessary to give hierarchies of shapes and to see a shape not as a physical object, but as a mental object.

Struve (1987) also analyses how concept images in geometry develop. He summarizes, that children in primary and lower secondary classes learn geometry like a natural science, they describe and explain and generalize phenomena. Thus for them geometry becomes an empirical theory.

For the author of this paper it is a miracle that we in physics can use mathematical formulae and even complex mathematical theories to predict future events. We trust - but we cannot prove - that events will occur tomorrow in the same mode as yesterday when there will be the same conditions. There are big similarities between physics and empirical geometry: Given certain assumptions we can predict events - by the use of mathematical theories.

What does that mean for the theory of procepts? When we analyze in "3+2" possible step-by-step procedures of the children we also observe "empirical mathematics" with real objects. And like in geometry the children generalize and learn to predict future events. We trust, but we cannot prove, that "3+2" always "will be the same", but we (as mathematics educators or researchers) avoid speaking of a miracle by introducing "counting principles" (like axioms in geometry). In this view an elementary procept in the meaning of Tall et al (2000a) just is the shift from the empirical stage to the theoretical stage. Following these ideas consequently there should be no fundamental obstacle also to find procepts in geometry.

3. Procepts in Geometry!

When we look for procepts in geometry we first need activities or procedures. In the beginning they may be "experienced one step at a time". After practicing them for a while the user perceives "the overall patterns and continuity and flow of the entire activity, the procedure itself becomes an entity - it becomes a *thing*" (Davis).

Gray & Tall (2000a) distinguish procedure and process. For them procedure is like a specific algorithm. Using the example "4+2" there are

lengthy procedures (as "count-all"), compressed into shorter procedures (like "count-on" or first "count-both" or "count-on-from-larger") or other techniques (i.e. "remembering known facts" or "deriving facts"). These different procedures all are used "to carry out essentially the same process in increasingly sophisticated ways". Our following list may describe the mental development (which later on also will include geometry):

- (a) carry out accurately the given one procedure/technique
- (b) several procedures/techniques are possible, select one
- (c) several procedures/techniques are possible, make an efficient choice
- (d) carry out the process flexibly and efficiently, i.e. determine and select an appropriate procedure/technique
- (e) discussing, arguing (process becomes an object)

Table 1: Development of an (elementary) procept

Table 1 especially also describes the development of the elementary procept "4+2". Tall et al (2000a) continue: "Soon the cognitive structure grows to encompass the fact that 4+2, 2+4, 3+3, 2 times 3, are all essentially the same mental object", that means that the procept "6" consists of a collection of elementary procepts.

We see the symbols used (4+2, 2+4, 6, etc.) as abbreviations to describe (and to evoke!) the according processes or objects mentioned. In general we think symbols are abbreviations to name or to recall a process or an object. They serve like key words. In arithmetic/algebra/calculus we use letters a, b, c, d, ... and other symbols like +, %, dy/dx, ... to evoke concept images. But other key words like "six", "field", "parallel", ... or " \square ", " \diamond ", ... or " \oplus ", " \otimes ", ... or " \perp " ... might do the same. Only important for mathematics education is the concept image evoked by that symbol or key word.

Tall et al (2000b) point out that "symbols occupy a pivotal position between processes to be carried out and concepts to be thought about. They allow us both to do mathematical problems and to think about mathematical relationships". Important, there is only one symbol with a dual meaning. And we like to add, it is not important what type of symbol or key word it is.

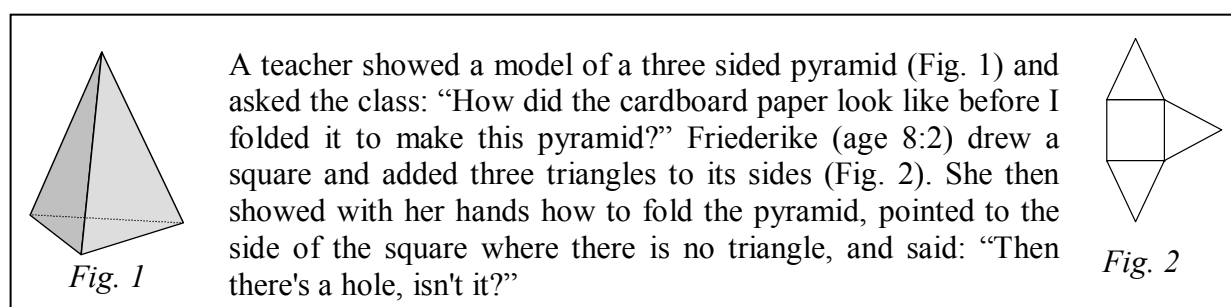
Thus we think we should expand table 1 by adding **consciously** the process of tagging or naming, that means **communication** is an essential part of developing procept images:

- | | |
|-----|---|
| (a) | carry out accurately the given one procedure/technique |
| (b) | several procedures/techniques are possible, select one |
| (c) | several procedures/techniques are possible, make an efficient choice
and explain |
| (d) | carry out the process flexibly and efficiently, i.e. determine
and select an appropriate procedure/technique
and discuss possible alternatives |
| (e) | use abbreviations (symbols, key words) for
discussing, arguing (process becomes an object) |

Table 2: Development of an (elementary) procept image

4. Examples of a Procept in Geometry

Analyzing video tapes from class room interactions (Meissner & Pinker-Nell 2000) we suddenly disagreed on the interpretation of aspects in the following situation.



We started discussing if it is necessary for everybody to fold mentally before deciding if this drawing is a net of a pyramid. By looking at Fig. 2 an experienced geometrician sees that it does not represent the development of a solid. He can decide without actually folding the net. In his reasoning the process of folding has been encapsulated to the static concept “development”. Friederike however has some notion of “development” in which she still needs to carry out the process of folding explicitly, as her hands indicate. Thus we consider “development” as a procept in spatial reasoning.

Let us discuss and analyse this case in more detail. First, Friederike gets, probably without realising it, two contradictory stimuli at the same time. The key word “pyramid” leads to a square because all pyramids Friederike knew till now had a square as the base. However the given solid says that Friederike only needs three triangles. She compromises and gets the hole. Obviously she is

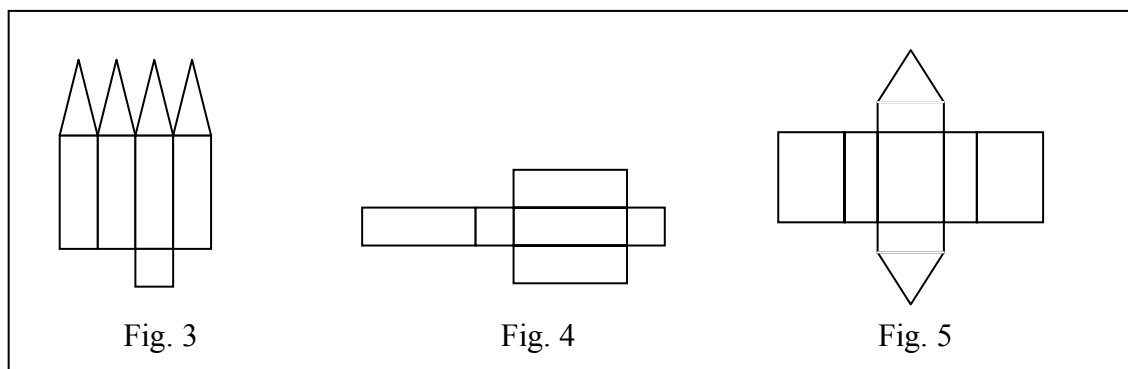
familiar with a mental folding-up procedure, but she has not enough experiences to bridge the gap immediately.

This episode was observed before we systematically introduced activities in the class room to draw developments for solids (pyramids, rectangular solids, houses): The children learned to make the net of a pyramid by placing a wooden model onto a sheet of paper and then repeatedly tilting it from its base onto one side and back to the base again, each side being encircled with a pencil. The resulting figure would be a star shaped net. Next, we have asked them to make the net of a rectangular solid. What we have experienced many times is that in strictly following the learned procedure they forget the solid's upper side and produce a net that would fold to an open box.

We then pointed onto the missing side of the given solid asking where this was drawn. Very often there was a laughter in the class room and immediately the drawing was completed correctly. The procedure "development" they had acquired so far was based on an activity of what could be called "tilting from and back to the base". With the rectangular solid this procedure of "tilting" had to be revised by extending it. This mental change is typical for the development of procepts. Proceptual thinking also includes the ability to revise an encapsulated procedure to meet new demands (Gray 1994, p.2). We saw a similar expanding of the procedures when we used solids with concave sides.

To draw the developments the children got a card board and a wooden solid. Some of the children just started tilting and drawing. Others first took the solid to find an appropriate starting position on the card board (by tilting without drawing) to make sure that their drawing will fit on the paper. Here the activity already becomes a flexible and efficient process.

The last lesson of that teaching unit (details see Meissner & Müller-Phillip 1997) started with an exhibition of about 20 different (plane) developments of buildings fixed with tape on the black board. There were only lines drawn where to fold later on (but not distinguishing if to fold inside or outside). The children (grade 3, age about 8 - 9) had to describe which net might become what type of building before they could choose one of the developments to verify their guesses. We are sure some of the children just identified simple nets without any mental folding. They just *saw* "that is a tower" (Fig. 3) or "that is a garage" (Fig. 4) or "that is a house" (Fig. 5). We think for them a simple net had become an elementary procept.

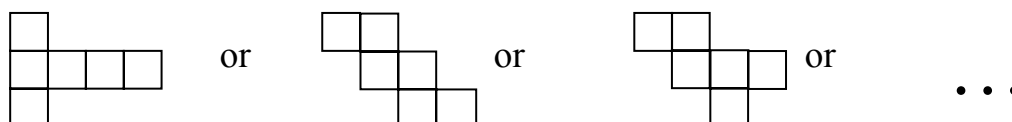


But where is the symbol, one of the characteristics of a procept? We think the net itself is the symbol. The one interpretation of that symbol is a procedural one, "folding up". The other view is static, "this is ..." (an object).

Symbols of procepts follow syntactical rules. Also from this point of view there are reasons to take (at least simple nets) as a symbol. In the following we will demonstrate this view by comparing procepts from arithmetic or algebra with the procept "net".

A process is a set of procedures:

We can describe "6" by "4+2" or "5+1" or "3+3" or ... And we can describe "cube" by

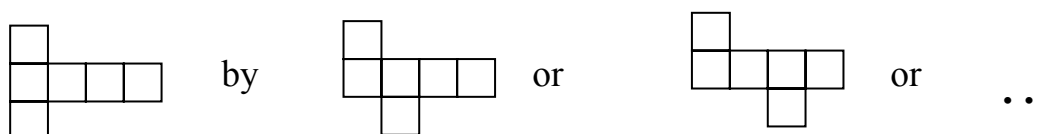


Each symbol belongs to a specific process:

This is true for "6" or "32" or "1/2" as well as for nets shown in figure 3, 4, or 5.

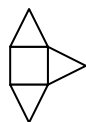
Symbols can be manipulated according to "syntactical rules":

Replace "3+4" by "4+3" ($3+4 = 4+3$) or replace " 2×8 " by " 8×2 " ($2 \times 8 = 8 \times 2$) or replace " $3 \times (4+2)$ " by " $3 \times 4 + 3 \times 2$ " or ... We also can replace

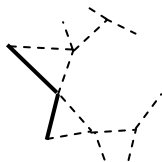


There are "syntax errors":

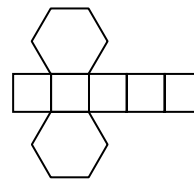
The notation of power does not allow symbols like " 2x " or " $_2x$ " or " x_2 " or " $_2x$ " or " 2x ". Or the notation of addition does not allow " $+2,4+$ " or " $2,4,+$ " or ... the notation "net" does not allow



(missing side)



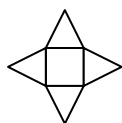
(neighbored lengths)



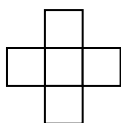
(number of side surfaces)

Procepts, described by a symbol, can be expanded:

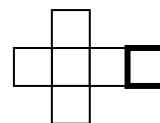
“ 3×4 ” (multiplication of integers) gets expanded to “ 3.5×6.9 ” (multiplication of decimals).



“tilting”



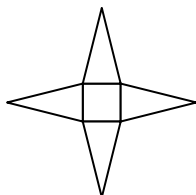
gets expanded to a



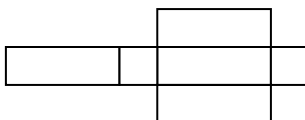
“conscious tilting”

Symbols can be variables:

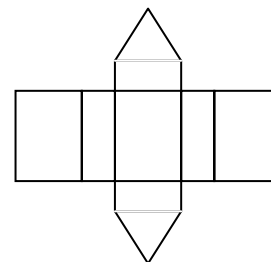
We use letters for variables in arithmetic or algebra. A “net” also may have the meaning only of a variable, i.e. by giving the spatial shape, but no geometric proportion of the specific solid:



“a” pyramid

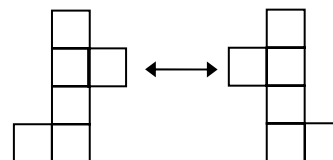
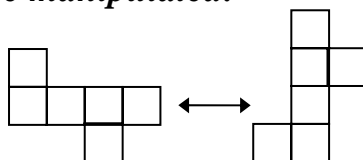
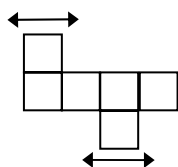


“a” rectangular solid

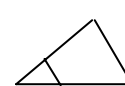
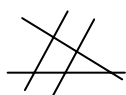
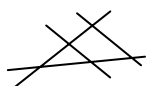


“a” house

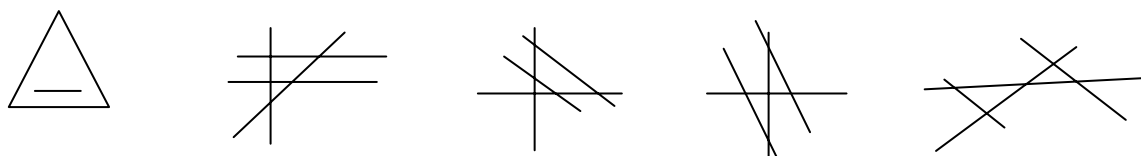
Symbols can be manipulated:



More briefly we will add two other examples. One main theorem in geometry is, what we call in German the Strahlensätze. A figure of four lines, where two intersecting lines cross two parallel lines, leads to three or four basic statements about the ratio of according lengths:

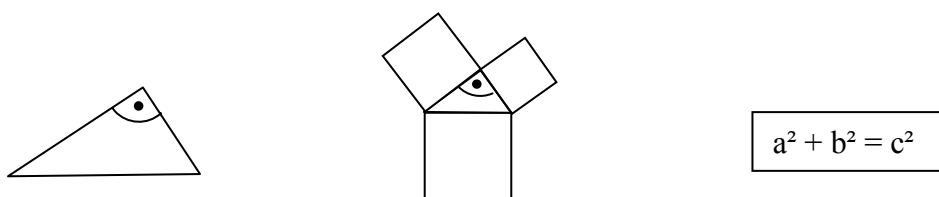


We think the symbol or tag of the Strahlensätze is one of the given figures. A proceptual thinking of "Strahlensätze" is only possible when we are able to regard the above figures as an entity (of related procedures). Then the procept "Strahlensätze" is encapsulated in each of these figures. The different types of figures can also be seen as manipulations of symbols according to syntactical rules. Some more manipulations may be the following:

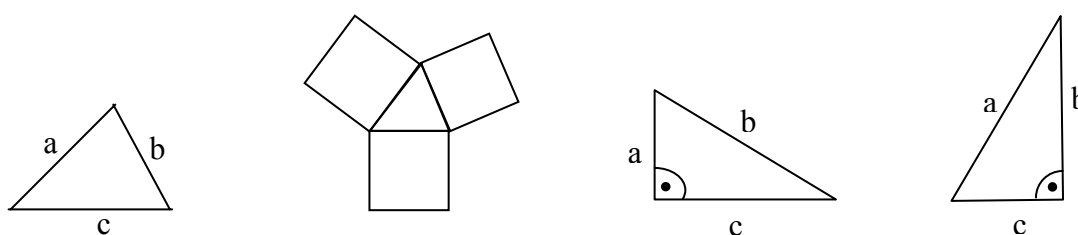


The last figures even indicate an extension of the original concept. Of course all these symbols also implicitly include variables: It is not important where the intersection point is in relation to the two parallels nor is the size of the angle of the intersecting lines nor the width of the parallels.

Another example is Pythagoras' Theorem. There are several types of tags:



Often our students do not achieve a proceptual "pythagorean" thinking. They ignore or they do not see the property "perpendicular" or they have fixed mental images of how to name the sides of a triangle:



5. Summary

The theory of procepts can be extended. We also have geometrical procepts like *net of a solid*, *Strahlensätze*, *Pythagoras' Theorem*, but also *triangle*, *polygone*, *circle*, ...

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RELATION BETWEEN ARGUMENTATION AND PROOF IN MATHEMATICS: COGNITIVE UNITY OR BREAK?

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Abstract: *The purpose of this paper is to analyse the relationships between argumentation and proof. My assumption is that argumentation and proof can be compared from two points of view: content and structure. Toulmin's model can be a tool to compare the two structures. This paper shows how Toulmin's model can be used to highlight the presence or absence of cognitive unity during the solution of geometric problems needing the production of conjectures and related proofs.*

1. Introduction

I will consider the solving process of geometric problems in which students interact with dynamic environments, in my case the Cabri-Geometry software. I consider a situation in which the student produces an argumentation during the production of the conjecture and then constructs a proof of this statement. The purpose of this paper is to analyse the relationships between argumentation and proof. In particular, my research aim is to analyse similarities and differences between the structures of the two processes, in order to analyse them from the point of view of cognitive unity (Boero, Garuti, Mariotti, 1996).

In general, in dealing with problems asking for a conjecture, the solution is not immediate. Then the production of an argumentation during the construction of a conjecture is expected. I gave some open-ended problems to 12th -grade students in Italy and in France. The students worked in pairs in order to favour an argumentation activity between them. They worked on a computer running the Cabri-Geometry software. I thought that the software could help the student to identify the geometrical proprieties which are beneath the figure construction and which are necessary to the production of proof.

2. Relationships between argumentation and proof

The relationships between the production of a conjecture and the construction of proof has been an object of study from a cognitive perspective. Actually, research studies showed the following:

“During the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organising some of previously produced arguments according to a logical chain” (Boero, Garuti, Mariotti, 1996).

This phenomenon is referred to by the authors as cognitive unity.

The hypothesis is that in the proof the student organises some of the previously produced arguments into a logical chain. The word “argument” refers to a reason given to support or disprove something.

In this paper the word “argumentation” refers to a discursive activity (cf. Grize, 1996) based on arguments.

During the solving process, which leads to a theorem, an argumentation activity is probably developed in order to produce a conjecture. When this statement expressing the conjecture is made valid in a mathematical theory, a proof is produced. This proof is a particular argumentation based on a mathematical theory.

A conjecture could be provided without any argumentation. A conjecture can be a “**fact**”, directly derived from a drawing, from an intuition and the like. In this case there is not an explicit argumentation justifying this fact. But, I am interested in the following kind of conjecture.

I define a **conjecture** a statement strictly connected with an argumentation and a set of conceptions¹ (Balacheff, 1994). The statement is potentially true because some conceptions allow the construction of an argumentation that justifies it.

The conjecture can be transformed into a valid statement if a proof justifying it is produced.

I define a **valid statement** a statement which is provided with a proof referring to a mathematical theory. The statement is valid because a mathematical theory allows the construction of a proof that justifies it.

I am interested in comparing the processes used to construct a conjecture and its validation: argumentation and proof.

¹ The formal definition of conception is the following : «... we call conception C a quadruplet (P, R, L, Σ) in which: P is a set of problems; R is a set of operators; L is a representation system; Σ is a control structure. ... » (Balacheff, 1994)

3. Cognitive unity in content and in structure

My assumption is that the argumentative process supported the production of a conjecture and the related proof can be compared from two points of view: content and structure. The presence or absence of cognitive unity can be studied in relation to this assumption.

It is possible to observe whether there are similarities or differences between argumentation content and proof content. Usually, there are many similar content elements in the argumentation and proof, and cognitive unity is frequently found with respect to the content (Pedemonte, 1998).

Beside similarities in terms of content, it is interesting to compare argumentation and proof from the structural point of view, i.e. to observe analogies and differences between argumentation structure and proof structure.

It is possible to classify (in order to compare) the argumentation and proof according to the classic structures like deduction, abduction and induction. In a deductive argumentation, the statement is obtained from the data by means of a principle (which permits the inference). In an abductive argumentation the statement is obtained before the data is identified (Arzarello, 1998). In this case a principle allows the assertion of a statement even if all the data are not available. In an inductive argumentation the statement is obtained as a generic case after research from specific cases.

Only deductive argumentation can be easily and directly transposed into a (deductive) proof. In order to transform an abductive argumentation into a proof its structure needs to be reversed. Inductive argumentation has a structure far away from the structure of an deductive proof; in this case, a link between argumentation and proof can be found only when argumentation is based on the “generic case” (Balacheff, 1988).

According to the previous analysis I can expect that even in the case of “cognitive unity” (which concern content) the transition from argumentation to proof may demand relevant (and sometimes difficult to perform) changes concerning structures –in particular those from abductive or inductive argumentation to deductive proof.

4. Duval’s answer

Differences between argumentation and proof have been deeply analysed in the work of R. Duval: despite the use very similar linguistic forms and proposition’s

connectives, there is a “gap” between the two processes. According to Duval (1991), the structure of a proof may be described by a ternary diagram: data, claim and inference rules (axioms, theorems, or definitions). Within proofs, the steps are connected by a “recycling process” (Duval, 1992–1993) the conclusion of a step serves as an input condition to the next step. On the contrary, in argumentation, inferences are based on the contents of the statement. In other words the connection between two propositions is an intrinsic connection (Duval, 1992–1993): the statement is considered and re-interpreted from different points of view. For these reasons the distance between proof and argumentation is not only logic but is also cognitive: in a proof, the epistemic value² depends on the theoretical status whereas in argumentation it depends completely on the content. Then it is easy to observe the cognitive distance between the two processes.

According to Duval, the distance between these two processes can explain why most of the students don’t understand the necessity of a mathematical proof: if there is an argumentation that justifies the statement the proof can be unnecessary.

Some doubts are expressed about the nature and the educational relevance of the gap between argumentation and proof, as described by Duval (in particular see Douek, 1999). I share these doubts. I think that there are some very similar elements between argumentation and proof. In particular, assuming that a proof is a particular argumentation, both argumentation and proof structures can be described by a ternary diagram. This is the reason why I need a tool to compare the structure of the two processes.

5. How to analyse or compare the structure of the argumentation process and the proof process?

I have built up a theoretical framework to analyse argumentation structure and proof structure. Toulmin proposes a model describing the structure of the argumentation (1958). I use this model as a tool to compare the structures relating to the two processes: argumentation and proof.

In any argumentation the first step is expressed by a standpoint (an assertion, an opinion). In Toulmin’s terminology the standpoint is called the claim. The second step consists of the production of data supporting it. It is important to provide the justification or warrant for using the data concerned as

² The epistemic value is the degree of certitude or conviction associated with a proposition (Duval, 1991).

support for the data-claim relationships. The warrant can be expressed as a principle, a rule and the like. The warrant acts as a bridge between the data and the claim. This is the base structure of argumentation, but auxiliary elements may be necessary to describe an argumentation. Toulmin describes three of them : the qualifier, the rebuttal and the backing. The force of the warrant would be weakened if there were exceptions to the rule, in that case conditions of exceptions or rebuttal should be inserted. The claim must then be weakened by means of a qualifier. A backing is required if the authority of the warrant is not accepted straight away.

Then, Toulmin's model of argumentation contains six related elements as showed in the following figure.

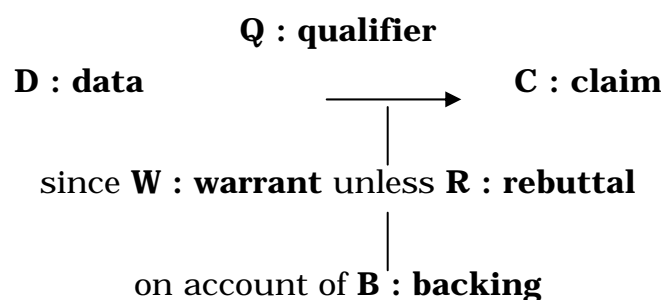


Fig.1. Toulmin's model of argumentation³

It is interesting to compare the idea of epistemic value (Duval, 1991) and the idea of the qualifier. The epistemic value of the claim is inherited by the epistemic value of the data. The claim's force is inherited by the data's force. On the contrary, the qualifier is given by the data and also by the warrant's force. The warrant's force is important because the warrant plays a basic role in the argumentation.

If we consider a proof as a particular argumentation, the warrant is an axiom, or a definition, or a theorem, in a specific theory.

Toulmin's model reveals a very powerful tool to compare the process of argumentation and the proof subsequently produced. It is possible to compare the argumentation warrants and the proof warrants. For example if the warrant in an argumentation is related to an intuitive conception, it is possible to see whether in the proof the warrant becomes a theorem of a theory or on the contrary if it remains at the level of conception.

³ Let us illustrate this model with the same example used by Toulmin (1958): *Claim* : Harry is a British subject; *Data* : Harry was born in Bermuda; *Warrant* : A man born in Bermuda will generally be a British subject; *Rebuttal* : No, but it generally is. If his parents are foreigners or if he has become a naturalised American, then the rule doesn't apply; *Qualifier* : True : its only presumably so; *Backing* : It's embodied in the following legislation :...

Within the theoretical framework described above, a research project has been designed and is still in progress. In the following section, I illustrate the use of this model as an example when the resolution process of an open-ended problem will be analysed.

Interview

The following example is taken from a set of data collected in four 12th-grade classes in Italy, and in one 12th-grade class in France. The students worked in pairs on a computer running the Cabri-Geometry software. The experiment lasted an hour and a half. The problem proposed was the following:

Problem

ABC is a triangle. Three exterior squares are constructed on the triangle's sides. The free points of the squares are connected defining three other triangles. Compare the areas of these triangles with the area of triangle ABC (see figure pg. 6).

I will transcribe a part of a solution protocol related to the proposed problem. This part is based on the transcriptions of the audio recordings and the written productions of the students.

According to the classification given in the previous section, different types of argumentation can be described according different structures.

A typical deductive argumentation could be the following. Suppose that, the student compares the base and height lengths between triangle ABC and one of the external triangles (ABC and ICD in figure pg. 6), in order to compare the two areas. It is possible to consider the sides of the same square as bases (BC, CD) for some triangles and compare the heights (AL, IM) considering the small triangles constructed on the heights (ALC, ICM). Observing that the small triangles have two equal angles and an equal side permit the conclusion that the two triangles are equal under the SAA congruence criterion. Then the large triangles have equal areas.

A typical abductive argumentation could be the following. The student, who wants to compare the two areas, realises that the two bases of the triangles have the same length, thus it is possible to prove that the heights have the same length in order to prove that the areas are equal. The view that the small

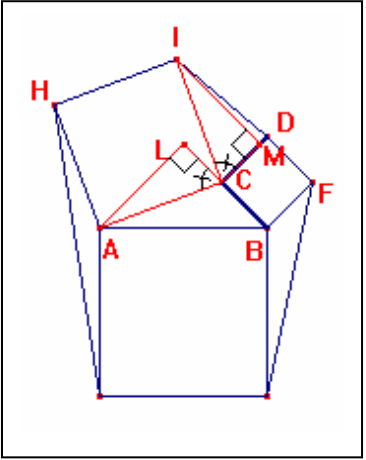
triangles constructed on the heights are equal can encourage the search for a theorem to prove this fact. The congruence criterions are remembered and the data needed to apply one of them is sought out.

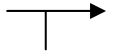
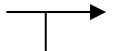
A typical inductive argumentation could be the following. The student may consider some particular types of the triangle ABC: right-angled triangle, equilateral triangle; or he may consider limit cases, for example when the points A, B, and C are on the same line. This is an “inductive search” moving from particular cases to general laws. One of the particular cases can evolve into a generic example (N. Balacheff, 1988) which can lead to a proof.

Example

Using the model described above, I have analysed an excerpt of the argumentation and the proof produced by students. My purpose is to show how Toulmin’s model can be used in order to compare the structures of the argumentation and proof.

In order to analyse the argumentation, I have selected the assertions produced by students and reconstructed the structure of the argumentative step: claim C, data D and warrant W. The indices identify each argumentative step. The student’s text is in the left column, and my comments and analyses are reported in the right column. The text has been translated from Italian into English. The analysis starts at claim C₇; at this point students are comparing the area of the triangle ABC and the area of the triangle ICD. Till now the students spoke about the construction of the heights of the two triangles. They decided to construct the heights in order to compare the areas of the triangles ABC and ICD.

<p>.... Students construct the heights of the triangles ABC et ICD</p> <p>31. L: I’m prolonging the straight line, yes, the straight line on the segment... what have I done?</p> <p>32. G: The straight line by the points B and C</p> <p>33. L: ah it’s true !</p> <p>34. G: now, we need to do the line perpendicular to this line</p> <p>35. L: ah there that’s it done but you know that it seems they are equal...</p> <p>36. G: almost equal !</p>	<p>The figure as represented from the students using Cabri-géomètre</p> 
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<p>37. L: not anymore, it seems that they are perpendiculars, I have observed this before</p> <p>44. Students together: hey, these are two equal triangles !</p> <p>45. L: it's true, ALC and ICM these are two equal triangles...what do they have?</p> <p>46. G: we realized... then AC is equal to IC because they are sides of the same square</p> <p>47. L: wait!</p> <p>48. G: AC is equal to IC because they are sides the same square, after</p> <p>49. L: LC...</p> <p>50. G: it's equal to CM, why ?</p> <p>51. L: Then... Because it's equal to CM... in my opinion, it's better to prove ... no wait this angle is right and this angle is right too.</p>	<p>C₇: The heights seem to be equal. C₈: The heights seem to be perpendiculars.</p> <p><i>The statements are "facts" where the epistemic value is joined to perception of the figure in Cabri-Geometry.</i></p> <p><i>The Cabri-Geometry drag allows them to see the small triangles. The students realize that the heights are the heights of two equal triangles. The statement is now a fact.</i></p> <p>C₉: The triangles ALC and ICM are equal.</p> <p><i>The structure of the speech of the students is:</i></p> <p>The triangles  to find equal sides are equal and angles congruence criterion</p> <p><i>The structure of the argumentative step is an abduction:</i></p> <p>D₉= ?  C₉</p> <p>W: CONGRUENCE CRITERION</p>
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The structure of the argumentation is that of an abduction. The students see that the small triangles constructed on the height (ALC and ICM) are equal and they search for a theorem to prove this fact. During the proof, students make data D₉ explicit in order to affirm that triangles ALC and ICM are equal. The abductive structure of the argumentation is transformed into a deductive structure in the proof. Once obtained, claim C₉ is used to deduce that the heights of the triangles ABC and ICD are equal and consequently that their areas are equal.

<p><i>The students write the proof:</i></p> <p>I consider the triangle ABC and the triangle ICD. At once I consider the triangles ALC et ICM and I prove that they are equal triangles for the SAA congruence criterion because we have:</p> <ul style="list-style-type: none"> • AC = IC because they are two sides of the same square • ALC = IMC because they are right angles (angles constructed as intersection between the sides and the heights) • ACL = ICM because they are complementary of the same right angle (- LCI) <p>In particular IM = AL. Then the triangles ABC and ICD have the same base lengths (as sides of the same square) and the same heights, then they have the same area.</p>	<p><i>The proof structure is a deduction:</i></p> <div style="display: flex; align-items: center; justify-content: space-between;"> <div style="text-align: left;"> <p>D₉: AC = IC ALC = IMC ACL = ICM W: SAA congruence criterion</p> </div> <div style="text-align: center;"> $\xrightarrow{\quad}$ </div> <div style="text-align: left;"> <p>C₉: the triangles ALC and ICM are equal</p> </div> </div> <p><i>If the triangles are equal then it's possible to conclude that the heights are equal, and finally then the areas are equal because the bases are equal.</i></p> <p><i>The conclusion C₉ of the previous step is the date D₁₀ to apply the inference to the second step.</i></p> <div style="display: flex; align-items: center; justify-content: space-between;"> <div style="text-align: left;"> <p>D₁₀: C₉</p> </div> <div style="text-align: center;"> $\xrightarrow{\quad}$ </div> <div style="text-align: left;"> <p>C₁₀: the heights are equal</p> </div> </div> <p style="text-align: center;">W: inheritance</p> <div style="display: flex; align-items: center; justify-content: space-between;"> <div style="text-align: left;"> <p>D₁₁: C₁₀</p> </div> <div style="text-align: center;"> $\xrightarrow{\quad}$ </div> <div style="text-align: left;"> <p>C₁₁: the areas of the triangles ABC and ICD are equal</p> </div> </div> <p style="text-align: center;">W: formula of area</p>
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At first glance, the protocol appears to be an example of cognitive unity. Indeed, students use the “SAA congruence criterion” both in the argumentation and proof in order to justify the statements. Words and expressions used in the two processes are often the same (“triangles ALC and ICM are equal”, “heights are equal”, and the like). But looking more carefully, we can observe a change between the structures of the two processes: we find an abductive structure in the argumentation (from D₉ to C₉) that is transformed into a deductive structure in the proof. We cannot undervalue the importance of the structure in the comparison between argumentation and proof; it is not unusual that the student tries to transform abduction into a deduction during a resolution process (sometimes successfully, sometimes without getting an acceptable solution).

6. Conclusion

In this paper, I have analysed some relationships between argumentation and proof; I have used Toulmin’s model as a tool in order to compare the structures of the two processes.

The analysis carried out on the student’s protocols, highlighted deep similarities between arguments provided during the construction of a conjecture and the proof subsequently produced. Such similarities mainly concern the

content of the arguments; on the contrary a careful analysis carried out in respect of the structure of the organisation of the arguments, may reveal interesting discrepancies. Toulmin's model clearly reveals the structure of both argumentation and proof facilitating the comparison between them. When students use abduction during argumentation (and this seems to be natural in the production of a conjecture), a structural change is needed and can be detected in students' protocols.

The study reported in this paper is still in progress. Further analysis will be carried out in order to clarify the nature of argumentation to find other analogies or differences with proof.

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EMERGING KNOWLEDGE STRUCTURES IN AND WITH ALGEBRA

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Abstract: *We take abstraction to be an activity of vertically reorganising previously constructed mathematical knowledge into a new structure. Abstraction is thus a context dependent process. In a previous publication, we proposed a model for processes of abstraction. The model is operational in that its components are observable epistemic actions. Here we use the model to analyse an interview with a pair of grade seven girls carrying out an algebraic proof. The analysis reveals how two kinds of knowledge emerge in the students: Knowledge of algebraic structures and knowledge about algebra as a tool for proof.*

Abstraction is a central process in learning mathematics; however, it is notoriously difficult to observe. Many researchers have taken a predominantly theoretical stance and have described abstraction as some type of decontextualization (see Dreyfus, 1991, for a brief review). In a previous paper (Hershkowitz, Schwarz & Dreyfus, 2001), we have taken a different approach and proposed a model for abstraction that is operational in the sense that its components are three observable epistemic actions. We are practitioners who are informed about recent theoretical research, but we are also deeply involved in a curriculum design, development, and implementation project. We have been considering not only what abstraction could mean in the framework of this curriculum project but also how processes of abstraction manifest themselves empirically in project classrooms. Thus, although our outlook is theoretical, our thinking about abstraction has emerged from the analysis of experimental data. In this paper, we briefly review our definition and model of abstraction, and then illustrate them by means of an interview with a pair of seventh graders dealing with a problem situation from this curriculum. The relevant theoretical background has been discussed in our previous paper. A more detailed description of the model can also be found there.

Mathematics educators have proposed that abstraction consist in focusing on some distinguished properties and relationships of a set of objects rather than on the objects themselves. Abstraction is thus a process of decontextualization. According to Davydov (1972/1990), on the other hand, abstraction starts from an initial, undeveloped form and ends with a consistent and elaborate final form. Similarly, Ohlsson and Lehtinen (1997) see the cognitive mechanism of abstraction as the assembly of existing ideas into more complex ones. Noss and Hoyles (1996) go even further. They situate abstraction in relation to the conceptual resources students have at their disposal and see it as attuning practices from previous contexts to new ones. Therefore, according to Noss and Hoyles, students do not detach from concrete referents at all. Leaning on ideas of these and other authors, we define abstraction as an activity of vertically reorganising previously constructed mathematical knowledge into a new structure. The use of the term activity in our definition of abstraction is intentional. The term is directly borrowed from Activity Theory (Leont'ev, 1981) and emphasises that actions occur in a social and historical context. It also stresses the inseparability of actions from goals, their meaning being perceivable only within the activity in which overall motives drive individual actions of participants. The reorganisation of knowledge is achieved by means of actions on mental (or material) objects: Mathematical elements are put together, structured and developed into other elements. Such reorganisation is called vertical (Treffers and Goffree, 1985), if new connections are established or some inaccessibility is overcome, thus integrating the knowledge and making it more profound.

According to this definition, abstraction is not an objective, universal process but depends strongly on context, on the history of the participants in the activity of abstraction and on artefacts available to the participants. Artefacts are outcomes of human activity that can be used in further activities. They include material objects and tools, such as computerised ones, as well as mental ones including language and procedures; in particular, they can be ideas or other outcomes of previous actions (knowledge artefacts).

This definition of abstraction in context becomes productive through a program of research to experimentally investigate processes of abstraction. As abstraction is an activity consisting of actions, the first step in the realisation of this program is to identify the kinds of actions involved in the activity of abstraction. The actions we identified in the first study belong to the general class of epistemic actions, actions relating to the acquisition of knowledge (Pontecorvo & Girardet, 1993; Schwarz & Hershkowitz, 1995). For example, appealing to a strategy or inferring a consequence from data are epistemic actions. In many social contexts, such as small group problem solving or teacher-guided inquiry in a whole class forum, participants' verbalisations may attest to epistemic actions thus making them observable. The three epistemic

actions we identified as related to processes of abstraction are Recognising, Building-With and Constructing, or RBC.

Constructing is the central step of abstraction. It consists of assembling knowledge artefacts to produce a new structure to which the participants become acquainted. *Recognising* a familiar mathematical structure occurs when a student realises that the structure is inherent in a given mathematical situation. Recognising may occur in at least two cases: (1) by analogy with another object with the same or a similar structure which is already known the re-cognising subject; (2) by specialisation, i.e., by realising that the object fits a (more general) known (to the subject) class all of whose members have this structure. In terms of actions, the process of recognising involves appeal to an outcome of a previous action and expressing that it is similar (by analogy), or that it fits (by specialisation). *Building-With* consists of combining existing artefacts in order to satisfy a goal such as solving a problem or justifying a statement. The same task may thus lead to building-with by one student but to constructing by another, depending on the student's personal history, and more specifically on whether or not the required artefacts are at the student's disposal. Another important difference between constructing and building-with lies in the relationship of the action to the motive driving the activity: In building-with structures, the goal is attained by using knowledge that was previously acquired or constructed. In constructing, the process itself, namely the construction or restructuring of knowledge is often the goal of the activity; and even if it is not, then it is at least indispensable for attaining the goal. The goals students have (or are given) thus strongly influence whether they build-with or construct. If they solve a standard problem, they are likely to recognise and build-with previously acquired structures. If they solve a non-standard problem, they might be faced with an obstacle that causes them to construct by vertically reorganising their knowledge to overcome the obstacle.

The three epistemic actions are the elements of a model, called the dynamically nested RBC model of abstraction. According to this model, constructing incorporates the other two epistemic actions in such a way that building-with actions are nested in constructing actions and recognising actions are nested in building-with actions and in constructing actions. The genesis of an abstraction passes through (a) a need for a new structure; (b) the construction of a new abstract entity; (c) the consolidation of the abstract entity through repeated recognition of the new structure and building-with it in further activities with increasing ease. We have argued in the previous paper that this model fits the genesis of abstract scientific concepts acquired in activities designed for the special purpose of learning. In such activities the participants create a new structure that gives a different perspective on previous knowledge. The model is then compatible with the dialectical theory of abstraction developed by Davydov (1972/1990). Moreover, the model is operational: It

allows one to identify processes of abstraction by observing the epistemic actions and the manner in which they are nested within each other.

In the remainder of this paper, we illustrate the model and its operational nature by means of an interview with a pair of students involved in an activity that presented a definite potential for abstraction to them. We will focus on a pair of girls who will be identified as Ha and Ne, or collectively as Ha&Ne. The students were part of a grade 7 introductory algebra class toward the end of the school year. Their algebra course consisted of activities based on problem situations that dealt with the generalisation of numerical and visual patterns, with the mathematical description of growth phenomena and with comparisons between different growth patterns (e. g., linear versus exponential). During these activities, the students also encountered some elementary algebraic manipulations, including the simple distributive law $a(c+d)=ac+ad$.

The interview with Ha&Ne was videotaped and transcribed. The transcript was analysed separately by each of the three researchers with respect to the occurrence of (combinations of) the epistemic actions that form the dynamically nested model of abstraction. Differences between the three analyses were resolved in discussion between the researchers. All such differences related to minor issues such as whether a constructing action that lasted over about ten student utterances should or should not include the tenth utterance (lines 140/141, see below); or how to classify a statement that included aspects of recognising as well as aspects of building-with (line 157, see below). The analyses of the three researchers agreed with respect to the kind and substance of all extended epistemic actions as well as with respect to the nesting of these actions.

The interview activity was designed for students from whom the use of algebra for proving properties could possibly be expected but who had never actually done it. The activity was intended to lead students into a situation, in which they felt the need to justify a property whose proof requires algebraic manipulation. Students were asked to investigate properties of rectangles of the same type as the following ones:

7	13
9	15

3	9
5	11

After creating (in Excel) a ‘seal’ that generates such rectangles upon input of any number into the upper left cell, and after discovering and investigating properties of such rectangles, the students were presented with the following (and other) tasks:

1. *Try to find as many properties as possible common to all rectangles of this type.*
2. *In each rectangle that you generated by means of the seal, compare the sum of the numbers in the diagonals. What did you find? Do you think the rule you have just formulated is always correct?*
3. *In each rectangle you generated by means of the seal, compare the product of the numbers in the diagonals. What did you find? Do you think the rule you have just formulated is always correct?*
4. *If you build more rectangles similar to the above, will the properties you formulated in (2) and (3) hold for all these rectangles? Justify your claims and try to convince others.*

The property that the sums of the diagonals are equal (question 2) will from now on be called the diagonal sum property or DSP. The property that the difference between the products of the numbers in the diagonals equals 12 (question 3) will from now on be called the diagonal product property or DPP. An easy and natural way of justifying the DPP is to use algebraic manipulation and compare $X(X+8)$ (the expression for the main diagonal) to $(X+6)(X+2)$ (the expression for the secondary diagonal). In addition to reorganising their knowledge so as to arrive at a proof of the DPP, this activity presented two more opportunities for abstraction to the students. The first such opportunity is the construction of the extended distributive law $(a+b)(c+d)=ac+ad+bc+bd$: The students had never used that law yet but needed it in the present activity to transform the expression for the secondary diagonal. The second opportunity for abstraction more global in that it concerns the entire proof task, namely the establishment of the general perspective that algebra can serve as a tool for the justification of general properties.

We now present our analysis of that part of the Ha&Ne interview, in which the girls attempted to justify the DPP (lines 111-175). We pay particular attention to processes of emergence of knowledge structures, which we deem to be new and to give a new perspective for Ha&Ne. We show how the nested epistemic actions of constructing, building-with and recognising are indicative for such processes. We present evidence that, at the highest level, this part of the interview constitutes for Ha&Ne the construction of the perspective that algebra can serve as a tool for the justification of general properties. This level will be called level 1 and the corresponding constructing action will be denoted C_1 . This C_1 will be seen to be composed of a sequence of epistemic actions at a lower level (level 2); these level 2 actions are thus nested in C_1 . The level 2 actions are a constructing action (denoted C_2) of the extended distributive law, an extended building-with action (denoted B_2) of the proof of the DPP) and a sequence of recognising actions. Similarly to C_1 , the actions C_2 and B_2 are themselves

composed of B and R actions at a still lower level (level 3) that are nested in the level 2 actions. This nested structure of the epistemic action is schematically presented in the accompanying figure.

In response to Question 1, Ha&Ne produced ten properties, most of them elementary. They also included a divisibility property (the sum of all four elements is divisible by 4) but not the DPP. Interestingly, in response to Question 2, Ha&Ne stated the DSP without showing any urge to explain why it holds for all seals. After they stated the DPP in response to Question 3, the issue of explanation came up, and they proceeded to prove the DSP rather than the DPP.

They came back to the DPP when reading Question 4, where they are explicitly asked to justify their claim that the property is universally true. They start by a discussion whether or not they had justified the DPP earlier. Then, they embark on the computation of the first (main) diagonal.

The extended distributive law. The students were familiar with the simple distributive law. When applying it to the main diagonal to obtain $X(X+8) = 8X+XX$ they expressed their awareness of applying the law: *So, like, one does the distributive law* (H121). Shortly later, the following exchange ensued:

H133 *And this [thinks] ...*

N134 *It's impossible to do the distributive law here. Wait, one can do ...*

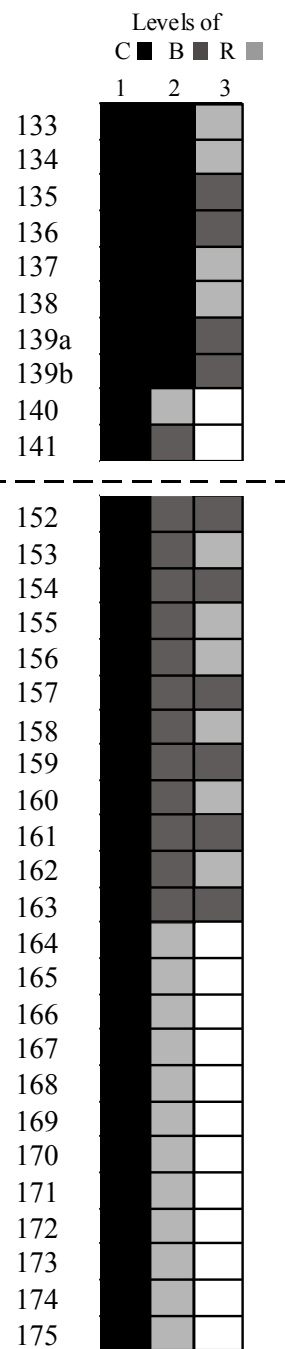
H135 *This is $6X$.*

N136 *This is $6X$ times X and $6X$ times 2.*

H137 *Wait, first, no ...*

N138 *Yes.*

H139 *No because this is X plus 6, this is not $6X$, it's different. Wait. First one does ... X ; then it's XX plus $2X$, and here $6X$ plus 24. Then ...*



In H133 and N134, the students focus on the second diagonal, recognising that it has a more complex structure than the first, and wondering whether it is possible to apply the distributive law here as well. While this is not something they know from earlier experience, they are motivated by the need to simplify this expression in order to progress toward their aim of justifying the DPP. The

recognition of the expression $6X$ (in H135), allows them (in N136) to use distributive law: $6X(X+2)=6X \cdot X+6X \cdot 2$. Although it is not clear from the transcript why the expression $6X$ arose, recognising it as a single unit gave the girls the necessary point of view to realise the applicability of the law, though not to the correct expression for the second diagonal. The correction of the mistake forces them to separate the 6 and the X (in H139); in spite of this added complexity, they are now able to obtain the correct simplification by building separately with the X and separately with the 6 and adding the results together. To the best of our knowledge, Ha&Ne had not seen the extended distributive law applied before this interview. Alternating Recognising and Building-With existing knowledge has thus allowed them to construct a new knowledge structure, the extended distributive law.

This process is represented in rows H133 to H139 of the accompanying figure as follows: At the lowest level (level 3), recognising and building-with actions alternate. There is no strict association of each statement to one of these actions since some statements (such as N138) cannot be categorised, whereas others (such as H139) include both, recognising and building-with. At the next higher level (level 2) these actions, taken together over the entire exchange from N134 to H139, constitute the construction of the extended distributive law. This is indicated in the diagram by C_2 : a constructing action at level 2. We note that no single action at level 3 has been identified as a constructing action; rather, constructing is a composite action in which (alternating) recognising and building-with actions are nested. We also note that what happens at level 2 cannot be fully understood without taking into account what happens at level 1; we have given a description of the students' actions without asking what drives these actions. We will relate to this issue below.

The proof of the DPP: After a brief digression (140-151), the students proceeded as follows:

- H152 *Ah, it's XX plus $8X$, but I don't know, like, how this will also be XX plus $8X$. Like, it has to be.*
- N153 *Is XX a square root?*
- H154 *I have the first part. This is XX , so this is OK.*
- I155 *Yes.*
- N156 *Is XX a square root?*
- H157 *... plus $8X$. Here I have $6X$...*
- I158 *Yes.*
- H159 *Ah, and $2X$, can I do this? Because $6X$...*
- N160 *Is XX a square root?*
- H161 *You can write this. Ah, yes, XX is X to the power 2, because it is X times X . Wait. XX is X to the power 2 plus $8X$, wait ...*
- N162 *Write this.*

H163 *Wait, it's X to the power 2 plus $6X$, plus $8X$, but there is also, like, plus 12. Ah, so, like, plus 12 because this is bigger by 12. Understand?*

At the start of this segment, most elements needed for the proof of the DPP had been attended to but the students may not have been presently aware of all of them. Some elements are technical such as the simplified expressions for the diagonals. Others are important for the flow of the argument such as the statement of the DPP and the plan to inspect the two diagonals, both of which had been explicitly mentioned earlier. In H152, the two diagonals again became the focus of attention. The plan is now more specific and detailed than was possible earlier. There is explicit reference to a comparison of the two diagonals and to the fact that the two corresponding expressions have to match. Ha systematically uses the above elements to build-with them the completion of the proof of the DPP. She alternates recognising (e.g., H154 and H161) and building-with (e.g., H157 and H159) actions at level 3, all of which are familiar to her. She combines them artfully but without the need to restructure her existing knowledge. The segment culminates in H163 when the number 12 becomes significant for the students as being the difference between the two diagonals. Since this corresponds precisely to the claim that had been made earlier, it completes the proof. According to our interpretation that this proof has been achieved by combining previously known elements without the need to restructure knowledge, we classified this segment as building-with at level 2.

The culmination of the Construction in H163 is followed by a segment of quite a different nature: The students review what Ha has developed during the previous segment. They arrive at a clear and convincing formulation of the proof. As a pair they are not constructing something new, nor even building-with the acquired elements but rather recognising, in the literal sense of recognising step by step the previously developed argument. As an individual, Ne might be constructing the proof of the DPP but we do not have enough evidence to make this claim.

Algebra as a tool for the justification of general properties. The culmination described in the previous paragraph completes the algebraic justification, and thus it also completes the C_1 construction in which it is nested. This C_1 construction, in spite of its dialectic nature, is carried out within the norms of mathematical proof through its algebraic and Excel notations. Hence, we claim that, in addition to the construction of the algebraic justification of the DPP itself, a more global and deeper construction occurs, namely the realisation that justifications may be expressed algebraically and that some algebraic computations (including the C_2 constructions) have to be done in the process. In this sense C_1 has a different nature from C_2 : From the beginning of their struggle to construct the DPP justification, the students are at the level of the C_1

construction. On this C_1 level their progress is controlled and monitored by their awareness and their need to accomplish the DPP justification. During this process, they face algebraic obstacles. Overcoming these obstacles necessitates the construction of new (to them) mathematical structures, which are the C_2 level constructions. These C_2 level constructions are controlled indirectly by the motive of the C_1 construction. They thus make the C_1 level into a deep holistic construction, which goes beyond the specific construction of the DPP justification, and in which the constructions of unfamiliar algebraic structures are nested. In this sense C_1 is an activity of vertically reorganising previously constructed mathematical knowledge into a new mathematical structure, the awareness of algebra as a tool for proof.

We now accumulate evidence to show that Ha&Ne have constructed this encompassing knowledge structure during the interview. There were three instances during the interview, prior to the segments we reproduced in this paper, in which Ha&Ne connected a justification to algebraic computations. In the first instance, just after the first explicit statement of the DPP, the students asked the interviewer whether they needed to explain. Although the interviewer did not give a clear answer, they produced an algebraic computation which, however, dealt with the sums rather than the products of the diagonal elements. The second and third instances occurred as responses to worksheet questions that explicitly asked the students to justify claims. To these requirements for justification, they reacted by starting algebraic computations. It is their need or wish to justify relationships that drives their computations, whether these computations amount “only” to building-with as in H152-H163 or whether they result in constructing as in H133-H139.

Interestingly, in the digression between these two segments, which are similar and somewhat technical in character, the students raise, and rather quickly discard again, the possibility that the computer could be used to convince the interviewer that the DPP is true. We interpret this as a sign that some uncertainty prevails as to what counts as a mathematically convincing argument. This reinforces our claim, based on the teacher’s report, that algebraic proofs are a novelty for the students.

The students complete the algebraic structure in H163, where they obtain two expressions that are identical except for the term 12 that appears only in one of them. The link between algebra and justification now works in reverse direction: The students realise that the term 12 is significant for the justification of the claim. Once again, the (result of) the algebraic computation is linked immediately and explicitly to the claim that had to be justified.

Finally, in the concluding segment of the interview, the link between explanation and algebraic formula is mentioned explicitly (see 173):

H173 *OK, we can explain each one and leave the formula.*

I174 Yes.

H175 [Writes down, with N's help, what they found.] *X to the power 2 plus 8X, and here, here it is, wait, these two are X to the power 2, this is 2 times X, plus 6 times X, plus 6 times 2. So together it is 8X and 2 times 6 equals 12.*

We thus infer that the design of the activity was successful for this pair of students and that their construction of the specific DPP proof led, in all likelihood, to the more general realisation that algebra is a useful tool for proving certain types of general mathematical claims. This realisation constitutes a new perspective that has arisen in the course of the interview and was driven by the students' desire to achieve the specific proof of the DPP. The links between justification and algebraic computation, which were acted out during the earlier stages, became conscious and explicit during the later stages. We also notice that because of the encompassing nature of the construction, this is a process going on during the entire interview rather than in specific identifiable segments. Nevertheless, we were able to identify a number of specific statements in which the students' awareness of the power of algebra for justification became more and more apparent. Almost all of the students' actions throughout the entire interview contribute to the justification and thus to the students' realisation that such a justification can be achieved by algebraic means. It is this realisation which led to the explicit statements at the end. Therefore, the epistemic action C_1 of constructing the power of algebra as a tool for justification comprises the entire interview and is composed of all the other epistemic actions nested within C_1 . These nested actions include not only recognising and building-with actions but also the lower level constructing action C_2 for the extended distributive law. All these actions taken together constitute C_1 , the construction of the power of algebra as a tool for justification.

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THINKING STRUCTURES INVOLVED IN MATHEMATICS LEARNING

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Abstract: *The structural cognitive learning process of a domain implies three distinct stages: internalizing what we call basic learning units, constructing the specific mental structures and practicing them in order to develop specific competencies. Every school subject appears as a logical structure, which needs to be learned as a structure, with its natural relationships among concepts. An effective learning needs specific training of the intellect capacities. Learning Mathematics as a school subject supposes the complex structuring of chunks of information to be internalized. Its focus is to create mental structures able to quickly generate significant models for problem solving strategies within the frame of the assimilation-accommodation process.*

During a long period of time, throughout the history of the mankind, the education system tried to create educational models for stable and rigid social-professional contexts. Today, the speed, the multitude, the complexity of changes do not permit the luxury to learn many things that do not have an immediate practical utility. This fact is much more dramatic as far as we do not know what knowledge will be useful in the next 10-20-50 years. The solution is to shift the relationship between information and formation by directly teaching mental abilities.

The study tries to superpose a cognitive perspective (Glaser, 1988) on recent constructivist approaches of learning (Glaserfeld, 1991). Connections between the piagetian assimilation-accommodation process, the theory of “functional organs” (Leontiev, 1981), the “everyday cognition” (Nunes, 1992), and the strategy of generating “powerful representational systems” (Goldin, 1987-2000; Kaput, 1987) have also been used in developing a learning model aiming learning at creating specific structures of thinking.

We call *structural cognitive learning (SCL) of a domain* the process by which abilities specific to the expert in that domain are created in the one who learns. This kind of learning implies a bipolar cognitive construction in the sense that, on the one hand, the domain which is to be taught is organically integrated in a *constructed structure* emphasizing clearly specified objectives, and, on the other hand, learning each subgroup of the structure implies an *active reconstruction* of its meaning (in the sense of the constructivist definition of

learning). Of course, the definition of the *structural cognitive learning* process should be taken into consideration with a series of precautions. The problem is not to teach the “novice” **everything** the “expert” knows; the focus is not at the informational level, but at the formational one: the one who is taught will be able to react when facing a problem according to his/her psychological age and knowledge, as the expert reacts when facing a theory in his/her field. The fact is possible because, while specialized information cannot be “lowered down” to young ages, beyond certain limits, the mental abilities **have no barriers**, they can be “lowered” from the expert level to any age, providing the accessibility of the information that “hides” those abilities and is operating with them.

Thus, for instance, in this paradigm, at the level of primary education in numeracy, the teaching – learning process is centred on sequences of natural numbers. The teacher and the student are creating, comparing, developing finite series of numbers with different starting points (from 1, from 2 etc. using the already known numbers), and different distances between points (counting twos, threes,... tens and so on). These series could be ascendant or descendent. The expert sees in the sequences of numbers the head stones of the numerical system. For the child, in this construction, the number appears both as a moment of the sequences and as a possible result of each elementary operation. In its turn, each elementary operation – as well as all the elementary operations considered together – can generate any number. In this case, operations are the nuclei around which the other information spins.

To make things clearer, in Fig. 1 it is described an example of a *basic mental learning unit*, which should be internalized at the level of grades I and II, according to the SCL process. The increasing sequence of even numbers could be generated starting from 0 by adding the same number (+2); but addition with the same number means multiplication. In this way, a passage from addition to multiplication and vice-versa is created through that sequence. Symmetrically, the decreasing sequence of even numbers could be generated starting from an even number by subtracting the same number (-2); but subtracting the same number till zero could be written as division. In this way, a passage from subtraction to division and vice-versa is created through that sequence. Together with the connection between addition and subtraction, respectively between multiplication and division through the inverse operation, the diagram is representing the four arithmetical operations with natural numbers in their interconnected relationships.

The experiments showed that if such a structure is built in the child’s mind at an early stage, the quality of her/his learning is spectacularly increased.

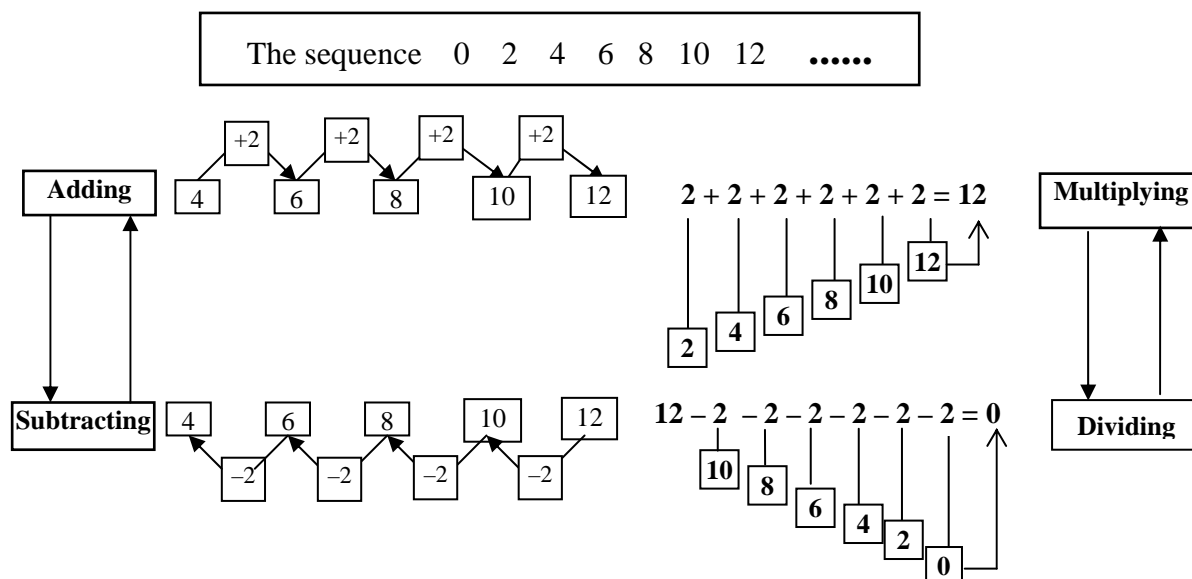


Fig. 1: An example of a basic mental learning unit

Finally, the primitive structure at the level of grade I essentially keeps a distant prototype of the expert thinking: the rapid reorganization of all the information from the perspective of any mathematical concept that is acquired. Such a pattern is very complex and its formation triggers special teaching problems – training mental abilities that permit a high mobility of concepts and shape their structure towards new ones, which are to be learned later on.

Cognitive science makes, in its way, the best use of the old learning theories, and this fact is positive, but not sufficient. In fact, human learning does not reduce itself to connections or perceptive level structures. The links (all kinds of connections taken into consideration by the classical theories of learning) express just superficial forms of learning because a new relation can be created, for example, not only between a known stimulus and an indifferent one, but also among already known stimuli, related in a way they have never been before. At the same time, a new piece of information can get into an informational context where it can take various positions: it can be added near the existing structure thus filling in a free valence; it can multiply, consequently supplementing valences, then it can partially or totally reorganize the existing informational system, lastly it can make up the basis for a new organizational system. In many cases, studied in a small degree yet, an organizational structure rejects the new information, obstructing in this way the learning process. If we take into consideration the above, then learning refers essentially, neither to information, nor to their mere connection, but to the development of structures.

The experimental application of the SCL process involved children from 6-7 to 14 years old with different abilities, from slow attainers to winners of national Olympiads. The research indicated to us that we could classify the mental structures generated by learning into: **rigid structures**, **mobile structures**, and **dynamic structures**.

To realise a description of these structures we take into account that a structure encompasses a *discrete component* representing the “nuclei” or the fixed, stable elements of the structure; a *contiguous component*, which could be “visualized” by a network and a *dynamic component* representing associations.

A **rigid structure** is characterized by over dimension, very stable nuclei, a poorly developed network, sometimes totally lacking, and associations that function in the area of recognition of a standard situation and its reproduction.

Such a rigid mental configuration is often exteriorized by the evolvment of fixations. The phenomenon frequently appears in classical geometry teaching; the student recognizes the isosceles or right-angled triangle only if it is in a certain position; any other position is perceived as a new learning element that requires a new nucleus in the structure. A rigid structure is usually generating the typical errors.

The emergence of such a structure in the learning process is the consequence, on the one hand, of teaching isolated information without spotlighting their connections in between and/or without allowing the necessary time for an internalizing process, that is required for creating a network and, on the other hand, of the excessive focus on already taught information, that also hinders the development of the network. A mental structure has a regenerative tendency to organize itself, a tendency that can be blocked only by the second above-mentioned aspect. In fact, the presence of that tendency actually explains the evolvment of learning even with the most inappropriate teaching.

A **mobile structure** is characterized by stable nuclei, a developed network, and by associations based on recognizing invariant elements in various environments. A mobile structure permits problem solving through analogy and inductive or deductive inferences when the context is partially familiar. Such a structure can enter into relations with other structures, ensuring a coherence of the reaction; it is typical for daily learning and could represent the ideal for learning school subjects such as history, geography, social sciences, etc. Nevertheless, such a model is no longer sufficient for learning mathematics in the post-industrial era.

A **dynamic structure** implies: flexible nuclei that are or could become structures in their turn; complex networks with ramifications and hierarchies; dynamic associations allowing the links between one structure and other ones, the relation to those structures, as a whole or in part, the self-development of the structure, the quick mobilization through the discovery of critical paths. Such a structure is extremely flexible; it can totally or partially multiply, it can temporarily migrate (while working out a task), or permanently migrate in other structures, by generating new stable functional configurations; it can become a part of some cognitive bodies together with fragments or multiplied aggregates

belonging to the old ones, within the self-creation of the intellect; it can relate any element of the structure with others from inside or outside, without disturbing the existing network.

The dynamics of the intellectual structures depends on their training to be mobile. Mobility training means to learn some special attributes of mobility: the ability to reconstruct the structure starting from any of its points; the ability to relate “a point” of the structure to any of the others; the ability to focus the whole structure on a given task in order to set new information in as many “points” of the structure as possible, or for the purpose of solving a given problem; the ability to easily reorganize the structure according to a certain working hypothesis and to a creative task; the ability to link it to another structure in the nearby or at a distance; the ability to give freedom to each “point” of the structure so that it might multiply and it could migrate into another structure (the maximum freedom of the elements in the structure); the ability to transfer the structure from an abstraction level to another one. The schemes contained in Fig.2 can suggestively express the differences among the three models of mental structures.

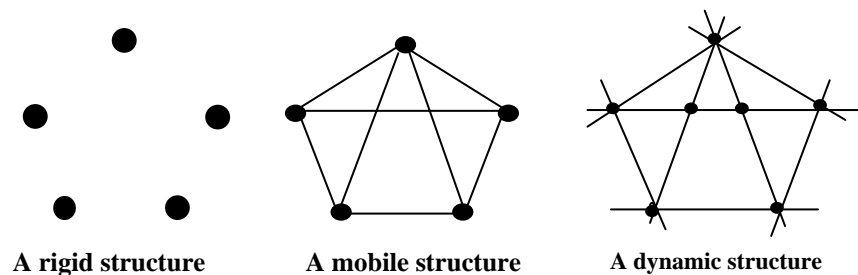


Fig. 2: A representation for different types of structures

In a first stage of concept learning, a mental configuration is created. Initially, this configuration is unstable; any new information could perturb it. The learning occurs if the information comes into the configuration and completes it i.e. if it is producing dynamic and contiguous connections in the brain, which generate an acquisition of knowledge and skills with a certain stability. The classical training, based on excessively relying on memory for learning, makes the stability stronger. Moreover, if the information transmission deepens the stability of the configuration, it becomes a negative factor; this is the critical point for training the mobility. Maintaining a balance between the stability and the mobility of the mental structures generated by learning becomes a fundamental problem of a well-driven training.

Since the SCL process of a domain was defined as the conceptual assimilation process of that domain with the purpose of creating, at the intellectual level of the one who learns, a similar behaviour to that of the expert, it is natural to explain what the *results* of that kind of learning are. By **output mental capacity** or, in short, by **competency** we understand the *intellect's*

potency to optimally mobilize its bio-psycho-physiological resources in order to solve a problem within a certain field, or having a specific feature.

According to the above terminology, it is a matter of the created mental structure's potency to **quickly generate and select significant and adequate models** for a problem-solving situation within the frame of the assimilation-accommodation process (Piaget, 1971).

To each organizing set of elements from the reality corresponds an organizing block of knowledge and understanding irreducible to its components.

A cognitive schema consists of structured knowledge simultaneously activated, corresponding to a real situation. Any type of learning generates mental models.

We differentiate three types of models, according to the degree of conformity to the outside realities (the degree of correlation with the real world).

- *“Trustful” models.* In the real world we find situations that correspond almost totally to the existent model. The role of these models is important for human life. If every object we came across during an ordinary walk was new, then we would either give up all attempts of advancement or we would stop each time to clear it up, consequently there would be no advance at all. Even during the most unusual trips, the things we regard as new are not numerous. Thus, in ordinary life we do not pay much attention to what we come across unless it is something really new to us, or it causes problems as never before (a tree fallen over the night, an acquaintance unusually dressed, something which raises our interest, etc.).

In education, it is the same: while reading a book in our field, we are not interested in what we already know; we stop and ponder only on the new elements.

- *“Adapting” models.* In most cases, the reality is not exactly the same as the model and therefore we need to correct the latter. The correction may be insignificant, temporary, but also important and permanent. In the second case, the intellectual structure acting as a model is completed or a new model is generated.

- *“Self-generating” models.* New situations create (temporarily or permanently) new intellectual structures. The existent intellect structure is amplified in two ways: one is by creating new structures with the elements of the old structures combined in a different way, or assembling particular structures according to new tasks; the other one is by assimilating unknown information which requires new structures.

The accommodation of mental structures through adequate models is expressed at the level of each individual by the manifestation of some specific competencies. In the case of the SCL process, the initial stimulus is neither a simple “S” (connectionism), nor an “S” chain (new-behaviorism), nor is it a chain of perceptive or intuitive structures (Koffka, 1935; Kohler, 1940). The stimulus is a structure of learning units of the nature of the one we presented in Fig.1. As far as the domain is assimilated, from an information group to another, the mental models extend and grow while being refined and connecting to each other in increasingly complex structures.

In real teaching, the limits of building structures are not so clearly marked. At the first contacts with a new field, the focus is laid on internalization, on primary development of the mental structures and on practicing them in simple situations. Later, the internalization continues, as new information is assimilated, but the focus lies on their integration in mental structures and on practicing them so that they might become extremely mobile in three directions: able to *multiply* (reproduce) on higher levels of abstraction, able to *integrate* in new structures having the same nature or different natures, and able to *mobilize* with great precision when there is a need of them in solving some practical tasks (by their appropriate reduction to the already developed intellectual models, or by their temporary changing for the purpose of solving totally new tasks). In the process of practical training, the models confront the external requirements and they are identified as being known or unknown; in the second case, finding a solution implies a creative effort.

In the external (physical) constructions, the critical paths are drawn up according to a series of constraints – a block of flats cannot be built starting simultaneously with the foundation, roof, rooms, stairs, etc., and this is not because it would not be economical (in fact it would be very economical), but because it is physically impossible. However, a construction on the mental plane is of a totally different kind; it permits developments, expansions; the construction may start and develop simultaneously in several points. Moreover, the accumulation of information, even a structured one, is not enough to create competencies. Simultaneously with the internalization of the basic structure, for developing a SCL process, one applies operations aiming at thoroughly fixing and continuously increasing the mobility of each constitutive element of the created structure. The attributes of the assimilated model will be in the end: the capacity to multiply (on the whole, partly, or only some of its elements) within any new structure to which it is related by sense, no matter whether it is a structure in the same domain or not, without affecting the basic model (without weakening or deconstructing it). The parts of the model will have the same degree of dynamics, that is the whole can be restored mentally from any point, one may apply to it the most varied thinking operations while taking over new information or solving problems.

In order to develop a SCL process, the focus on the internal structures that are created as a result of learning becomes compulsory. The specific training included in the teaching practices leads us to the emergence of an **over-learning** (efficient, effective and creative learning) phenomenon.

While the physical time cannot be expanded, the didactical time can be expanded, or, on the contrary, contracted according to the method that teachers use, the philosophy they start from, the quality of the didactical technologies they apply. The SCL strategies dilate very much the didactical time by the mere fact that thus permit the learning of intellectual capacities, that there is no need to wait for their spontaneous formation with a waste of time, because these - the mental capacities - become, from the very beginning, instruments for assimilating information in what we call *Learning thinking*.

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BUILDING A FINITE ALGEBRAIC STRUCTURE¹

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Abstract: *The paper deals with the process of building an inner mathematical structure. This process is illustrated by the example of a non-standard arithmetic structure, through reflections by the first of the authors. The key aspects of the process are identified and the specifics of a structure built as an analogy to the existing structure are presented.*

1. Introduction and framework²

The paper focuses on the process of building an inner mathematical structure (hereinafter IMS) investigated via the construction of a new structure as an analogy to an existing structure. The process of constructing an IMS is a mental activity, i.e. it is not directly observable and thus “presents a methodological problem because construction is a relatively rare event [...] these events might often occur when students sit alone and think hard about mathematics” (Hershkowitz, Schwarz, Dreyfus, 2001). Concept maps have been used, for instance, for investigating how the inner mathematical structure grows and changes. In our research, we used (i) methods of introspection, (ii) think-aloud interviews, for this purpose.

It is widely acknowledged by mathematics educators that the cognitive processes of concept acquisition do not have to follow mathematical logic. It is quite possible that while mathematicians and mathematics teachers consider the structure of mathematics logical and “beautiful”, students find it “fragmentary and discontinuous” (Merenluoto, Lehtinen, 1999). Nevertheless, university mathematics is often taught in the sequence: definition – theorem – proof – illustration, “with little opportunity for developing a full range of advanced mathematical thinking” (Tall, 1995).³ There is a substantial body of research concerning the transfer between elementary and abstract mathematics whose

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²The contribution reports on the research which has been going on in the framework of research into the internal mathematical structure presented in more detail in Milan Hejny's paper within this working group.

³In his dissertation, Lakatos (1976) tried to show by analysing in great detail the process of formulating and solving certain mathematical problems that formalism is at variance with reality, that formal structure of mathematics only plays a secondary role in the process of doing mathematics. The present teaching of mathematics does not seem to account for this.

essential part, “the move from the object \rightarrow definition construction to definition \rightarrow object construction” (Gray et al, 1999), presents a serious cognitive difficulty. Our experience confirms some research findings that this is also a problem in group theory (see e. g. Dubinsky et al., 1994, Almeida, 1999, Hazzan, 1999). In this paper, the illustration will be given of how “parrot-like” knowledge (i.e. knowledge which was gained only formally without real understanding) is vitalised when it enters a creative mathematical process in an active way. This idea confirms the need to adhere to the ‘necessity’ principle when presenting new knowledge (Harel, Tall, 1989).

2. The tool of research

The tool of our investigation of an IMS is an arithmetic structure $A_2 = (\underline{A}_2, \oplus, \otimes)$ which we call *restricted arithmetic*. It was elaborated by Milan Hejny especially for the purpose of investigating an IMS on the basis of *analogy*. There is a strong analogy between A_2 and an “ordinary arithmetic” of integers $(\mathbf{Z}, +, -, \cdot, <, |)$. All numbers in \underline{A}_2 will be called *z-numbers*⁴.

The gate to the restricted arithmetic is the mapping $r: \mathbf{N} \rightarrow \mathbf{N}$, which we call the *reducing mapping* and which can be introduced in at least two ways. The first one is more “mathematical” and the second has been used in our experiments.

1. Reduction r is mapping $r: \mathbf{N} \rightarrow \mathbf{N}$, defined as $r: n \rightarrow n - 99 \cdot [n/99]$, where $[y]$ is the integer part of $y \in \mathbf{R}$.
2. Reduction r is introduced as an instruction:
 - (i) for $n < 100$, $r(n) = n$,
 - (ii) for $n \geq 100$, split n into two parts: the last two digits and the rest. Add these two numbers. Repeat the process until you get a number from 1 to 99.

For instance, $r(7305) = 73 + 05 = 78$,
or $r(135728) = r(1357 + 28) = r(1385) = 13 + 85 = 98$, etc.

Let us have the set $\underline{A}_2 = \{1, 2, 3, \dots, 99\}$. The reducing mapping r is used to introduce binary operations of *z-addition* \oplus and *z-multiplication* \otimes in \underline{A}_2 as follows: $\forall x, y \in \underline{A}_2, x \oplus y = r(x + y)$ and $x \otimes y = r(x \cdot y)$.

For instance, $72 \oplus 95 = r(167) = 68$, $72 \otimes 95 = r(6840) = r(108) = 9$.

⁴Prefix *z* will be used for concepts / operations in restricted arithmetic. It comes from the Czech language.

3. Methodology

During the last three years, a number of ‘think-aloud interview’ experiments have been conducted (partial results have been reported in Stehlíková, 1997, 1998, 2000). Subjects have been presented with A_2 as shown in part 2 and then asked to solve some tasks, mainly linear equations. Moreover since 1997, the first of the authors herself has investigated the structure of A_2 . The results presented in this contribution originated mainly through her introspection. Introspection was chosen because we believe that by studying ourselves from the inside we can make inferences about the mental processes of other people, we “develop sensitivity” (Mason, 1998). “By introspection we mean constantly seeking to discern our individual perceptions of experiences, both past and present, and our reactions to them” (Duffin, Simpson, 1997).

While investigating A_2 , the first of the authors made introspective notes and collected all her solutions of the tasks, which she mainly posed herself. Then her whole investigative process was divided into smaller parts according to the main topic studied. It was sometimes difficult to do so as all topics are interconnected. These parts were then decomposed into phases and in order to get a clearer picture of the process, they were described in the form of a table. A part of the table can be seen in section 5. Each phase was described by one line. The table has been reorganised several times mainly because it transpired that it was not detailed enough. Via the process of building the table, various phenomena have been identified. Some of them will be presented below. Where applicable, reference to experiments with students will be made. Introspective remarks will be written in first person singular form.

4. Phenomena of the process of building an inner mathematical structure

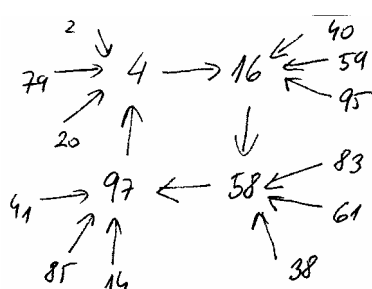
In this section, some results will be presented which we gained via introspection. Because it is the purpose of this paper to focus on phenomena particular to the building a structure as an analogy, we will only present one phenomenon which is attributable to the process of building an IMS as such.

4.1. Looking for an organisation principle

Throughout the process of investigating A_2 , the need frequently arose to reorganise the work when the amount of information grew too large. For instance, the task “solve quadratic equations” naturally led to the necessity to determine which z -numbers were squares. The list of squares was made which brought about much different information that had to be organised somehow or linked together. In this case, the organisation was achieved through determining

regularities and anomalies in the list of squares and investigating them in detail afterwards (see the table below, C1, C2).

Visualisation belongs among organising principles as a way of getting insight into the whole structure (Gestalt). *When investigating a list of squares I wanted to make it more informative so I made an arrow diagram of all squares and hence square roots. Its repeated redrawing yielded an instructive picture consisting of nine suggestive clusters (one of them is in the figure below – $2^2 = 4$, $79^2 = 4$, $20^2 = 4$, $97^2 = 4$ etc.) which both motivated and facilitated further investigation, particularly of important subsets (see the table below, lines F – M).*



Note: In March 1999, about a year earlier, my tutor showed me a similar visualisation but I was not ready for this information yet. I had not done much experimenting with squares and I did not feel the need to draw a diagram. I could not understand what he was talking about. I had to construct it for myself.⁵

Discovering and applying a suitable organisation principle can be, and usually is, a lengthy process.

Two approaches to the organisation can be distinguished:

1. utilitarian approach – we use the organisational principle which is most convenient for the given purpose, e.g. the original simple list of all squares is more convenient if we want to solve a given quadratic equation. The diagram, however, is more informative if we want to get an insight into the structure of squares.
2. axiomatic approach – we want to find as concise a picture of the situation as possible; take the diagram of squares for example, we can go even further and describe squares in an even smaller number of rules provided that we discover the rule of how to infer from one equality $2^2 = 4$ all four equalities (i.e. $2^2 = 4$, $79^2 = 4$, $20^2 = 4$, $97^2 = 4$) or six equalities (when zero divisors are involved).

Let us give one more example of this phenomenon. The sets of numbers (clusters) from the diagram of squares were put under close scrutiny as to the properties of their elements, closure with respect to basic operations, etc. This study yielded much information of a different kind, which again brought about the need for a new organisation of pieces of knowledge. This time, the objects which were to be organised, were not just numbers, but sets of numbers or even some regularities. Subsets of A_2 which were additive or multiplicative groups

⁵Apparently I had not reached a sense of intimacy with the problem yet (see Carlson, 2000).

were identified and needed to be summarised. Visualisation of these subsets was achieved through a table and they were completed using mathematical theory.

4.2. Specifics of building a structure as an analogy

When building a new structure as an analogy to an existing structure, the new structure is usually richer in concepts and ideas and enriches the original structure. The connections between the two structures are being made at the level of objects, operations, strategies, problems, etc. while the differences and similarities are being recorded. Our experiments have shown that students' success in their investigations of A_2 depends to a great extent on their ability to distinguish between these two structures.

4.2.1. Regularities in the new structure

Regularities are facts which are true for a greater set of objects. Regularities in the new structure (i.e. the structure which is being built) can be divided into regularities (a) parallel to ordinary arithmetic (for instance, additive inverses have the same square in A_2), (b) non-parallel to ordinary arithmetic (for instance, the fact that $(AB)^2 = (10 \otimes AB)^2$, where A, B are digits such that number AB is not zero).

4.2.2. Anomalies in the new structure

Anomalies in the new structure are rules, properties, numbers, etc. that contradict what we know from ordinary arithmetic, we can describe them as a violation of the parallel. At first they seem to be isolated. For instance, number 55 appeared like a single anomaly first due to its property that $55^2 = 55$. Then later it appeared as one possible unit element in a subgroup of the multiplicative group. A similar thing happened with number 45. Thus numbers which equal their squares and can be a unit element of a multiplicative group make a set $\{1, 45, 55\}$.

If an anomaly is a number, it can become an element of some set of other numbers / anomalies with the same properties. If an anomaly is a property or a rule, it can be later perceived as a non-parallel regularity (see the example above). Non-parallel regularities usually take a long time to emerge, for instance, *it took me several weeks before I suddenly understood that the two rules $(AB)^2 = (BA)^2$ and $A^2 = (A0)^2$ can be expressed by one rule $(AB)^2 = (10 \otimes AB)^2$ where number AB is not zero.*

Anomalies are important phenomena of the process of structuring because they are usually perceived more sharply than regularities which confirm the parallel. First, they are the source of a **cognitive conflict** for a student and of the

need to re-arrange his/her IMS. Second, anomalies are accompanied by **surprise** which greatly contributes to the person's motivation – he/she wants to learn more and investigate the causes of anomalies. It is one of the driving forces of the investigation of the new structure.

4.2.3. Broadening intuition

Illustration: After discovering the fact that anomalies do not have to be singularities, that they may bunch together and thus create their own structure, anytime I encountered any new anomaly I half expected to find a whole set of anomalies which had something in common.

Both introspection and experiments with students⁶ confirm our hypothesis that while at the beginning the only intuition which students probably have is that concerning ordinary arithmetic (intuition of regularities in ordinary arithmetic), after several similar experiences, they broaden this intuition. As a consequence, the emotional power of anomalies diminishes as they are already expected to occur.

4.2.4. Analogies as obstacles

If the new structure is built as an analogy to an existing structure, which the student is familiar with, the connections between these two structures can either help or hinder the work in the new structure. It usually takes quite some time and needs some erroneous processes before the solver realises that he/she must distinguish precisely whether the method he/she is using is based upon analogy or upon structural investigation within A_2 . For example, students often infer that because in ordinary arithmetic the neutral element for addition is 0 and 0 is not an element of \underline{A}_2 , hence there is no neutral element for addition in A_2 .

4.2.5. Developing new strategies

Our experiments have revealed that when students start investigating the new structure, they usually stick to the strategies they know from ordinary arithmetic and use them without scrutinising them. Only after they had been put in a situation in which a certain strategy failed to produce the correct answer, did they begin to realise that they had to be careful with the transfer of strategies from ordinary arithmetic (they developed a new intuition). For instance, such a situation invariably occurred when they were presented with the equation of the following type: $x \oplus 68 = 4$. Their answer was that either $x = -64$ or that it had no solution because there were no negative numbers in A_2 .

⁶By students, we mean those who have no or little experience of creating a parallel structure. Our experiments have been carried out with students – future mathematics teachers who were in their first year mainly.

The situation can be put to good use by challenging students that there certainly is an answer. It often leads to their deeper analysis of what it means to solve a linear equation, that it is not the rule “take number 68, change its sign and put it on the right side of the equation”. Moreover, the student’s knowledge is being restructured in that a negative number is not perceived only as a number with the minus sign but rather as a structural element which has certain properties which can be transferred to a different structure. In other words, they realise that even though negative numbers in \mathbb{Z} and additive inverses in A_2 differ in appearance, they are the same on the structural level. In some cases this led to the discovery of zero element and additive inverses in A_2 . Thus, restricted arithmetic often enables new concepts to be introduced when needed (necessity principle (Harel, Tall, 1989)).

On the level of metastrategy when working in the new structure, one gradually builds new ways of working. For instance, *illustrated in the table below is how I developed a way of investigating general powers, through the investigation of squares.*

5. Time sequence of building an IMS – illustration

In this section, we will describe, in the form of a table, a part of the investigation of A_2 , namely the study of powers and its consequences. The table has been constructed via introspection. The process of investigation has been decomposed into phases and each phase is described by one line in the table. *In the second column, the task I gave myself is described, whilst in the third column, the process of its solution with selected results⁷ is presented.*

	Task/challenge	My activity
A	Solve quadratic equations	<i>I applied the standard formula and this led me to the necessity to find square roots and hence squares. (1997)</i>
B	Make a list of squares.	<i>I made a list of squares and investigated it. (1997)</i>
C	Investigate the list	<i>I noticed (a) anomalies, (b) regularities. (1997)</i>
C1	Find all anomalies	$45^2 = 45$, $55^2 = 55$, $22^2 = 88$ and $88^2 = 22$. Hence $45^n = 45$ (for $n > 0$), $22^3 = 55$.
C2	Find all regularities	1. $(AB)^2 = (BA)^2$, 2. $A^2 = A0^2$, where A, B are non-zero digits, 3. $AB = 10 \otimes BA$, at least one of the digits A, B is non-zero, 4. additive inverses have the same square.
D	Justify the results	<i>I proved the regularities above. (1998)</i>

⁷Only the results which we consider, at this stage of research, fundamental with respect to investigating IMS will be presented. The year given in each line roughly corresponds to the time when I worked on the task. It does not mean that it took a year to prove regularities, for instance, but rather that I felt the need to prove them only after some time.

E	Organise the list in a more meaningful way	<i>I made an arrow diagram of all squares and hence square roots (see the figure above). The diagram consists of nine suggestive clusters. What does it mean? (March 2000)</i>
F	Investigate the diagram	New questions arose. How can the individual clusters be characterised? What is the property of numbers with the loop ($45^2 = 45$, $55^2 = 55$, $1^2 = 1$, $99^2 = 99$)? What is the connection among numbers belonging to one cluster? having the same square? (March 2000)
G	Work on questions which emerged in F	<i>I divided the diagram into several subsets of numbers, I called them important subsets (hereinafter IS). These were numbers which are (are not) zero divisors, numbers with the loop, numbers from one cluster. (May 2000)</i>
H	Study IS of squares	<i>I realised that the main characteristics of IS is the closure under addition and/or multiplication and neutral elements. I investigated it for all identified subsets and identified some groups among them. (May 2000)</i>
I	Find other groups	<i>I decided to study third powers in the same way. (August 2000)</i>
J	Study third powers and their subsets	A diagram similar to the diagram of squares, IS of third powers and the identification of subgroups. <i>I felt the need to make the study more systematic and it occurred to me that the study of general powers might help. (August 2000)</i>
K	Study general powers	<i>I investigated general powers of zero divisors and non-zero divisors. I classified all z-numbers according to the length of the period and found out that the sets $M = \{a^k, a \neq 99 \text{ is a non-zero divisor, } k \in \mathbb{N}\}$ form the group under multiplication. (1998, August 2000)</i>
L	Summarise all subgroups	A table of subgroups of the additive group (\underline{A}_2, \oplus) of the order 1, 9, 33, 99 and a table of the subgroups of the multiplicative group (\underline{G}, \otimes) of the order 1, 2, 3, 5, 6, 10, 15, 30 was created where \underline{G} is a subset of non-zero divisors without 99 of \underline{A}_2 . (September 2000)
M	Look for theory	Lagrange's theorem ⁸ used for determining the order of possible subgroups. Subgroups of some orders are still missing. (Sept. 2000)

6. Concluding self-reflection

As a university student, future mathematics teacher, I successfully passed through all main mathematical subjects and I was given a lot of deep concepts, difficult theorems and proofs and complex theories. But I have never before had such a complex and long-term experience of doing mathematics. My experience with restricted arithmetic completely confirms that “it is the relationship

⁸I already knew Lagrange's theorem but I had never used it in as sensible way as this. I saw for the first time that this theorem could play such an important role in the process of doing mathematics.

between the learner and a problem that is of significance, not the perceived level of the problem as viewed within some hierarchy of abstraction” (Geiger, Galbraith, 1998). My cognitive and meta-cognitive behaviour exhibits nearly all the phenomena described in the study of the mathematical behaviour of mathematicians (Carlson, 2000).

If I analyse my three-year experience, the most important matter which emerges is that my attitudes towards mathematics both as a branch of science and a school subject have changed. My prejudice that mathematics is a rigid, ready made, never changing subject has been violated. What are the key factors which contributed to this change?

- a. The context is suitable for me with respect to my mathematical knowledge and abilities, it is easy enough to be able to carry out real mathematical procedures nearly on my own, and on the other hand, it is difficult enough to bring sense of real achievement and joy.
- b. The analogy with ordinary arithmetic allows me to pose questions and develop solving methods myself.
- c. The context is rich and open enough. The topic in this form cannot be found in literature; it is generally elaborated as a part of the complex theory of modular arithmetic, which would mean that first the theory of congruences, modulo a composite number, would have to be mastered (in the way “definition – theorem – proof”).
- d. I can see the usefulness of some concepts and theorems of abstract algebra, previously learnt formally and now understood. If such knowledge enters the investigation in an active way and helps to solve a problem, this brings satisfaction and joy with motivating consequences.
- e. My fear of making a (stupid) mistake diminished and I used mistakes to diagnose my problems.

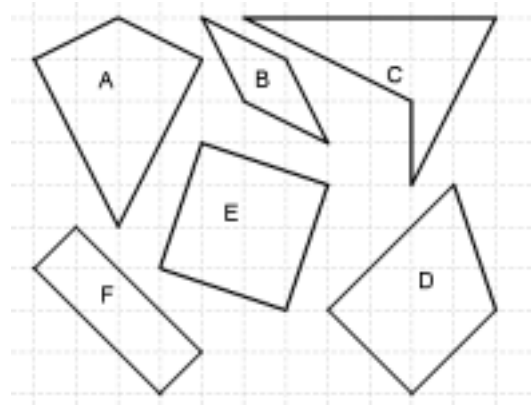
7. Structuring in geometry

Our research in building IMS has been extended to the geometry of figures. This is the preliminary information of our approach and the tool used.

Contrary to arithmetic where the operations are the centre of attention, the basics of geometric structuring are geometric objects, mainly figures and their properties. The first experiences have revealed that the effective tool of both research and teaching is a game in which one of the players secretly chooses one of the figures from the given set. The other players ask yes-no questions concerning geometric properties of the figure and try to guess which one it is.

For instance, the game related to the figure below went as follows. It was played with students – future elementary teachers.

	Question	Answer
1.	Does it have four vertices?	Yes, all have four vertices.
2.	Is it non-convex?	No.
3.	Does it have a right angle?	Yes.
4.	Are the two opposite sides equal?	No.
5.	Is it a trapezium?	No.
6.	Does it have precisely two right angles?	Yes.
7.	Is it symmetrical?	Yes.
8.	It is a deltoid.	Yes.



The quality of questions can be judged from the point of view of both geometry and strategies. For instance, question 1 points to the erroneous image of the inquirer – she only saw three vertices in the figure C. The concept of vertex is connected with a tactile perception of pricking. From a strategic point of view, only question 4 is good. It divides the remaining set of four figures A, D, E, F into two equally numerous parts – (A,D) and (E,F). Deeper analyses via the two presented phenomena enable to diagnose students' geometric images.

Under the structure in the geometry of figures, we mean the correspondence

1. property \leftrightarrow a suitable subset of figures,
2. negation of properties \leftrightarrow complement of a suitable subset of figures,
3. conjunction of properties \leftrightarrow intersection of a suitable subset of figures,
4. disjunction \leftrightarrow union of a suitable subset of figures.

The investigation which at present concerns finite sets of figures will concentrate on infinite sets, too.

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INTUITIVE STRUCTURES: THE CASE OF COMPARISONS OF INFINITE SETS

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Abstract: *This paper investigates the intuitive structures (i.e., primary intuition and secondary intuition) of three groups of prospective teachers by examining their responses to comparison- of- infinite-sets tasks. It reports on their intuitive tendencies to regard one-to-one correspondence, inclusion and single-infinity as acceptable criteria for the comparison of infinite sets. It also investigated their tendencies to claim that these methods could be used interchangeably for such comparisons. The findings show significant differences between participants who had studied a traditional or an enrichment Cantorian set theory course and those who had not participated in any course. Those who had taken a course frequently accepted one-to-one correspondence as a general method for comparing infinite sets and usually stated that one method should be used for such comparisons. The participants in the enrichment course performed better than the ones in the traditional course. Still, a non-negligible number of those who had studied set theory still claimed that inclusion and single infinity are also suitable for comparing infinite sets.*

This paper examines secondary-school prospective mathematics teachers' intuitive structures when dealing with infinite sets. That is to say, the paper investigates prospective teachers' primary intuitions and their secondary intuitions regarding infinite sets. It presents several aspects of Fischbein's theory of intuitive cognition (see, for instance, Fischbein, 1987), and relates to Fischbein's claim that the structure of "appropriate" secondary intuitions can be created by "good" instruction. The paper focuses on two issues: (1) what are the primary intuitions of prospective secondary-school teachers of mathematics regarding the comparison of infinite sets? (2) What is the impact of different types of instruction on participants' intuitive structures?

Intuition plays a crucial role in mathematics education. It is highly recommended to take into consideration students' intuitive ways of thinking when teaching (e.g., NCTM, 1991; 2000). In his book: "Intuitions in Science and Mathematics: An Educational Approach", Fischbein (1987) presented a theory wherein he defined the notion of intuition and the essential role it plays in

students' mathematical and scientific thinking processes. According to Fischbein, intuitive knowledge is a self-explanatory cognition that we accept with certainty as being true. It is a type of immediate, coercive, self-evident cognition, that leads to generalizations going beyond the given.

Fischbein differentiated between primary intuitions and secondary intuitions. Primary intuitions were defined as intuitions that “develop in individuals independently of any systematic instruction as an effect of their personal experience” (Fischbein, 1987, p. 202). Secondary intuitions were defined as “those which are acquired, not through natural experience, but through some educational intervention” (Fischbein 1987, p. 71). Secondary intuitions were defined as evident when formal knowledge becomes immediate, obvious, and accompanied by confidence. Secondary intuitions about a certain concept or process are often inconsistent with the related primary intuitions about the same concepts. Questions that arise naturally are: How can we identify students' primary intuitions? What intervention can be used for the acquisition of secondary intuitions? (How does something we learnt become intuitive?) What can we do in order to help our students develop them?

The present study addresses the mathematical notion of actual infinity. Actual infinity plays an important role in various mathematical theories, and it is one of the mathematical notions that was identified by mathematicians, philosophers and psychologists as being counter-intuitive. The study examined the impact of two types of interventions on students' primary intuitions. Research findings indicated that the methods that students applied for the comparison of infinite sets were largely influenced by methods they had used when comparing finite sets. Still, they usually did not use one-to-one correspondence, the criterion that should be used to determine the equivalency of two infinite sets within Cantorian set theory (Borasi, 1985; Duval, 1983; Fischbein, Tirosh, & Hess, 1979; Fischbein, Tirosh, & Melamed, 1981; Martin, & Wheeler, 1987; Tall, 1990; Tsamir, 1999; Tsamir, 2000; Tsamir, & Tirosh, 1994; 1999; Yehoshua, 1995). It was also found that when students used more than one method for comparing infinite sets, they reached contradictory conclusions, of which they were usually unaware (e.g., Tirosh & Tsamir, 1996; Tsamir & Tirosh, 1999).

The findings described so far related to the structure of students' primary intuitions of actual infinity. Fischbein's assumption, however, was that, under instructional intervention it is possible to develop a new structure of logically based interpretations, i.e., secondary intuitions, and that these would supersede the primary intuitions. In this context, one may wonder what the interventions are that could be considered for developing students' “appropriate” secondary intuitions of actual infinity.

A number of studies examined the influence of different interventions on students' acceptance of one-to-one correspondence as the single method for comparing infinite sets (e.g., Sierpiska & Viwegier, 1989; Tirosh, 1991; Tsamir & Tirosh, 1999; Tsamir, 1999; Tsamir, 2000; Yehoshua, 1995). Nevertheless, these interventions were tried with secondary school students, who were not obliged to study Cantorian set theory as part of their curriculum. Nowadays, in Israel, Cantorian set theory is customarily taught at the college level to mathematics majors and to prospective secondary mathematics teachers. It is usually presented in a traditional, lecturing manner. The present study examined the impact of two types of courses on prospective teachers' intuitions of actual infinity. One was the traditional course (T-ST), and the other, an enrichment course, that took account students' intuitive tendencies to overgeneralize from finite to infinite sets (E-ST).

THE RESEARCH

Participants

Participants were prospective secondary school mathematics teachers, who studied at Israeli State Teacher Colleges. Seventy-one of them had never studied Cantorian set theory [N-ST], 110 had completed a year-long, traditional set theory course [T-ST], and 125 had completed a year long enrichment set theory course [E-ST].

The T-ST and the E-ST Cantorian Set Theory Courses

The T-ST course consisted of 24 sessions of 90 minutes each. The syllabus of this course included the following topics: the notion of 'set', finite and infinite sets, the cardinal numbers of infinite sets, axiomatic development of set theory, relations and operations between sets, functions, the comparisons of the number of elements in infinite sets, cardinal numbers, well-ordered sets, ordinal numbers, axiom of choice, Zorn's lemma and paradoxes in set theory. This course was presented in a traditional, formalized manner with little or no emphasis on intuitive aspects and usually ignoring "peculiarities of the infinite".

The E-ST course also consisted of 24 weekly class sessions of 90 minutes each. The first sessions were devoted to discussing connections between mathematics and reality; the axiomatic, independent nature of mathematical systems and the crucial role consistency plays in determining mathematical validity. The other sessions of the course related to the topics studied in the T-ST course. The transition from finite to infinite sets was introduced via several activities, aimed at promoting prospective teachers' awareness of their own

alternative conceptions (Tsamir & Tirosh, 1999; Tsamir, 1999). The historical development of the notion of actual infinity and its connection to students' intuitive conceptions were also discussed (e.g., Borasi, 1985). The discussions were then extended to other intuitive beliefs about mathematical notions (e.g., “zero is nothing”, “a set includes at least three elements”) illustrating their impact on common responses to related tasks.

Instruments

The prospective teachers were asked to answer, in writing, a five-part questionnaire. Here one part, which consisted of two tasks, is discussed. The first task provided an explanation illustrating the notions of one-to-one correspondence, single infinity and inclusion, written as if presented by different students in a class.

In a class, students presented the following claims:

Betty: Consider a huge dance hall, where couples (a man and a woman in each pair) are dancing. There is nobody in the hall, but the dancers. We can easily claim that the number of men in the dance hall is equal to the number of women, since each man is paired with a single unique woman and each woman is paired with a single unique man.

Similarly, when each element of set A can be paired with a single unique element of set B , and every element of set B can be paired with a single unique element of set A , then the sets have the same number of elements (to be called “one-to-one correspondence”).

Danny: When set A is included in set B , i.e., set B consists of all the elements of set A and at least one additional element, then the number of elements in set B is greater than the number of elements in set A (to be called “inclusion”).

Tom: All infinite sets are equivalent, all have the same number of elements (to be called “single infinity”).

Participants had to declare in writing, whether they viewed each of the above mentioned methods as suitable for comparing infinite sets.

The second task asked the participants to express their opinion on whether a specific one of the three above-mentioned methods should always be chosen for all for the comparison of infinite sets; or whether each problem should be examined separately, and any applicable method be chosen.

Process

The participants were given about 30 minutes to answer this part of the questionnaire in writing. After completing the written assignment, 10 N-ST, 10 T-ST and 10 F-ST participants were interviewed orally, in order to get a better insight into their thinking. Primary intuitions were investigated via the reactions of the N-ST students to the presented tasks. The nature of the post-intervention reactions, and the examination whether they have become secondary intuitions, were judged by analysis of the responses of the T-ST and E-ST participants.

RESULTS and DISCUSSION

Primary intuitions regarding the comparison of infinite sets

As can be seen from Figure 1, more than half of the students who had not studied Cantorian set theory agreed that one-to-one correspondence could be used for the comparison of infinite sets. Still, 65% of the N-ST participants accepted inclusion, and about a third of them accepted single infinity. Only about a third of the N-ST participants responded that “a single method must be used for all comparisons” (Figure 2).

Figure 1: Frequencies (in %) of accepting one-to-one correspondence, inclusion, and single infinity for the comparison of infinite sets

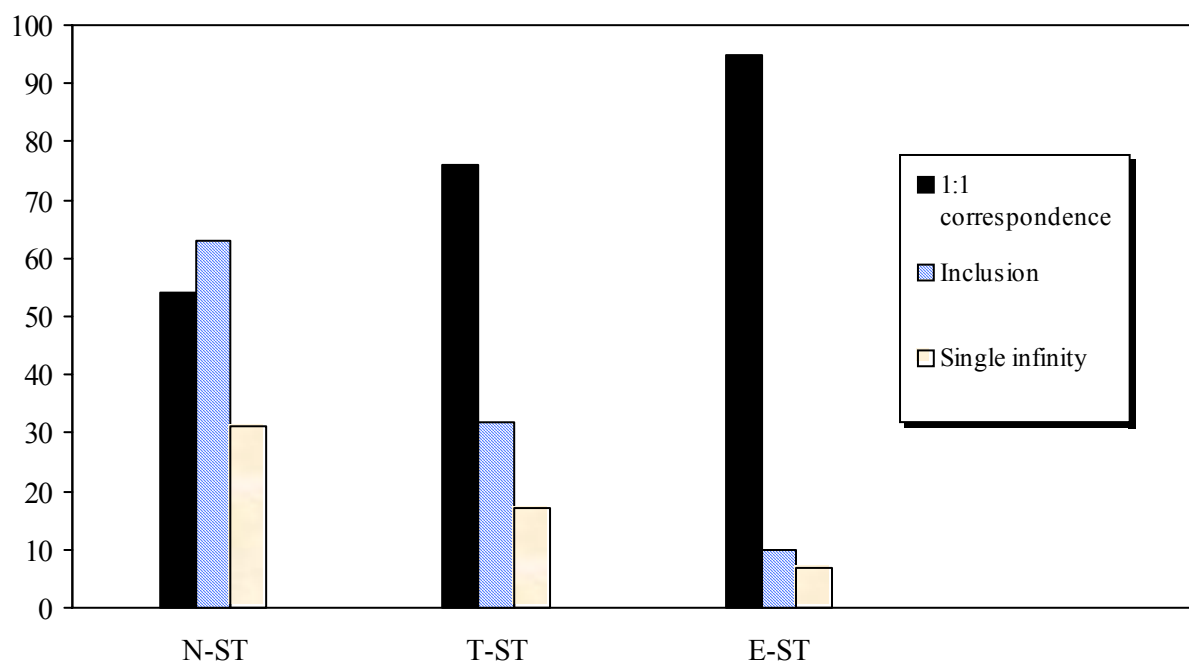
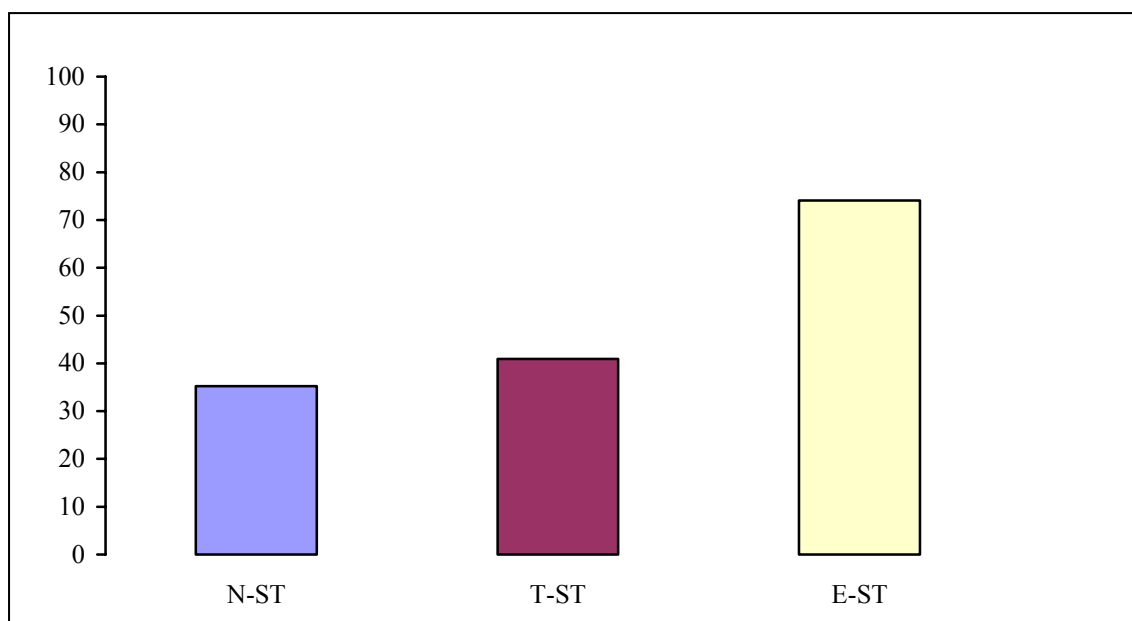


Figure 2: Frequencies (in %) of declaring the need to use a single method for the comparison of infinite sets



The findings regarding students' claims that more than one method could interchangeably be used to compare infinite sets is consistent with findings of previous studies, indicating students' tendency to use several methods for such comparisons (e.g., Tirosh & Tsamir, 1996).

The impact of two different interventions on prospective teachers' intuitions

The study of Cantorian set theory either in T-ST or E-ST courses assisted prospective teachers in becoming familiar with the role of one-to-one correspondence in Cantorian set theory (Figure1 and Figure 2).

When asked to judge the acceptability of one-to-one correspondence, inclusion and single infinity for the comparison of infinite sets, participants exhibited a marked tendency (76% of the T-ST and 95% of the E-ST) to accept one-to-one correspondence (Figure 1), which was significantly higher than that of the N-ST participants ($\chi^2=47.60$, $df=2$, $p<.001$ $\Phi=.39$). They also exhibited a marked tendency to reject the other methods, inclusion and single infinity (for inclusion $\chi^2=61.24$, $df=2$, $p<.001$ $\Phi=.45$, and for single infinity $\chi^2=18.84$, $df=2$, $p<.001$ $\Phi=.24$), which was again significantly higher than that of the N-ST participants. Figure 2 shows that there was also a significant increase ($\chi^2=29.6$, $df=2$, $p<.001$ $\Phi=.31$) in prospective teachers' tendency to claim that a criterion should be chosen in advance for all comparison-of-infinite-sets tasks.

Nevertheless, even after the interventions, traces of primary intuitions based on experiences with finite sets could be identified. In spite of intervention, about 17% of the T-ST participants and about 10% of the E-ST

participants still regarded single infinity as suitable for comparing infinite sets, and about 32% and 10% of them respectively, viewed inclusion as suitable for such comparisons. Part of them explicitly claimed that it is possible to use these three methods interchangeably.

It is clearly noticeable that the E-ST course was more beneficial than the T-ST course in promoting prospective teachers' awareness of the need to use only one method when comparing infinite sets. In their declarations, significantly more E-ST than T-ST participants claimed that it is necessary to use a single method for all the comparisons of infinite sets ($\chi^2=49.3$, $df=1$, $p<.001$ $\Phi=.51$; see Figure 2), and pointed to one-to-one correspondence as the method for such comparisons (See Figure 1; $\chi^2=36.5$, $df=1$, $p<.001$ $\Phi=.54$; see Figure 1). While most T-ST participants declared that either inclusion or single infinity could be applied, the E-ST participants usually rejected these options.

Since both, T-ST and E-ST participants had been introduced to the formal Cantorian theory and to the use of one-to-one correspondence, it could be expected that all would identify this method as the acceptable one for comparing infinite sets. Still, there was evidence of the continued influence of intuitive ideas associated with finite sets, which interfered with the ability of the T-ST participants to reflect on their judgments and be aware of possible contradictions. The E-ST participants exhibited a greater tendency to work within the framework of the Cantorian theory.

These findings are consistent with Fischbein's claim that "one has to bear in mind that intuitively based conceptions cannot be eliminated simply by mere verbal explanations... The development of new... mathematical and scientific intuition implies, then, didactical situations in which the student is asked to evaluate, to conjecture, to predict, to devise and check solutions... It is certain that mathematics education cannot be successfully achieved by simply bypassing the intuitive obstacles through purely formal teaching" (Fischbein 1987 p. 38; 213).

Final Comments

This study discussed two types of intuitive structures that is, primary intuitions and secondary intuitions of actual infinity. We have discussed prospective teachers' primary intuitions and their responses after two types of interventions. The findings showed that, as expected, the primary intuitions of the participants before studying Cantorian set theory were similar to those found among young students, as reported in the literature. The findings also showed that participation in a traditional T-ST course, and even more so in an enrichment

course (E-ST) promoted prospective teachers' awareness of the need to use only one method for the comparison of infinite sets, preferably one-to-one correspondence.

Still, the data here indicated that commonly, even after intervention, traces of primary intuitions based on finite experiences continued to linger in the majority of the prospective teachers. They persistently continued to declare single infinity and inclusion as suitable for the comparisons of infinite sets, claimed that a number of methods could be used interchangeably and remained unaware of the contradiction this led to. This problem was also foreseen by Fischbein: "Primary intuitions are usually so resistant that they may coexist with new, superior, scientifically acceptable ones. That situation very often generates inconsistencies in the student's reactions depending on the nature of the problem" (1987, p.213). Although a change was exhibited in the E-ST participants' post-intervention comparisons of infinite sets, and most of them declared the need to stick to one-to-one correspondence and power when comparing infinite sets, fragmented remains of finite notions could still be identified in their justifications.

Moreover, participants' answers in the interviews, both before and after instruction, were immediate. In a number of cases they included words like "certainly", "obviously", "sure" and "clearly", which indicated feelings of intrinsic certainty and self-evidence. The blend of out-of-class daily, finite experiences, and in-class studies of set Theory, seemed to give rise to a new type of intuitions. However, while the participants seem to have acquired new intuitions regarding the comparison of infinite sets, these intuitions were not consistent with Fischbein's definition of secondary intuitions. As mentioned before, for Fischbein, secondary intuitions were only those completely in line with the formal theory. He explained, for instance, "if for a mathematician the equivalence between an infinite set and a proper sub-set of it becomes a belief – a self explanatory conception – then a new, secondary intuition has appeared" (1987, p. 68). There is clear evidence that for most prospective teachers the post-intervention intuitions did not comply with Fischbein's definition of secondary intuitions. However, such responses could no longer be regarded as primary intuitions, as they had been engendered as result of an intervention. One may wonder, in terms of primary vs. secondary intuitions – what kind of intuitions are these? Or is there a need to define a third type of intuition, which is post intervention, but not perfectly consistent with the formal theory.

In light of the findings, there seems to be a need to define a third type of intuitions, "transitory intuitions", which evolve after intervention but are not perfectly consistent with the formal theory. In a way, transitory intuitions can be regarded as "advanced primary intuitions", consisting of a mix of primary intuitions and theorems derived from the formal theory. Such intuitions arise

when, through intervention, which becomes part of the learners' experience, new elements are added to, but do not wholly replace primary intuitions. These new intuitions have the "intuitive nature" of secondary intuitions but do *not* have their "formal perfection".

The term "transitory intuitions", pointing to the range between primary and secondary intuitions, insinuates a certain direction going towards the secondary intuitions. It may be interpreted as suggesting that secondary intuitions must necessarily be achieved. This, however, is not my claim. In many cases, even after quite extensive interventions, one may still hold transitory intuitions, perhaps with a higher portion of formal knowledge, but never reach the perfection of secondary intuitions. Since the blend of transitory intuitions may consist of various portions of primary intuitions and formal knowledge there is a need to further investigate this nature of transitory intuitions, and to examine the impact of different didactical approaches on students' transitory intuitions regarding various mathematical notions.

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WORKING GROUP 2

Tools and technologies in mathematical didactic

Group leaders:

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INTRODUCTION TO WG2

TOOLS AND TECHNOLOGIES IN MATHEMATICAL DIDACTICS

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This set of papers and posters is the work of the Thematic Group Tools and Technologies in Mathematical Didactics. The papers build on the work of the group at CERME1, where three embedded levels were distinguished when analysing the use of tools and technologies:

- the level of interactions between tool and knowledge;
- the level of interactions between knowledge, tool and the learner;
- the level of integration of a tool in a mathematics curriculum and in the classroom.

Bearing in mind these three embedded levels of analysis, the Thematic Group, as represented by this collection of papers, worked on the following questions:

- How can ideas of representation, metaphor and tool help us to understand how learners interact with technologies?
- How do tools and technologies mediate learning?
- What might be the parallels and contrasts between computer algebra system (CAS) use in algebra and calculator use in arithmetic?
- How does CAS use compare with use of such other tools as dynamic-geometry software or statistics packages?

Theoretical Ideas

Theoretical ideas, such as representations, metaphors, mediation and tools, proved useful when looking at how students link representatives (artifacts-in-use in an activity) to the mathematical function they are meant to represent, through actions of *interweaving* representatives and rejecting those that are incompatible with their concept image. The paper by Landa illustrates the mediation of a spreadsheet, showing how students misunderstanding can be clarified not only for the teacher but also for the students. In Cerulli's work can be seen, in the software that he designed, the equivalence of a button (a software functionality) and a statement in algebra. Algebraic knowledge can then be build and used in

the same way that a theory is built from axioms by way of proof. Across these papers, technology can be seen as means to allow the use of a wider variety of metaphors and representatives (see Jones, and Schwarz & Hershkowitz).

Algebraic Knowledge when using Spreadsheet and CAS

The papers by Chiocca and Hošpesová show two contrasting cases of the educational use of spreadsheet. The former case showed conceptual difficulties shifting from the mathematical content at stake (statistics) to a “spreadsheet” concept (the distinction between absolute and relative references). In the latter, the spreadsheet was reported as facilitating the access to mathematical concepts and fostering positive attitudes. In this apparent contradiction, technologies can be seen as facilitators or obstacles, depending on the point of view. Spreadsheets are, in a sense, quite intuitive. Nevertheless, their use implies understanding more or less their operation, especially when the problem is not the direct equivalent of a paper and pencil task. Obstacles reported in a task using such software do not necessarily imply that this task is to be banished. Educators have to reflect on the students’ actual difficulties and on the knowledge involved in their resolution.

Routitsky and Lagrange both present the results from large-scale surveys. Routitsky’s is about teachers’ attitudes towards the use of calculators. This, it seems, does not have a straightforward relationship with the period of time over which teachers have used calculators. When the period is short, the attitude is generally good. Then, over a longer period, it declines. Then it grows slowly again. This implies that teachers are generally inclined to use calculators at the beginning, then they come up against difficulties and they need time to overcome these. Lagrange’s paper surveys the literature on the use of technology, especially Computer Algebra Systems, to teach and learn mathematics. It shows a variety of works and trends, some very optimistic, others more aware of difficulties that students and teachers might meet.

This raises the question of the “instrumental” and “institutional” approach to the use of technology. The instrumental approach refers to a technological tool as a mental construction by the user. The institutional dimension considers the tasks, techniques and theories in a given institution (classroom, educational system, ...) and the impact of the introduction of technology on these. These approaches are complementary to the “epistemological approach” that relies on the study of the knowledge in relation to the introduction of technology.

Dynamic Geometry Systems (DGS)

The paper by Gallopin and Zuccheri focuses on how to improve the teaching of deductive reasoning. They give an example from their work of how to use as didactical instruments, strictly linked, technological tools and mathematical theories and concepts. The work of Mogetta relates to the forming of conjectures from visual images in DGS, what she called ‘dynamic’ definitions. Olivero and Robutti give examples from their research on the role of measuring in the proving process.

Open Questions

The Thematic Group finished with the following open questions:

1. The influence of the use of technology on proof.

It appears that we have to look at the process of proving as a specific theoretical activity. General software like DGS and CAS can tend to encourage empirical activity. Specific settings are necessary to reach a more theoretical level. Proving is often difficult, as students may be easily convinced by empirical evidence. Working on proof probably implies the use of more specific software like Cerulli’s “L’Algebrista”.

2. The notions of “tool”

A technology can be used by the teacher as a “didactical tool” or by the learner as a means to do tasks and learn mathematics. The notion of “tool” is also used for a host of different “immaterial” entities like concepts or theories. We have to distinguish these notions when analysing the use of technology. Looking at a technological tool for the learner as an “instrument” is beneficial because it accounts for the links between the appropriation of the tool and the learning of mathematics.

Concluding Comments

Overall, the work of the Thematic Group covered a variety of software technologies (DGS, spreadsheet, CAS, multimedia, distance education, calculators), school levels (from primary to university students) and methodologies (small scale case studies, software design, big surveys...). The outcomes for the participants were that a range of common notions were developed and a variety of concerns shared. We hope that this variety is captured in this set of papers.

IS WHAT YOU SEE WHAT YOU GET? REPRESENTATIONS, METAPHORS AND TOOLS IN MATHEMATICS DIDACTICS

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Abstract: *This paper is exploratory in character. The aim is to investigate ways in which it is possible to use the theoretical concepts of representations, tools and metaphors to try to understand what learners of mathematics ‘see’ during classroom interactions (in their widest sense) and what they might get from such interactions. Through an analysis of a brief classroom episode, the suggestion is made that what learners see may not be the same as what they get. From each of several theoretical perspectives utilised in this paper, what learners ‘get’ appears to be something extra. According to our analysis, this something ‘extra’ is likely to depend on the form of technology being used and the representations and metaphors that are available to both teacher and learner.*

Introduction

“What you see is what you get” (WYSIWYG) was a catchphrase on the 1960s US TV show, *Rowan and Martin's Laugh-In*. In the 1980s it became a byword in computer-based desktop publishing, referring to any technology enabling the user to see images on-screen exactly as they appear when printed. The development of graphical user interfaces (for forms of software that are proving useful in mathematics didactics, such as Logo, spreadsheets, dynamic geometry, graph plotting and statistical modelling software) has raised questions about how learners’ interactions with these interfaces mediate their understanding of mathematical ideas. This paper seeks to open up discussion about how the theoretical resources of representations, metaphors and tools can assist in an examination of what learners in classroom interactions with technology ‘see’, what understanding they might get from such interactions, and the implications for the theoretical ideas.

Following an overview of some of the main theoretical ideas, a brief extract from a piece of classroom research is considered from several different perspectives. The aim is to see in what ways these perspectives might both

illuminate and constrain interpretation of the classroom incident. The paper concludes with some commentary on the relationship between the various perspectives and what they might mean in terms of practical classroom responses.

Theoretical Preamble

There is extensive literature on representations, metaphors and tools in mathematics education yet this literature often focuses on one of these theoretical ideas and, in the main, does not seek to examine the relationship between them. Indeed, each of the terms is reasonably complex in itself, with discussion continuing as to what each one is and how the idea might be useful. In this section we identify some of the main features of each theoretical idea, as a precursor to using them in an analysis of a classroom interaction.

In the literature on representation, a distinction is often made between internal and external representation. An internal representation is a hypothesised mental construct; an external representation is a material notation of some kind (such as a graph, a table or an equation). As Kaput (1998) observes, such a distinction is cognitivist in essence and does not necessarily take account of other perspectives on thinking and learning. These other perspectives, including socio-cultural viewpoints, for example, raise questions about whether learner interactions with screen images are usefully described in terms of internal and external representations, or whether such images constitute a new class of representation.

The literature on representation currently makes little reference to notions of metaphor even though work on the latter (for example, Lakoff and Nuñez, 2000) shows how it is possible to point to deep metaphors which are implicitly embedded in our language and which are therefore part of the way we talk/think, whether we like it or not. Metaphor, in the general sense, characterises the substitution of one similar concept for another one. Metaphor is a widely-used idea in software design (examples being the desk-top, menus, windows, etc) and in human computer interaction. In mathematics, metaphor occurs as translation of structure from one domain to another and has been posited as crucial for our sense of understanding mathematical ideas. For their part, images (such as screen images) are posited as part and parcel of the metaphorical mappings that bring new mathematical concepts into existence (see, for example, Sfard, 1997). This raises the question of the relationship between ideas of metaphor and representation, about which we hope to stimulate discussion through this paper.

The notion of tools is widely used in mathematics education. At its most straight-forward, the term refers to physical implements. But the use of the term has expanded to include not only physical implements but also technical procedures (like the algorithms of arithmetic), symbolic resources (such as those of natural languages and mathematical and musical notation), and, most recently, cognitive processes. Such use of the word tool can be considered as metaphorical, as a way of understanding the use of technical procedures, symbolic resources and cognitive processes. Computer environments (such as microworlds) and electronic calculators are frequently referred to as both technological and cognitive tools. These tools, as well as being physical artefacts, encompass technical and symbolising capabilities and become objects to think with. It is widely recognised that tools change the way that activities are carried out and can shape the conceptions of the user (Gutiérrez, Laborde, Noss and Rakov, 1999; Lajoie, 1993).

The wide use of the terms representation, metaphor, and tool in mathematics education highlights the complexity of trying to understand and describe what may be happening when learners (and their teachers) interact with mathematics when using computer software, calculators, or other technology. In an attempt to begin to try to clarify the relationships between these terms, and perhaps their interactions, a segment of classroom interaction has been taken as a catalyst for producing various perspectives on the role of the technological imagery in learning.

An Interaction from the Mathematics Classroom

The task described below was set within what Ainley, Nardi and Pratt (2000) call an *Active Graphing* approach. With this approach, children are encouraged to make a scattergraph as soon as they have a few pieces of data. The children are then expected to discuss the graph, perhaps with their teacher, and make conjectures about any patterns that emerge before deciding what data to collect next to test these conjectures. The data extract in Figure 1 is taken from the work of two 9-year-old children, Laura and Daniel (both pseudonyms). The children were working on a task (introduced verbally by the teacher) in which each group was given a 75 cm length of ribbon, and challenged to make a rectangular frame which had the largest possible area. The children collected initial data by pinning a length of ribbon on to a display board to make the frame, then measuring the length and width of the frame. They entered these results on a spreadsheet, and, with help, set up a third column with a formula to calculate the area of the frame.

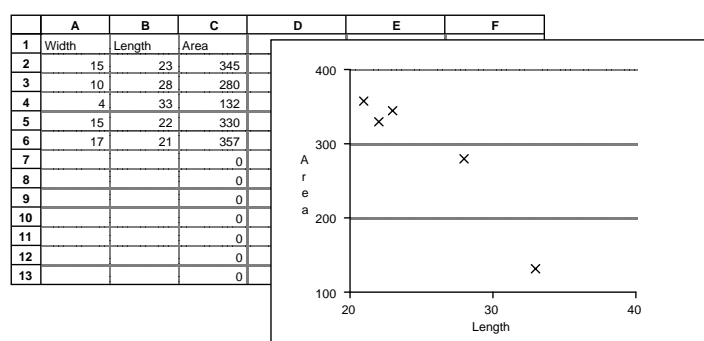


Figure 1. Children's spreadsheet work on the rectangle task

The data extract below is taken from their discussion of their first graph (see Figure 1).

Researcher: Okay, so what is this graph saying?

Laura: It's a hill.

Daniel: It's like a mountain there.

Laura: I think it's going to come down again.

Daniel: and go back to nought.

The first thing to note is that the spreadsheet technology available to the children, while supporting the rapid display of their data, did place constraints on aspects of their interaction. For example, the children had to organize their data in a particular way, since the software only allows graphs to be made from adjacent columns. Further, while the graphing facility within the particular software (*ClarisWorks*) creates a window containing a graph that can be dragged to different sizes and proportions, such dragging changes both the appearance of the graph and the scaling on the axes. By default, the software selects scales and ranges of values on the axes which display the data points centrally in the window (as in the example shown in Figure 1). Thus the axes may not start from zero with the consequence that the full range of possible values of a given variable may not be visible. The recognition of this shaping of student activity by the technology is an important prerequisite for a theoretical discussion of their learning (see, for example, Jones, 1999).

Analysis

In this section, the above data extract is subject to analysis from four different perspectives, including modelling, multiple representations, co-construction, linguistic, etc. in order to examine the convergence, or otherwise, of these

viewpoints and to see what they reveal about the roles of representation, metaphor, and tool.

§1 Representation as tool and symbol

At first sight it seems that the children do not respond directly to the researcher's invitation to read information from the graph. Their first two comments suggest that they are seeing the graph as a picture. The fact that they do see a picture, rather than a series of separate points, is likely to be significant. They seem to be looking *through* the individual points of the graph to construct a coherent image that takes in the whole of the data set.

The latter two comments are even more interesting. Laura's comment suggests that she is extrapolating to imagine data which has not yet been collected, and what is more she is doing this 'backwards' to a part of the graph (to the left of the current position of the y axis) which does not yet exist on the screen. This suggests that she is using the graph as *a tool* that she can (mentally) manipulate to make conjectures about the outcome of the experiment.

Laura's use of the words 'come down again' may also link to the overall purpose of the task. The children are trying to find a maximum area, and the 'hill' for Laura seems to contain the idea of a value increasing and decreasing. The experimental data the children have collected shows that they started with a width of 15, tried smaller widths, tried 15 again (though without seeming to notice that they get a different area!), and then tried a larger width. This sequence reflects a sense of 'going up and down' that links closely with the hill metaphor.

The graph *as a symbol* has as its referent the tabulated data, and the rectangular frames which have been created. Nemirovsky and Monk (2000) talk about symbolizing as *the creation of a space in which the absent is made present and ready to hand*. This seems quite a useful way of seeing what is happening for Laura and Daniel as they look at the graph. The graph symbol allows them to hold all the data in one space, so that they can see something about the overall pattern of what is happening. The symbol contains all the complexity in a more manageable way than the data, and so allows them to talk about how 'it' is changing. Notice that when Laura says she thinks 'it' is going to go down, 'it' might be any or all of:

- the trend in the graph,
- the value of the area in the data,
- the size of the space within the frames they are making.

§2 *Representations as tools*

The tabular representation can also be viewed as a symbol in the sense that it is a counterpart for the conceptual object of function (actually of several functions in this case), via the input-output process. That is, it stands for function in a metaphorical sense but may not name or point to it. This representation-symbol contains a number of other symbols, including those for specific numbers, and the symbolisation of variables in the table using natural language, namely ‘width’, ‘length’ and ‘area’. For the learner, a tabular representation can assist in the construction of understanding of properties of a function, such as its one-to-one nature (a problem with the data in this table!). However, if a student decides, for example, to use the table values to interpolate or extrapolate *other* values for the function, or to calculate the perimeter of the rectangle by adding a further column with its associated symbols, then such activity has moved them beyond *looking through* to a stage of *acting on* the representation.

Once the table is complete the focus shifts to using the software as a graphing tool, for drawing the graph of area against length. In this case the students have taken the route the tool directs them in and have drawn a discrete set of points to represent the functional relationship with each point symbolised by a little cross. One can see this graphical picture either as a counterpart symbol, or a representation of a function, comprising other symbolic objects. These include the counterpart symbols which are described as axes and the language symbols ‘Length’ and ‘Area’, both of which stand for the independent and dependent variables. The little crosses are also symbols pointing to ordered pairs in the function, etc. A student can pay attention to this representation and construct some properties of a function as a process or an object (Tall *et al*, 2000) as with a table, but it becomes more interesting in some sense when the student interacts with it and uses it as a tool. The comments such as those of Daniel who says of the graph that “it’s like a mountain there.” and of Laura who describes it as a hill appear to require a global modelling strategy. They may or may not have seen a continuous model of the function in their mind’s eye when making these statements, but their interaction with the representation has comprised more than *looking through* it. They see the graph as an entity, an ‘it’. They are imposing a global model on the graph and *construing* properties of the process or an object underlying the model. Later Daniel again identifies a local property of the graph, namely that it appears to head “back to nought”. This may have been inspired by Laura’s comment about the trend of the values saying “...it’s going to come down again”, again paying attention to a local property. This brief encounter with the activity demonstrates, when modelling functions in a computer environment for building understanding, the students’ interaction with the tabular and graphical representation as tools is crucial.

§3 *Multiple representations of a real situation*

Tools can be said to both aid and initiate thinking. Before the data are entered, the table is a tool to organise the data collected and the graph is a tool to organise the table data, whereas once the data are entered the table and graph are both representations (models) of the real situation. When the students are confronted with a representation, a dialogue with, or interrogation of, the representation is operationalised. In the vignette above, that three representations (models) can be found infers that students had to think in order to change from one representation to another. Wild & Pfannkuch (1999) call the thinking that is required to move between representations, or to change representations to engender understanding, *transnumeration*. Overall, using the approach of Wild & Pfannkuch would mean characterising the dialogue in terms of five fundamental elements – recognition of need for data, *transnumeration*, consideration of variation, reasoning with statistical models, and integrating the contextual and statistical.

The first question confronting the students is what measures should be captured from the real system. The children must think how to capture the notion of area so they decide to make a rectangle and measure the width and length to the nearest cm (*transnumeration*). These are determined to be the relevant measures for the problem. They then represent these measures in a table of data as a way of systemising their thoughts (*transnumeration*). They calculate the area using a spreadsheet tool much as they would use a calculator or pencil-and-paper. The table-of-data representation has no order that easily allows the students to interact with the relationship between the variables. Whatever was noticed or not noticed by the students, the table of data must be changed in some way to convey new or increased meaning. The students have to think that perhaps a graphical representation will allow them more insight into the data. What variables should they graph? What graphical tool should they choose? When they obtain a graphical representation (*transnumeration*), a dialogue between them and the data ensues.

In this episode only two elements of statistical thinking – reasoning with a statistical model and recognition of the need for data – are activated. What features can the students see in the data? First they see a hill. The students perhaps do not have the language to discuss trends and therefore use the metaphor of a hill to describe the pattern they are ‘seeing’. When they further describe the pattern they imagine what the representation might look like if there were more data. The statistical-system tools allow multiple representations of the real situation to be seen so that students can engage in a dialogue with the data in a search for meaning and ultimately understanding about the real situation.

§4 *Co-construction of representational relationships*

A critique of some of the work in mathematics education on multiple representations (generally numerical, graphical, and symbolic) is that the various representations can be no more than (external) representations of each other with no grounding for the learner in any experience. For the learner, they are not ‘representations’ since they are not representing anything known to the learner. The data extract above illustrates that this does not necessarily have to be the case. Here the representations are linked to concrete, experientially-real data; in this particular example the construction of rectangular picture frames from a fixed length of ribbon. So it could be said that the phenomena of making picture frames is at the centre of the activity, and the representations are means of understanding and reasoning about the phenomenon.

The mathematical relationships hidden in the spreadsheet formula used to calculate the area of each rectangular frame of ribbon, and, indeed, in the model of space that is the Euclidean plane that controls the phenomenon’ are also models of the phenomenon. So there is a two-way (at least) representational relationship. It is a form of co-construction. The forms of representation available to, or, more particularly, used by, the learners control, or influence, the exploration of the phenomenon just as the phenomenon influences, or controls, each representation.

The forms of representation permitted in the software environment are given, or, perhaps more accurately, proscribed, as the learners are not free to create their own representations but can only make use of those representations provided by the software. The representations that are available, in turn, generate imagery which is intimately connected with the metaphor of the cognitive tool. The representations available to the learners are not static. The *active graphing* approach exploits the potential of computer-based environments for the *active exploration of phenomena*. The children in this example have experience of this approach and, in the last two lines of dialogue, are making predictions based on their interpretation of the graph. They seem to be using the representations, particularly the graphical representation, as a means of building up a sense of the quadratic relationship that models the phenomenon they are exploring.

As Leont’ev (1981, pp. 55-6) argues, “the tool mediates activity and thus connects humans not only with the world of objects but also with other people”. Thus the process by which learners create meaning is embedded within the setting or context and is mediated by the forms of interaction and by the tools being used. Here the argument is that the learners create representations, albeit limited by the forms of representation available via the tool, and the (available) representations create the learner’s ideas of those representations.

Discussion on Tools, Representations and Metaphors

In terms of tools, and taking the spreadsheet in its totality as a tool, in this example of classroom interaction the tool is not used here to its full potential. For example, for these children in this case, ‘length’ is not taken to be dependent upon ‘width’. Tools are like this –their full potential is rarely used. In addition, tools are not mathematical in themselves. They are only *used* mathematically.

In terms of representations, it could be said that there are three representations in this example: a table, a graph, and a dialogue. Yet the word representation carries an implication that *a thing* is being represented. Here the representations are not just different aspects of a mathematical relationship, the relationships are different kinds of thing in each case. For example, in the classroom example analysed above, the table of data has no ‘shape’, no sense of increasing or decreasing. It is raw, unordered data. Yet it implicitly contains the formula relationship between the sides of a rectangle and its area. In contrast, the graph does not contain that information: the points on the graph cannot be read to the accuracy of the formula. What it does contain is an ordering of the data. Finally, the pupil dialogue is about a trend, not about the formula, nor about the data as individual points.

These three things, the table, the graph, and the pupil dialogue, are different things in kind. In fact, it could be said that there is not *a thing* being represented at all. When we talk about representations, we talk *as if* there were something to present. This is a metaphor. The metaphor is that mathematics is *like a thing*. Nominalising in mathematics is a metaphor whereby mathematics is likened to objects in the world. There are other options, for example we could talk about mathematical ideas as actions.

If talking of mathematical relations as objects is a metaphor, what is it a metaphor of? To ask that question is to fall into the same trap: it implies that there has to be a metaphor of anything. We talk in metaphors because there is no other way of doing it. Mathematics is created by talking about a relationship, tabulating it, graphing it, describing it. These are all (Wittgenstein, in Shanker, 1987) *normative activities*: the communication lays down the ways it makes sense to talk about, describe, or illustrate these ideas. The benefit of having many re-presentations is that this mathematical idea has a lot of different aspects – no one representation embodies the entirety of the idea.

In the above analysis, the terms representation, metaphor and tool were each given a variety of roles. For example, ‘tool’ was interpreted as a function of a representation (§2); ‘representation’ was used as a model (§3), and as a mode of description (§4); ‘metaphor’ was used as a linguistic feature (§1 and §3), as a relationship between a function and its representation (§2). The lesson here is

that we must be careful not to assume the same functionality for our use of such terms as well as not assuming consonant interpretations.

Concluding Comments

Perhaps there is the basis of productive discussion arising from people's different uses of the same theoretical concepts. In terms of our actions as researchers, we are reminded that it is essential to be very clear, very early in any writing or discussion, that the meaning of our conceptual constructs is evident. The other side of this coin is that, when reading the work of others, we should not to jump to conclusions about what these constructs mean when used by other authors.

But mostly we are reminded that constructs are just that. They are constructed by us, and are therefore useful or not useful. They are not true or false. There is no unequivocal thing that can take the name 'representation'. The consequence of this is that constructs must be judged for how they speak to the readers or listeners. Do they help teachers understand learning experiences or teaching behaviour? Do they help researchers frame useful questions? Do they add to the analytical tools available to mathematics educators? And so on.

This paper is titled "Is What You See What You Get?" because we want to focus on our ability to understand the relationship between technology (particularly visual technology) and mathematics learning. Can we make use of the ideas of representation, tool and metaphor to discuss what is "seen" and what is "got"? At the risk of being glib, the various perspectives used above return the following different answers to these two critical questions.

- §1 What is seen is a picture, much of which is able to be inferred through the use of technological tools. What the learner gets are symbols that can be given meaning (added value?) through metaphors.
- §2 What is seen are representations generated through the use of a tool. The representation is a metaphor of the mathematical relation. What the learner gets are properties of these representations that have been construed from them as objects.
- §3 What is seen is data transformed in different ways. What the learner gets is the power to ask questions and to reason.

§4 What is seen are representations generated from real experiences. What the learner gets is the ability to co- (and/or re-) construct the representations in response to the questions they raise.

In total, what are seen are tool-generated representations of different, yet related, things. What the learner gets is a way of communicating mathematics. In none of the above is what the learner sees the same as what the learner gets. In every case, the learner gets something extra. Perhaps that is the power of (all) technology.

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INTRODUCING PUPILS TO THEORETICAL THINKING: THE CASE OF ALGEBRA

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Abstract: *Within the theoretical framework of Vygotsky, this paper presents the results of a teaching experiment concerning the introduction of pupils to theoretical thinking in algebra. Starting from the results of a previous study concerning with the use of Cabri-géomètre to introduce pupils to geometry theory, the experiment is based on the use an algebra microworld called “L’Algebrista”. An outline of the classroom experimentation is followed by an analysis of some protocols, according to the Vygotskian theory of semiotic mediation.*

Introduction

As clearly shown by previous research studies, the evolution of algebraic symbolism can be described in “procedural-structural” terms (Sfard, 1991). The procedural character of pupils’ conceptions that relates to literal terms and expressions tends to persist; at the same time, although symbolic manipulations of literal expressions is largely present in school practice, the absence of “structural conceptions” appears evident (Kieran, 1992, p. 397).

In Italian high schools, pupils begin fairly quickly to be trained in simplifying expressions (first numerical and then literal) and this training is intensively practiced at grade 9, when the first months of the school year are devoted to pupils’ introduction to ‘Algebra’.

The limitations of a procedural approach to symbolic manipulation have been often pointed out. This demonstrates the need for a “structural-relational” approach in order to master symbolic manipulation in a productive way (Arzarello, 1991).

Poor strategic decisions made by students with extensive algebra experience have been described. Even these students seem unable to identify the right transformation to be accomplished. For example, when the task does not

explicitly indicate what transformation has to be performed, pupils are unable to take a decision and “go around in circles” (Kieran, 1992, p. 397) carrying out transformations without any clear goal.

A key point to a structural approach is the notion of “equivalence relation” between expressions. Actually, the manipulation of expressions means substituting one expression for one that is equivalent. The meaning of the words “*expression*” and “*equivalent*” are not univocally, and *a priori*, determined, but it shall be so, once a set of axioms is accepted. We consider “*symbolic manipulation*” as characterised by activities of transformation of expressions using the rules given by the assumed axioms and definitions. Thus, symbolic manipulation makes sense within a theoretic system. Certainly this perspective is not very common in school practice (at least in Italy), yet it is exactly the perspective we assumed.

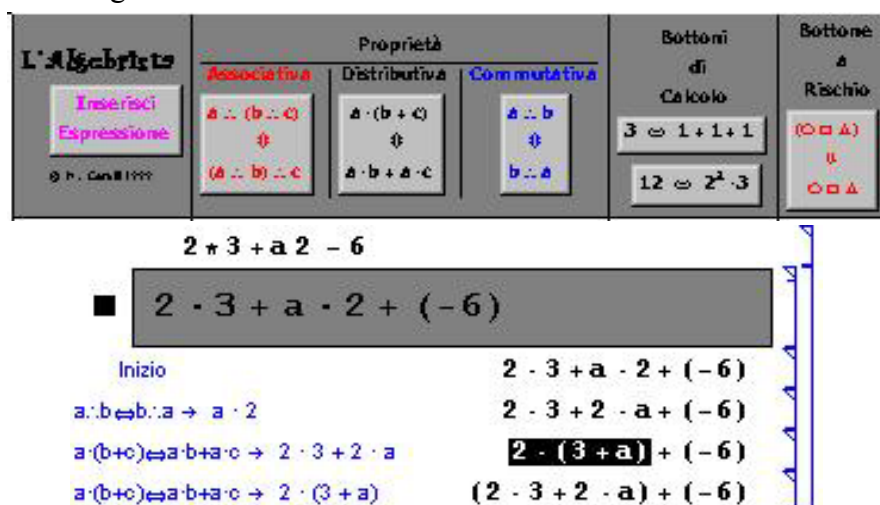
A previous research study, concerning pupils’ introduction to geometry theory (Mariotti et al., 1997, 2000), clearly showed how a computer environment may offer a support to overcome the well-known difficulties related to the theoretical perspective. In particular, Mariotti analysed the semiotic mediation that can be accomplished by the teacher using specific instruments offered by the Cabri-géomètre environment (Mariotti, in press).

Using a similar approach, a research project, still in progress, has been set up and a computer microworld, *L’Algebrista* (Cerulli 1999, Cerulli et al. 2000) designed that incorporates the axioms defining the algebraic equivalence relation. A prototype of the software was realised and used in a teaching experiment with ninth grade classes. The basic hypothesis in the project is that the axioms, definitions and theorems of algebra are the main elements involved in the transformation of expressions. In *L’Algebrista*, expressions on the screen can be manipulated using buttons. Such computational objects may be interpreted as signs referring to expressions and axioms (or theorems) within algebra theory; the manipulation of such signs corresponds to symbolic manipulation. In other terms, in the microworld a physical counterpart of expressions and axioms allows the user to visualise and make explicit the mathematical entities and relationships which are involved in symbolic manipulation.

Activities within L’Algebrista

In order to help the comprehension of the discussion in the following sections, we describe an example of interaction with *L’Algebrista*. The user writes

Fig. 1: In a notebook the user writes the expression to work with (« $2 \cdot 3 + a \cdot 2 - 6$ » in the example), then after selecting it the button *Inserisci Espressione* is clicked, thus L'Algebrista creates a new working area where the buttons are active.



(Fig. 1), in a notebook, the expression to work with (“ $2 \cdot 3 + a \cdot 2 - 6$ ” in our example), then selects the expression and clicks *Inserisci Espressione* (“Insert Expression”). L'Algebrista creates a new working environment, where the original expression is marked on its left with the label *Inizio* (“start”). The operation of *inserting* the expression is fundamental because it proclaims the entrance into the microworld where it is possible to act only using the buttons offered by L'Algebrista. It is interesting to remark that when an expression is *inserted*, its new instance comes out with some changes, in particular every subtraction is transformed into sum and every division is transformed into multiplication. This follows from a precise didactical choice because we want pupils to work in a “commutative environment”.

Interaction always happens by selecting a part of an expression and clicking on a button. The selection was designed so that it is not possible to select parts of expressions which are not sub-expressions from an algebraic point of view. For instance, given the expression $a \cdot b + c$ it is not possible to select $b + c$, if one tries to do it the software will automatically extend the selection to $a \cdot b + c$; on the other hand one can select $a \cdot b$ or c or a etc. This feature corresponds to fact that the expressions of this microworld incorporate a fundamental algebraic characteristic of mathematical expressions: their tree structure.

In the example, the expression is transformed by selecting the term $a \cdot 2$ and clicking the *commutative property button*; a new expression is produced (written just below), the term $2 \cdot a$ is substituted by the term $a \cdot 2$, while on the left a label indicates the button used and the sub-expression it was applied on. In the next step one part of the expression is transformed using the *distributive property button*, and finally, using the same button, such a transformation is

inverted. Coherently with our didactical hypothesis, the buttons incorporate all the functions of the properties of operations without advantaging any peculiar direction, and furthermore they produce only correct chains of equivalent expressions.

Outline of the classroom experimentation

A first experiment was carried out during the school year 1998/1999 in a 9th grade class (Cerulli 1999), and permitted the development of a second version of the prototype, which has been experimented in another class at the same level during the school year 1999/2000. The second teaching experiment represents a junction point between our research concerning algebra and the previously mentioned study concerning pupils' introduction to geometrical theory (Mariotti et al., 1997, 2000). The idea is to introduce pupils to theoretical thinking at the same time in geometry and algebra with the support of the environments offered by *Cabri* and *L'Algebrista*. A research project on the effectiveness of the joint use of such microworlds has been planned for the following school year (2000/2001) in 9th and 10th level classes.

A detailed description of the study is beyond the scope of this paper, which aims to analyse some aspects of the experiment within the framework of semiotic mediation; here we just indicate the basic ideas inspiring the sequence of activities concerning algebra.

First of all we recall that our **educational goal** concerns:

- *to introduce pupils to symbolic manipulation;*
- *to introduce pupils to a theoretical perspective.*

According to our hypothesis, the concepts of *equivalence relationship*, and of *transformation of expressions* (by means of axioms), are the basic principles underling symbolic manipulation (Cerulli *et al.* 2000). Thus, they represent the **starting points** of pupils activities. Furthermore, any transformation of an expression into an equivalent one is based on axioms, definitions, and theorems. As a consequence, it is possible to introduce symbolic manipulation taking a theoretical perspective. In our approach this is done considering the activities of expression transforming as activities of theorem proving (Cerulli *et al.* 2000).

In our experiment we introduce the problem of comparing expressions, taking into account the fact that, at this school level, pupils consider numerical expressions as equivalent when they give the same number as result. Thus it is

not difficult to negotiate the interpretation of numerical expressions as computation schemes, which will be equivalent if they give the same result.

The idea of interpreting expressions as computation schemes allows one to introduce the properties of sum and multiplication as principles (theory axioms) that determine *a priori* whether two computation schemes lead to the same result: if two expressions are equivalent on the base of such properties then the computation of the two expressions must lead to the same number. Thus a new equivalence relationship between expressions, based on the iterated use of the buttons, is introduced:

if one expression can be transformed into another using the properties of sum and multiplication (our axioms), then the two expressions are equivalent.

In the microworld *L'Algebrista* this corresponds to:

two expressions are equivalent if it is possible to transform one into the other using the given buttons (representing our axioms).

Once this equivalence relationship is accepted, pupils are asked to compare expressions (see an example in Fig. 2). A new terminology is introduced: one says that the equivalence of two expressions is **proved** if one expression is transformed into the other using the axioms; vice versa one says that the equivalence is **verified** if the calculation of both the expressions leads to the same result. With literal expressions the difference between *proof* and *verification* becomes even more definite: the use of axioms becomes the only way to state the equivalence between two expressions, whilst numerical verification (substituting the letters with numbers and computing the expressions) becomes the main way to prove that two expressions are not equivalent. Coherently with our theoretical perspective, Il Teorematore (Cerulli *et al.* 2000) can then be used to add a selected choice of proven equivalencies to the set of buttons that can be used for new proofs (see Fig. 3).

<p>1. Considera le seguenti espressioni: $a*b-b*b$ $a*(a-b)$ $(a-b)*(b+a)$</p> <p>a) Quali di esse pensi che siano equivalenti? E quali pensi che non lo siano? Perché? Sapresti dimostrarlo?</p> <p>b) Analizza la dimostrazione che hai fatto nel punto precedente ed indica per ogni passaggio fatto se hai utilizzato un teorema o un assioma</p>	<p>1) Consider the following expressions:</p> <p>a) Which of them do you think are equivalent? Which do you think are not? Why? Can you prove it?</p> <p>b) Analyse your proof and specify, for each step, if you used a theorem or an axiom.</p>
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Fig. 2 – An example of activity concerning expressions comparison

■ $13 \cdot m + m \cdot 17 == 30 \cdot m$

Inizio $13 \cdot m + m \cdot 17 == 30 \cdot m$

$a \cdot b \Leftrightarrow b \cdot a \rightarrow m \cdot 17$ $13 \cdot m + 17 \cdot m == 30 \cdot m$

$a \cdot (b + c) \Leftrightarrow a \cdot b + a \cdot c \rightarrow 13 \cdot m + 17 \cdot m$ $(13 + 17) \cdot m == 30 \cdot m$

$3 \Leftrightarrow 1 + 1 + 1 \rightarrow 13 + 17$ $(30) \cdot m == 30 \cdot m$

$(O + \square O) \Leftrightarrow O + \square O \rightarrow 30$ $30 \cdot m == 30 \cdot m$

Somma Termini Simili
 $2 \cdot \Delta + 3 \cdot \Delta \Rightarrow 5 \cdot \Delta$

Fig. 3 – In the case of the task “Prove that $13 \bullet m + m \bullet 17 = 30 \bullet m$ ”, Federico, using L’Algebrista, produced the proof on the left. After such a task a class discussion led to the acceptance of the rule for summing monomials, and the below button was added to the set of available buttons in L’Algebrista.

To conclude this section we remark that the concept of equivalence between algebraic expressions is basic in our approach, but we observe that it does not come straightforward from the use of the software. Working within *L’Algebrista* may reinforce the concept of equivalence by means of transformation rules, whilst it might weaken the concept of equivalence based on the substitution of numbers to letters; actually in L’Algebrista is not possible to substitute letters with numbers. As a consequence, the teacher has the delicate role of guiding the construction of a correct relationship between the two kinds of equivalence.

Semiotic mediation

Within the Vygotskian framework of semiotic mediation theory, the signs used to mediate mathematical meanings play a central role. *L’Algebrista* was designed as a microworld which could mediate the idea of theory in algebra, and the process of theory building. Algebra theory, as far as imbedded in the microworld, is evoked by the expressions and the commands available in *L’Algebrista*. According to the Vygotskian theory (Mariotti, in press), expressions and commands may be thought as external signs of the Algebraic theory, and as such, they may become instruments of semiotic mediation (Vygotsky, 1978).

The process of building a theory, by proving, accepting and using new theorems, can be evoked by specific activities within *L’Algebrista*. Proving that two expressions are equivalent in algebra corresponds to proving a theorem. Thus, in the microworld, *transforming an expression into another*, using the

available buttons, corresponds to *proving a theorem*. Furthermore, creating using Il Teorematore, a button corresponding to a new *equivalence relationship*, and adding it to the collection of the available buttons, corresponds to accepting a new theorem. Finally *using a button* created with Il Teorematore corresponds to *using a new theorem*.

In summary, the main instruments of semiotic mediation, offered by *L'Algebrista*, and related to the theoretic aspects of algebra, are:

- *expressions in L'Algebrista* are signs of *algebraic expressions*;
- *given buttons* are signs of *axioms* and *definitions*;
- *transforming an expression* into an other using the buttons corresponds to *proving* that the two expressions are equivalent, the produced *chain of justified steps* (the justification of each step is reported on its left) corresponds to a *proof*;
- *new buttons*, built using Il Teorematore, are signs of *theorems*;
- *adding new buttons* to the set of available buttons is a sign of the meta-theoric operation of *adding new theorems* to a theory.

Some comments, on how such signs can function as semiotic mediators, are included in the analysis of some protocols.

Signs derived from *L'Algebrista*

The representation of an expression in *L'Algebrista* incorporates its mathematical tree structure, and this structure becomes explicit, “tangible”, thanks to the *selection function*, when the user interacts with the environment.

In the case of a comparison task, an example of how pupils may use the *selection function* as an external sign of control of the algebraic structure of an expressions, is provided by the protocol in Fig. 4: Lia (9th grade) tries to prove that the two expressions are equivalent, and at each step she underlines (*selects*) a sub-expressions and transforms it using an axiom that applies. This behaviour recalls the interaction between the user and *L'Algebrista*: when transforming an expression, one first has to select a sub-expression and then to click on a button representing an axiom that applies. Furthermore, in the example, Lia refers clearly to the buttons of *L'Algebrista* using the word *button* (Ita.: *bottone*) and reproducing the iconography of the *buttons of neutral elements* (Ita.: *elementi neutro*) and of the *computation buttons* (It.: *bottoni di calcolo*) that she is using. In particular she refers to the following buttons:

- $0+A \Rightarrow A$: this button transforms an expression of the kind “ $0+A$ ” into the expression “ A ”, where “ A ” can be any expression. This button corresponds to the axiom defining the neutral element of the sum operator.
- $0*A \Rightarrow 0$: this button transforms an expression of the kind “ $0*A$ ” into the expression “ 0 ”. This corresponds to one of the properties of the “zero” element concerning the multiplication operator; such a property, in our experiment, is assumed to be an axiom.
- $3 \Leftrightarrow 1+1+1$: converts a number into its decomposition as a sum of *ones*, and, if applied on a sum of numbers, transforms it computing its result. This button corresponds to the definition of sum between numbers, it does not apply on letters.

Handwritten derivation of the distributive property $(a+b) \cdot c = a \cdot c + b \cdot c$ using algebraic rules and calculator button sequences:

$$\begin{aligned}
 (a+b) \cdot c &= a + a \cdot b = (a+b) \cdot b + (-1) \cdot (b \cdot b) + a \cdot (a+b) \quad \text{commut.} \\
 &= (a+b) \cdot b + (-1) \cdot (b \cdot b) + (a+b) \cdot a \quad \text{commut.} \\
 &= (a+b) \cdot b + (a+b) \cdot a + (-1) \cdot (b \cdot b) \quad \text{commut.} \\
 &= (a+b) \cdot a + (a+b) \cdot b + (-1) \cdot (b \cdot b) \quad \text{distrib.} \\
 &= (a+b) \cdot a + (a \cdot b + b \cdot b) + (-1) \cdot (b \cdot b) \quad \text{both. e. nach.} \\
 &= (a+b) \cdot a + a \cdot b + b \cdot b + (-1) \cdot (b \cdot b) \quad \text{commut.} \\
 &= (a+b) \cdot a + a \cdot b + (b \cdot b) + (b \cdot b) \cdot (-1) \quad \text{distrib.} \\
 &= (a+b) \cdot a + a \cdot b + (b \cdot b) \cdot (1 + (-1)) \quad \text{both. di. abels} \\
 &= (a+b) \cdot a + a \cdot b + (b \cdot b) \cdot (0) \quad \text{elemento neutro} \\
 &= (a+b) \cdot a + a \cdot b + 0 \quad \text{elemento neutro} \\
 (a+b) \cdot c &= a \cdot c + a \cdot b
 \end{aligned}$$

Fig. 4 - In the case of a comparison task, performed in paper and pencil environment, the protocol shows that pupils use signs clearly derived from L'Algebrista. In particular the *selection function*, or the iconography of the buttons.

Making conjectures and proving

Let us consider the problem reported in Fig. 2, which was given in class, with no computers; pupils are asked to compare three expressions and to find out which of them are equivalent; the produced conjectures are required to be proved.

1) $a \cdot a - b \cdot b$

2) $a \cdot (a - b) = a \cdot a - a \cdot b$

3) $(a - b) \cdot (a - b) + 2 \cdot (a - b) \cdot b =$
 $= (a - b) \cdot (a - b) + 2 \cdot (a \cdot b - b \cdot b) =$
 $= a \cdot a - a \cdot b - a \cdot b + b \cdot b + 2 \cdot a \cdot b - 2 \cdot b \cdot b =$
 $= a \cdot a - 2 \cdot a \cdot b + b \cdot b + 2 \cdot a \cdot b - 2 \cdot b \cdot b =$
 $= a \cdot a - b \cdot b$

Penso che la 1^a e la 3^a siano equivalenti, e non la 2^a, perché la 1^a e la 3^a, applicandoci delle proprietà vengono uguali, mentre la seconda no.

b) $(a - b) \cdot (a - b) + 2 \cdot (a - b) \cdot b =$ Ho applicato la proprietà distributiva.
 $= (a - b) \cdot (a - b) + 2 \cdot (a \cdot b - b \cdot b) =$
 $= a \cdot a - a \cdot b - a \cdot b + b \cdot b + 2 \cdot a \cdot b - 2 \cdot b \cdot b =$
 $= a \cdot a - 2 \cdot a \cdot b + b \cdot b + 2 \cdot a \cdot b - 2 \cdot b \cdot b$
 $= a \cdot a - b \cdot b$ a questo punto la 3^a espressione è uguale alla prima.

Fig. 5 - Silvio first of all checks the equivalence using his computing skills, once he made his conjecture he uses the properties of the operations (i.e. axioms) and a theorem to prove it. A translation of each statement is reported on the right of the image.

I think the 1st and the 3rd are equivalent, but not the 2nd, because applying the properties they become equal, while the 2nd does not.

I applied the distributive property.

I applied the distributive property on these two pieces.

I summed the two equal terms $-a \cdot b - a \cdot b$ and I cancelled its result with its opposite obtaining 0 for the 1st theorem.

I cancelled also $+b \cdot b$ with its opposite and as it was $-2b \cdot b$ I obtained $-b \cdot b$.

At this point the 3rd expression is equal to the 1st expression

Silvio (Fig. 5) begins reducing the second and the third expression in a form that makes easier comparing them with the first. This part of the protocol looks like typical protocols produced by pupils when asked to *compute* (ita.: “calcolare”) expressions. In this case Silvio is not required to compute expressions, but he uses his computing skills to produce a conjecture: as a result he finds out that the third expression is equivalent to the first. Note that Silvio’s explanation of how he produced his conjecture anticipates its proof; the properties of the operation, the *axioms* previously introduced, are used by the pupil already during the heuristic phase as *tools* to accomplish the specific task.

In the last part of the protocol, as required, Silvio writes a correct proof of the equivalence of the two expressions referring to axioms and theorems. In particular he refers to the “1st theorem” that the pupils proved on their own, such theorem states that “ $a-a=0$ ”. From a formal point of view the chain of equivalent expressions of the second part of the protocol represent a real proof, while the chain reported in the first part does not because steps are not explained referring to algebra theory.

Q) Sono equivalenti la prima e la terza? Mentre la seconda non è equivalente alle altre due perché attribui gli stessi valori ad a o b non dà lo stesso risultato.

$$(a-b) + a = a-b+b = a-a-b+b$$

proprietà della moltiplicazione (assioma)

$$(a-b) + a = (a-b) + a = a-b+b$$

$$(a-b) + a = (a-b) + a = a-b+b$$

Commutatività moltiplicazione (assioma)

$$(a-b) + a = (a-b) + a = a-b+b = a-a-b+b$$

secondo il nostro Teorema

$$(a-b) + a = a-a-b+b = 0$$

Fig. 6 - Marta substitutes letters with numbers to find out which of the three expressions are equivalent; nevertheless she uses axioms and theorems (as she remarks) to prove the equivalence of the first and the third expression.

The first and the third expressions are equivalent, while the second is not because giving the same numerical numbers to **a** and **b** the result is not the same of the other two.

Distributivity of multiplication (axiom).

Commutativity of multiplication (axiom).

Following our theorem this is **0**.

Differently from Silvio, Marta (Fig. 6), does not use the properties of the operations to produce her conjecture: she substitutes numbers to letters and computes the obtained expressions. Nevertheless, when proving the equivalence between the first and the third expression, Marta produces a correct formal proof. She reports, at each step, the axiom or theorem she is using and underlines the sub-expression to which each specific axiom/theorem is applied.

In particular, she correctly specifies (as required) whether any equivalence relationship is an axiom or a theorem. This distinction corresponds, in our teaching experiment, to the distinction between given principles (axioms) and relationships that were discovered and proved by the students (theorems); it finds its counterpart in L'Algebrista: axioms are represented by given buttons, theorems are produced by pupils with Il Teorematore.

Finally, the fact that Marta uses the words “our theorem”, referring to the “1st theorem” mentioned by Silvio, shows how she is conscious that she is using a theorem she produced together with the other pupils.

The last example we consider is the case of Marco (Fig. 7); he does not give any explanation of how he produced his conjecture and doesn't seem to be sure of what he found out: he says that he “thinks that” ... and he is going “to try to prove” the equivalence between the second and the first expression. What he does is to transform the second expression into the first one referring to the properties of the operation and to the buttons of *L'Algebrista*. Although he doesn't produce a correct proof, as the two expressions are not equivalent and he doesn't use correctly the axioms he mentions, Marco has taken a theoretical perspective: he is conscious that he has to produce a proof and he tries to base his reasoning on the given axioms and theorems represented by the buttons of *L'Algebrista*.

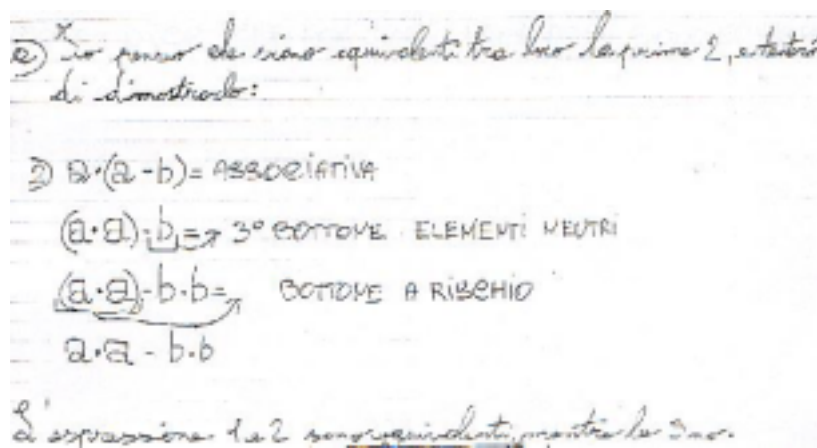


Fig. 7 - Marco tries to prove a wrong conjecture arriving to a wrong conclusion. It is notable how he is conscious that he is trying to produce a proof and how he tries to base his reasoning on the given axioms and theorems represented by the buttons of *L'Algebrista*.

I think that the first two expressions are equivalent, and I am going **to try** to prove it:

Associative property.

3rd button of neutral elements.

Risky button.

Expressions 1 and 2 are equivalent, while number 3 is not.

Conclusions

The development of information technologies raised many issues, one of those concerns the revision of school curricula taking into account the changes brought by this development. The ideas we presented in this paper give an example of a new way to approach symbolic manipulation (Ita. "calcolo

letterale"). Our proposal is to be considered in the broader perspective of the introduction of pupils to theoretical thinking. Thus symbolic manipulation has been interpreted taking a theoretical perspective and the particular software environment has been designed as embedding Algebra Theory.

The axioms incorporated in the buttons of *L'Algebrista* become tools that pupils can learn to use to transform expressions in order to attain activities' goals, and as such they can function as semiotic mediators. The distinction between buttons representing axioms, and buttons for computations, helps distinguishing the terms "proof" and "verification"; and may contribute to build the meaning of *proof* as well as the idea of *theory*. Furthermore the possibility of creating new theorems and making them usable, offered by "Il Teorematore", lets the student take part in the activity of *theory* evolution.

The presented protocols highlighted how some features of *L'Algebrista* can mediate some specific concepts related to algebra. In particular it is worth to observe that in the presented examples a central role, seems to be played by the particular set of activities: pupils refer explicitly to the history of the construction of their *theory* by using expressions like "*our theorem*" or "*1st theorem*".

Thus the following questions rise: what kind of activities may a teacher set up to exploit a tool to facilitate processes of semiotic mediation? Which of such processes may happen merely by using the specific tool? And which of them may happen and be effective only thanks to its integration in social interaction with peculiar activities? A research project was set up in order to study such questions. In particular the triangle Teacher-Microworld-Pupils will be studied in terms of semiotic mediation in the case of the joint use of *L'Algebrista* and *Cabri-géomètre*.

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A DIDACTICAL EXPERIENCE CARRIED OUT USING, AT THE SAME TIME, TWO DIFFERENT TOOLS: A CONCEPTUAL ONE AND A TECHNOLOGICAL ONE

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Abstract: *The authors have tested on a group of secondary school students (15-16 aged) a didactical path which has the aim of develop logical reasoning and to point out the fundamental elements and axioms of classical Euclidean Geometry. The basic methodological idea of this path is the use, strictly linked, of an unusual geometrical environment (the Mascheroni Geometry) as conceptual tool, and of the software Cabri-géomètre as technological tool. The path has been obtained, extending and completing a didactical proposal that the authors have already published in Italy. In this paper we describe and discuss this experience.*

1. Introduction

The authors, and in particular Luciana Zuccheri, who teaches on course for mathematical degree of the Trieste University, have observed a decrease of logical reasoning abilities in the students that begin the University. This observation is validated by mathematical tests proposed to fresher students of scientific disciplines (e.g. see: Boiti and Fiori 1997). This fact seems to be more serious than a lack of contents, which could be more easily remedied.

At pre-university level, the method of hypothetical-deductive reasoning may be mainly learned studying Euclidean Geometry and in Italy this matter must be treated in the two first years of secondary school (14-16 years old pupils). Nevertheless generally the teachers devote to Geometry few time and deal with this subject superficially. In opposite trend to this teaching praxis, recently many authors have underlined the importance, to formative aims, of the rational method in Geometry teaching (e.g. see: Bernardi 1997, Mammana and Villani 1998) and the pedagogical and didactical value of the geometrical constructions (see: Avron 1989). We think that one of the reasons of misunderstanding of the rational method by the pupils is that, at the beginning of secondary school (14-15 aged pupils), the teacher cannot point out in formal way the axioms of Geometry and must carry out the proofs partially using intuition, so at the end the pupils get an unsatisfactory idea both of Geometry

and of rational method. Therefore starting from the second year of secondary school (15 or more aged pupils) would be fruitful propose to the pupils a deeper analysis of the fundamental geometrical concepts, that they have learned earlier. But at this stage it is not possible to get back to the beginning and it is necessary a motivation, which excites the curiosity and the intuition of the pupils.

To reach this goal, we have tried to link an unusual (and old) geometrical environment (the Mascheroni Geometry) and a modern technology (the computer with the software Cabri Géomètre), using them as didactical tools, the first as a conceptual tool and the second as a technological one. The reasons of our choice are the following:

- In the Mascheroni Geometry, even if we remain in Euclidean environment, we must consider any geometrical object in a new aspect and we had the conviction that this could force the pupils to discuss again all the necessary theory related with the fundamental entities and axioms¹.
- Sharing the opinion of several authors (e.g. see: Noss and Hoyles 1996), we were convinced that the new didactical technologies and the new media could be very useful for suggest more effective ways of teaching the rational Geometry. In particular, we believe that a dynamical Geometry software like Cabri makes easier the understanding of geometrical constructions, it is useful to stress dynamically mathematical aspects and to explore new possibilities, and it could create problems related to computing approximation, which are interesting to discuss. Further the realization of “macro-constructions” points out the limit-situations and, writing the instructions, the pupils improve their language. These convictions come from experiences realized by the authors in pre-service and in-services courses for teachers, and by teachers of Didactics Research Group of Trieste which use Cabri for several years at middle school level (see: Rocco M. 1996). The software Cabri is well known and there is a very large literature about it (e.g. see the website: <http://www-cabri.imag.fr/>; for the Italian literature see the references contained in: Boieri 1996, Pellegrino and Zagabrio 1996).

Using the software Cabri, in a first time we have produced a didactic proposal for secondary school teachers (see: Gallopin and Zuccheri 1999). This contains a didactical path for 15 or more years old pupils, which leads step by step to the proof of Mohr-Mascheroni Theorem, which states that "Every geometrical problem, which can be solved by means of ruler and compasses, can be also solved by compasses only". To give the proof of the Theorem we constructed a little set of linked propositions, which is easy to review globally at the end.

¹ We refer, at this stage, only to the classical axioms of Euclid's Elements.

In a second moment, in the last months of the school year 1998/99, we have planned and realised the experience that we describe in the following.

2. Aims of the experiment

The principal aim of this experience was extend and complete a part of the didactical proposal contained in Gallopin-Zuccheri 1999, and to test it, observing in particular:

- a. About the mathematical contents, if they could excite the interest and if they are understandable from 15-16 aged pupils.
- b. About Cabri, if in this case it is useful to improve the comprehension of the mathematical contents.
- c. About Cabri, if in this case it is useful to improve the intuition, the formulation of conjectures and to suggest strategies for their proof.

Furthermore, we wanted that the pupils who participate to the experiment achieved learning objectives regarding the contents, and we wanted to give them a first approach to the university environment.

3. Organisation of the experiment

In order to realise the experience, we involved eight² students (15-16 aged) of a secondary school of Trieste (a "Scientific Lyceum"); half of them came from the third class of a traditional course, while the others came from the second class of an experimental course. The sample was composed in the same number of boys and girls; they participated of their own free will.

At the beginning we asked the pupils to fill a short questionnaire about their familiarity with computer, their knowledge of Cabri, their interest, and school issues in Mathematics. We got the following answers:

- Three pupils declared a good familiarity with computer, three a sufficient familiarity, two a not sufficient familiarity.
- Nobody knew Cabri.
- Six pupils declared a lot of interest in Mathematics, two a sufficient interest.

² We considered this a good number to permit that two researchers make a complete observation.

- Three pupils declared very good issues in Mathematics, three good issues, and two sufficient issues.

We developed the experience in three sessions of two hours, after school hours. In each of them we carried out a lesson using the following method: problem posing and solving, by means of collegial discussion guided by us, and individual work.

The experience took place in computer laboratories of the Trieste University. We have used: 9 personal computers (one for each student and one for us), a projector connected to our computer, a camera to record the sessions, a whiteboard, floppy disks and paper at student's disposal (they could save their Cabri paints and macro constructions, and could draw and help themselves in reasoning using traditional tools; at the end of the experiment, we got these to support our analysis). Cabri Géomètre version 1.7 for MS Dos, and Cabri Géomètre II were loaded in every computer.

At the end we proposed a new questionnaire with the aim of understand what they thought about the experience, the method and the matter they dealt with. We will explain and analyse its results in the last section of this paper.

4. Description of the didactical path

In the following we briefly describe the activities we carried out in the three sessions in which the experiment was divided. We analyse in particular the first, to explain how we tried to link the conceptual and the technological didactical tools. For more details from mathematical point of view, we refer to Gallopin and Zuccheri 1999.

4.1. First session. One researcher (Zuccheri) explains the subject using a whiteboard, the other (Gallopin) shows the aspects related to Cabri, using a computer connected with the projector. Each student has a computer, white paper and pencil.

Problem posing. We start with a brief introduction about the constructions by ruler and compasses, and their importance from historical and technical point of view. In particular, we deal with the three classical problems and we talk about the impossibility to solve them with ruler and compasses (*the pupils of the experimental course just knew it*). We explain the rules of these constructions. Any construction is formed by a finite number of steps, in each of them we can: a) trace a straight line, b) trace a circle, c) intersect two lines, d) intersect a line and a circle, e) intersect two circles. We formulate the Mohr-Mascheroni

Theorem (*it surprises them very much*) and propose them to proof it almost in part. We suggest that we must consider a), b), c), d), e) by compasses only.

Discussion. We discuss with the pupils about the meaning of the problem. What does "*to solve a geometrical problem by ruler and compasses*" and "*to solve a geometrical problem by compasses only*" mean? The main problem seems to be the construction of a straight line, which seems to be impossible using compasses only.

Deepening. The discussion leads to consider the difference between the ideal geometrical straight line and the line that we trace by the ruler. We conclude that a straight line is given if are given two its points (*the students remember the postulate "through two points passes one - and only one - straight line" and recognize that it is not trivial*³). We discuss about the possibility to find, starting from two initial points, other points of the same straight line.

Problem posing. Now we propose to find a way to carry out d) with compasses only.

Cabri. We explain the fundamental tools of Cabri⁴. We can use only point and circle (by centre and radius).

Deepening. We stress the connection between this way to assign a circle and the third Euclidean postulate ("*it is possible to trace a circle with any centre and any radius*").

Cabri. We show that the initial objects can be moved. Each pupil traces: a) two points O, Q; b) the circle of centre O passing through Q; c) two points A, B not belonging to the circumference, which represent the straight line AB intersecting the circle.

Individual work. Each student has the following tools: Cabri, paper and pencil.

Suggestion. We suggest to remember some well-known situation in which circles and straight lines are related.

First step. Many pupils remember the radical axis of two (or of a sheaf of) circles.

³ This is important, because this postulate is so intuitive that it seems obvious. For the didactical implications of this fact in learning processes see E. Fishbein 1998.

⁴ In this session we have used the version 1.7 for MSDos.

Suggestion. We suggest to analyse the situation of any two intersecting circles with their radical axis, and to find how we can use it to solve our problem.

Individual work. Each student has the following tools: Cabri, paper and pencil.

Second step. One pupil (A.⁵) says that, if we could trace a circle "*equal*" to the given circle, with respect to A and B (*we correct: "symmetric"*), we are sure that the straight line AB is the radical axis of the two circles. The intersections of the two circles are the intersection points of the given circle with the line AB.

Problem posing. We point out now the new problem: we have to find the construction of the symmetric of the given circle, with respect to the axis AB.

Discussion. Two symmetric circles must have symmetric centres and the same radius. The pupils try to find first the symmetric point of the centre O.

Individual work. Each student has the following tools: Cabri, paper and pencil.

Third step. A pupil (A.) solves the last problem, without suggestion, with paper and pencil. He traces a circle of centre A and radius AO, a circle of centre B and radius BO and intersects them. Their intersection is O', symmetric point of O with respect to AB. He explains the construction drawing on the whiteboard.

Individual work with Cabri. Now, in a part of the worksheet of Cabri, the pupils must individually realize the construction of the symmetric point. We explain the command intersection of two objects and its importance (*in the following we don't need to repeat this explanation*).

Cabri. We explain the macro constructions. We realize together the macro construction of the symmetric point (*the greatest difficulty for the pupils is formulate clear and concise instructions for users; they don't use a direct speech, like their textbooks*). Then we explain how to save the macro (*somebody has technical problems*). Finally we use the macro to get the symmetric of the centre O.

Problem posing. We pose the problem to find the symmetric of the given circle, stressing that they must have the same radius.

⁵ Here we put in evidence only the answers relevant to solve a problem.

The solution. A girl (*F.*) says that we have to find the symmetric of *Q* with respect to *AB* (*we think that this elegant solution has been forced by the way we used to assign the circle in Cabri; in general it is more natural to "transport" the radius, but it needs a special construction*). She applies the macro construction to *Q* and gets *Q'*. The intersection of the two circles of centre *O*, *O'* and radius respectively *OQ* and *O'Q'* produces the two intersection points of *AB* and the given circle (*we emphasise the solution drawing, and then erasing, with Cabri the straight line passing through AB*).

Discussion at the white-board. We discuss with the pupils the solution, drawing on the whiteboard.

Cabri. We use “aspect of the objects” to adorn the figure (*the pupils like to paint the figures*). We use the reconstruction step by step to overview our work (*all the pupils are very interested and satisfied, and, considering the numerous circles which appear, they recognize the utility of the macro construction*). We start to produce the macro construction of the intersection of a straight-line and a circle (*the pupils continue spontaneously by themselves*); at the end a pupil (*M.*) dictates the instructions for the user (*it is late, but they will finish the work*).

Discussion drawing with Cabri. With the pretext of testing the macro construction, we explore with the pupils if the solution is complete or not. We observe that the macro construction produces correctly the intersection points, if they exist (*we move the initial points until the straight line AB doesn't intersect the circle*). Finally we show that, if the initial points *A*, *B* are aligned with *O*, the construction doesn't produce the intersection points. We assign to the pupils, for the next session, the task to explain it.

4.2. Second session. In this session⁶ we propose construction problems by compasses only: to transport a segment, to double a segment, to bisect an angle. Nobody has solved the task proposed in the previous session: we discuss it (*theoretical and approximation problems arise*). The work is individual, with final collective discussion (*we observe the work style of anyone and give individual hints to perform the constructions with Cabri, when it is necessary*). The pupils can use the computer in the resolution process (*we observe that for somebody the computer make easier this process, but for others the technical difficulties hinder the process, or Cabri give no hints for the solution*).

4.3. Third session. We have 4 computer and the pupils work in pairs (*initially we had proposed to work without computer, but the pupils were disappointed*). We consider more formally the problem to intersect a circle with a straight line

⁶ In this session we use Cabri II, version for MS-Dos.

aligned with its centre (we need it for bisecting an angle). The construction is more difficult in this case. We explain the proof on the whiteboard, but it is too long and few pupils understand the explanation until the end. Then we lead the discussion about the way to establish with compasses only if three points are aligned (*a property that usually seems to be trivial*). This problem interests the pupils and they try to find a solution. With a little help they find a method and test it using Cabri.

5. Comments and conclusions

The questionnaire we proposed at the end contained the following questions:

1. Why did you participate in this experience?
2. What did you expect of this experiment?
3. Did you like the experience? Explain why.
4. Should you like to be involved in other experiences of the same kind?
5. Do you think that it could be interesting also for your classmates?
6. Did you meet any difficulty? And, if the answer is “yes”, did it arose from the subject or from the use of computer or from both?
7. Do you think that the computer helped you in the resolution process of the problems or, on the contrary, caused other complications?

The answers were the following:

1. Three students declared that they participated in the experiment because they were interested in the argument, five students in the use of computer.
2. Their expectation was the same of their motivation.
3. All pupils liked the made work; the reasons were very different: four pupils considered the matter interesting, two were enthusiast about the use of computer in geometry, two were satisfied to be actively involved in the lesson.
4. All answers were affirmative. The pupils were very satisfied of the experience (successively many of them asked if was possible repeat a similar experience the next school year).
5. Five pupils thought that the classmates could be interested in a similar experience; three didn't give a sure answer.
6. Three pupils didn't meet any difficulty, five declared they met very few difficulties, regarding the treated subject (not regarding the use of Cabri).

7. One pupil declared that the use of computer caused him more complications. Seven declared that the computer helped them, "*because it makes easier drawing the figure*" (6 similar answer) and "*because it makes easier the solution process*" (1 answer).

Furthermore, we have verified by individual interviews, during the experience, the learning of the principal contents, with satisfactory results (we didn't test it again in a second time, to verify the keeping of the knowledge after a long time, because of the end of the school year). On the basis of this, of their answers and of their interest during the work sessions we can conclude that we have reached the aim we proposed in Section 2 at a).

About the aim proposed in Section 2, at b) and c), we observed during the experience that Cabri is a very good didactical tool for the teacher *to explain and to give hints*, using all the possibilities (drawing and moving the figures, carrying out and verifying macro constructions, repeating the construction step by step), *with the aim to improve the comprehension and promote the formulation of conjectures and problem solving strategies*. The teacher plays an important role in this process.

About the possibility of improving the *pupils autonomous ability* of formulating conjectures and strategies for their proof, first of all we want to stress that, in our opinion, this sort of abilities need a long time of training and maturation to be developed. Indeed, in this short experiment we hoped only to get some indications. Observing the pupils' work-styles in problem solving activities which took place during the whole experiment, we noticed different behaviours. Now we restrict our analysis to the solution process of 5 problems of Mascheroni Geometry which we formulated very carefully and which the pupils had to solve using compasses only. They are the following:

1. Find the symmetric of a given point with respect to an axis.
2. Find the symmetric of a given circle with respect to an axis.
3. Duplicate a given segment.
4. Transport a given segment in a given point.
5. Bisect a given angle.

With respect to these problems, we observed the following four situations:

- I A. (a boy with very good results in Mathematics) solved 3 problems and F. (a girl with good results in Mathematics) solved 1 problem. They belonged to the same experimental course. Both worked intensively making various conjectures. We observed they worked using generally

first paper and pencil, and only in a second moment the computer. Therefore we can't get any conclusion about the possibility that Cabri helped their intuition in a relevant way.

- II *M.* (a boy with sufficient results in Mathematics belonging to the experimental course) and *B.* (a girl with good results in Mathematics belonging to the traditional course) solved 2 problem, using only the computer, without paper and pencil. *M.* had a good familiarity with the computer and liked it. *B.* declared an insufficient familiarity with the computer, but at the end declared that she liked very much working with the computer and that the computer helped her in the solution process. In this case we can say that Cabri helped their intuition. We think that the common reasons of success in these two cases were the strong motivation and their immediate familiarization with Cabri.
- III *L.* (a boy with good results in Mathematics belonging to the experimental course) declared a good familiarity with the use of computer (and we can confirm it), but he didn't realize autonomously any solution using it or using paper and pencil, throughout he seemed to be very happy to use the computer. We observed that his interest in computer was so strong that he didn't put attention to the geometrical problems. This hindered him to get any autonomous result.
- IV *D.* (a boy with very good results in Mathematics, belonged to the traditional course, with a sufficient familiarity with computer), *R.* (a girl with good results in Mathematics, belonged to the same traditional course, with not sufficient familiarity with computer), and *P.* (a girl with sufficient results in Mathematics, belonged to the same traditional course, with a sufficient familiarity with computer), didn't use paper and pencil to solve the problems, but they wanted use only the computer. Cabri didn't help them because they didn't solve autonomously any problem. Nevertheless at the end of the experience, they declared that the use of Cabri helped them: we think that they refer to the comprehension of the geometrical explanations given by us and by the other students. In our opinion, in these cases, the little knowledge of the use of computer has played a decisive role. In fact we observed that they concentrated their attention on the use of Cabri rather than on the geometrical problems.

As we have seen, someone had intuitions also without computer, others working with Cabri, others (having or not familiarity with the computer) weren't helped by Cabri. On the basis of these observations and considering also our personal experiences, we mind that, about the aim proposed in Section 2.c), we can conclude that, if the geometrical intuition is sufficiently developed and if the difficulties linked to the use of the software are overcome, the use of Cabri became conscious, useful and productive also in the direction of formulating

conjectures in problem solving activities.

The difference in the results obtained by *B.* and by the student of the group described in IV (in particular by *D.* and *R.*, which had results in Mathematics and initial familiarity with computer comparable with *B.*), puts in evidence that the familiarization with the software could need different training times. We think that it may be interesting to investigate if, after a longer training period, pupils with the same characteristics of the group IV can improve their abilities in formulating conjectures and in general in problem solving activities.

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WHAT BRINGS USE OF SPREADSHEETS IN THE CLASSROOM OF 11-YEARS OLDS?

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Abstract: *This paper describes the results of a pilot study realized in the fifth grade of school attendance in the Czech Republic (children aged 11) focused on use of spreadsheets in the classroom. The paper discusses some children results illustrating how Excel can influence children's reasoning.*

1. Background, aims of the study

The development of the use of modern technologies in last decade in the Czech Republic has been so dramatic that nobody could have expected it 10 years ago. From the initial idea, that only young people would be able to handle the computer, computers have become, for many people, the daily reality. This development should lead not only to the changes in the content of school subjects, but also, undoubtedly, large changes in the philosophy of education as a whole. This “living” technology could provide the pupil with flexible and rational knowledge, skills and competencies. That is why information technologies should be implemented into the education as an integral part of many activities (compare with global education) not only as a special subject. The usage of computer is a preparation on the period when the pupil leaves the school and would be expected to work and live in the environment that is not more possible without computers.

In Czech curricular materials it is explicitly said that: “The knowledge and skills obtained in mathematics are the prerequisites for understanding in sciences, economy, technologies and usage of computers. ... The pupils should learn simple statistical tables and diagrams, use of formulae and understand it, express in words and graphical tools relationships of the quantities in the society and nature, ...”(Mathematics – curriculum for 1st – 9th grade Czech School, 1996) To fulfil this target is possible with the support of computer but does not result from any activity with technology automatically. As far as the usage of computers will stress the procedures and sequences of buttons, it can lead to the stress and actually has opposite effects.

With the task to obtain a view how can the possibility to use computer with specific program – spreadsheets – influence the strategies of solving problems we conducted a pilot study (Tržilová - Hošpesová, 2000) with a group

of 10-years olds. Children solved 4 problems with the help of computer and program Excel. In this study the results of two problems will be discussed.

We created specific learning environment, constructive, in my opinion, that enables the children use computer or not for solving problems. In the course of experimental education we perceived the ability of adaptation of the learners to this new environment. Excel seems to us more flexible and user-modifiable so that the children can use this environment for various purposes. There are sets of problems for which spreadsheets are a useful problem-solving tool (activities that focus on data that can be collected from variety of sources, explorations into relations between numbers). There appears to be a minimum level of syntax, which children need to know before they can start to use them. The advantage is that a special long training is not necessary. In order to use Excel as mathematical tool pupils need only to be able:

- to orient in the table,
- to understand the way in which rules can be constructed from values,
- to understand what happen when the rule in one cell is replicated into other cells (compare: Ainley, Nardi, Pratt, 1999).

The aims of the study:

- To gain broader understanding of how the experience fit into the larger field of current mathematics education.
- What type of interactions goes on between a child and a computer?
- Impact in different mathematical domains

What is the influence of the use of computer on problem solving strategies of a child?

It is possible to use a spreadsheet to do mathematics after only a brief introduction. In our study children had no difficulty handling Excel. They seemed to understand what they were doing and they frequently could explain it well orally often by pointing at the screen, writing down their findings.

The use of computer strongly motivates the children to solve the problems (the outputs have good graphical quality, a lot of computation was done in short time).

Every pair of children solved the problem on their own level of understanding (finding some results, finding all results, and/or creating generalisation).

Method

In this contribution the study involved four lessons of experimental education. In first two lessons the children were given the opportunity to use/or not use the computer to solve the projects. In 3rd and 4th lesson were possible to solve the problem only with help of computer. Explanation of each spreadsheet's skill or concept was performed via demonstration at the computer or in discussion format. To solve the problems children entered numbers to spreadsheet and they used them for calculations (they created formulas and used a graphing facility). They noted the results in the worksheets. We observed these activities of pupils and their discussions in pairs. We focussed on next abilities of pupils:

- To predict further development according gained results (entering a value, examining the results, choosing another value based on newly calculated display).
- To decide what calculations is necessary to be done.
- To generalise the rule.

In the study children solved four projects (Tržilová - Hošpesová, 2000) with the support of program Excel. Here the results of two of them will be discussed.

Multiplication tables and their patterns

Problem:

Hana is the 2nd grade pupil. She learns the multiplication facts. There are some pattern in the rows of multiples which can help her to remember them, e. g. in the row of multiples of 5 there are only figures 0 and 5 as a number of units. Can you help Hana to find more patterns dealing with multiplication facts?

The main task of the project was to introduce the basic skills of using a computer with spreadsheets programme: to orient in the table, to enter number and formula in a cell, to copy a formula containing relative references to another cells). We suppose that children will solve the problem without the help of computer or that they would generate sequences of numbers and look for patterns.

Tables

Children solved two problems focused on creating and using formulas, reverse the formula.

Problems:

Fill in the table; firstly you have to increase each number for 7 and then to make it smaller seven times.

Number	735	959	2 345	3 654					
Changed number					3 787	259	6 594	3 598	4 221

Magic numbers are the products of two following numbers, e. g. 56 is magic because it is the product of numbers 7 and 8. Find out the numbers what must be multiplied to obtain these magic numbers.

1 560	4 830	3 782	552	...

Discussion of children's results*Multiplication tables and their patterns*

All involved children used computer to solve the problem. They created multiplication tables of various numbers and expressed results of their observations on different level of abstraction:

1. Children only created the multiplication tables on computer and rewrote them in the worksheet without comments.
2. Pupils realised the occurrence of different digits in the multiplication tables, as in the example given in text of the problem. Some examples of children's solutions are shown in fig. 1.

V násobcích čísla 5 jsou na konci čísla - př. 25 a 50 vždy 5, 0, 5, 0 atd.

V násobcích čísla 78 jsou na konci vždy čísla 8, 2, 6, 0, 4 a opět 8, 2, 6, 0, 4 a

V násobcích čísla 98 jsou na konci vždy čísla 8, 6, 4, 2, 0, což znamená vždy - 2 př. $8 - 2 = 6$ atd.

Translation:
 In the multiples of number 5 there are in the end numbers, e. g. 25 and 50 always 5, 0, 5, 0 etc.
 In the multiples of number 78 there are in the end always numbers 8, 2, 6, 0, 4 and again 8, 2, 6, 0 4 and in multiples of number 98 are in the end always numbers 8, 6, 4, 2, 0, that means always - 2, e. g. $8 - 2 = 6$ etc.

Fig. 1

3. Children made the rows of multiples of greater numbers and found out the similarities between them and basic number facts (fig. 2).

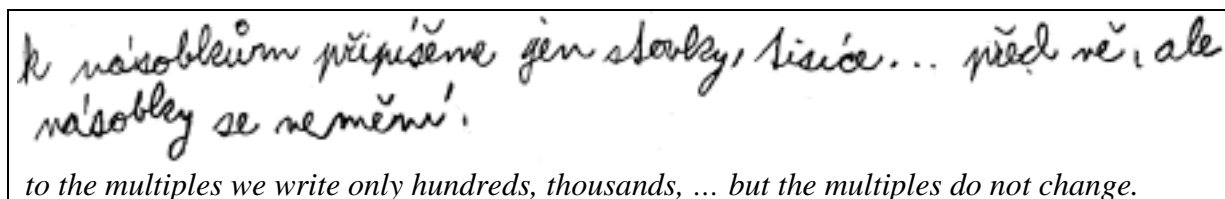


Fig. 2

4. Children did not create the multiplication table. They put down multiples of some numbers and try to find a pattern (fig. 3).

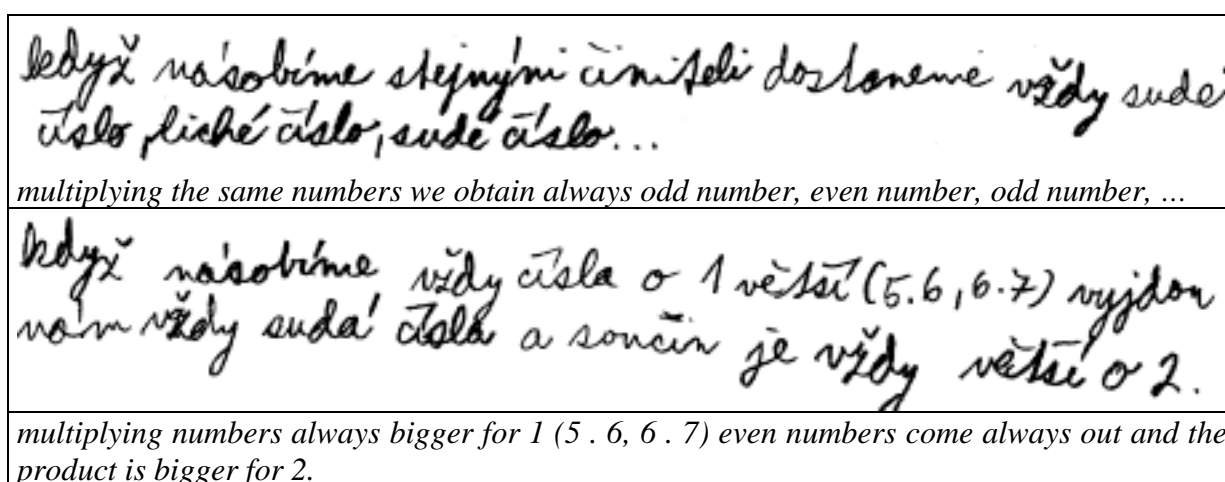


Fig. 3

Tables

All of participating children used computer to solve both problems. First of all the children create in Excel the same table as they have on work list. After some trials they were mostly able to solve the tables, even in case where the output numbers were given. The strategies again varied according the level of abstraction.

1. Children put in the formula for each number as in example given in fig. 4.

		A	B	C	E	F
1	Number	735	959	2 345	=E2/7-7	=F2/7-7
2	Changed number	=(A1+7)*7	=(B1+7)*7	=(C1+7)*7	3 787	259
3						

Fig. 4

2. Children put in the formula for the calculation in cell A2 and then copied it to other cells. For the computation with given output they put down the new formula in cell E1 and copy it for next columns.

3. Children put in the formula in cell B2 and copy it into some cells in line 2 (An arrow in Fig. 5 shows it.) In cells F2, G2, ... figure 0 occurred at the beginning. Children changed numbers in cell F1 (G1, ...) as long as a good result occurred.

	A	B	C	E	F	G	
1	Number	735	959	2 345			
2	Changed number	106	138	336	0	0	...
	Formula written in line 2	$=(B1+7)/7$					→

Fig.5

To solve the problem with magic numbers three strategies occurred:

1. children put in each cell formula for multiplication of concrete numbers, they seemed to manage their trials according to previous experience;

4 556	68*67
4 970	
4 692	70*71

2. children put in cells the formula for multiplication the numbers in columns, they managed their trials according to previous obtained products;

A	B	C
72	73	5 256
53	54	2 862
35	36	1 260
14	15	210
15	16	240
...
369	370	136 530
370	371	137 270

3. children used formulas to create two columns of following numbers and third column of their products, and then they picked up the demanded "magic" numbers.

Fig. 6

While the first and second strategy reminds the use of pocket calculator or written multiplication, the third strategy used fully the opportunities of computer and it is the starting point to introduce the iterative method of problem solving.

Working with spreadsheet children demonstrated a relatively high level of skill in handling data in all its forms (collecting information, drawing graphs, interpretation of collected information and graphs). This seems to be successful way in which referenced types of graphs can be introduced to children.

The move to advanced mathematical thinking involves a difficult transition, from a position where concepts have an intuitive basis founded on

experience. Where to obtain an experience with move, changes, and relationships between them?

Looking at numbers more from the point of view of relationships and structures, work with spreadsheets can be used to support pupils in the transition from arithmetic to algebra (Sutherland, Rojano). This transition involves manipulating general relations, operating on the unknown, working with functions and inverse functions, and developing formal algebraic methods. Pupils from primary to secondary education can learn to use the spreadsheets language to solve mathematical problems, and these experiences form a basis from which more traditional algebraic knowledge can be developed. In this sense, spreadsheets provide access to the potential of the algebra language, thus removing one of the main barriers to learning algebra. In addition the computer frees pupils from the arithmetic activity of evaluating expressions, thus enabling them to focus on the structural and algebraic aspects of a problem.

Conclusions

The adaptive process of the learners, it means the way the learners make sense of the feedback and derive from this an understanding of their own activity shapes the meaning they construct of the mathematics involved.

Acknowledgement

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A MULTI-DIMENSIONAL STUDY OF THE USE OF IC TECHNOLOGIES: THE CASE OF COMPUTER ALGEBRA SYSTEMS

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Abstract: *In the field of the educational use of CAS, a great body of research papers and reports on experimentation has been published. Still, questions exist about the uses of CAS and about the specific contribution of CAS. A meta-study of papers on IC Technologies covering research studies and experimentation carried out in various countries and published in the years 1994 to 1999 has been conducted by a team of researchers. A set of papers on the use of CAS has been included. Considering the position of these papers in the meta-study provides interesting insight into the specificity of this technology.*

Introduction

Computer Algebra Systems (CAS) have been available for more than 15 years and early research studies emphasised their potentialities for enhancing the students' practice of mathematics and for providing opportunities for a change in teaching. A lot of research studies, innovations, experimentation has been done and published. Still, interrogations remain about the uses of CAS and about the position of CAS among the other Information and Communication technologies.

- What would be a real integration of CAS into the into students' practices, learning and understanding of mathematics?
- What specific view of the use of Information and Communication (ICT) exists in research about CAS?

An opportunity for investigating these questions arose from a research project funded by the French Ministry of Education which offered to make use of existing published works to address questions on the use of ICT.

A team of didacticians belonging to five research teams worked on these questions (see notes 1 and 2). We were from various field of the use of ICT in the teaching and learning of mathematics. A first idea was the great diversity of the approaches and findings of papers in the field of ICT reflecting the

complexity of the integration of technology. Thus, as a difference with classical meta-studies which “translates results to a common metric and statistically explores relations between study characteristics and findings” (Bangert-Drowns and Rudner, 1991), we did not try to define a single metric. It appeared to us that looking at the integration of technology requires a plurality of perspectives or approaches and that a first stage of this “meta-work” would consist in a precise identification of these perspectives.

For this reason, our methodology was to conceive of possible perspectives or factors (that we named “dimensions”). Our aim was then to analyse writings on the introduction of ICT. As a writing focuses on a necessarily limited number of perspectives, we wanted to be able to identify in which dimension(s) its contribution is and what specific approach and results it brings into this(these) dimension(s).

Thus, our method was made up of two stages.

1. Considering a corpus of papers representative of experiments and research on the use of technology, to specify our set of dimensions.
2. Classifying a collection of papers through these dimensions, to determine clusters representative of typical perspectives of the use of technology and, inside the clusters, to relate the dimensions of analysis to the specific results obtained by the papers.

For us, the analysis should cover the varied fields of the use of computers into the teaching of mathematics: (dynamic) geometry, computer algebra and other algebraic software, arithmetic and graphical calculators, computerised learning environments... Computer Algebra is one of these fields. In the prospect of the above questions, I will focus in this paper on this field, drawing on the analysis of writings on CAS included in the general corpus.

A representative set of papers

The method consisted of starting with a corpus as large as possible to cover the whole range of publications regarding the nationality of the author(s), the type of work (report on innovation, presentation of software, research report, general reflection...), the domain of knowledge (numbers, algebra, geometry, calculus...) and the level of the students (primary, high school, college and university, pre and in service teachers).

We decided to cover the years 1994 to 1999, which appeared to produce enough diversity. We used a variety of international sources (The " Zentralblatt

für Didaktik der Mathematik" database with the entry "Computer Assisted Instruction", four international journals on mathematical education, seven international journals on computers for mathematics learning, books with chapters on technology and math education...) as well as French works (professional and research journals, philosophical dissertations, research and official reports...)

This resulted in a corpus of six hundred and sixty two published works. We analysed these papers first to establish the different dimensions in the approach of the use of technology (first stage), then to select a sub-set for a more detailed analysis of the dimensions and of the outcomes (second stage).

Among these six hundred and sixty two publications, one hundred and forty six (22%) were about CAS use. Among the other fields, only geometry has an approaching percentage. This high percentage could be surprising, because CAS technology is offered to a limited number of students of the upper secondary, college and university level and it not so widespread in the actual practice as compared with, for instance, numerical and graphical calculators.

This high percentage could be related to our choice of including all the papers issued of a journal dedicated to CAS educational use. This journal was first published from 1994 under the name "International Journal DERIVE". In 1997 it changed to the name "International Journal for Computer Algebra in Mathematics Education" (IJCAME)... As stated in the editorial of the first number after the change of name, "it exists to provide a medium by which a wide range of experiences in the use of symbolic algebra in mathematics education can be presented, discussed, criticised and best practice assimilated into the new curricula of schools, college and universities..." (Berry 1997). So, this journal offers a great interesting variety of papers and our choice to include all papers of this journal helped to consider a whole range of works, making sure that no approach of CAS use is neglected.

The IJCAME was not our single data source: forty eight publications came from a great variety of other journals or books. For instance, eight papers came from the proceedings of a European congress held in Montpellier in 1998 (Guin, 1999), four from the book of the Nato series "Advanced Educational Technologies for mathematics and Science" and three from the IFIP 98 book "IC technologies in school mathematics" (Tinsley, Johnson, 1998).

With regard to nationality and type of work the corpus of papers about CAS use is close to the general corpus. CAS use is relevant at pre-university and university level. Teachers at this level are generally also researchers used to writing papers in research journal. Another fact is that symbolic computation raises much interest in Math Education, even when not so much classroom use is

observed. So the high percentage of the CAS corpus inside our general corpus might reflect the great number of papers written in the field. From these considerations, a relative representativeness of this corpus in the field of CAS use seems reasonable (see note 3).

The first stage analysis was done by looking at the issues addressed in the publications. An abstract of these issues was established for each of the six hundred and sixty two publications. From this data, I did a qualitative classification of the hundred and forty six papers about CAS educational use. It is presented and analysed in the next paragraph.

Then we selected a sub-corpus for the second stage analysis. This second analysis had to enter more precisely into the study of relationship between specific questions about the integration of technology in schools and specific results obtained. So we identified, in the first stage corpus, publications which had sufficiently developed set of questions, methodology and findings. We did another selection to avoid papers too close in their analysis and findings. Finally, to avoid biases and to respect the diversity of approaches, we did arrangements to have a distribution of nations, types of hard and software, domains and levels similar to the initial corpus. So, we selected seventy nine papers. Among these, twenty one were in the field of CAS use. In a second part of the paper I will present this second stage analysis and the findings related to CAS.

A qualitative classification of papers about CAS educational use

We classified the papers of the CAS field into five great types.

Technical descriptions and optimistic postulates

Seventy seven papers (53%) started from a technical description of possibilities of CAS. We classified these papers into three subclasses with regard to the object they present and analyse.

1. Twenty eight papers offer just a presentation of technical capabilities of Symbolic Computation in a given software or calculator. They generally emphasise the potential of these capabilities for visualising, modelling or programming.
2. Sixteen papers refer this presentation to behaviours of students. This presentation is generally short and the technical capabilities presented in the paper are seen to make students' use of CAS very beneficial.
3. Thirty three papers are centred on the description of lessons. In some cases, they are parts of a new curriculum. Sometimes, data is given from the authors' experience of teaching these lessons.

Argumentative presentations of innovative classroom activities

Thirteen articles (9%) argued about innovative classroom activities using CAS. Contrasting with the above class, these papers gave reasons why CAS might be beneficial in these activities. They were teaching issues rather than research papers, and so their evaluation was generally based on the author's experience.

Papers assuming that CAS use will improve teaching and learning

Eighteen articles (12%) started clearly from assumptions of improvements resulting of CAS use by students. We distinguished two sub-classes with regard to the specificity of the assumptions

1. Eight papers started from general assumptions and offered little empirical evidence. Assumptions were about understanding, conceptualising, problem solving. They offered CAS use as means to save time and allow more emphasis on modelling. Authors also often saw CAS as a solution to learning problems.
2. Ten papers started from more specific assumptions, addressing one or more of the following issues: improvements of students' mathematical abilities by CAS use (problem solving, calculations...), enhancement of conceptions, better attitudes towards mathematics, change in the teaching strategies.

The assumptions were tested through the observation of a curriculum modified by the introduction of CAS. Papers starting from assumptions about students, generally looked for external evidence, comparing "experimental" and "control" students. The statistical procedure used in these papers supported not always strongly the assumptions. Papers starting from assumptions about teaching generally used indicators like the time devoted to the teacher's talk with regard to the time of the students' autonomous refection and compared observations of classroom situations with and without CAS. Changes were observed towards students working more individually or in small groups.

Paper centred on questions

Thirty one papers (21%) started from questions about the use of CAS. In contrast with the above papers, they did not presuppose advantages of this use. The observation of innovations, experimentation or examples of use were not presented for themselves, but as a help to address the questions. We distinguished six types of questions:

1. General questions (eight papers): limits and constraints of CAS, problems resulting from the use of CAS, types of understanding and thinking promoted by CAS use, tasks that CAS more easily tackles, procedures of use of CAS, software design and interface, tutorials and teacher training.

2. Theoretical questions (three papers): relationship between cognition and culture, as seen through the use of CAS, interaction of register of expression, discussion of theoretical concepts to help understanding CAS use (specific to this use, or general in Math Education).
3. Questions about students (four papers): use of DERIVE as a tool to 'diagnose' and remedy students' difficulties, exploitation of the knowledge developed through CAS use to help farther learning, difficulties of students associated with the use of DERIVE itself or in relating mathematical ideas to CAS output, differential attitudes towards CAS and other software.
4. Questions about teaching (three papers): teachers' behaviour or difficulties, abilities requested for using new technologies.
5. Questions about the instrument (two papers): comparison of the TI-92 and DERIVE's interface, comparison of calculation procedures and conceptions of number with paper/pencil and the TI-92.
6. Questions about situations of use of CAS (twelve papers): use of CAS functions versus menu commands, co-existence of students using DERIVE at home and other students, determinants for the resolution of non-standard problems using CAS, comparison of procedures using various software for a given task, changes of the 'didactical structure' of lessons, exams.

The integration of CAS into the school institutions

Seven papers (5%) were centred on the use of CAS in the everyday practice of teaching and learning in the existing school institutions (we name "integration" this introduction of a technology). The upshot is that they considered this use generally in the long term. These papers addressed the issue of the "ecological" aptitude of CAS to exist in the school institutions rather than the issue of "improvements". They got evidence through experimental data but generally not with a comparative methodology.

Our interpretation

The above classification shows a wide range of approaches to the use of CAS and specific contributions of these approaches for an integration of CAS into teaching and learning.

Papers with a technical approach of possible use of CAS, prevail. In these papers, real references to students' behaviour are scarce. So they cannot directly provide a support to the use of CAS in the classroom although they are a basis to create and experiment classroom situations.

Another approach is to present and argument innovative classroom situations. This approach is especially interesting when it reports on long time experiment of students' use of CAS. They could advantageously be followed by a discussion on the teacher's options.

Papers starting from assumptions stress on improvements that can be expected from the use of CAS. General assumptions on CAS support to students' conceptualisation and remedy to difficulties are often followed by optimistic conclusions. Other papers specify their assumptions into research hypothesis that they try by external comparison often completed by a statistical procedure. Ten years or so ago, classroom use of CAS was unusual and not easy to organise. So, innovators had to offer strong justifications for this use. The comparative approach was certainly consistent with this goal. Its drawback is that it gives little real insight in what is really going on when students use CAS.

It is probably a reason why more papers start from questions, implicitly admitting that students' use of CAS is not so straightforward. Very general questions have few answers. More precise questions on students, teachers, instrumental settings or situations of use bring to a sharper view. They help to specify dimensions of analysis of the introduction of technology.

The last class (type 5) gathers a small minority of papers explicitly addressing the issue of integration, which also implies to study questions, but with a specific approach to get insight on an ecologically sustainable use of ICT. Questions on tasks, procedures and conceptualisation, as well as on CAS as an instrument appear as not to be missed. They generally build specific methodology to account for the complexity of the integration.

The 'second stage' analysis

A set of dimensions and questions about the introduction of technology

In our prospect to look at the introduction of technology with a multi-dimensional framework, eight dimensions for the analysis of the introduction of technology were derived from the study of the papers of the corpus, including the above classification of papers in the CAS field. I will restrict in this paper my analysis to four dimensions that I found the most present in CAS papers.

1. the general approach of the introduction (“problematics”, type of hypothesis, of methodology, validation processes...)
2. the epistemological and semiotic dimension (Considering the mathematical knowledge at stake in technological settings, the possible

effects of its computer implementation. Looking at both the paper/pencil and the technological registers of expression and their connections).

3. the cognitive dimension (The theoretical framework used to analyse the student's functioning and learning processes)
4. the instrumental dimension (We distinguish here between artefact and instrument. While the artefact refers to the objective tool, the instrument refers to a mental construction of the tool by the user. Computer based technologies for doing mathematics are not just tools or artefacts. As instruments, they shape the mathematical activity and thinking. The instrument is not given with the artefact, it is built in a complex instrumental genesis).

We specified each of the dimensions by a set of questions. So we obtained a questionnaire of 96 questions. The answers to the questionnaire for each publication were assumed to give a picture of how this work focuses on each specific dimension.

CAS papers

In the seventy nine papers selected for this second stage analysis, twenty one were about CAS use (26%). Like in the other fields, we selected papers with a clear analysis of aspects of the use of technology. Thus, the twenty one papers come generally from the three last groups of the above qualitative classification (papers assuming improvements, papers investigating questions, studies of the integration into a curriculum).

Statistical procedure

For each of the seventy nine "second stage" papers, we established a detailed form with its theoretical framework, its set of questions, its methodology, its analysis and its conclusion. These forms were used first to answer the questionnaire, then to establish the outcomes and insights issuing from the papers. We designed a statistical procedure based on cluster analysis to get partitions. The classes (clusters) in these partitions gather publications focusing typically on specific entries of the questionnaire. We applied the procedure first to the whole set of questions of the questionnaire, then to subsets of questions based on one or two dimensions. We got 8 partitions and 45 clusters. Among these 45 clusters, 19 appeared significant.

Results

I look now at the four dimensions mentioned above, summing up the interpretation we gave for these clusters then considering the CAS papers belonging to each cluster in order to have insights on the specificity of analysis on CAS among other fields of the use of IC technologies.

1. the general approach of the introduction

A huge cluster gathers nearly half of the papers. The papers in this cluster generally focus on the students and investigate questions on effects of the introduction of technology ("does technology enhance student's achievement, conceptions, views of mathematics..?") with an experimental approach more frequently internal (comparing a priori analysis and classroom observations).

A smaller cluster (10%) considers also the students, but starts from assumptions of a better learning that they try to prove through comparative (external) evidence (like in the third class of our first stage analysis).

Papers in the CAS field are 9 in 35 of the first cluster, and three in seven of the second cluster. Discourses about the use of CAS (see for instance Pérez Fernandez, J., 1998) often put great emphasis on the advantages of CAS as appearing in these studies. Our statistical study shows that papers like this, once typical of the CAS field, tend to be not so many in the recent years.

2. the epistemological and semiotical dimension

A significant cluster in this dimension gathers 12 papers paying much attention to the relationship between the content knowledge at stake and the new means provided by technology. They look in particular to the changes that technology might bring in the mathematical practices, to the possible obstacles and to the semiotic aspects.

A bigger cluster gathers 29 papers with a weaker epistemological approach: they analyse the mathematical knowledge without looking precisely to the consequences of the introduction of technology.

We interpreted this as an evidence of the attention paid in the majority of papers to the mathematical knowledge at stake when learning with technology, but also of the difficulty of considering the consequences of the use of technology on this knowledge.

CAS papers are ten in the big cluster. In contrast, there is only one paper about CAS in the other cluster. Thus a weakness of the research on CAS might be that it does not consider more acutely the epistemological and semiotic influence of the use of symbolic computation. Probably the changes in the knowledge, in the registers of representation and the practices are difficult to see in CAS use.

3. the cognitive dimension

A big cluster (27 papers) in this dimension reflects a constructivist frame emphasising the potentialities of IC technologies for visualisation and action. A smaller cluster (9 papers) refers to the cognitive sciences and stresses on the support of new technologies to bring new cognitive tools for the students activity. Another small cluster (8 papers) draws on situated cognition and on the potentialities of IC technologies for connecting knowledge from varied experiences.

We observed that in many papers the constructivist approach of the big cluster works more like a formal reference than as a functional theoretical frame. In contrast, more specific concepts tend in the smaller clusters to help focus sharper on potentialities of technology.

Not many CAS papers are in these clusters (nine). Most of CAS papers present no explicit or implicit cognitive framework. Eight in nine are in the big cluster and some are good representatives of the above mentioned tendency to use a cognitive approach as a formal reference.

As a comparison, only two papers about geometry appear in the first cluster and nine appear in the small clusters. So, as a difference with geometry, the cognitive framework used by research on CAS seems to have not yet evolved to more functional concepts.

4. the instrumental dimension

In this dimension, we have three clusters. The first one (fourteen papers) focuses on a conjunction of several instrumental aspects of technology: the specific constraints of a technological tool shaping the action of the learner, the influence of instrumental settings (use in a computer room or on privately owned calculators), the organisation of classroom activity by the teacher.

Only two CAS publications are in this first cluster. One is Artigue et al. (1998), addressing the question of integration. The other (Drijvers, 1994) is a comparison of students procedures to solve the same problems with a graphing calculator and with CAS. This study provides insight on the specific constraints of different tools and on the role of students' familiarisation with the computer. Studies of this kind are rare, maybe because of the (wrong) idea that CAS would be directly the best mathematical instrument.

A second cluster (six papers) considers the importance of the time necessary for students to transform a technological tool into a mathematically

productive instrument. It is done mainly of papers about geometry and has no CAS paper.

A third cluster (eight papers) focuses on time in a different way. Papers in this cluster address the issue of the time that technology could save. Five papers in eight are about CAS. The emphasise of time that technology could save for the sake of students' conceptualisation is common in general discourses about CAS and the five papers address this issue. The interesting thing is that they divide in their approaches and findings. Papers like Kutzler (1997) draw perspectives for future trends in mathematics teaching from the assumption that CAS will save time for experimenting, problem solving ... Other papers, like Mayes (1994), question this assumption through an experimentation. They show that solving problems with technology is not so straightforward for students. More complex problems bring heavier cognitive load, and technology does not solve by itself all difficulties. Thus time and techniques are necessary to investigate problems.

Conclusion

Was does technology change in the learning? Is it better with technology? Obviously, innovations, experiments and research provide no direct answers. All depends on how we look at their approaches and findings. Research has to deal with the difficulty of accounting for the complexity of the phenomena related to the integration. From our meta-study, we offer a multi-dimensional framework as a method to look at this complexity from varied perspectives. In this paper, I tried to show how this method could help to consider a specific field -computer algebra systems– in the integration of technology and confront its approaches and findings with the other fields. Looking at the position of this field raises questions for the evolution of research.

1. The great number of "teaching issues" offering examples of educational use of symbolic computation is representative of the interest for CAS among a part of the teachers. Because they look at the new applications appearing day after day, these papers are potentially interesting contributions on the use of up to date technology. On the other hand, the diffusion of ideas among researchers and teachers is problematic when authors do not tackle the complexity of the educational situations. So, what research would be necessary to combine this wealth of propositions with rigorous and acute analysis?
2. Postulates on time saving and better opportunities for conceptualising marked early research on CAS use and are still a big trend in discourses about symbolic computation in education. Our study of papers in the years

1994 to 1999 shows a different picture of recent research. Mere external comparative studies based on postulates become uncommon and papers tend to question the manifold aspects of the changes that CAS brings, with less optimistic preconceptions. The issue of time remains a concern, although in research results CAS doesn't look different from other technologies like "graphing calculators" or "dynamic geometry"... This focus on time to be saved could be a disadvantage when it leaves other important instrumental issues in the dark.

3. More generally, research on CAS use illustrates properly the difficulty to tackle the multiple dimensions of the introduction of technology. In the little addressed dimensions (situational, institutional, teacher, man-machine interaction) isolated CAS papers exist like in other fields. In more common dimension, like the semio-epistemological, instrumental and cognitive dimension, research on CAS use seems to be behind the other fields' evolution. In my opinion, this is a consequence of a certain isolation of CAS research. So being aware of other's field approaches should help CAS research. Reciprocally, innovation and research on CAS educational use appears to be a very active field, and the wealth of experimentation and research done in this field should benefit the general understanding of the integration of technology.

Notes

1. The five research teams were DIDIREM (Université Paris 7), ERES (Université Montpellier II), Laboratoire Leibniz (IMAG Grenoble), LIUM (Université du Maine), Equipe TICE (IUFM Bretagne).
2. A complete documentation on this research, including the material for the statistical analysis is available on the Internet:
<http://www.maths.univrennes1.fr/~lagrange/cncre/rapport.htm>. See also Lagrange, Artigue, Laborde, & Trouche (2001).
3. Among the seventy four talks offered to the congress ICTMT4, twenty three were about CAS use (31 %).
<http://www.tech.plym.ac.uk/math/ctmhome/ictmt4.html> /Presentations at ICTMT-4, 9-13 August 1999.htm

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MEDIATION OF THE SPREADSHEET: COMPOSITION OF THE ARGUMENT

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Abstract: *In this paper we show how six pairs of students gave meaning to the formal notation of composition of functions through the mediation of a spreadsheet. The meaning for mathematical notions involved in activities was constructed and shared, mediated by the spreadsheet environment. The mediation of the spreadsheet allowed students to realize that one previous step—the composition of the argument—is necessary in the composition of two functions. Supported by the numerical nature of the spreadsheet and the interaction with the teacher, the task, which at the beginning was only arithmetical production for the students, acquired a deeper mathematical sense.*

Research question and framework

The use of technology to deal with mathematical notions is a trend that is awakening growing interest because of its great potential for facilitating and improving learning. Particularly, research reports that spreadsheet environments may support the processes of conceptualization and use of algebra (Dettori et al., 1994; Rojano and Sutherland, 1997; Rojano and Ursini, 1997). However, there are no studies focusing on the mediation of the instrument in the development of algebraic concepts and its associated use.

From a Vygotskian perspective, activities are *mediated* by tools (Werstch, 1991, Moll, 1990) because it is considered that when a tool is used, ‘it affects the object which is acted upon as much as it does the subject that uses it since it changes the type of activity and its cognition’ (Kozulin & Presseisen, 1995).

In this paper, we present a part of a wider study in which we analyze the mediation of a spreadsheet when it is used to work with activities related to the composition of functions. The main aim of this study was to analyze how students give meaning to the formal notation of composition of functions through the mediation of a spreadsheet.

Here we present an analysis of the process of how students give meaning to the formal notation of composition of functions through the mediation of the spreadsheet.

It is known that, with the usual approaches in paper and pencil environments, students experience serious difficulties when working with composition of functions (Ayers et al., 1988). In these environments, it is necessary to resort to analytical representation, which often creates problems for students (Kieran, 1992). In contrast, when the electronic spreadsheet is used, composition of functions is worked at a numerical level, resulting in an operation with the same level of complexity as addition, subtraction, product or division of functions. The spreadsheet permits students to compose functions using a process whose complexity they are not always aware of. The task of labelling columns by writing an analytical expression can induce students to reflection and allows them to become aware of the operations they are carrying out with the computer (Landa & Ursini, 2000)

Methodology

The study was conducted with a group of 12 students (14-15 years old) working in pairs. These students had not yet had formal instruction on the notion of function and had had no experience in using the electronic spreadsheet. After a brief introduction to the use of the spreadsheet, students were given tasks that involved the idea of composition of functions. A total of 8 hour-and-a-half sessions were conducted in the computer laboratory of the school. The researcher was the group's teacher. The students worked in pairs on the computer. The data consisted of the students' written reports of the results they obtained: the macros of Excel recorded during the sessions, the notes taken by the researcher during the sessions and the record of verbal exchanges between students and between students and the researcher.

Activities and Results

The nature of the spreadsheet is arithmetic. Visual priority is on numeric values obtained either with arithmetic operations on numbers or with formulas that relate the contents of cells to others. The automatic way of producing columns of numbers—sweeping the mouse down—eclipses the formulas. Students do not spontaneously pay attention to the formulas produced with relative references and to the relationship between cells. It is necessary to lead students through a

process of reflection in order to help them consider the relationship between formulas and the number produced in each cell.

It is this relationship we are concerned with because it may imply composition of functions. Either a column or a row is generated by sweeping a formula with the mouse. For example, to generate the sequence 1, 2, 3, ..., n in column A, we can sweep down the formula $=A1+1$ if the content of cell A1 is the number 1. Thus, we could consider $f(0)=1$; $f(f(0))=2$; $f(f(f(0)))=3$; ... $f(\dots f(0))=n$ for $f(x)=x+1$. On the other hand, if we write the formula $=\text{sqrt}(A1)$ in B1 and sweep the formula with the mouse to generate the row, in each cell we obtain numbers corresponding to the formulas $=\text{sqrt}(B1)$; $\text{sqrt}(C1)$;... which could be interpreted as $=\text{sqrt}(\text{sqrt}(A1))$; $\text{sqrt}(\text{sqrt}(\text{sqrt}(A1)))$; ... respectively.

In our research we tried to lead students to this kind of reflection by asking them to generate columns of numbers using formulas and produce analytic expressions to label the generated columns. This requirement should lead students to deduce an algebraic expression representing the general formulas from the observation of the structure of the particular formula, for example, to generate the analytical expression $(x+1/x)$ from the formulas $=A3+1/A3$; $=A4+1/A4$; $=A5+1/A5$; ...

After generating the sequence of numbers 1, 2, 3... in column A by 'adding 1 to the cell above' and to label this column with x, students had to produce numbers in column B obtained by subtracting 5 from the numbers in column A. They were asked to label column B with an analytical expression that would reflect the operation performed. Although they had no difficulties in generating column B, a group discussion was needed in order to help them produce the expression $x - 5$ to label the column. After this experience they were required to generate column C by 'adding 10 to the numbers in column A and dividing all by 2'. Without exception, the six pairs of students wrote the formula $=A3+10/2$ (the number 1 was in cell A3). The numeric result obtained, 6, did not match the expected result, 5.5. We realized that they were using their arithmetic knowledge to control their interaction with the spreadsheet. After mentioning that parentheses could be used, all the students wrote $=(A3+10)/2$, swept the mouse down and labelled column C with the expression $(x+10)/2$.

The meaning of the symbolic expressions $x-5$, $(x+10)/2$ are directly related to the corresponding column of numbers. These kinds of expressions hold a description of the operations performed in order to generate the columns. In order to produce them, it is necessary to focus on the particular operations, to deduce the general operation and to express this through an analytical expression. Producing these expressions can be considered a first level of abstraction in which students no longer pay attention to the specific numbers in the cells, but instead they focus their attention on the process performed to

obtain them. The symbolic expressions ‘says’ which procedure was done (e.g. $x-5$), and at the same time this notation is a generalization of the specific formulas $=A3-5$, $=A4-5$ In this way, labelling columns with symbolic expressions induces students to take into account all these aspects.

In a subsequent activity, after generating a sequence of numbers in column A, students were asked to generate column B with the square roots corresponding to these numbers and column C with the square roots of the numbers obtained in column B. Once more they had to label columns A, B and C with appropriate expressions. In this occasion students were able to label the three columns. They wrote x for column A, $\text{sqrt}(x)$ for column B and $\text{sqrt}(\text{sqrt}(x))$ for column C. They identified the last expression with the notation $f(f(x))$ (Landa and Ursini, 2000). Moreover, students were able to generate columns of numbers with the functions $f(x)=\text{sqrt}(x)$ and $g(x)=\sin(x)$ when they were proposed. After generating a sequence of natural numbers, labelled x , they generated columns of numbers corresponding to $\text{sqrt}(x-5)$, $\sin(x+4)$, $\text{sqrt}(\sin(x))$ and $\sin(\text{sqrt}(x))$ and they labelled them with $f(x-5)$, $g(x+4)$, $f(g(x))$ and $g(f(x))$, respectively.

Although the notation used by students suggests composition of functions, there is no evidence that students were actually taking this notion into account. Their activity during this task was reduced to identifying the symbols f and g with the operations performed with numbers. This suggests that students were identifying the functions f and g with the operators sqrt and \sin respectively according to the schema:

$$\begin{array}{c} f \quad (\quad g \quad (x)) \\ \uparrow \quad \uparrow \\ \text{sqrt} \quad (\quad \sin \quad (x)) \end{array}$$

The main idea of composition of functions, that is, to act on an argument created by the application of the same or of another function, was missed. These results suggest that composing functions in a spreadsheet environment could leave out the notion of composition, at least when the functions involved have notational names.

In order to help students work with composition of functions, activities involving functions without notational names were designed. Our aim was to avoid students’ identifying a function with an operator. A worksheet containing the following indications was given to students:

1. Use a formula to produce a sequence of numbers in column A. Name this column x .
2. Use a formula to fill column B by adding its reciprocal to each of the

numbers in column A. Name this column $f(x)$. Additionally, label this column with the analytical expression corresponding to the formula used.

3. Produce column C according to the expression $f(x)+9$. Label this column with the analytical expression corresponding to the formula used.
4. Produce column D according to the expression $f(x+9)$. Label this column with the analytical expression corresponding to the formula used.

Points 3 and 4 were aimed at helping students focus on the argument of the function.

Students produced column A using a formula. They produced column B by sweeping down the formula $=A3+1/A3$ using the mouse (table 1). They labeled it by writing $x+1/x$. Column C was generated by the formula $=B3+9$ and was labeled $x+1/x+9$.

	A	B	C	D
1		$f(x)$	$f(x)+9$	$f(x+9)$
2	x	$x+1/x$	$x+1/x+9$	
3	1	$=A3+1/A3$	$=B3+9$	10.1
4	2	2.50	11.50	11.0909091
5	3	3.33	12.33	12.0833333
6	4	4.25	13.25	
7	5	5.20	14.20	
8	6	6.17	15.17	
9	7	7.14	16.14	
10	8	8.13	17.13	
11	9	9.11	18.11	
12	10	10.1	19.1	
13	11	11.0909091		
14	12	12.0833333		

Table 1

It seems that students were identifying the function $f(x)$ with the column B (see Table 1, cell C3). This suggests that $f(x)$ was not seen as a process but as an entity to which they were adding the number 9.

Difficulties arose when they had to work with $f(x+9)$ to fill column D. For some students this functional notation evoked multiplication. Several of them considered that $f(x+9)$ indicated that the content of column B had to be multiplied by $(x+9)$. Therefore, to fill column D they used the formula $=B3*(A3+9)$. There were other students asking each other ‘But who is f here?’ or claiming ‘There’s no f here!’.

We realized that an obstacle was emerging with this activity. When generating numbers in column C according to the expression $f(x) + 9$, the argument of the function was not the focus of their attention. Both, the notation $f(x) + 9$ and the analytical expression they wrote were expressing the addition of two well-defined entities: the function identified with column B and the number 9. Students were realizing now that $f(x+9)$ was something different from $f(x) + 9$. They noticed that parentheses were not in the same position, but they did not understand the implication of this difference.

Until that moment, the spreadsheet had been a useful tool helping students to make sense of functional notation. But faced with the notation $f(x+9)$, this tool alone did not provide enough support to help them to make sense of the functional notation. The researcher's intervention was needed. The following is a dialog sustained with a pair of students. There were similar verbal exchanges with the other pairs.

Researcher: *What does $f(x)$ do?*

Student: *It adds its reciprocal to x*

Researcher: *Which is the value of x considered here?* (pointing at cell B3 where number 2 was displayed)

Student: *Number 1.* (pointing at cell A3)

Researcher: *Now, tell me which value of x was considered here.* (pointing at cell B12 which displayed number 10.1)

Student: *Number 10.*

Researcher: *$f(x+9)$ means that you add 9 to x first and then you apply the function $f(x)$ using this new value. What value of x must be considered to fill this cell?* (pointing at cell D3)

Student: *Number 1.*

Researcher: *1 plus 9. How much is it?*

Student: *10*

Researcher: *What is the value of $f(x)$ when x is 10?*

Student: *10.1* (looking at cell A12 and pointing at 10.1 in cell B12)

Researcher: *Therefore, what is the value we expect to have in this cell?* (pointing at cell D3)

Student: *10.1*

After verbal exchanges like this, students started to look for a way to produce the number 10.1 in cell D3. In this way they were identifying the function not only with column B, but with the process leading to its generation. They were focussing as well on the argument $x+9$, which did not have a column as reference.

The verbal exchanges seem to have been effective because students started to work in order to overcome the detected obstacle and they followed

different strategies in order to make sense of the expression $f(x+9)$, although all of them were determined by the constraints of the spreadsheet. Three main strategies were observed and these are described below.

Strategy 1

Using an arithmetic approach, students introduced arithmetic expressions $=10+1/10$, $=11+1/11$, $=12+1/12$... in cells D3, D4, D5, ..., respectively. The goal was to obtain the same numbers in cells D3, D4, D5, ... as those that appeared in cells B12, B13, B14, ... (see table 1). The arithmetic expressions introduced suggest that to produce them they were considering the argument $x+9$ mentally calculating the numbers on which the function was acting: number 10 corresponding to $1+9$, 11 to $2+9$, 12 to $3+9$,

Asked to use formulas in order to generate the same numbers, they wrote the formulas $=A3+9+1/10$, $=A4+9+1/11$, $=A5+9+1/12$ in cells D3, D4 and D5, respectively, obtaining the expected results. These formulas show the students' first attempt at expressing the mental operations which they had previously carried out. However, the formulation of the expressions continued to include a specific number that they had already calculated mentally, expressing it by writing $1/10$, $1/11$, $1/12$ in their formulas.

After this they swept down the formula $=A5+9+1/12$ written in cell D5. When they did not obtain the expected results in cells D6, D7, D8..., they reviewed the formulas used and switched to the formula $=A3+9+1/A3+9$ in which the operation leading to the argument is explicitly displayed but still incorrectly written. In the next step they introduced parentheses and swept down the formula $=A3+9+1/(A3+9)$. They had no difficulty in producing the analytical expression $x+9+1/(x+9)$ to label column D.

Strategy 2

Students using this strategy produced first a column of numbers using the formula $=A3+9$ in cell D3 (table 2). In this way they were working directly on the argument as an isolated entity assigning it a place on the spreadsheet in a well-identified column. They labelled it $x+9$. After this they generated column E with the formula $=D3+1/D3$ and wrote $x+9+1/x+9$ as its symbolic expression. Because they did not need to use parentheses to produce the column of numbers, they did not realize that it was necessary to include them in their symbolic expression.

	A	B	C	D	E
1		$f(x)$	$f(x)+9$		$f(x+9)$
2	x	$x+1/x$	$x+1/x+9$	$x+9$	$x+9+1/x+9$
3	1	=A3+1/A3	=B3+9	=A3+9	=D3+1/D3
4	2	2.50	11.50	11	11.0909091
5	3	3.33	12.33	12	12.0833333
6	4	4.25	13.25	13	
7	5	5.20	14.20	14	

Table 2

Strategy 3

When using this strategy students used the content of cell A12 as the argument of the function. They wrote =A12+1/A12 in cell D3 (table 3) and swept this formula down with the mouse. They got the expected values and they labelled column D with the analytical expression $(x+9)+1/(x+9)$.

This strategy suggests that, for these students, the argument $x+9$ was already a well-defined entity and that it was already displayed on the spreadsheet, in column A. In fact, the contents of cell A12, A13, etc. correspond to $1+9$, $2+9$, etc. When using this strategy the argument was composed ‘visually.’

	A	B	C	D	E	F
1		$f(x)$	$f(x)+9$	$f(x+9)$		
2	x	$x+1/x$				
3	1	2	11	=A12+1/A12		
4	2	2.5	11.5	=A13+1/A13		
5	3	3.33333333	12.3333333	=A14+1/A14		
6	4	4.25	13.25			
7	5	5.2	14.2			
8	6	6.16666667	15.1666667			
9	7	7.14285714	16.1428571			
10	8	8.125	17.125			
11	9	9.11111111	18.1111111			
12	10	10.1	19.1			
13	11	11.0909091	20.0909091			
14	12	12.0833333	21.0833333			
15	13	13.0769231	22.0769231			

Table 3

There were pairs who used only one of the strategies mentioned above and pairs who used a combination of two of them.

Conclusions

How do tools and technology mediate learning? This is an example of mediation of a spreadsheet, showing how students' misunderstanding can be clarified for the teacher but also for the students.

The results obtained show that the spreadsheet environment helped students become aware of the importance of the argument on which a function is acting. This environment offered them the possibility to separate and to make the different entities involved explicit in the composition of functions: the variable x ; the function acting on this variable, $f(x)$; the composed argument, $x+9$; and finally the function acting on the composed argument, $f(x+9)$. In this environment the necessity of becoming aware of the *composition of the argument* as a step previous to the composition of functions emerged quite clearly. This suggests a temporal order for the composition of functions in which argument is dealt with first and then used as the argument of the function.

Finally, we want to emphasize that the spreadsheet was not only mediating students' construction of the meaning they gave to the formal notation of the composition of functions, but also the researchers' observations of this process. The spreadsheet became a magnifying glass that allowed us to detect the relevant moments involved in this process.

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MATHEMATICS, TECHNOLOGY AND EXAMINATION IN DISTANCE EDUCATION

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A teacher says: It is the examination that controls everything, and it is the examination that directs the unofficial syllabus. ... The examination is so important that it is meaningless to do anything in terms of new pedagogical methods if you don't do something about the examination at the same time. The examination represents the entire pedagogy; it is the examination that decides what and how people learn. If I want students to learn more deeply — and I really do — then I have to change the examination; that's the only way that really works.

(Högskoleverket [National Agency for Higher Education], 1997, p. 21)

Introduction

It is important to realize the weight and importance the examination has when discussing possible changes to a course or a program. If the examination is not changed, then nothing is really changed in a course. The students rapidly interpret the important survival code in a course and adjust their learning and performance strategies according to it. We will discuss some experiences from a program in teacher education and point out situations, which become more visible when teaching and learning mathematics over the Internet. At the University of Gothenburg, we offer, among other programs, programs for elementary teachers who want to take an examination to become mathematics and natural science teachers for grades 1 to 7. We also offer a program for mathematics and natural science teachers for grades 4 to 9 to become teachers for the Gymnasium (grades 10 to 12) in mathematics and physics. Both are three-year programs (which are conducted at half the usual pace) and are given entirely by means of distance media, including e-mail, Web pages, video films, and “classroom discussion” through the computer conference system First Class. The first program started in fall 1997, and every fall since then a growing number of students have been accepted into the program. The program for mathematics and natural science teachers for grades 4 to 9 started in the fall of 2000.

To teach and learn mathematics over the Internet

In today's education an increasing emphasis is put on students' learning through problem-oriented or problem-based educational methods. The underlying idea is to improve the quality of students' learning about complex problems or phenomena in the world through assignments that give rich opportunities for active investigation, analysis, and reflection (Bowden & Marton, 1998; NCTM 2000). Such methods entail an increased use of a wide variety of different information sources. When studying mathematics, students also use tools like graphing and symbol-manipulating calculators and a variety of computer programs like MS-Excel, Graphmatica, and others. Students use assorted textbooks and other reference books, and many of them also use their family members, friends, colleagues, or maybe neighbours as a reference group.

One could argue that the way distance students study is closer to the way people ordinarily work than traditional in-class study. In many walks of life, people are valued for the everyday jobs or projects they do, their ability to work with others, their responses to problem situations, and their capacity to find tools or information that will help them to fulfil an assignment. In that kind of occupation as well as in distance studies, it is important to be open and flexible in one's learning. It is desirable and would be natural if the examinations in distance-education courses in mathematics could mirror that fact. Surprisingly, many distance education courses in mathematics, in Sweden as well as in other countries, seem to lose their flexibility when it comes to the issue of examinations. Students are encouraged to use graphic calculators and suitable computer packages when they are studying, but that way of working is not taken advantage of in examinations. When it comes to an examination, the students must be identified and sit in a guarded and controlled room in a school close to where they live.

During their schooling, students inevitably try to identify, interpret, and follow authority (Lingefjård, 2000; Lingefjård & Kilpatrick, 1998; Cobb, 1986). One interpretation of this social behaviour is that the search for trustworthy authority is part of the human survival instinct. That instinct does not disappear when students begin their university studies, although the search for authorities or survival structures may be more hidden the older and more sophisticated they get. Nevertheless, there is always a didactical situation. In the case of distance education, there are naturally other sources of knowledge besides the student's personal knowledge that the student can rely upon.

In the complexity of a situation in which students are always away from the teacher when reflecting and learning mathematics at their own convenience, using calculators and computers from time to time, it is hard to describe all the relations that occur. The discussions in which the students take part nearly

always have a third, silent partner: the calculator or computer software and its result. The third partner in the discussion changes the didactical contract between the students and the instructor at the same time that the situation of distance learning also changes the didactical situation.

Assessment & Examination

Contrary to past views of learning, the cognitive psychology of today (Marton & Booth, 1997) suggests that learning is not linear but proceeds in many directions at once and at an uneven pace. People of all ages and ability levels constantly use and refine concepts. Furthermore, there is tremendous variety in the modes and speed with which people acquire knowledge, in the attention and memory capabilities they can apply to knowledge acquisition and performance, and in the ways in which they can demonstrate the personal meaning they have created. Current evidence about the nature of learning makes it apparent that instruction that strongly emphasizes structured drill and practice on discrete, factual knowledge does students a major disservice (Ramsden 1992, Marton & Booth, 1997). Acquisition of knowledge skills is not sufficient to make one into a competent thinker or problem solver. People also need to acquire the disposition to use their skills and strategies, as well as the knowledge of when and how to apply them. These are appropriate targets for assessment.

If one adds the component of existing technology, assessment becomes even more complicated. The support to be provided by technology when students are being assessed is a difficult issue and the subject of ongoing discussion in several places around the world. An essential consideration is whether students using, say, a computer program when they are learning should therefore be allowed to interact with that program when being assessed in mathematics. We have to find ways of assessing what is looked upon as important, rather than assessing what is easily measurable. In other words, we have to deal with the truism that, in mathematics education, what is assessed is what is valued, and what is valued is what is assessed (Arnold, Shiu & Ellerton, 1996).

If mathematics teachers allow group work, discussion, and information gathering in libraries and over the Internet, and also want students to learn more mathematics in collaborative work, then they face great demands on what types of problems they should pose. Silver and Kilpatrick (1989) argue for the use of open-ended problems in the assessment of mathematical problem solving, thereby moving from facts and procedures to concepts and structures. A relevant problem should encourage students to make various assumptions and use various strategies in which technology can serve as an aid but the technology

part in the solving process should never be a goal in it self. The problems teachers choose also need to provide the students with opportunities to express what they have learned in the course and in previous courses. At the same time that the problem should remain nontrivial in the presence of technological tools, their use should not be the only performance component that is essential and leads to success (Lingefjård, 2000; Lingefjård & Holmquist, 2001).

An essential part of the work inside the assessment process is how the teacher delivers student support. It is not always easy to handle the transition from conventional tuition to flexible and distance learning tuition. When feeding back to students some sense of the progress they are making in the course, a teacher is working in the field of informal assessment. Despite the fact that some of this is carried out by course material the teacher has the most important role in informal assessment. Nevertheless, our focus is more on formal assessment carried out by teachers. This is a very important part of students' feedback; it could be that such feedback is the most obvious teaching contact they have with their teacher during the coursework. Giving such feedback is a difficult task – the students will not understand, neither too kind nor too direct feedback in an assessment situation at a distance (Simpson, 2000).

In order to deal with the assessment questions in reality and to investigate potentials and obstacles in alternative assessment in mathematics education, we are at present running a research project funded by the Distance Education Authority (*Distansutbildningsmyndigheten* [DISTUM]) in Sweden. The study is strongly connected to the program for mathematics and natural science teachers for grades 4 to 9 to become teachers for the Gymnasium (grades 10 to 12) in mathematics and physics.

The first course in the program, a one-year course in mathematics at the University of Gothenburg known as MAL610, has a variety of objectives, all related to the main aims of learning and teaching mathematics and to the didactics of mathematics (for a definition of “aims” and “objectives,” see Rowntree, 1994, p. 50). The fact that the course plan is the first document to be accessed on the homepage for the course should, hopefully, give the students an overview of the course: <http://ma-serv.did.gu.se/matematik/mal610/mal610.htm>.

Course Structure

The course involves practice in mathematics by means of textbook problems, larger project-oriented assignments, and a final examination (Bowden & Marton, 1998; NCTM, 2000). The course is based upon working by means of information and communication technology (Webb, 1992). Nearly all communication is done in that way: individual work, group discussions with other course participants, and discussions with advisors (Blomhøj, 1993).

Course Model

To characterize the model used in this course, we use the classification of course models by Richard and Rohdin (1995). This course can be classified as being close to a third-generation course, with a dominating duplex communication that uses a number of facilities to enable maximal communication without interfering with independence over time and space (see media below). According to Bååth (1996), the course could be classified either as tutorial guidance, where the students control the learning but the teacher is available for guidance, or as a course with good structure and good opportunities for dialogues.

Course Media

The course contains both face-to-face lectures at the start of the semester, virtual discussion groups (by way of First Class), e-mail contact, and a telephone evening once a month when the students are offered online guidance. Most of the course material (except the related literature) is available on the course's homepage. On the homepage, students can access general course documents, exercises, and assignments. This Web page also contains tutorials (via Power Point) on the use of graphing calculators and Graphmatica (freeware used in the course). The students are also encouraged to make sure that they have MS-Excel installed on their computer and to start using it. The Web page is in one sense the most important "location" of the whole course: Even though the course activities are spread out across different media, groups, and locations, the students can always come back to this page to find their way out of the task jungle.

Participants in the Course and in the Study

In the fall of 2000, the program for natural science teachers for grades 4 to 9 to become teachers for the Gymnasium (grades 10 to 12) in mathematics and physics enrolled 23 students. They ranged in age from 26 to 49 years, with a median of 35.7 years and a mode of 32.5. Twelve were women, and most of them were experienced teachers. They all started the program with a one-year mathematics course that may be seen as consisting of four parts: calculus of one variable; discrete mathematics; integrals, curves, and series; and calculus of several variables.

Structure of Course Parts

Based on the idea of constructing a variety of project problems for the course, we constructed different problems according to the different content areas above. We present one such problem and some reflections connected to this kind of assessment.

A Project Problem

Meta cognition

Meta cognition or what Piaget once called Reflective intelligence is something central for many teachers of mathematics as well as researchers in mathematics education. To expand ones own thinking on a mathematical problem from “just solving it” to actually understand how one solve it is of major importance for research on how students learn mathematics.

See for example *The psychology of Learning Mathematics* by Richard Skemp (1978).

Solve the following problem and try at the same time to do a careful and throughout analysis of how you were thinking when you solved it.

Problem: Let $f(x)$ be an arbitrary cubic polynomial with the real roots a , b & c . Identify $(a+b)/2$ and draw a tangent to the curve in $((a+b)/2, f((a+b)/2))$. Is this tangent always passing through c ?

Try to solve the problem in at least two ways and carefully examine your own thinking during the process. With this in mind, suggest why and how this problem would fit into the teaching of mathematics in the Swedish gymnasium.

The mathematics teachers need to understand the interplay between the field of study that we call mathematics and the activity of using the mathematics they study in the problem-solving process. If we see the two as equally important, it is logical to encourage students to make the best use of technology like graphing and symbol-manipulating calculators and computer programs. Hence, it is also important to try to find appropriate assessment tools and to try matching assessment to the teaching and learning process. As is clear from the above discussion, the need for appropriate assessment will be especially evident in a distance-learning situation.

Views from preliminary findings

The course activities are monitored by following the discussions in the virtual classroom in First Class, as well as by using the tool *History* in First Class. The History tool enables an observer to view which student who has been viewing and possible reading a message and if they have answered or not. It should be stressed that we naturally are unable to investigate the learning process in a direct way; we can only try to observe the learning outcome in terms of external

characteristics such as students' written communication, attitudes, and performance skills. One conclusion so far is that the students focused much more on the traditional final exam than on the written assignments, thereby revealing a possible strong opinion that it is the mathematics you can do by paper and pencil that is really important.

One way to shape the educational process is to involve students in the assessment procedures. Any advice or instruction to a student on how to express the intended outcome will undoubtedly affect the way in which that student and his or her peers present the solution. When students become more involved in the process of evaluation, it may be seen as a substantial part of the didactical contract being negotiated between student and teacher (Brousseau, 1997). Through this interplay, the students can learn to identify the criteria for qualitatively good performance. It makes sense to give learners opportunities to analyze strong and weak answers to more open-ended problems (Moran, 1997).

Many of the participants seemed to view the projects as something less important. When for instance a specifically more didactical task were given to the participants, including to compare ones own solution to a problem with a "model" solution, this was treated as not serious or not real mathematics. It is notable that the participants are all aiming at getting a teacher certificate for teaching at the Swedish gymnasium (grades 10-12), and that the mathematics courses at the gymnasium all stress the importance of communicating mathematics in a variety of ways: in writing mathematics, in oral presentations, and by traditional paper and pencil methods. Yet, the majority of the participants acted as if they were socialized back to the study situation they took part of some 20 years ago, with a more traditional and limited view of mathematics.

One of the students in the class decided to spend most of his study time on the problems in the textbook, which were more traditional. When the dead line for the Meta cognition task was approaching, the student discovered that he couldn't solve the problem, and consequently not reflect on the way he solved it. When he complained that the problem was too difficult, one of the instructors told him that several students in a class at gymnasium level had solved the problem. Furthermore students in the pre-service program for mathematics teachers had studied the same problem and at least solved it in one way as well as analyzed their thinking when doing so. The student became very upset with this information and actually demanded a tip that could help him trough, which he eventually got. This communication could be seen as an example of how the awareness of one's own perspective on mathematics, teaching, and learning may become clearer or at least more visible for both students and teachers. The changes in assessment may provoke a discussion of different learning and teaching perspectives, a discussion that should take place in all university

courses in mathematics taken by both prospective and experienced teachers (Romberg, 1993).

The direct feedback is something that many students seem to see as a one-way agreement. Even though several of the students have failed to meet the deadline for assignments or exercises that they were accountable for, they at the same time have demanded direct feedback from the teachers. The mathematician who had the main responsibility for the mathematical content managed to take active part in daily discussions. He also presented exercises and assignments as early as possible, he offered solutions to several problems from the textbook, and he scheduled course meetings and exams well in advance. Nevertheless students often expressed criticism because they considered the virtual classroom silent over a weekend or during holidays.

The appeal for direct feedback was evident in the course when procedures had resulted in correct answers or when there was an error in a long calculation. Even though the students all are teachers and some have long experience in teaching mathematics, most of them demanded guidance when they became students themselves. Consequently, they were not so interested in examining and judging their own solutions to mathematical problems. It seems that even experienced teachers need a long time to take full responsibility for their own studies and that the dominance of procedures over concepts when learning mathematics is hard to abandon.

An unexpected problem was the difficulty of communicating mathematics in written form that many students demonstrated. While some students showed expertise in writing mathematical formulas with their word processor's equation editor and in pasting formulas directly into First Class, others persistently communicated with inadequate handwritten reports by fax communication. This variety in expertise and in communication proficiency makes it very hard for teachers and students to agree on norms and expectations.

In accordance with the course structure that was used, we observed how the students made use of the available information and communication technology. At present we are doing our analysis of all the information we have about the experiences of students and teachers in the course. From the students we have gathered: opinions about individual work, group discussions with other course participants, and discussions with the teachers in the course. From the teachers we have information about: how they viewed assessing all the different responses from the students, arranging examination situations including the choice of relevant problems, and delivering student support. From these perspectives, our intention is to describe and analyze the observed potentials and obstacles in flexible, alternative assessment in mathematics. The results will be published in a more detailed project report during the fall of 2001.

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STRUGGLING TO ‘PROVE’ MOTION: FROM DYNAMIC PERCEPTION TO STATIC THEORY

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Abstract: *Proving a statement conjectured during the solution of a problem is a manifold process, made up of different phases and involving three main components: the problem, the agent and the context of solution (including all the tools available, other individuals and the situation for devolution of the problem). When the context incorporates a dynamic geometry software such as Cabri-Géomètre, dynamic tools may be used in interaction with conceptual and theoretical tools.*

This paper illustrates and analyses the role of the individual in selecting, using and organising the tools available in the different phases of the solution, with a special focus on the transition from the production of a conjecture to the construction of its proof. Possible conflicts between tools of a different nature may arise and affect the proving process: an example of the possible tension between ‘dynamic’ concepts and ‘static’ theory is discussed.

Introduction

The process of proving a statement while solving a mathematical problem is a complex one, made up of a number of phases and sub-processes that are not linearly linked. Previous studies focusing on the proving process, and specifically on the possible continuity between argumentative and deductive processes in the solution of a problem, have highlighted that the continuity is possible and theoretical constructs have been introduced to better describe and analyse such continuity.

The process of producing a conjecture for a problem and that of constructing a proof for it may be very close to each other and involve argumentative activities that might be linked. An Italian research group (Boero et al. 1996, Garuti et al. 1998, Mariotti et al. 1997) has elaborated a theoretical construct that tries to describe a possible relationship of continuity between the two above mentioned processes. The definition of cognitive unity (CU) reads as follows:

CU: during the production of a conjecture, the student progressively works out his/her statement through an intensive argumentative activity, functionally intermingled with the justification of the plausibility of his/her choices.

During the subsequent statement-proving stage, the student links up with this process in a coherent way, organising some of the previously produced arguments according to a logical chain. (Garuti et al, 1998)

In the present paper cognitive unity will be considered as an analytical tool to interpret and explain some of the processes students engage in when striving to organise the informal arguments produced during the solution process into a logical chain, that corresponds to mathematical rules. The argumentative activity will be discussed in connection with the operations and actions performed by means of the available tools in the different moments of the proving process.

The general hypothesis underlying the discussion is that the individual naturally tends to work toward a harmonisation of the conflicts that may arise in the transition from the production of the conjecture to the production of its proof.

Using tools to solve problems: the construction of a toolkit

In any problem solving situation three main components may be identified: the problem, the agent, i.e. the individual-acting-with-mediational-means (Wertsch, 1991) and the context of solution (including all the tools available, other individuals, be they peers or teachers, and the situation for the devolution of the problem). Both discursive and concrete or mental operations are enacted in the process of solving a problem and proving the conjectured solution. The focus of the present paper will be on the agent as well as on the operations performed by means of the available tools: the specific context considered is provided by a setting that includes the software Cabri-Géomètre (Baulac et al., 1988). Within this context, the word ‘tool’ incorporates many different meanings and refers to both concrete and psychological tools. Drawing on the seminal work of Vygotsky, tools and signs (i.e. psychological tools) may be distinguished according to their function:

The tool’s function is to serve as the conductor of human influence on the object of activity; it is *externally* oriented; it must lead to changes in objects. [...] The sign, on the other hand, changes nothing in the object of a psychological operation. [...] the sign is internally oriented. (Vygotsky, 1978, p. 55)

In some cases the distinction cannot be neatly drawn: within a certain activity (for instance the process of solving a geometrical problem, as in this paper) some of the externally oriented tools may be internalised and function as psychological tools. The internalisation process as well as the relationships among the tools used within a certain context are complex and manifold. In order to describe them I will introduce the idea of toolkit as an organised set of tools (both internally and externally oriented) that each individual develops and uses in a particular context (in this case a problem solving situation). The idea of toolkit is meant to account for:

- the diverse and manifold nature of its components: mediated action is viewed as involving different types of tools (such as language, symbolic systems, drawings, constructions and changes of configurations, dynamic manipulation, gestures etc.)
- the relationships among the components, that change and evolve continuously: the individual toolkit available to each agent affects the processes enacted during the solution and shapes the mental activities through a continuous development and re-conceptualisation of previously acquired tools.¹

The individual has a crucial role in the management of his/her personal toolkit: the evolution occurring within the context of a problem situation is basically subjective and does not follow specific rules. Wertsch (1991) refers to ‘privileging’ as a strategy to select one mediational means as more appropriate or efficacious in a particular socio-cultural setting. The process of privileging is assumed to be a dynamic one and one that follows patterns accessible to conscious reflection and hence to self-generated change (ibid. p. 124). The management of tools chosen as more efficacious in the particular context of the action is peculiar to each agent and depends heavily on the features of the socio-cultural context where the actions take place.

Such model has links with the idea of instrumental genesis, introduced by a group of French researchers in mathematics education (Rabardel, 1995, Verillon & Rabardel, 1995) in order to analyse the use of tools in doing mathematics. The model describes a process that transforms an **artefact**, i.e. “the particular object with its intrinsic characteristics, designed and realised for purpose of accomplishing a particular task” into an “**instrument**, that is the artefact and the modalities of its use, as are elaborated by a particular user”

¹ For instance, a student might use a theorem as an exploration tool in an initial phase of the solution, seeking possible properties of the configuration at hand, and later reuse the same theorem in order to justify the conjecture. In this case the same conceptual tool ‘theorem’ is functionally related to the phase of solution and assumes the value of argument when a proof is constructed.

(Mariotti, to appear). Schemes of use are individually developed and shape and organise the actions performed by an agent within any mediated activity.

The re-organisation of tools within the specific context of the solution of a problem brings about a re-interpretation of tools that had been previously appropriated² by the agent³ as well as the development of suitable modalities of use. In the case of conceptual tools the role of the environment providing the context of solution becomes crucial. The process of re-interpretation is not always successful, since there might be conflicts between the phenomenological and the theoretical worlds (Balacheff & Sutherland 1994), which coexist in the context where the problem is tackled. Cabri is a microworld that incorporates the basics of Euclidean geometry as well as tools that allow a dynamic exploration and that give visual and conceptual feedback to the agent: once they have been internalised, such tools may control behaviour and shape the process of solution.

In the specific case of this paper, tools may be classified in three main groups:

- technical tools (including tools of Cabri, like constructions, dragging, menu functions etc. as well as drawings, additional constructions, measurements)
- theoretical tools (i.e. axioms, definitions and theorems of Euclidean geometry)
- conceptual tools (i.e. the tools internalised by the individual to function as organisers of his/her mental processes. This set includes the “personal” theorems and definitions as they have been appropriated by the agent).

Turning ‘dynamic’ perception into geometrical facts: the case of Giulia

In order to illustrate some of the theoretical ideas discussed in the previous section, I will present the analysis of a particular protocol, highlighting the use of diverse tools in different moments of the solution. I will focus on some of the tools of Cabri, as well as on some of the conceptual tools⁴ used by a student, in

² The term appropriation draws on the theories developed by Leont’ev and Bakhtin, who expanded and refined the idea of internalisation as introduced by Vygotsky. For a discussion see Wertsch (1998).

³ It is in this sense that I will refer to personal or subjective definitions and theorems, to indicate results from the theory that have been previously conceptualised and are re-interpreted and used in the specific context of the solution of a certain problem.

⁴ In this case theorems, definitions, that acquire a subjective connotation when used in context, changes of configuration, generic examples and counterexamples will all be included among the conceptual tools.

order to better describe and define the ‘role’ of such tools as components of the student’s toolkit, in the transition from the conjecturing to the proving phase of the solution process.

The excerpts reported here refer to the case of Giulia (12th grade, Liceo Scientifico), tackling the following problem⁵:

Two intersecting circles C_1 and C_2 have a chord AB in common. Let C be a variable point on circle C_1 . Extend segments CA and CB to intersect the circle C_2 at E and F respectively.

What can you say about the chord EF as C varies on circle C_1 ?

Which is the geometric locus of the midpoint of EF as C varies on the circle?

Justify the answers you provide.

First episode: harmonisation of technical and conceptual tools

After an initial dynamic exploration of the figure through dragging, that takes into consideration particular configurations corresponding to extreme cases, Giulia ends up with the formulation of a general conjecture:

G: I would need ... what can I do here? Now ... it is the segment which moves ... it seems that ... the distance of the segment from the centre of the circle does not change because it seems that the circle rotates... the segment rotates and the distance between segment and centre of the circle on which I take the segment does not vary ... therefore they should be equal ...

C: hmm ... in those four positions ...

G: in all the positions ... (see Fig. 1)

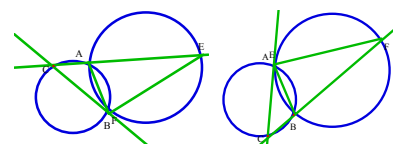


Fig. 1

In this case, the dragging facility is the tool that supports the heuristic process leading to the generalisation of a conjecture previously elaborated for a specific position of C . Perceptual judgements about rotation and variation seem to be the most relevant elements in formulating the conjecture. The perceived rotation of chord EF and the invariance of its distance from the centre of the circle provide all the empirical elements needed in order to inductively state the

⁵ The problem was assigned within the context of an interview carried out for a PhD project. The student was allowed to use Cabri and encouraged to think aloud while solving the problem. The interview was recorded and fieldnotes were taken to trace the operations performed.

invariance of EF in length. The verbal expression of such inductive process assumes a sort of “deductive” form:

<p style="text-align: center;"><i>the segment rotates</i></p> <p style="text-align: center;">+</p> <p style="text-align: center;"><i>the distance between segment and centre of the circle does not vary</i></p>
<p>⇓ therefore</p>
<p style="text-align: center;"><i>they should be equal (all the chords in the different possible positions)</i></p>

The causes of the invariance of EF are thus identified in two perceptually evident facts: the dynamic behaviour of the figure suggests that the property of invariance is an “effect” of the rotation of the chord.

Giulia seems to establish a sort of relationship of cause-effect between rotation (motion) and invariance in length (geometrical property), although she stays on a semi-empirical plane. Once spotted, the perceptual invariant must be turned into a geometric invariant: the process of argumentation enacted in order to realise such change makes use of a number of tools, to be chosen among those related to the Cabri world and those related to the theory (i.e. Euclidean geometry).

In the case under exam, the visual evidence of the property (the chord EF *looks* equal for all positions of C) does not find an immediate justification in a geometrical theorem. Further exploration through slow dragging and the drawing of an additional element characterise the process of refining the conjecture into a more suitable form:

G: I could ... AB is constant ... therefore what varies.... well.. if I take the triangle CEF what varies are the sides CE and CF, but why is EF constant? EF is constant....

[she is dragging point C slowly along the minor arc]

G: I can't think about anything .. do you think that drawing something would help me?

C: Well.. maybe drawing AB can help you....

G: drawing AB... it's always equal lengthed because the triangles which are formed are similar...

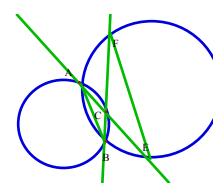


Fig. 2

The dynamic exploration is temporarily abandoned, to leave room for reasoning on a specific static configuration (see Fig. 2), that ‘shows’ two similar triangles when the chords AB and EF are parallel. The general validity of such property is tested through dragging and a more generic configuration (see Fig. 3) is chosen as the basis for a justification of the new conjecture (i.e. triangles ABC

and CEF are similar). At this point Giulia draws on well known theorems about chords and angles in circles and the following justification is easily produced:

G: if I think about arcs.. well.. let's see.... it's true! one angle subtended ... well these are angles subtended by the same arc and thus they are equal .. this one [$\angle BAE$] and this one...[$\angle BFE$] [...] well ... in this case [see Fig. 3] the triangles are similar and therefore if this is constant this other one has the same... how do you say that ... times the same scale factor... therefore it is equal ... and therefore constant ...

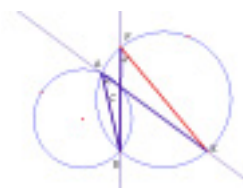


Fig. 3

Dynamic tools and theoretical/conceptual tools are employed jointly by Giulia, whose toolkit seems to evolve mainly in terms of the relationships established between the actions enacted to find a suitable configuration and the actual identification of geometrical properties related to that configuration. The process is refined with a continuous movement from a dynamic change of the figure to the conjecture of a property: the subsequent testing of its general validity goes with the search for a configuration that may express generality (an example is provided by the dragging from a situation with parallel chords to a generic situation that may show the existence of similar triangles in general).

Dynamic and conceptual tools seem to be harmonised and lead to a correct, although partial⁶, justification of the conjecture, although the initial statement based on the idea of rotation has completely been abandoned in favour of a theory-oriented exploration of the problem situation.

Second episode: conflict between dynamic causality and static deduction

Later in the solution process the conjecture, proved in the most general case, becomes an assumption for the second part of the problem, requiring to find the locus of the midpoint of chord EF. The conjecture is expressed after a dynamic exploration of the situation:

⁶ The justification provided by Giulia proves the conjecture only partially: in order to complete the proof she would have needed to show that the scale factor between triangles ABC and CEF is invariant. Giulia seems to implicitly assume that the rotation of the chord's endpoints around the circle C2 "automatically" ensure the fixedness of the scale factor as C moves along circle C1.

G: When C moves on the circle the midpoint forms a circle around the centre, another circle with the same centre as this bigger circle ... shall I say why? [...]

Yes ... if this segment has to be always the same ... if it is always equal ... it has always ... the same distance from the centre, because when you vary the segment ... I mean ... equal segments are those ... I don't remember ... they have the same distance from the centre ... equal segments on the same circle have the same distance from the centre, therefore, if I prove ... actually I have proven that that segment over there is always constant [...]

I didn't prove it because I didn't prove that this one rotates, or something like that...

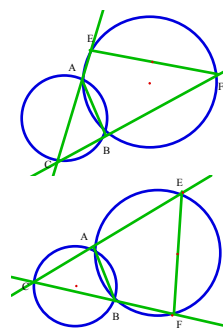


Fig. 4

A personal theorem is stated by Giulia at this stage: ‘equal segments on the same circle have the same distance from the centre’. Such ‘theorem’ draws on the properties of angles subtended by chords in circles, but its use in this context leaves such elements implicit. Giulia seems to establish a deductive link between the hypothesis, i.e. ‘ EF is constant’, and the stated theorem in order to deduce that the locus is a circle. The argumentation is not concluded, though, because of a difficulty met in harmonising such theorem with a dynamic (personal) definition of locus: Giulia makes explicit her need to prove that the chord (and therefore its midpoint) actually rotates, thus generating the locus. Three elements of a different nature seem to be conflicting: the concept of locus, its representation through the dynamic software and the synthetic proof requiring a reasoning on a static configuration and the application of definitions and theorems which do not involve any movement.

After an exploration of the figure aimed at matching her personal definition of locus with the properties of the figure, as perceived through the Cabri feedback, Giulia reformulates the conjecture, saying that the radius of the circle (the conjectured locus), has always a fixed length, in relation to the length of EF . The result is easily deduced after constructing the triangle (see Fig. 5) and using properties of angles and arcs, analogous to those used in the preceding part of the solution.

Once again Giulia makes explicit the conflict between the static proof and the dynamic cause of the generation of the locus (i.e. the movement of rotation):

G: Now I must also say why the geometric locus is a circle, mustn't I? Shall I prove it? [...]

G:... it crossed my mind that I had to prove also ... no ... maybe it is stupid ... that I had to prove that it was rotating around the centre....

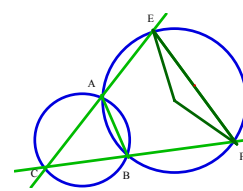


Fig. 5

C: Oh, right... that it was rotating....

G: [...] I can see that it forms a circle when I grab the point C and drag it around the circle ... but if I take a random point it might not form a circle, how can I be sure it forms a circle in any case?

Different possible interpretations may be given to the need to prove the rotation of the midpoint of EF to generate the locus. The key point is that Giulia seems to have developed (or maybe only made explicit) a personal dynamic definition of locus, within the context of this problem, and to have established a link between such definition and the initial conjecture, expressed in terms of motion (rotation of EF as cause of its own invariance in length).

Although a synthetic proof is achieved correctly, the conflict generated by the use of dynamic tools brings about a sort of rupture: the dynamic definition of locus and the traditional geometrical definition are not conceived of as the same object. Hence the conjecture is related to a dynamic ‘world’, while the proof is linked to a separate static ‘world’: for Giulia the proof provided seems to be a list of deductions, which do not actually prove the conjecture, since they do not cover its dynamic aspects.

The toolkit and the CU construct: some concluding remarks

The case of Giulia illustrates how exploring the problem situation by means of a combination of technical (dynamic) and theoretical/conceptual tools may be a productive strategy, in terms of achieving a successful management of the toolkit. The bulk of the activity of exploration during the conjecturing phase is generally carried out through the use of tools such as dragging, drawings (in this specific case within the Cabri environment), constructions, changes of configuration (either through dragging or through additional figural elements), heuristic techniques (for instance the examination of extreme cases). Although the use of such tools may bring about the production of arguments based on a perceptual judgement, theoretical aspects may be underlying them and offer room for developing arguments into a logical chain. In actual fact, in the phase of justification of the conjecture some of these tools may be reused and re-interpreted with a stronger link to the theory: the toolkit evolves and the newly incorporated tools need to be harmonised with and related to other existing tools.

The paper attempts to illustrate the fact that the different phases of the solution process present distinct features in terms of the evolution of relationships among tools linked to the phenomenological rather than to the

theoretical world. In the particular case examined, at the beginning of the solution process dynamic exploration serves the purpose of finding relationships between figural elements as well as conjecturing the reasons of the underlying geometrical properties. As the solution develops, dynamic exploration seems to be guided by the identification of theorems and properties that hold for particular cases: there is a progressive interaction between technical and conceptual tools at this stage, and the properties are still tested by means of the dragging test. In the final part of the process the theorems, previously identified as holding in the general case, are finally used to (partially) prove the conjecture on a static figure, on which the initial relationship of cause-effect between motion and theorems disappears.

A conflict between the dynamic causality, as perceived from the observation of the figure, and the static deduction only emerges in the second part of the solution process, when the notion of locus is introduced. The personal appropriation of such notion by Giulia seems to be originating the conflict: the idea of locus seems to have been conceptualised in dynamic terms, and to be conflicting with the known static definition. In this case a dynamic management of the problem does not prove to be sufficient to make progress in the construction of the needed proof, thus making other conceptual or theoretical tools necessary.

Personal theorems and definitions, either previously appropriated or developed and contextualised along the solution through the use of dynamic tools, may be substantially different in nature from ‘official’ theorems and definitions. A possible hypothesis, illustrated here by the case of Giulia, might be that a harmonisation of the two types of tools is difficult and a possible rupture of the cognitive unity may occur. The argumentative processes enacted during the elaboration of the conjecture, and drawing on the idea of dynamic causality, may possibly be abandoned when a deductive process is enacted in order to produce a proof.

Hence the individual needs to reorganise the toolkit and to choose a different “set” of tools in order to achieve a reformulation of the conjecture and prove it after a new exploring process. When a sufficient theoretical control is kept along the whole solution process the management and evolution of the toolkit might be successful in supporting the process of selection and (more or less systematic) organisation of the arguments produced. The issue of the achievement of a good theoretical control of the operations and actions performed while solving a problem is strictly linked to the issue of the appropriate construction of meanings by the individuals within the context of solution and, more broadly, within the context of their mathematical experience.

Some of the ideas explored in this paper open up avenues for further research: more evidence is needed in order to establish whether an appropriate management of the personal toolkit each individual develops in particular contexts, is linked to the idea of cognitive unity. A germ of hypothesis may be formulated:

there is a tendency to establish or re-establish CU once it is broken. In correspondence with possible different causes for the rupture to occur, an appropriate management and reorganisation of the toolkit may help the agent overcome the rupture.

Some reasons explaining a possible rupture of the CU have been implicitly or explicitly suggested: a particular stress has been given to a possible conflict between the dynamic nature of the exploration in Cabri and the essentially static nature of Euclidean geometry as theoretical system. More reflection is needed and further evidence needs to be provided in order to evaluate the actual impact of possible ruptures of the CU on the construction of a meaning of proving.

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AN EXPLORATORY STUDY OF STUDENTS' MEASUREMENT ACTIVITY IN A DYNAMIC GEOMETRY ENVIRONMENT

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Abstract: *This paper reports on students' measurement activity in a dynamic geometry environment (Cabri-Géomètre), in the context of open geometry problems. In particular, we explore the cognitive role of measurement in the production of conjectures and proofs, with respect to the evolution from empirical to theoretical practices. Different ways of using measurements have been identified. The protocol we present shows how the use of measures constitutes a delicate and key point within the evolution towards theoretical thinking.*

Introduction

New technological tools are widespread in most classrooms now. However, the presence of technology cannot in itself bring about educational change. Simply making software available does not mean that people will more or less automatically take advantage of the opportunities that it affords (Perkins, 1985). Generally speaking, using new technologies in the classroom implies the redefinition of contents, methodology and of the role of the teacher (Noss, 1995; Bottino & Chiappini, 1999). The role of technology is being widely discussed in the debate, currently pursued in many countries, aimed at the development of new mathematics curricula (e. g. Crem, 1995; AAVV, 2000; Abrantes, 2001; Robutti, 2001). At the same time Mathematics Education research has addressed this issue from different points of view, for example investigating from a cognitive point of view how the introduction of new technologies change learning in the classroom (e.g. Schwartz & Yerushalmy, 1992; Laborde, 1993; Hanna, 1996; Artigue, 1997; Mariotti & Bartolini Bussi, 1998; Sutherland & Balacheff, 1999; Lagrange et al, 2001).

Contributing to this discussion, our interest concerns the impact of new technologies in the curriculum and is focused on the study of the consequences of this impact on the teaching and learning processes. Particularly, our aim in the paper is to report on students' approaches to measurement in a dynamic

geometry environment and to analyse the cognitive role of measurement in the production of conjectures and proofs. First, we set the research problem in the perspective of a wide research project focused on the use of dynamic geometry in the context of an approach to theoretical thinking. Second, we outline a framework for analysing students' measurement activity within that context. Finally, we illustrate our thinking with an example of the solution process of a pair of students and we draw some preliminary conclusions.

The Research Problem

The ongoing research project we are currently involved in, is centred on the teaching and learning of proof in geometry at secondary school level within a dynamic geometry environment, namely Cabri-Géomètre (Baulac & al, 1988). Teachers and researchers have developed and implemented classroom activities in an attempt to integrate geometry teaching and new technologies, drawing on the perspectives of Hölzl, 1996; Laborde, 1998; Hoyles & Healy, 1999. The focus of the research is the analysis of students' cognitive processes within the proving process, i.e. the process of exploring, conjecturing and proving in an open problem. In particular, we observe the students' evolution from an empirical approach to mathematical activity, which involves experiences such as observing, modifying and identifying invariants, to a more theoretical one, which involves the construction of definitions and proofs and the understanding of the deductive structure of mathematics, made of axioms and theorems. This evolution is to be considered at two levels. As a long term process, it is the evolution from an empirical mathematics to the systematisation of some aspects of a theory¹. As a short term process, the evolution concerns the organisation of the perceptual aspects, which emerge from students' exploration of the problem in Cabri via an empirical approach, in the construction of a proof.

In a previous study we analysed the use of the dragging function in Cabri. We identified different modes of dragging which students use according to different purposes, during the solution process of open problems, and we analysed these modes from a cognitive point of view (Arzarello et al, 1998b; Olivero, 1999). Recently, we have been studying the different ways in which students use measures in Cabri (Olivero & Robutti, 2001). These measures are dynamic, i.e. they change on the figure as you drag points. Measuring is a powerful tool of Cabri (and of other dynamic geometry software), and it can be used by students with varying degrees of confidence.

¹ The extent to which the theory can be developed depends on the age of the students, the school level and the type of school.

In this report we will analyse students' approaches to measurement and the cognitive role of measurement in the production of conjectures and proofs.

A Framework for Analysis

Drawing on the previous study about the use of dragging, we developed a framework for analysing students' approaches to measures in Cabri.

The ways of using measures that we observed can be classified in two types.

1. Use of measures within an ascending process, in order to explore a situation, i.e. looking for regularities, invariants, etc. We say there is an ascending process (Arzarello et al, 1998a; Arzarello et al, 1999) when students move from drawings to theory. Drawings are seen as source of exploration and discovery: they are to be 'read' in order to get information.

In this first case measurements are used as a heuristic tool (cf. *mesure exploratoire*, Vadcard, 1996) and have a perceptual connotation. The students 'read' measures in order to get ideas about properties, invariants, and relationships of a figure. Measures can be used on a static figure, aiming at identifying the properties of one particular configuration (it is not very common). Otherwise measures can be used together with dragging (this modality is very common), in order to observe how the properties of a figure change, e.g. the length of the sides of a quadrilateral, or to identify relationships in a figure, e.g. how the length of one side depends on another side, or to discover invariants.

2. Use of measures within a descending process, in order to validate or refute conjectures, to check properties, to construct proofs, etc. We say there is a descending process (Arzarello et al, 1998a; Arzarello et al, 1999), when students move from theory to drawings. Drawings are no longer seen as source of discovery, but of validation. As such, they embody the already discovered geometrical properties, that is bits of a theory.

In this second case measurements are used as a control tool (cf. *mesure probatoire*, Vadcard, 1996). They can be used in order to check a prediction; for example, if the students perceive a regularity, but cannot judge if this is really a geometrical property of the figure, they use measures to check this. Or measures can have the status of experimental validation of a theoretical statement, when the students have already formulated conjectures or even proofs: the subjects use

measurements in order to either validate conjectures or to find logical relationships which can contribute to the construction of a proof.

Methodology

The students involved in the project are Italian Secondary School students (15-17 years old). All the classroom teachers participate in the Mathematics Education Research Group² at the University of Turin as teacher-researchers (Arzarello & Bartolini Bussi, 1998).

The tasks used in the project are open problems (Arsac et al, 1988), which differ from traditional tasks of the form "prove that", in that the students are asked to explore a geometric situation, make conjectures and finally prove them. They allow students to investigate a geometric configuration using different modalities: static, dynamic (Goldenberg, 1995), transformational (Simon, 1996), and so on.

The technological environment chosen is Cabri, which contains a number of different tools students may use to tackle the problems: constructions, dragging, measures, calculations, etc. Particularly, in this report we concentrate on the measurement tools, which allow students to take measurements of segments, distances, angles, areas and perimeters of constructed figures.

In the classroom sessions, the students were presented with an open problem and were asked to work in pairs at the computer (with Cabri), trying to formulate conjectures and proofs, in a two hours period.

Two observers were usually present in the classroom and observed one pair of students. The data collected are fieldnotes, a videotape of the work of this pair and their written production.

A Preliminary Analysis: *"What do you trust more: this drawing or your proof?"*

To illustrate our thinking, we will analyse some excerpts³ of a protocol taken from a 2nd year classroom (15 years old students) of a *Liceo socio-psico-*

² In this group teachers and researchers collaborate in the development, implementation and evaluation of new materials for the classroom.

³ We chose the parts of the protocol which show students' use of measurements. All the lines in the protocol were numbered for easier reference.

*pedagogico*⁴. These students (A and T) are medium achievers; they have used Cabri a few times over the year before this activity, doing construction and exploration problems.

The students were given the following problem (Varignon's problem):

Draw any quadrilateral ABCD. Draw the midpoints L, M, N, P of the four sides.

1. Which properties does the quadrilateral LMNP have?
2. Which particular configurations does LMNP assume?
3. Which hypotheses on the quadrilateral ABCD are needed in order for LMNP to assume those particular configurations?

At the beginning the students construct a generic quadrilateral ABCD in Cabri and the inside one LMNP with the vertices in the midpoints of ABCD (Fig.1).

[...]

16. A: *Try to take measurements of the sides, it seems that that equals that* (she points at LM and NP) *and that equals that* (PL and MN).
17. T: *yes, it is a ...*
18. T moves C and stops to observe the figure.
19. T: *it is ... a parallelogram*
20. T puts measures on the sides of LMNP (Fig.2).
21. T: *so...*
22. A: *so the opposite sides are...*
23. T: *congruent...eh...*

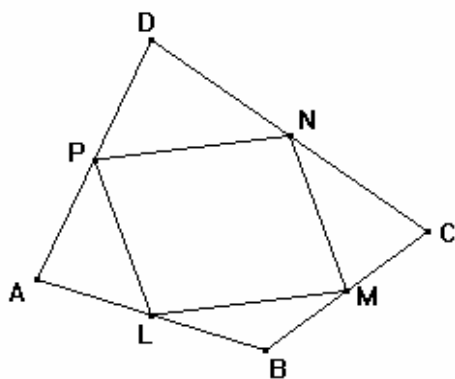


Figure 1

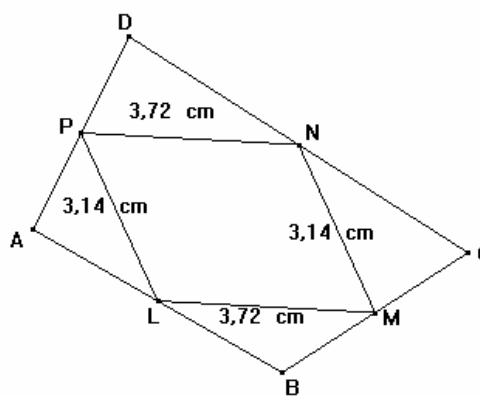


Figure 2

At the beginning, the students start dragging the vertices of ABCD, observing that the opposite sides of LMNP seem to be congruent (16⁵) and they want to check this observation by taking measurements of the sides of LMNP (16). However, first an observation is made about LMNP being a parallelogram (19), then measures are used in order to have more information supporting this

⁴ The students have 4 mathematics classes per week.

⁵ The numbers quoted in the analysis refer to lines in the protocol.

conjecture (22-23). Measures are used within an ascending process (type 1), together with dragging, in order to discover a conjecture. They add further information to an initial perception (of a parallelogram).

[...]

38. *T: so... it may be a rhombus*

39. T drags B slowly on the left and slows down when it seems she is getting a rectangle.

40. *T: it may also become a rectangle!*

41. *A: try to make this (ABCD) a square. Let's see which quadrilateral it (LMNP) becomes.*

42. *T: So I need to put measures.*

In the second phase of the solution process, the students observe some more regularities in LMNP while dragging: it may be a rhombus (38) and a rectangle (40). They pay attention to ABCD as well and try to relate the two quadrilaterals (41). In order to make ABCD a square they first measure the sides of ABCD (42). They use measurements together with *guided dragging*⁶ in order to achieve a square.

After having explored the situation and formulated some conjectures, they start proving one, which is: "*if ABCD is a square then LMNP is a rhombus*".

[...]

178. T draws on paper a square ABCD with the sides 12 squares length, puts the midpoints of the sides and gets the internal quadrilateral LMNP (Fig. 3).

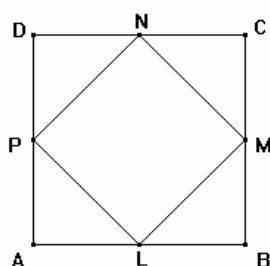


Figure 3

The conjecture is discovered in Cabri, thanks to the use of measures, which seem to represent an essential tool in the ascending process, together with dragging. In order to get started with the proof, one of the students makes a drawing on paper. The procedure to get a square is the same as the one used in Cabri: taking measurements of the sides. This time measures are read in terms of the little 'squares' the paper they are using is divided into (178).

⁶ *Guided dragging* is dragging the basic points of a figure in order to give it a particular shape (Olivero, 1999).

[...]

182. *T: all this stuff...these...they are congruent (the halves of the sides of ABCD).*

183. Then T writes down the thesis: LM equals MN, equals NP, equals PL. Meanwhile A uses a ruler to measure the sides of LMNP.

[...]

194. *T: so PL equals MN. The same for ...PDN triangle and LBM triangle \Rightarrow PN equals LM...Should I do a cross comparison? PDN triangle and PAL triangle \Rightarrow PN equals PL. What's missing? These two are done, these two are done....*

[...]

198. *T: They all have equal angles. So it is a rhombus! Ok!*

In the third phase, the students attempt to construct a proof. The proof is produced with a rigorous reasoning, using the congruence of the triangles in which the figure is decomposed (182). At the end of this reasoning the students are convinced of the validity of their conjecture (198) from a logical point of view, because they proved it. They work within the theory, as they construct the proof according to the deductive rules of mathematics: they deduce the thesis (rhombus) from the hypothesis (square), connecting all the steps with logical consequence. However, "there is profound difference between determining that something is true and explaining why it is true" (Nunez, 2000, p.3).

[...]

208. *T: try to make ABCD a square*

209. A drags A and B trying to obtain a square with a side of 6.17

210. *T: is it a rhombus?*

211. *A: well...(A points at the sides of LMNP). A rhombus has got equal sides...oh no! ...It's not a rhombus! (Fig.4)*

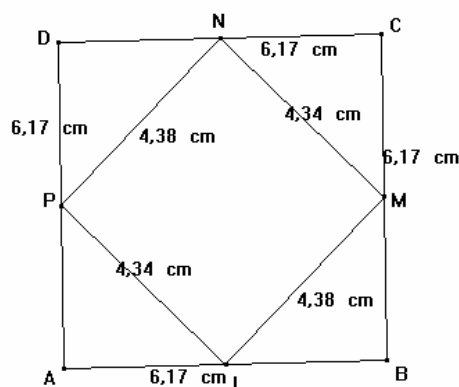


Figure 4

The students feel the necessity of validating their deduction in Cabri. So they go back to Cabri and drag ABCD into a square (208), in order to check if LMNP is a rhombus and they measure the length of the sides of LMNP. The

students are now using measures in a descending process (type 2). They have a theoretical deduction, which holds within geometry; however, they are seeking an explanation for that and they want to understand why is that so. They are unhappy when they see that the measures do not corroborate their thesis (211). A conflict arises, as from a logical (theoretical) point of view they came to one conclusion, but the empirical point of view does not corroborate it. So the experimental validation provokes a conflict.

[...]

225. *T: the problem ...is that this is not a square (ABCD).*

226. *A: it's impossible.*

227. *T: look... no... Because if you say that this equals this (PD and DN) and you say they have an equal angle (D) and then this equals this (PN and LM) and this and this (PL and MN)...then this becomes a square (LNMP), but we've just seen that it is not a square. So it's all wrong!*

These students' attitude towards measurement seems to be of complete trust⁷; in fact they immediately go back to their proof and check each deductive step very carefully, thinking there may be something wrong there. And they conclude by rejecting that proof: *So it's all wrong!* (227). The motivation for this is that the figure does not look like a square (227): at this moment perceptual aspects are very strong and prevail over theoretical considerations (the Cabri figure is not a square because the angles are not right angles). This seems to be a final decision, however the conflict is not yet solved for them.

[...]

251. *I⁸: what is the conjecture you're proving?*

252. *T: if ABCD is a square... then the other should be a rectangle...because the measures (in Cabri) show that the opposite sides are congruent, so it is a rectangle.*

[...]

257. *T: however I do not understand this, it doesn't make sense.*

The conflict is recognised by the students (257). It is interesting to notice that T starts explaining the teacher what the figure and the measures in Cabri (252) show (LMNP rectangle) and not what they got in the proof (LMNP rhombus). The use of conditional sentences (*if...the other should be...*-252) shows that she does not really give the status of truth to this fact.

⁷ "I feel equally convinced that our most prevalent notions both about the function of measurement and about the source of its special efficacy are derived largely from myth" (Kuhn, 1977, p. 179).

⁸ I is the teacher.

258. *I: why? What puzzles you?*
 259. *T: because...if this is the midpoint (she points at P) then it divides this side in two equal parts (she points at AD and AP and PD).*
 260. *T: so it should be: if it is a square, the quadrilateral inside is a square too.*
 261. *I: right!*
 262. *T: why the figure doesn't show that?*
 263. *I: what do you trust more: that drawing or your proof?*
 264. *T: my proof! (laughing).*

T goes back to the proof again; this time she reconstructs it on the figure without reading the one they got before. Her conclusion is a conjecture (260), which is different from the one the figure shows (262). This time she seems to be more convinced of the proof, as she is asking why the figure does not show what she has just proven true (262). The teacher's intervention focuses the attention on the two aspects of empirical validation and theoretical proof.

Discussion of Results

The overall evolution of the proving process the students carried out in the previous example can be summarised as follows. First, the students explore the situation in Cabri, make a conjecture, they are convinced of that conjecture and they prove it: the conclusion is that the conjecture holds. Second, they want to better understand the situation and they want an explanation for their conjecture; so they go back to Cabri and use measurements. This experiment does not confirm their proof. So, at first they refute the proof; then they re-construct the proof, and they seem to be convinced by that. However, the measures in Cabri still do not validate it. What is the problem?

A conflict (262) between the theoretical result (proof) and the empirical answer given by Cabri (the figure does not look like a rhombus) arises. This happens because the students, in their descending process, try to validate the proof at an empirical level. This is not wrong, but it requires looking at the figure from another point of view, not only empirical, but also theoretical. In fact, during the ascending process, the figure is looked at from an empirical point of view, because the aim is to explore and look for properties and conjectures. In the descending process, instead, the geometrical properties of the figure need to be accounted for: the figure must be seen as a generic example (Balacheff, 1992). In our example, when validating the proof the pupils 'read' the figure at an empirical level (211), in the same way as at the beginning of the ascending process. They 'read' the properties of ABCD from the measurements on the figure, concluding that ABCD is a square because it has got equal sides. And in the same way they 'read' the properties of LMNP from the measurements: it does not have equal sides therefore it cannot be a rhombus.

The students do not consider that their hypothesis is *ABCD square* while the Cabri figure is not a square because the angles are not right angles. Instead of 'reading' the Cabri figures, they should have looked at them from a theoretical point of view, according to which ABCD is an 'approximation' of a square and LMNP is an 'approximation' of a rhombus. The proof would have then been validated.

Continuous back and forth shifts from ascending to descending processes can be frequently observed as cognitive activities during the proving process and they reveal rich mathematical activities. However, in order for the evolution to theory to happen, it is necessary that every time you go to the figure you see it from a different level, adding the information provided by the geometric properties that were discovered before as conjectures.

Issues for Further Research

The analysis of students' measurement activity in software environments such as Cabri sits within two major research strands: the study of the mediation technological tools offer in the context of problem solving and the study of students' cognitive processes. The purpose of our research is to analyse the mediation of Cabri (or other tools), linking the cognitive aspects discussed in this paper with other elements playing a role in the interaction between students and tools. For example, epistemological and conceptual aspects related to the knowledge, which comes into play in problem solving activities within technological environments, need to be considered. These aspects are to be tackled from two points of view: the point of view of technology and the point of view of the subject knowledge, i.e. mathematics.

In the general framework aiming at linking research and practice aspects (Robutti et al, 2000), understanding the relationships between epistemological and cognitive aspects proves relevant to define the role of the teacher in the management of the classroom activity and in the curricular planning.

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GRAPHICS CALCULATORS USE IN MATHEMATICS IN VICTORIAN SECONDARY SCHOOLS

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Abstract: *In 1995, the Victorian Board of Studies accepted a recommendation from the VCE¹ Working Group to allow students to use graphics calculators in VCE mathematics examinations as from 1997 (Victorian Board of Studies, 1995). In 1997, Victoria became the first state in Australia to permit the use of graphics calculators in final external examinations. The use of graphics calculators has become quite widespread. With the support of the Board of Studies, a survey of secondary schools was undertaken to gauge the response of teachers to these tools. This paper provides information on choice of mathematics topics for which the graphics calculators were used and how teachers view graphics calculator use in secondary mathematics courses.*

Introduction

Mathematics instruction traditionally involves teaching of a procedure or algorithm which training makes routine ‘so that the process can be suppressed to a lower level of consciousness’ (Tall, 1993). In theory this gives the possibility to move on to the next stage in which the clichéd procedure is used to explore new mathematical objects that are more general. In practice, however, many students are not trained in basic manipulations to the extent that allows them to use these processes automatically and concentrate on the meaning of more abstract mathematical objects. With the calculator and/or computer carrying out some of these processes even those students who did not master enough their basic skills can concentrate on the conceptual knowledge while technology performs for them routine algorithms. For example, while teaching Statistics, technology gives the opportunity to concentrate on interpretation of statistical graphs, while statistical software or graphics calculator performs the routine procedures of graphing. A comprehensive review of previous research on all aspects of teaching and learning with graphics calculators is provided by (Penglase and Arnold, 1996).

¹ Victorian Certificate of Education (VCE) is awarded to students on a satisfactory completion of two final years (Year 11 and 12) of secondary school in Victoria (Australia).

However, despite all positive features, that technology could ideally bring to the classroom, one could imagine that, if this technology is not easily available for both teachers and students, or, if teachers did not have time to familiarise themselves with this technology, then theoretically useful tool becomes useless in practice. Therefore, the purpose of this paper is, to examine

- *How the availability of graphics calculators associates with teachers' attitudes toward this tool?*
- *Are there any other factors like gender, age, or teaching experience that associates with the attitudes toward graphics calculator?*
- *How attitudes and availability associate with implementation of graphics calculators into curriculum?*

Background

In Australia graphics calculators became available at the end of 1980's and Victorian educators began to pay attention to this teaching and learning tool. From the early 1990's staff from Swinburne University of Technology, The University of Melbourne, and some other educators began to introduce graphics calculators to the Victorian teachers (Jones, 1997; Tynan and Dowsey, 1997). However, until 1997 the ban on the use of graphics calculators in external examinations applied universally in Australia (see Tobin, 1997, for a discussion of the impact using a graphics calculator would have made on VCE mathematics papers prior to 1997).

In 1995, the Victorian Board of Studies accepted a recommendation from the Mathematics Key Learning Area Committee to allow students to use graphics calculators in VCE mathematics examinations as from 1997 (Victorian Board of Studies, 1995). Concerns were raised by public and media on equity issues and appropriateness of the use of this technology in external examinations.

To underpin the Board's policy decision a state-wide survey of schools was initiated in 1997 by Swinburne University of Technology with the support of the Board of Studies. The survey aimed to assess the level of availability of calculators in the classroom and to determine teacher attitudes to this tool. There had been little previous research published on teacher attitudes and all in the USA (Chamblee, 1995; Jost, 1992; May, 1995).

A previous paper (Routitsky and Tobin, 1998) raised the issue of teacher support for the decision to use graphics calculators in VCE examinations and found that there was broad agreement for the policy across all sectors, regions and school types. This support level ranged from 64% to 70%, depending on the

VCE subject, and this occurred, despite there being a similarly common perception (about 73% of respondents) that the use of calculators raised serious equity issues in terms of student access. The policy implications of the survey were that 'for 1999 examinations in Mathematical Methods and Specialist Mathematics setting panels will assume that all students will have access to the approved graphics calculator. For Further Mathematics, this requirement will come into effect in 2000' (Victorian Board of Studies, 1999).

Taking up this theme, Routitsky, Tobin and Stephens (1998) analysed further the data on teachers who disagreed with the Board's policy. The purpose of this investigation was to determine if this were linked to their personal level of access to graphics calculators, or whether it related to the level of access which their students had, either through ownership or school access. The results of that analysis demonstrated, perhaps unsurprisingly, that teachers who disagreed with the Board policy tended to come from schools where they and/or their students had limited access to a graphics calculator. This is consistent with a previous study on teacher attitudes to use of graphics calculators in a college algebra course in the USA, where it was found that the only significant variable on level of teacher support was degree of familiarity of the user (Chamblee, 1995). Reduced familiarity is an immediate consequence of limited access.

Methodology

At that time our main concern was about availability of graphics calculators. We recognised that this problem had two sides: school policies and how these policies were implemented from the teachers' point of view. That is why the survey was organised in two parts. It had two aims. The first to determine from school Mathematics Coordinators what policies were implemented by schools to make graphics calculators available for teaching. The second to determine teachers and students access to the graphics calculators, teachers attitudes to its use, and actual use of graphics calculators in the classroom. The main target group of the survey was teachers of VCE mathematics subjects. This spans years 11 and 12 and includes the five subjects, Mathematical Methods 1 & 2, General Mathematics Units 1 & 2, Mathematical Methods Units 3 & 4, Further Mathematics Units 3 & 4 and Specialist Mathematics Units 3 & 4. It is only the unit 3 & 4 subjects which have external examinations affected by the new policy - these are ninety-minute papers involving multiple choice, short answer and analysis tasks.

According to its aims, the survey had two different target populations, Victorian Mathematics Coordinators² and Victorian Mathematics Teachers, and two different instruments. The brief coordinator questionnaire included information on school sizes (indirectly), calculator models, booklisting³ policies, class sets, student access and estimations of student calculator ownership. The focus of this questionnaire was school penetration of the graphics calculators.

The focus of the teacher survey (four pages) was on attitudes to the Board's calculator policy and attitudes to the graphics calculators themselves as well as on the availability of the graphics calculators in the classroom. This teacher survey enables us to assess any variations in responses between school regions, types or sectors as information on these was gathered also.

There are seven educational regions classified by the Department of Education in Victoria. The three sectors are Government Secondary (including TAFE), Catholic, Independent. The three types of educational providers classified were boys' schools, girls' schools, and coeducational schools.

As it was said before, there has been little previous research undertaken on teacher attitudes towards graphics calculator. In the USA, Jost (1992) examined interrelations between teaching style and the frequency of use of graphics calculators during instruction. She found that teachers that use the inquiry-oriented methodologies used the calculator more during instruction (and this was the rationale for the question about teaching style). Chamblee (1995) investigated the association between teachers' attitudes toward graphics calculator and various personal characteristics on a comparatively large scale. May (1995) examined teacher concerns, teacher training and curriculum issues related to the integration of graphics calculators. From the survey of 33 teachers, May found that teachers' greatest concerns were 'outdated curriculum', lack of professional development, 'lack of time, and inequities of ownership'. This work supported our initial intention to include some questions about professional development and topics taught with graphics calculator alongside with questions about ownership and attitudes toward graphics calculator.

² Mathematics Coordinator is usually a senior experienced teacher in Victorian Secondary school, who supervises mathematics teachers, coordinates curriculum policies in Mathematics in the school, has similar responsibilities to the head of department, if school is big.

³ Booklisting policies in this case are the school policies that either recommend students to have a certain type of calculator or do not do so.

Every Victorian school and TAFE⁴ college which provided Mathematical Methods 3&4 to its students (the most popular Year 12 mathematics subject) received one Coordinators' Questionnaire and five Teachers Questionnaires with the covering letter in which we asked Coordinators to answer his/her Questionnaire and to give the teachers their questionnaires. This study design had its limitations. Teachers were not sampled, because there was no sampling frame to do so at the time of the survey. To reduce the impact of this limitation the comparison of population distribution of different types of schools and the achieved sample of schools was performed and post-stratification techniques were used to reduce possible bias.

Both surveys had good response rates. 73% of all Victorian Mathematics Coordinators returned their questionnaires after two mail-outs and a follow-up telephone survey. The achieved sample was representative across all regions, school sectors and school types. 46% of teachers returned their questionnaires. It made more than 1000 teachers from all Victorian regions, school types, and sectors. However, there was slight over representation of teachers from Independent schools and rural regions. Specifically the methodology of data collection, response rates, profile of respondents, and possibilities of bias of estimates have been discussed in previous papers (Routitsky and Tobin, 1998). This paper is based mostly on the results from the second (Teachers') survey and tried to answer the questions listed in the introduction.

Results

1. Teachers' and students' access to graphics calculators.

The majority of Mathematics Coordinators or 99.7% (334 out of 341) answered 'Yes' to the question: *Are graphics calculators used in the school?*

Teachers were asked a similar question but about themselves rather than about their school. The question was: *Do you use graphics calculators in your teaching?* The majority of teachers answered 'Yes' to this question 77.6% (831 out of 1071).

However, the aim of analysis was to investigate this issue deeper than that. We were interested whether school policies on the one hand, and students' and teachers' access to graphics calculators on the other hand, varied across school sectors (Government Catholic, and Independent Non-Catholic), school types (Boys', Girls' and Co-educational), or regions. Overall 53.4% of schools

⁴ Institutes of Tertiary and Further Education (TAFE) are vocational institutions in Victoria. Some of them also provide VCE subjects for those who left school earlier.

had at least one class set of graphics calculators and 78.6% prescribing ('booklisting') graphics calculators. Information on booklisting by schools shows that Independent Non-Catholic schools were much more likely to prescribe calculators than Government schools. Catholic schools were also more likely to 'booklist' calculators than their Government counterparts - though to a lesser degree. Coordinator results demonstrate that 9% of all respondents (10% after weighting to assist over representation of Independent schools) neither booklisted calculators nor provided any class sets for students in 1997. This figure is higher than the proportion of schools that do not use graphics calculators. Possibly some schools used graphics calculators for demonstration purpose, but neither have a class set, nor prescribe graphics calculators for students.

Teachers were asked two questions about accessibility of graphics calculators for them and their students in the lesson. In this section the answers to the following questions will be analysed: *Do **you** have access to the graphics calculator for use in your teaching?* *Do your **students** have access to a class set of graphics calculators in your lessons?*

The answers were graded from 1- 'NEVER' through 'RARELY', 'SOME TIMES', 'OFTEN' to 5- 'ALWAYS'. In Year 11 and especially in Year 12 the permanent availability of graphics calculators in the classroom is essential for developing students' skills in their preparation for external examinations.

Research shows (Steel, 1996) that mean improvement in assessment performance (for example, in such topics as Functions and Graphs) of students, who use calculators every lesson is twice as higher compare to those of them who use calculators every second lesson. Therefore, even those students who can use and whose teachers can use graphics calculators 'OFTEN' are disadvantaged against those who can use and whose teachers can use graphics calculators 'ALWAYS'. As for those students who are able to use the class set of graphics calculators 'NEVER', 'RARELY', or 'SOME TIMES' (more than 35% of classes), these students most likely do not gain much from the use of graphics calculators especially if they don't have their own graphics calculators. That is why categories 'ALWAYS', 'OFTEN', 'NOT OFTEN' were used, where category 'NOT OFTEN' combines three categories: 'NEVER', 'RARELY', and 'SOME TIMES'.

The trend we have in Table 1 was not unexpected. The percentage of those teachers who could not regularly use graphics calculators varied from 16% in Government Sector to 8% in Independent Sector. Statistically differences in the availability of graphics calculators across sectors were significant ($\chi^2 = 40.19$ DF = 4, $p < 0.001$ for teachers, and $\chi^2 = 24.17$, DF = 4, $p < 0.001$ for students).

It is interesting that although Independent Schools tended to purchased class sets of graphics calculators less than Government schools, students of Independent schools were more likely to use them on a regular basis than their Government counterparts. 50% of Independent schoolteachers reported that their students could use graphics calculators 'ALWAYS' against 34% Government school teachers.

Table 1 Access to Graphics Calculators across School Sectors

Sector	Teachers' access (%)				Students' access (%)			
	GOV.	CATH.	INDEP.	Total	GOV.	CATH.	INDEP.	Total
ALWAYS	61%	73%	83%	68%	34%	47%	50%	40%
OFTEN	24%	17%	9%	19%	29%	21%	17%	25%
NOT OFTEN	16%	10%	8%	13%	37%	33%	33%	35%
Sample size	527	194	218	939	520	193	207	920

It was not expected, that differences across school types in the availability of graphics calculators would also be significant ($\chi^2 = 19.97$, DF = 4, $p = 0.001$ for teachers and $\chi^2 = 18.20$, DF = 4, $p = 0.001$ for students). The percentage of those teachers who can not regularly use graphics calculators varies from 16% for Boys' schools to 7% Girls' schools (see Table 2). For students these figures are 43% and 25% consequently. We can clearly see that availability of graphics calculators is much higher in Girls' schools than in Coeducational or Boys' schools (both types are mostly private schools in Victoria).

Table 2 Access to Graphics Calculators across School Types

School Type	Teachers' access (%)			Students' access (%)		
	GIRLS'	BOYS'	COED.	GIRLS'	BOYS'	COED.
ALWAYS	82%	69%	65%	55%	34%	37%
OFTEN	10%	15%	21%	20%	23%	26%
NOT OFTEN	7%	16%	14%	25%	43%	37%
Sample size	165	62	712	161	61	698

Both Tables 1 and 2 suggest strong association between teachers' and students' access to graphics calculators in the classroom. There were the same trends in students' ownership of graphics calculators as they were for availability of graphics calculators in the classroom (Routitsky and Tobin, 1998).

2. Teachers' attitudes toward graphics calculators

In the survey teachers were asked to respond to several statements about use of graphics calculators for teaching and learning even if they have not used the graphics calculator. Responses were collected on a scale from 1 'NEVER' to 5

‘ALWAYS’. In this section, we will look at the responses to the following statements (the name of the corresponding variable is given in the brackets):

- *The graphics calculator is useful in your teaching (USEFUL)*
- *Teaching with graphics calculators reduces the time needed for explanation (SAVES TIME)*
- *Teaching with graphics calculators makes learning mathematics easier for students (EASIER)*
- *Graphics calculators improve students’ understanding of mathematics (UNDERSTAND)*
- *Usefulness of graphics calculators in your lessons depends on your teaching style (TEACH STYLE)*

The preliminary analyses showed that extreme attitudes ‘NEVER’ and ‘ALWAYS’ have smaller percentages compare to categories ‘RARELY’ and ‘OFTEN’ in responses to all statements. That is why all attitudinal variables were recoded from five categories to three: ‘NEVER or RARELY’, ‘SOME TIMES’, ‘OFTEN or ALWAYS’. The recoding did not lead to the loss of information but made this information clearer. For further analysis these variables were used in the three categories form.

Figure 1 Teachers’ Attitudes toward Graphics Calculator Use for Teaching

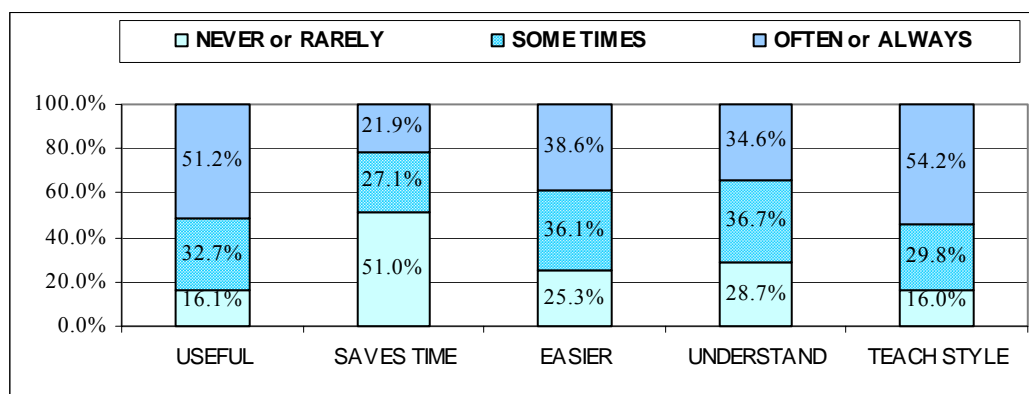


Figure 1 shows some similarity in responses to the statements about *usefulness* of graphics calculators and dependence of it on *teaching style*. Teachers are more positive about these two statements than about others. More than 50% of teachers, who responded to these statements, believe that graphics calculator is ‘ALWAYS’ or ‘OFTEN’ useful in their teaching and that usefulness depend on their teaching style. Teachers overall are not so positive about the statement that teaching with graphics calculators *makes learning mathematics easier* for students, nor so positive about statement that graphics calculators *improve students’ understanding* of mathematics. More than a quarter of all teachers who responded to these statements believe that this happens ‘NEVER or RARELY’ and around one third of them think that it

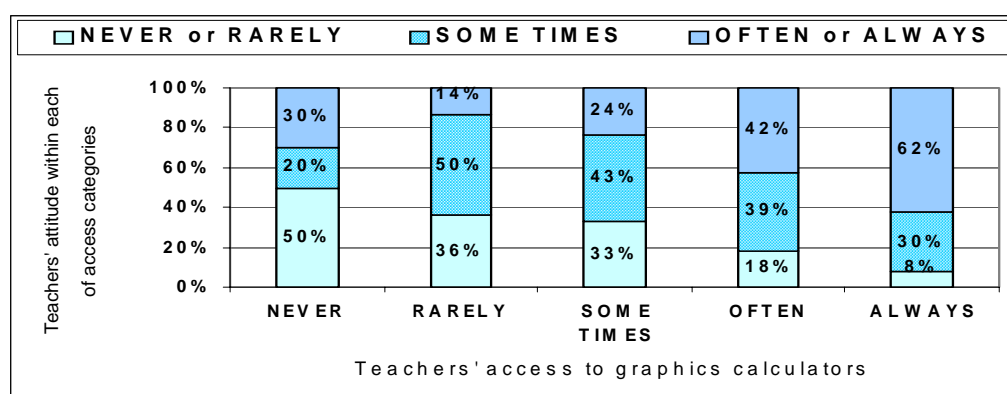
happens only ‘SOME TIMES’. As for *saving time for explanation*, more than 50% percent of all teachers responded to this statement believe that it happens ‘NEVER or RARELY’.

3. Interrelations between teachers’ attitudes toward graphics calculators and access

Figure 2 shows that there was an inverse relationship between teachers’ access to graphics calculators and negative attitudes to usefulness of graphics calculators in teaching. The better access to graphics calculator the less proportion of teachers’ responded that graphics calculator was useful ‘NEVER or RARELY’. This is consistent with a previous study on teacher concerns about the use of graphics calculators in a college algebra course in the USA, where ‘for total concerns score, only graphing calculator expertise rating was found to be a significant predictor’ (Chamblee, 1995). Limited access is a major reason for reduced expertise rating.

However, positive attitude interrelates with access in a more complicated way. 30% of teachers who never had access to graphics calculators supposed that it was useful ‘OFTEN or ALWAYS’. Between those teachers who had access to graphics calculators ‘RARELY’ this consideration dropped to 14% and then began to increase with the increase of availability of graphics calculators. Yet, those teachers who had access to graphics calculators ‘SOME TIMES’ (hypothetically more often then ‘RARELY’) still are not so positive about usefulness of this tool for teaching as those who can not use it at all. This attitudinal behaviour corresponds to the well-known psychological scheme: Unconscious Unskilled → Conscious Unskilled → Unconscious Skilled → Conscious Skilled. Usually when we unaware of difficulties we are more optimistic then when we begin to use a new method. However, when skills increase teacher confidence increases as well.

Figure 2 Graphics Calculators Useful in Teaching (interrelations between teachers’ access to graphics calculators and teachers’ attitudes).



Accessibility is a major factor for teachers to improve skills in the use of graphics calculator. Similar trends are found for interrelations between students' access to a class set of graphics calculators and teachers' perception to its usefulness. Interrelations between accessibility of graphics calculators and other attitudinal variables are similar but not so distinctive, although for all attitudinal variables general trends are the same: more access, better attitude.

4. Interrelations between teachers' attitudes toward graphics calculators and teacher's personal characteristics and experiences

The following teachers' personal characteristics were examined into relations to teachers' attitudes toward graphics calculators use for teaching and learning (a) *Gender*, (b) *Age group*, (c) *Years teaching Mathematics*, (d) *Time teacher use graphics calculator*. Differences between male and female teachers were significant in their attitudes toward only two statements: *Teaching with graphics calculators makes learning mathematics easier for students* (variable EASIER $\chi^2 = 9.17$, DF=2, $p=0.010$); and *Graphics calculators improve students' understanding of mathematics* (variable UNDERSTAND $\chi^2 = 6.68$, DF=2, $p=0.035$). However these differences were due to men being more certain than women. Women more often responded 'SOME TIMES' and men more often took both ends of the scale: 'NEVER' or 'ALWAYS'.

Teacher attitudes were significantly different ($\chi^2 = 14.23$, DF=6, $p=0.021$) across age groups only toward one statement: *Graphics calculators save time needed for explanation*. Although, there were approximately half of teachers in all age groups who thinks that graphics calculator saves time 'NEVER or RARELY', positive attitude toward this statement 'OFTEN or ALWAYS' monotonously increases from 15% of teachers age 21-30 to 28% of teachers who are older than 50. One would think that these differences were due to teachers' experience in their job. However, this was not the case. Analysis showed no significant differences between teachers who taught mathematics for a different number of years, in their attitudes toward different aspects of usefulness of graphics calculators in teaching.

The important issue was the teachers' experience with graphics calculators. Their attitudes toward all statements strongly interrelated with the time they use graphics calculator.

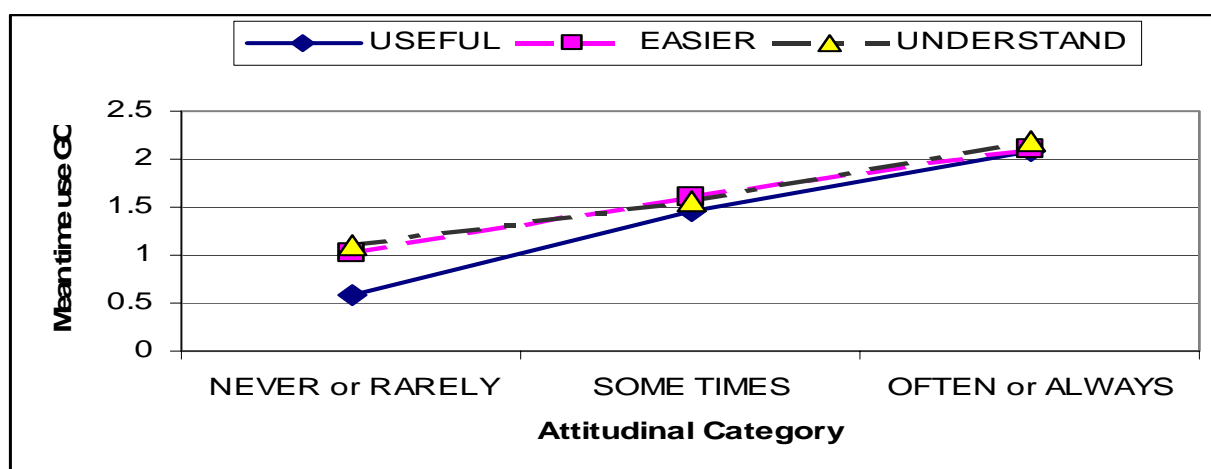
Table 3 Time Teachers Use Graphics Calculator (in Years) vs Attitudes

Mean time use GC	USEFUL	SAVES TIME	EASIER	UNDERSTAND	TEACH
NEVER or RARELY	0.59	1.44	1.03	1.10	1.50
SOMETIMES	1.45	1.63	1.60	1.56	1.45
OFTEN or ALWAYS	2.09	2.16	2.10	2.18	1.78
F ^{*)}	87.11	19.47	47.04	50.01	6.02

^{*)} (For all variables $p < 0.005$, between groups $df=2$, within groups $df>900$)

It is especially clear for three variables: **USEFUL**, **EASIER**, and **UNDERSTAND** (in bold). For these variables, differences in mean time teachers use graphics calculators between the three attitudinal categories (which are differences between second and first row and then between third and second row) vary from nearly half a year to nearly a year. For example, for **USEFUL** (first column), difference between mean time for **USEFUL** SOME TIMES (1.45 years) and mean time for **USEFUL** NEVER or RARELY (0.59 years) is 0.86 years. To improve her/his attitude further to **USEFUL** OFTEN or ALWAYS an average teacher needs another 0.64 of a year. Figure 3 illustrates these relationships.

Figure 3 Time Teachers Use Graphics Calculator (in Years) vs Attitudes



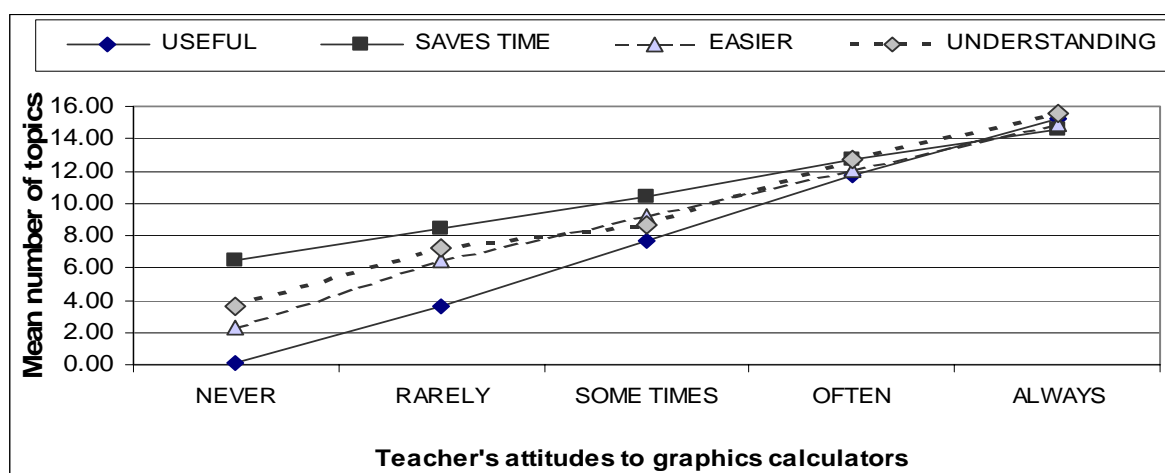
This suggests that an average teacher has to use graphics calculators at least half a year to improve her/his attitudes, and as we saw before, graphics calculators must be well accessible during this time. It also shows that when an average teacher uses graphics calculator more than two years her/his attitude becomes positive for all attitudinal variables used in this study.

5. *Interrelations between number of topics taught with graphics calculator by teachers and their attitudes toward graphics calculators*

Teachers were asked a number of questions about topics in which they possibly use graphics calculators for teaching. Mathematics topics for Years 9-12 were listed. Although the list of topics was not all-inclusive, it was quite comprehensive. Less than 8% of teachers confirmed that they use graphics calculators in 'OTHER TOPICS'. There were favourites like Year 12 Statistics (nearly 78% of all teachers who use graphics calculator in Year 12) and less popular topics like Linear Programming (30.6%). Out of 77% (831 teachers) of teachers who use graphics calculators in any of 9-12 Year levels, the number of topics in which they use it varies from 0 (sic! 12 teachers) to 38 (one teacher). The median number of topics across all year levels was 10 per teacher with quartiles around 5 and 16 topics. There was some association between the *Number of Topics Taught with Graphics Calculator* (TOPICSN) and attitude toward this tool, although these associations were different for different attitudinal variables.

Figure 4 illustrates a stronger association between number of topics and positive attitude to general usefulness (USEFUL), compare to association with SAVES TIME. For example, teachers who think that GC is 'NEVER' useful in average don't teach any topics with GC (mean number of topics is very close to 0), teachers, who think that GC is useful 'RARELY' in average use it approximately in 4 different topics, and so on. The increase in average number of topics is not so rapid for variable SAVES TIME. While TOPICSN increases by four with each attitudinal category for USEFUL, it increases only by two for SAVES TIME.

Figure 4 Association between Number of Topics taught with Graphics Calculator and Attitude towards it.



One can imagine the cyclic process: the more a teacher uses the tool the better attitude she/he has towards it because of experience. On the other hand, the better attitude teachers have the more topics in average they try to use the tool in. However, compared to such variables as *Time teacher use graphics calculator* and availability, particularly ownership, association between attitudes and *Number of Topics taught* was weak.

6. Interrelations between number of topics taught with graphics calculator by teachers, their experience in use of it and the level of students ownership

Multiple regression shows that the best predictor of *Number of Topics taught with Graphics Calculator* (TOPICSN) is *Time teacher use graphics calculator* and *Average Level of Ownership* across all year levels (OWNG1), which together explains 55.5% of variation in TOPICSN. When attitudinal variables are added to this regression, none of them improves it and none has a significant coefficient as well as the combined attitudinal variable USEFULG (the sum of four attitudinal variables). However, when USEFULG is entered into regression alone, it explains nearly 20% of variation in *Number of topics taught with graphics calculator*, and regression coefficient is significant. Since both *Time teacher use graphics calculator* and level of students' ownership (OWNG1) are highly correlated with attitudinal variables, the latter does not add to regression when OWNG1 and TIME USE are entered.

Summary

Teacher overall attitudes to usefulness of the graphics calculators positively associates with the level of access to the calculators in the classroom and with time teacher use graphics calculator but generally does not correlate with gender, age or teaching experience in Mathematics.

Number of topics taught with graphics calculator, is best predicted by *Time teacher uses graphics calculator* and by level of students' ownership (OWNG1). There is also some association between teacher attitudes toward the tool and number of topics they teach using it.

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PRODUCTION AND TRANSFORMATION OF COMPUTER ARTIFACTS: TOWARDS CONSTRUCTION OF MEANING IN MATHEMATICS

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Abstract: *Artifacts both mediate our interaction with the world, and are objects in the world that we reflect on. As computer-based artifacts are generally intermingled with multiple praxes, studying their use in praxis uncovers processes in which individuals, the community and tools are involved. In this paper, we examine a now common computer-based artifact in mathematics classrooms, the representative. This artifact is often in continual transformation in the course of action during school activities. We document how several praxes with representatives mediate the construction of meaning. We show that the ambiguity of computer representatives regarding the examples and concepts they represent, boost this construction. The construction of meaning of functions is described as a process occurring through social interaction, and the interweaving of the ambiguous artifacts through teacher's creation of states of inter-subjectivity.*

It is now commonplace to point to the role of artifacts as mediators of human activity (e.g., Kaptelinin, 1996). Bødker (1997) recently considered the special role of computer-based artifacts in human activities. She claimed that computer-based artifacts need to be studied as *artifacts-in-use-in-a-certain-praxis*, as they are often involved in various intermingled actions or activities. In other words, activities with computer tools involve the computer tool itself as an inherited artifact, and material outcomes produced by this tool and transformed by participants engaged in such activities. Trivially, (educational) software enables users to “produce” prints or displays. But in addition, it generally affords multiple forms of automatic transformation of outcomes (e.g., stretching, shrinking, cloning, merging). We argue in this paper that the teacher can direct the production and transformation of computer artifacts towards the construction of meaning.

Computer artifacts in the case of the function concept

The concept of mathematical function is central in mathematics education. It is taught in schools using at least three notation systems, the algebraic, graphical and tabular representations. Actions or operations that an individual can undertake in each representation depend on material tools available. Kaput (1992) calls the algebraic representation an action notation system, meaning that it involves calculations and transformations. In contrast, he calls graphical and tabular representations display notation systems, meaning that the activity of the user is generally confined to interpretation. This theoretical distinction between action and display notation systems does not hold any more when one uses tools providing, in addition to the representations themselves, passage among representations and user based manipulations, or what is often called multi-representational software. All representations become action notation systems: it is possible to “walk” on a graph, to stretch graphs, to rearrange a table according to a particular criterion, or to refine the step in a table. The most common visible outcomes of actions mediated by multi-representational software are the *representatives* (Schwarz & Dreyfus, 1995), that is displays within the representations such as specific “windows” for graphs, or specific tables of values through which problems are solved. For example, Figure 1 shows three graphical representatives of the same function, $f(x)=x(30-2x)$.

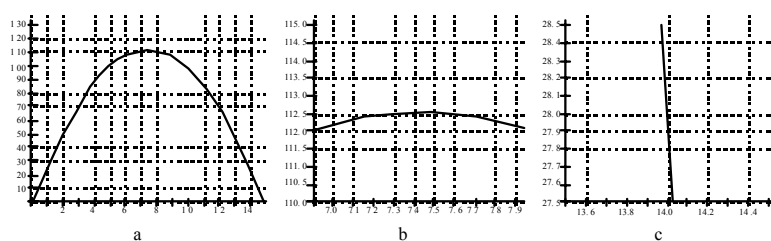


Figure 1

Classroom activities are often intended to encourage students to produce representatives as visible outcomes of actions, to interweave them while solving problems, and to use them as artifacts in order to talk about mathematical entities. A central psychological concern is the nature of the mathematical entity constructed through extensive production and interweaving of representatives. In environments that are not based on computerized tools, representatives produced and used during classroom activities are limited. Graphs or tables are often *presented* to students or *constructed according to prescriptive instructions* (Kaput, 1992). All students in the same class use identical representatives. These representatives are often chosen by the teacher to exhibit as many properties of the function as possible. For example students are likely to be asked to construct a graph such as in Figure 1a, because it displays domain, range, sub-domains of increase and decrease, maximum, and axis of symmetry. Instruction often concentrates simply on translation skills between representations, and mastery of these skills tends to become the central goal of teaching. As a consequence, students do not have the opportunity to talk about functions as abstract entities.

The use of multi-representational tools turns many of the manipulations of representatives into automatic operations. It is then not surprising that students produce a variety of representatives. Figures 1 displays 3 representatives of the many visible outcomes of actions produced by students in an activity described in the next section. Such representatives are generally ambiguous when taken in isolation in the sense that they are partial, and thus can refer to different entities. For instance, as mentioned above, Figure 1c may refer to a linear function. Also, a representative may display properties that are not possessed by the function. Intuitively, it would seem that such ambiguity is detrimental to learning and or development. However, an analysis of the role of ambiguity in development suggests quite the contrary.

Ambiguity and development

Philosophers of language and psychologists who adopt a dynamic, and pluralistic view of linguistic communication (Wittgenstein, 1968; Voloshinov, 1973; Rommetveit, 1985), have recognized that ambiguity plays a central role in the development of language and thought. This socio-cultural pluralistic view disputes the constructivist assumption that the child learns about significant features of her immediate surroundings in exploration and manipulation of material objects. Rommetveit makes clear that there is no such thing as a “*literal meaning*” and that construction of meaning is then a social construct that emerges from the ambiguity of states of affairs and from the experiential alternatives at a particular moment. External reality is apprehended through cooperation and social validation. Meaningful aspects of the adult world are brought into joint focus of attention in the child’s interaction with an adult and negotiation of the adult’s “truths” and linguistic structuring of attention (Vygotski, 1981). Rommetveit (1985) identified the interactions in which such restructuring takes place, as “*states of inter-subjectivity*”. For Rommetveit ambiguity creates states of inter-subjectivity possibly leading to the construction of a shared social reality: In dyadic interactions, verbal utterances have multiple subjective meanings; one participant brings some aspect into focus, it is attended by both of them, and each assumes that they mean the same. Similar to words in verbal interaction, computer representatives are an inherent part of communication in mathematical activities. Our working hypothesis in the present paper is that computer representatives often mediate the creation (by the teacher) of states of inter-subjectivity in the mathematics classroom leading to a shared social reality.

Mathematical entities, such as the function concept or geometrical concepts, are constructs that have been elaborated by the mathematical community. These entities are not directly accessible to children but are known

through their properties, which are the invariants of this mathematical entity across their possible instances. But these invariants can be approached only through examples, about which students and teacher talk. Similarly, direct manipulations on mathematical entities are impossible. The examples of the mathematical entity cannot be manipulated directly either: manipulations are possible on representatives only. The outcomes of these manipulations can be new representatives, interpretations, problem solving strategies, etc. The relations between concept and examples, examples and representatives, or concepts and representatives are inherently ambiguous: it is often not clear what the examples/representatives mean, and to what they refer. We review here the kinds of ambiguities that stem from these relations.

Examples of a mathematical concept contain all the invariant properties of the concept (also called the *critical attributes* of the concept). In addition to the invariants of the mathematical entity, examples also have self-attributes, which may change from one example to another. Examples are then ambiguous regarding the concept to which they refer. In some cases, students identify such self-attributes as critical attributes of the concept. This is because the example is the only one the student is familiar with or the one the student prefers to reason about the concept (the example is then called a prototype). The self-attributes of the prototype are then imposed on all other examples. We designate this ambiguity a *prototype ambiguity*.

Another type of ambiguity refers from an intrinsic property of *representatives*. Representatives are mostly parts of concept examples (e.g. Figure 1). As examples they may bear *prototype ambiguity*, but because they are partial, they are also often ambiguous in the sense that only some of the critical properties of the entity are displayed in the representative. This kind of ambiguity we designate *representative ambiguity*. It may lead to two different phenomena: (i) difficulty in constructing or seeing the mathematical entity through its partial representative; (ii) linking a representative to another entity with critical attributes compatible with the representative. Hence, the same representative may refer to more than one mathematical entity. Psychological findings fit our epistemological analysis.

Schwarz and Hershkowitz (1999) showed that some students, who learned about functions in an interactive computerized environment, took advantage of the *prototype ambiguity* to *build or analyze* new examples of the function concept. For example, they used the prototypical graph of $y = x^3$ to construct the graph of $y = -2x^3$. Other students, unfamiliar with graphers could not build additional examples from the prototypes they learned, and related to the prototype as an exclusive example. For example, they claimed that one function only “passes” through two given points, thus imposing this self-attribute of

linear functions on all functions. Thus it appears that *prototype ambiguity* may lead to beneficial psychological outcomes (construction of new examples) or to detrimental ones (*imposition* of prototypical attributes).

Schwarz & Dreyfus (1995) undertook a study of the function concept in Grade 9 students, using a questionnaire with items exhibiting ambiguity of many kinds. They measured the effect of the use of multi-representational software on problem solving involving functions, by comparing the achievement of an experimental and a control group. The most significant differences between the two groups were on items in which *ambiguity* was relevant. The study showed that intensive production and use of representatives (with multi-representational software) improved the learning of the function concept.

The two studies reported that the gains of students using computerized tools originated from the fact that such tools enabled the students to produce more examples of functions and that students could act on representations and pass from one to another. However, such interpretations did not focus on mechanisms that may explain such gains. We claim that the students' gains may originate from the fact that representatives can be *produced*, transformed, and *interwoven* easily while students solve problems. In the following section we describe and analyze an activity in which a teachers took advantage of the *representative ambiguity* to create states of inter-subjectivity that boost shared social construction of mathematical meaning.

The research: Representatives as artifacts-in-use in an activity

We document several praxes in a parochial school Grade 9 class studying an introductory course on functions. The course was based on a curriculum designed for the upper 60% ability level of the population, organized around problem-situations. Students had graphic calculators at their disposal. Many interactions took place in the classroom: among students in small groups, between the teacher and students, or between individuals/groups and computerized tools. Typically, activities consisted of a few phases as following: (i) problem solving (in groups of four); (ii) the writing of group reports on the ideas raised during the problem solving; (iii) synthesis in the form of debate orchestrated by the teacher; (iv) a homework assignment based on ideas raised in the group reports. Ideas raised in phases (ii) and (iii) were evaluated, approved or refuted. Therefore, when students (as individuals or in small groups) reported on their work, they knew that the teacher would use some of their reports to discuss issues further in the class. The teacher was explicit about the fact that writing accurate reports eventually helps the class to learn better. Hence, when reporting on their collective or individual work, students knew that

it would not serve exclusively for assessment and they grasped “reporting” as a communicative act that invests them with some of the responsibility for their own learning. The fact that the teacher generally did not use fictitious reports, also harmonizes with the fact that meaning was negotiated in the class. Both teacher and students took it for granted that each of them assumed the other to hold the same belief about the construct under consideration.

The fence: A typical task

The Fence was given at the beginning of the year as a group evaluation task, after the three first sessions of instruction during which students performed a series of tasks on functions with graphical calculators. The three sessions were all about problem situations in which the mathematical objects were cubic, quadratic and linear functions (in that order!). In each problem situation, students were invited to construct and/or interpret representatives in the algebraic, graphical, and tabular representations. The approach in these initial activities was informal and students were not led to systematically articulate properties of specific functions. Rather, students passed to a new representative in a new or same representation (what we have called “*to interweave*” representatives) when they felt it was needed. The formulation of the Fence was then given to the students:

Oranim school received a 30m long fence to enclose a lot to serve as a rectangular vegetable garden. The lot is contiguous to the school wall, so that the fence has three sides only (see Fig. 2).

- Find four possible dimensions for the lot, and the corresponding areas.*
- For which dimensions does the lot have the biggest area?*
- If one of the dimensions is 11m, what is the area of the lot? Can you find another lot with the same area? If yes, find its dimensions, if not, explain.*
- How many lots with the following areas are there: $80m^2$, $150m^2$?*

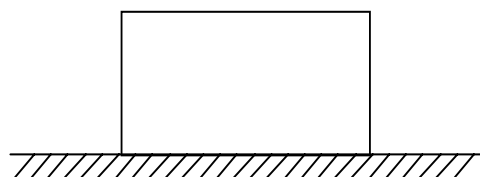


Figure 2

Eight groups of four girls solved the Fence, resulting in eight group reports. One week later, the teacher gave a homework assignment on the basis of the group reports. 32 individual homework assignments were collected and analyzed. The reporting and the whole design were communicative actions by means of which representatives were shared by the whole class. In the following, we study how the teacher capitalized on the ambiguity of the representatives in homework assignments in which individuals constructed the meaning of mathematical objects.

The role of computer artefacts in the fence

Diverse strategies were evidenced in the group reports. For example, several groups used an inductive strategy that led to generalization: they organized their numerical examples of the dimensions of the lot as a table with three entries per row: first side, second side, area. Then, they generalized the numerical values in algebraic terms: x , $30-2x$, $x(30-2x)$. In other reports, students used a trial-and-error strategy, constructing a table and finding the maximum by successive approximation. Yet other groups used a modelling strategy: they first constructed a formula and used the calculator to draw a suitable graphical representative and read the maximum of the function by walking on it. The diversity of strategies evoked was accompanied by an even more diverse collection of representatives and by various strategies for interweaving them. The following are some examples from the group reports.

Example 1: This excerpt exemplifies a group who was “in a hurry” to find a formula, inserted $30 - 2x$ (the formula for the second side) as the formula for the area, and reported on the surprise caused by the graphical representative obtained.

We drew the graph on the calculator, according to the formula $30-2x$, with a range of 0-15 for x and a range of 0-150 for y (Fig. 3a). It did not seem OK because we had to find the area and the formula fitted the second side; so we understood that the formula was wrong and we decided to replace it by $(30-2x)x$ because this is the area formula (two arrows pointing to x and to $30-2x$ and labeled first side and second side, appeared in the group report). We looked for the largest area. We got a graph that seems to us a more correct one (Fig. 3b). Note: we did not change the range, only the formula.

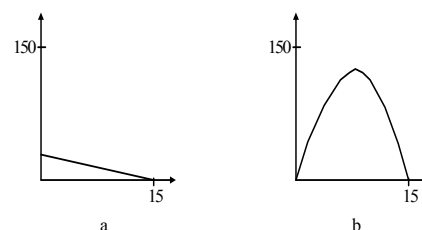


Figure 3

The students were surprised by the first graphical representative they produced. So, they turned back to the algebraic representation, and corrected the formula to obtain what they called a “*more correct graph*”. It seems that the students rejected the first two representatives because the second one (the graph) did not make visible the intrinsic attributes of their mental representation of the situation. Consequently, they produced two more representatives and accepted the second one, because it made visible the requested property of the corresponding function, having a maximum at some point. This example seems to show that students link representatives to the function they are meant to represent, through actions of *interweaving* representatives and rejecting those, which are not compatible with their concept image.

The teacher collected all representatives and “redistributed” some as artifacts for subsequent homework. For example, she used the representative in Figure 4a in the following question:

A few students drew this straight line as the area graph (see Fig. 3a). Most students said that this graph does not fit the area. Explain why. Give as many reasons as you can.

This question could be answered directly and quite simply by invoking only local properties of the linear graph. For example, it is clear that when $x = 0$, the area is 0, but for the linear graph, when $x = 0$, the area is not 0. However, only 7 students gave such answers. The remaining 25 students preferred to *compare* the linear graph presented in the homework sheet with representatives they had already constructed, as shown in Example 2:

I know that for the side zero, there is no area. Also for the side 15, there is no area. So I know that between zero and 15, there must be an arc ... and so for 1, 2, it increases until it must go down in order to get to 15 (where there is no area). And on this graph, (Figure 3a) it always goes down.

In her group report, this student and her peers had first constructed a table of values (with 0 and 15 as x -values), then modeled the formula of the area, and drawn the graph in Figure 1a. She could have used this graph as a whole to dismiss the proposed linear representative. Instead, she used the properties represented by the representative produced by her group, to show that the linear representative does not have the required attributes: “her” graph showed no area for the two extreme values $x=0$, and $x=15$, a fact that matches the Fence. Thus, such a graph must increase and then decrease. It may seem strange that the student used such a sophisticated justification. But, students were accustomed to handle ambiguous outcomes-representatives whose meaning was clarified by comparing them with other representatives, and by deciding whether they refer to the same meaning. This example shows an additional type of *interweaving of representatives* leading to constructing mathematical meaning - by comparing them.

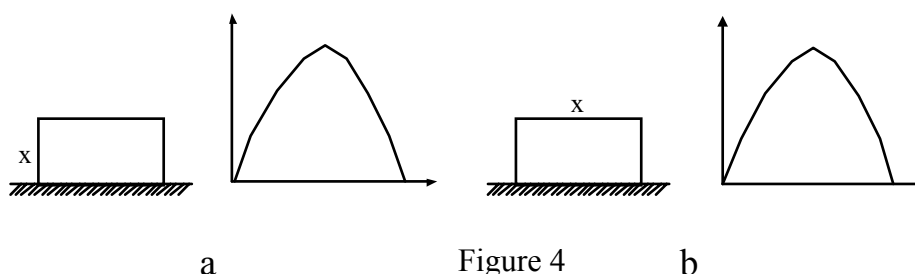
In sum, when the teacher found the students with the linear graph, she created a state of inter-subjectivity in which the ambiguous linear representative led students to ask whether it may refer to what they solved the problem in their groups. The decision relied on a critical attribute of the function, the domain of increase/decrease. It appears then that the teacher’s selection of an artifact produced by groups in the class, and its distribution to create a state of inter-subjectivity within this class, led students to cope with a mathematical construct. Inter-subjectivity in this example was attained through *representative ambiguity*.

The next example shows a similar phenomenon, where both prototype and representative ambiguity are capitalized on.

Example 3: In an additional homework assignment, the teacher took advantage of the fact that one group used another variable to express the side of the lot, and area of the lot, to ask:

Some students chose the side perpendicular to the wall as x ; the formula for the area is then $(30-2x)x$ and the graph looks like this (see Fig. 4a):

Other students chose x for the side parallel to the wall; the formula for the area is then $x(30-x)/2$ and the graph looks like this (see Fig. 4b):



If you draw the two graphs on the same coordinate system, do they coincide? Give as many reasons as possible.

The teacher makes it clear that she redistributed two representatives produced by two different groups. The answers to this question were interesting. For example:

Yes. The graph is the same. The second side is with jumps of two, whereas the jumps for the first side are of one. So the graph of the side 1 is like that (Fig. 4a) and the graph for the second side is the same (Fig. 4b), the jumps on the x -axis being of two (Fig. 4b).

Again, students know that many representatives have been produced by members of the community, but here, in contrast to other examples, they are asked whether the representatives referring to two different mathematical objects can be identical as material entities. Ambiguity is thus clearly *prototypical*, meaning that the same prototypical example (a parabola) may represent two different entities. This ambiguity is also a *representative*, because the teacher drew the two representatives without scaling. Again, the teacher brought students to a state of inter-subjectivity in which they were invited to construct a higher mathematical construct; i.e., that replacing x by $2x$ in a function “shrinks” its graph by half. Expressing this property was mediated by the two artifact-representatives redistributed by the teacher. And indeed, most of

the students did not rely on discrete considerations to differentiate between the two representatives or to decide whether they might overlap. Rather, they usually appealed to general properties (critical attributes). As each student had produced representatives referring to one of the two functions, $x(30-2x)$ and $x(30-x/2)$, she could take advantage of her representatives and of the graphs redistributed by the teacher and interweave them by comparison in order to respond to the challenge introduced by the teacher. Therefore, the two artifacts constructed by different groups and selected by the teacher to cope with the issue of whether they can coincide led students not only to answer in the affirmative, but also to identify a new attribute. But there is even more. The attribute quasi-articulated in the present homework assignment -- replacing x by $2x$ has the same effect on the graph of a function as shrinking the scale by two, leads students to link two different mathematical objects belonging to the same family. The two functions having a common (and “perfectly ambiguous”) representative relate to the same family of quadratic functions. Of course, at the beginning of the year, the relation between the two examples of quadratic functions is still embryonic, but the same ambiguity was used later to lead students to construct the meaning of the concept of quadratic functions.

Discussion: Computer artefacts and construction of meaning

The present study has shown that representatives produced via the mediation of the calculators, helped in the social construction of meaning of functions. We showed that, although the relations between concept and examples, examples and representatives, or concepts and representatives, are inherently ambiguous, dealing with and relating to many examples or acting on representatives did lead to the construction of the meaning of mathematical entities. We showed that *representative ambiguity* the *production* of various representatives and the need and ability to *interweave* them, helped to extract invariant properties of functions. We also showed that the plethora of examples and representatives produced by the graphic calculators upon need, and the actions facilitated by them to produce new representatives, *led students to clarify ambiguities*. More specifically, we showed that students took advantage of the partiality of representatives because they were put in a situation of dilemma in which they had to reconcile apparently different representatives. The computerized tool enabled students to compare them, to produce new representatives and to integrate or reject (partial) representatives. Advantage was taken of partiality *to view the intrinsic attributes* of the function by *refining* through interweaving representatives (e.g., by passing to another representation and/or by manipulating representatives in order to reduce ambiguity).

Resolving ambiguities was not only an outcome of the manipulation of computer representatives. The teacher had a central part in leading students in a dialectical activity leading to the resolution of ambiguities. Figure 5 shows the different phases of the Fence from the perspective of the teacher. In a first stage, groups of four students (shown as “clouds”) solve the Fence and report on their actions during the solution.

The reports include numerous representatives. Some of these representatives may reflect “correct” actions (such as the algebraic representatives $y = x(30 - 2x)$ and $y = x(30 - x)/2$ and all graphical representatives derived from these formulae), some may reflect incorrect ones (such as the formula $y = 30 - 2x$ and the linear graph obtained). As they are the footprints of the idiosyncratic moves undertaken by each of the groups, most of the representatives are different in form. A short analysis of the reports (as in the first and second examples) shows that most of the groups coped with the diversity/ambiguity of the representatives to solve the Fence.

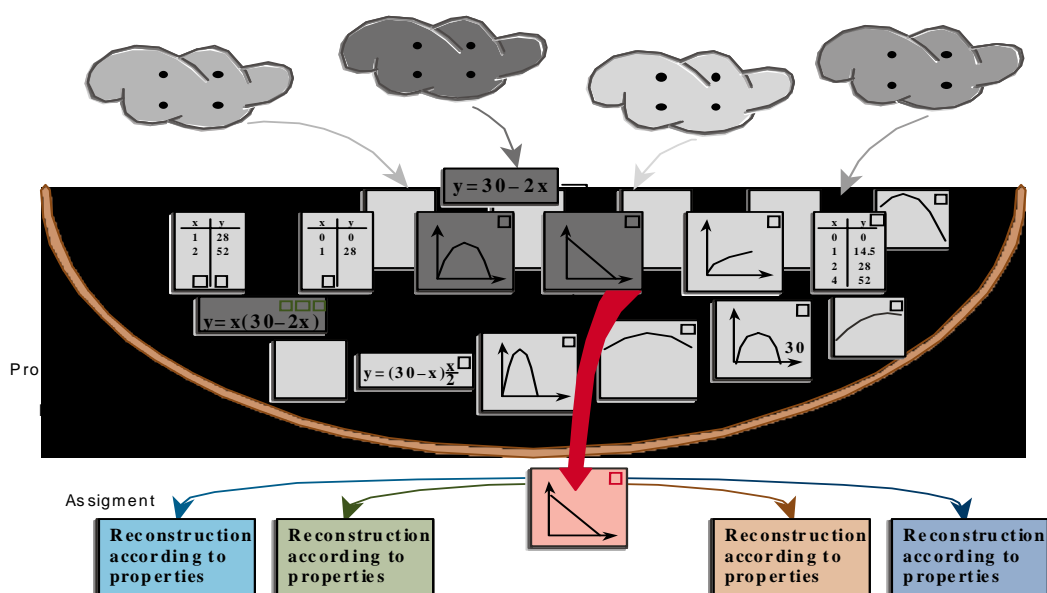


Figure 5

The role of the teacher in the home assignment was indirect. After collecting representatives in the reports as raw material, she selected some of them having the potential to lead to the construction of further meaning. It was clear to students that these representatives were private outcomes of particular groups. Their redistribution in homework assignments created then states of inter-subjectivity in which students were faced with representatives from same or different representations, possibly referring to different functions. The construction of shared social reality followed states of inter-subjectivity created by *prototypical ambiguity*.

The teacher first imposed an implicit community by forming groups of students at the beginning of the year to solve problem situations. She supported reporting and criticizing as *communicative actions* that gave the actors a visible role: The teacher redistributed some of the representatives in the homework assignment, to be the property of the whole class. Although we do not report here on the actual interactions that took place in the class during the Fence activity, we suggest that the class began then to function as a new community whose overall motive moved gradually from reaching the solution to reflecting on acceptable solutions. In this community, students progressively became encultured to the fact that any report can be shared by the community. Artifacts were integrated-transformed or discarded by the community according to their compatibility with previous outcomes/representatives. Critique was generally done at the global level of all the transformations a particular artifact/representative can undergo without altering what it means (in other words, the properties of the function). Our suggestion relies on one study undertaken in the same class at the end of the year (Hershkowitz & Schwarz, 1999) in which we reported on the reflective practices in that same class and on how previous outcomes (hypotheses, methods, graphs, etc.) were transformed during successive phases of an activity.

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**POSTERS AND ABSTRACTS
OF ADDITIONAL PAPERS LINKED
WITH WORKING GROUP 2**

DEVELOPMENT OF INTELLECTUAL SKILLS OF THE PUPILS WITH COMPUTER TECHNOLOGIES

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Keywords: *Mathematics teaching; real-life problems; computer technologies*

Practical usage of computer technologies for mathematical modeling in Excel environment in education is considered (on example of science course at high school level). For this purpose an amount of real-life problems are proposed: the shortest way problem, Teplitsky's problem (problem about glider), problem about a defeat of the moving purpose, problem about a pursuit.

Numerical methods for solving these problems and their visualization offer to explore solutions for great variety of parameters what stimulates interest to math, developing of math intuition, and forms intellectual skills of the pupils.

Our results show that Excel is suitable environment for designing the course of computer modeling, which integrates all the science courses together. It seems to be beneficial to arrange the international collaborations in this area under ERME.

STUDENT'S PROJECTS ARE A TOOL FOR THE FORMATION OF INVESTIGATING SKILLS

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Keywords: *Mathematics teaching; pupil research skills; computer technologies*

In this paper some problems of formation of pupils' research skills are considered. For this purpose the teacher organizes the study researches during mathematics lessons and scientific projects using information technologies.

Solution of such great problems now not under a force to all comprehensive schools of Ukraine, but school of a new type: educational complexes, lyceum, gymnasiums, the colleges, so and number of comprehensive schools already have begun work in this direction and already have achieved noticeable outcomes. The use of modern information technologies and computing experiment of realizations of primary study researches at mathematics lessons is on our sight actual. This work receives the further development in student's projects conducted by the Small Academy of sciences, international school projects, including telecommunicative ones.

Discussion of the most interesting results of such researches to our mind would be useful for ERME group.

BASIC STRUCTURIZATION AND INTERACTIVE ALGORITHMIZATION IN MATHEMATICAL EDUCATION

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The offered technology of structurization and algorithmization in mathematical education is based on extraction from an educational material of those components, which are basic (main) for chosen course: definitions, theorems, problems. On the basis of selected components it is offered to create the teaching, interactive environment, which is founded on creation computer files (word-files): basic didactic materials (BDM); algorithmic didactic materials (ADM). Creation of the BDM and ADM will give the opportunity to achieve the following results: to make more active feed-back in educational process with the use of dialogue; to give pupils models of methods and algorithms of proofs of theorems and ways of solutions of problems us; to create of computer files with didactic materials, which are free from language barriers; to organize collaboration of teachers for creation of database BDM, ADM for any school and university mathematical course; to use BDM and AMD in Internet technology for education: HTML files; Java files, didactic testing, algorithmic testing.

THE MATHEMATICS OF THE BOYS/GIRLS: EXCHANGE OF EXPERIENCE AMONG BOYS/GIRLS OF THE SAME AGE

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Keywords: *Mathematics teaching; student interaction in mathematical learning situations; social interaction in mathematical learning situations*

We present a very complex work, which can be examined from various points of view. In this poster we describe the methodology, which is, in our mind, its most original side. The experience was planned and carried out already three times (years 1996, 1998, 2000) by a university researcher (L. Zuccheri) and a group of ten teachers of different pre-university school level. It consists in various phases. At the beginning of the school year, each schoolteacher plans a didactical project in Mathematics, which will be developed with their pupils/students during the following months. This project is discussed and focused in the research group and involves actively the pupils, leading them to produce autonomously a presentation of a mathematical topic. In a second moment the pupils/students participate to a meeting in which they explain to other pupils and students these topics, with the methodologies that they think to be more appropriate. At this meeting, called "*The Mathematics of the boys/girls: exchange of experiences among boys/girls of the same age*", is also invited, as visitor, any class which requires it. It takes place in a school. The organisation of the meeting is quite different from that of a meeting among adults: each "speaker-class" organises a laboratory activity and carry out it in a classroom, where it receives one "visitor-class" at once, to work with it during a fixed time. The visitors can be also older or younger as the speakers can. During the three editions of the meeting the number of involved pupils (6-16 aged) increased from 500 to approximately a thousand. This experience has reached various positive goals, as we have seen in the valuation given by the teachers, and analysing 94 reports made by pupils. Some information is available at the website: www.nrd.univ.trieste.it

A VIDEO ABOUT MATHEMATICS

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Keywords: *Mathematics teaching; popularisation of mathematics; audio-visual media*

Mathematical thought has reached results that are unthinkable for non-specialists and has opened a way that became the main road for the cultural, scientific and economic development of human society. Owing to the importance of Mathematics in youth education and for the development of rational thinking, the UNESCO supported the proposal of the I.M.U. to declare 2000 as the World Year of Mathematics; still, only a few people are aware of the role played by this discipline. Today's resorting to computer science makes many people unaware even of the Mathematics they actually use, and the image of Mathematics is overshadowed by prejudices and commonplaces that have taken root in the collective imaginary. The reasons for it are many, and it is not easy to contrast them, also because many people think that it's impossible to popularise Mathematics (see ICMI Study 1989). On the basis of these considerations, we have produced a video which is an attempt to explain in a simple way methods and aspects of Mathematics. It is addressed to all people and it doesn't include the latest research result; but by *using the game of Tangram as a metaphor*, it aims at going against the main commonplaces about Mathematics and at explaining aspects that have always characterised its development. With these intentions we have developed and extended the basic idea of a didactical path experimented at middle school level. The video, called: *"What are we playing: Tangram or Math?"*, has been realised with the co-

operation of the Trieste University Television Service. It was presented during many lectures and meetings, and shown experimentally in various schools. It is now available (by streaming) at the website: www.univ.trieste.it/~nirtv/tanweb, where you can find its text and its references.

MATHEMATICAL PACKAGES AS A TOOL OF A CONSTRUCTIVE APPROACH IN MATHEMATICAL EDUCATION

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Keywords: *Mathematics teaching; computer technologies; teaching materials*

The possibilities for supporting a constructive approach in plane geometry course in the environment of the original dynamic geometry package DG (the original package designed in KSPU) at the high school level are discussed.

The problem-posing lesson is discussed as a form of implementing a constructive approach in educational practice (on example of a problem field described in Japan materials on the ICTM-9). Advantages of using DG in this context are discussed as well as common methodological questions about using DG in solving different kinds of geometric problems.

The conclusion about the most perspective kinds of mathematical packages for IT support of Constructive Approach in Math Education is proposed: the pedagogical adaptations of the most powerful professional mathematical packages: CAS and DGS (Computer Algebra Systems and Dynamic Geometry Systems).

The crucial point for implementing Constructive Approach in real pedagogical practice is designing the high quality didactic materials (the top level of which are distance courses integrated with appropriate packages). One of the main tasks of ERME is to arrange the European Communications, Collaborations and Cooperation in this field. The KSPU is opened for this.

WHAT KIND OF OBSTACLES MAY BE EXPECTED IN THE SIMULTANEOUS LEARNING OF MATHEMATICS AND COMPUTER SOFTWARE?

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Abstract: Official instructions from the Ministry of Agriculture advise the simultaneous teaching of certain mathematical concepts, such as sequences or statistical distributions, and of the EXCEL spreadsheet software.

With reference to the works of P. Rabardel and C. Laborde, the question arises as to the possible effect of the ‘artefact’ on the learning of these mathematical concepts.

Résumé: Les instructions officielles de l’enseignement agricole préconisent l’enseignement simultané de certains concepts mathématiques, tels que les suites ou les distributions d’échantillonnage, et de l’utilisation d’un tableur.

En se référant aux travaux de P. Rabardel et de C. Laborde, on peut s’interroger sur l’influence de l’artefact tableur dans l’apprentissage de ces concepts mathématiques.

USING COOPERATIVE LEARNING TO TEACH PRIMARY MATHEMATICS TO AD/HD CHILDREN IN A COMPUTER-BASED ENVIRONMENT

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Abstract: This research project took place at a Primary school in southern England. The Project set as a target the teaching of some mathematical concepts (addition of two two-digit numbers and that of fractions) to children with attention deficit and hyperactivity disorders AD/HD. The whole project was based on multisensory approach in teaching mathematics in which a particular multimedia package was used, amongst other tools, in order to reinforce and consolidate current knowledge. An integral part of the project was to investigate what collaborative aspects were being developed amongst the teacher, the children and the computer program, and whether this way of working enabled them to better understand the concepts being introduced. The project found that, with careful planning, implementation, and evaluation, cooperative learning activities can be achieved successfully by most students.

WORKING GROUP 3

**Theory and practice of teaching from pre-service
to in-service teacher education**

Group leaders:

Fulvia Furinghetti

Barbro Grevholm

Konrad Krainer

INTRODUCTION TO WG3

TEACHER EDUCATION BETWEEN THEORETICAL ISSUES AND PRACTICAL REALIZATION

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The title of the WG3 was “Theory and Practice of Teaching from Pre-service to In-service Teacher Education”. We received 18 papers, which were reviewed by at least three reviewers. All the authors took part in the process of review. Giuliana Dettori (CNR, Genoa, Italy) and Salvador Llinares (University of Sevilla, Spain) also acted as reviewers. Three papers were considered not to fit the themes of WG3, three were rejected, two participants were not able to attend CERME 2. After the conference the authors submitted a revised version of their papers, six full papers and two summaries were accepted. All addressed themes consistent with the issues which emerged in the Working Group.

Altogether there were 24 participants to WG3 plus 4 people, who joined our groups of special interest. In the conference there were nine presentations, which acted as a starting point for developing the discussion. The list of the questions raised reported below outlines the themes touched during the conference. Participants to the WG3 came from 15 countries. This meant that different approaches and theoretical considerations were presented. This heterogeneity is also caused by the fact that teacher education is a subject strongly influenced by social context, state policy, and so on. More than in other fields, the researcher in the field of “teacher education” subject has to balance what is suggested by the theoretical considerations and what it is possible to realise in practice. The discussion reflected this position and the themes touched fluctuated between these two poles.

Questions raised in WG3 (report by F. Furinghetti and B. Grevholm)

A lot of different and complex issues came up but no issue stood out as of primary importance. Here are some questions that focused our discussion:

- Relations between pre-service and in-service training. Is the difference going to disappear when we get life-long learning?
- The role of mathematical knowledge? How can we keep a good balance between subject knowledge and pedagogical knowledge?
- How do issues with regard to primary and secondary level differ? What do they have in common?
- Would the discussions in our group have been different if we were educators only and not researchers?
- What are the relationships with institutions, and the political influences on our behalf?
- How precisely should we define (in our papers etc.) central concepts like reflection, improvement, changes, development?
- What are the effect on teachers of stimuli like writing, reading, and technology?
- How to promote teachers' awareness, teacher students' awareness?
- What is the role of discussion, conversation, communication in promoting professional growth?
- Is the role of technology less strong than in earlier phases?

LONG LASTING ISSUES

- What is important for future teacher training?
- How to organize the crucial role of recruitment?
- How to foster the development in student teachers?
- What is the relationship between drastic changes in society and consequences for new models for teacher education?

SPECIAL INTEREST GROUPS

1. The future of WG3 (Report by K. Krainer)

Mission statement

- To investigate the professional growth of pre-service teachers, qualified teachers and teacher educators.

Philosophical goals

- To investigate the relationship between theory and practice.
- To investigate teacher development in the classroom.
- To investigate the connection between pre-service and in-service training.
- To investigate the development of teachers' subject knowledge.

Strategic goals

- To elect a group board.
- To engage in collaborative research.

- To expand the group.
- To exchange email addresses.

2. Classroom Reality and Teachers' training (Report by H. Sakonidis)

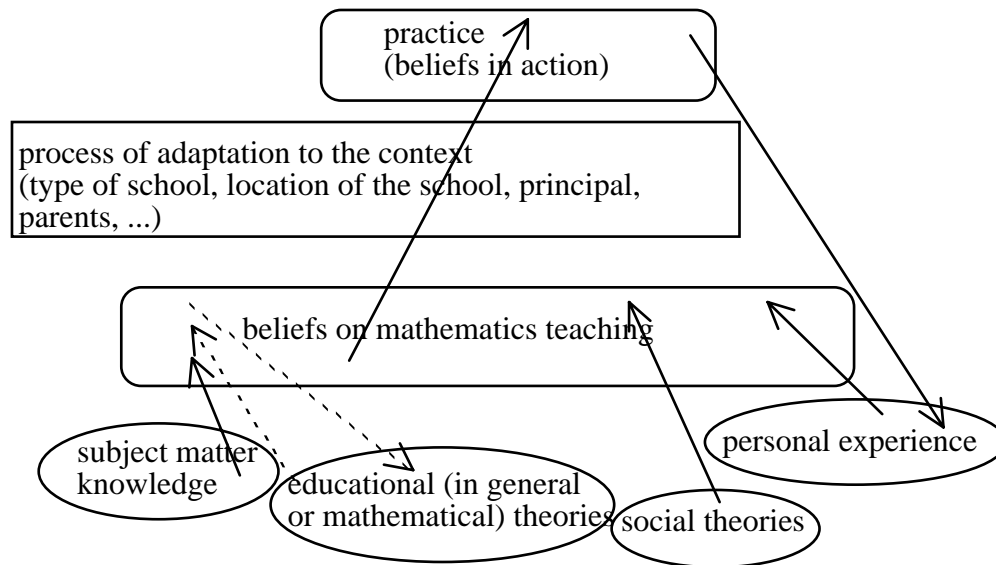
During the two sessions, a number of points were raised, discussed and analysed in relation to the topic of the group. Among them, the following were considered as most important:

- Mathematics teaching practices influence teachers' training and vice versa. This is because teachers' training programs all over the world draw upon these practices in order to help student-teachers acquire a better understanding of the mathematics classroom reality. This, in turn, shapes the teaching strategies these students are likely to adopt in their future school career as well as the relevant research.
- Learning how to *observe* a mathematics classroom at work is of particular importance for a future teacher. This is because the teachers to be should become conscious of the situations and actions needing to be taken by a teacher as well as of their effects on the pupils' understanding of and attitude towards the subject matter.
- In order for the relevant observational skills to be developed, a number of means and tools were suggested: videos, transcripts of lessons, schemes of analysis / reading of the classroom events, etc.
- In a number of countries, a person based at school is often appointed by the organisers of the teachers' training course and properly trained (usually a well-trained and/or experienced teacher). This person acts as a *teaching supervisor* and her/his responsibility is to attend to the teaching of a group of student teachers, discuss with them adjustments and working plans, etc. S/he also constitutes the link between the schools and the University/Institution where the training is taking place. This was considered as a very useful and functional idea.

Finally, the members of the group agreed that in the future, a cross-analysis of some observational data (videotaped lessons) could provide very useful insights into the teaching practices in different social, economic and cultural contexts. The results of such an analysis would allow for a better understanding of the mathematics classroom teaching practices and could be used in the teachers' training to show the diversity and richness of these practices.

3. Relation between theory and practice (Report by A. Somaglia)

The discussion was developed around the schematic frame provided by the figure below. The aim was to study how theories may affect the practical knowledge of teachers and how teachers themselves perceive that their practice fits with theoretical issues of mathematics education.



DEVELOPING AND RESEARCHING PROFESSIONAL KNOWLEDGE WITH PRIMARY TEACHERS

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Abstract: *This paper presents a formative-investigative process within the context of in-service education of primary teachers. We show how the professional development of three teachers and the gathering of the research data on the part of the teacher educators-researchers are complementary. The degree of co-operation between the group members is also analysed. The study takes as its starting point classroom practice as a source of problems, and emphasises the role of reflection as a key component of the professional development process. The analysis of the professional knowledge of the teachers provides several suggestions regarding desired professional knowledge and models of pre-service teacher education.*

1. Introduction

A university mathematics teacher educator finds it difficult to meet the demands of the two main areas of concern: the education of future teachers and research. In this article, we assert the importance of university teaching in this area and the need for research to locate its object of study and to draw its conclusions in the area demarcated by the relation between teaching and research. This relation should not be confused with the relation between theory and practice: both teaching and research have their theory and their practice. Hence, the frequent associations, theory-research and practice-teaching, are imprecise, although not devoid of meaning. We hope that the following sections make it clear that our study goes beyond such associations.

One of our major areas of activity is teacher education. Indeed, the research presented here was carried out within a project called *Professional Development Through Collaborative Research on Problem-Solving*, funded by the Ministry of Education of Andalucía (Spain). This study involved three experienced primary teachers (V, P, J) and the two authors of this paper (who work in the field of mathematics education and are in charge of primary and secondary mathematics teacher education).

The present paper concerns the practice of research into inservice education. On the part of the researchers, this practice is backed up by theory, to which we shall refer later. On the part of the primary teachers, their professional practice is the point of departure for reflection upon their professional

knowledge. Their aim was to articulate their opinions on primary teaching, whilst all involved sought for desirable characteristics of the professional knowledge of the mathematics teacher at primary level¹. This also allows us to draw conclusions with respect to teacher education models. In this way, what started as research into inservice education, became linked to the teaching on preservice teacher education courses, through a network of theories and practices.

2. Theoretical framework

As already mentioned, one of our major areas of activity is teacher education. We understand it as “an interaction process (embedded in a social, organizational, cultural... context), mainly between teacher educators and (student) teachers, but also including systematic interactions among teachers aiming at professional growth. At the same time, we can see teacher education as a learning environment for all people involved in this interaction process.” (Krainer & Goffree, 1999, p. 295)

Another major area of concern is professional knowledge, taxonomies of which can aim either for exhaustive detail, following Bromme (1994), or for concision, following Carrillo, Coriat & Oliveira (1999). Independently of the chosen taxonomy, one can organise the components of professional knowledge into three groups. The first one concerns general psychological and pedagogical knowledge (some authors add sociological knowledge to it); in the second group we consider those components which are linked to subject matter knowledge; last, the third group includes the components that are related to subject teaching and learning knowledge². Matched to each group we consider teachers’ related beliefs as part of their professional knowledge.

To the aim of amplifying scientific knowledge can be added that of stimulating the professional development of the participating teachers. This professional development should not be interpreted as just a modern version of ‘professional change’. Whilst this latter encompasses an attitude of superiority on the part of the researcher, the former makes it clear that the most important thing is to make available opportunities for reflection on one’s own knowledge

¹ In Spain there are not specialists in mathematics at primary level. When writing “mathematics primary teachers” we mean that we are focusing on primary teachers when dealing with mathematics teaching. For these teachers mathematics is only one of several subjects (language, social and natural environmental knowledge, arts) they teach.

² The frontiers between the second and the third groups are fuzzy. Some research lines consider these groups as a joint one. In particular, didactic of mathematics is sometimes understood as including the mathematical content, which we include in the second group.

and beliefs, this being a possible point of departure for a change chosen by the teacher.

The aim of this paper is to offer a model of professional development as well as a means of researching certain aspects of the teachers' professional knowledge. We outline the role of reflection in the process of construction of professional knowledge (professional development) following Cooney & Krainer (1996), who emphasise "a reflective component of inservice programs in which teachers explicitly consider the implications of their own learning experiences for their teaching" (p. 1162). We endeavour to supply data which haul reflection out of the ambiguity in which it typically finds itself. Likewise, we seek to specify the concrete nature acquired during the project of the co-operative research initially proposed. Finally, we make suggestions for primary preservice teacher education.

Our research covers various aspects of this professional knowledge of the teachers. However, faced with the impossibility of presenting the complete study, we have decided to focus our attention on the educating/researching process which has been carried out, more than on its results. We believe that it is possible to extract from this process models of preservice and inservice education of primary mathematics teachers, and that, furthermore, the richness of the results obtained is due largely to the richness of the process.

3. The process

Goals

The project arose from the interest demonstrated by the teachers to undertake a process of research on their classroom practice with the aim of modifying it. They wanted to create an environment of professional learning, a sub-community of practice, in which to develop the four dimensions that, according to Krainer (1998), characterize professional practice: action, reflection, autonomy, and networking. In fact, as it usually happens, they tried to develop, above all, reflection, and also networking. They wanted to enlarge the environment in which they build their professional knowledge: the interdependence between the two autonomous systems that model the teaching and learning of mathematics (the students' learning processes and the interactive teaching process (Steinbring, 1998)). For their part, the teacher educators/researchers were interested in studying the professional knowledge of these teachers.

In the preliminary design, to which all the participants contributed, it was agreed to focus the study on the teachers' beliefs about the teaching and learning

of the subject and on their classroom performance. As far as for both the teachers and the researchers are concerned, the aim was to (i) contribute to the theoretical and practical development of a proposal for the professional knowledge desirable for a primary school mathematics teacher and the role of problem-solving in this knowledge, (ii) initiate a process of professional development in these teachers, and (iii) obtain outcomes to facilitate the professional development of teachers who are involved in processes of inservice training. Likewise, on the part of the researchers, the aim was to obtain ideas which would contribute to the improvement of preservice teacher education courses.

Preliminary steps

Following this design, in the first few months of the project, several group discussions were held on different aspects of the teaching and learning of mathematics at primary level, and the role of problem-solving in the aforementioned process. Included in these sessions were discussions of articles and research papers relevant to the field. This initial work allowed, in the first instance, the exploration and articulation of each teacher's beliefs to get under way, and in the second, the creation of a common language within the group, and the familiarization with what would be the tool of analysis of the beliefs: the tool of the analysis of the beliefs of the teacher with respect to the teaching and learning of mathematics (Carrillo, 1999)³.

Once the project was underway, we set about gathering data on the teachers' performance. To this end, a video recording of one class by each of them was made, and the participants also kept a diary of their classes (always in relation to the teaching of mathematics). The sessions were recorded by one of the teacher educators, whose presence in the class was that of non-participating observer. The camera remained throughout in a fixed position, such that the performance of the teacher and their interaction with the children could be recorded. The focus of attention, in accordance with the aim being pursued, was the teacher herself. With each teacher the following procedure was carried out: 1) once a recording had been made, this was analysed jointly in a group session; this analysis was conducted on the basis of a previous analysis of the recording by the trainers, also considering the aspects which the group members were to highlight during the group viewing; 2) the diary was analysed (first individually,

³ This tool of analysis was originally designed for secondary teachers, and some of its descriptors were now adapted for the primary level. A summary of this instrument is organised as a table, which is divided into 4 tendencies: traditional, technological, spontaneous, and investigative. Each tendency is described in terms of six categories (methodology, subject significance, learning conception, student's role, teacher's role, and assessment), with their correspondent descriptors. This instrument is similar to Contreras, Climent, & Carrillo's (1999) for the analysis of the role that teachers give to problem-solving in the classroom.

then in the group); 3) finally, contrasting the analyses and taking into account what the teacher had explicitly stated in the sessions that took place before the project begun, a profile of this teacher's beliefs with respect to the teaching and learning of mathematics was drawn up (again, individually in the first instance and then in joint session).

It is difficult in this process to separate the points of data collection from their analysis. During data analysis we were also collecting data. For example, in analysing any one of the videos each teacher reflected her personal beliefs, such that her corresponding declarations (noted down) formed part of the data collected about her verbalised beliefs. This is the purpose of the previous analysis of the recordings by the teacher educators: to use the viewing of these and their analysis in order to collect data.

Example: Watching V's video, V, P and J's analyses provide information about V's performance, as well as about the beliefs of all of them, which are new data.

V: One must give a pattern to the children.

P: But you do not have to give it before they think.

J: One must reach a balance with respect to when setting a more open activity is important and when it is better that the teacher says everything.

With respect to the drawing up of the profiles of the teachers we would highlight two aspects. The first of these reiterates, as in previous research (Contreras, Climent, & Carrillo, 1999), that the individual analysis by each researcher, followed by joint discussion, appears to us a mode of analysis which reduces subjectivity of the process and confers greater rigour, resulting in more reliable data. On the other hand, the fact that the very teacher who is subject of the analysis is at the same time one of the members of the research group turned out to be an extremely important plus in the analysis. The degree of reflection and the interest of the teacher in the results led, on several occasions, her to be placed, in the joint analysis, in "less desirable" positions than those in which she had been categorised by the other members.

Example: Obtaining V's profile.

P: I think that V applies a structured, but not closed, program.

V: I think the program must be closed. For me the most important thing is to close it...In the session which was videotaped I planned a particular goal and everything was driven to get it.

Data Collection	Data Processing	Results concerning ...
Recording of classroom lessons and lesson diaries of	Discussion of recordings (group), analysis of diaries	MTLB (mathematics teaching and learning

teachers	(group) and discussion of profile of each teacher with respect to MTLB	beliefs), SMB (school mathematics beliefs)
Field notes from the group discussions about maths and its teaching and learning, and about the classroom recordings	Serve as evidence in the discussion of the profile of each teacher	MTLB, SMB, PCK (pedagogical content knowledge), Knowledge <i>of</i> mathematics, Knowledge <i>about</i> mathematics

Table 1: A study of the teachers' beliefs

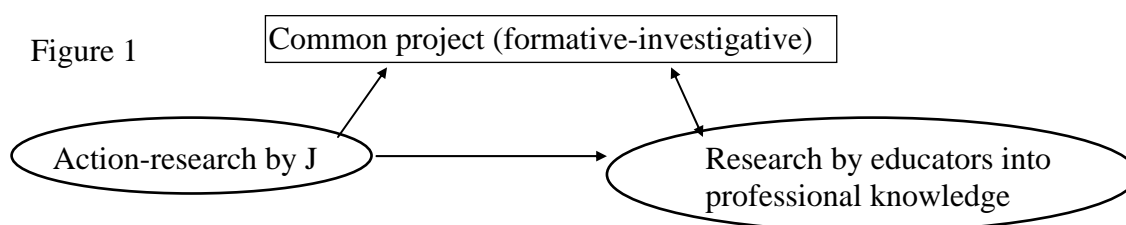
Deepening in the reflective feature

The events explained above coincide with what had been previously planned, but during the realisation of the work various aspects arose which required the initial design to be modified and the focus of interest amplified. One of these aspects was the unequal involvement of the teachers in the project, and the varying degrees of conviction that their professional development might come as the result of reflection in and on their classroom practice. The amount by which they distanced themselves from such presuppositions was in direct proportion to the amount by which they aligned their beliefs about teaching and learning along a traditional axis⁴. As a result, the role of J (with investigative beliefs) stood out; in this case the analysis and discussion of her beliefs and practice showed that her professional development should come from converting her *espoused beliefs* into *in action beliefs*⁵. Thus, this teacher began to be submerged in a process of action research on her classroom practice, in which she herself collected data on her performance, reflected on them, and drew her own conclusions, which caused her to modify her previous practice. It struck the researchers as interesting to amplify the observation of the performance of this teacher from a single class (as it was originally done with the three teachers) to a teaching sequence, with the aim of studying the process of professional development and the professional knowledge of J “in action” (during a longer time). J made very clear the advantages of the teacher herself taking on the role of directing the research according to her interests (Jaworski, 1998). In principle it was a case of two projects, with distinct interests and methods, parallel to the common project of the group, in which the interested parties were the teacher and the teacher educators. But the discussion which ensued when the questions that this

⁴ In the terms defined by Schön (1983), this project is based on the teachers' reflection on action. In this way, their reflection in action has been promoted; at the same time, we have approached some of the limitations of the reflection in action pointed out by Schön. It is the case of the teachers' didactical tendencies.

⁵ According to Ernest (1989), we consider relevant to differentiate the model a teacher shows in action (in action model, beliefs...) from the one which is inferred from their declarations (espoused model, beliefs...).

recording had raised were put to the group appeared to us a good means of collecting more information about the professional knowledge of the three teachers (going deeper into aspects which had hardly been scratched up to that point, such as the pedagogical content knowledge and the content knowledge, aspects which in the analysis of J had stood out as the “weakest” of her professional knowledge⁶), and a means of furthering their professional development. In this way, it became a question of interest for all the members of the project; what was intended to be the study by the researchers of the process of professional development of J, outside the work of the research project, and J’s own process of action research, become inter-linked, separate but mutually beneficial (like a kind of symbiosis) (figure 1).



The notes taken by the researcher making the recordings⁷, along with the subsequent analysis of the videos by the two researchers, permitted them to formulate questions concerned with mathematics and pedagogical content knowledge, which had been afterwards put to the three teachers in the project’s working sessions. These questions derived directly from situations arising in the recorded classes, or from problems arising in these, or from the responses of the teachers to the questions. The questions were put to the teachers by the researchers, generating different opinions, which were discussed amongst

⁶ In (subject) content knowledge we include knowledge *of* and *about* mathematics (Ball, 1990). On the other hand, we understand pedagogical content knowledge (Shulman, 1986) as closer to subject content knowledge than to a specification into mathematics of general pedagogical knowledge (Marks, 1991, uses the name *specification* for the latter, and *interpretation* for the first). Nevertheless, in the case of primary teachers their understanding of the mathematical content can not be taken for granted, at least in some topics, in order to build their pedagogical content knowledge on it. For this reason, being aware of the weakness of their mathematical knowledge and convinced of the necessity of starting from their demands, in our analysis of their pedagogical content knowledge we focus on the knowledge of specific aspects of the teaching and learning of some mathematical topics. In particular, topics such as pupils’ misconceptions, learning difficulties, use of materials, etc...can not be derived from a *specification* of general pedagogical knowledge, but at the same time they do not correspond to a transformation of mathematical content into forms that can be used in teaching (which is the usual understanding of Shulman’s pedagogical content knowledge).

⁷ All subsequent recordings were made under the same conditions as the previous ones (given that the aim remained the same).

themselves. The role of the researchers in this case was to guide the discussion and to provide questions to ensure that the distinct aspects of their professional knowledge related to the situation came into play, as well as to note the responses of the teachers. After intervening, the researchers went on to participate in the discussion, indicating the aspects of professional knowledge that they had made clear and debating with them their deficiencies.

Example: Situation coming from J's practice.

The pupils are asked to divide circles in halves. They draw the following *different* possibilities:



Pupils: One can draw many, many lines as the above, just rotating the line a little.

A pupil: If the circle is bigger, one can draw more lines than if it is smaller. If we go on rotating the lines mm by mm, in the biggest circle there will be more mm.

J decides not to deal with this conjecture, “because this is a different issue”.

In one session of the project one asks the teachers to deal with this conjecture.

J: I support that pupil's conjecture.

P: If you open the circle you will find an infinite number of points for the diameters.

J: I understand your point, and in this case the amount of halves would not depend on the size, but I do not see it intuitively.

This stage had two facets: educational and investigative. For the trainers and J it was investigative, for all the teachers it was educational. Each of the teachers was to be profiled according to our plan, both with respect to their beliefs and their subject matter knowledge and pedagogical content knowledge. At the same time, in the same way that the group had discussed the teachers' beliefs, which were now being reconsidered, we aimed at raising their awareness of their subject matter knowledge and pedagogical content knowledge, and to stimulate their development (via formative processes). The discussion of the questions raised accomplished this goal, as it was through the discussion process, along with the results of the analysis by the researchers of the group sessions data, which allowed the profiles to be drawn with respect to subject matter knowledge and pedagogical content knowledge (this, in conjunction with the process of action-research by J, comprised the investigative facet of this stage of the process).

Data Collection	Data Processing	Results concerning...
Video recording and observation by one of the researchers of a series of lessons by J	Analysis by researchers	(Knowledge <i>of</i> and <i>about</i> mathematics and PCK) of J
Discussion of classroom situations arising from observation of J	Observation and analysis by researchers	Knowledge <i>of</i> and <i>about</i> mathematics ⁸ and PCK
Mathematics problem solving	Observation and analysis by researchers	Knowledge <i>of</i> and <i>about</i> mathematics

Table IIa: Investigative process (with respect to subject matter knowledge and pedagogical content knowledge)

1st Phase	2nd Phase	3rd Phase	Results
Classroom situations and mathematics problems raised by researchers	Discussion by teachers	Discussion with the trainers	Consciousness-raising and restructuring of Knowledge <i>of</i> and <i>about</i> mathematics and PCK

Table IIb: Formative process (with respect to subject matter knowledge and pedagogical content knowledge)

Summarising

The resettling of the original project design not only meant the study was broadened, but that the roles of the participants, and even the kind of research, were also altered. The project was originally conceived as collaborative research (Feldman, 1993), in which all the participants would be researchers, and some (the teachers) would also be informants. This distribution was respected so far as the beliefs of the teachers was concerned. With respect to the study of the teachers' pedagogical content knowledge, and knowledge *of* and *about* mathematics, the teacher educators acted as researchers, and the teachers only acted as informants, although they participated in the discussions about the results of the analysis by the teacher educators.

In the case of the beliefs, the discussion of papers relating to the teaching and learning of mathematics and of research into teachers' beliefs allowed them to receive training as researchers. However, in the case of the study of their subject matter knowledge and pedagogical content knowledge, circumstances did not permit for an appropriate medium-term training programme from which the necessary data could be derived.

⁸ Unless otherwise specified, reference is to all three teachers.

4. Conclusions

The process described here become a way of initiating professional development for the participating teachers which we are sure will be continued in an autonomous fashion (although not isolated, as a result of the improvement of networking). We propose that the process be adopted as a model for inservice teacher education. Group work is also of particular value, whereby small numbers of teachers and researchers/teacher educators share a process of research-professional development⁹. The process should favour critical thinking on the part of the teachers, encouraging them to reflect on the subject matter, its teaching and learning, its characteristics and professional requirements and, above all, their own awareness of and decisions about the process of professional development. A process, in short, which takes the teacher's own practice as the source of challenge, and which enhances the role of self-reflection.

With respect to preservice teacher education, the cases which arise in the formative-investigative process comprise, on the one hand, a good point of departure for the elaboration of materials for use in the classroom, and on the other, a source of information about the teachers' knowledge, both the knowledge that is really brought into play, and the knowledge that is considered desirable for the successful management of the primary mathematics teaching-learning equation. As we mentioned at the beginning of this paper, it is difficult to fully attend to both teacher education and research; this kind of research opens the way for university teaching (preservice teacher education) to nourish itself from the process and from the results of the educational research in the context of inservice education.

Among the results that were obtained concerning the professional knowledge of the project's teachers, we found that for them the only mathematics which made sense was school mathematics, and their only point of reference was everyday mathematics, and that of the textbooks; there was no knowledge *about* mathematics (about the value of examples and counterexamples, systems of proof, the accuracy of a definition, and so on); teachers' knowledge of mathematics was restricted to the primary curriculum and their depth of knowledge was not greatly deeper than that acquired by some of their students on finishing at this level. They were particularly lacking in the area of geometry, and their knowledge of arithmetic (where primary teaching has traditionally focused its efforts) was mechanical, based on the mastery of the algorithms, and showing difficulty in the manipulation of non-integers real numbers. They lacked strategies for solving mathematical problems, and tended to avoid them rather than face up to them. The three teachers showed different

⁹ This model includes the case that educators have only educating goals, and no research ones.

degrees of pedagogical content knowledge. J displayed greater knowledge, but as this was the result of her own experience and self-development, it was limited and somewhat superficial when broaching the kind of questions related to these aspects.

Finally, we would like to close this paper by reaffirming our conviction, as teacher educators, of the need to take these results on board in the design of our teacher education programmes, both initial and inservice.

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INVESTIGATION INTO PRACTICE AS A POWERFUL MEANS OF PROMOTING (STUDENT) TEACHERS' PROFESSIONAL GROWTH

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Abstract: *More and more initiatives in teacher education in Europe establish learning environments where (student) teachers are supported in planning, carrying out and analysing investigations into practice. At a second level, researchers investigate the professional growth of their (student) teachers. The research reports usually contain a variety of arguments for the success of the initiative, putting an emphasis on the importance of investigations. This paper takes three teacher education initiatives as an example, looks at the reasons for the success and tries to find out supportive factors that seem to be crucial for (student) teachers' professional growth. The three initiatives are analysed through the lens of a four dimension-model of teachers' professional practice. The analysis shows that the freedom of defining one's own research questions, professional support by teacher educators (e.g. through clearer goal definition or bringing in theoretical considerations) and rich opportunities to reflect on and to share experiences with others are decisive factors promoting (student) teachers' growth. The paper concludes with a list of success indicators that might be taken into account when planning, carrying out and evaluating teacher education initiatives that focus on (student) teachers' investigations into practice.*

1. Background

The recent development of European mathematics teacher education shows a trend towards closer interconnections between teacher education as a field of practice and as a field of research (see e.g. Krainer, Goffree & Berger 1999, Krainer 2000). Above all, more and more initiatives in teacher education establish learning environments where (student) teachers are supported in planning, carrying out and analysing investigations into practice. At a second level, researchers investigate the professional growth of (student) teachers. The research reports usually contain a variety of (data-enriched) arguments for the success of the initiative, putting an emphasis on the importance of (student) teachers' investigations and the development of an investigative attitude. In the following the term "investigation" is used with a broader meaning, ranging from

the first systematic reflections by student teachers to more elaborate research by experts. However, in all cases the goal of the activities is a better understanding of practice.

This paper aims at looking more deeply at the reasons for the success of teacher education initiatives and at finding out supportive factors that seem to be crucial for (student) teacher's professional growth. Based on that, the considerations will lead to a list of success indicators that might be taken into account when planning, carrying out and evaluating teacher education initiatives that focus on (student) teachers' investigations into practice.

As an example, two pre-service initiatives and one in-service education initiative are briefly described and discussed. The analysis of these three cases is based on the papers describing the initiatives and the research results. In all three cases at least one author also had the role of the teacher educator, therefore the papers have to a large extent aspects of self-evaluation of their teacher education initiative, reflecting explicitly or implicitly on the success of the initiative with regard to the professional growth of the participants.

All three initiatives are regarded through the lens of the four dimension-model of teachers' professional practice (see Krainer 1998). This means that they are analysed how they relate to the following four dimensions:

Action: The attitude towards, and competence in, experimental, constructive and goal-directed work;

Reflection: The attitude towards, and competence in, (self-)critical and one's own actions systematically reflecting work;

Autonomy: The attitude towards, and competence in, self-initiated, self-organised and self-determined work;

Networking: The attitude towards, and competence in, communicative and co-operative work with increasingly public relevance.

In the following, a brief description of the interaction between these four dimensions of teachers' professional practice is given. The situation of teachers at schools is mostly dominated by *action and autonomy*, there is a lack of *reflection and networking* in the sense of a critical dialogue about one's teaching with colleagues, mathematics educators, etc. In many cases, their work in the classrooms shows the same pattern: the students neither get enough time to reflect on what they are learning nor share their experiences with their co-learners, thus joint construction of meaning is not adequately promoted. One of the most important reasons for this pattern (action predominates reflection, autonomy predominates networking) is that traditional (pre- and in-service) teacher education itself causes and contributes to that situation: joint reflections by (student) teachers' are not seen as important features of the learning process.

Therefore, it seems to be crucial to promote more reflection and networking in teacher education. Experience shows that a further development with regard to these two dimensions also has a positive impact on the other two dimensions. For example: an increased competence in reflection raises the quality of action, and the knowledge of views of others enlarges the view of one's own situation. Summing up, more reflection and networking contribute to a higher quality of autonomous action. This in turn improves the quality of reflection and the construction of shared meaning. This means that teacher education has to look carefully at the interaction between these four dimensions.

In the following, the three initiatives are described. Then – taking the four dimensions as criteria – the analysis is focused on working out factors that lead to the initiatives' success in fostering (student) teachers' growth.

2. Description of the three teacher education initiatives

The three initiatives stem from two different countries in Europe, namely from Portugal (APOA) and The Netherlands (MILE), and from Israel (Tomorrow 98) which has close connections to European research in mathematics education. The APOA-participants are (secondary) student teachers who have had no teaching practice before, the ones from The Netherlands are advanced (primary) student teachers with some teaching experience, and in the case of Tomorrow 98 the participants are practising (secondary) teachers. However, also the further development of teacher educators is explicitly seen as a goal of this project.

	<i>Kind of teacher education / Primary or Secondary</i>	<i>Number of (student) teachers / teacher educators</i>	<i>Intended main activity of (student) teachers</i>	<i>Main research focus of teacher educators / researchers</i>
<i>APOA: Portugal</i>	Pre-service / Secondary	12-15 / 1 (+1 researcher)	Observing and analysing teaching	Stud. teachers' formation of profess. identity
<i>MILE: The Netherlands</i>	Pre-service / Primary	2 / 1 (+1 researcher)	Analysing video fragments of teaching	Stud. teachers' knowledge construction
<i>Tomorrow 98: Israel</i>	In-service / Secondary	120 / 20 (+1 researcher)	Implementing innovative teaching	Profess. growth of teachers and teach. educators

APOA: Student teachers' investigations as an entrance door to a discourse community concerned with professional practice (Portugal)

APOA (standing for Pedagogical Activities of Observation and Analysis) is a regular course for secondary mathematics student teachers in their fourth year of study at a university in Portugal (the fifth year of their study is dedicated to an internship at a school). It aims at (further) developing student teachers' investigative attitude through using and reflecting on some skills of observation and analysis. The outcome of this course is investigated by a researcher and the teacher educator of this course (Ponte & Brunheira 2000).

The course is based on three assumptions: the observation of practical situations is important to provide student teachers with the opportunity to reflect, question, and theorise about school and mathematics teaching and learning; the observation needs planned and constant inquiry by student teachers in an environment that encourages free expression of opinions; the identification of specific questions for observation and reflection, the choice of appropriate methods, the collection of relevant data, and the presentation of conjectures and conclusions constitute an investigative activity for student teachers.

APOA classes are run by a mathematics educator and have at most 12 to 15 student teachers in order to allow a close teacher-student relationship.

The research shows that the student teachers regard field work and its analysis as valuable activities and highlight the importance of observation and reflection. The course also contributed to the formation of their professional identity. The student teachers stated that the activities had broadened their views on teaching practice. Sharing and comparing different points of view led partially to a change of their own views. The authors conclude that the discussions in the group in particular fostered the growth of a “discourse community” of student teachers concerned with professional practice.

The project leaders argue that almost all important theoretical issues appear in practice, and it is immediately possible to indicate relevant theories. Even in later more theoretical seminars it would be possible to refer to practical experiences within the project. The internal collaboration among teacher educators as well as the collaboration of the teacher education college with the practice school is promoted.

MILE: A multi-media environment as a research data base for student teachers' learning about teaching (The Netherlands)

The MILE (Multimedial Interactive Learning Environment) project in The Netherlands (see e.g. Goffree & Oonk 1999a, b) aims at developing a multi-media environment which can be used in mathematics teacher education in order to promote student teachers' learning about teaching. This approach is based on the assumption that "practical knowledge" and "knowing about practice" cannot only be acquired in real life situations, but also – and maybe even better – through the investigation of digital representations of real life situations in classrooms. In this case the student teachers are not restricted to one teacher as a supervisor and a few classes, and thus can see a broader spectrum of teaching and they are freer in their observations and analyses.

The MILE environment so far consists of lessons, split narratives in the form of video fragments (from mathematics teaching grade K1 to 6) with an average length of 1 to 2 minutes, including transcripts. The fragments and transcripts can be accessed with a search engine.

Research was conducted at two levels: at the level of (pairs of) student teachers' investigation into classroom practice via MILE-narratives, and at a meta-level in terms of a university team's systematic research on student teachers' knowledge construction in a computer supported collaborative environment. The student teachers were supported by a teacher educator having the roles of a coach and an expert, for example stimulating them to view a situation from a different perspective or providing theoretical considerations.

Dieneke and Hayet, two student teachers in the MILE project, highlight in their reflections on their activities that the investigations broadened their view of students' learning, led them to new questions and, also to the insight that teachers must be able to deal with the subject matter with (more) flexibility.

The research on student teachers' investigation processes led to the observation of four levels of student teachers' knowledge construction: "assimilation" (student teachers "copy" teachers' actions and beliefs from the video), "accommodation" (student teachers "modify" teachers' actions and beliefs and adjust them to their own purpose), "seeing new links" (student teachers establish (new) links between events on a video and a theoretical discussion or their own training experiences), and "theorising" (student teachers design their own "local theories"). The investigation process of student teachers in MILE manifests itself as a cyclical process of planning, searching, observing, reflecting and evaluating.

Tomorrow 98: Interweaving the professional development of teachers and teacher educators (Israel)

The Tomorrow 98 project in Israel is a five-year professional development programme for about 120 high school mathematics teachers (Zaslavsky & Leikin 1999). It aims at a variety of goals, among others, at preparing teachers for innovative and reform oriented approaches to mathematics teaching and at changing from a transmission metaphor to a constructivist perspective to teaching. The teachers are supported by about 20 teacher educators who themselves are seen as learners (in their role as teacher educators) and are supported by the project director.

The teacher educators provide learning experiences for the teachers, which they apply in workshops for teachers or in their classroom. The experiences gained in the workshops and the classrooms are reflected in joint meetings. In addition, the teacher educators investigate their own practice within the project, write down their experiences and discuss their findings with their colleagues in several meetings.

The research conducted within the framework of the project investigated the professional growth of teachers and teacher educators. The methodology followed the paradigm of grounded theory, and refers to several theoretical considerations by other researchers.

The results indicate that both groups, the teachers as well as the teacher educators, show progress in their professional growth. However, Zaslavsky & Leikin (1999, 155) highlight differences in the extent and nature of their growth. Whereas the progress of the teacher educators is stressed through expressions like “*become more competent with respect to innovation*”, the progress of the teachers is described far less spectacularly, like “*become more aware of the potential of innovative approaches*”. The researchers also point out that the teachers felt less committed and accountable for their role in the project.

3. Looking at the three initiatives through the lens of the four-dimension-model - supportive factors for (student) teachers’ professional growth

In the following, the three teacher education initiatives are briefly analysed concerning the dimensions action, reflection, autonomy and networking. For each dimension those factors have been selected that appeared to be most supportive for (student) teachers’ professional growth.

APOA:

Action: The authors put much emphasis on carefully planning and guiding the student teachers' observations, indicating that these "*must be led by a problem and be carried out with some instruments*" (Ponte & Brunheira 2000, 6). Concerning the analysis, the student teachers were able to choose topics that attracted their attention for developing a further inquiry. There are several quotations showing that the student teachers found the observation goals and tasks challenging.

Reflection: Student teachers' observations led to interesting (partially written) reflections on their observation activity and on their (future) professional identity, for example: "*With this experience I could put myself in the 'place of the teacher' and imagine how I would react in very different situations ...*" (Dora in Ponte & Brunheira 2000, 7). This is also an indicator that the APOA student teachers developed an investigative attitude.

Autonomy: The quotation above (like many others) also shows that observations gave the student teachers enough freedom to refer their reflections to their own situation, reflecting the transition from a student via a student teacher to a teacher.

Networking: It is a main goal of the initiative to foster the development of "*common meanings*" or establish a "*discourse community*". Among others, this helped the student teachers to see how different people can observe the "same" events and interpret it in different ways, and it broadened their views on teaching practice.

MILE:

Action: The student teachers have a clear task, namely to analyse MILE-narratives under an individually chosen research question. The teacher educator (as a coach and expert) gives support by stimulating them to view a situation from a different perspective or providing theoretical considerations.

Reflection: The student teachers' investigations lead to interesting reflections, for example: "... *the interview [with one teacher on the video] made the most impression on me. It was looking in the head of the teacher and finding out secret information.*" (Hayet in Goffree & Oonk, 1999b, 43)

Autonomy: The student teachers have great autonomy in devising their own research questions (e.g. "*How does Minke [a teacher on a video] build his*

lessons up? Which situation does he take into account in the beginning and what does he want to achieve?”). This enhances an investigative attitude (e.g. “MILE is addictive ...”, Hayet in Goffree & Oonk, 1999b, 62) and supports the wish for getting new views and sharing experiences.

Networking: The student teachers share their experiences and thus enrich their personal views, for example: “*It was useful to discuss our different viewing perspectives.*” (Hayet in Goffree & Oonk, 1999b, 62). “*The discourse motivates and encourages the students [student teachers], especially stimulating them to undertake further research.*” (Goffree & Oonk, 1999a, 197). The search engine and the support by a tutor facilitates the process of generating new views.

TOMORROW 98:

Action: Whereas the teachers had the task of implementing innovative approaches to teaching, the teacher educators had a double role: on the one hand they had to facilitate the teachers’ activities, on the other hand they also were learners (intensively supported by the project director).

Reflection and Networking: The teacher educators had more time and opportunities to reflect on and to share their experiences than the teachers. Tami (one of the teacher educators), for example, stated: “*Today, I am much more open to many different ways of facilitating teachers’ and students’ cooperative learning in mathematics. I now accept and use methods that are not very structured, that seem to give way for many different kinds of cooperation to different extents.*” (Zaslavsky & Leikin 1999, 156)

Autonomy: There is some evidence that the teacher educators had more freedom to define their investigation process in the project than the teachers (e.g. Ronit in Zaslavsky & Leikin 1999, 156: “*I found myself dealing with questions that interested me, because I asked the questions ...*”).

Some short remarks concerning the dimension of *networking*: Whereas the APOA course mainly builds on the heterogeneity of joint experiences and reflections (e.g. including questions concerning the collaboration between teachers, organisational issues or educational reform by student teachers) and aims at giving the student teachers a first systemic insight into the complexity of teachers’ tasks at their school, the investigations within the MILE project are mainly limited to a set of video-recorded situations in classrooms, but give the student teachers much scope to define their own questions and to go deeper with

their research. In the case of APOA we might use the term “horizontal networking” and with regard to MILE “vertical networking” (see e.g. Cooney & Krainer, 2000). In the case of Tomorrow 98, horizontal and vertical networking happened. The description of the project seems to indicate that the teacher educators had more time and opportunity for both levels of networking.

Summing up, the following factors supporting (student) teachers’ professional growth have been found in all three cases:

Action: There are challenging goals and tasks for (student) teachers and helpful theory-based support by teacher educators. Thus the (student) teachers found better ways to link theoretical and practical aspects of knowledge.

Reflection: Student teachers’ investigations lead to interesting reflections on their own professional identity and role and those of others. This increased their repertoire for (future) action.

Autonomy: The (student) teachers had some autonomy in defining their observation or research interests and thus were able to relate their findings to their own situation and background. This autonomy allowed them to start where they are, namely at their own situation and pre-knowledge.

Networking: The (student) teachers share their experiences with other participants or with the teacher educators, they use electronic means and research literature, and thus enriched their personal views (and those of others).

An additional aspect: Model function of teacher educators: In all three cases, the teacher educators not only promoted (student) teachers investigations, but – at a second level – also investigated the professional growth of their (student) teachers. The fact that the teacher educators themselves have a research interest in (student) teachers’ professional development not only leads to a clearer focus on goals and design of the teacher education initiatives but also leads to an enrichment of teacher educators’ actions based on the research results and the reflections on their actions. Both aspects seem to be essential for improving theory and practice of teacher education. Furthermore, this investigative attitude of the teacher educators might contribute to (student) teachers’ views that investigations are really decisive for promoting professional growth and that writing – as an additional process of reflection – is an important factor that allows a bigger community to share and discuss the findings.

4. A check list of success indicators as an instrument for teacher education initiatives (aiming at investigations into practice)

In this text, it was only possible to sketch some essential factors promoting (student) teachers' professional growth that can be found in all three initiatives. However, the analysis also showed a variety of additional – explicit and implicit – views of how professional growth of (student) teachers can successfully be promoted. In the following, a list of ten indicators is presented that both builds on the analysis of the three cases and on the authors' own experiences in the field. The list might be taken into account when planning, carrying out and evaluating teacher education initiatives that focus on (student) teachers' investigations into practice. Of course, such a list can never be complete. However, it might be used as a starting point to reflect our goals, both for our teacher education practice and for our investigations into that practice.

List of success indicators

The (student) teachers...

- ... are able to explain (and critically reflect on) the goals of the investigation (or more general, the whole teacher education initiative).
- ... are able to explain (and critically reflect on) the important outcomes of the investigations, in particular through identifying parts of (practical and theoretical) knowledge and abilities that have been developed further.
- ... are able to explain (and critically reflect on) the design of the investigations.
- ... are able to explain (and critically reflect on) their own role in the investigations and those of others.
- ... are able to point out research questions that might be worth investigating.
- ... are able to express (and critically reflect on) their own professional growth (personally, socially, content-related) and those of others, in particular through indicating situations that promoted their learning process.
- ... are able to sketch their (new) views on teachers' practice.
- ... are able to express (and critically reflect on) consequences for their (future) practice, in particular through sketching situations where they might apply things learned throughout the investigations.
- ... are open to critical reflection on the learning climate, the quality of support, the consideration of their interests, wishes and critique.
- ... are willing to recommend participation in such an investigation to other people and are able to explain its specific strengths and weaknesses.

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PRETTY (GOOD) DIDACTICAL PROVOCATION AS A TOOL FOR TEACHER'S TRAINING IN GEOMETRY

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Abstract: *In this paper, we present what we call in French PPD (Petites Provocations Didactiques) translated in English by Pretty (Good) Didactical Provocations. These teachers' training situations try to provoke and destabilise the representations of future teachers on geometry. So they must realise the diversity of the geometrical paradigms.*

We suggest three examples intended to clarify the nature of the P(G)DP and the pertinence of this type of training situation. All the examples are taken in the geometrical frame: it is caused by the fact that the setting up of Didactical Provocations needs an epistemological and didactical analysis that we have made only for Geometry.

Introduction

The IUFMs, Instituts Universitaires de Formation des Maîtres (French University Training Colleges), have been in charge since 1991 of the formation of pre-service teachers. Before the creation of the IUFMs, the teachers for primary school (children from 3 to 11) were trained in Normal Schools ("Ecoles Normales"). Nowadays, the IUFMs accept, after a first selection, graduate students from any University (three years of study). During one year, these students prepare in the Institutes a competitive examination to become pre-service teachers. The mathematical examination part is composed of classical mathematical questions and also of questions about the teaching of mathematics in primary school (study of pupils' errors, compared analysis of textbooks). The successful candidates receive a theoretical and practical education of one year (the "second year") in all the matters of the primary school; they receive a salary and are almost sure to become effective primary school teachers the year after.

The study of teachers' training in mathematics in that institutional system is our main theme of research.

In this article, we introduce the notion of Didactical Provocation as a consequence of our researches on the teaching of Geometry for pre-service teachers. In our former studies, we have (Houdement. and Kuzniak, 1996) introduced a classification of the different kinds of approaches used by teacher trainers in the IUFMs to convey a professional education in mathematics to their students.

Then, we focused our attention (Houdement and Kuzniak, 1999) on a particular and problematic mathematical theme in the pre-service teachers' training: the teaching of Geometry. In this work, we developed an approach of geometry based on a play between different geometrical paradigms.

The relation between a particular strategy and the different paradigms led us to suggest the idea of special training's situations, the P(G)DP whose presentation is the main aim of our present paper. But, to understand this approach, we think that it is necessary to give a brief overview of our theoretical construction.

Our theoretical background

On training strategies

Three types of knowledge: mathematical knowledge, didactical knowledge and the "third" knowledge.

Teaching Mathematics is obviously connected to *mathematical knowledge* but also to other ones that are not automatically owned by a specialist of mathematics and that are more or less close to mathematics like history, epistemology, didactics, psychology or pedagogy. We classified this large set of knowledge in two parts. The first one, that is made explicit and structured clearly within the frame of didactical theories, constitutes *didactical knowledge*. The second one, that is not explicitly written and theorised, but exists in the professional action of each teacher is what we called "*third knowledge*".

We were a priori more interested in the teaching of the last two types of knowledge which are more specific of what is teachers' training. But the following additional questions seemed to be very important: how to combine the three types of knowledge and how to give, at the same time, to students who are not (and sometimes far afield) specialists of mathematics, a minimum level in mathematics. Another very important question is how to transpose the theoretical knowledge? This transposition is connected with the conception about teaching that the teachers own.

The different strategies

The combination between the three types of knowledge can take different forms: they can be suggested to or developed by the students; they can be juxtaposed or connected; the connection can be explained or not...

So we have distinguished four main strategies:

- the cultural ones,
- the strategies based on monstration,
- the strategies based on homology,
- the strategies based on transposition.

These strategies differ concerning the explanation of knowledge, the combination between these three types, the position they give to the students. They also depend on the knowledge considered as dominant and on the transposition made by the teacher trainer. Let us summarise these strategies.

In a *cultural strategy* the didactical and the “third” knowledge is only presented and exposed without special connection with the mathematical one. The students receive; they are in charge of the connection between the three levels and of the possible applications into the classroom .

The *monstration strategy* favours the “third” knowledge and supposes students will extract themselves this knowledge if for example they observe a mathematical activity conducted by an expert directly in a classroom (or on a video). Seeing is privileged. The students are considered as fellow-teachers.

The *strategies based on homology* use the lack of mathematical knowledge of the students as a pretext for the teacher trainer to build a learning situation for their students: this situation is conceived as an epitome of the trainer’s conception of teaching. He hopes that the students, by impregnation, would utilise the same model in their future own classes. The students are considered as pupils, but they are supposed to analyse the situation to pinpoint elements of didactical knowledge and the “third” knowledge.

The *strategies based on transposition* favour didactical knowledge. The trainer tries to control the phenomena of transposition of knowledge which is a bias of every teaching situation (Chevallard) In the facts, the control of the transposition often follows a first phase of homology: the teacher trainer explains and justifies the didactical aids he used during the situation. He explicitly refers to elements of didactic theory. The students are considered as pre-teachers. The reader could see examples later.

On Geometry

On the other hand (Houdement and Kuzniak 1999), we have insisted on the complexity of the term Geometry: for us a coherent definition of the word Geometry is necessary. In our conception, Elementary Geometry is a particular theory of space, which tends to represent the local properties of the real space. Its more elaborate form is \mathbf{R}^3 with the structure of an Euclidean space. We have set and studied two hypotheses:

1. Different and coherent paradigms are determined by the same term of Geometry. This explains, partly, the problems we face at the junction of the different institutions: primary school, secondary school and university.
2. Students (future teachers), teachers and pupils take place implicitly in different paradigms; this is a source of misunderstanding.

To study these hypotheses, we have introduced three kinds of knowledge needed to consider Geometry in its relation with space: intuition, experiment and deduction. In this conception, Geometry appears as a dialectical synthesis of this knowledge that evolves and takes into account the mathematical progress.

We have distinguished three paradigms:

Natural Geometry (Geometry I) is the basic starting point and constitutes the first synthesis: we want to name here Geometry deduced from reality and intimately related to it. The legitimacy for validation in the Natural Geometry is the correspondence between intuition and the conclusions of an experiment or a deduction. The backward and forward motion between the model and the real is permanent and is allowed to prove the assertions: the most important thing is to convince.

Next, we meet Natural Axiomatic Geometry (one model is Euclid's Geometry) where the axioms are as close as possible to the intuition of the ambient space. This Geometry (Geometry II) is a model of the reality. But, once the axioms fixed, the demonstrations inside the system are necessarily requested to progress and to reach certainty.

Last, we have Formalist Axiomatic Geometry (Geometry III) where the most important thing is the system of axioms itself with no relation with the reality. The system of axioms is complete and independent of its possible applications to the world.

The last contact with geometry of our students before IUFM took place in secondary school or at University, they learnt Geometry II or III. In the primary

school, Geometry I is predominant. We think that a necessary condition for the students to succeed in teaching is to realise that geometry covers different paradigms, each paradigm owns its coherence and takes a dominant place in a school level. From our perspective, knowledge of the different paradigms could be a didactical knowledge if it is introduced with a teaching point of view.

Pretty (good) didactical provocation

In this presentation, we try to articulate the two precedent approaches (training strategies and geometrical paradigms) in the frame of the real teachers' training.

From our studies on the training strategies, a cultural approach of mathematics is inefficient for almost all the students. It appears disconnected from the practice and from their main preoccupations about geometry in the classroom. A monstration approach is here inappropriate considering the nature of the knowledge to teach (didactical and mathematical). Then, an only homological approach is also inefficient because it can't enable the students to keep enough distance with their anterior geometric studies.

Our idea is to introduce specific and brief situations for the students that are geometrical situations near as soon as possible to the level of the future pupils of our students (its homological character) and sufficiently astonishing for them (its provoking character) to produce a real interrogation on the nature of Geometry. It would be interesting if these situations were at the junction of the different paradigms and lead the student to question the situation itself (and not to stay in the situation).

We give now three examples of these kind of situations.

- 1) Construct a circle whose an arc is given
- 2) The construction of a trapezium
- 3) Straight Lines on a cylinder.

Construction of a circle whose a small arc is given.

In a first phase, four arcs of circle (280° , 180° , 125° and 55°) are given on a sheet of paper. The students should construct the whole circle.

Mathematical analysis:

The problem is clearly located in the real space of objects and the given task is practical. The circle, partially known, appears as a drawing on the paper.

To solve this problem, the students could refer at different kinds of knowledge taken in our paradigms of Geometry.

In Geometry I, it could be practical know-how not specifically related to mathematics like folding, construction step by step in reporting several times the arc of circle copied with a tracing paper.

In Geometry II, it is possible to draw the centre of the circle by using the properties of the perpendicular bisector of a segment or to use the properties of the inscription of the right angle in a circle.

In this case, the geometrical knowledge gives drawing techniques with ruler, compass and square. Geometry II plays the role of a technology for techniques (Chevallard 1999).

Didactical analysis and development:

The problem is not difficult and could appear as a revision problem for the students. In this first phase, we are confronted with a kind of social conflict between students referring implicitly to different paradigms of Geometry. This situation is particularly interesting in the training of future teachers for primary school; they have, indeed, very different levels in mathematics and groups of such students are generally very heterogeneous.

Some students try to inscribe a right angle (obtained by a fold paper) in the arc: they move the piece of paper to succeed to put the vertex on the arc and the sides cutting the arc; it works to find the centre of the arcs bigger than 180° . They could know these techniques without any idea of the justification of them: they work in Geometry I. At this moment of the situation, students referring to paradigms of Geometry II are sure of their position, just surprised by the procedures used by the other students.

When all the students have understood the usual construction, it's time to introduce our Didactical Provocation with this new instruction: compare your circle drawn on the smallest arc (55°). There exist a great variety of lengths for the radius of the circle. The students to justify these differences use different arguments but they conclude that theory is inefficient for this kind of practical task. The construction of the perpendicular bisector of a segment appears as a theoretical fact in the context of Geometry II where constructions are idealisation efficient on the ideal figures of Geometry II.

Conclusion

In our precedent studies, we saw that a word appeared commonly when we asked in-service or pre-service primary school teachers the meaning of Geometry: it was rigor. But the sense of this word was not the same for all these teachers. For a part of them (principally the in-service teachers), it was the synonym of precision of the drawing and of the measures, for the other it was related to the reasoning and to logical thought. This kind of situation permits to point this fact and to explain in the frame of a general definition of Geometry the limits and the particularities of each approach.

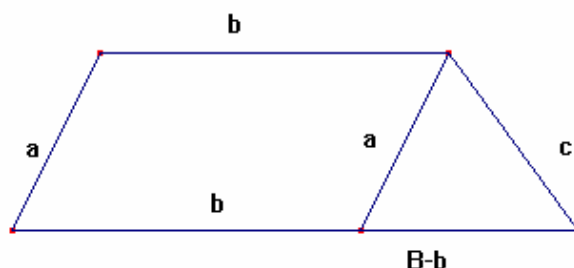
Construction of a convex trapezium with imposed lengths

The following instruction is given for the students.

Construct a convex trapezium with lengths of sides: 8 cm, 7 cm, 5 cm and 2 cm. How many solutions are there? Give justifications.

Mathematical analysis

A rapid draw permits to find conditions of existence.



It is possible to construct the trapezium if the lengths a , $B - b$, c are those of a triangle, B and b corresponding to the sides parallel with $B > b$.

The condition of existence is $|a - c| < B - b < a + c$ where B and b , with $B > b$, represent the lengths of the parallel sides.

Length for B	Length for b	Lengths for the triangle	The triangle exists
8	7	1 2 5	no
8	5	3 2 7	no
8	2	6 5 7	yes
7	5	2 2 8	no
7	2	5 5 8	yes and it's isosceles
5	2	3 7 8	yes

There are three solutions if one considers equivalent the figures that are connected by an isometry positive or negative.

Didactical analysis

At first this situation appears to students as a usual construction problem: they have to draw an object respecting conditions of lengths and form and they could use instruments such as ruler and compass. The problem is set in Geometry I and the students would begin to work in Geometry I, considering the problem solved if they exhibit a “good” trapezium. The drawing instruments are not really imposed, but the usual contract of the course recommends ruler and compass and uses the tracks of the auxiliary construction as justification. The form is well known and characterised by the parallelism of two sides. The difficulty (impossibility) of the construction of some trapeziums may induce students to go out of the techniques, to begin to think of possibilities of construction.

This change corresponds for us to a passage from Geometry I to Geometry II: working about the drawing was not sufficient, it was necessary working about the figure and thinking about its constructibility before its realisation with ruler and compass. The students are in face of different paradigms for geometry: it's necessary to precise the characteristic of each of them.

How it progressed with students

As anticipated, students tried to draw a trapezium with ruler and compass, some of them only with graduate ruler. Three reactions existed: some of them didn't succeed; others just succeeded to draw one with luck, as they said, or by using graduate ruler or parallelism of the lines of their paper. It produced a short interrogation about the utilisation of instruments and a discussion about how to determine which trapeziums exist.

The main difficulty resulted in the fail of synthetic analysis. None of them thought of drawing an approximate trapezium to study it. The teacher suggested this idea and helped them going out of this paradigm to think theoretically about the construction of a polygon. Progressively the students thought of using triangles. Then some of them tried to cut the trapezium in a rectangle and two right-angled triangles, but renounced because of the impossibility deducing or calculating some lengths. Others even tried to inject Thales theorem, but vainly (procedure possible but long). A few cut the trapezium into a parallelogram and a triangle (the two last decompositions are valid in geometry II because the new lengths can be deducted of the sides). None of them used directly the triangular inequality to conclude.

Conclusion on the pertinence of the situation as a Didactical Provocation

This situation presents several advantages. First, it seems to be a usual construction problem to those of primary school, when the teacher asks pupils to draw a figure with certain conditions, under the control of the instruments. But

under a usual form, this construction is not a question for pupils of the primary school: the students realise the necessity of analysing the construction problem, particularly concerning the paradigm at work. That justifies the homological side of the situation.

This problem can not be solved in the habitual framework of the primary school: it is necessary to introduce another (more theoretical) point of view. This new point of view is interpretable as a change of paradigm (Geometry I to Geometry II); this new paradigm contains the definitions of geometrical objects and the rules of production of properties and new objects. The definitions seem natural and near reality, but they are formal. That justifies the character of transposition of this situation: a new knowledge, the vision of geometry as three coherent paradigms, is necessary to analyse and foresee some geometrical problems in a teaching perspective.

Lines on a right cylinder

We have set the students the following problem: determine the straight lines on a right cylinder. They could use rolls or sheets of paper to shape right cylinders.

Mathematical analysis:

Here we consider a classical right cylinder. We can, in the way of the English writer Abbott in *Flatland*, imagine an inhabitant on this cylindrical surface. This inhabitant will develop a natural geometry which could be based on the notion of distance. Here are some of the questions set on this surface:

Which is the shortest distance from one point to another?

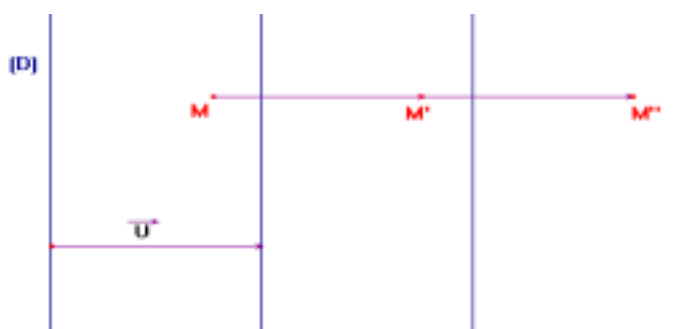
What are the properties of a triangle and of the other figures? In particular, is the sum of the angles in a triangle equal to two right angles?

What becomes of the theorem of Pythagore? Etc.

In this context, we define a line on a cylinder as the shortest distance between two points. We explain this approach in the didactical analysis.

To solve these different problems, it is very useful to have various models of the right cylinder. We need to determine the normal space associated with the natural geometry of the cylinder: we refer to a particular space, possibly drawn in the microspace, which realises and contains the properties of the whole space. Its concrete realisation makes possible an intuitive reasoning and allows easier vision for people who try to do geometry.

To obtain this space in the case of the cylinder we could roll, several

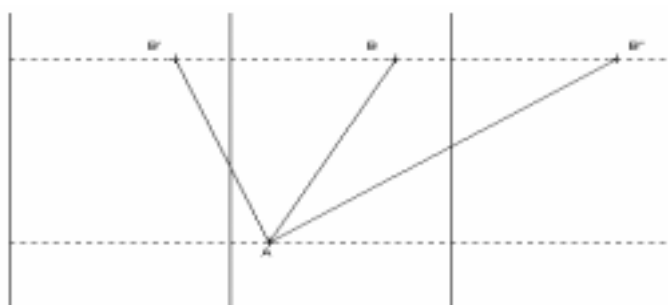


times, the cylinder up on the plan. So we obtain a network made of parallel lines to D. In this network, two points M and M' are equivalent if there exists a number n and a translation t of vector $n \cdot \vec{U}$ such that $M' = t(M)$ where \vec{U} is a

vector whose length is the perimeter of director circle of the cylinder (Model 1).

A strip of the plane determined by two parallel lines makes a second model (Model 2), but only one of the lines belongs to the model.

Now, we can solve our problem of geodesics on a cylinder using the first



model. The geodesic on a cylinder is the image of a straight line on the plane model. But there is a small difficulty, on a cylinder there are two ways to join A to B; from the left side or from the right side of B. Which is the shortest?

In the plan network (Model 1) associated to the cylinder, the point B is equivalent to points B', B'' etc.

The shortest way, AB or AB', depends of the relative position from A to the perpendicular bisector of BB'.

The straight lines drawn on the Model 1 make a constant angle with the generator: they are helices on the cylindrical surface in \mathbb{R}^3 . By extension straight lines of the cylinder are the lines prolonging the segment of the shortest length between A and B.

Hence, near the helices we have the generators joining two points on a parallel to D. Finally, if two points are on a perpendicular to the generator D, the line is a director circle of the cylinder. In this case this straight line is bounded and is not unlimited as a straight line on the Euclidean Plan.

There exist three kinds of straight lines: these helices, the director circles and the generators. These different types of lines should pose questions for the students and demonstrate the necessity of definition to produce geometry.

Didactical analysis and development

In a precedent study (1999), we have asked students, teachers and pupils for possible definitions of straight lines. For all these people, the question clearly refers to the line into the Euclidean space and to the intuitive shape of straight lines in the ambient space. Here, the same question set in an unusual context, cylinders' one should provoke the students to re-examine familiar facts made obvious by their use.

Rolls of cardboard and sheets of paper are given to suggest to the students to make effective constructions and to use the link between space and plan (cylinder and the model 1). It is important for us to introduce the plan model of the cylinder to justify the constructions and to avoid staying in an experimental approach.

During the session in the classroom, two main problems have appeared:

The difficulty of transforming the notion of the straight line in the context of the geometry of the cylinder. The students are perplexed and avoid finding definitions of straight lines; they try to draw straight lines on the rolls of paper. One part uses a rule and makes roll the cardboard under the rule: we find here a definition in action of the line as a stroke made with a ruler (as pupils said in our study). An another part tries to make section of the cylinder by a plan to bring the problem in a familiar context.

The second difficulty, perhaps more fundamental and not expected by us, is that the students don't make any relation between the plan and the cylinder, even if we only give sheets of paper: the isometric application between Model 1 and the cylinder is not evident. This observation implies that we must study this situation deeper.

Conclusion

This situation gives prominence to the difficulty and the necessity of definitions in a mathematical approach and is very disturbing for the students. In this sense, it constitutes a Didactical Provocation. But, we meet here a fundamental fact in a training strategy; the difficulty for the students to think and to speak about pedagogy or didactical knowledge when the mathematical task is too important for them. Only several Didactical Provocations set on a theoretical frame could avoid this problem, as we shall develop in our general conclusion.

General Conclusion

We come back to the main purpose of this paper, the training of the future teachers in mathematics within transposition strategy: explain an explicit didactical knowledge showing its pertinence and its straight forward relation to mathematics. In our opinion, it is very important to give pre-service students knowledge about the nature of mathematics. The Didactical Provocations could constitute training situations to transmit this knowledge in a close link between theory and classroom practice: in fact with these situations the students construct for themselves questions that ask for mathematical and didactical answers. In this way it seems possible to evoke a certain character of “adidacticism” (Brousseau 1986) in these Didactical Provocations.

To produce the most efficient situations, it is necessary to find crucial notions and for this to develop theories on the nature of the mathematical objects that take into account a didactical and epistemological perspective as we have tried to do on the geometry taught at school.

This fundamental frame, explicitly presented to the students, would be used to explain and to understand the different phenomena and difficulties encountered during the Didactical Provocations. Without this frame, or another of the same kind, students stay fixed on the action when the mathematical problem is not easy to solve. However, it is important that they meet this kind of situations in their training.

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A STUDY OF PRIMARY ITT STUDENTS' ATTITUDES TO MATHEMATICS

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Abstract: *In this report I attempt to analyse the responses to a questionnaire administered to 124 final year ITT (initial teacher training) students. The broad aim of the survey was to elicit the students' attitudes to mathematics. The majority of the students will be required to teach mathematics in UK primary schools (5-11), and consequently, any negative attitudes could be detrimental to the education of future generations of British children. The study has its theoretical foundations in the work of McLeod (1988), who linked attitude to emotions and beliefs, and was inspired by recent research conducted in the UK by Rowland et al (1999) and Green and Ollerton (1998) on the attitude and mathematical abilities of primary ITT students.*

1. Introduction

All students intending to teach in UK primary schools have to demonstrate mathematical competence before they can be awarded Qualified Teacher Status (QTS). The Standards they have to meet are dictated by the government (DfEE, 1998, annex D), but assessment arrangements are determined by individual institutions. Mathematical knowledge is required which goes well beyond that normally taught to 11 year old British children (e.g. simultaneous equations, algebraic graphs). Students at my university (following one year postgraduate and 3 year undergraduate courses) were audited and tested in the final term of their final year. This process was very intensive and involved some trainees undergoing remedial support and taking several tests. I became aware that this was causing some anxiety and resentment. I was concerned that if trainees were leaving with negative attitudes to mathematics, this could be detrimental to pupils in their charge. I decided to try and investigate the extent of the problem. How widespread was it? Did it extend to all areas of mathematics?

2. Literature Review

McLeod (1988) defines *attitudes to mathematics* within the general context of the affective domain. He firstly defines *affect* as all the feelings connected with

mathematical learning. *Emotion* he defines as a kind of passionate response to a particular mathematical situation. Typically, emotion is intense, but short lived and subject to change. When I administered my questionnaire, most of the students had just taken an audit test under exam conditions. (They completed the questionnaire immediately following the examination). They were undoubtedly emotionally charged. McLeod uses *attitude* to refer to affective responses which are more consistent, long term and stable.

Mandler (cited in McLeod, 1988) attempts to formulate a theory based upon a cognitive science approach to problem solving adapted to the affective domain. According to Mandler, *emotion* arises during a planned session on problem solving in which intellectual activity is *interrupted*, usually, but not exclusively, because of some mental blockage. This leads to *physiological arousal*: increased heartbeat, perspiration, muscle tightening. The resulting emotions can be panic, frustration, confusion etc. The cognitive evaluation that follows the interruption can lead to either positive or negative emotions, leading to the formation or reinforcement of positive or negative attitudes. For some students, the process of sitting an examination in mathematics could pose a whole series of problems as they struggle to answer the questions. The resulting negative emotions could lead to reinforcement of negative attitudes.

Several researchers have reported on their experiences with Primary students and their attempts to “meet the standards”. Rowland *et al* (1999) found a strong link between students’ performance on a maths audit and performance when teaching number on teaching practice; students who were strong mathematically tended to be stronger in the classroom and conversely, mathematically weaker students tended to be weaker at teaching number on teaching practice. If this is the case then it is clear that the auditing process needs to begin much earlier: in the case of one-year students, before the course starts.

Green and Ollerton (1999) collected data on students’ attitudes using a mixture of interviews and written statements. They conclude that students’ negative attitudes are a result of their earlier experiences in a ‘traditional’ setting: “...mathematics which is centred on fixed rules, procedures and repetitive examples, learnt predominantly from textbooks...” (p48). This contradicts the finding of Clute (1984) that anxious students do less well in so called “discovery” lessons than with expository teaching. The remedial sessions arranged for my students were definitely in the latter category (one lecturer with 80 students).

3. Research Methodology and Design

The main research question was concerned with students' attitudes to maths. This was sub-divided in two ways. In the first place, attitude was broken down into emotional responses or *feelings* and beliefs or *perceptions*. This distinction is important because students could well be of the opinion that whereas mathematics may well be a useful, challenging and logical subject, nevertheless it causes them great anxiety and they could be fearful about teaching it. Secondly, I felt it would be useful to explore any differences in attitude to the various areas of mathematics (e.g. number, algebra, shape and space, measures, mathematical reasoning and statistics and probability). I felt that students would accept the need to be knowledgeable about and confident to teach number, shape and space and measures, but much less so about the other areas. This question also has a link to feelings and perceptions, because negative attitudes to a particular topic could be caused by feelings of inadequacy, and conversely, positive attitudes could arise out of beliefs about the perceived usefulness of the topic.

I decided the best way to collect this information was via a questionnaire. I had a fairly large population (152) and very little time. There are problems, of course, associated with this method. In the first place, responses could well be unreliable, partly because respondents could be less than honest in their replies, and partly because questions could be misinterpreted. Secondly, the extent to which respondents actually reflected over their answers is also questionable. The majority had just finished an examination and were undoubtedly keen to leave. Thirdly, the limitations of questionnaire design leave very little room for respondents to express themselves fully. It is difficult to ask the question "Why?" The other option, interviewing a small, selective sample would have been preferable, because it would have overcome some of these difficulties, but was not viable because of time constraints.

The questionnaire consisted largely of tick-boxes with yes/no responses or a 5-point Likert scale (from strongly agree to strongly disagree) (Oppenheim, 1992) and was therefore quick and straightforward for the students to complete.

4. Analysis

The questionnaire was administered anonymously to the students immediately following the second maths audit test. I received a total of 124 replies from a total population of 152 (some were absent due to job interviews). This represents a response rate of almost 82%. Almost 90% of the population were

females. In terms of other characteristics, such as age, 70% were in the 20-24 age group. Only 4 students were over 40 years old.

The early questions sought students' opinions on maths itself and their perceived abilities at maths. Typically, they were statements to which the respondents were required to indicate their agreement using the 5-point Likert scale described above. For the purposes of simplifying the analysis, I propose for the most part, to group the responses as positive (strongly agree, agree), neutral, and negative (disagree, strongly disagree). To the statement "I like maths", 44% responded positively, which I thought was encouraging (though clearly this means that 56% were either ambivalent or negative). When I qualified this by adding "...when I can see a point to it, this figure rose to 78%. This statistic illustrates that respondents' beliefs about mathematics were generally positive. The next statement "I like maths when I can do it" produced a similar positive response (79%). This demonstrates a possible link between achievement and attitude. In response to the statement "I'm hopeless at maths", 27% strongly agreed or agreed; this suggests a negative attitude to mathematics was held by nearly $\frac{3}{4}$ of the population.

I then went on to look at students' attitudes to the various topic areas within maths covered by the UK government standards (DfEE, 1998, annex D). These were described above. I felt that there could be significant differences. A student might accept that it was important to understand and to be confident about number, because of the need to teach it, but could well feel differently about algebra or mathematical reasoning, because it doesn't appear in the 7-11 curriculum.

I scored students' responses using 1 = "strongly agree" through to 5 = "strongly disagree" and computed the mean responses to the statement "I like (*topic name*)". A low score (below the midpoint of the scale, 3) indicates a positive response. Analysis reveals that, using the mean score for "I like maths" (2.89) as a comparison, students were more positive about *number* and *shape and space* (2.17 and 2.26 respectively). In contrast, they were rather negative about *mathematical reasoning* (3.22) and *statistics and probability* (3.24). Surprisingly (to me at least), *algebra* scored 2.93, i.e. slightly positive. *Measures* scored 2.53. Using the paired samples *t*-test to compare the means of pairs of variables reveals some highly significant results (the appropriate *p* values are given in parentheses): the difference between *number* and *maths* is highly significant ($p < 0.001$). Similarly the difference between *maths* as a subject and the topic *statistics and probability* is highly significantly (0.001), as is the difference between *maths* and *measures* (0.004). There are no statistically significant differences between *number* and *shape and space* (0.495), or *mathematical reasoning* and *statistics and probability* (0.935), or *maths* and *algebra* (0.586).

Summarising these results, it would appear that maths as a subject lies in the middle of a continuum, with students claiming to prefer *number, shape and space* and *measures* to *mathematical reasoning* and *statistics and probability*. Maths and algebra lie in the middle of this continuum, students' opinions being generally neutral. Unsurprising results in some respects, because one might expect *number, shape and space* and *measures* to be popular (traditional primary school topics). *Algebra*, not normally considered a primary school topic, wasn't disliked by the students. Conversely, *statistics and probability*, firmly established in UK National Curriculum for over 10 years now, appeared to be disliked.

Other related statements were “(*maths topic*) is hard”, “I feel competent to teach (*maths topic*)”, “(*maths topic*) is a useful topic for primary students to study” and “I wish I was better at (*maths topic*)”. They were scored using the same Likert scale. The results, which are not detailed here due to lack of space, are broadly in line with those given above. In general, respondents were positive about *number, shape and space* and *measures*, and negative about *algebra*, *mathematical reasoning* and *statistics and probability*. Students generally had more negative attitudes towards *mathematical reasoning* and *statistics and probability* than *algebra*. It is not possible to be certain from a survey analysis why these results occurred, but one could speculate that the topics which feature in the primary school curriculum were popular because (a) they were more familiar, and after 24 weeks in school, the majority would have had experience of teaching them, and (b) they are conceptually less demanding. This doesn't explain why statistics and probability, which until recently featured strongly in the UK National Curriculum, was so unpopular. Again, speculation leads one to conclude that maybe if descriptive statistics had been separated from probability in the questionnaire, the results would have been very different.

5. Conclusions

In response to the questions I posed in section 3, I should like to offer the following tentative conclusions:

1. Over one third (36%) of respondents claimed to dislike maths, and could therefore be described as holding negative attitudes. Adding the qualifier “...when I can see a point to it” to the statement “I like maths” produced a much more positive response: 78% either strongly agree or agreed. I would argue that this qualifier has a link with beliefs, with respondents connecting this statement with their beliefs that mathematics has a utility value. This, I would argue, consists of the mathematics teachers need in order to be successful teachers at key stage 2 (5-11). It demonstrates that

respondents' beliefs about mathematics were generally positive. On the other hand, 79% responded positively to the statement "I like maths when I can do it". This suggests a link between attitude and perceived achievement. Negative feelings emerge when subjects' subject knowledge is challenged, as it was in this case. The consequence of this is that the negative emotional responses expressed by the majority of the respondents could well be temporary. This supports the findings of both McLeod (1988) and Mandler (cited in *op cit*)

2. The issue of attitude to various mathematical topics has some similarities with the above analysis. Three of the topics provoked generally positive attitudes (*number, shape and space, measures*), whilst the remaining three (*mathematical reasoning, algebra, statistics and probability*) generally provoked negative responses. It seems reasonable to conclude that positive attitudes occurred for topics considered to be less conceptually demanding and with a perceived utility, whilst negative attitudes were provoked by topics which were seen as difficult and not useful. It is surprising that *statistics and probability* was considered to be difficult and not useful. This may be because descriptive statistics was linked to probability and respondents' views may well have been referring specifically to probability.

It would be interesting to conduct a further survey in 12 months time in order to establish whether or not respondents' attitudes have changed. If the negative attitudes encountered were caused by short-term, but intense emotional responses, then the outcomes could well be very different.

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INFORMATION TECHNOLOGIES AND THE DEVELOPMENT OF PROFESSIONAL KNOWLEDGE AND IDENTITY IN TEACHER EDUCATION

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Abstract: *This paper addresses the preparation in ICT of preservice mathematics teachers. We describe the aims, assumptions, and work carried out in a course offered at the University of Lisbon and briefly discuss the processes involved in the development of professional knowledge and the formation of professional identities. Using a qualitative methodology, based on the administration of free-response questionnaires to the student teachers enrolled in the course in 1999-2000, we analyze their general perspectives about ICT, the implications of the use of ICT in their view of teaching methodologies, and their development of a professional identity. This work provides suggestions for teacher education practice and for further research.*

Introduction

Mathematics teachers may use in their practice a great variety of educational ICT (Information and Communication Technologies) materials and resources, including general-purpose tools and educational software (NCTM, 2000). The World Wide Web may be regarded as a “metatool” where one can find information about new developments in mathematics and mathematics education, software, sample tasks, classroom ideas, reports of experiences, news about meetings and other events, etc. Preservice teachers need to be acquainted with these resources and to develop confidence in using them (Bottino & Furinghetti, 1999; Ponte & Serrazina, 1998). In Portugal, this is quite problematic since they often arrive at this stage of their professional preparation with previous little contact with new technologies. Not surprisingly, they are rather suspicious regarding the role of ICT in education and have little confidence using it.

Since ICT is an ever expanding world some choices need to be made regarding what is most important to know. In addition, learning about ICT and its uses in mathematics education must assist preservice teachers in their process of developing professional knowledge regarding this domain as well as concerning teaching and learning mathematics, since all these aspects are

interrelated (Berger, 1999). Also, it may help to develop a professional identity, stimulating the adoption of a standpoint and values of a mathematics teacher. The purpose of this paper is to discuss the effects of a university course dedicated to the use of ICT in mathematics teaching in student teachers' perspectives about ICT and classroom methodologies and their development of a professional identity.

The ICM Course

In Portugal, the school system is made up of several “cycles”. Basic education is compulsive and includes three cycles (cycle I for pupils aged 6-9, II for pupils aged 10-11, and III for pupils aged 12-14). Secondary level education is optional, has different strands and is attended by pupils that may be 15-17 years old or older (if previously retained in one or more grades). Mathematics is a separate subject taught by a specialist teacher from the beginning of cycle II, that is, from grade 5 onwards.

The mathematics teacher education program at the University of Lisbon prepares teachers for grades 7 to 12 (that is, to teach cycle III of basic education and secondary education). This program includes three years of mathematics studies, followed by one year of education studies and a last year of practicum in a school. Usually, 110-120 new candidates are admitted every year in the program.

The ICM course¹ which constitutes the focus of this paper integrates the 7th semester of this program—the first semester dedicated to educational studies. We focus on the work done in 1999-2000. The aims of ICM are to facilitate the acquisition of competencies in ICT and to promote the development of new perspectives about its use in mathematics teaching. The main idea is that pupils can learn mathematics by doing explorations and investigations and that the processes of discovery and proof are at the core of mathematical activity. The use of ICT in mathematics teaching can be very useful to emphasize those processes. We decided to work with a limited number of pieces of software (The Geometer's Sketchpad [GSP] and Modellus), as we intended student teachers to explore them in depth. They got acquainted with these software and their educational applications within present mathematics curricula in Portugal.

Student teachers used GSP in several activities to see its possibilities for studying geometry, always through an investigative perspective. Their

¹ ICM stands for “Interdisciplinaridade Matemática-Ciências”.

mathematical preparation leads them to use a deductive approach to solve all kinds of problems, and we used GSP as a powerful tool to show that the perspective of mathematical creativity has many forms other than deduction (de Villiers, 1997). Student teachers started with simple mathematical questions about properties of triangles and quadrilaterals, then they explored certain features of conics and later they investigated invariant properties of some geometrical transformations. Finally, we explored Javasketch's possibilities for including animations with GSP in webpages.

Modellus is a software for constructing mathematical models of all sorts of phenomena, showing their evolution in time through different types of representations. Some student teachers used it in their group project. And the Internet was explored in many of its features, research techniques and elementary publication techniques and, at the same time, we reflected upon its use in schools.

At the beginning of the year most preservice teachers have limited experience with ICT, especially with the Internet. Besides lacking competencies in this area, they also lack confidence to work with computers. This prompted us to opt for two main characteristics for the course's methodology: (i) extensive practical work with computers and (ii) collaborative group work. The role of the instructors was to create stimulating learning situations, to challenge student teachers to think, to support their work, and encourage the diversification of learning routes.

The course was attended by 95 preservice teachers. They were divided in four classes that met twice a week for two hours. The classroom was equipped with 9 computers connected to the Internet that permitted the work in small groups of two or three preservice teachers at each computer. They had free access to the computers when there were no classes in the room.

The main component of the work carried out in ICM was the development of a project, consisting of the creation and publication of a group homepage focusing on a mathematical theme that could be of interest to teachers and preservice teachers. Each group was responsible for the choice of the theme and for searching materials. The preservice teachers were encouraged to do their research on the Internet and to find other sites related to their theme, mainly in Portuguese, so that they could have some public impact. In their research, the groups also made extensive use of traditional materials such as books, textbooks, and journals.

The Web pages produced cover a large variety of topics in geometry, functions, numbers, algebra, probability, and combinatorics, presenting theoretical information and practical work. There is a general concern for

presenting historical background about the theme and to show some applications to the real world and other sciences².

The development of professional knowledge and professional identity

Mathematics teachers' professional knowledge may be regarded as a blend of declarative, procedural, personal and strategic knowledge that is used in situations of practice (Shulman, 1986). This includes classroom teaching, but also other professional roles such as tutoring pupils, cooperating in schools' activities and projects, interacting with members of the community, and working in professional groups. Teachers of mathematics need (i) to know about educational theories and issues, (ii) a good foundation in their subject, and (iii) a strong preparation in the specialized field that concerns their activity: the didactics of mathematics. This includes perspectives about curriculum, students' learning, and classroom instruction and assessment (Boero, Dapueto, & Parenti, 1996). Teachers' knowledge is rooted in such beliefs and conceptions about mathematics and its teaching (Ponte, 1994; Thompson, 1984). To challenge such ideas can be regarded as a central aim in teachers' professional development (Carrillo, 1998).

Teachers' professional knowledge may also be viewed as mainly tacit and originating from practice through a process of personal reflection (Elbaz, 1983; Schön, 1983). Therefore, it is not sufficient merely for student teachers to have knowledge of mathematics, educational theories and didactics. Since professional knowledge is deeply personal and related to action, its development requires diversified working contexts and the experience of situations as close as possible to professional practice.

ICT is increasingly important in the activity of mathematics teachers, (i) as educational media to support pupils' learning, (ii) as professional productivity tools, to prepare materials for classes, to carry out managerial duties, and to search for information and materials, and (iii) as an interactive *medium* to interact and collaborate with other teachers and educational partners. Teachers need to know how to use the new equipment and software and also what their potential strengths and weaknesses are. These technologies, changing the environment in which teachers work and the way they relate to pupils and to other teachers, have an important impact on the nature of the teachers' work, that is, on their professional identity.

² The pages can be found at the address: <http://www.educ.fc.ul.pt/icm/pagalunos.htm>.

Developing a professional identity involves assuming the essential norms and values of the profession and an attitude of commitment to improve oneself as an educator and the educational institutions. A mathematics teacher must carry out the proper professional activities of a teacher and identify personally with the teaching profession. That means assuming a teacher's point of view, internalizing the teacher's role and ways of dealing with professional issues in a natural way. For example, choosing to decide about the value of a variety of resources available for teachers and learning to use them is an increasingly important part of being a teacher. It requires knowledge of exploring software and Web sites. It requires an attitude of openness and confidence in using computers (Berger, 1999).

Berger & Luckman (1973) regard the development of a professional identity as an aspect of the development of secondary socialization. According to these authors, primary socialization refers to the introduction of the individual to the society, becoming part of it. The child internalizes the roles, attitudes and values of significant others, with little possibility for critical distance. Secondary socialization comes later on, as the internalization of "institutional worlds" and involves the acquisition of specialized knowledge (including professional knowledge). Such specialized knowledge is constructed with reference to particular fields of activity that draw on specific symbolic universes.

The construction of social identities is seen by Dubar (1997) as involving two complementary processes. One, the biographical process, is the internal construction by individuals through time of the social identities using the different categories offered by the institutions in their environment. It involves a transaction between inherited and desired identities. The other, the relational process, involves external transactions between the individual and significant others. It concerns the recognition in a given moment and legitimizing space of the identities related to knowledge, competencies, images and values expressed in the underlying action systems.

Methodology

In this study, data was gathered through a written questionnaire completed anonymously on the last day of classes by the preservice teachers who attended this course. The questionnaire included six free-response questions and ample space was provided to answer them:

1. How do you define your current relationship with ICT? What evolution occurred in this regard during this semester?

2. Did this course provide you with the development of perspectives about the role of ICT in mathematics teaching? Specify.
3. How do you see the future of ICT in schools?
4. How do you evaluate the work that you carried out in this course?
5. Comment on the working methodologies used in this course.
6. What suggestions can you give to improve this course?

A set of categories, subcategories and subsubcategories was developed to code the answers. The data analysis software NUDIST (version 4.0) was used to classify them and to provide reports. In this paper only a few categories are used, those most related to the questions we discuss.

Analysis

General perspectives about ICT. The first question that we want to discuss is the contribution of the course to the development of a general perspective about the role of ICT in mathematics teaching.

1. Evolution of student teachers' perspectives. They recognize that this course made a difference to their professional preparation regarding the potential of ICT for mathematics teaching. Many of them probably had heard the media talking about the importance of ICT in society and in schools, nowadays, but not much more than that. According to their answers, it is possible to conclude that they evaluate the new perspectives that the course brought to them very positively, especially because they think the school system expects teachers to be well prepared in this area:

“In fact, since the beginning of the semester, the educational issues that I learned enabled me to move on regarding the use of new technologies, somehow. So, if in the beginning I did not understand very well what was the purpose of the computer in the classroom, today this opinion not only changed as it enriched a lot, through discovering software and techniques to use it in mathematics classrooms.”

2. ICT in the classroom. Most student teachers refer to the Internet, GSP, and Modellus as facilitators of the teacher's role. Many of them regard these instruments as sources of motivation: “it is indispensable to use new technologies in the mathematics classroom. This is the only way we can make mathematics accessible and attractive to our future pupils”. Others consider the software explored in the course useful to support learning specific themes, such as geometry. As one student teacher says: “The use of GSP showed me that

when pupils use it they understand geometry better. Therefore I find it important to use GSP with all age level classes”. In fact, this software is referred to by many student teachers as having a lot of possibilities in mathematics teaching.

Another aspect revealed by the data is that many preservice teachers consider that the role of ICT in mathematics teaching goes beyond motivating pupils. They see it offering the possibility of promoting a new vision of mathematics for them, on the one hand, because ICT can make applications of mathematics more visible and, on the other, because the use of ICT stimulates pupils’ autonomous work. One preservice teacher commented, in this respect:

“By using new technologies it is possible to provide a smoother perspective of mathematics, so that pupils feel motivated to “discover” mathematics, since today any youngster can have access to a computer.”

Student teachers regard the work carried out on searching information and publishing pages in the Internet highly. This is considered to be an activity with a lot of potential for teachers and pupils. As one of them says: “by using the Internet we can easily have access to information from all over the world, which enables us to expand our knowledge, also in mathematics”. Regarding the pupils, one student teacher says that they “may learn a lot searching in the Internet”. Student teachers regard the research activity on the Internet as inquiry and point out the possibility of drawing a parallel with pupils’ learning processes. One comments that: “it was also important to discover the Internet in a more “intimate” way, since that enabled me to see its application in research projects that is easy to develop in a mathematics classroom”.

3. Perspectives about learning with ICT. Some preservice teachers show strong evidence of having developed a perspective of ICT use that values experimentation and exploration, and pupils’ active role in learning. One of them put it very nicely:

“We can use the computer, Internet, and GSP to do several activities through which our pupils may explore mathematics themselves, since as they made the discoveries, the classes become active and the pupils become autonomous and only in that way may they construct their own learning.”

This vision about mathematics teaching, emphasizing exploration and investigation, that permeated ICM’s classes, has a strong influence on student teachers’ perspectives. As one of them says:

“The work undertaken with the Sketchpad was important since it helped us to think, to discover by ourselves geometrical properties regarding the topics

that we were given. It is a good methodology that we, future teachers, may use in mathematics teaching, if possible.”

Impact of working methodologies. The second question that concerns us is the contribution of this course towards the development in preservice teachers of an appreciation for working methodologies that stress an active role of the learner, inquiry, collaboration, and group work.

1. Preservice teachers’ involvement. At the beginning of the course, most of them found the tasks very challenging, mainly due to their lack of knowledge and familiarity with computers. They consider that to face the challenges they needed much determination and hard work, individually and in group. The active involvement of the learner is an indispensable condition to significant learning; two student teachers testify how much they were committed to their work:

“I can say that personally I felt much involved. (...) Looking back, I think that I learned so much, so much, that it was very useful.”

“It was a very interesting work, that required much devotion, but that ended up as something very gratifying.”

The preservice teachers’ evaluation of the level of involvement required reveals that some of them ended up with a sense of personal development, namely, having a more positive attitude regarding new learning situations. As one comments:

“First of all, and speaking for myself, it was a mixture of fun (when we solved some problem) with deep disgust (when the computer decided to ‘play’ at improper hours). But, most of all, it was positive to sweat until we got where we wanted (...) I think that the fact that I attended this course was a lot of fun and taught me things that will stay for all my life.”

2. Project work. The projects and the research carried out in this context emerge out as the most relevant aspects of the course activity. Some preservice teachers were pleased with the opportunity they had to choose the theme for their project and to learn more about it, and clearly emphasize the inquiry process. For instance, one evaluates the work carried out in the following way:

“An interesting work on an interesting theme that is still little known (...) There was a research work at several levels and after that information was collected, it had to be ‘filtered’ and presented in the form of a Web page.”

Resulting from this research, carried out within their projects, some student teachers comment that they started to do Internet searches more often and that they developed a disposition to investigate new software by themselves.

3. Group work. Many student teachers consider group work as a very positive aspect of the course. For some, it increases the quality of the final product: “I think that with my colleagues’ cooperation, with the work that we developed together, the result was a very successful page”. In some cases, there is also a positive reference to the discussions within the group:

“Group work is a fundamental working methodology. Of course, we may have a big mismatch of opinions... However, such mismatches lead to a “discussion” and intensive exchange of opinions until we reach a consensus.”

Besides, group work is regarded as a preparation for professional activity in schools. One student teacher says that his experience in this field will be of great importance in the future “as the collaboration of teachers is indispensable for the evolution (...) of mathematics teaching.”

Development of a professional identity. A third point of interest is the impact of this activity on the development of a professional identity. Student teachers’ responses show aspects of this process, especially as they assume new perspectives and values that they connect to their future professional role.

1. Biographical process. Let us consider the following statements:

“Looking back, I think that I learned so much, so much...”

“All along this semester (...) I abandoned a wrong idea that I had...”

“This course... taught me things that will stay for all my life.”

“[My] opinion (...) enriched a lot, through discovering software and techniques to use it in mathematics classrooms.”

“The use of software in ICM classes helped me to have the notion of the diversity of means that we have available to teach our pupils mathematics contents using new technologies.”

Explicitly or implicitly, these sentences have, a projection of future activities and roles as well as assessments of past ideas and perspectives that are no longer valued. They mark aspects of student teachers’ biographical identity defining processes, involving transactions between inherited and envisioned identities as they reflect about past ideas and conceptions and show appreciation regarding what will be their future work as mathematics teachers.

2. Relational process. In other responses we see influences of relational processes, involving interactions of student teachers with others, including their teachers and other preservice teachers:

“I enjoyed the instructor’s method, always available, and giving us freedom to work.”

“The relation between teacher and student could not have been better. Whenever we needed, the teacher ‘ran’ to help us, requiring, however, that we first try to solve the problem that we faced. The work carried out with Frontpage developed us as researchers.”

“Of course, we may have a big mismatch of opinions [in group work]... However, such mismatches lead to a ‘discussion’ and intensive exchange of opinions until we reach a consensus.”

“The collaboration of [school] teachers is indispensable for the evolution of (...) mathematics teaching.”

This relational process led preservice teachers to appreciate the value of group work, despite all its inherent difficulties, and value the teacher-student relationship as a complex interplay of supporting and challenging. They recognize the need for negotiations involving different people to reach some level of agreement. They also indicate their appreciation of collaboration, an important aspect of mathematics teachers’ professional identity.

Conclusion

Work carried out with ICT based on sophisticated software raises many technical problems that may jeopardize the development of the classes according to the instructor’s plans. This requires much capacity for on the spot decisions. In this course, such problems were common since preservice teachers wanted to include rather complex visual effects in their Web pages. Consequently, the production of pages tended to take more time than they expected. All this strongly suggests the need of careful planning in such a course.

Despite these problems, data collected in this study shows that the experiences provided in this course led student teachers—who often begin their educational preparation with rather negative attitudes towards computers—to develop confidence in their use of ICT. They also developed new perspectives about the use of ICT in mathematics education and an appreciation for working methodologies that foster students’ learning. Both are important aspects of the

professional knowledge necessary for mathematics teaching. Student teachers also took important steps in assuming professional values and attitudes, such as the need to discover and investigate by themselves and the constructive role of discussions and collaboration in undertaking professional tasks.

ICT is not just a simple auxiliary tool. It is an essential technological element that shapes the social environment, including mathematics education. Future teachers need to develop confidence in using this technology and a critical attitude regarding it. They need to be able to integrate ICT within the goals and objectives for mathematics education. The task of preservice teacher programs is not just to help student teachers learn how to use this technology in an instrumental way, but to consider how it fits into the development of their professional knowledge and identity. The design of this course was intended to provide student teachers with deep experiences of working in ICT projects. Other working contexts need to be created, taking into account the myriad features of this expanding technology, especially its potential for long distance interactions and working in a collaborative way.

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REFLECTIONS ON TEACHERS' PRACTICES IN DEALING WITH PUPILS' MATHEMATICAL ERRORS

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Abstract: *In many mathematics classrooms today, the teacher is the ultimate source and authority. Current research suggests that pupils should learn to use sources of information other than the teacher in order to become independent learners. The present study examines the way teachers deal with pupils' errors in secondary mathematics classrooms. The results show that the teacher's dominant attitude does not allow them to deal with pupils' errors other than as anomalies to be eradicated.*

1. Teachers' role in the construction of the mathematical meaning

School mathematics instruction is often ruled by a material-quantitative representation of mathematical knowledge, with the teachers devoting themselves entirely to the service of the subject matter and to the duty of conveying it. They know what the result has to be, and the pupils also know that potential results are already predetermined. As a consequence, knowledge is introduced only by the teacher, who determines the scope of the knowledge taught, as well as what counts as valid mathematical knowledge. Hence, mathematical knowledge cannot take on "autonomous corrective or modifying function within the process of teaching and learning" (Steinbring, 1991).

A number of studies have questioned the basic assumption of the 'transmission model' according to which knowledge can be conveyed by the teacher to the pupils. For example, Glasersfeld (1983) argues that "mathematical knowledge cannot be reduced to a stock of retrievable 'facts'... (it) is constructive and, consequently, is best demonstrated in situations where something new is generated, something that was not already available to the operator". The 'transmission model' entails a view of how form and content are related in mathematics education: the predominant role of the teacher is justified on the basis "of the clear-cut features of mathematical content. This allows them to classify student productions easily as 'false' or 'true' ", whereas

“the dichotomy of ‘true’ and ‘false’ is enforced via ‘telling’ the students the correct ways to perform on content” (Seeger, 1991). Overcoming this model implies a change in the very notion of content, which goes beyond the dualism of ‘true’ and ‘false’. This is contradictory to teaching mathematics as a subject to be created and explored (Cooney, 1988).

The above raise once again the old question about whether mathematical knowledge can be objectively given or only subjectively constructed. Steinbring (1991) argues that “... the special epistemological prerequisites of mathematical knowledge which may have an impeding effect on understanding and which cannot be communicated, conveyed, or methodically evaded by the teacher, require that ... students must be enabled to cope with the knowledge ‘on their own’; (the knowledge) ... does not ‘belong’ exclusively to the teacher” (Steinbring, 1991).

2. On dealing with pupils’ errors in mathematics teaching

Current research on teaching and learning mathematics, independently of the perspective taken, accepts the premise that pupils are not passive absorbers of information, but rather have an active part in the acquisition of knowledge. Furthermore, it emphasises the need for mathematics teaching to be much more than a study of ready-made mathematics, which is still so prevalent. Since learners always construct their knowledge, the critical issue is the nature of the socially and culturally situated constructions (Cobb, 1994). Thus, the teacher’s task is to challenge pupils by introducing effective mathematical activities, and maintain a classroom culture that encourages and facilitates independent learning.

In a traditional teaching approach, the teacher presents the intended outcome, distinguishes ‘right’ from ‘wrong’ and asks the pupils to follow directions, while s/he checks their answers and corrects their mistakes. In a non-traditional approach (e.g. constructivist), the pupils are called to take responsibility for their learning, to think and express their opinion freely. It is the discussion within the classroom and the validation procedure that give rise to the correct answer. For this reason, the teacher provides hints and directions, examines errors rather than judges for accuracy, and the pupils work in groups, and discuss and argue. This process gives rise to a number of questions concerning the role of the teacher; the mode of intervention and its appropriateness; the questions s/he poses and the answers s/he provides, and the way s/he organises the results of the pupils’ work (Jaworski, 1994). This has a number of consequences for teacher training which are mainly concerned with (a) how to guide experienced teachers towards new practices, without allowing

'old habits' to invalidate the new approaches and (b) how to ensure that the new practices are not invalidated and, at the same time, lead to desirable results. With regard to the first issue, a number of approaches have been suggested, e.g. socio-constructivist (Cobb, Wood & Yackel, 1993) while, in relation to the second, a number of criteria for the investigation of teaching practices have been put forward, e.g. Jaworski's (1994) 'teaching triad' (management of learning, sensitivity to students and mathematical change) or Frank et al's (1997) examination of classroom practices (providing opportunities for children to solve mathematical problems, and listening to children's mathematical thinking for making decisions about instruction).

We believe that in examining teaching practices in mathematics, the validation procedures used and the management of the pupils' errors by the teacher are of great importance. This is because the validation procedures are related to the widely accepted view that the individual construction of mathematical meaning takes place within the classroom while, at the same time, it contributes to the constitution of the classroom's mathematical culture. Therefore, in elaborating a mathematical meaning, "it is not acceptable to attribute the authorship...to anyone in particular...the meaning is elaborated through negotiation" (Cobb & Bauersfeld, 1995). As a consequence, when the teacher imposes an answer or a meaning, the individual and cultural constructions are torpedoed. On the other hand, the management of the pupils' errors (i.e., mathematically false outcomes) by the teacher is significant as it might allow him/her to identify their possible source and proceed to suitable interventions so that the students recognise them. This realisation by the learners is absolutely crucial in the process of the development and reorganisation of their mathematical knowledge.

Research findings support the crucial role of the validation procedures and management of pupils' errors in the teaching and learning of mathematics. Thus, it has been found that pupils often accept the results of a mathematical activity without evaluating them, especially when there are no indicators about it or the result 'looks' satisfactory and close to their expectations (Bellemain and Caponi, 1992). Furthermore, they sometimes misinterpret external indicators and adapt them to their existing system of knowledge as this requires less effort; this is analogous to the effort needed to accept a ready-made piece of knowledge (Brousseau, 1997). They also frequently attempt to conform to the teacher's expectations or they look for an answer or a sign from him/her indicating the expected response. Jaworski (1994) reports the following characteristic episode:

Pupil. *Why don't you say one and tell us to do it?... We are going to be here all day...*

Teacher. *That's passing responsibility onto me and ...*

Pupil. *Does it really matter? You are the teacher, aren't you?*

Both the validation procedures used and the management of the pupils' errors by the teacher are closely related to the notion of the transfer of control from the teacher to the pupil. This notion is a decisive factor in the development of autonomy and substantial thinking by the children as it allows classrooms to become less judgmental and shifts responsibility for making sensible contributions to the children. Mathematical reasoning and thinking from this point of view allows personal sense making rather than depending upon teachers and textbooks. We thus believe that this element is crucial and we focus on it in the investigation of teaching practices.

In the Greek educational system, teaching mathematics is rather traditional. All Greek state schools (primary and secondary) use the same textbooks, distributed free of charge by the Ministry of Education. These textbooks were first published at the beginning of the 1980s and there have been a very small number of minor changes since then. The secondary mathematics teachers are graduates of Mathematics Departments (four year courses, with great focus on the discipline of mathematics with few, if any, subjects concerning education) and they have limited in-service training.

In earlier studies (e.g. Kaldrimidou, Sakonidis & Tzekaki, 2000) we examined how the epistemological features of mathematics emerge in both the primary and the secondary Greek classroom, as well as in different branches of mathematics, i.e. algebra and geometry. The results showed that, in all cases, teachers, independently of the teaching practice followed, dealt with these features in a unified manner. We argued that this could not be attributed to either the teachers' subject knowledge or to the nature of the mathematical knowledge itself. A possible explanation could be sought in their interpretation of their leading role in teaching mathematics, as well as in their attempts to manage mathematical meanings through means that are accessible and comprehensible to the pupils. In this dominant role which teachers attribute to themselves, in both traditional and non-traditional teaching approaches, the way they deal with pupils' errors and manage control appears to be, as argued above, crucial. The study described below focuses on these two aspects of mathematics teaching.

3. The study

The data reported here come from a large project¹ that focuses on the teaching of mathematics in the nine years of the Greek compulsory educational system (6 – 15 year olds). The study aims to investigate the possibility of applying

¹The project is financed with resources from the operational plan "Education and Initial Vocational Training" of the 2nd Community Framework Support, European Commission, European Social Fund, Directorate General V.

alternative, pupil-centered mathematics teaching approaches in the Greek school. However, for the purpose of the present study, only the data concerning the three years of the gymnasium (12 – 15 year olds) are considered.

The research problem addressed here focuses on teachers' practices in dealing with the pupils' errors and the validation procedures in the teaching of mathematics. This, as argued above, determines to a great extent the degree to which the teachers allow the children to take control.

The data collected consists of 22 mathematics lessons (from 11 teachers) observed in various classes of the three gymnasium grades for over a month in the northern part of Greece. For each teacher, two 45 minute sessions on different topics, most of them algebraic, were observed; these were then videotaped and transcribed. Teachers were strongly advised to "work as usual". In addition, in order to acquire a wider perspective of teachers' ideas about teaching mathematics, a questionnaire was constructed and answered by a sample of 231 teachers from the same area. There were two items in the questionnaire concerning pupils' errors: "why do pupils make mistakes?" (item 28) and "how do you deal with them?" (item 29).

4. Presentation of the data

The analysis of the answers to the two (open) questions showed that 98.5% of the sample attributes pupils' errors to a lack of knowledge, e.g. "...some children cannot grasp certain mathematical ideas"; "...they do not study enough"; "sometimes, pupils cannot remember the right mathematical knowledge needed to respond correctly". In trying to deal with these errors, 50.3% of the subjects repeat or remind, 39.2% provide some examples and 5.4% do something else (e.g. anticipate or adjust the lesson accordingly). These results suggest that teachers use pupils' errors as a tool for diagnosing learning difficulties and attempt their eradication mostly through direct methods, such as repetition or applications.

The transcripts of the mathematics lessons were analysed in relation to the treatment of the pupils' errors by the teacher in two phases: before the error was made and after. For each of the two phases, four categories of treatment were identified from the data, which are as follows:

Before the error was made (BE): warning about the possibility of an error (1), intervention concerning morphological or procedural remarks (in order for the error to be avoided) (2); drawing attention to the error through questioning

(in order to be realised by the pupils themselves and avoided) (3); dictation of the answer or provision of clear indications (in order to be avoided) (4).

After the error was made (AE): rejection of an error and correction by the teacher without explanation (1); correction by the teacher with explanation (2); drawing attention to the error and correction either by the teacher, or the pupil him/herself or another pupil (3), and finally questioning and putting forward criteria for the location of the error (4).

Below, two characteristic examples for each of the categories of each phase are reported in order to provide an overall view of the management of the pupils' errors by the teachers in the sample.

Before the error was made (BE):

A. Warning about the possibility of an error (BE1)

Example BE1.1 (The class is trying to simplify an algebraic expression)

T(eacher). *In the worksheet I gave you, I have included a case where ... there is a minus in front of the expression (describes). What will I do?*

P(upil). $-2a$

T. $-a-1$ or, if I don't want to change it immediately $-(a+1)$. *You should be very careful. This is where most of the errors appear. Solve as many equations as you can, starting from the simplest ones.*

Example BE1.2

T. *It is about one fraction, then. It is obvious that these two fractions are united. It is a good idea to first put them together and then start cancellations.*

B. Intervention concerning morphological or procedural remarks (BE2)

Example BE2.1 (The pupils attempt to simplify an algebraic expression)

T. *Be careful, we need to multiply here, wait! Yes, but when we multiply.. put a dot here .. when we multiply, children, an algebraic expression by a term, where shall we put this expression?*

P. *In brackets*

T. *In brackets. That is, the x^2y^2 in brackets.. Good, ... let's see what Alexandros is going to write now. Well done! You put it in brackets and we now carry on as we know.*

Example BE2.2 (The teacher asks the children to decide which angles are equal in a figure with two parallel lines intersected by another line)

P. *When two parallel lines are intersected by ...*

T. *No, shall we say it in a simpler way? We decide which angles are acute and which are obtuse and then we know which ones are equal.*

C. Drawing attention to the error through questioning (BE3)

Example BE3.1 (The pupils are trying to find the values of a in order for the denominator to be $\neq 0$)

P. $a^2 - 1 = 0 \Rightarrow a^2 = 1$

T. *This is one of the ways. Are we sure that we will not lose the root? In the denominator? And what do we say then?*

P. *Will we say $a=0$?*

T. *Square root of 1. Thus, we get the 1. How will we get the -1?*

P. *We don't get the -1, because -1 times -1 equals plus ...*

T. *We are fishing in unclear waters. Any safer way?*

P. *Shall we put plus?*

T. *No. The $a^2 - 1$ can be written as $a^2 - 1^2$. Does this expression remind you of anything?*

P (upils). *Difference of squares*

Example BE3.2 (The children are trying to factorise an algebraic expression)

T. *Ah! Thomas has a good idea. $a^4 + b^4$. Do you agree? How could we write it?*

P. ??

T. *If it was $a \times b$ and all squared, then it would have been $a^4 \times b^4$. Does this hold in our case too? For which operations are the properties of the powers true?*

P (another). *Multiplication and division.*

T. *Multiplication and division, not addition... Thus, we cannot apply the above property here.*

D. Dictation of the answer or provision of clear indications (BE4)

For this category, the teacher often tells the student(s) exactly what to do, e.g. “take minus out of the bracket, .. write now .. ‘minus, bracket..’”. In other occasions, the teacher points out to what the children should pay attention to. Furthermore, linguistic interaction is very frequently replaced by nuances in voice and gesture, which were detected in the videotaped lessons, e.g. the teacher gestures negatively when the pupil hesitantly starts writing or saying something that is in the wrong direction.

After the error was made (AE):

A. Rejection of an error and correction without explanation (AE1)

Example AE1.1 (The pupils work with a problem of factorisation)

P. *Madam, in $x^2 - 2x$, if we write x times x equals $2x$? The x is cancelled and then $x=2$.*

T. *Be careful! Which x 's are going? ... priority of operations ... first we multiply.*

P. *Madam, we will do $x^2 = 2x$... $x * x = 2x$*

T. *But you have a root! It is not allowed! All right? You lose a root. Don't do this kind of cancellation, because you lose roots. All right? However, when we take out the common factor, we don't lose the root.*

B. Correction by the teacher with explanation (AE2)

Example AE2.1 (The class works on solving equations with denominators)

T. *Different from zero. Which is the denominator before cancellation and after? The factorised denominator. Good! And what do we conclude from that? x is different from what?*

P. *From zero.*

T. *No! watch, $x-3 \neq 0$. This is why. You are right! We haven't done this before. Write down clearly: $x-3 \neq 0$ implies $x \neq 3$ and $x-5 \neq 0$ implies $x \neq 5$. You are right. I haven't shown you a similar example before.*

Example AE2.2 (The class attempts to solve an equation)

T. *It is, thus $x=2$ is a solution. But even if we substitute x by -2 , is it again true?*

P(pupils). *Yes*

T. *It is again true. Therefore, we shouldn't lose a solution. We must write $\pm\sqrt{4}$*

P (Kostas) *When we say $x^2=4$, isn't it $x^2 = 2^2$? Therefore, since the two exponents are equal, isn't it the case that their bases must be equal?*

T. *Well, look, you will learn in the lyceum ... (explains). Our problem here is to solve the equation. What do we notice? Both 2 and -2 squared make 4. Thus, both these values make the equation true. Therefore, we shouldn't lose -2. From now on, we will always write it this way... In geometry it was not necessary to use both signs. Here it is. Because we have got the root. When we factorise, it becomes clear which is the one solution and which is the other. When we use this way, it's not clear. So, be careful, don't get carried away because you will lose a root. That is, the negative root.*

C. Drawing attention to the error -correction by one of the classroom partners (AE3)

Example AE3.1 (The error is corrected by the teacher)

T. *Hmm, haven't you missed something outside of the square root?*

P. *Yes*

T. *Ah, well done! \pm , otherwise we lose a root. Now, finish with it. How many solutions does this equation have?*

Example AE3.2 (The correction is made by the pupil)

T. *Which one? Lambrini says that we should cancel out 3 with 6 and x with x. Olga?*

P. *You cannot cancel because they have not been factorised.*

T. *They must have been factorised, eh? We haven't learnt about this type of factorisation.*

Example AE3.3 (The correction is made by another child)

T. *You suggest a separation of terms. We can't do this because both terms are unknown. George expressed his opinion. Anyone else?*

D. Questioning and putting forward criteria for the location of the error (AE4)

Example AE4.1 (The children need to decide about the value of a , for the denominator to $\neq 0$)

T. *Good! How did you understand it?*

P. *1^2 is 1*

T. *1^2 is 1. Well done, $a \neq 1$. Is there anyone who wants to add something?*

P (another). *...and $a \neq -1$*

T. *Great! How did you think of it?*

P. *Because $(-1)^2$ becomes $+1$*

T. *Right, now ... this was a bit like fishing in unclear waters without knowing. Is there any safer way to find which values ... in the denominator?*

Example AE4.2

T. *Now, we are going to do it straightforwardly ... Do you like what Kostas wrote?*

P. *No*

T. *What is the problem?*

P. *The brackets ...*

T. *Well done Kostas! Brackets and now?*

5. Discussion and conclusions

The analysis of the episodes above shows that teachers keep for themselves the control over errors by warning and directing pupils or making the corrections themselves. That is, they believe that errors are something to be avoided (they often use strong terms as '*not allowed*' with a strong emphasis in their voice, '*lose*', '*safer way*', etc). In this context, they often do not pay attention to pupils' contributions, thus missing opportunities for a fruitful interaction in the construction of mathematical meaning. As a result, teachers invariably use morphological or procedural rather than conceptual elements for the elaboration of mathematical meaning. This practice allows them to keep intact their leading role in the construction of mathematical knowledge.

An interesting question raised concerns whether teachers show the same attitude in different teaching paradigms. Examining other teaching approaches (e.g. problem-solving, activity-based classes) similar tensions from teachers' attempts to keep control of the pupils' outcomes can be identified. This appears in Jaworski's (1994) attempt to introduce a constructivist approach to teachers "(there was) a tension coming from a desire for students to discover particular mathematical facts, and the reluctance of the teachers to tell these facts when it seemed that the students were not going to discover these...". Even when the teacher was trying to pass responsibility onto the pupils, s/he emphasised the word 'accurate'. In a similar way, Arsac et al (1992), during an experiment in which they were trying to introduce a debate among children about their solutions, recorded one teacher-intervention per minute during an 80 minute session, despite the fact that the teacher was assigned to only present the problem and simply regulate the discussion. Moreover, the interventions were directed to "some special word or meaning that maintained the students' dependence and the idea that the teacher is responsible for the validity of their answers" (Arsac et al., 1992).

The view that learning mathematics is a personal construction of meaning, shaped by context, purpose and social interaction, challenges the teacher's role as the ultimate source of knowledge and truth. It assumes a shift of control and authority from teacher to pupils. This implies that pupils' errors should be considered as a natural force for promoting questions, reflection and exploration on the part of the children themselves, rather than as an anomaly to be eradicated. Such a change in teacher attitude, although crucial to both teaching and teacher education, seems difficult and complex to achieve. We believe that a deeper and more systematic study of the related issues is required.

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**ABSTRACTS OF ADDITIONAL PAPERS
LINKED WITH WORKING GROUP 3**

SIMULATORS IN MATHEMATICS TEACHER EDUCATION

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There are serious difficulties that have to be overcome during preservice teacher education. The point is that students perceive the task of teaching as revisiting classroom settings they themselves experienced; their beliefs about teaching are well established by the time students go to the university, and, moreover, such beliefs act as a filter through which new information on teaching is sifted. It is highly important that initial teaching education programs are actually unsuccessful in bringing about beliefs changes unless teachers use a new approach which they subsequently find successful. What we presented cannot be treated as a result of study; we simply wanted to discuss an interesting phenomenon we have observed, which, to our opinion, leads to important conclusions and, furthermore, indicates a field of research. An idea of “simulator courses” in mathematics teacher education aiming in shaping and developing teacher knowledge is suggested. The point is that a teacher must be accustomed to cope with situations of uncertainty. Every teacher must have an experience in giving right-minded answers to questions that, at first glance, seem to be meaningless or foolish, moreover, he or she must be able to respond instantaneously. Examples of the discussion in a classroom based on the so-called “silly-and-wise” questions are given and possible outputs are indicated. Connections with the notion of the substantial learning environment and the four-dimensional model of teachers’ professional practice are established.

DEVELOPMENT OF CRITICAL THINKING

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This research concerns the results of the International Project on development critical thinking and active learning (“Reading and Writing for Critical Thinking”) that we have realized in 2 years term. We carried out special courses for undergraduate students of «mathematics-computer science» and workshops

for in-service teachers. The Ukrainian “face” of the project realization (democratic transformations in education and person/child centered study) includes the best traditions of Soviet pedagogical school. The peculiarities of a new system of education lies in the fact that it should not only give knowledge, but also form the need for its constant and independent acquisition, to mould skills and habits of self-education as well as the independent and creative approach towards knowledge during whole active life. It means that the level of a personality’s social defense is changing now. Only a widely educated person is able to flexibly restructure the direction and essence of his/her activity. The main purpose of this collaboration is to bring to classroom effective methods of instruction, which promote ideal thinking among students of all ages and across all course content. In higher education we create new courses that can be used for teacher education as well as for freshmen students any faculty, besides it is a content and a tool of discussion and inquiry-based courses.

WORKING GROUP 4

Social interactions in mathematical learning situations

Group leaders:
Götz Krummheuer
Gérard Sensevy

INTRODUCTION TO WG4

THE COMPARATIVE ANALYSIS IN INTERPRETATIVE CLASSROOM RESEARCH IN MATHEMATICS EDUCATION

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Abstract: *Qualitative classroom research in mathematics education is often criticized because of its poor and unreflected methodological standards. Especially with regard to the attempt of the empirically grounded development of theory it will be shown that the comparison of interpretations of different classroom episodes is essential for a methodologically sophisticated research standard.*

I really haven't thought too much about actual topics, but I do know that I will do a case study (Kilpatrick 1999, p. 57).

The epigram chosen from Kilpatrick's plenary address for CERME 1 is part of a harsh criticism of the recent practice of qualitative research in mathematics education. One of his arguments is the poor scientific standard as exemplified above in a quote of a doctoral student about his/her research intentions. He finishes his comments with a call for more quantitative research. I agree with him that often theoretically and methodologically unelaborated research projects try to justify their work by referring to the grounded theory approach or similar approaches based on the qualitative research paradigm, and will not refer to any example of this project.

However, I think that there is a need for qualitative research in mathematics education. Unfortunately methodological considerations are mostly quite underdeveloped. Obviously, the pure attraction of a qualitative research approach or the supposed school related relevance of the research questions often leads to an underestimation of methodological problems, thus dismissing them as an aspect of minor importance (see Bruner 1990; Krummheuer & Naujok 1999). Qualitative research also requires an elaborated discussion of its methodological basis, especially with regard to common sense about empirical research among the rather “quantitatively” thinking majority of classroom researchers.

My interest in this introduction is to stress the methodical principle of comparison which in my view is central for qualitative research. I am not intending to lead a meta-theoretical debate about the pros and cons of qualitative in comparison with quantitative methods. Although the following considerations are the result of methodological reflections within a concrete research project¹, I am trying to describe rather general issues according to the qualitative classroom paradigm.

Characteristic of interpretative classroom research is a "praxeological" understanding of methods, which in the sense of Bohnsack 1996 describes the applied research practice as "reconstructive" in two respects:

- The analysis of the observed classroom episodes is oriented towards the reconstruction of the processes of interaction among the students and the teacher, which is geared to theories of interaction, argumentation and participation.
- The steps of analysis, which are employed in this action usually have been partly developed throughout the concrete research work. The reflection of this method of analysis and evaluation is based on the second use of the term "reconstructive social science".

(for both aspects see Bohnsack 1993, p. 8). In the following I mainly would like to deal with the second aspect, that is to reconstruct the research practice of a concrete project and to reflect these activities from a methodological point of view. In the following I formulate these activities in a rather general way: It will be explained that as a research scientist dealing with the domain of social interaction in primary education, one is usually not in the position to cope with an a-priori stock of theories, which are sufficiently developed in order to adequately understand a certain classroom situation. In such cases the researcher is facing a specific methodological problem: the necessity for *constructing* elements of a theory that claim to generate a theory-consistent interpretation of the *selected* part of reality. The issue of plausibly generating novel elements of a theory takes priority over the valid rejection of a theory. This approach will be methodologically reinforced in the second section by bringing out the "experimental" character of comparative analysis.

¹ The project had been supported by the German Science Foundation (DFG) for three years. The final report is published by Krummheuer, Götz und Brandt, Birgit, 2001 .

1. The unavoidability of theory construction

It is widely accepted that research in social sciences in contrast to natural sciences deals with domains, which are distinguished by autonomous acts of interpretation of the actors:

The intellectual objects, which are created by social scientists, refer to and are based on intellectual objects, which are created by people who are, in the sense of a everyday life, living among their fellow human beings (Bohnsack 1993, p. 23; translation by Götz Krummheuer).

Central concepts of work in social science are therefore those of “meaning” and of “negotiation of meaning” (see Bruner 1990, p. 1 - 10). Meanings emerge from the definitions of the situation, from the subjectively and intentionally shaped attributions of sense by the persons involved. Furthermore they are coordinated through interactive negotiations between the involved persons. Interpretative research aims at understanding these individual meanings, as well as their interactive generation- and clarification-processes.

Within the established practices of interpretation by the persons involved, it is possible that *permanently novel* interpretations and patterns of interpretation are brought up, which for their part again define new social ways of living. The setting up of a theory in these cases cannot take place in the sense of the hypothetico-deductive model *before* the empirical phase. It can only be performed *during* the process of empirical analysis. From the perspective of the "interactive paradigm"

... the research on ways of social living can only succeed with the help of methods, which allow the discovery of the point of view and of patterns of interpretation of the observed without having formulated exactly specified and precisely operationalized hypothesis in advance (Kelle 1994, p. 54; translation by Götz Krummheuer).

Processes of interaction in today's primary education are of such a kind:

- From the point of view of developmental psychology, children of grade school age are in a phase in which they, qualitatively speaking, still go through structural changes in their ways of interpretation. Even though one is usually able to describe the changes in the child's ways of interpretation with the application of corresponding cognitive models of development, a prediction which is specific to both the individual and the situation in the context of classroom observation is usually not possible.
- With regard to subject matter primary students in their role as learners do not yet (per definitionem) have certain ways of interpretation shaped by structures of the subject matter. They still have to be developed and

according to all of what is known about these processes, they do not linearly proceed in ratified ways of interpretation. On the contrary many winding and diversified learning paths are traveled which underlie unique change.

- Historically speaking we are in a phase of rapid and on-going change in culturally defined patterns of interpretation. With regard to the persons involved these changes continually lead to new ways of interpretation and patterns of action which therefore cannot sufficiently be logically deducted from already existing theories. The further multicultural differentiation of our societies plays an important role at this point. With regard to the everyday classroom situation with a multi-ethnic student population this leads to an almost complete loss of stable, prognostically applicable patterns of interpretation during the course of the lesson.

Frequently in debates about this "special way" of the social sciences, the phenomenon of the impossibility of an a-priori creation of hypotheses is seen as a general characterization of such research. As Kelle 1994 convincingly demonstrates this is, however, only valid for domains with permanently changing definitions of the situation, for:

As long as definitions of the situation of the persons involved consist of culturally stable patterns of interpretations, [...], and as long as the sociological researcher for himself has access to these patterns (of interpretation), causal models of the structure [of the social reality] for the explanation of social action [...] can be formulated with the help of a hypothetico-deductive model, that is before contacting the empirical field (ebenda p. 16; translation by Götz Krummheuer).

In the following, however, I will focus on the research practice of domains in which "culturally stable patterns of interpretations" are not available. In these cases the logical basis of the construction of a theory can be configured in the context of interpretative classroom research with the help of the inference of "abduction", as it has been developed by Peirce 1978 and Hanson 1970. Regarding abduction the following conclusion is drawn:

The surprising fact, C, is observed;

But if A were true, C would be a matter of course,

Hence, there is reason to suspect that A is true (Peirce 1978, 5.189).

Solely the fact that an observed phenomenon is regarded or described as surprising, points out the need for the development of a theory. Current theories would have expected something else or wouldn't have made a substantial explanation possible at all. By the formulation of an adequate theoretical alternative it becomes possible to regard the observed phenomenon as less of a

surprise, that is to explain it. Therefore Peirce characterizes the result of abduction as an "explanatory hypothesis" (ebenda, 5.189).

The abductive conclusion is logical and has a definite form (ebenda, 5.188). The established hypothesis is an ex-post-facto-hypothesis, which gains its plausibility from the fact that it is based on empirical analysis, as well as on the reflection of relevant theoretical pre-knowledge. Hence, it does not only spring from creative intuition and brilliant speculation. In case that these hypotheses are successful such conclusions lead to the formulation of empirically saturated theoretical explanations of the observed phenomena (vgl. Kelle 1994, p. 22). Abduction is different from induction as usually applied in quantitative research.

Logically speaking it is always possible to construct a multitude of alternative rules A to the surprising phenomenon C, so that C is a matter of course of these A's. From a methodical point of view the question about criteria for a rational practice of choice between alternative explanatory hypothesis arises. Heuristics for an empirically based construction of a theory and an evaluation of its plausibility are needed. The term "heuristics" should stress the fact that there is no schematism or algorithm for generating a hypothesis. The construction of a theory in accordance with the abductive research logic is based on a theoretical pre-knowledge and a "local methodology of discovery" (ebenda, p. 361).

In the *comparative* analysis such a local methodology of discovery can be seen. By comparing interpretations of different episodes on the one hand certain constructions of theory can be ruled out in case they do not match the interpretations. On the other hand such a comparison gives direction to a novel theoretical construction, as a confrontation of the initially employed theories shows their deficits. Here, the efforts of generating a theory aimed at overcoming these deficits are more essential (see Strauss & Corbin 1990, p. 176 ff).

In principle such a heuristic generation of theory is speculative in part. However, on the basis of comparative analysis, empirically controlled and theoretically oriented conditions of such abductively created theoretical elements are produced. In comparison to induction and deduction, abduction is a relatively weak conclusion which bears the danger "of extremely fallible insight" (Peirce 1978, 5.181(3)). Historically, however, according to Peirce, abduction is the type of inference by which most of our current solidly grounded theories are developed.

2. The experimental character of comparative analysis

In the context of interpretative methods, comparative analysis represents a central activity. Already the Chicago school emphasized this point of view and proceeded accordingly while neglecting a systematical methodological review at the same time (see Bohnsack 1996). Glaser & Strauss 1967 re-emphasized this point and speak of a "Constant Comparative Method of Qualitative Analysis" (p. 101). As it especially becomes clear in Strauss & Corbin 1990, the comparison of interpretations of different observed parts of reality represents a main activity on nearly every level of analysis: from the first interpreting approach to the later more theoretical reflection (p. 176 ff). By means of comparison two things can be made possible and controlled:

- the estimation of the ability to generalize the claim of the developed theory and
- the documentation of the complexity of the reality, which could be made understandable through the corresponding development of a theory.

The first aspect deals with the validity of the "conceptual representativeness", a description from Strauss & Corbin 1990. They want to make clear that in qualitative research the aim is to find the representativeness of the developed theoretical concepts within the *interpretations* of the selected parts of reality, unlike quantitative research, which aims at representativeness on the level of the *sampling* (p. 190 f). If the generated new theoretical concepts convey a sufficient understanding of (strongly) differing parts of reality, a derivation of a relatively global claim is possible based on this contrasting empirical development of the theory:

.. there will be wider applicability of the theory, because more and different sets of conditions affecting phenomena are uncovered (Strauss & Corbin 1990, p. 190 f).

The second aspect emphasizes the fact that comparative procedures aim at grasping the selected part of reality by their specificity. These efforts strive to theoretically describe the unique conditions of the case in the most exhausting way (s. ebenda p. 191). This is only tentatively possible, due to pragmatic reasons. From a methodological point of view the comparison of interpretations of different parts of reality however represents a possibility to identify the specific characteristics of these particular cases in *relation* to each other (see Bohnsack 1989, p. 135 ff and Glaser & Strauss 1967, p. 35).

Thus, interpretations of episodes which *contrast* strongly with each other are especially promising. Such an analysis of contrasting episodes provides the

conditions for the possibility of generating an ex-post-facto hypothesis in which its generation and its proving coincide (s. Kelle 1994, S. 367).

Studies, which are performed in the sense of a comparative analyses do not represent case studies in the usual sense. Several cases are rather analyzed and compared to each other in a reconstructive interpretative way. This point of view is easily disregarded in the usual debate about quantitative versus qualitative research. If one follows the historical delineation of Hammersley 1989, this restricted understanding of qualitative research as a "performing of case studies" has already infiltrated the methodological controversies of the 50's in the USA.

As indicated in Krummheuer 2000 several qualitative research projects in mathematics education can be subsumed under this approach of comparison according to their methodological basis. Also the following papers of Working Group 4 will show the strength of comparative analyses and the flexibility of this methodological approach with regard to specific research questions as well.

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SOME NUMBERS ARE STRAIGHT AND SOME ARE ROUND: CONSIDERING MEANING AND FOCUS IN CLASSROOM TALK

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***Abstract:** This paper will explore ideas about meaning and focus through detailed analysis of the transcript of a lesson in an elementary mathematics classroom. This analysis will consider Anna Sfard's (2000) ideas about focus in relation to examples taken from the lesson. Sfard's analysis is powerful and has a strongly reasoned conceptual basis. In her paper she uses her concept of discursive focus to explore a lesson on statistics with seventh grade American students. It is my suggestion that her analysis is even more significant when applied to the teaching of basic mathematical topics to young children and it is from such a setting that my examples are taken.*

Meaning

The commonsense, folklore notion that words are containers holding meaning inside them so that the meaning of each word is clear and unambiguous seems to me to be misguided. I would suggest that meaning is instead constructed by the participants in an interaction through their interpretations of each other's contributions. The meaning of the talk of the classroom is bound up with what is going on in the lesson, other people's contributions and the task at hand. During the course of the lesson the teacher is involved in strategies that help to interpret the meaning of different participants' utterances. I would consider meaning to be developed out of interaction rather than intrinsic to particular words or phrases:

‘the meaning of an utterance is considered, not as a property of the utterance itself, but rather as a relation, called a ‘*refers-to*’ relation, between the situation in which someone makes an utterance and a situation to which the utterance is interpreted as referring’. (Barwise and Perry 1983)

In this quotation Barwise draws attention to the speaker and the situation to which the speaker is referring but leaves the participants' interpretations as implicit rather than overt.

Thinking and Communicating

Anna Sfard (2000) suggests that thinking can be viewed as communication with oneself. She describes this as a development of Vygotsky's belief in the developmental priority of communicative public speech over inner private speech. This in turn leads to the realisation that the better one understands public discourse, the better one will understand dialogue with oneself. This implies that the investigation of communication between people is the best route to discovering the mechanisms of human thinking.

The idea that communication has primacy over thinking has important consequences for investigations into mathematical thinking. As Sfard says:

Instead of being merely helpful in constructing and sharing the knowledge of preexisting mathematical objects, communication and its demands must now be regarded as the primary cause for their existence. (p.297) (2000)

Her work then is organised around the idea of thinking as communicating but she is also well aware of 'the process-object duality of mathematical concepts' (p.298). Even though we may not believe that the world of mathematics is made of Platonic mathematical objects, we may act as though they do just because it works to do so.

Sfard then goes on to explore what communication is. She is not prepared to accept the idea of communication as the exchange of meanings, ideas, feelings and so on because she is wary of the tendency to reify these terms and treat them as essentially separate from the people who experience them. Instead she adopts Wittgenstein's focus on the 'language-game' and the meanings, ideas and feelings as interpretation. She defines communication as:

An activity in which one is trying to make his or her interlocutor act or feel in a certain way. (p.300) (Sfard 2000)

I would also suggest that the speaker may also be trying to make her listener think in a certain way. In her paper, and this one as well, the principal form of communication under consideration is verbal and the activity expected from the listener will be giving a verbal response to an utterance, although it might also be engaging in an activity or behaving in a specific way. The effectiveness of the communication will be assessed by the participants by comparing the listener's actual response from the one that was expected. In this evaluation an observer is likely to make just as valid an assessment of effectiveness as one of the participants but the proviso to any assessment of effectiveness is that the

perspective from which it is made must be made explicit and the assessment must be regarded as provisional.

Discursive Focus

It is at this point that Sfard's analysis becomes powerful and useful as a tool to analyse classroom talk. She points out that talk is always about something and for it to be effective it is essential that the participants are talking about the same thing. This 'talking about the same thing' can be termed, she suggests, the clarity of the discursive focus. As Sfard says:

The communication will not be regarded as effective unless, at any given moment, all the participants seem to know what they are talking about and feel confident that all the parties involved refer to the same things when using the same words. (p.303) (2000)

Her elucidation of the idea of *discursive focus* is very helpful. She suggests that, although discursive focus seems intuitively clear, there can be a discrepancy between the words that we use to identify the subject of our attention and what we are attending to, looking at or listening to, when speaking. For this reason she separates the discursive focus into components. The first two of these are the *pronounced focus* which is what we say, the words we use, and the *attended focus*, which is what we are looking at or attending to.

The third component to the discursive focus is less tangible than the other two. Sfard calls it the *intended focus* and suggests that it is the speaker's interpretation of the *pronounced* and *attended foci*. It involves that whole cluster of experiences evoked by the attended and pronounced foci as well as all the statements the speaker would be able to make about the focus in question.

These three aspects of focus deal with different degrees of public and private: the pronounced focus is publicly spoken, the attended focus may be explicit but usually remains hinted at and the intended focus is mainly private.

Sfard starts with a consideration of effectiveness and it is this that she is concerned with in developing her analysis. She suggests that there are occasions when there is a good fit between the pronounced and attended foci but ineffective communication or the opposite but that the effectiveness of communication depends primarily on the intended focus. As this is usually kept private, the effectiveness of the communication can only be assessed through interpretation. In this interpretation Sfard suggest that it would be wise to use phrases such as 'compatible in the eyes of the interpreter' or compatible rather

than to suggest that the focus for two or more participants is ‘the same’. I would suggest that it may be helpful to use the phrase ‘interpreted focus’ to describe the listener’s interpretation of the discursive focus.

Sfard also considers how focus might function in school mathematics. She suggests that the lack of readily available attended foci in many mathematics topics tends to make mathematics more difficult and inaccessible. This difficulty is further compounded by the fact that the pupils have little if any experience outside school of the objects with which school mathematics is concerned. As she says:

The scarcity of perceptual mediation in mathematical discourse may be a principal reason many people find mathematics prohibitively difficult, almost inaccessible. The students’ task is further complicated by the fact that most of the mathematical objects discussed at school, instead of being known in advance and tightly related to children’s former experiences, are built through the discursive activity itself. (p.308) (Sfard 2000)

I would suggest that this is particularly important in considering how children develop understanding of some of the most basic mathematical concepts and procedures. When Sfard refers to the ‘*scarcity of perceptual mediation*’ she seems to be referring to mediation that is not oral.

Using this idea of discursive focus, I would like to define the meaning of an utterance as the individual’s interpretation of the discursive focus of the utterance. This suggests that there are a number of meanings of any given utterance as there are a number of speakers and listeners involved in the talk of the classroom. The effectiveness of the communication will depend on the compatibility between the different interpretations of the discursive foci. This begs the question that since we can never see an interpretation how can we judge the compatibility of different interpretations? The only possible judgements that can be made are on the basis of actions and speech by the participants.

Counting On: The Lesson

I will now consider Sfard’s ideas about focus in relation to the talk of one mathematics lesson about ‘Counting On’. The lesson was with a class of 25 children aged 5-7 years of mixed ability, sex, race and social background. The class were used to my presence as a participant observer as I spent one lesson a week with them over one academic year. The lesson was typical of this class’

lessons with this teacher and had not been devised especially for the purpose of the recording. The teacher was not confident with mathematics as a subject and welcomed me for the advice and support she felt that I could give her. She was in her second year of teaching at the time of the recording. The data is taken from a transcript of the lesson taken from an audio recording.

The excerpts are taken from the initial phase of the lesson when all the children were sitting on the floor at the front of the room and the teacher was talking to the whole class. She had fixed a strip of paper to the wall measuring 30cm by 2m and began the lesson by writing the numbers from 1 to 20 from left to right on this paper. This gave a number track although there were no divisions between the numbers. The track looked like this, although on a larger scale:

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

The activity of writing the numbers on the paper strip linked back to work done in a previous lesson that aimed to get the children to draw the numerals accurately getting seven the 'right way round' and so on. The lesson then went on to consider 'Counting On' problems using the number track as an aid.

The first excerpt begins as the teacher was about to write the number ten on the track.

Example 1:

- T: That's right. Number 9 (*teacher writes '9'*) and number 10, two numbers or one number? (*teacher pauses and looks at pupils*)
- Sev. Ch: Two
- T: What number do we need to go first?
- 40 Sev. Ch?: 1 (*teacher writes '1'*)
- T: and zero (*teacher writes '0'*) Fred's group, which bit is the zero? (*teacher points to '0'*)
- Ch: Units
- T: Units. No units (*teacher points to '0'*) and 1 ten (*teacher points to '1' in '10'*). Right.
- T & Ch: 11, (*teacher writes '11'*) 12, (*teacher writes '12'*) 13, 14, 15, 16, 17, 18, 19, 20 (*counting together while teacher writes on board*)
- T: So. Right. Let's think of a number between zero and 20. Put your hands down, I'll choose someone. Andrew would you like to think of a number for us?
- Andrew: Two numbers between zero and 20?

50 T: No just one number
Andrew: 10

The first anomaly occurs in the opening question: two numbers or one number? The pupils' first task in interpreting this is to consider what the attended focus is: does the teacher mean 'are 9 and 10 different numbers?' or does she mean something else? She had just written '9' and paused until the question was answered. So the pupils at this point are engaged in trying to interpret her question.

Another interpretation would be that 'two numbers or one number?' refers to the digits in the number 10. In this case there is a discrepancy here between the pronounced and the intended foci. The pronounced focus is 'number' but the intended focus would be the digits in the number 10. It is easy to see the teacher's reasoning behind using 'number', as the pronounced focus, rather than 'digit' here as the children are very young and the teacher is probably trying to avoid technical vocabulary. However this leads to lack of clarity in the discursive focus and possibly more confusion for the children.

Despite this lack of clarity, the interlocutor's response is the intended one as several children say 'two'. The teacher follows this up with 'what number do we need to go first?' and here the pronounced focus is again 'number' with the intended focus 'digit' but here the clarity would seem to be greater if only by virtue of the repetition. The response '1' does match the intended one which implies some degree of compatibility of focus.

The next question follows on after the teacher finishes writing the complete number '10' and asks 'which bit is zero?' Here the pronounced focus is 'bit' whilst the attended focus, the number to which the teacher was pointing, was the whole number '10'. In this the 'bit' becomes quite clear to the pupils as there are quite apparently two 'bits' to the number '10'. In this utterance I would suggest that there is likely to be a high degree of compatibility between the discursive focus of the teacher and her pupils. However at line 43 the teacher then says: 'No units and 1 ten.' At this juncture she did point first at the zero and then at the one in ten but there is a conflict in this working from right to left, when she was trying to draw attention to the order in which the number '10' is written from left to right. This constitutes a conflict between the attended focus, the pronounced focus and the intended focus. In this example it seems as though none of the aspects of the discursive focus are working together to improve its clarity.

At line 43 the teacher and children work on through the list of the numbers from eleven to twenty in order writing each in turn on the number track. Here the pronounced focus of the number and the attended focus of the

written number are highly compatible and the intended focus of a list of the numbers emerges for all to see. I would suggest that here the clarity of the discursive focus is about as good as it gets.

Line 46 opens with two ‘markers’: ‘So. Right.’ Through which the teacher draws attention to a change of emphasis or change of phase of the lesson. The pronounced focus acts as a signal to pay attention. The use of markers to signal shifts in the focus of the lesson has been explored by Voigt (1985).

The teacher’s next request to the pupils to think of a number between zero and twenty produces a response from most of the pupils. Instead the teacher decides to choose Andrew to answer. Her question to him: ‘Would you like to think of a number for us?’ does not permit ‘no’ as an answer! The intended focus is that Andrew will supply the class with a number even though the request was couched as a question.

Surprisingly Andrew responds with a question: ‘Two numbers between zero and twenty?’ Trying to interpret the intended focus of his question here is difficult. He could have been asking whether the teacher wanted a two-digit number or two separate numbers. However the response that the teacher gives does elicit from him the response she was looking for which suggests that the communication was effective. If he had been talking about the digits rather than the number one would have expected his suggestion of ‘just one number’ to be a single digit number rather than ‘10’.

This analysis demonstrates the complexity of the different interpretations of talking about mathematics that are open to the participants in an elementary classroom setting. This hardly seems to be a straightforward scenario in which everyone knows what everyone else is talking about. The whole question of the compatibility of foci between teacher and pupils is highly problematic.

The above episode is immediately followed by the introduction of the main focus of the lesson. The teacher wants to use the number track, which she has now made on the wall with the class, to tackle problems about counting on and counting back. It is at this point that the possibilities for misconception multiply as the teacher starts to confuse the use of numbers for cardinal and ordinal purposes. The number track on the wall amounts to a number frieze with the numbers listed in order but without any marked boundary between them and without any sense of the value that each number has. The teacher goes on to use this frieze to describe counting on without linking this to addition and concentrating on counting on as being about ‘jumps’ along the number track:

Example 2:

T: ...when you're doing counting on and counting back you always start on the first number. So if we say let's do another one 5 count on 4.... Let's do 5 count on 4. You always start on 5 (*teacher points to 5*) the first number OK? And then you do a little jump and count on 4 ready? (*pupils join in counting with teacher*) 1, (*teacher moves finger to 6*) 2, (*teacher moves finger on to 7*) 3, (*teacher moves finger on to 8*) 4, (*teacher moves finger on to 9*) What's the answer?

70 Ch: 9

This whole scenario is intensely problematic for a number of reasons that are highlighted by an exploration of the discursive foci involved. Firstly there are some problems with actual intended focus of 'counting on'. If we consider the activity of 'counting on' itself and take the pronounced focus: 'Let's do 5 count on 4.' How would we tackle it? Would we take a number track and find 5 and then count on 4 to arrive at 9? No, we would not. We would say 5 then use our fingers, putting up one finger at a time but counting on '6,7,8,9' until we had raised four fingers and arrived at nine. There seems to be a fundamental problem here with the setting up of the task as a way of looking at 'counting on'. We are arriving at the right answers but the fit between the attended focus of the track and the intended focus of the activity of 'counting on' seems poor.

In this excerpt there is compatibility between the pronounced and attended foci but little compatibility with the intended focus of 'counting on'. The teacher's first pronouncement: 'when you're doing counting on and counting back you always start on the first number' is a straightforward instruction that she makes clearer in referring to the example. Here then the pronounced focus of starting on the first number is compatible with the attended focus demonstrated by pointing at the first number but the problem arises with the intended focus when 'you do a little jump'. In fact what the teacher and pupils are counting are the 'little jumps' and the first count coincides with the first 'jumps' as the teacher moves her pointing finger on to the next number. So the idea that 'you always start on the first number' is at odds with the attended focus of counting the moves.

The following excerpt comes further on through the lesson once the teacher has moved on to examples of 'counting back' using the same model. Sequences of this type occurred throughout the lesson and the analysis of the interaction can be carried out in a similar way to that in the above example.

Example 3:

- T: What's special about the first number? It always has to be
 Ch: Bigger
 T: Bigger than the second number. If we're always counting
 back
 100 the first number has to be bigger than the second one. So
 let's do 12, you can do this one for me, count back 3. Put
 your hand up if you can do it. Don't shout out just put your
 hand up. 12 count back 3. Amy?
 Amy: 9
 T: Is she right? Start with 12 (*teacher points at 12*) count back
 3. 1, (*teacher moves finger to 11*) 2, (*teacher moves finger to 10*) 3, (*teacher moves finger to 9*) Well done! Let's do 20
 count back 9. Are you ready? (*Teacher points to 20*) 1,
 (*children counting with teacher, teacher points to 19*) 2,
 (*Teacher points to 18*) 3, 4, 5, 6, 7, 8, 9 (*teacher pointing to*
each successive number 17, 16, 15, etc and counting with
children)

This shows the teacher using the same attended focus of the numbers on the number line as before but the intended focus now is the activity of 'counting back'. The initial comments in the exchange focus on the pronounced focus 'bigger than', specifically the first number needing to be bigger than the second number when 'counting back'. Why this should be so remains unaddressed. How will this be interpreted by the pupils? It seems to be taken as given that the pupils do have an understanding of 'bigger than' although the talk in this excerpt gives no evidence of this because the participation of the pupils is minimal. Sfard stresses the importance of speaking about objects in order to construct them and it is difficult to see how pupils might be helped to construct 'bigger than' from this exchange.

In the 'counting back' example the attended focus is the numbers on the number track as before with the pronounced focus of the list of numbers. The 'counting back' difference is in the direction that the 'little jumps' are made, from right to left instead of from left to right, although there is no pronounced focus on this.

In the initial introduction to the lesson the teacher concentrates on drawing her pupils' attention to their writing of numbers. In this part of the lesson she refers back to ideas, which the pupils have already explored, about the shapes of the number symbols themselves:

Example 4:

- T: ...Some numbers are straight and some numbers are round.
Ch?: Another way of doing the 4 you do a sort of triangle and a line at the right hand side and a line at the bottom
20 T: That's right. This (*pointing at 4 on board which is written in the traditional hand written way- like an L with a mark at the bottom*) is the easiest way to do it. Now 7, 7 is a number people always get the wrong way round. Go along here and down there

Here the intended focus is on the signifiers, the symbols used to represent the numbers rather than on the meaning of the numbers themselves. The first comment by the teacher: 'Some numbers are straight and some numbers are round' has the attended focus of the list of numbers from 1 to 4 on the number track but a pronounced focus that is quite non-specific. The interpretation that I would make of the intended focus is that it is on the shapes of the numbers as they are written. This is a focus on the signifier rather than the signified. The reference to 'straight' and 'round' may or not make sense to the pupils but there is no indication from their responses about this. My concern about these pronounced foci would be that they add nothing to the child's concept of the meaning of the numbers that they represent. There is nothing about '6' that is related to its 'roundness'. Sfard comments on the nature of mathematical objects, such as numbers e.g. in this case 6. She says that:

The existence of some special beings (that we call *mathematical objects*) is essentially metaphorical and, further, that the phrase 'construction of objects' does not mean creation of any tangible, self-sustained, mind-independent entities. These expressions merely signal that people who speak about such 'virtual things' as numbers, functions, or sets go through experiences similar to those generated by the 'actual reality' discourse and use linguistic forms reminiscent of those usually applied to material objects.

If the teacher's intended focus is ultimately the nature of numbers themselves then it is not clarified by pronounced and attended foci that make reference to the shapes of numbers as they are written.

Summary and Conclusions

Sfard's paper concentrates on the ways in which discursive foci help to construct 'mathematical objects'. The evidence from my transcript highlights

the complex ways in which pronounced, attended and intended foci might work together or in conflict to create or fragment the individual's construction of meaning. Even at a very elementary level mathematically the opportunities for confusion and lack of compatibility between participants abound. If the teacher is to develop ways of talking to children about mathematics that enable them to construct mathematical objects that are 'stable, permanent, self-sustained and located beyond the discourse itself' (p. 326) (Sfard 2000) then she needs a great awareness of the discursive foci that she is using.

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NARRATIVE ORIENTATION IN THE CONSTRUCTION AND SOLUTION OF WORD PROBLEMS BY ENGLISH ADDITIONAL LANGUAGE (EAL) LEARNERS OF MATHEMATICS

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Abstract: *This paper reports some early analysis from a study designed to examine how English additional language (EAL) students participate in the discourse of mathematics learning in a UK primary school. Taking the discursive psychology of Edwards (1997) and others as a theoretical and methodological framework, the study will examine how small groups of Year 5 students participate in the joint construction and solution of mathematical problems. In this paper I examine a transcript of an EAL/non-EAL student pair jointly writing and solving a word problem. There is evidence that orientation to a personal narrative frame plays an important part in the EAL student's meaning making, leading her to make sense of the problem which emerges, going against research which suggests that students find it difficult to relate word problems to 'real life'.*

Introduction

Despite concern that children from minority ethnic backgrounds under-achieve in mathematics in British schools, particularly if English is an Additional Language (EAL) [1] (Gillborn and Gipps, 1996; OFSTED, 1999) there has been little research with a dual focus on mathematics learning and EAL. Within mathematics education research has generally sought quantitative relationships between language and attainment (e.g. Clarkson, 1992; Cocking and Mestre, 1988). As quantitative instruments are necessarily embedded in language there are difficulties with this approach, in particular leading to problems of validity, since it is difficult to untangle mathematical attainment from linguistic effects. Research in applied linguistics has mainly focused on Halliday's (1978) elaboration of the "mathematical register" and the acquisition of mathematical language rather than on mathematical learning. There seems to have been little research which has investigated the *process* of learning mathematics when English is an additional language: how do EAL students learn mathematics? Where such research has taken place (Moschkovich, 1999; Setati, 1998) it has generally examined interaction between students and teachers or other

professionals. Such interaction tends to be heavily dominated by the adult participant(s) and therefore affords a restricted view of EAL students' use of language and participation in classroom discourse. The study reported here looks specifically at interaction between students as a way of gaining insights into the learning process.

Methodological and theoretical perspective

Working in multilingual, multicultural classrooms challenges assumptions of common understandings, raising questions about how the learning process can be investigated. If meanings are seen as subjective and situated within each learner's cultural background, the languages they speak and their previous experiences of education and of mathematics, difficulties arise in assuming that 'we all know what we're talking about'. The discursive psychology developed by Derek Edwards (1997) and others (Potter and Wetherell, 1987; Edwards and Potter, 1992) offers both a theoretical perspective and a methodological approach which avoids the problem of making inferences about children's psychological states (including meanings) based on what they say when as researcher I have limited access to their language, culture and experience of the world. Edwards' (1997) approach draws on conversation analysis and ethnomethodology to develop a discourse analysis which entails an important shift in focus. Instead of taking utterances as evidence of what the speaker thinks or knows, to be tested against objective reality, the analyst examines how reality is constructed in discourse, focusing on the business performed by utterances in context. Analysis of classroom discourse asks "not *what* do children think but *how* do children think" (Edwards, 1993: 216). Language is reconceptualised as primarily "a medium of *social action* rather than a code for representing thoughts and ideas...or a grammatical system" (Edwards, 1997: 84, original emphasis). This is not to deny that language does not represent ideas or cannot be analysed grammatically. Rather it is to foreground social action as the primary function of language, which is seen as having evolved through social interaction, and therefore as being structured both by and for social interaction (ibid.). Discursive psychology seeks to carry out psychological inquiry which takes account of this primarily social conceptualisation of discourse.

Edwards and Potter (1992: 28-29) outline five distinctive aspects of the discourse analysis of discursive psychology:

1. Discourse analysis is of naturally occurring talk and prepared texts, rather than talk or text produced for the purposes of analysis, such as in psychological experiments, which can be analysed as instances of

- ‘experimental talk’, but not as a way of gaining access to participants’ psychological states.
2. Discourse analysis is concerned with the content of talk and its social organisation, rather than linguistic approaches to structure, for example. This includes seeing talk as sequential and analysing utterances within the sequential context in which they occur, rather than as isolated snippets of the mind (Potter and Wetherell, 1987: 93).
 3. Discourse analysis is concerned with action, construction and variability. Different ways of talking are used in different circumstances and for different rhetorical purposes. For Potter and Wetherell (1987: 67) looking for such variation is an important part of preliminary analysis, since instances of variation may be examined to see what is achieved by varying the way in which things are said.
 4. The rhetorical organisation of talk and thought is designed to counter potential alternative versions which may arise. As rhetoric serves a purpose, its use is seen as systematic. Analysis therefore proceeds from variability to looking at “the patterning or organization of different versions and the way they are constructed” (Potter and Wetherell, 1987: 67).
 5. It is the consideration of such ‘cognitive’ issues as knowledge, truth, reality or mind in terms of how they are dealt with in discourse that leads to this approach being characterised as ‘psychological’: “intentions, goals, mental contents and their intersubjective ‘sharing’ are analysed as kinds of business that talk attends to, rather than being the analyst’s stock assumption concerning what is actually going on” (Edwards, 1997:107). The focus has shifted to looking at how participants use psychological states in interaction. This is not to deny that people have intentions or meanings, but to argue that we can only examine how notions of intention or meaning are employed in interaction as a form of social action.

In this paper I offer an analysis of two students jointly constructing mathematical word problems. The purpose of such analysis is neither to find out what the students think a word problem is like, nor to see how much mathematics they know. Instead the aim is to examine how the participants jointly negotiate and construct the problems, how they do ‘writing word problems’. Before commencing the analysis, however, some background is necessary.

Research context

As part of my doctoral research into the learning of mathematics by EAL students I have been visiting the Year 5 (aged 9-10) mathematics lessons in a multicultural urban primary school in the UK. The school has approximately

150 students from a variety of cultural and linguistic backgrounds. Most EAL students in the upper classes have a reasonable level of functional English. In Year 5 there are six students recognised as EAL. Initially I had hoped to record students as they worked in order to obtain records of naturalistic interaction. As this proved impractical the approach was modified: small groups of students were withdrawn from the classroom and recorded while they worked on a task together. Although not identical to classroom situations, the teacher frequently asks students to work together in this way. Furthermore, the task selected was one which the teacher uses during her mathematics lessons. Thus although the interaction was not completely natural, neither is it particularly artificial.

The research design involves selecting a topic from the teacher's schedule for the term. In this case the topic concerned calculators, including some work on using calculators in the context of money. Six pairs or threes of students were recorded both before and after the calculator topic working on the task of writing word problems which could be used in the money part of the topic. A calculator was provided. The students were also asked to solve their problems. The primary data consists of audio recordings of the interaction which was fully transcribed. The analysis offered in this paper is of one pair of students working on the task after the lesson sequence. 'Cynthia' comes from a Cantonese speaking background and has recently come to the UK from Hong Kong where she previously went to school. She has been in the school for less than two years and in that time has learnt virtually all her English. She is a lively, enthusiastic student. 'Helena' is an African-Caribbean student, hard-working and reliable.

Writing word problems: unfolding and orienting

From preliminary readings of this and other transcripts, it is striking how the word problems emerge from the students' deliberations. They do not appear to start out with a clear problem in mind, leaving only the work of finding a suitable form of words. The first move tends to be the selection of a name. The path from the name to the completed problem can be fairly direct or can meander slowly, taking sudden changes of direction. The outcome is always recognisably a word-problem - recognisable to me and their class teacher, at least. A related observation is that it is difficult to look at the transcripts of problems in-the-making and attribute aspects of the development to one or other of the participants. The problems emerge from the combined efforts of the students. Cynthia and Helena's second problem provides a particularly interesting example: the problem which emerges concerns Cynthia buying presents for her mum. What is intriguing is how in the final question there is no mention of either presents or Cynthia's mum, only of the items Cynthia

purchases. Through the course of producing the problem, a story develops which then does not appear in the final problem.

These observations are in accord with Edwards (1997) when he argues: “The intelligible orderliness of social life stems not from a set of updateable knowledge structures in a sense-making cognitive being, but from how social actions flexibly unfold, as situated performances” (Edwards, 1997: 165). Cynthia and her peers are producing word-problems through social (inter)actions through which the problems unfold. Although there is no clear idea of what the completed problem will be, however, there is a sense that the students ‘know’ what kind of thing they are going to end up with. This idea is also expressed by Sacks (1987) who shows how in question-answer exchanges participants show a preference for agreement. This preference is also evident in the form of disagreement answers, which tend to be couched in agreement terms. Agreement is not just exhibited by the answerer, however: “If there is...an abstract or formal preference for agreement, then we have to see that the questioner is designing the question with an orientation to getting agreement” (p63).

The notion of orientation has proved valuable in analysing the interaction of Cynthia and Helena. In an earlier paper (Barwell, 2000) I examined the second part of this transcript in which the two students jointly prepare their second word problem. As part my analysis, I identified a strong orientation towards a coherent story or situation which guided the students’ discussion. This orientation is evident, for example, in segment 1 (see [2] for transcription conventions).

Segment 1

- 347 H okay then/ Cynthia has fifty pounds/ to buy her mum a present
348 C (*laughs*)
349 H and she gets her/ a big dress/
350 C big dress/ no/ my mum doesn’t like dress/ I get her ahhh/ big music box
351 / if you open it/[it’s music/

Cynthia’s name has been chosen as that of the protagonist in the problem. Helena has offered a first line for the problem (347) which sets up a scenario about buying a present. Although the emerging problem is couched in third-person (she, her) terms (347, 349), Cynthia responds in a way which suggests she is oriented to a story which makes personal sense to her. She rejects Helena’s suggestion of a dress as a present on the grounds that “my mum doesn’t like dress” (350). The point here is not whether Cynthia’s mum really likes dresses or not. Rather it is Cynthia’s apparent orientation to a guiding narrative frame (Bruner, 1990), which is still emerging from her interaction with Helena. Furthermore, there is a tension through which the narrative

orientation emerges: a strong personal narrative on Cynthia's part compares with Helena's more generic, impersonal narrative. In segment 1 this is apparent in Helena's use of 'she' or 'her' when talking about the Cynthia in the problem (lines 347, 349), which contrasts with Cynthia's use of 'I' or 'my' (350). Cynthia's more personal orientation to the problem led her subsequently to solve the problem with relatively little difficulty at the end of the recording.

The purpose of the analysis in this paper is to determine whether there is evidence for Cynthia's more personal narrative orientation in the first half of the transcript in which the two students construct their first word problem, and in which Helena is the protagonist.

Helena's pocket money

Cynthia takes charge and elects to write. She chooses Helena as the name of the protagonist in the problem, which Helena is a little surprised to see (49: H: what?/ you're writing my name). Cynthia develops the first part of the problem with tacit acceptance from Helena who does not challenge or disagree. There is no evidence of any kind of personal narrative during the first few minutes of work, with both students referring to the Helena in the problem as 'she', although there is a more generic narrative sense in Cynthia's question. Gradually Helena becomes more involved in the production of the question, taking over the writing from Cynthia and suggesting and defending alternative forms of words. In this way they produce their first question (see [4]) which takes about ten minutes and is completed by line 266. The next sixty lines of the transcript record their discussion as they solve their problem. Helena contributes "one pound eighty" (i.e. 45p x 4) and there follows a discussion about how to proceed:

Segment 2

- 278 C how much you spend/ eight pound eighteen innit/
 279 H times
 280 C it's add
 281 H it's **times**
 282 C one pound eighteen/[add/ **add**/ add/ seven/ equal
 283 H [times
 284 no it's times seven
 285 C add
 286 H no it's times seven
 287 C is it?/ how?
 288 H cause you/ multiply it by seven
 289 C oh yeah
 290 H yes it is times

- 291 C yes times times times times times done it wrong/ one eight/ oh/ times/
292 seven/ wow/ twelve pounds sixty I haven't got enough money to spend
293 then/ I change/ you got/ you got/ oh wait/ twelve/ is that twelve pound/
294 yeah/ twelve pound sixteen/[add

Cynthia and Helena dispute whether they should add or multiply the daily spend and the number of days in a week. Suddenly (291) Cynthia becomes very animated, accepting Helena's argument for multiplication. She performs the calculation and gets the result "twelve pounds sixty". There is a problem here as the problem states that Helena has £10. Cynthia's reaction to this is intriguing: "I haven't got enough money to spend". In this moment, Cynthia appears to personalise the problem. Her use of 'I' is evidence that there is a personal narrative orientation in her work, in her thinking, which guides her participation in the construction of problems with Helena. In the first part of this transcript, it is rarely visible, but at this instant it appears to break the surface for the first time. At this point Cynthia's orientation to a personal narrative frame seems to become more prominent. She begins to identify explicitly the Helena of the problem with the Helena sitting next to her (segment 3, lines 293, 296).

Segment 3

- 293 C then/ I change/ you got/ you got/ oh wait/ twelve/ is that twelve pound/
294 yeah/ twelve pound sixteen/[add
295 H [seven times eight
296 C and you got twenty pound/
297 H (*laughs*)
sixteen = sixty [3]

Indeed when she says to Helena "and you got twenty pound" Helena's response is to laugh. Laughing here seems to have the effect of accepting what Cynthia is doing but at the same time as marking it as out of the ordinary in some way, suggesting that for Helena the personal orientation is less prominent. A little later, Cynthia increases Helena's pocket money again, identifying herself as the 'donor' (segment 4, 307: "*gave* you some more") and this time Helena acknowledges the more personal narrative orientation, entering into negotiations with Cynthia about how much her pocket money should be.

Segment 4

- 307 C oh you have thirty pound gave you some more
308 H no/ twenty five
309 C no
310 H no cause it's going to be lots/ of money/[alright then
311 C [Parveen got loads of money in
312 a week
313 H how much did she get

- 314 C sometimes she got fifty pound
 315 H fifty
 316 C five oh
 317 H fifty
 318 C yeah she said
 319 H yeah right
 320 C fifty that is not/ (...) work/ cause I don't know/ how much is dat work/
 321 maybe she's joking then right just (right) fifteen/ um right/ fee (*three*)
 322 voh (*oh*) voh voh/ take away one two six oh/ one two six zoh (*oh*)/
 323 equal/ I got/ seventeen/[pound/ um/ [fourteen pence/ left
 324 H [(...) [forty pence

As Helena moves into displaying a more personal orientation, Cynthia opens up the discussion, justifying her position of a £30 rate of pocket money by referring to another member of the class, 'Parveen' (311). Cynthia constructs a short characterisation of Parveen as getting £50 pocket money, with the result that £30 for Helena seems more reasonable. She reinforces her position in response to Helena's questioning repetition by giving the amount in figures (316), clarification being also a way of underlining. When Helena continues to seem somewhat incredulous (319), Cynthia carefully softens her characterisation by suggesting that Parveen might have been joking. The outcome of this carefully occasioned discussion about Parveen which Cynthia has instigated, is that it is her figure of £30 which stands and appears in the final version of the problem, while Helena's offer of £25 (308) is dropped. This discussion completed, Cynthia has no difficulty in completing the problem (lines 321-323). There is a sense, although this is speculation, that Cynthia could have substituted any amount for the pocket money and been able to complete the calculation with as much ease as she displays here.

Discussion

The moment at the end of segment 2 is pivotal in the students' work on this problem. Prior to this point, their work seems oriented to producing a canonical, typical word problem. Afterwards the students' orient to a more personal narrative. This appearance of the personal coincides with a change in Cynthia's position regarding the solution to the problem. She is in dispute with Helena. She manages her change of position to one of agreement by also accomplishing a shift in the balance of orientations towards a more overtly personal narrative. The coincidence of these two shifts seems significant. The orientation to the personal is an important part of Cynthia's process of making meaning in the word problems. This is an interesting finding: Lave (1992) argues that one of the difficulties of word problems for students is the separation of school and 'real life' experience. In this analysis, Cynthia clearly evades this separation

through her orientation to a narrative which crucially is personal to her. She makes use of this guiding orientation in the social actions involved in jointly writing and solving a word problem. The problem is meaningful to her and she is able to solve the problem with ease. Cynthia's highly competent participation in the task is all the more remarkable given that she has been learning English for less than two years.

Notes

1. English additional language (EAL) refers to any learner in an English medium learning environment for whom English is not the first language and for whom English is not developed to the level of a native speaker.
2. Bold indicates emphasis. / is a pause < 2 secs. // is a pause > 2 secs. (...) indicates untranscribable. ? is for question intonation. () for where transcription is uncertain. [for concurrent speech. ^ ^ encloses whispered or very quiet speech. = for latching (no gap between words).
3. Cynthia sometimes mixes up /ty/ and /teen/ at the ends of numbers like sixty and sixteen.
4. The final version of the problem, unedited, is: Helena has £10 pocket money in 1 week, everyday she buys 4 packet of chrisps each costing 45p. How much does she spend in 1 day, How much does she spend in 1 week and How much does she have left?

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STUDYING VALUES IN MATHEMATICS EDUCATION: ASPECTS OF THE VAMP PROJECT

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Abstract: *Values are a powerful force in mathematics education. They are involved in both the teaching and the learning of mathematics, and yet they are also largely hidden. They are rarely taught explicitly and many teachers of mathematics do not even believe that they are teaching values. This paper describes some of the work of the Values in Mathematics Project, its rationale, methods, and some of its findings. The authors hope that other researchers will also become interested in this topic so that collaborative work can be done in different countries on what is a crucial but neglected area of research.*

1. Teachers' decisions and values

Imagine that you are a Grade 4 mathematics teacher. It is the first day back at school after the Christmas holiday, and you are talking with your class before getting down to work. You ask if anyone had any 'mathematical' presents. One boy says that he had been given a mathematical game from his uncle's country. He says it is very interesting, it has many variations, and he asks if he can show the class how it is played.

What would you do? Would you let him show the class and see what develops? Would you say something like: "Well that would be nice, but we don't have time now to do it, maybe later" or maybe: "Excellent, show me after the class, and I'll decide then if we can play it."

Are mathematical games a part of your teaching ideas? Would this game fit within your curriculum? Does that matter? In any case, you would probably make your choice in the way that you normally do, and not think much more about it. But the fact remains that you must make a choice, and that choice depends on your values.

All teachers must make decisions in situations like this, and the decisions they make relate in some way to their values. Sadly, little is known or has been

written about the values which mathematics teachers think they are imparting, or how successful they are in imparting them. In our research study, the Values in Mathematics Project, we are examining teachers' awareness of what values they intend to teach in their mathematics classrooms, how (and what) values teaching takes place, and most importantly, whether teachers can gain control over the values they are teaching.

2. Mathematics, culture, and values

Human beings everywhere and throughout time have used mathematics (Bishop, 1988). The mathematics typically can be observed as behaviours illustrating the following six 'universal' activities (i.e. every cultural group does them): counting, measuring, locating, designing, explaining, and playing. These behaviours are reflective of the culture of the people demonstrating them and are necessarily influenced by what that cultural group values.

Moreover it is clear that all teachers cannot help but teach values, although most values teaching and learning in mathematics classes appears to happen implicitly. Many teachers, who believe that mathematics learning has value for their students, have never considered the particular values they are imparting.

Although current developmental policies, through their statements of intent, often mention the encouragement of 'desirable' values, the curriculum prescriptions, which follow, have little to say about their development. For example, the Goals of the Australian school mathematics curriculum include these statements (Australian Education Council, 1991):

As a result of learning mathematics in school all students should:

- *realize that mathematics is relevant to them personally and to their community;*
- *gain pleasure from mathematics and appreciate its fascination and power;*
- *appreciate:*
 - *that mathematics is a dynamic field with its roots in many cultures; and*
 - *its relationship to social and technological change.*

It is clear from these statements, which are typical of educational goal statements, that valuing mathematics has entered into their choice. Secondly they all contain implications for values teaching and for cultivating what we might term 'mathematically informed valuing'.

Also there is now a great variety of proposals from research, and ideas for improving mathematics teaching being generated internationally. In particular in the areas of information technology (see Noss and Hoyles, 1996), ethnomathematics (see Barton, 1996, Gerdes, 1995) and critical mathematics education (see Skovsmose, 1994), the role of mathematics teachers is being critically examined. What is of special interest about these kinds of developments however is that there is a strong concern both to question, and also to try to change, the values currently being taught. But there is still no researched knowledge on how to help this happen in mathematics education.

Values exist throughout all levels of human relationships. At the individual level, learners have their own preferences and abilities that predispose them to value certain activities more than others. In the classroom there are values inherent in the negotiation of meanings between teacher and students, and between the students themselves. At the institutional level we enter the political world of any organization in which issues, both deep and superficial, engage everyone in value arguments about priorities in determining local curricula, schedules, teaching approaches, etc. The larger political scene is at the societal level, where the powerful institutions of any society with their own values determine national and state priorities in terms of the mathematics curriculum and teacher preparation requirements. Finally, at the cultural level, the very sources of knowledge, beliefs, and language, influence our values in mathematics education. Further, different cultures develop different values.

After examining the research literature in preparation for the empirical part of the Values in Mathematics Project, our initial analyses reveal that there are two main kinds of values which teachers seek to convey: the general and the mathematical. For example, when a teacher admonishes a student for cheating in an examination, the values of 'honesty' and 'good behaviour' derive from the general socializing demands of society. In this case, the values are not especially concerned with, or particularly fostered by, the teaching of mathematics. However when we think about the incident at the start of this paper, we very soon involve mathematical values. Bishop (1988, 1991), argued that the values associated with what can be called Western mathematics could be described as follows:

Rationalism - involving ideas such as logical, and hypothetical, reasoning.

Objectivism - involving ideas such as symbolizing, and concretizing. Mathematicians throughout its history have created symbols and other forms of representation, and have then treated those symbols as the source for the next level of abstraction.

Control - involving aspects such as rules, predictions, and applications to situations in the environment.

Progress - involving ideas of exploring and progression, through abstracting and generalizing.

Openness - meaning the 'public' verification of their ideas by proofs and demonstrations.

Mystery - involving the mystifying, and surprising side of mathematics, including the origins of mathematical ideas.

It seems from the research literature that over the last centuries these six values have been fostered by mathematicians working in the Western culture, and it is these values that teachers in Western cultures are probably also promoting when they teach mathematics.

However, we have also recognised that culture is a strong determinant of mathematical values, and research shows us that not all cultures share the same basic values. So it is likely that mathematics teachers working in different cultures will impart different sets of values to their students, even if they are teaching to the same basic mathematics curriculum. This is one reason why we are establishing collaborative research projects with other countries. There is already a similar project taking place in Taiwan (see Chin and Lin, 1999), and other colleagues have also expressed interest.

3. The VAMP research project

In 1999 we obtained funding for our three-year research project which had the following goals:

1. To investigate and document mathematics teachers' understanding of their own intended and implemented values.
2. To investigate the extent to which mathematics teachers can gain control over their own values teaching.
3. To increase the possibilities for more effective mathematics teaching through values education of teachers, and of teachers in training.

For Goal 1 we are studying both teachers' intentions, and their actual teaching behaviours. To begin the research we ran a series of inservice workshops with teachers, which enabled us to gain some initial insights into the kinds of values teachers were considering. As a result of these workshops, we developed a detailed questionnaire, which we gave to 30 volunteer mathematics teachers in

Victoria. The questionnaire was used to identify the teachers who were willing to participate in the research and whose views about values were sufficiently, and interestingly, different. We also wanted teachers in both primary and secondary schools.

For Goal 2 we began our first school-based work with eight selected volunteer teachers, to clarify via initial interviews their ‘intended values’, and through classroom observation and post-observation interviews, the ways in which they implement these in the classroom. Through this process, the teachers were encouraged to identify the role that they want values teaching to play in their classrooms, and to identify in which areas they are achieving what they want, and in which areas they desire change.

Following a number of group discussions with the eight teachers, a joint plan will be devised to attempt to implement certain specified values different from those normally emphasised by the teachers. The principal aim of the group discussion sessions is for the teachers to be able to support each other during what could be a challenging experimental period. The joint plan will be implemented over a similar observational period to that used in the first approach. The researchers’ tasks will be to observe and document the extent to which the implementation takes place. Following the observations and teacher interviews, further group discussions will be held. The teachers will be asked to keep journals with weekly entries and these journals will be particularly important documents for analysis and discussion during this phase.

It is our contention that improving and making values teaching more explicit in mathematics classrooms will make mathematics learning more effective. Hence the need for Goal 3 above. We anticipate that we will be generating in-service activities for teachers, based around the topics arising from the research. The interest and concern is not with the particular choices the teachers might make but with the values underlying their decisions.

Through activities based around questions such as these, it is our hope that we shall be able to make mathematics teachers not only more aware of the different values that they are teaching, but also that they will be more in control of their own values teaching. By this means we intend teachers to develop a greater range of teaching techniques, and to be able to offer a more rounded mathematical education to all their students, based on considered value judgements.

4. Ideas arising from the study

The study is only in its second year but already some interesting ideas have emerged. In our preliminary discussions with the teachers we learnt about their general feelings concerning values, and also what they considered to be mathematics education values, as can be seen in Table 1.

Table 1: *Aspects of meaning either used in or arising from discussion with teachers*

General meanings of 'value'	Mathematics educational values
<p>To value:</p> <ul style="list-style-type: none"> • to command • to praise • to heed • to regard <p>A value is:</p> <ul style="list-style-type: none"> • a thing regarded to have worth • a principle by which we live/act • a standard by which we judge what is important • something we aim for • qualities to which we conform 	<p>Clarity</p> <p>Flexibility</p> <p>Consistency</p> <p>Open mindedness</p> <p>Persistence</p> <p>Accuracy</p> <p>Efficient working</p> <p>Systematic working</p> <p>Enjoyment</p> <p>Effective organization</p> <p>Creativity</p> <p>Conjecturing</p>

One of the foremost questions to be addressed in the questionnaire was whether teachers actually saw a place for values teaching in mathematics education. By expressing an interest in being involved in this project, respondents have tacitly indicated support for the concept. However, while there was agreement or strong agreement by many with the statement: “There is a place in mathematics teaching for the teaching of values,” a number of responses made us question whether this statement was interpreted in terms of actual or possible behaviours.

The results from questions aimed at influences on the portrayal of values in mathematics teaching included the following. The teacher’s personal value framework rated consistently highly, sometimes in concert with religious/spiritual values but sometimes these were diametrically opposed, with the latter ranked last or near last. Although it was generally agreed that curricular resources (e.g. curriculum guides, textbooks, etc.) portrayed values, there was an equivocal response to the degree of influence exerted by the kind(s) of pupils in the particular class, the school ethos and culture, and the particular topic being taught. That is, some teachers claimed to portray values

consistently across classes, topics, or both, whereas others stressed the need to respond to different students' needs. An example of the dilemmas in making generalizations here is given by the following comment:

The kind of students I have in my classes does not change the values I portray. I consider it important to provide a realistic consistent modeling of my own values, especially to the low socio economic students I teach who express cynicism concerning, and often feel betrayed by, teacher "masks."

Research on mathematics teacher beliefs, particularly in relation to teachers' actions in the classroom seems to demonstrate that teachers' actions frequently bear no relation to their professed beliefs about mathematics and mathematics teaching (Thompson, 1992). Other research has shown striking inconsistencies between different belief statements given by the same teachers (Sosniak, Ethington and Varelas, 1991). A section of the questionnaire therefore consisted of items with contextualised classroom situations. It asked for teachers' open-ended feedback regarding (a) their response to each situation, (b) the contextual factors guiding their respective responses and (c) the underlying values underpinning their actions.

Some respondents did in fact demonstrate the same kinds of inconsistencies, but these were explained in terms of contexts. The teachers' indication has been that the kinds of values being represented were influenced predominantly by their own personal value framework. So it may be expected that with such personal involvement, preferred values were translated into portrayed ones in the classroom.

Table 2: *Comparison of descriptors associated with preferred and portrayed mathematical values*

Preferred	Portrayed
1. Logical thinking (1.3)	1. Logical thinking (1.6)
2. Creativity (2.2)	2. Systematic working (2.3)
3. Systematic working (2.5)	3. Puzzling (2.7)
4. Puzzling (2.8)	4. Creativity (2.8)
5. Beauty (3.9)	5. Beauty (4.6)
6. Improving career prospects (5.8)	6. Improving career prospects (5.8)

Note. Average rankings are denoted in brackets.

In the case of mathematical values (Table 2), this expectation held true for the highest-ranked value corresponding to logical thinking, that is, rationalism (Bishop, 1988), as well as to the two lowest-ranked values which

corresponded to mathematics improving one's career prospects, and to beauty. However the value of creativity, in particular, appeared to be under-emphasised despite strong teacher intentions.

5. Further analyses

The research is still progressing and we are now in the process of analyzing the data from the first part of the intervention study. We have classroom observation and interview data from the eight teachers concerning the values they are teaching both explicitly and implicitly in their classrooms. It is clear that the teachers are becoming more aware and articulate about their values teaching through this research.

For example, one primary teacher at a Catholic primary school was certainly aware of teaching values explicitly in her lessons, through the school's system of having a 'value for the week' to be interpreted and emphasised by all teachers (for example, 'strength of character'). But she confessed that she didn't think about teaching these kinds of values in her mathematics lessons. Nevertheless, it was clear from observing her mathematics teaching that she certainly did teach values both explicitly and implicitly, including for example, what she called "being independent, taking risks, and being creative". She encouraged these behaviours, spoke about them and rewarded students who demonstrated them.

It will be interesting to see whether the primary teachers, who teach a range of subjects, will find it easier than their secondary colleagues to address values explicitly in their mathematics activities with their students. This is only one of several intriguing ideas to have emerged from our study. Others will follow as we explore further this widely neglected aspect of mathematics education.

6. Note

The 'Values and Mathematics Project' (VAMP) is a three-year (1999-2001) Australia Research Council funded project jointly conducted by Monash University and the Australian Catholic University. Its web-site is <http://www.education.monash.edu.au/projects/vamp>

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CLASSROOM INTERACTION AS MULTI-PARTY- INTERACTION: METHODOLOGICAL ASPECTS OF ARGUMENTATION ANALYSIS

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Abstract: *At CERME 1 the first results of the research project "Reconstruction of formats of collective argumentation" were presented.¹ The project is an empirical study of mathematics education in which elements of an interactional theory of learning are investigated (Krummheuer & Brandt 2001). It reinforced the concept of learning by participating in collective argumentation (Miller 1986; Krummheuer 1995; Brandt & Krummheuer 1999). We analyze classroom interaction as multi-party-interaction (Sacks 1995; Sahlström 1996). The aim of this paper is to demonstrate some aspects of our methods of analysis, which were developed during the research process.*

The project is conducted in the tradition of the German interpretative classroom studies, which were initiated by a group of researchers led by Heinrich Bauersfeld (1978, 1980) in the seventies. Following Cobb & Bauersfeld (1995) and their idea of combining cognitive constructivism and social interactionism, the aim is to contribute to a sociologically orientated interactional theory of learning and teaching in mathematics classrooms. The theoretical background and views of learning and argumentation have been developed in subsequent research projects (Krummheuer 1992, 1995, 1997), in which the research methods *interaction analysis* and *argumentation analysis* have been introduced (see Krummheuer 1995). These methods were modified for the project and a third method was added – the *participation analysis*, which consists of two parts: the *reception design* and the *production design* (Brandt 1997, 1998). The following exposition concentrates on the combination of the argumentation analysis and the production design. The goal of this composed analysis is to describe learning processes in an appropriate way for our research questions – the "formats" (Bruner 1983) of collective argumentation (Miller 1986) in classroom interaction. Further the method of analysis is a result of our research process, developed during the research process (see Bohnsack 1993 for aspects of "reconstructive social research").

¹ It was based at the Freie Universität of Berlin and supported by the German Research Foundation (DFG) from October 1996 to June 2000.

First, I will outline the main concepts of learning and argumentation, the argumentation analysis, and the participation analysis (production design). In the second part, the combined analysis will be demonstrated by an interaction example. The paper concludes by a brief description of the interaction as a learning situation with respect to our research questions.

1. Learning by argumentation in multi-party-interactions

In the current discussion about learning mathematics, it is well accepted that learning and teaching cannot be described by the transmission of knowledge model. Going back to Piaget's theory of cognitive development, the learner as a creator of her or his cognitive schemes is a central idea in the comprehension of mathematical learning.

Learning processes mainly occur in social interaction and cannot be separated from them. Hence, the constructivist point of view from (cognitive) psychology is only one side of the development of the individual. We focus on the relevance of social interaction processes for the individual learning process. The way in which a student participates in a classroom lesson initiates and constrains her or his individual learning process. Thus, for describing individual learning processes, it is appropriate to supplement the cognitive point of view by the interactionist perspective from sociology. The individual constitution of meaning (as an aspect of learning processes) is strongly related to the participants' collective constitution of meaning in the interaction.

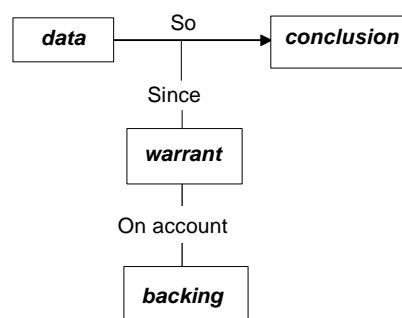
"[...] the individual's reasoning and sense-making processes cannot be separated from their participation in the interactive constitution of taken-as-shared mathematical meanings" (Yackel & Cobb 1996, p. 460).

Taken-as-shared meanings emerge in the interaction process through negotiation, as do the forms of interaction. We consider "collective argumentation" (Miller 1986) as of major importance to the process of negotiating meanings.

1.1. Functional analysis of argumentation

Negotiation by "collective argumentation" is not aligned to logical argumentation but to rhetorical forms of argumentation (Krummheuer 1995, 1997). In an "analytical argumentation" (Toulmin 1969) – like a mathematical proof – the validity of a conclusion is deduced from the basic premises using deductive inferences. In contrast to that, the participants of a classroom interaction try to present their actions as valid and convincing, which can be

described as "substantial argumentation" (Toulmin 1969). Hence, his functional analysis of argumentation is adequate to analyze argumentation in classroom interaction (see Krummheuer 1995). With Toulmin the *data*, the *conclusion*, the *warrant* and the *backing* are the four functional categories of an argumentation. The general idea of an argumentation is to infer the actual demand (conclusion) from a shared assertion (data), in the sense of transferring the joint agreement. So, data and conclusion built the minimal form of an argumentation. The warrant can be produced to legitimate the step from the data to the conclusion – so the warrant defines the shared assertion as data. Backings are global statements and convictions and supports the warrant in the argumentation. Often, this functional element of an argumentation does not appear in collective argumentations. It is a feature of negotiating processes, that several participants contribute to the generation of an argumentation.



Following the concept of symbolic interactionism (Blumer 1973), such process is more than the sum of the participants' individual contributions:

- The individually uttered arguments are reflecting the dynamic of the interaction process.
- No single participant could produce all foundations or reasoning in the emerging connection and combination of statements.

Toulmin's functional analysis is too narrow to describe these dynamics in an appropriate way. Hence, we enlarged the approach so that the personal engagement of a single pupil in an argumentation can be described in detail.

1.2. *Speakers' roles in a multi-party-interaction*

When researching various forms of participating in social interaction, Bruner's (1983) concept of "formats" can be used. "Formats" are described as a

"... standardized, initially microcosmic interaction pattern between an adult and an infant that contains demarcated roles that eventually become reversible" (p. 120).

The participation in such formats includes the change of roles within the learning process. Consequently, the learning process could be described as the increasing autonomy of the child in such stable interactional structures.

The mutual attention, which is typical in such dyads (adult – infant) cannot be assumed for classroom interaction with 20-30 children. So, the concept of learning by participating in formats must be modified with respect to the peculiarity of an interaction with three or more participants. Such interaction is commonly called a "multi-party-interaction" or "multi-party-conversation" (s. Sacks 1998; Levinson 1988). Goffman 1981 offers an useful approach to multi-party-interactions.² In his general criticism of the dyadic model for conversation he demands:

- the dissolution of the speaker-hearer dyad, and furthermore
- the decomposition of the everyday terms speaker and hearer.

For the speaker aspects he invents the term *production format* and comments:

"Plainly, reciting a fully memorized text or reading aloud from a prepared script allows us to animate words we had no hand in formulating, and express opinions, beliefs, and sentiments we do not hold" (p. 145).

We take this concept up in our term *production design* for classroom interaction (see Brandt 1998 for the hearer aspects). Following Levinson's (1988) critical examination, each utterance or contribution to a collective argumentation can be split up into three analytical aspects:

1. the gestical/acoustical appearance ("sounding box"),
2. the syntactical construction with certain words and expressions ("formulation")
3. the thematic/semantic contribution to the negotiation of meaning ("content/ thought").

Each utterance contains all aspects. We base our description of the speaking person on her or his responsibility for one, two or all of these aspects (Levinson 1988; Brandt 1997; Krummheuer & Brandt 2001). In case of a speaker talking with restricted responsibility, there is a silent participant, who is responsible for the missing aspects:

speaking person				<i>non speaking person with responsibility</i>		
	sound. box	formulation	content		formulation	content
creator	+	+	+	/		
traducer	+	-	+	formulator	+	-

² See Sahlström 1997 for a detailed examination of the relevant literature.

paraphraser	+	+	-	initiator	-	+
imitator	+	-	-	inventor	+	+

We want describe learning processes as participation on collective argumentation. Thus, the argumentative function of an utterance is taken as the content in the production format. This argumentative content of an utterance is called the *explanative idea*. Using this relation, the interactive genesis of a collective argumentation can be described.

2. Empirical data and analyses of an collective argumentation

The empirical basis of our research are videotaped lessons of two classes: a first grade class and a multi-age class (1st–3rd graders). Particularly chosen video recordings were transcribed and analyzed. In this paper the example is a lesson from the first grade class. It is selected for the purpose of demonstrating the analysis method and will not be discussed as special type of collective argumentation (see for more detail Krummheuer & Brandt 2001).

2.1. Transcript: Mister X

The first graders are concerned with a kind of mathematical game, called "Mister X": One boy writes a number between 10 and 20 on the back of the blackboard (hidden) and the other children have to guess this secret number. On the front of the blackboard, the boy draws a big X and notices the guesses: the numbers smaller than the secret on the left and the bigger ones on the right side of "Mister X"). One child proposes the right solution 13 and the following collective argumentation emerges by promotion of the teacher:³

T why could it only be thirteen in the end (...) David \
David because fourteen was too big
T stop \ that's very important now \ start again \
< David because fourteen was too big because \ *lies down on his table*
< T *showing the fourteen* yes \ . and *showing the twelve*
David the. *speaking faster* twelve was too small \
T repeat this Efrem \ Efrem - . repeat this \ David told us something
very clever \ you can keep it in your minds \ who can repeat
what David told \ Petra

³ "<" marks speakers, which speaks simultaneous; / and \ a pitchraising respectively a pitchdropping.

teacher with a claim, which is to be justified. So, she is the creator of the conclusion. David's contribution because the 14 is too big is the first accepted answer. He is oriented to the interaction process, of course. Nevertheless, he offers this first data as creator. The teacher forces him to repeat his answer and in this repetition he is an imitator of his own. The gesture of the teacher is subordinated and she takes David's thought. She supports the first data as a paraphraser. David stops his explanation and the teacher takes over the guidance with her gesture: Showing the 12, she determines the content and presents the second data gestically as a creator. David puts it in the final (acoustical) form as paraphraser, with the teacher as an initiator.⁵

Finally, Petra repeats the words uttered by David, also supported by the gesture of the teacher. Repeating the datas, she is speaking as an imitator without responsibility for all aspects of her utterance. David is the initiator of the first part (because because the fourteen was too big /) For the second part (an and the twelve was to too small \), the content was initiated by the teacher. David is the formulator of this part. Now, the teacher starts with a sentence (and in between is only a /), which is finished by Petra (thirteen). In the functional analysis of argumentation, this assertion was seen as backing. So, the teacher offers the backing as creator and Petra completes this explanatory idea as an imitator. The dynamics of the collective argumentation is illustrated by the following representation. This layout combines the two demonstrated analyses steps:

speaking person/ function	utterance	explanatory idea (argumentative category)
(function of non-speaking person)		
T: creator	why could it only be thirteen in the end (...)David \	Definiteness of the solution (conclusion).
David: creator	because fourteen was to big	Blackboard: 14 is too big (data).
David: imitator (inventor: David)	because fourteen was to big	
T: paraphraser (initiator: David)	<i>showing the fourteen</i> yes \.	
T: creator	and	Another data is necessary (warrant).
T: creator	<i>showing the twelve</i>	Blackboard: 12 is too small (data).
David: paraphraser (initiator: T)	the. <i>speaking faster</i> twelve was to small \	

⁵ This is only one version of interpretation. It is also possible to describe David as an imitator. The decision between different versions must be done in the preceding interaction analysis – which is left out here. For the purpose of this paper the decision is not existential.

Petra: imitator inventor: David	because because the fourteen was too big /	Blackboard: 14 is too big (data).
T: imitator (initiator: David; formulator: T)	<i>showing the fourteen</i>	
Petra: imitator (initiator: T; formulator: David)	an and the twelve was to too small \	Blackboard: 12 is too small (data).
T: imitator (inventor: T)	<i>showing the twelve.</i>	
T: creator	and in between is only a /	Natural Order of the numbers (backing). (Definiteness of the solution as conclusion)
Petra: imitator (inventor: T)	thirteen	

3. Learning in collective argumentation

In our research project, a main goal is to describe learning by participating in formats of argumentation (see Krummheuer 1995, 1997). With respect to this aim and focussing David as learner, the layout above can be interpreted in the following way: Offering the conclusion (in form of a question) the argumentation is introduced by the teacher as creator. David can act as creator for a first data. The teacher takes this data as the upper bound of delimiting the interval. So, from her point of view, the step from the data to the conclusion is not complete. The missing data – the lower bound – was forced by the gesture of the teacher, who is the initiator of David's utterance *the. twelve was too small *. Here, David is a paraphraser. So, the teacher integrates David in a complex interaction pattern. In this process emerges a complex argumentation with two dates. Taking the delimiting of the interval as "formatted argumentation" (Krummheuer 1995), it is not sure, that David could complete his first data by the lower bound without the integration in the interaction process by the teacher. And it is not discernible, if David recognizes the logical necessity of linking the two datas. The teacher contributes this linking. Nevertheless, David acts in a relative elaborated form of autonomy (at least as paraphraser). So, he has the opportunity to catch the explanatory ideas of this (formatted) argumentation.

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LEARNING IN MATHEMATICS DURING GROUP DISCUSSIONS OF SOME RICH PROBLEMS

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Abstract: *In this paper, I discuss the interaction between pupils and between pupils and their teacher as contrasted with the pupils' own reflection and active construction of knowledge, when they were working in groups with so called rich problems. The problems were also taken up jointly in a class discussion. I observed a reciprocal combination of and co-operation between one individual's reflection and conveying of ideas to group and class mates and the interpersonal joint effort in the group. The teacher seemed to reside in a hazy background, but a closer analysis showed that her disposition of the discussion as well as her suggestions, hints, and enthusiasm were of paramount importance.*

Introduction

How do the following factors interact and support each other when a pupil learns mathematics: her co-operation with her teacher and with her class mates on one side and her own effort to reflect on mathematical concepts and problems and create her own knowledge on the other? I do not anticipate getting a full answer to this question in a short study. However, it is my hope to be able to shed a little light on it in a project, where pupils were doing group work on rich problems which was afterwards discussed in a whole-class setting.

Social Constructivism

I used social constructivism as a background for my study. As was mentioned above, I especially wanted to study the duality and the connection between the pupil's own reflection on the one hand and the co-operation between pupil and teacher and between pupils on the other. For a long time there has been an ongoing discussion between advocates of sociocultural theory with Vygotsky as precursor and those of social constructivism with Piaget as forerunner on this issue. (See e. g. Steffe & Thompson, 2000; Lerman, 2000).

It is clear that sociocultural theorists stress the social connection more, while social constructivists prefer to emphasise the role of the individual. Cobb and Yackel (1998) maintain that individual students actively contribute to the development of the classroom microculture, a culture that both allows and constrains their individual mathematical activities. They continue

This reflexive relation implies that neither an individual student's mathematical activity nor the classroom microculture can be adequately accounted for without considering the other.

(Ibid p. 161.)

This means that the individual pupil will to a great extent be helped to reflect on the mathematical content by the common discourse and will try to fit new experiences into her own knowledge structures. But the pupil herself has to construct her own knowledge. Jaworski (1998) expresses the same thing in the following way: "Individually students were developing their own meanings, but their understanding was strongly dependent on the classroom discourse". (Ibid p. 19.)

Cobb (1995) expresses the difference between Vygotsky's emphasis on the social connections and the stronger stress of (social) constructivism on the individual.

Thus, in contrast to Vygotsky's focus on the social and cultural basis of personal experience, this constructivist analysis highlights the contributions that actively interpreting individuals make to the development of local social and cultural processes ...

(Ibid p. 380.)

I believe it is clear that a useful theory for mathematical understanding, learning, and teaching "should recognize the dialectical duality between the personal and institutional facets of knowledge and its understanding" (Godino, 1996, p. 421.) An individual can never construct her knowledge in a vacuum, but, in the final step, she, herself, shapes and builds up her own knowledge, her own mathematical understanding.

Rich Problems

Quite a lot of research has been carried on into problem solving and pupils' learning of mathematics in connections with mathematically rich problems or tasks. (See e. g. Gravemeijer, McClain & Stephan, 1998; Jaworski & Potari, 1998; Lubienski, 2000; Stigler & Hiebert, 1998.) One example is the research

done in the Netherlands on Realistic Mathematics Education (see e. g. Treffers, 1987). I quote Murray et al:

Initiating and sustaining mathematical development through posing problems that students have to work on has been found to be a successful way of learning mathematics, *but only if* the problems are well-designed and well-sequenced, and the classroom culture in its full complexity supports learning.

(Murray, Olivier & Human, 1998, italics in the original.)

In recent years the problems used in this field of research have often been called "rich problems", and I will use this term in the present paper. There exist several different definitions of a rich problem, but two colleagues in my school of education and I have agreed upon the following one:

- The pupil should develop his mathematical knowledge through working with the problem.
- The problem should be easy to understand, and every pupil should have the capability to work with it.
- The problem should not possess a given solution strategy, known to the solver.
- The problem should be experienced as a challenge, demand an effort, and take time.
- The problem should be solvable using several different representations.

It should be recognised that for many of these points, it will be difficult to decide beforehand if they will be fulfilled or not. This can only be settled by a thorough analysis carried out after the pupils have been working with that which was intended to be a rich problem.

The Problems Used

In my study, I used the following problems:

Problem 1

Lisa buys an ice-cream cone with two scoops with different tastes. She has a choice between three different tastes for the scoops; vanilla, strawberry, and bilberry. In how many ways is it possible for her to make her ice-cream cone? It does not matter in what order the scoops are put on top of the cone.

The problem was afterwards extended to deal with four, five, six, and ten different tastes. Lisa could still buy only two scoops. Eventually, the pupils were asked for a general method for deciding the number of possibilities, when the girl has a certain number of tastes to choose from, a number that is, however, not numerically given.

Problem 2

Three pupils meet and shake hands with each other. Everyone shakes hands with everyone else. How many handshakes will there be?

This problem was extended in a way similar to the one above to deal with four, five, and ten handshaking pupils. In this case, too, the pupils were asked to find a general idea that could be used for computing the number of handshakes, if a certain, but not numerically given, number of pupils meet and shake hands.

Beforehand, I tried to find different methods that could be used for solving the two problems. I have categorised them in the following way and also tried to order them in an ascending degree of abstraction. I got the following five categories:

- Unsystematic search with the help of counting out, i. e. for example v s, v b, b s or figure, where each letter might be shortening for a taste, e. g. v for vanilla.
- Systematic search with the help of counting out or figure.
- A recursion formula
- Creation of a formula for the calculation of a result already written down.
- A general formula, which could be used directly for all numbers.

These categories formed a theoretical network in my research, and were not conveyed to the pupils. However, it will be shown later that the pupils spontaneously used most of these possibilities.

Purpose and Questions

According to social constructivism there are three important factors affecting the individual pupil's learning:

- her own active creation of knowledge
- the interplay in the group of peers, where she is working
- her interplay with her teacher.

In this project I wanted to study a special problem-solving situation, where the pupils were working in groups with two related rich problems. I wanted to get to know what the pupils learnt by working with these problems but above all, as far as possible, find out how the above mentioned factors co-operate, when the individual pupil creates her new knowledge.

To sum up, I hoped to get answers to the following question:

- How do the pupils learn mathematics when working with rich problems?

To be able to answer this question I first looked for the answers of two sub-questions:

- What do the pupils learn through working with rich problems?
- How do they learn this according to the pupils' own understanding?

Method

I carried out the study in one class in year 5, which means that the pupils were about eleven years old. They had previously been working with their own methods for computation in the four arithmetic operations instead of using the standard algorithms. They had often formed spontaneous groups when working with exercises from their text book, but they were not used to organised group work or common class discussions about problems they had been working with.

The whole project was organised in the following way. The steps are shown in chronological order. With the class teacher's help I selected a group of three girls, Ada, Janet, and Lisa, and a group of three boys, Bertil, Gustav, and Martin, whom I intended to follow more closely. During point 5, Martin had to be replaced by another boy, David. In my role as observer, I also more or less acted as a teacher.

1. An individual written test for all pupils in the class.
2. A short survey of the text of the first problem. The first three parts of the problem were given as homework.
3. Group work with the first problem for about 40 minutes. The three last parts of the problem were not handed out until the first three had been treated.
4. Pupils in the selected groups were interviewed at the end of the group work session.

5. Group work with the second problem for about 40 minutes. This problem was not given as homework, and all parts of the problem were handed out at the same time.
6. Pupils in the selected groups were interviewed at the end of the group work session.
7. A common class discussion about the two problems lasting about 40 minutes. The discussion was led by the class teacher.
8. An individual questionnaire was handed out to all pupils.
9. An individual written test for all pupils.
10. An individual clinical interview with all the pupils in the selected groups. These pupils were also interviewed about their attitudes to mathematics, problem solving etc.
11. An interview with the class teacher.

I used several different evaluation methods:

- Clinical interviews
- Interviews with pupils, individually and in groups
- Interview with the class teacher
- Observations when the selected groups were working with problem solving
- Observation during the common class discussion

All observations were audio-taped, and I collected copies of all written documentation during group work and wrote down what was written on the board during the class discussion. The records from the observations were transcribed and analysed together with the written documentation in the way described in the results section.

During the clinical interviews, I discussed the pupils' solutions to the second test and gave them some new problems. The pupils were asked to explain their solutions and give reasons for them. All the interviews were tape-recorded and transcribed.

Results

During the group work that I followed, there occurred, of course, many different kinds of episodes. It should be noted that an episode is supposed to be finished and another one to start when something new happens: a new problem is discussed, a new strategy is suggested etc. I have tried to categorise the episodes in the following way. I will later use the words which are underlined to identify the different types.

1. Pupil(s) get stuck in a line of thought, think in a wrong way or return to a more primitive method.
2. Pupil(s) get on with the help of their peer(s).
3. Pupil(s) get on with the teacher's help.
4. A pupil takes an initiative, has got a mathematical idea.
5. Co-operation.
6. Different opinions.
7. A cognitive conflict.

In an episode classified in category 2 (peer), we can see a direct effect on the line of thought of one or several group members. An episode assigned to category 4 (idea) might, of course, also lead to an effect on the other group members' way of thinking, but it is not shown as clearly as in an episode from category 2.

A cognitive conflict (category 7) is to me a situation where one or more pupils get a result which differs from one that they have calculated in another way, and they cannot see the reason for the incongruity. In a way you might say that they get stuck (category 1), but I think it makes a great difference if they realise that there are several but mismatching solutions or if they see only one primitive or even wrong solution. Pupils having different opinions might also be regarded as a cognitive conflict, but again I want to separate this case from the one described as category 7.

First of all I will show in a table how often the different types of episodes occurred during the group work in the selected groups.

Table 1. *The number of times the different types of episodes stated above occurred during the four periods, when the selected groups were working with problem solving.*

Episode	Get stuck	Peer	Teacher	Idea	Cooperation	Different opinions	Cognitive conflict
Number of times	13	5	9	32	8	3	4

I will give some examples of the episodes. The most common type was when a pupil took an initiative and/or got an idea. This occurred 32 times among the 74 episodes shown in table 1, i.e. in almost one half of the occasions. As an example, I will show when Janet explained the recursion formula which she used in the first problem to her peers. The girls had found the number of possibilities for six tastes, 15, and Janet stated that the number for seven tastes will be 21. R H means the observer or interviewer.

R H Do you know why you have to add six there, then, when you go from six to seven tastes?

Janet I think that then you can take seven, that taste, if you have [unclear irrelevant talk], you have that taste.

R H Yes.

Janet If you say that here is, *you can say* (unclear), six first, and then you have another one, sure, then you can, then, then you have combined those, you know, so you can take, this taste you can take with that, you can take it with that, you can take it with that \...

Ada It will be six extra, like.

(Observation 23.11.99, my translation.)

Shortly after the girls had used this recursion formula, however, they returned to mark the tastes as circles and draw lines between these circles to mark the different combinations. I have classified this episode as an example of the pupils returning to a more primitive method (get stuck).

I also wish to show an example of a cognitive conflict. It occurred when the girls had solved the hand shaking problem for 20 pupils by writing the sum $19 + 18 + \dots + 1$. I helped the girls by writing the sum in the inverted order underneath and making vertical sums like this

$$\begin{array}{r}
 19 + 18 + 17 + 16 + 15 + 14 + 13 + \dots + 3 + 2 + 1 \\
 \underline{1 + 2 + 3 + 4 + 5 + 6 + 7 + \dots + 17 + 18 + 19} \\
 20 + 20 + 20 + 20 + 20 + 20 + 20 + \dots + 20 + 20 + 20
 \end{array}$$

When the girls computed the sum on the last line they got 20×19 . However, they had already got the answer for the corresponding case in the first problem, and they knew that the two problems had corresponding solutions. The girls said:

Janet Yes, but you can't take twenty times nineteen or something.

Ada Well, he had written twenty nineteen times, hadn't he.

Janet OK.

Ada And he said that he was going to do it, then it will be twenty times nineteen, and that isn't right.

Janet OK

R H No, it isn't right (pretending hesitating), how come?

(Observation 30.11.99, my translation.)

Later, however, the two girls solved the conflict in co-operation, observing that I had written the solution to the problem twice.

The class teacher led the class discussion, and I classified the episodes that occurred in the following categories. As before I underline the words that I will use to indicate a special category.

1. Suggestion from pupil/group of pupils.
2. The teacher intervenes with a suggestion, a correction or other important remark.
3. A pupil agrees.

The different types of episodes were distributed in the following way:

Table 2. *The number of times the episodes stated above occurred during the class discussion.*

Episode	Suggestion pupil	Teacher	Agrees
Number of times	21	3	1

As we can see, the class discussion was fully dominated by suggestions from pupils and from groups of pupils. There were only three instances where the teacher intervened in a decisive way except for administering the flow of solution suggestions. Once she clearly showed that she had difficulties in understanding a solution that one of the pupils suggested and asked her to explain better. On another occasion she just wanted to make sure that the class had understood another pupil's solution. Finally, it also happened once that she uttered some doubts about a suggested method of drawing lines in an unsystematic way between circles. She asked very humbly: "Won't that be very difficult?" A little later she went on: "OK. Have any of you seen any pattern in this, when you draw lines, do you see any pattern in this, so you could compute it?" (Observation 02.12.99, my translation.)

When I studied the results the pupils showed in tests and in clinical interviews and also the pupils' own opinions in questionnaires and interviews, I could estimate that they had learnt the following mathematics and problem-solving strategies, at least temporarily. Some of these may, however, overlap.

- A way to compute an arithmetical sum.
- A recursion formula for the solution of the given problems.
- To solve problems in combinatorics and to think systematically.
- To co-operate.
- Not to give up.

- To express their thoughts in words.

The pupils were also asked how they thought they learnt what they learnt. They were given the following alternatives:

By reflecting yourself
From your peers
From your teacher.

In the questionnaire, they were allowed to tick more than one alternative. The result there was:

Table 3. *Number of ticks that girls and boys, respectively put for the different alternative answers to the question: What do you think you learnt most from? The number of girls was 16, the number of boys 14. The pupils were allowed to tick more than one alternative.*

Alternative	Number of ticks, girls	Number of ticks, boys
By reflecting yourself	10	7
From your peers	12	6
From your teacher	6	2

In an interview, Gustav told me very clearly why he thought that he learnt more from his peers than from his teacher:

Gustav Yes, but he, like, if I don't know an exercise,

R H Mm.

Gustav and he knows it,

R H Mm.

Gustav Then, he writes the answer, 'cause we were both supposed to write it on the paper.

R H Yes, but do you learn anything, then, then,

Gustav Eer.

R H if he only writes the answer?

Gustav If he only writes the answer. Perhaps he has to work it out in a good way, and then, maybe, he writes it down, and then I see what he is doing, och then I learn how to do it.

R H Yes, but if Mrs Andersson (the teacher) writes how to compute it then?

Gustav Yes, then I will maybe learn a bit, then.

R H It is better, when can, when your pals work it out, do you think?

Gustav Yes.

R H Tricky this, isn't it?

Gustav Mm. Though, Mrs Andersson doesn't do the sums, Mrs Andersson only shows us a way to work it out.

(Interview with Gustav 20.12.99, my translation.)

Discussion

In the group discussions, the pupils were inventive and lots of different strategies appeared in the different groups. As I see it, this variety of ideas were, to a great extent, due to the individual pupil's initiative, partly caused by the possibility she had had to immerse herself in the first problem during the homework.

In the two groups which I followed the work developed in quite different directions. The girls worked with a recursion formula in the first problem, while the boys used combinations of letters, each letter being shortening for a taste. During the class discussion still other solution methods turned up. As mentioned, this was often caused by an initiative from one single pupil, but, at the same time, I saw a lot of examples how the pupils managed to co-operate and support each other in the solution process.

In very few instances, at least in the two groups mentioned, the teacher, i. e. me, transferred his ideas to the pupils. It happened, however, when I showed both the girls and the boys a way of computing an arithmetic sum, but none of the pupils spontaneously used this method. In spite of that, several of them used another way, which was discussed during group work and class discussion.

In the class discussion, it was also very noticeable that the teacher used her power to influence only to ask a pupil to explain her idea a little more thoroughly or to make sure that the other pupils understood a given solution. On one occasion only, the teacher gave vent to a little doubt about the efficiency of a suggested solution method and asked the other pupils for a more systematic way of tackling the problem. I also noticed that the class teacher exhibited a willingness to let the pupils voice their ideas and suggestions. She created a classroom climate, where the pupils became used to putting forward their solution strategies and, above all, to listening to their peers' suggestions and ideas.

On the other hand, it is very interesting to see how one pupil valued his peers' contributions compared to those of his teacher. Of course, his perspective is another one than the researcher's, but nevertheless the pupils' points of view have a bearing on their learning process. However, in my opinion, the study shows that the teacher's behaviour paid, because her pupils readily showed that

they could meet her expectations and give their own ideas and at the same time co-operate with each other in the groups.

To sum up, I believe I have found that the individual pupil in this study brought her thoughts and ideas to her group, where they were worked up jointly and the group members, generally, agreed upon one solution which was then accounted for in front of the class as the group's joint solution. Finally, the individual pupil built up her own mathematical knowledge in connection with the items that had been brought to life by the problems, using her own previous knowledge and experience as a starting-point.

It might be that the teacher seemed to reside in a hazy background, but I would argue that the closer analysis that I tried to make, reveals something else. Her attitude to the way her pupils learn, her disposition of the discussion as well as her slight suggestions and hints, and, not least, her enthusiasm were of paramount importance.

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SOLVING AN ALGEBRA PROBLEM IN A TRIADIC SITUATION IN TENTH GRADE

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Abstract: *A social interaction setting is used in this study in order to elicit the possible transition from an arithmetic procedure to an algebraic one. The collected corpus is mainly (but not solely) made up of the written transcription of verbal exchanges. It is analyzed according to criteria that essentially relates to the conceptual task analysis. But certain linguistic aspects are taken into considerations as well. The case study presented in this paper shows that an interactive process with positive effects upon the individuals is likely to take place.*

Résumé: *Un dispositif d'interaction sociale est utilisé dans cette recherche afin de favoriser la mise en place chez les élèves d'une procédure algébrique en substitution à une procédure arithmétique. Le corpus recueilli est composé principalement (mais pas uniquement) de la transcription écrite des échanges verbaux. Il est analysé selon des critères qui renvoient essentiellement à l'analyse conceptuelle de la tâche, mais certains aspects langagiers sont également pris en compte. L'étude de cas présentée dans ce papier montre qu'une dynamique interactive ayant des effets positifs au niveau des individus est susceptible de s'établir.*

Whatever didactical approach is chosen to introduce algebra – problem solving, generalization, modelling, functional approach,... - we notice learning difficulties among students. Several studies suggest that many of these difficulties can be attributed to the conceptual change which occurs during the shift from an arithmetic mode of thinking to an algebraic one (Bednarz et Janvier, 1994 ; Schmidt et Bednarz, 1997 ; Booth, 1985 ; Filloy et Rojano, 1989; Kieran, 1990).

The purpose of the present study is not only to observe, but also to elicit the possible transition from an arithmetic procedure to an algebraic one, during a problem-solving situation. The task to be worked on, as well as the

experimental design has been chosen with care to serve the purpose of this project. Hence,

- The given problem (Ref. Appendix 1) can be partially solved with arithmetical procedures, however the complete solution requires the use of algebra (the conceptual task analysis is not presented in this paper)
- The experimental design is based on a social interaction situation, which we think will trigger positive cognitive evolution at least among some students.

Our work fits, in fact, within the general framework of the socio-constructivist approach (an approach largely inspired by the works of Piaget and Vygotski). This approach has been described in many works, of which we mention only three recent ones directly concerning didactics: Dumas-Carré and Weil-Barais, 1998; Joannaert et Vander Borgh, 1999; Gilly et al., 1999). Considering that inter-individual cognitive functioning influences the individuals' cognitive behaviour, and in order to understand in what ways students (or some students) participating in a group work can gain cognitive advantages, we will try to analyse the socio-cognitive interactions.

Experimental procedure

The experiment consists of three phases organised as follows:

- During the first phase, three students must individually solve the problem. The problem is presented in a written format. (Ref. appendix 1 for the problem).
- The individual work is interrupted thirty minutes later. The triad is then formed and during this second phase, the students should work together and agree upon a common solution.
- Then the experimenter presents to the triad a graphical representation modelling the situation. This third phase, during which they must interpret the graph, is not reported in this paper. It corresponds, in fact, to an aspect of our research problem which is not covered here.

Population

The experiment was designed as a case study and aimed at highlighting certain processes. We examined eight groups of three students each from two tenth

grade classes. However, in order to avoid over-complicating the presentation of the analysis, we will only refer here to one of these groups.

Methodology for data collection

During the first phase, instructions were given to the students to write down on their papers, step by step, all the stages of their work as well as all the calculations (papers were numbered and calculators were forbidden). The corresponding analyses for this phase of individual work were based upon the examination of these writings.

Concerning the second phase, all the interviews were audio taped and later transcribed. In addition to these transcriptions we collected the written work of the group (papers were given one by one and when asked for).

Situational context

All interviews were conducted in the school. The experimenter and the students were alone in the classroom. The mathematics teachers introduced the experimenter to the students as a colleague doing research in mathematics education. The interview varied in length depending on the groups, and reached a maximum of two hours (only the time for the first phase was fixed by the experimental procedure).

Data analysis methods

To analyse the data collected during the second phase, we divided the corpus into a chronological sequence. Every sequential element was denoted an episode. It is the conceptual dimension that plays the most important role in the division of the episodes and in making sense of what is happening among the participants. In other words, they are the ones that essentially determine the description and the explanation of the episode. So, even though some linguistic aspects were taken into consideration during the interpretations, it is the conceptual dimension which guided mainly the interpretative reading of the interactions. In appendix 2, we present an extract from the protocol of the chosen group. It regroups together several consecutive episodes (episodes 11, 12, 13).

Criteria for the analysis

Five criteria have been used to analyse the corpus corresponding to the second phase.

- The status of the letter. We adapted to our situation the categorization proposed by Janvier (1996) and Kücheman (1981) and kept three categories: Letter as a label or as an object (coded LE); Letter as an unknown (LI); Letter as a variable (LV).
- The register that the students have been using is one of the criteria (Ref. Duval 1995). Four categories were chosen: Numerical register (coded RN); Algebraic register (RA); Graphical register (RG); Natural Language (RL)
- The procedure, being a criterion which is not independent from the other preceding ones, is in fact characterised by a register, the status of the letter, and by the meaning given to the equality sign as well. This latter characteristic, classically cited in works about algebra (Ref. Vergnaud et al., 1987), comes in two categories: result announcement (SA) and equivalence (SE). In fact we consider four possibilities for the procedure: Arithmetic: (RN, SA, absence of the letter or LE); Algebraic: (RA and/or RG, SE, LI or LV) ;Arithmetic/Algebraic: (RN and RA, LE or LI, SA or SE) ; Arithmetic/Graphical: (RN then RG, LI or LV, SE).
- The rationality, we adopted the categorization of Grugeon (1997) with a slight modification and we considered four possibilities: common, pre-scientific, scientific, and school.
- The relation, with two possibilities: the presence of a relation (which can manifest itself by the use of a function, a graph...) and the absence of a relation (which can manifest itself by the use of a common argumentation).

Note: For the division of the protocols into episodes, we agreed that every change of episode corresponds to a change in the value of at least one of the criteria.

Analysis and interpretation of the extract presented in appendix 2

We can summarize the individual work of each student in the group (phase 1) as follows: each student used an arithmetical procedure, but while student G could not solve the end of the problem (Fanny's case, Ref appendix1), student D did come up with a satisfactory answer. As for student F, he/she did not tackle this part of the problem.

The extract cited comes at the end of phase two. In the preceding ten episodes the observed procedures are in general arithmetic procedures (yet there are two episodes during which the students used an Arithmetic/ Algebraic procedure after D's initiative). The four types of rationality are successively called upon.

At the end of episode 10, students agreed on a solution and wished to conclude their work by writing a sentence. The group dynamics that took place were as follow:

- Upon a formulation made by D (“...if Fanny wants to spend...”), which suggests an enumeration of possible cases, G reacts (“But she can ...”) by shifting the speech towards a common (everyday) logic. G reiterates this argumentation but, despite the strength of G's last speech (in which having recourse to the use of the pronoun I, G gets involved personally), F and D pursue their initial project, without taking into consideration in any way G's intervention. Thus at the end of episode 11, F's intervention (“Do you all agree with this?”) refers to the content of Fs debate with D.
- It is the need for a proof, expressed by D (not satisfied with the fact that the solution had been found by successive trials), which signals the beginning of episode 12. It is around this new issue that G resumes the group work (G admits having strayed from the collective project during some time (“...now I speak logically”). We can notice that G will go back to a common rationality (“you cannot take 22.5 videos...”), to claim the pertinence of the solution using successive trials. F, then D, suggested this solution once again in this episode.
- Upon a direct questioning by F (“What do you need to be able to prove?...”), D suggests a formula which he had used in a previous episode. This formula was also used previously by the three students. Then D launches into finding a relation between two formulae and writing an equation (the letter has the status of an unknown and we are in an algebraic procedure). The three students participate actively in the resolution of the equation and are convinced they reached their goal (To D, this goal is explicitly the proof: “that's it, it's proven”).

Conclusion

G gets always involved in his/her speech expressing himself/herself in a personal mode; whereas D uses only generic expressions, sometimes assorted with modal deontological expressions (“We should have tried with every number!”) or existential expressions (“there is for sure one equation which can

help us”). This opposition confirms the distance already noticed in the mastery of the task between the two students G and D.

Student F rarely expresses himself/herself in a personal mode; instead, F uses generic expressions, using pronouns such as “nous (we)” and “on (one)”, showing that he/she often expresses himself/herself on behalf of the group. What is striking concerning F, is that he directly questions D and G “Do you want to sketch a graph?”, “what do you need to be able to prove?”, spurring them on or always urging to determine where the group stands. It seems that F plays the role of the social leader while D that of the cognitive leader.

The interaction seems positive for all the students including D who found the satisfactory answer since the beginning of the first phase. In fact, in the second phase D tackles explicitly the question of the need for a proof and moves from an arithmetic procedure (successive trials) to an algebraic procedure.

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APPENDIX 1 - The problem

A video rental store offers three options for renting videos per year:

Option A: Normal rate, 30FF for each rented video.

Option B: a subscription rate of 160FF per year, and 20FF for each rented video.

Option C: a super subscription rate of 400FF per year, and 10FF for each rented video.

Annie wants to spend 420FF on video renting per year.

Bernard wants to spend 480 FF on video renting per year.

Claude wants to spend 520 FF on video renting per year.

Didier wants to spend 560 FF on video renting per year.

Emily wants to spend 600 FF on video renting per year.

Fanny wants to spend between 620 FF and 680 FF on video renting per year.

What advice would you give each one of these persons concerning the choice of the most advantageous option for them. Explain your answer.

APPENDIX 2 - The Episodes

Episode 11

F¹: So, do we add a final sentence?

G: OK

F: For option B no? How do you want me to ... (*Express it*)²

¹ These letters are made to recognize the students in the triad:

F is "Face": the student facing the experimenter

D is "Droite": for the student at the right hand side

G is "Gauche": for the student at the left hand side

² The expressions in italics are the experimenter's notes

- G: For more than 24 videos, if Fanny ...
D: For more than 640, if Fanny wants to spend more than 640
G: But she can spend, she wants to, it is in her budget
D: Yes she wants to
F: But you say: for more than 640
D: But she wants to spend between 620 and 680, more than 640 is option C
G: You know what, when somebody wants to spend between 620 and 680, at the end he/she will choose 680
F: And for less than 640 it is option B
D: B
G: I think that, for example if I want to spend a certain amount of money, you see, I fix a sum and it is clear that I will take the last one, so I take option C very easily
F: Do you all agree with this?
D: But
F: But what?

Episode 12

- D: We did not prove how we get 640
F: Yes we did
D: It is not proven
F: But yes it is
G: Yes of course
F: Yes, you have that for more than ..., here you did try for 20
G: Yes, why did we try for 24
D: Why for 24! We just bumped into it by chance
F: No on the contrary, oh yes, no but we wanted to see if..., but with option B it is 24 videos it is the same price
G: Let me see, a little bit
F: We should have tried with every number!
G: Frankly speaking, very frankly, now I am speaking logically
F: Do you want to sketch a graph?
G: No, no not at all, look when you say I want to spend between 620 and 680, how much do you spend? You cannot spend in between the two; it depends on the number of videos.
F: We must try all the solutions!
G: There are not one million solutions, you know; because it is between 22 and 28 videos, you cannot take 22.5, see! So we must try with 21, 22, 23, 24, 25, 26
F: We tried with 23 and 24
-

- D: This is it! We bump by chance, there is for sure an equation that can help us, this is it
- G: This is why I want to sketch a graph
- F: What do you need to be able to prove? Try with all the videos from 22 till 28

Episode 13

- D: Wait! $(x - 400)/10$
- F: This is option C
- D: Yes, must be equal to $(x - 160)/20$, isn't it? And we will find the price is it so?
- G: Didn't you do this before?
- D: No, so
- F: $20x$

They are trying to solve $(x - 400)/10 = (x - 160)/20$

- G: Bring it (*the 20*) to the other side, you cross multiply, and you put $20x$
- D: Yes cross multiply, $20x - 800$
- F: 8000
- D: Yes 8000, must be equal to
- G: We bring the denominator to the other side
- D: $10x$ equals
- F: Minus
- D: 6400
- F: Yes 6400
- D: So $x = 640$ this is it, it is proven
- F: OK we are done
- G/D: yes

PRAGMATIC PERSPECTIVES ON MATHEMATICS DISCOURSE

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Abstract: *Within the psychological tradition, analysis of students' utterances has focused on the 'transactional' function of language in the expression of 'factual' propositional content. The clinical interview is an effective means of eliciting data with this (though not only this) objective (Rowland, 1999a, b). The purpose of this paper is to emphasise that language also fulfils an essential 'interactional' function in expressing social relations and propositional attitude, and to argue for the significance of such dimensions in the analysis of discourses of a mathematical nature. The paper explicates the nature of 'pragmatic'¹ meaning, reviews some approaches to discourse rooted in or related to speech act theory (Austin, 1962; Searle, 1969), and concludes with analysis of a fragment of text for illustrative purposes.*

Meaning

There exist a number of accounts of 'meaning' within natural language. Accounts differ partly because of different perspectives deriving from linguistics, philosophy, psychology, sociology, literary criticism, theology, and so on. In discussing meaning in this paper, I have in mind a view of communication that focuses on human intention, and my starting point is what the philosopher Paul Grice (1957) called 'speaker meaning'. Grice distinguishes between two kinds of meaning: natural and non-natural. The first of these might be called the semantic or truth-conditional meaning of sentences such as:

if $x > 2$ then $x^2 > 4$

This is meaning of a conventional, literal kind. Of course, such a notion is not unproblematic, pre-supposing as it does that words and symbols refer to things in an unambiguous way, and that the syntax of the sentence then takes care of the meaning.

¹ It should soon become clear that 'pragmatic' is used here, and throughout in this paper, in a technical sense, so that it means something more specific than just 'expedient' or 'dogma-free'.

Non-natural meaning is associated with (human) communication which is intended to be received in a particular way by a recipient. This is the ‘pragmatic’ meaning, which *may* be conventional, but certainly need not be in the case, for example, of ironic, metaphorical and indirect communications. A critical feature of non-natural meaning is the particular way that it is *intended* to be recognised and interpreted by a recipient. The distinction between the truth-conditional, semantic meaning and the pragmatic meaning of an utterance is demonstrated by B’s turn in the following interchange.

A: Do you think you’ll get back home in time to cut the grass?

B: Well, I’ll try, but there are road works on the A14 this week.

The semantic content of the reply is to the effect that B will endeavour to get home on time, and that there are works on a particular road. In practice, we read much more into it. Even in the absence of ‘well’, we might infer *inter alia* that:

- B will travel by road,
- the road works on the A14 might cause B to be delayed,
- B is not confident that s/he will be home in time to cut the grass.

The inclusion of ‘well’ adds an additional dimension to the reply: that is, B’s anticipation that his or her reply is not the one that A would like to hear (Wierzbicka, 1976; Brockway, 1981).

In an account of the ‘transactional’ and ‘interactional’ functions² of language, Brown and Yule touch on a distinction which seems to parallel that between semantic and pragmatic meaning.

That function which language serves in the expression of content we describe as *transactional*, and that function involved in expressing social relations we will describe as *interactional*.

Whereas linguists, philosophers of language and psycholinguists have, in general, paid attention to the use of language for the transmission of ‘factual propositional information’, sociologists and sociolinguists have been particularly concerned with the use of language to negotiate role-

² This binary subdivision of the functions of language is adequate for our present purposes. A finer taxonomy of speech functions due to Roman Jakobson (1960) identifies six functions: referential, emotive, conative, phatic, metalingual and poetic.

relationships, peer-solidarity, the exchange of turns in a conversation, the saving of face of both speaker and hearer. (Brown and Yule 1983, 1-4)³

The importance of the transactional function of language in the teaching and learning of mathematics is self-evident. Michael Halliday, a linguist, leaves us in no doubt as to the educational significance of the interactional function.

If we consider the language of a child, there is good evidence to suggest that control over language in its interpersonal function is as crucial to educational success as its control over the expression of content, for it is through this function that the child learns to participate, as an individual, and to express and develop his own personality and his own uniqueness. (Halliday, 1976, 197-8)

Pragmatic meaning is the means frequently (though not necessarily consciously) used by speakers to convey affective messages to do with social relations, attitudes and beliefs, or to associate or distance themselves from the propositions they articulate. That is to say, pragmatic meaning is an important tool in fulfilling the interactional function of language.

Shiffrin (1994) and Jaworski and Coupland (1999) give excellent surveys of a wide range of approaches to discourse which are particularly sensitive to pragmatic meaning and interactional function. The following sections of this paper give a necessarily brief overview of three contributions to pragmatic analysis. Each points to possible means whereby non-natural meaning might be conveyed by a speaker, and each will play a part in the scrutiny of a teaching episode towards the end of this paper.

Speech Acts

In the late 1950s, the Oxford philosopher John Austin gave some lectures on how speakers “do things with words”, and so invented a theory of ‘speech acts’ (Austin, 1962) which now occupies a central place in pragmatics. The essential property of speech acts is that they bring about (or have the potential to bring about) a change in some state of affairs. Paradigm examples include the naming of a ship, the joining of two persons in marriage, and the sentencing of a criminal. The name, the marriage and the sentence are what they are because an authorised person has declared them to be so. Austin distinguished between the *locution* of a speech act (the words uttered), its *illocution* (the intention of the

³ Grice and Austin demonstrate that philosophers of language are by no means exclusively preoccupied with transaction.

speaker in making the utterance) and its *perlocution* (its effects, intended or otherwise).

Whereas declarative utterances typically have truth conditions, speech acts must satisfy certain ‘felicity conditions’ in order to ‘count’ as an action. For example, the felicity conditions for a *question* include the expectation that the enquirer doesn’t know the answer, that s/he would like to know it, and has reason to believe the hearer is able to supply it. Questions in classroom situations are curiously exempted from these rules (Labov and Fanshel, 1977).

One aspect of speech act theory with significant pragmatic implications concerns *indirectness*. Three broad illocutionary categories are normally identified - statement, question and command/request - having typical realisations in declarative, interrogative and imperative verb forms. These agreements between intended function and realised form break down in ‘indirect speech acts’, in which the outward (locutionary) form of an utterance does not correspond with the intended illocutionary force of the speech act which it performs (Levinson, 1983, pp. 263ff). Common forms of this are to declare a preference or to use an interrogative form in order to convey an order or request. For example:

Teacher: I’d like to take in your exercise books.

Diner: Can you bring me the wine list?

These are both instances of how speakers frequently accomplish an indirect speech act by stating or questioning one of the felicity conditions (Gordon and Lakoff, 1971). The teacher explicitly *states* his wish to receive the books i.e. that s/he meets the felicity condition to do with speaker sincerity; the diner *questions* the ability of the waiter to provide the list i.e. s/he questions one of the preparatory pre-conditions.

Politeness

So why should speakers in general, and teachers in particular, be indirect in this way?

One insight into such indirectness in classroom mathematics talk is provided by a sociolinguistic theory of ‘politeness’ developed in the late 1970s. This theory claims that speakers avoid threats to the ‘face’ of those they address by various forms of indirectness, vagueness, and so on, and thereby ‘implicate’ (hint at) their meanings rather than assert them directly. Politeness theory (Brown and Levinson, 1987) is based on the notion that participants are rational

beings, but with two kinds of ‘face wants’ connected with their public self-image:

- positive face - a desire to be appreciated and valued by others; desire for approval;
- negative face - concern for certain personal rights and freedoms, such as autonomy to choose actions, claims on territory, and so on; desire to be unimpeded.

Now some acts (‘face threatening acts’, or FTAs) intrinsically threaten face. Orders and requests, for example, threaten negative face, whereas criticism and disagreement threaten positive face. The perpetrator therefore must either avoid such acts altogether (which may be impossible for a host of reasons, including concern for her/his own face) or find ways of performing them whilst mitigating their FTA effect.

Brown and Levinson identify and catalogue a number of related linguistic strategies, including quasi-interrogative commands (such as that of the Diner, above) which redress the threat to the addressee’s negative face, their autonomy, respecting their right to refuse. These include conventionally polite, indirect speech acts such as “Can you pass the salt, please?”.

Hedges

The linguist George Lakoff coined the term ‘a hedge’ for a word or phrase that makes a proposition “fuzzy” or vague in some way (Lakoff, 1972). A hedge can be categorised either as a ‘shield’ or as an ‘approximator’.

Maybe, *probably* and *possibly* are examples of so-called ‘plausibility shields’, which stand outside a substantive proposition (e.g. ‘[Maybe] we should call a taxi’) and point to something less than complete commitment to it. One of the functions of shield-hedges is to protect the speaker from accusation of being committed to a false proposition (Channell, 1994). Incidentally, teachers more often use ‘attribution shields’ such as “so-and-so says that ...”, distancing themselves from a proposition by attributing it to someone else. This is a pedagogical strategy which avoids ‘closing’ on a problem, in order to sustain discussion and invite a variety of proposals. For example:

Teacher: John says you can’t divide 739 by 9. What do other people think?

Approximators such as *about*, *around* and *approximately*, as well as *sort of*, *kind of* and *basically*, can, like shields, also have the effect of withholding

commitment to a proposition. They achieve this by inserting vagueness into the substantive proposition itself (e.g. ‘The taxi will be here in [about] ten minutes’).

Here, a 14-year-old boy makes a prediction, but the vagueness of his answer suggests that it was far from secure:

Allan: The maximum will probably be, er, the least’ll probably be ‘bout fifteen.

Allan hedges his *prediction* in two ways: the shield ‘probably’ is reinforced with the approximator ‘(a)bout’. The very act of complying with the teacher’s request for a prediction is a threat to Allan’s positive face, since he could be thought foolish if his prediction were subsequently found to be in error. The shield makes his lack of commitment explicit; use of the approximator ‘about’ is a more subtle protective strategy, for it renders Allan’s answer “almost unfalsifiable” (Sadock, 1977, p. 437).

There follows a fragment from one mathematics lesson. The discussion of the interaction here *focuses* on the pragmatic meanings of some of the utterances within the text, with concern for the beliefs and attitudes of the three participants towards the subject-matter and each other. The choice of this fragment is to some extent determined by the ‘conjecturing atmosphere’ (Mason, 1988, p. 9) that permeates the episode. It is this factor, I believe, that makes the ‘conversation’ below a special kind of discourse. As I have argued in Rowland (1999b), such an atmosphere involves the student (and possibly the teacher) in taking risks, in articulating generalisations on the basis of partial evidence. Evidence of the management of such risks, by teacher and student, is to be found in pragmatic discourse analysis. The discussion which follows also involves reference to aspects of justification and proof which have not been previewed in this introduction, but will be familiar to researchers in mathematics education.

Discourse analysis: Hazel

Hazel, an elementary school teacher, describes 10-year-old Faye and Donna as able mathematicians who often work together. Her discussion with them is essentially an exploration of the following problem.

Take three equally spaced numbers, such as 10, 13, 16. Find the product of the outer pair [$10 \times 16 = 160$] and the square of the middle term [$13 \times 13 = 169$]. The difference is 9. What will happen if you take other

similar number-triples? What if you take a common difference other than 3?

Hazel's conversation with the two girls falls into four episodes, the first of which is the main focus for this analysis:

Episode 1: Investigation of the case when the common difference is 1
[turns 1-61]

Episode 2: Investigation of the case when the common difference is 2
[62-105]

Episode 3: Investigation of the case when the common difference is 3
[106-120]

Episode 4: Search for a higher-level generalisation which includes the three generalisations arrived at inductively in the previous episodes as special cases [121-160].

In every case Hazel's instructions and requests to the two girls are presented as indirect speech acts, for example (there are many):

17 Hazel: Shall we try it out and see what happens? Do you want to each choose your own set of consecutive numbers?

66 Hazel: Right would you like to try out with ten, twelve and fourteen one of you and the other one can try another jump.

130 Hazel: Can you tell me what the difference in the answers of the two sums that, the two multiplications you're doing would be when you have a difference of four between each number?

17 and 66 are on-record FTAs, 'orders' presented as questions out of respect for the children's negative face, as Hazel imposes on their personal autonomy of action. These are conventionally indirect. She believes that the investigation will be a worthwhile, educative experience for them with a potentially stimulating outcome. Nonetheless she recognises the risk-taking which is inherent in her quasi-empirical approach, and that she requires their cooperation as active participants in the project as they generate confirming instances of generalisations-to-come. In [17] she says "Shall we try it out?", the plural form including and identifying herself as a partner in the enterprise. In [130] she probes for a prediction, and realises the threat to the girls' positive face - what if they fail to make a correct prediction, will their reputation as "good mathematicians" be dented? [130] respects their positive face, and the indirect modal form redresses the on-record FTA. These features of Hazel's language are manifestations of her 'sensitivity to students' (Jaworski, 1994). Fallibilistic teaching, inviting conjectures and the associated intellectual risks is unimaginable if the teacher is not aware of the FTAs that are likely to be woven into her/his questions and 'invitations' to active participation. Redressive action

dulls the sharp edge of the interactive demands that this style places on the learner. For Hazel, notwithstanding her authority in her own classroom, the indirect speech act has become a pedagogic habit.

Early in the conversation, Faye [9] observes a difference of 1 between 10×12 and 11^2 . Somewhat precipitately, perhaps, Hazel asks:

- 10 Hazel: One number difference ... do you think that will always happen when we do this ...?

Faye readily agrees, but Hazel, perhaps realising that she has not probed but has 'led the witness' seems to want to give them more of an option to disagree.

- 12 Hazel: What makes you think that? Just 'cos I asked it ... or ...?

Donna gives hedged agreement [14]: Hazel invites her [15] to account for her provisional belief.

- 14 Donna: I think so.

- 15 Hazel: Why?

Arguably this is a tough question - to account for a belief that one is not really committed to anyway. Donna's justification [16] is phenomenological rather than structural.

- 16 Donna: Well if um ... if it's after each other like ten, eleven, twelve ... um ... it will be one more because it's one more going up.

It is the basis of a subsequent higher-level generalisation at the beginning of Episode 2.

- 62 Hazel: Okay. Right, what would happen if you had numbers that jumped up in two instead of one, so you had ten, twelve and fourteen?

- 63 Faye: I think the answer is a two number difference. So two.

- 64 Donna: Yeah, yeah. So do I.

The substantive proposition in [63] - that there is a two number difference - is, in fact, false. By prefacing it with a Shield, Faye marks her utterance as a conjecture, withholding commitment to it.

Returning to Episode 1: Hazel encourages the children to try out two more examples with three consecutive integers. They obtain a difference of 1 in

each case and Faye [27] affirms her belief (unhedged) that, as Hazel puts it [26], “that will always happen”.

- 26 Hazel: Do you think that will always happen then?
27 Faye: Yes.
28 Hazel: How can you say for certain ‘cos you’ve only tried out three examples?

Donna offers a brief diversion:

- 35 Donna: I don’t think it will happen if you do like eleven, fourteen, twenty-two.
36 Hazel: But you’re talking about the one that ... if you always have a set of three consecutive numbers will it work?

Her “like eleven, fourteen, twenty-two” is a delightful example of a vague generality; what like-ness does she intend to point to with this single example? It is difficult to judge how Hazel interprets it, except that she takes it to exclude “three consecutive numbers” - and perhaps this is precisely what Donna intended to convey through her example. Evidently ‘consecutive’ is a useful but neglected item in the mathematical lexicon.

Faye brings the discussion back on course with a request for a what philosophers of science might call a ‘crucial experiment’ (Balacheff, 1988) - testing the conjecture with an example well outside the range so far considered, to explore the extent of its validity.

- 38 Faye: I’d like to try it out in the hundreds.

Donna’s choice for the experiment seems to be guided by Hazel:

- 39 Hazel: [to Donna] You want one difference between each of those. If you’re going to start with a hundred you could have a hundred and one, a hundred and one and a hundred and two. Would you like a calculator ...?

Faye makes an independent choice [60] of 110, 111, 112:

- 51 Faye: I still get one number different.
52 Hazel: So that ... so do you ... will it always work d’you think?
53 Faye: Yeah ... I think.
54 Hazel: How can you be sure?
55 Donna: Umm
56 Faye: [laughing] Well ...
57 Hazel: Are you sure?

- 58 Faye: Well not really, but ...
59 Donna: Quite yeah.
60 Faye: I think so. Yeah quite sure. Because it has worked because we've done ten, eleven ... Well I've done ten, eleven, twelve, nine, ten, eleven which are quite similar and then I've jumped to, um, um ... a hundred and ten, a hundred and eleven, and a hundred and twelve. It's quite a big difference. So yeah?
61 Donna: Yeah so do I.

By this stage Hazel seems reluctant [52] to influence their commitment to the generalisation (the 'it' that 'always works'). Faye's intellectual honesty is very evident here. Her crucial experiment [60] provides another (presumably weighty) confirming instance of the generalisation [51] yet her assent to it is still hedged, partial [53]. One senses that Hazel has created, or nurtured, a 'Zone of Conjectural Neutrality' (Rowland, 1999b) in which Faye understands that it is the conjecture ('it always works') which is on trial, not herself. She is free to believe or to doubt. Nevertheless, her 'well's [56, 58] indicate that she senses, perhaps, that it would be easier if she agreed - that agreement would better respect Hazel's positive face wants - for Hazel would gain satisfaction from Faye's coming-to-know.

Conclusion

Classroom talk is a rich resource for the analysis of students' cognitive structuring of mathematics, in which student errors are a particularly rich basis for conjectures about fundamental mathematical misconceptions. Such analyses provide essential diagnostic insights into individual knowledge construction. Reports of such analyses are typically set within a framework of knowledge about mathematical cognition. This may be of a general kind, to do with concept formation, abstraction and so on, or related to knowledge about the construction of knowledge in particular topic areas such as fractions or functions.

Concern for more interactional features of the classroom, such as students' propositional attitudes and teachers' sensitivity to their students' self-esteem, necessitates rather a different set of lenses and analytical tools from which to view texts. Linguistic tools which focus on pragmatic meaning have significant potential for text analysis in mathematics education, especially in research into social and affective factors in the teaching and learning of mathematics. Further evidence of this potential is given in Rowland (1999b). A recent paper by Bills (2000) explores the prevalence, purpose and effect of a

range of politeness strategies in mathematical dialogue involving a teacher and two 17-year-old students.

Any analysis of classroom interaction involves the selection and application of analytic perspectives, and pragmatic tools are as yet novel in the field of mathematics education. It will be interesting to see what further insights they yield for researchers as they come into more general use, what the pedagogical application of such insights might be, and whether teachers perceive them as valuable.

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A MODEL FOR EXAMINING TEACHERS' DIDACTIC ACTION IN MATHEMATICS, THE CASE OF THE GAME "RACE TO 20"¹

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Abstract: *This paper is based on an exploratory study aimed at providing a better understanding of the teaching process. By analyzing interactions between teachers and students in a particular situation (Race to 20), we developed a model for examining teachers' didactic action. This model is rooted in a number of theoretical concepts in French didactics, i.e. topogenesis and chronogenesis management, monitoring the didactic contract, the devoluting and instituting processes. The teaching process is considered from a theoretical standpoint involving three levels of description: the fundamental structures of the didactical relationship, the types of tasks teachers have to meet the demands of that relationship, and the classes of techniques they have to produce in order to carry out those tasks. Through the use of this model, we emphasize the necessity of considering these different types of techniques as mixed techniques that are shaped by the different constraints of the teacher's action inside the didactic relationship.*

Introduction

1. Teachers' action

1a. General purpose

This paper is based on an exploratory study, aimed at providing a better understanding of the teaching process. The research published on this topic in the past few years is plentiful, and has been conducted from different points of view. In this paper, attention must be paid to two distinctive features of our work. First, this exploratory study is based on an empirical observation of student-teacher interactions. These interactions are scrutinized using a fined-grained analysis method, and they are characterized both in "natural" language,

¹ For a description of a didactical use of this game, see Brousseau (1998), chap. 1.

and in the theoretical language of mathematics didactics. Second - and this is a crucial point - we do not consider the teachers' action from a *prescriptive* viewpoint. We try to account for teachers' specific rationality. This means we do not evaluate and deplore the gap between their actual practices and some "right way of teaching" pronounced by the "authorities". Our point of view is a descriptive, comprehensive, explanatory one.

1b. Human action, the teachers' action

We cannot mention all of the general features of our theoretical framework for analyzing the human action. Only, an outline of such a frame can be given. We consider human action, and more specifically a teacher's action, to be as follows. Action is expressed in terms of *dispositions*, which are elaborated by adaptation to particular *situations*. *Didactic institutions* are settings for these situations.

In order to understand the teacher's action, let us first state some features of its general structure as it is modelled in this paper. In paragraph 4b, this structure will be described in greater detail. In a general way, we shall say that the teacher's work consists of initiating, establishing, and monitoring the didactic relationship, which is a ternary relationship between the teacher, the students, and knowledge. In doing so, the teacher has to produce and apply numerous techniques, which are usually communicative techniques. We call these techniques *didactic techniques*. So, if we want account for the teacher's action, we have to describe, understand, and explain the techniques used to teach, i.e., to enable students to appropriate knowledge.

2. The theoretical concepts of the model

In order to describe the teacher's work, we will use the theoretical framework in French didactics, developed by Brousseau (1998) and Chevallard (1999). As Brousseau (1998, p. 19) says: "[...] what is at issue... is to describe certain kind of human relationships in such a way the concepts of *didactique* are made to appear in order to serve as useful means to description [...]". These concepts will be presented as Brousseau or Chevallard defined them.

2a. The didactic contract

In the didactic system (built from three subsystems: the teacher sub-system, the knowledge content subsystem, and the student subsystem), "[...] a relation is

formed which determines - explicitly to some extent, by mainly implicitly - what each partner, the teacher and the student, will have the responsibility for managing, and, in some way or other, be responsible to the other person for. This system of reciprocal obligation resembles a contract. What interest us here is the didactical contract, that is to say, the part of the contract which is specific to the 'content', the target mathematical knowledge." (Brousseau, 1998, p. 31). Brousseau suggests that "[...] this interplay of obligations is not exactly a contract, [because] it cannot be made completely explicit. There are no known, recognized, sufficient ways of allowing the construction of new knowledge of or ensuring, against all resistance, the student's appropriation of the target knowledge." (Brousseau, 1998, p. 32). So Brousseau emphasizes the following point: "The theoretical concept in *didactique* is therefore not the contract (the good, the bad, the true, or the false contract), but the hypothetical *process of finding a contract*. It is this process which represents the observations and must model and explain them" (Ibid.). In our modelling of the teacher action, we have to use the dual concept of *adidactic situation* and *setting* (named "milieu", in Brousseau, 1998) which is related to the didactic contract.

2b. *The adidactic situation and the setting*

An adidactic situation is a learning environment designed by a teacher. Three criteria for this. First, the student must not be conscious of the teacher's intentions about the knowledge. Second, the student is engaged in a game "bringing together a 'milieu' and a 'player', with this game being such that a given piece of knowledge will appear as *the* means of producing winning strategies" (Brousseau, 1998, p.57). Brousseau thus presents the milieu (the setting) as "the system opposing the taught system" (Ibid). We emphasize these point: the setting in a non didactical situation is incapable of provoking learning at all, most of didactical virtue being contained in the didactic contract; the student interaction with a setting can be expected to provoke the expected adaptations and learning when imbedded in a didactical contract i.e. in an adidactic situation. Thus, and this is the third criterion, the students must be aware of this: the setting is designed by a teacher, and knowledge acquisition is the expected effect of playing the game. Such an adidactical situation (the Race to 20), and its setting are presented later in this paper, so that these definitions will no longer be abstract.

We'll use these concepts in a description of the teacher's action. In fact, the teacher can be considered as someone who, at all times in the didactical relationship, is trying to build a setting for the student's action and to ensure that the student has elaborated the right relationship to that setting. We will see that

a very large set of didactic techniques are ones we can call the *setting establishment* techniques.

3. The teaching tasks

3a. *Monitoring adidactic situations: the devoluting and the instituting processes*

In order to learn, the student must become engaged in the learning situation and build a relationship with the setting that defines this adidactic situation. “Devolution is the act by the teacher makes the student accept the responsibility for an (adidactical) learning situation or for a problem, and accepts the consequences of this transfer of this responsibility.” (Brousseau, 1998, p. 230). In our model, the devolution process is one of the fundamental structures of the teacher’s action. Another fundamental structure of this action, symmetrical to devolution, is the instituting process, when “[the teacher] defines the relationships that can be allowed between the student's 'free' behaviour or production and the cultural or scientific knowledge and the didactical project; she provides a way of reading these activities and gives them a status.” (Brousseau, 1998, p.56).

Devoluting and instituting processes are two sorts of teachers' tasks in monitoring adidactic situations. They are subtle processes in the didactical relationship, running through the whole learning-teaching process.

3b. *Managing the didactical contract: topogenesis and chronogenesis*

The core of a learning-teaching process can be viewed as follows:

- a) At any time in this process, the teacher and the student have a specific set of tasks to carry out their mathematical works. This division of didactical labour has been called *topogenesis* (Chevallard, 1991). Each participant in the didactical relationship has a *topos*, i.e. a specific set of tasks to accomplish, which define his position in the didactic system. These reciprocal positions develop throughout every learning-teaching processes. *Topogenesis monitoring* is synonymous with producing and managing teacher's and student's topos.
- b) In order to describe the development of the teacher and the student mathematical works, Chevallard (1991) proposed the concept of *didactic time*. That is the time of the teaching progression through the study of

knowledge. In fact, the teacher's action is constrained by the necessity of presenting to his students a body of knowledge, part by part, shaped for teaching. So, in their action, teachers have to give a certain amount of time to pieces of knowledge, in order to cover its content. When teachers give up some item of knowledge, replacing it by a new one, they produce an unit of didactic time. This type of monitoring implies an efficient pacing (Mercier, 1992, 1995; Sensevy, 1996). *Chronogenesis monitoring* is synonymous with producing and managing didactic time.

We can now consider the didactic contract as rooted in two dimensions of teachers' action. First teachers have to ensure the running of didactic time, i.e. monitor the chronogenesis; second, at every moment in this chronogenesis, they have to make the reciprocal positions of the student and the teacher clear, i.e. monitor the topogenesis.

4. Empirical study

4a. Race to 20

Brousseau designed and used it as a paradigm to introduce most of the main features of the Theory of Didactical Situations. The game is played by two players. The first player says "1" or "2" (for example, "1"); the other continues by adding 1 or 2 to this number ("2" for example) and saying the result (which would be "3" in this example); the first person then continues by adding 1 or 2 to this number ("1" for example) and saying the result (which would be "4" in this example); and so on. Each player tries to reach "20". The winning series for one player (2, 5, 8, 11, 14, 17, 20) is 20 modulo 3 i.e. the numbers $p=20-3*n$. The quotient of the division $20 \div 3$ gives the number of stages which is necessary to win ($20 \div 3 = 6$ means there are 6 steps to reach 20); the rest of this division gives the "starting" number (for the race to 20, 2). The race to 20 (by adding 1 or 2) is a peculiar case of the general race to n (by adding (p , or $p - 1$, or $p - 2$, ..., or 1). Brousseau organized several phases in the teaching process based on this game, referring to different types of didactical situations (action, formulation, and validation).

4b. Methodology

This paper is based on a study of 5 lessons in the Race to 20, given in the 5th grade by 5 teachers (4 regular teachers, 1 student teacher). The methodological device and procedure was as follows. The researchers gave the teachers a text

that described the Race to 20. It was a simple description of a few lines, without the "solution" and without any theoretical or mathematical terms, similar to the description we presented above at the beginning of paragraph 3a. In order to make sure the teacher understood the meaning of this text, the researcher played the game with the teacher, without giving any information. It was the first time the teachers came in contact with this situation.

Then the researcher asked teachers to organize a mathematics lesson based on this game in their classroom. Teachers were free to design the lesson as desired. The methodology of this study, mainly designed by Schubauer-Leoni and Leutenegger (1997), and Leutenegger (1999), can be summarized as follows: a) Each teacher engaged in a "preactive" interview called the "anteinterview" (before the lesson), in which he/she was asked about his intentions regarding the lesson. b) The lesson was videotaped, and transcriptions were made of the audio part of the videotapes. c) Each teacher engaged in a "postactive" interview (immediately after the lesson). This interview was conducted "blindfold" i.e., by a researcher who had not seen the lesson. d) A final interview was conducted several weeks after the lesson. It was used a self-confrontation interview based on videotape (or the transcription) of the lesson, for the teacher confrontation with his/her own teaching.

In the present paper, we describe this methodology to situate the general context of this study, but we do not utilize all its possibilities. Rather, we focus our analysis on a single lesson, produced by a particular teacher.

4c. Three classes of techniques

In this paragraph, we present three fundamental classes of techniques. They are grounded on what we considered the three fundamental functions of the teacher's didactical action: elaborating the *topogenesis*; managing the *chronogenesis*; establishing the *setting*.

The description of the teaching techniques is organized as follows: the left part of the table is devoted to the transcription of classroom interactions and the right part is devoted to the simple description of this action in "natural semantics" (i.e. in the words of someone "familiar enough" with the action). After this, the same action is analyzed from our theoretical viewpoint. The teacher is an experienced teacher who has elaborated the lesson in alternating phases of pair work, group work, and whole-class debate.

A topogenetic technique: a move of devolution

In a general way, the devolution produced in this lesson (and in the other "Race to 20" lessons studied) is a devolution of the *correct action to play the game*.

	So you have to play two rounds and then, w'll try to figure out, uh, can we try to figure out who will be the winner... and when we know that?	Indication of the number of rounds to play (re-using of a relevant feature in a student's sentence). Incentive to predict (anticipate the winner) and identify (when we know that).
68	Well I don't know	Affirmation of ignorance.
	W'll try to find the rule... Ok, it works, we try to go...	Indication: use of the term "rule".
150	Ah! You think it comes from the starting number.	<i>Monitoring a dialogue (you think) and indication:</i> introduction of a meaning specific to the game, "starting number".
152	Therefore, you think it's this way?	Incentive to develop a logic argumentation (therefore). Asking for a confirmation.
154	Well, I don't know, we have to see.	Affirmation of ignorance.

Devolution is a process. This process runs through the whole learning-teaching process. Here, the technique of devolution was characterized by a "topogenetic boundary" change: the teacher said, "well I don't know", or "well I don't know, we have to see". Thus, by affirming his "ignorance" (true or false, it does not matter), he emphasized a certain symmetry with the students in this didactical work. We can consider this as an *up-down topogenetic move*: the distance between the students' topos and the teacher's topos is reduced.

On the other hand, the topogenetic move could be a *bottom-up* one, and increase the topogenetic distance. For example, when the teacher said "for me, I would like to listen to the Reds" (the "Reds" are one of the groups into which the teacher has split the class), he increased the distance between himself and the student: he acted as an expert who focused the student's attention on a particular point. In contrast when he said, "well I don't know", or "well I don't know, we must see", the teacher stood in a (fictitious) researcher's position. He acknowledged a certain kind of ignorance for himself, which legitimized the students' ignorance and urged them to take on a searcher's position. In saying that, the teacher meant "I, who most of the time knows very well what I am teaching, I move to a searcher's position, and I seek an answer which I don't possess *a priori*. Therefore we have to search." *We*, i.e. *you*. Of course, teacher's ignorance was fictitious, for he announced, at the beginning of the lesson, that he was able to win "all the time". This up-down topogenetic move is a technique for managing the didactic contract in a devolution process: the

students have to search because the (winning) "rule" or the "starting number" are new pieces of knowledge; when the teacher indicated them his declaration of ignorance "gave some space" to the students.

We can analyze this technique in the following way: the teacher made a sign to the students, and this sign had to be interpreted inside the didactical contract specific to the didactical situations. It is a weak form of the Topaze Effect, which we can call Topaze Indication. The teacher did not suggest the right answer directly, but at least for the students able to understand the specific codes of this didactical contract, suggested the proper way to work: search for the "rule" and the "starting number". This point raises an interesting issue, the question of the epistemological habitus² the students need to communicate successfully with the teacher.

We would like to emphasize a last point concerning this technique, a crucial point: it occurs when the students are confronted with a question (the "starting number") which is fundamental for mathematical understanding of the game. Thus, didactical techniques are not only linguistic and communicative, but are rooted in mathematical setting. We can conjecture that such topogenetic techniques are produced at mathematically critical times.

A chronogenetic technique

In the ante-interview, when we asked the teacher about the difficulties the students might encounter, he (and the other teachers interviewed in this study) stressed on the problem of a students' hypothesis about evenness of the winning series. Actually, the students did not have many means of acting on numbers.

When they were asked to characterize a number, they could think of evenness, because for these students, this is the only relevant property for speak about numbers (for example, they do not master the notions of multiple and factor). Therefore, before the lesson, the teacher was waiting for students to focus on this topic. His strategy for dealing with this point was elaborated in three steps.

First, at the beginning of the lesson, when certain students were searching in the direction of even numbers, and when some among them proposed this winning series (2, 4, 6...) they were ignored by the teacher, who seemed to listen to them with only one ear and did not take up their argument.

² On the notion of habitus, see Bourdieu (1990); on the notion of didactical habitus see Mercier (1986, 1992), Schubauer-Leoni (1988), Schubauer-Leoni and Perret-Clermont (1997), Sensevy (1998, 2000), Mercier, Sensevy, Schubauer-Leoni (1999).

Second, after several minutes of the lesson, the teacher was working among the groups (the class was broken down in 4 student groups) and he reacted specifically to the problem: inside a group, a student thought that 11 was a winning number but an other student in the same group was an advocate of the rule “every even number is winning”. Then the teacher brought out the contradiction, as follows:

185	You, you are thinking of even numbers, he, 11; it's not an even. I don't know	<i>focus on the contradiction</i>
189	You, you use even numbers.	
191	To go to 14, you are playing only pair numbers?	
193	And he is playing 11, thus...	

Third, later in the lesson, the problem was brought up again. Conducting the debate, the teacher gave more room to the even series proposal, as follows:

203	They have, wait, they have another theory. It's the even number theory, that's right.	<i>A competing theory is named. A virtual confrontation is organized, the term "theory" is enacted</i>
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The first time, the teacher did not react. The second time, the teacher drew the student's attention to the contradiction that involved evenness. The third time, the "conjecture" was named a theory. In order to understand this, we can propose the following interpretation.

The third time, in the teacher's mind, the discovery of the winning series was almost made: so he could bring up the even numbers idea. Contrary to the situation at the beginning of the lesson, there was no risk of weakening the "true theory". Instead of that, the teacher could conjecture that the discovery of the winning series would be strengthened by the discussion.

Obviously, the "resonance" of the teacher i.e., the way the teacher seizes or does not seize the students' answers (Comiti, Grenier, Margolinas, 1995; Comiti & Grenier, 1995) depends on the chronogenesis. In other words, a given student's statement is not taken into account at time t1, is considered in a certain way at time t2, and in an other way at time t3: these different reactions are grounded on the progress of the didactic time.

A setting establishment (mesogenetic) technique

In the learning-teaching process, the teacher has to establish the setting in which the student works. The following episode can be analyzed as a particular way of doing so.

155	Student: Yes yes, look at this! This is the infallible numbers! When we were beginning ... The teacher: So, wait, wait, stop stop stop, you are engaged already... So, shh! Please... Thus you already have some series and for, of, you...	<i>A new task is designated</i>
156	You try to use the infallible strategy by comparing your, your series.	<i>Seizing of the term "series" Proposing the term "infallible strategy"</i>

We previously used the notion of "Topaze Indication", in this case we can use the notion of "Jourdain Indication". This notion was illustrated when the teacher seized and used a word produced by the students ("infallible" numbers), and used it - this is the actual Jourdain indication - together with a meaningful teacher's word ("strategy"), in order to create the new relevant meaning of "infallible strategy".

This episode is illustrating the concept of "mimetic postulation" (Sensevy, 2000). Communication between two persons is grounded in the necessity, for a person, of considering the interlocutor as the same person oneself. If the teacher allows himself to cover the student's meaning (infallible) by his own meaning (the infallible strategy), it is because he postulates that the conceptual distance between the two expressions is small enough to be travelled by the student.³ Thus, the didactical relationship seems to increase the fundamental constraint of all types of communication: if I want him to learn, I must consider the student I teach as a rational person, who shares already with me the rationality that I want him to build.⁴

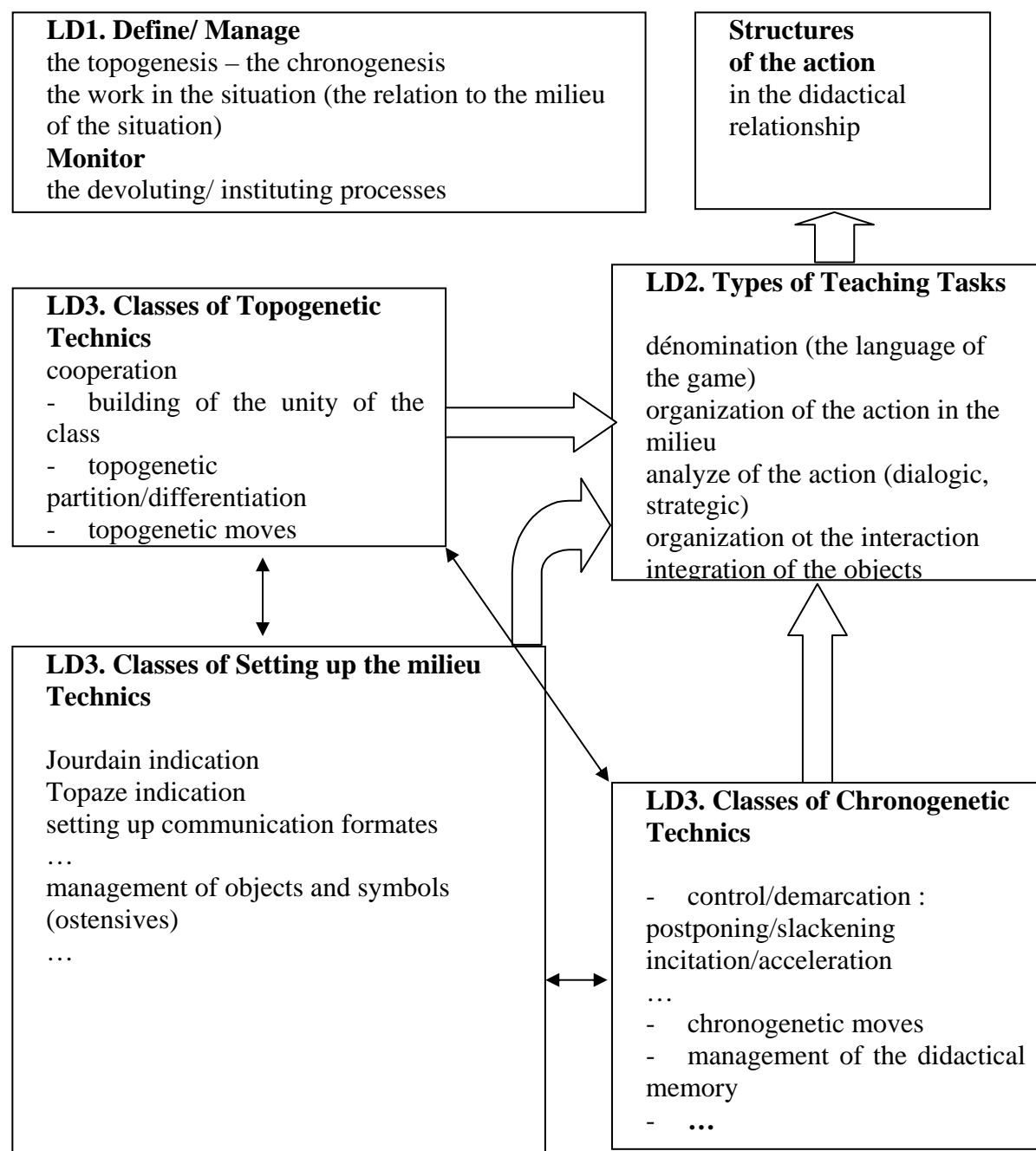
From a didactical viewpoint, this technique allows the teacher to organize the confrontation of the students' statements and monitor it (Schubauer-Leoni, 1997). If this technique is powerful enough, the setting changes: the teacher brings in the milieu some meanings that the students need to consider, and that they have to evaluate. Therefore, the *establishing the setting* consists, among

³ In psychological terms, this communicative form can be explained by using the concept of the zone of proximal development. (Cf. Vygotsky, 1962).

⁴ This is an anthropological and didactical use of the "principle of charity" described in the analytical philosophy (see for example Davidson, 1984). In this way, the Platonian paradox of the Meno can be overstepped in the temporal action of teaching (Mercier, 1996).

other features we cannot mention in this short paper, of diffusing the relevant statements of the students, if need by changing them!

4. The model



This model is produced from three levels of description of the teacher action. The first level of description (LD1) concern what we call the structures of the didactical relationship. The second level (LD2), according to Chevallard (1999), is the level of what we call the teaching tasks. The third level (LD3), which is the main topic of this paper, referred to the different classes of

techniques. This model is grounded on the study of "race to 20" lessons, but it could be, and it aims to be, a more general model of the teacher's didactical action in mathematics.

It is not possible to describe neither the prominent feature, nor the wholeness of teaching techniques or monitoring tasks. However, we want to emphasize two points:

- a) There is no bijection techniques/tasks. Several techniques of different natures (chronogenetic, topogenetic, setting-up the milieu) may converge towards a specific task. We have to remind that the tasks themselves are not isolated, but are included in a functional system of tasks, a way of teaching.
- b) Perhaps the more crucial point of this model: the techniques specify themselves reciprocally. This feature is a consequence of the very nature of the didactical action: teaching involves the pacing of didactic time (chronogenesis), the monitoring of topogenesis, and managing the effective pupils' relation to the setting. Most of times, it is not possible to isolate one of these types of action: the relevance and the efficiency of didactical processes are grounded on "mixed techniques", oriented by the functional structures of the didactical relationship.

Conclusion

In conclusion, we would like to recall the prominent features of our way of modelling the teacher's didactical action.

In order to understand and to explain the teacher action, we think that we have to:

- clearly identify the teacher action as functionally structured by the necessities of the didactical relationship
- describe his activity in terms of types of tasks responding to the fact that the didactical relationship is grounded on communicative acting
- understand that these tasks are most of time accomplished by mixed techniques, in which a particular status of the chronogenesis and the topogenesis (status that the teacher produces in cooperation with the class) specify (and is specified by) the work in progress about the knowledge.

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FORMS OF INTERACTIVE CONSTRUCTION OF NEW MATHEMATICAL KNOWLEDGE

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Abstract: *The acquisition of new knowledge represents a theoretical and practical problem for the learning and understanding of mathematics. The central meaning of the interactive construction of actual new knowledge will be discussed referring to the theory of “collective learning processes” (Max Miller). This concept will be used for a qualitative analysis of episodes [of elementary teaching] under an epistemological perspective. The analysis will lead to three forms of interactive construction of new mathematical knowledge.*

1. Introduction: Concrete visualizability or structural universality as the basis of mathematical understanding?

The common conception is that elementary school mathematics should be taught in a visual manner, with concrete material and direct references to the students' experiences. It is naturally proceeded on the assumption that the concrete visualizability of the learning material is the very thing to stimulate a sensible understanding of mathematics. But which particularities can be observed when carefully analyzing the actual course of interactive learning and teaching processes and when comparing them with regard to “concrete visualizability” and “arithmetic-structural universality”?

An important question of a research project*) tries to understand elementary school students' possibilities of constructing new mathematical knowledge in common interactions, of accomplishing exemplary generalizations as well as of explaining new mathematical relations with their own words.

Elementary school students are not able to construct new mathematical knowledge including the necessary generalizing reasons with the common concepts of elementary algebra and to operate with the yet unknown knowledge. For them, the new mathematical knowledge is tied to situative learning and experience contexts. When trying to generalize arithmetical relations, the children have to develop their own situative descriptions. Still,

they are able to see the general in the particular and to name it with their own words.

Geometrical-intuitive and arithmetical-structural learning environments (with questions to "figurate numbers" (triangular, rectangular, and square numbers) as well as, among others, to "crossing out – squares" and "number walls") have been conceived in order to stimulate these interactive construction processes. Both types are substantial, i. e. profound problems that are rich in relations (cf. Wittmann 1995), which make many activities and forms of reasoning possible for the children. The (eleven) teachers participating in the research have constructed teaching material of their choice, out of the conceived learning environments, and have carried out short teaching units (of maximally 5 lessons; 47 lessons have been documented in 3rd and 4th grades in the area of Dortmund).

2. New mathematical knowledge in interactive learning processes

For every theory of learning and acquiring knowledge, the question what the core of the new knowledge consists of respectively how the newness of the knowledge yet to learn can be characterized is of central importance. The sociologist Max Miller (1986) makes this problem of the emergence of the new the central starting point of his learning theory of "collective argumentation". He says: "Any learning or developmental theory can ... legitimately be expected to give an answer to the question how the new in the development can emerge. ... Every answer to the question ... is... bound to the following validity criterion: it has to show that the new in the development presupposes the old in the development and still systematically exceeds it, otherwise there can be no new respectively the new is already an old, and the term »learning« or »development« loses any sense" (Miller 1986, p. 18).

Miller adds three important questions: "How can ... the validity of his already acquired (old) knowledge ... be shaken or relativized for the single individual? How can the single individual make new experiences that systematically exceed his present knowledge? And how ... can there be an obligation for the single individual to further develop his knowledge ...?" (Miller 1986, p. 18/19). On the one hand, these questions aim at the problem of actual new knowledge which is not yet present in the old knowledge – for example as a mere schematic or logic deduction –, and on the other hand, they trace the social conditions under which it is possible for young people to construct this pretentious knowledge.

In entirely individualized "learning processes" that are limited to the single student and his cognition, the individual can but add knowledge derived from his present knowledge basis, which – according to Miller – therefore is no new knowledge, but identical with the old. Only interactive processes make it possible to potentially develop new knowledge by contrastings, contradictions and new interpretations. "Only in the social group and because of the social interaction processes between the members of a group, the single individual can make those experiences that make fundamental learning steps possible" (Miller 1986, p. 20/21).

In the following it will be represented, with reference to teaching episodes, how, in interactive knowledge constructions, socially constituted epistemological qualities of mathematical knowledge have consequences to the effect if real new knowledge is constructed or if, after all, it is a matter of knowledge identical to the old one. The central problem lies in Miller's statement: " ... it must show that the new in the development presupposes the old in the development and still systematically exceeds it ..." (Miller 1986, p. 18).

3. Differences of interactive knowledge constructions in the frame of geometric and arithmetical learning environments – exemplary episodes

Visual material plays a central role for the acquisition of mathematical knowledge in elementary school. Attention must be paid to the fact that they do not work automatically, but have to be actively interpreted and structured by the children. "There is no direct way from the visual material to the student's thinking, at best different difficult detours. The property of the number 3 is not visible at three smarties or three Lego bars, as if the child through simple contemplative observation could derive it. It is an abstraction, which ... does not work by merely leaving out the supposedly unimportant" (Lorenz 1995, p. 10). This abstraction has to be managed by the child itself after all, by reading new – yet invisible – relations and structures into the visual material.

In traditional mathematics teaching, arithmetical exercises are mainly assessed under the perspective of a mathematically correct procedure (Winter 1982). In the frame of substantial learning environments, not the algorithmic procedure, but the underlying algebraic structure is stressed as the essential basis of the development of arithmetical relations. Separated packets of calculation tasks or "multi-coloured dogs" (Wittmann 1990) do not contribute to that; productive exercises in which an operative structure or a substantial problem organizes the arithmetical structure are needed. Then arithmetical

exercises can be filled with meaning and illustrated in exemplary relations by a wired arithmetical structure.

The sketched orientation to the essential aspects of geometric–intuitive and arithmetic –structural learning environments constitutes an important starting point for the analysis of the interactive construction of new knowledge in exemplary episodes.

3.1. Pierre and Caroline determine the ninth square number with dot patterns

Patterns of dots are to be used structurally as geometrical-intuitive reference domains for the interpretation and development of arithmetical relations. In contrast to this, the following empirical procedure of the children could be observed in many episodes to the topic "figurate numbers": the dots usually served the children as concrete marks to directly count numbers. The collection of many single dots dominated, and it was counted step by step with partly more effective strategies; the intended structure was not actively constructed.

Pierre and Caroline, for instance, determine the demanded numbers by direct counting strategies with the help of consecutive concrete, square dot fields – which they draw themselves – in one episode. Together, they have constructed the 5., 6., 7. and 8. (square) dot field completely and always found the demanded number by counting dot by dot. The determination of the number of dots in subsequent patterns shows a refinement of the strategy from a "direct counting of all dots in the completely drawn pattern" to an "additive determination of all dots from the number of the inner (old) dots and the outer (new) border dots."

An optimization of the strategy of further square numbers can be recognized which can be classified as "empirically – counting procedure". The use of the dot field remains bound to the old knowledge; the single points are directly counted exclusively. A real new relation in the numbers or a geometrical structure in the dot field is neither constructed nor used.






3.2. Christopher and Nico determine the sixth rectangular and triangular numbers

In this lesson, the children examine the relation between triangular and rectangular (oblong) numbers. The dot fields for the first five rectangular

numbers (divided by color into two triangular configurations) have been noted on a big poster on the blackboard together with the numbers of each triangular or rectangular number. Now it is about determining the numbers and the configuration for the sixth position.

91 Ch I noticed something.

92 Ch Up there it goes four. Then it goes six. Then it goes eight. And then it goes ten. [At this point, T indicates on the left side of the board first the number 20 and then the number 30.] Then it goes twelve. [T now points at the empty field below the number 30.] Therefore there should be thirty-two on the other seventeen [2 seconds break] ohm, forty-two should be on that an on the one twenty-seven.

		Picture	
1.	2		1
2	6		3
3	12		6
4	20		10
5	30		15

Christopher names the number series 4, 6, 8, 10, 12. He apparently refers to the second column [that the teacher points at]. He seems to have in mind the respective growth between the numbers, out of which he deduces the new numbers 32 and 17 (93). In the left number column, he has constructed a number bigger by 2, in the same way as he does in the right column: from 15 to 17. He corrects the numbers to 42 and 27; both numbers are raised by 12.

94 T I see. You mean ..., that's quite an interesting idea, Christopher. You mean, here should be a forty-two? [points at the empty field below the number 30]

The teacher confirms and enters this number.

96 T Yes. And there? [points at the empty field below the number 15 at the right side of the table]

97 Ch Twenty-seven.

Christopher repeats once again that the "27" should go in the other spot. Then teacher asks for a justification, but Christopher cannot give one. Then the teacher calls Nico.

100 T No? ... Nico.

101 N Twenty-one.

102 T Why do you think twenty-one?

103 N Because twenty and twenty are forty [points at the ten's place of "42"] and one and one are two [points at the unit's place of "42"]

Nico claims "21" and reasons with a "calculation": 20 plus 20 equal 40, and 1 plus 1 equal 2.

The analysis of Christopher's knowledge construction shows that he develops a continuation principle for the 6th rectangular and triangular number out of the present arithmetic pattern. By enumerating the series 4, 6, 8, 10, 12 is meant exemplarily that the difference between the rectangular numbers always grows by "2", and therefore "12" has to be added to the 5th number now. This addition of "12" is transferred to the triangular number, and "27" is determined for the 6th triangular number. Christopher constructs a general arithmetic relation between the rectangular numbers and transfers it directly to the triangular numbers. This connection is inferred solely from the arithmetic structure; it is not proved, for instance by referring to the geometric pattern of the rectangular numbers.

In Nico's knowledge construction, the rectangular number "42" is halved exemplarily, first the ten and then the one. The connected intention – the triangular number is half of the matching rectangular number – is pronounced explicitly; the argument is limited in a tight form only to the procedure of the arithmetic halving.

Both students actually construct new knowledge, which cannot be derived from present one; they produce new arithmetic, structural relations, which were not used or known in this form before. It is striking that these knowledge relations are limited to the arithmetic number symbols, without using a connection to the geometric configurations.

Using the geometric relation between triangular and rectangular numbers, the question why the locally observed structure is really general and why it could be "always" continued could be answered. The produced new knowledge constructions are largely detached from the geometric problem; it is apparently only about arithmetic particularities and structures.

3.3. Kim justifies why the magic number is always "66"

In this teaching episode, the children work at in an arithmetic – structural learning environment on "crossing out number squares" that are constructed by adding given border numbers of a table; they have the following quality: one can circle three arbitrary numbers so that there is one and only one circled number in each row and each column. The sum of such three numbers, no matter which choice, is constant. In this example, the constant "magic

+	13	9	4
10	23	19	14
12	25	21	16
18	31	27	22

number” is “66”. The children are to do a worksheet and try to give a reason why the “magic number” is always “66”.

10 Ki [shows the teacher the first page of her copy-book with the “magic formulas” and explains the solution] Mhm, we know it now -, now. One can divide the six-, one can divide the sixty-six into three different things #

Magic Formula 1:

Calculate and fill the table!

+	13	9	4
10	23	19	14
12	25	21	16
18	31	27	22

Compare with your magic square!

12 Ki # Always. And one can do that very many times.

14 Ki So that there are nine solutions. And if one, like, takes the twenty-two, the twenty-one and the, whatever, then the result is always sixty-six. Or if one takes the thirty-one, sixteen and the nine -, ... nineteen, then the result is sixty-six again. [Kim, her partner, and the teacher appear on the screen]

Kim comes to the teacher and tells her reasons. She imagines that the “66” is divided into three different “things”; this could be done several times, until one had nine “things”. The teacher agrees and then asks why the result is not 77 or 100. Kim answers that in this particular square, there were the very numbers to obtain the 66; and if one wanted the magic number to be 77, one would have to choose other numbers for the square. Later, she exemplarily gives numbers to prove this.

Kim’s construction of the particular number square contains the idea of an “inversion”: The magic number is not the “subsequent” result of an arithmetic rule; it is put at the beginning of the construction of a number square. Therefore the sum of three circled numbers is always “66”. The “equation” $\bigcirc + \bigcirc + \bigcirc = 66$ was only understood as a rule to calculate the magic number so far. But Kim now interprets this connection so that the circled numbers can be found out by dividing the number “66” into three numbers.

This construction represents actual new knowledge. Even if the construction done by Kim is not directly realizable, and requires the division of the magic number into 6 border numbers, it can be interpreted as a construction of real new knowledge which is not merely detached from the given problem but which extends the arithmetic structure of the given magic square and makes it more understandable.

4. Different forms of constructing mathematical knowledge: rule or fact knowledge, problem-oriented relational knowledge and problem-separated relational knowledge

In the research project, 39 comparable teaching episodes with interactive constructions intended meanings as well as generalizations of mathematical knowledge in elementary school were analyzed thoroughly. In doing so, three different types of interactive knowledge constructions were determined: (1) knowledge constructions which occur almost exclusively in the frame of already present fact knowledge in situative, empirical contexts; (2) constructions of actual new knowledge, maintaining an interrelation with the mathematical problem context; (3) constructions of new general knowledge relations which are detached from the present problem connection.

In the first example, Pierre and Caroline do not produce real new knowledge; they draw bigger and bigger concrete dot fields and count the amounts of dots; in doing so they produce further fact knowledge on the basis of present fact knowledge [type (1)].

In the second example, Christopher and Nico develop new arithmetic relations in the frame of existing number series; they leave the geometric-structural context in which a justification would have been possible. They produce problem-separated relational knowledge [type (3)].

The third episode exemplarily shows the construction of new, problem-related knowledge [type (2)]. Kim constructs a new relation between the numbers in the magic square and the magic number, which is based on the present problem context, produces a new structure and therefore conveys new insights.

From an epistemological perspective, the construction of real new mathematical knowledge presupposes the production of a new *relation* or *structure* between elements of already familiar knowledge (cf. Steinbring 1999, 2000). This fact represents an important link between the social-interactive constructions and the epistemological conditions of mathematical knowledge. The kind of knowledge construction is determined only by interactively interpreting the epistemological nature of the mathematical knowledge elements. Dot patterns have been dealt with ambiguously in the interaction: On the one hand, they were often interpreted as concrete, empirical objects, to directly and schematically count demanded numbers where it was not recognizable that the "... new in the development systematically exceeds the old ..." (Miller); on the other hand, geometric configurations often were not paid attention to, so that the construction of a relation in arithmetic patterns could

lead to new knowledge which did not show how the “... new in the development presupposes the old in the development ...” (Miller).

The arithmetic symbols are often seen as mere parts of calculation problems which are to lead to certain results; but when those problems are calculated, arithmetical symbols can get new interpretations in the interaction: They can embody structures and relations in an exemplary way. Such changed interpretations have happened several times in the interaction. In doing so, it is shown in an exemplary way how real new knowledge can emerge interactively in elementary teaching, and how “... the new in the development presupposes the old in the development and still systematically exceeds it ...” (Miller). For the mathematical knowledge – epistemologically consisting of structures and relations – this dualism of a “presupposition with simultaneous exceeding” can be realized by socially constituting a new knowledge relation in the old knowledge context.

Remark

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WORKING GROUP 5

**Mathematical thinking and learning
as cognitive processes**

Group leaders:

Inge Schwank

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ALGEBRAIC EXPRESSIONS AND THE ACTIVATION OF SENSES

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Abstract: *This paper deals with some issues concerning the activation of senses in algebraic expressions. Given the theoretical distinction between sense and denotation of an algebraic expression, the activation of senses can bring one to correct or incorrect solutions, depending on several factors. In particular we stress the relevance of the “sign component”, as observed in most common errors in solving inequalities.*

1. Introduction

There is evidence that the use of symbolic expressions can be a relevant cause of difficulties to students at different school levels. If we look at the historical evolution of algebra, we see that rhetoric and syncopated algebra (i.e. algebra totally expressed by words and algebra expressed by a mixture of words and symbols) have been quite easy to use and understand. On the contrary, in a symbolic system the meaning of words and operations can stay behind the scene, since the symbolic language has the power of taking away most distinctions that are preserved by the natural language. Because of this specificity, the symbolic language expands its applicability but induces a sort of semantic weakness. It seems that such language is suitable for many contexts, without belonging to any one in particular. Hence the origin of the gap between symbols and meanings, which is confirmed by the rigid (stenographic) use of the algebraic code by many students. It often happens that some students are able to control the underlying meaning only when they make use of rhetoric or syncopated algebra.

The “rhetoric” method seems to be used spontaneously (Harper, 1987) and does not depend on the level of instruction.

2. The theoretical framework: an outline

The introduction and use of symbols and symbolic expressions have been widely discussed by semiologists. In previous papers Arzarello et al. (1994, 1995) have referred to Frege's semiotic triangle (Sinn; Bedeutung; Zeichen) to distinguish between sense and denotation of an algebraic expression.

By algebraic expression we mean:

- a) terms i.e. constants and/or variables and their combination by means of the symbols for operations;
- b) propositions and/or propositional functions constructed by the use of terms and predicates (for instance $5+3=8$ is a proposition, $x-2>0$ is a propositional function).

The **denotation** of an expression is the object to which the expression refers, while the **sense** is the way in which the object is given to us (see the example by Frege concerning the two different senses of Venus, namely as *Esperus*, the night star, and as *Phosphorus*, the morning star: the two expressions have the same denotation, but different senses).

An algebraic expression incorporates in its writing the mathematical object involved (the denotation) and the way in which such an object is expressed (see Fig. 1)

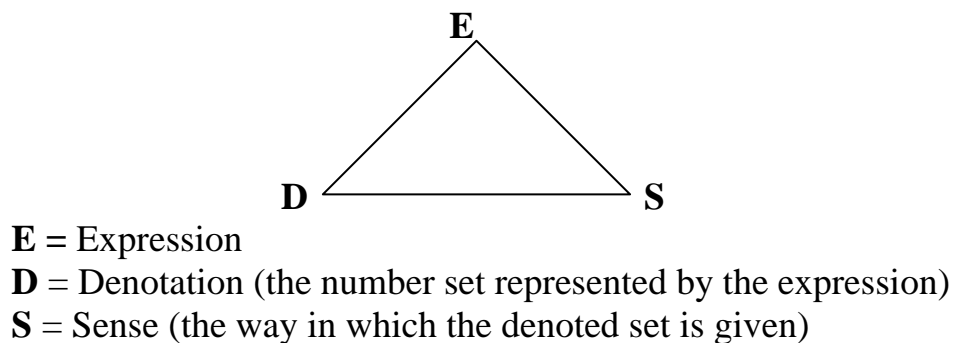
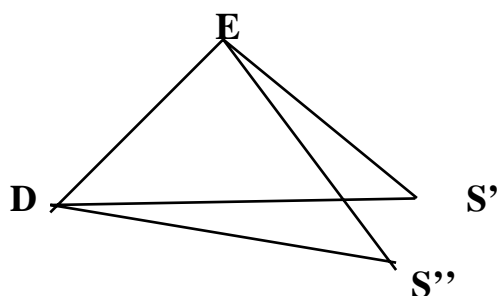
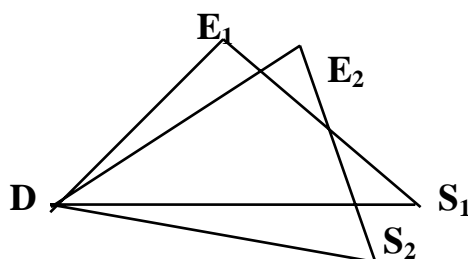


Fig.1

For example the expression x^2-1 , when considered in \mathbb{N} , denotes the numbers which immediately precede a squared number: the expression x^2-1 can activate the sense of computing x^2 and subtracting 1, but also, for example, that of considering the area of a square, decreased by one. Hence, the same expression, denoting the same object, can have different senses (see Fig. 2)

**Fig.2**

Two expressions (for instance x^2-1 and $(x+1)(x-1)$) can denote the same object but emphasize two different ways to calculate the denoted object, i.e. two different senses. So, there is one denotation and two different senses expressed by two different symbolic expressions (see Fig.3).

**Fig.3**

So mathematics, as well as natural language, has plenty of expressions which have the same denotation. Moreover, many times in mathematics and particularly in algebra, expressions incorporate their sense in a concise way.

Algebraic transformations can produce expressions with different senses and the same denotation (i.e. $x(x+1)$, which becomes x^2+x , or x^2-1 which becomes $(x+1)(x-1)$). They are invariant with respect to the denotation of the symbolic expression they act upon, because they can change the sense but not the denotation of the symbolic expression itself.

The converse is not true. If two different expressions have the same denotation, they are not always reducible to each other by algebraic transformations (for example the two equations $x^2+2=0$ and $x^2+5=0$, when considered in \mathbb{R} , denote the same object, i.e. the empty set, but they are not reducible to each other by algebraic transformations). Sometimes, it may be very difficult for students to conceive this invariance and difficulties in grasping the equivalence of two given equations (inequalities) can be interpreted as difficulties in recognizing the invariance of denotation (the truth set of an equation) with respect to the variance of sense (Bazzini, 1997).

The capacity to master sense and denotation seems to be at the very core of algebraic thinking. Furthermore, the passage from one sense to another is fundamental in doing algebraic manipulations.

Interesting insight into the dynamics of algebraic thinking are provided by the analysis of students' behaviour when facing the activation of senses.

Examples of good activation of senses are described in Arzarello, Bazzini and Chiappini (1994). On the other hand, several errors observed in students at work show very clearly the activation of incorrect senses, as witnessed by the example below.

3. Example of activation of incorrect senses

Stefania is a student attending the second year of the Liceo Scientifico, a scientifically oriented high school, in Italy. She has received traditional teaching. She gets good scores (7 or 8 out of 10), but, notwithstanding her success, she does not feel very confident with mathematics and regularly goes to do mathematics with Rossella, a middle school teacher, who is deeply involved in maths education research. We do not know if her insecurity is the only reason why she likes being in contact with Rossella, or if she understands that there is something else behind the mathematics she learns at school.

Here a dialogue between Stefania and Rossella, in a non-classroom situation:

R. writes

$$y = x^3 + 6x$$

$$y = x^3 + 3x^2$$

and says: "Compare these two functions, that is, try to say when the value of one function is greater than the value of the other one."

S.: I would make a system (and she puts {)

$$\{ \begin{array}{l} y = x(x^2 + 6) \\ y = x(x + 3) \end{array}$$

$$y = x(x + 3)$$

R.: Solving a system means finding the common solution of the two equations, I have asked you just to compare, that is saying when, for example, $x^3 + 6x > x^3 + 3x^2$

S.: And y, where does it go?, Ah, it is the solution.

R.: It doesn't matter

S.: So, why is there a y and then it disappears?

R.: If you consider the system, you have $y=y$, thus also $x^3+6x = x^3+ 3x^2$, do you agree?

S.: Yes

R.: So, go on.

S.: I should solve the inequality

$$\begin{array}{l}
 x^3+6x > x^3+ 3x^2-3x^2+6 > 0 \\
 -3x(x-2) > 0 \quad -3x > 0 \rightarrow x < 0 \quad x-2 > 0 \rightarrow x > 2 \\
 \quad \quad \quad 0 \qquad \quad 2
 \end{array}$$

+	-	-
-	-	+
-	+	-
$0 < x < 2$		

R.: What does that mean?

S.: Perhaps, if I substitute a value which is between 0 and 2, the equality is true. But I don't know if it is true, or whatever. That is, maybe the inequality is true.

We observe that, at the beginning of the interview, the writing

$$\begin{array}{l}
 y = x^3 + 6x \\
 y = x^3 + 3x^2
 \end{array}$$

activates the incorrect sense of solving a system, according to previously acquired experience.

Later, during the whole interview, Stefania tries, without success, to make sense of the symbols in front of her.

Incorrect behaviours like those of Stefania have induced us to guess that, given the theoretical framework outlined above, the **sign component** can be the origin of mistakes.

Following this hypothesis, we have tried to investigate the most common errors, with special attention to inequalities. The study has included interviews with a sample of thirty secondary teachers and the analysis of students' behaviour.

The interviews aimed at pointing out different approaches in introducing inequalities and noticing the common errors students make when meeting inequalities.

4. The case of inequalities: approaches and behaviours

A wide range of approaches in introducing inequalities emerges from the interviews, but it is worth noticing that equations and inequalities are always introduced **in sequence**, i.e. firstly equations and secondly inequalities.

It is important to recall that in the Italian school system, inequalities are taught to students of 16 or 17 years of age.

Furthermore, the Italian secondary schools are divided among many different specialisation fields. Therefore, we interviewed a group of teachers, belonging to different types of high school (for details see Bazzini and Ascari, 2000).

The different methodologies adopted by these teachers seem to offer a quite complete scenario of the different possible approaches.

The teachers were also asked to illustrate the most common difficulties of students. They reported that the typical errors are the following:

1. *Eliminate the common denominator.*

$$\frac{x-3}{2x+1} < 0 \rightarrow x-3 < 0 \rightarrow x < 3$$

2. $x^2 < 4 \rightarrow x < \pm 2$

3. $-x > 3 \rightarrow x > -3$

4. *Question: "How many solutions does the inequality $x-3 > 0$ have?"*

Answer: "One, $x > 3$ "

All these answers can be interpreted in terms of incorrect relation between sense and denotation of the given expressions. Such an incorrect behaviour probably derives from the role of the "sign" component, which, if taken just on the surface, can induce the activation of the sense of "equation". All the teachers agree that most of the difficulties arise from treating inequalities as equations (we call it "the ghost of equation"). Research findings indicate that students tend to multiply both sides of an inequality by a negative number without changing the direction of the inequality. In a wider sense, several researches suggested that students apply procedures that are valid in the context of equations for solving inequalities even though they are not necessarily applicable (e.g., Tsamir, Almog & Tirosh, 1998).

There is also reported evidence that students are inclined to reduce algebraic rational expressions without checking the limiting zero cases. They also tend to multiply by the denominator in both equations and inequalities without taking into consideration the zero cases or the negative cases.

Following these suggestions, additional investigation was planned. About one hundred thirty students attending high school in Italy were requested to fill in a questionnaire, in the framework of a research study involving Italian and Israeli students (Bazzini-Tsamir, 2001).

The analysis of the tasks reported below (n. 6, 7, 12 in the questionnaire) provide further data about students' performances.

Task 6

Check the following implication:

$$ax < 5 \Rightarrow x < \frac{5}{a} \quad \forall a \in \mathbb{R}; a \neq 0$$

Task 7

Check the following implication:

$$ax < 5 \Rightarrow x < \frac{5}{a} \quad \forall a \in \mathbb{R}$$

Task 12

Solve the inequality $(a-5)x > 2a-1$, x being the variable and a the parameter.

In both tasks 6 and 7, students were asked to determine the equivalency and to justify their claims. Task 12 dealt with the same issue in a different manner, asking the students to solve a similar given parametric inequality.

As far as Task 6 is concerned, about a quarter of the Italian ones gave the correct answer; "false". In their justifications, about 15% provided a justification, with reference to the sign of 'a'. Most students, while correctly limiting the range for 'a', said nothing about zero or negative 'a's. The other participants who justified these "false" responses explained, for instance, that "for negative 'a's the direction of the inequality should change, for $a=0$ it is impossible to divide by 'a', but for positive values the given implication is correct."

The answers relating to equations are mainly concerned with $a \neq 0$, like in the case of equations.

Most Italian participants (around 90%) correctly responded to Task 7. However, on a closer look, there is evidence that most students gave the correct answer with wrong justification. In particular, justifications such as, "the answer is 'false', because 'a' might be zero and it is impossible to divide by zero", or "there is no restriction about the need to have non-zero 'a'" were most prevalent. The high percentage of those who related only to $a \neq 0$ confirms the persistence of the "ghost of equation".

Task 12 was solved correctly by a low percentage of Italian students (around 15%). The analysis of incorrect solutions gives evidence that in most cases students did not consider the range of value of the parameter 'a' at all. A non-negligible number excluded only non-zero values, claiming, for instance, that “for $a \neq 5$ $x > (2a-1)/(a-5)$ ”. They solved the inequality with routine procedures, the same as the ones they adopted for equations.

5. Discussion and implications

Our data confirm the results by Tsamir et al. (1998) and Maurel and Sackur (1998).

Tsamir et al. point out wrong analogies between the solution processes valid for equations and those for inequalities, on the basis of a questionnaire submitted to 160 high school students, after three months of training on inequalities.

They observe that the structural similarity between the two entities (equations and inequalities) creates a strong intuitive feeling that the strategies held for solving equations should hold for inequalities as well. They claim that intuitive beliefs successfully compete with the formerly acquired knowledge.

From our perspective, such kinds of errors are deeply rooted in the complex relationship between the sign, sense and denotation of a given expression. Furthermore, such errors may be influenced by the sign, which induces the students to activate a sense stored in memory on the basis of previously acquired knowledge, but not in accordance with the intended meaning of the sign in that given situation. For example, when a student asserts that

$$x^2 \geq 4 \rightarrow x^2 > \pm 2$$

there is evidence that the graphical sign recalls the procedure adopted for solving equations, which is stored in memory, but is not appropriate in this case. Here the sign activates a familiar procedure, which is valid in situations like $x^2 = 4$, which resembles $x^2 \geq 4$ as far as sign is concerned. In this case ($x^2 \geq 4 \rightarrow x \geq \pm 2$) there is a distorted relationship between sign and denotation.

In this perspective, we can also include the errors which identify the equivalence of equations (inequalities) with their algebraic transformability. The student activates a known procedure without checking its applicability. We

could speak of routine procedures, which in this case are not the first step towards solution, but rather a repetition of routine mechanisms.

At this point two main questions arise:

1. Which introduction of equations and inequalities should be adopted to avoid (or at least to limit) such kinds of errors
2. Which control should be applied in the procedure.

As far as point 1 is concerned, we believe that the first approach to equations and inequalities is fundamental. In our view, equations and inequalities should be treated simultaneously and not in sequence. In fact, introducing firstly equations and secondly inequalities usually implies that solving procedures valid for equations, remain predominant also for inequalities. As a consequence inequalities are considered as a sort of “pathologic equations” and treated as such. The link between sign and denotation is totally distorted.

Moreover, we suggest that equations and inequalities should initially arise from problems of modeling.

Coming to point 2 (the problem of control), Maurel and Sackur (1998) suggest that procedures for solving inequalities can be controlled in two ways: by using a different representation register and by assigning specific numerical values to check the validity of the given inequality. In addition, Chiappini (1998) points out the emergence of difficulties in keeping a close control of the symbols $>$, $<$, $=$. In his view, the main difficulty is in the passage from working with propositions (typical of arithmetic) to working with propositional functions (typical of algebra). In fact, in the former case the “truth-judgement” can be easily obtained by looking at the proposition itself; in the latter, such judgement has to be outstanding until the number set which allows such judgement has been identified. The didactical problem consists in developing the awareness of the outstanding judgement.

In conclusion, we recognize that in situations where the teacher develops a suitable discourse, the pupils verbal and mathematical abilities are thereby developed.

This helps students to switch from one conceptual frame to another and to activate suitable senses.

The ability to do this is a very important part of the development of mathematical thinking.

As a consequence, it is vitally important for the teacher to appreciate the underlying theoretical concepts and to be able to arrange the didactical situations to take advantage of students present knowledge in order to develop it further.

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INDIVIDUAL DIFFERENCES IN THE MENTAL REPRESENTATION OF TERM REWRITING

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Abstract: *The project forms part of our long-standing efforts to investigate cognitive processes with respect to the formation of mathematical concepts and problem solving. The project being presented here is to involve the examination of the mental processes undergoing in term rewriting (algebraic manipulations). Term rewriting forms the central mathematical technique which represents a significant source of difficulty in understanding and errors of schoolchildren in (school) algebra. Looking at it from the aspect of basic research, the way in which a mental model of the problem to be solved is formed and subsequently applied by the person being under examination is to be investigated. It will be differentiated whether a static model formation is being aspired to or whether the mental model and the processing tools relating to it should model the process of the formula manipulation. Here, we refer to the theories of Schwank (1993a) in which predicative and functional cognitive structures are differentiated. The examinations are to be carried out in the form of clinical interviews. The problems are constructed in such a way that the conclusions on the cognitive structure of the person being under examination can be drawn from the interpretation of the interviews, as well as from the performance in the specific problems.*

1. Theoretical Background

Even though computer algebra systems (as for example Derive, Mathematica) advance into mathematics lessons, the capability of recognising structures in complex formulas, developing strategies for formula manipulation and successfully converting them into solutions is still a basis for the handling of demanding components in algebraic school mathematics. In spite of decades of efforts in mathematical education, a considerable number of pupils in mathematics lessons in grades 7-10 (at school) does not achieve the necessary skills in formula manipulation as needed in the field of term rewriting, equations and fractions. Our examinations belong to the field of analysis of processes of the formation of mathematical concepts. This demands an interdisciplinary starting-point for our research, where methods and results of mathematical logic

and foundations of mathematics, as well as of mathematics education and cognitive psychology have to be combined.

The analysis of pupils' behaviour when carrying out term rewritings reveals two directions in which this can be theoretically backed up. One direction refers to a mathematical subject analysis. Two different viewpoints can be taken here, either a semantic one, emphasising the algebraic structure, or a more syntactic one, focussing on the form of the terms. Another line of reflection refers to mental processes.

According to the above-mentioned differentiation of the two viewpoints either the process of understanding as regards content can be looked into, or the way in which a mental model of the formula manipulation task to be solved develops in a person's mind and which intellectual mechanisms have to interlock with one another in this complex thinking performance.

Research in mathematics education has up to now been more concerned about the aspect of (semantic) understanding and less about the cognitive processes, how formula representation and formula manipulation work in people's minds. As our project is meant to contribute to this neglected aspect, we will put the main emphasis on it when explaining the theoretical background.

1.1. Semantic Aspect

Subject analysis in mathematics education in the field of term rewriting is mainly executed under the semantic aspect and this matter is dealt with in the field of didactics of algebra. Vollrath (1995) provides a good overview of these didactic approaches. He also gives a list of psychologically oriented work in mathematics education based on the concept of understanding. For a few years, there has been an increasing number of work which analyses the link between formula manipulation and reference of meaning (Vergnaud et al., 1988, Kieran, 1991). Considerations of mathematical logic are also included in the didactic analysis. The differentiation of "sense" and "meaning" according to Frege (1892) is, for example, used in order to describe difficulties of pupils (Arzarello et al., 1994; Drouhard, 1992).

1.2. Syntactic Aspect

Contrary to the widespread opinion in mathematics education that pupils' activities as regards formula manipulation were to be attached to the frame

“abstract”, we are convinced that the primary problem for the pupils is to work with formal objects in a *concrete* and *combinatory* manner (Cohors-Fresenborg, 1979).

If you analyse the difficulties which pupils have in rewriting complex terms according to the rules, several partial problems can be isolated: First of all, it involves to locally choose a partial term from the complex term, for which a rewriting rule exists. Secondly, the rule has to be applied correctly and finally, taking strategic points of view into consideration, it has to be decided which rule has to be applied for which part of the term. So-called formulas for term rewritings in solving equations, as pupils in grades 7 to 10 have to use and understand at school, can be analysed from the basic point of view by those means which are subsumed to the concept “calculus” (in various special forms). The dealing with term rewritings when solving equations can be reconstructed by means of canonical Post calculi (Cohors-Fresenborg, 1977, pp. 78). The central technique when using calculi consists of replacing one particular partial word in a complex expression - regarded as a word within a specific alphabet - by another according to a calculus rule.

Lowenthal (1985) carried out investigations with children in mathematics education under the aspect of “nonverbal communication devices” as regards the use of formal rule systems. The calculus was represented as definite playing material. He was able to prove that even primary school-children were highly productive in using this type of mathematics.

1.3. Cognitive Aspect

The significance of the cognitive point of view in mathematics education was mainly founded by Davis (see Davis & McKnight, 1979). His work “Understanding ‘Understanding’” (Davis, 1992) gives a summary of cognitive oriented mathematics education. Basic research in mathematics education in the field of the formation of mathematical concepts is still in its initial stage. We think that the theory of Schwank (1986, drawn up 1993a; as regards EEG-examinations see Moelle, Schwank et al., 2000) considerably contributes to this. As our investigations are mainly based on this theory it is briefly described in the following:

1.3.1. Predicative versus Functional Cognitive Structures

The theory proposed by Schwank states that when constructing a cognitive structure the (syntactic) type of the means of expression has a special cognitive significance. Dualism is taken into consideration in the type of mental model in

the theory construction. She differentiates a predicative and a functional cognitive structure:

A preference for a predicative cognitive structure in a given situation means the expression of a static relation or the focussing on the structure and its description. A preference for a functional cognitive structure only has few relations and analysis of structures at its disposal, but a distinctive consciousness for processes and thinking in ways of effects. As relations and descriptions are not fundamental for such a cognitive structure, the interest in exact and precise descriptions is not very well developed.

When first looking into Schwank's theory, it is often presumed that this theory only uses different expressions than that of Anderson (1983), who differentiated between declarative and procedural knowledge. Schwank's theory, however, describes the logical nature of the objects of knowledge representation, according to which Anderson's theory can be used on a meta-level. She therefore demands to expand Anderson's theory by a dual part. There is also declarative knowledge about actions (not only about relations) and, vice versa, procedural knowledge about logical relations (and not only about operations) (Schwank 1993b, p. 252).

Examples from Mathematics Lessons

Teachers who are familiar with this theory told us that they often find examples where the distinction "predicative versus functional" helps to understand the differences in mathematical argumentation of their pupils.

Example 1:

Pupils of grade 10 were asked to write down their imaginations of the function which is given by $n(x) = 0 = x - x = 0 \cdot x$.

A girl wrote:

$$n(x) = x - x$$

This is my imagination as $x - x = 0$, and according to my opinion, there should be an x in a function.

A boy wrote:

I think $n(x) = 0$ is the most sensible as the number x disappears in the function and 0 is the result. The function does not need x . Even without any number x , the machine could produce the correct result.

The girl's reply concerns the level of formal presentation, not the process of computing. The boy claims that x has not got any effect on the result and

therefore should not occur in the definition of the function n . The teacher assumes that the girl is predicative and the boy functional (Sjuts, 2001).

Example 2:

The following was part final examination before leaving school for university:

Given three points ABC which form an equilateral triangle. Let M be the centre of the circumcircle. The triangle ABC is completed to a regular tetrahedron by the point D . Give reasons why the point P , which has got the same distance from all four corners of the tetrahedron, lies on the line through M and D .

A girl wrote:

The point P , which has got the same distance from all the four corners of the tetrahedron, must lie on the line through M and D , as this is a regular tetrahedron. If this tetrahedron was to be drawn into a sphere, the points D, C, A, B would be points that would lie on the surface of the sphere. The centre of the sphere would also be the centre of the tetrahedron. This centre of the sphere lies on the line through M and D , as this line goes exactly through the centre of the sphere.

Another girl wrote:

As M has got the same distance from A, B and C – as already mentioned above – the process for the construction of P can be imagined in such a way that M is pulled on a string, which describes the line through M and D , in the direction of D . The distance from M to A, B and C therefore increases regularly and the distance from M to D decreases. The “pulling of the string” ends when the distance from D to M is exactly as big as that from M to A, B and C . This point, however, is not longer M , as it describes the centre of the circumcircle of the triangle ABC , but P . Therefore, P lies on the line that goes through M and D .


1.3.2. Predicative versus Functional Use of Calculi

Let us now analyse how the process of the use of rules as regards calculi looks like when different cognitive structures (predicative versus functional) are taken as a basis. The decisive question when looking for a suitable conversion is what is used for orientation. We supposed that, in general, people with a preference for a predicative cognitive structure had an advantage when using calculus rules: This obviously concerns the case that, when using a rule, a given sample of symbols has to be recognised. This means finding a rule which changes the structure and using it syntactically correct. In case of a predicative mental model formation, attention is to be directed to the differences in the characteristics which apply to the single symbols before and after the use of the rule.

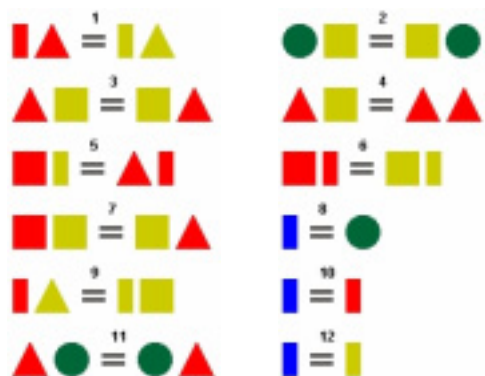
Comments during working will contain a lot of exact descriptions of the features which denominate the object of interest. The use of a rule causes the replacement of the characteristics of single symbols. The subjects' attention has to be directed to the exact understanding of the predicates and their replacement relations. We imagine a mental model following Klix' considerations (1984) on the hypothetical structure of a natural concept (see Schwank's analysis, 1993a).

For the situation described here, i.e. rules have been completely formulated by elements of object language, there is also a possibility for intellectual processing which manages without the described way of concept formation about characteristics as a necessary prerequisite. The aim is not primarily the changing of symbols in their predicates, but the organisation of the shifting, acquisition and modification of symbols. For the intended transport of symbols, you do not have to look for a suitable rule by understanding all predicates of a chosen part term and then by looking for a suitable term, but by taking the existing rules as patterns which are pulled over the word that is being worked on, e.g. from left to right, until a part sample fits to a premise of a calculus rule in question. The conclusion of fitting is from the logical point of view also the conclusion of applicability of a two-digit predicate, but in this case a very elementary predicate is concerned, i.e. the syntactic equality. Perhaps it is only regarded as a one-digit predicate, i.e. if the attention is only directed to a specifically chosen symbol pattern. When an applicable rule has been found in this way, term replacement is possible by only using manipulations on the level of object language. Comments during working will contain less exact descriptions of the features which denominates the object of interest, but will often be satisfied with pointing formulations, such as "this there", "that there", "over there" while the mouse is being moved.

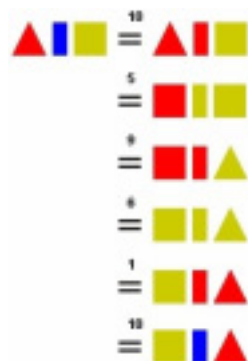
This description is to show how a functional cognitive representation of problem formulation and its processing can be imagined.

Statement: 

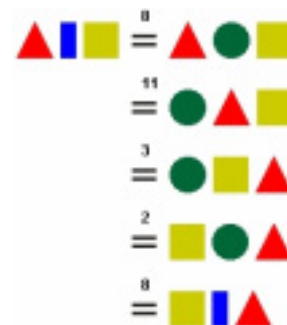
Rules:



Predicative Proof:



Functional Proof:



The problem shown above has been designed by us in such a way that it gives rise to a resonance from a predicative or functional cognitive structure and then produces a predicative or functional mental model. The rules suitable for the processing in the model are also available:

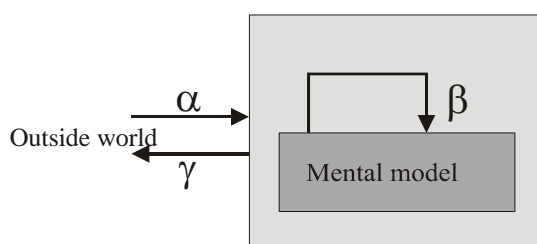
In a predicative mental model of this problem, the subject thinks that the demand of this problem is to work on the characteristic of the object in the first or last position respectively. The subject furthermore mainly looks for such rules by the use of which characteristics of an object can be replaced:

The first symbol of the starting term is triangular and red, the first symbol of the last term is square and yellow. Rules have to be found which replace triangles by squares and red symbols by yellow ones. R5 is one of those. Before it can be applied the colour of the second symbol has to be changed using R10. Before the colour of the first symbol can then be replaced by using R6, the prerequisite for the application of this rule has to be created by means of R9. The application of this rule also changes the shape of the last symbol at the same time. R1 is used to change the colour of the last symbol, R10 that of the middle one.

A subject with a functional mental model regards the problem as a problem of movements. Attention is drawn to rules by which movements can be organised:

If you look from the starting term to the last term, you realise that the first symbol has moved to the back and the last one to the front. Rules have to be found which enable such movement. R11 is one of those, but it needs a green circle as a tool. The introduction (and later on the deletion) of this tool allows the application of R8. After use of R11, the red triangle has to move on. This is possible with the help of R3. Now the yellow square is on its way which is continued by means of R2.

1.4. Mathematical Modeling



The interaction of a preference for a predicative versus functional cognitive structure, an external form of representation and a mental model is described by the concept of $\alpha\beta\gamma$ -automata guided by preferences. These have been introduced in the discussion about an adequate mathematization of the concept of action by Walburga Rödding (1977). This theory was then expanded in order

to describe learning processes (Cohors-Fresenborg & Schwank, 1983; Schwank, 1987). In case of an $\alpha\beta\gamma$ -automaton guided by preferences, a preference for predicative versus functional structures is concerned, if out of possible α , β , γ -transitions predicative versus functional transitions are preferred.

2. Execution of Pilot Studies

2.1. Development of Test Design

A computer supported test design has been developed, which offers term rewriting problems for processing by subjects. In the problems, the terms have not been made up by letters and arithmetic symbols (in order to factually and emotionally move away from school mathematics), but consist of graphic symbols which differ in several characteristics (e.g. colour, size, shape).

The subject has to solve every problem by term rewriting using the given rules. During this process, the part term which has to be substituted first has to be marked in the term. After that, the intended rule has to be clicked. The rule is automatically implemented. If the chosen rule cannot be applied, the subject is informed correspondingly on the screen. Rule applications can also be cancelled. The computer programme records the individual steps having been made by the subject (movement of the mouse, selection of the part term which has to be worked on by clicking, clicking and implementation of rules, time point of these activities).

2.2. Examination Design

The examination is to be carried out as a video documented, clinical interview with single subjects of grades 7/8. 45 minutes are planned for the processing of the problems on the computer. For the evaluation, transcripts are being produced from suitable parts in order to analyse the verbal comments.

In order to assign a cognitive structure to each subject independent from these analyses, the test QuaDiPF (Schwank, 1998) is additionally applied in a second examination lesson, also as a video documented, clinical interview. Assignment is made by the analysis of verbal comments. QuaDiPF (version A) was also used for EEG-examinations (Moelle, Schwank et al., 2000).

2.3. *Consequences from Pilot Studies*

The implementation of several pilot studies, both with pupils of grades 7/8 and with university students, has shown that as per the problems employed the original idea cannot be maintained, i.e. the subject does not consequently use the rules of a certain type according to his/her preference. The analyses proved that subjects in spite of the mentioned strategy – allowing conclusions regarding a preference for one cognitive structure – use rules in certain situations which are also possible but which we have assigned to another cognitive structure. As a result the problems have been revised. Furthermore, the problems and the rules included therein have been changed in such a way that no situation can arise in which the order of the use of rules is virtually determined (as with Markov-Algorithms) and therefore the “choice” of the use of rules would not have any diagnostic value.

The question how, according to the analyses of a use of a rule, an assignment “predicative versus functional” can be made, has considerably changed: originally we had a one-digit predicate in mind which applied to the rule, now it is a two-digit one, which is applicable to the use of a rule in a situation. The consequence for evaluation is that regarding the planned modelling of the subject’s behaviour as an $\alpha\beta\gamma$ -automaton guided by preferences, the preference does not assess the rules, but the transitions (rules in situations).

This change has consequences for the evaluation software: There is a modulus which enables a determination of preference in four levels in an expert rating for each problem and each use of rule: predicative and functional, indifferent and nonsensical. This is based on the presumption that there are obvious consequences resulting from some real decisions when working on a problem, which – looked at in isolation – does not allow any assignment to “predicative versus functional”. In a situation, a rule can be particularly useful for both cognitive structures, but with different objectives. The assignment “indifferent” is used for that.

When rating on the screen, the new software allows to show the use of rules in every situation in a tree-structured diagram and then to evaluate the potential decision. In this diagram, the levels of preference are automatically assigned to the colours of the following branch. After each evaluation, the branches can be put into another position so that this results in a top to bottom grading from “predicative” via “indifferent” to “functional”. Both measures result in a visual support of the taking effect of the preferences. During the evaluation, the way of solution of a subject can be represented in the coloured tree (see Striethorst, 2001). It is planned to state the assessment criteria more precisely in so far that they can be formally represented in an algorithm. The

algorithm assesses the possible alternatives in every step of decision according to the four categories mentioned above. The implementation of the algorithm in a Java-programme then supplies a formal representation of preference for predicative versus functional decisions. This algorithm can be taken as a decision-support-system for the $\alpha\beta\gamma$ -automaton described in chapter 1.4.

A study is being executed at the moment with 21 pupils of grade 7 (grammar school) with the help of the above-mentioned improvements. 6 problems have to be solved on the computer by each of the subjects within a time limit of 45 minutes. First evaluations of the data have shown that the above-mentioned improvements have considerably increased the power of prediction: The analysis of the verbal comments from the video records mostly match the classifications which result from the assessment of the single decisions when the subjects choose the rules. The second part of the study – use of the test QuaDiPF (Schwank, 1998) – is still to come. Data as regards eye movement are to be ascertained by means of an eye-tracker (see Schwank, 2001).

3. Objectives

The subject of this project is an analysis of mental processes, which take place in term rewritings, which have to be handled syntactically. Attention is directed to the way in which a mental model of the problem to be solved is developed in the subject's mind. It is to be differentiated whether a more static modelling by means of a local change of features is made or if a dynamic point of view is used which concentrates on the moving of symbols. The theory of differentiation between predicative and functional cognitive structures offers us an instrument which allows the analysis of this difference.

The objective of this project is the construction of problems for term rewriting so that the evaluation of the processing by the subjects reveals the following:

- Proof of a subject's preference for one of the two cognitive structures predicative versus functional (due to the classification of verbal comments).
- Proof of the use of a predicative versus functional model for the problem processing (from the assessment of the decisions made at the individual processing steps).
- Modelling of subject's behaviour by preference-controlled $\alpha\beta\gamma$ -automata.
- Comparison of the preference shown through the term rewriting tasks with the preference shown through the solving of the QuaDiPF tasks.

4. Perspective

The proof of postulated differences in mental modelling when dealing with term rewritings will give cause to further research and development in mathematics education in order to attach a higher importance to the up to now neglected syntactic aspect of school algebra and possible individual differences of the learners than has been regarded necessary and useful so far.

An interesting question of further research may be to investigate whether the assumptions of the teachers concerning the preference for predicative versus functional cognitive structures, which they appointed to their pupils (in the example in chapter 1.3.1), can also be found when the pupils deal with our term rewriting problems or those in QuaDiPF.

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COGNITIVE FACTORS AFFECTING PROBLEM SOLVING

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Abstract: *We are interested in the cognitive factors affecting problem solving. In this work we tried to find empirically relation between variables depending both on the surface structure and the underlying structure of the word problem. Hence, the relation between the text given in natural language and the mathematical expression employed by the solvers.*

Introduction

The research reported here is a continuation of the study (Kubínová, Novotná et al., 1994) on three formulations of the same comparison problems. Their pioneer work dealt with various strategies employed by students. A further analysis revealed that such strategies are affected by certain cognitive factors. Moving from the given text to the mathematical expression (the equation) is a process that can take several direct and indirect transformations. This route depends both on the surface structure of the text and the underlying schemes. We assigned a measure of complexity to this route.

Theoretical Background

Relevant to our work are the works by Kintsch who introduced notions such as: “text base” and “situation model” or Nesher’s “semantic analysis” of arithmetic word problems (Nesher and Teubal 1975; Kintsch, Kozminsky et al. 1975 April; Nesher and Katriel 1977; Kintsch and van Dijk 1978; Kintsch 1986; Nesher 1998). Yet these works raise more questions than they answer. Using “text base” in its propositional form and not addressing the surface structure phrasing, leaves out most of the difficulties of problem solving - the interpretation of the text. It may well be that the crux of solving a problem lies in that stage of interpretation and avoiding this stage means avoiding the real difficulty.

Next in line of vagueness is the concept of a “situation model”. As we demonstrate in the next section, this term is also not fully defined. Do we mean a world without mentioning any relations needed to be focused on, or is this an imaginary world constructed by the reader by comprehending a given text?

A second cognitive approach, not always separated from the first one, emphasized as its point of departure the schematic analysis of arithmetic problems, (Greeno 1978), (Fischbein 1997), (Hall, Kibler et al. 1989). Those who adopted the schematic approach were influenced by notions such as “frames”, “structures”, “analogies” (Mayer 1991), (Reusser 1992), (Rumelhart 1980), (Thompson 1985), (Vergnaud 1983).

The study described an attempt to analyze, just one problem in terms of its linguistic surface structure, its underlying schemes, and the mathematical model selected by the solvers in their attempt to solve it. We hope that the empirical evidence will shed light on some more cognitive aspects concerning problem solving.

The experiment problems and their formulations

Imagine a world consisting of: David with his 22 marbles, Jirka with his 44 marbles and Peter with his 132 marbles.

Which questions can be asked about this situation?

For reasons known only to mathematicians the problem they present will always miss some information (probably in order to teach how to deduce further information by mathematical tools). One of the goals could be demonstrating that if one knows only one absolute quantity and two relations he can find out and reconstruct the above entire situation.

If the formulation will be as follows:

***A1:** Peter, David and Jirka play marbles. They have 198 marbles altogether. Peter has 6 times more marbles than David, and Jirka has 2 times more Than David. How many marbles have each boy got?*

It means that the relation between Peter and David, and the relation between Jirka and David are given and the total sum is given. However, the relation between Peter and Jirka is not mentioned at all, neither are the quantities of either of them. Moreover, the verbal description already took a further step as to who will be the referent in a given relation (in this case – David), and who will be compared to him (Peter and Jirka), which will determine the lexical choice of

the word “more” instead of “less”. We could of course describe David as having less than Jirka or Peter and then get another formulation for the same relation:

B1. *Peter, David and Jirka play marbles. They have 198 marbles altogether. David has 6 times less marbles than Peter, and he has 2 times less marbles than Jirka. How many marbles have each boy got?*

Actually we have found 12 possible formulations for the same situation. These texts vary by the lexical use of “more” and “less”, and by what will appear as the subject or the predicate of the sentences in the text.

At the other end the mathematical expression defines the reference and the compared by means of independent and dependent variables. This is actually determined by the choice of the X of the equation.

We analyzed only algebraic solutions and took notice of the selected X by each of the solvers.

The Measure of Complexity

In order to understand better that there are real options of selecting a strategy let us examine Problem A2 as a detailed example:

A2. *Peter, David and Jirka play marbles. They have 198 marbles altogether. Peter has 3 times more than Jirka, and Jirka has 2 times more than David. How many marbles have each boy got?*

The surface structure of the given text presents two ordered comparison relations **f** and **g**. The first relation mentioned in the text says: “*Peter has 3 times more than Jirka*”. In short $P=f(J)$. The second relation in the text is: “*Jirka has 2 times more than David*”. In short: $J=g(D)$.

In selecting the equations for solving this problem one could choose David¹ for the independent variable and write:

David is X; Jirka is 2X, and, Peter is $3(2X)=6X$ (which involve an intermediate calculation).

The equation would finally be: $X + 2X + 6X = 198$

¹ The use of the name (e.g. David) in a formula or in an explaining sentence is an abbreviation for “the numbers of marbles which the person owns”.

In general terms: if D is the independent variable, the equation will be:

$$D + \mathbf{g}(D) + \mathbf{f}(\mathbf{g}(D)) = N$$

For the same problem, however, one could select Jirka to be the independent variable, thus: Jirka is X; Peter is 3X; David is $\frac{1}{2}$ X. ($D = \mathbf{g}^{-1}(j)$). Note, we have marked the second function \mathbf{g} with a $^{-1}$, because it is not the direct function given in the text. In the text Jirka was compared to David, thus David was the reference, but in writing an equation with Jirka as the independent variable, the actual formal writing is transformed and is the inverse of the original text (\mathbf{g}^{-1} marks the inverse function of \mathbf{g}). Although the text states that “*Jirka has twice as many as David*”, what was written formally was the translation of “*David has two times less than Jirka*”.

The equation in this case will be:

$$X + 3X + \frac{1}{2} X = 198$$

The general form of the choice of Jirka as an independent variable in the equation will be: $J + \mathbf{g}^{-1}(J) + \mathbf{f}(J) = N$

In this case the equation will include rational numbers making it more complicated for some solvers.

If Peter is selected to be the independent variable, the equation will be:

$$X + \frac{1}{3} X + \frac{1}{6} X = 198$$

In general terms: $P + \mathbf{f}^{-1}(P) + \mathbf{g}^{-1}(\mathbf{f}^{-1}(P)) = N$

The choice of different independent variables leads to different transformations executed in the transition from the given verbal text to the selected equation which is the measured complexity level.

We will score these transformations as follows: If we decide that each direct function is considered as a single complexity score, an inverse or compound function will be considered to be two scores. The level of complexity of a solution is defined as the sum of all the scores. For example, in the problem above (A2) selecting David as the X of the equation results in a complexity level of 4 (see Table 1), selecting Jirka, the equation is of a complexity level of 3; and selecting Peter, we arrive at a complexity level of 8. Table 1 presents the level of complexity in our study and their scores:

Table 1: Possible Complexity Levels

Possible Transformations	Level of Complexity
$f(x) + g(x)$	2
$f^{-1}(x) + g(x)$	3
$f(x) + g(f(x)); f^{-1}(x) + g^{-1}(x)$	4
$f(x) + g^{-1}(f(x)); f^{-1}(x) + g(f^{-1}(x))$	6
$f^{-1}(x) + g^{-1}(f^{-1}(x))$	8

The Research Hypotheses: The solvers of each problem choose a strategy that leads to the minimal level of complexity.

The Experiment

The set of problems:

- A1.** *Peter, David and Jirka play marbles. They have 198 marbles altogether. Peter has 6 times more marbles than David, and Jirka has 2 times more Than David. How many marbles have each boy got?*
- A2.** *Peter, David and Jirka play marbles. They have 198 marbles altogether. Peter has 3 times more than Jirka, and Jirka has 2 times more than David. How many marbles have each boy got?*
- A3.** *Peter, David and Jirka play marbles. They have 198 marbles altogether. Peter has 6 times more than David, and 3 times more than Jirka. How many marbles have each boy got?*
- B1.** *Peter, David and Jirka play marbles. They have 198 marbles altogether. David has 6 times less marbles than Peter, and he has 2 times less marbles than Jirka. How many marbles have each boy got?*
- B2.** *Peter, David and Jirka play marbles. They have 198 marbles altogether. David has 2 times less than Jirka and Jirka has 3 times less than Peter. How many marbles have each boy got?*
- B3.** *Peter, David and Jirka play marbles. They have 198 marbles altogether. David has 6 times less than Peter, and Jirka has 3 times less than Peter. How many marbles have each boy got?*
- C1.** *Peter, David and Jirka play marbles. They have 198 marbles altogether. Peter has 6 times more marbles than David, and David has 2 times less marbles than Jirka. How many marbles have each boy got?*

- C2.** *Peter, David and Jirka play marbles. They have 198 marbles altogether. Peter has 3 times more marbles than Jirka, and David has 2 times less than Jirka. How many marbles have each boy got?*
- C3.** *Peter, David and Jirka play marbles. They have 198 marbles altogether. David has 6 times less marbles than Peter, and Peter has 3 times more marbles than Jirka. How many marbles have each boy got?*
- E1.** *Peter, David and Jirka play marbles. They have 198 marbles altogether. Jirka has 2 times more marbles than David and David has 6 times less marbles than Peter. How many marbles have each boy got?*
- E2.** *Peter, David and Jirka play marbles. They have 198 marbles altogether. Jirka has 3 times less marbles than Peter, and 2 times more marbles than David. How many marbles have each boy got?*
- E3.** *Peter, David and Jirka play marbles. They have 198 marbles altogether. Jirka has 3 times less marbles than Peter, and Peter has 6 times more marbles than David. How many marbles have each boy got?*

The set of 12 problems was given to 167 teachers in an in-service workshop in Israel and 42 teachers in the Czech Republic. The problems were also given to 132 15-year old students who already studied **equations with one variable**. Each problem was solved by about 30 teachers and 20 students. It took less than 40 minutes for each to complete the task.

Findings

Table 2 presents the distribution of selected strategies in all problems by complexity level for teachers and students (In each cell the teachers are first followed by the students).

Table 2: Distribution of Selected Strategies in All Problems by Complexity Level – (Teachers, Students)

Complexity Problem #	2	3	4	6	8
A1	D=(83,91)			J=(0,0) P=(0,0)	
A2		J=(11,8)	D=(70,87)		P=(0,4)
A3			P=(7,31)	D=(62,38) J=(7,6)	
B1			D=(59,63)	J=(0,0) P=(24,13)	
B2		J=(21,18)	P=(7,18)		D=(55,36)
B3	P=(46,44)			D=(46,26) J=(0,0)	
C1		D=(46,43)	J=(36,43)		P=(0,0)

C2	J=(74,65)			D=(19,22) P=(0,0)	
C3		P=(0,4)	J=(48,65)		D=(32,17)
E1		D=(38,39)	P=(42,44)		J=(0,0)
E2			J=(46,18)	D=(29,48) P=(11,24)	
E3		P=(30,13)	D=(52,52)		J=(11,4)

The fact that the distributions of strategies within each problem are similar for teachers and students means that we are dealing with a distinct cognitive task in each problem. Each problem elicited different distribution of strategies (very similar for teachers and students). Thus, despite the difference between teachers and students, we can consider the two samples to be a replication of the same tasks.

Conclusions and Discussion

The presented findings are only a part of a wider study. We realized that two main factors affect the choice of strategy (the choice of X): the complexity level – namely choosing the lowest possible level of complexity; and the surface structure variable with preference for expressions containing “more”. One should remember that choosing an expression with the term “more” means that the referent is a smaller quantity leading to an equation with whole numbers. There are, however, cases in violation of this finding. We noticed that in such cases, surface structure variables such as “the order of the information” and an easy transformation from “less” to “more”, dictate the selection of the strategy.

After solving the problems we asked the teachers to reflect on the reasons for their choice of strategy. The replies were: “Working according to the order”; “Choosing the smallest quantity as the X”, “I chose the X according to the order of the information”.

In reply to the question **why did you choose this strategy?**, some said it was according to the actual text: “The text leads to a certain strategy”; “It was more convenient”; “My prior knowledge about the world”. The last question was **what makes a question easy or difficult?** “Finding out which is the smallest quantity to compare the others to”; “the identification of the given sets”. In some Czech solutions another reason for the choice of strategy occurred. Those who use the graphical representation of the problem structure (the use of line-segments, see e.g. (Novotná, 1997)), this representation supports the choice of David – the smallest part – for X.

As mentioned before, teachers and students mostly behaved similarly. However, in cases of discrepancy, students demonstrate clearly that they prefer a

direct translation of the text with low complexity level, even if they have to write an equation with fractions. Teachers prefer to arrive at an equation with whole numbers and probably are more at ease in making the linguistic transformations from “less” to “more”, arriving at a complexity level of 6.

We based our work on theories of solving word problems such as that of Kintsch, which distinguishes between the text base and the situation model (Nesher and Teubal 1975), (Riley and Greeno 1988), (Kintsch and van Dijk 1978), (Kintsch 1986).

Now, we would like to return to clarify what was actually the “situation model”. In Kintsch’s terms a “situation model”: “is a mental representation of the situation described by the text”, while our claim is that the same worldly situation can be described by different texts as exemplified by our 12 texts describing the same worldly situation. By saying “a different text” we mean different not merely on the surface structure level, but also in its propositional structure with its coherent macrostructure (Kintsch, p.89).

Observing the strategies employed empirically by teachers and students we saw that although the text implies certain relationships, the solver adds other relations that are not mentioned in the text. The ability to add such relations comes from the comprehension of the entire world situation. For example, in Problem A3 there is a description of the relation between Peter and David, and Peter and Jirka (A3). The solvers who chose “D” as their reference (X) did not hesitate to solve the problem by bringing in the relation between David and Jirka (not mentioned at all in the text). Moreover, this was even the preferable strategy used by 62% of the teachers.

Our study introduced a manifold variable namely the “complexity level”, to which we can attribute most of the variance in choosing the independent variable (The X of the equation).

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DRAWING IN THE PROBLEM SOLVING PROCESS

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Abstract: *This paper analyses and develops some aspects emerged from the first stage of a research project aimed at investigating the solving strategies of open-ended problems. In particular, the analysis we propose focuses on the formulation process of a conjecture from the point of view of figural-conceptual dialectics. A particular behaviour concerning the activity of formulation of a conjecture is highlighted, for which a reading key interpretation is proposed.*

1 Introduction

Drawing, intended as a system of graphic signs, has always accompanied thought, establishing a close relationship with the objects of geometry (Mariotti, 1995).

The role that drawings play in the geometrical activity and in particular in problem solving is the subject of many researches (Duval 1995, Fischbein 1993, Laborde - Capponi 1994, Mariotti 1995). Despite the different points of view from which these researches have been carried out, all of them acknowledge the importance of such role. However there is not a universal agreement on what is meant by "drawings" or "figures" and on their relations with the geometry. For instance, Duval (1995) does not make any distinction between "drawings" and "figures". Whereas Laborde and Capponi (1994) describe the relations among drawing, figure and geometrical object in terms of the classic triad significant, meaning and referent.

More precisely, according to Laborde and Capponi a drawing may be considered as a significant of a theoretic referent (the object of a geometrical theory), whilst the geometrical figure may be defined as the set of all the couples constituted by the referent and one of the drawing representing it (ibidem, p.168).

In the following we adopt this point of view.

2 Our research

The study presented in this paper has developed on the basis of a research project, still in progress, carried out with the aim of investigating on the solution processes of open-ended problems (Maracci 1998, Maracci - Mariotti 1999). The research involved 17 students (11th and 12th grades) selected from different scientific high schools and evaluated by their teachers as medium - high achievers. These students were presented with 4 selected open-ended problems to be solved in individual interviews during which they were asked to think aloud. The problem solving sessions were videotaped and the transcripts of the interviews analysed (Maracci, 1998).

In this report we will focus on the specific moment of formulation of a conjecture. The analysis of the transcripts reveals some different behaviours on students' part. Here we will describe and discuss the following one:

- after a period of investigation conducted with the aid of drawings, students are quite confident in getting the correct conjecture (or, in any case, the relations to be used); and they face the task to formulate it precisely (e.g. in the form "if ... then ...");
- the conjecture is achieved with clear and explicit reference to one or more specific drawings;
- students formulate their conjecture explicitly, producing at the same time a *new drawing very similar* to those, to which they previously referred.

With respect to such behaviour the following question may be posed:

When students seem to possess all the necessary informations why do they feel the need of producing a new drawing so similar, if not identical, to the previous?

In this report we will carry out our study referring only to the following of the 4 proposed problems:

Problem: *A convex angle rOs , where the two rays r and s are not on the same straight line, and a point P internal to the angle are given. Determine a line segment which has its ends on the arms of the angle and P as its midpoint.*

That is a construction problem. It presents some difficulties in the identification of the solution, but it may be successfully approached by assuming the segment given and looking for some characterizing properties.

The theory of figural concepts (Fischbein 1993), which we briefly present below, provide us with the suitable theoretical framework according to which

we will develop and propose some hypotheses to discuss and explain the highlighted behaviour.

3 Figural concepts and satisfactory drawings

As Fischbein (1993) clearly pointed out, when dealing with geometrical reasoning neither pure concepts nor pure images are involved but rather there is a fusion between figural and conceptual aspects. What we deal with are not pure concepts since they possess figural properties and reflect spatial relations, but they are not even pure images, since their properties are "completely fixed - directly or indirectly - by definitions in the frame of a certain axiomatic system" (Fischbein, 1993 p.160).

We are concerned with a mental construct which simultaneously and intrinsically possesses figural and conceptual properties. Fischbein calls *figural concept* this particular kind of mental construct.

The harmony between the figural and conceptual components of a figural concept appears rather frail; although it is a single mental construct, "it (the figural concept) potentially remains under the double and sometimes contradictory influence of the two systems to which it may be related - the conceptual and the figural one" (ibidem p.160). Consequently the perfect fusion between the two components remains only an ideal situation.

Within this theory a drawing is the material, concrete representation of a figural concept, reflecting the tension between the figural and the conceptual component. Such view of drawings is coherent with that of Laborde and Capponi mentioned in § 1. The connections between Fischbein's theory and the perspective of Laborde and Capponi might be not immediately clear, but unfortunately we cannot deepen this point here. For a more detailed discussion on the subject see Laborde-Capponi 1994 (especially pp. 168/9)

3.1 The process of drawings production

Coherently with Fischbein's theory, the following hypothesis can be stated: *when producing a geometrical drawing a student tries to reach a harmony between figural and conceptual aspects.* The search for such harmony might be not conscious, or at least not in these terms. One could, perhaps, speak of a generic feeling of satisfaction, from the students' point of view, and of the

observance of specific requirements learnt from school practice (e.g. the need for generality).

Which are the elements able to reveal whether a drawing adequately reflects the figural - conceptual balance of the figural concept it represents? In other words: what does it make a drawing *satisfactory* for students?

According to our hypothesis, at least three factors may be identified:

- a drawing should represent "correctly" the geometrical situation described in the problem, that means that the student's understanding of a given situation and her interpretation of the produced drawing should be consistent;
- a drawing ought to be recognized as sufficiently generic, for instance: a student will hardly accept the drawing of a rectangular triangle as representing a generic triangle;
- a drawing should possess a good gestalt, it should satisfy the fundamental laws which control the basic processes of perception: it is unlikely that a student will draw a square whose sides are not along the horizontal - vertical directions.

These conditions can appear and combine in many different ways and, as mentioned before, they correspond to needs which are not always identified clearly and consciously by the students.

3.2 *Satisfactory* drawings in solving open ended problems

Solving an open-ended problem is an activity which often requires students to explore a particular geometrical situation. Drawings play a fundamental role in activities of this type and for this reason the management of drawings assumes great importance. In particular, we are faced with the following question:

Once a student has produced a *satisfactory* drawing, which conditions can induce her to produce a new one significantly different from the previous and which conditions, on the other hand, can induce her to produce drawings - copies of the former?

The considerations we developed and the experimental results of our research led us to identify some elements which can influence the students' decisions on the management of drawings.

Producing similar drawings may be useful because:

- once a *satisfactory* drawing has been made, reproducing it without modifying its basic structure avoids that students need to control correctness, generality and good gestalt of the new drawing and allows them to focus all the necessary attention on other aspects of exploration;
- it allows to relate drawings one to each other so that the entire activity of exploration of the problem may be more easily controlled;
- it prevents loss of information when passing from one drawing to another.

On the contrary it can be necessary to make a drawing different from the previous ones if:

- one of the factors which made the drawing *satisfactory* fails;
- ambiguity or conflict emerge for some reason among the drawings, or within a particular one: this may occur when new elements are added;
- students realize that an exploration conducted on similar drawings may turn out to be limited and incomplete. We can find this type of behaviour at the beginning of exploration when students are looking for as much information as possible with no precise strategy in mind, or after an unsuccessful search when they try to grasp new information from new drawings.

Let us remark that a drawing may be perceived satisfactory even only in relation to specific moments and not throughout the problem solving session. We will better explain this point in the following paragraph.

4 Drawings and their *history*

As mentioned above, the activity of solving open-ended problems often requires that students explore a certain geometrical situation. In particular, this exploration is coupled with a more or less intense activity aimed at producing drawings which can provide material support for the manipulation of the concepts involved. According to the development of the activity the resultant drawings can be modified, by the students, e.g. by adding new graphic elements or even reinterpreting those which have been already drawn.

We think that independent of the way in which they appear on paper, in students' eyes the drawings produced during this type of activity could embody their *history*, i.e. their successive transformations and reinterpretations.

So a drawing may be interpreted as follows:

- it may be inserted in a *history*, i.e. considered with reference to the activities of its production and of its further manipulation. A typical example is that of a drawing produced autonomously by the student, who has drawn its constitutive elements in a certain order, in response to particular stimuli or needs;
- it may be considered achronically, without history. This is the case of a drawing produced by others, when the student was not present, and whose elements are considered all together.

These two interpretations are not mutually exclusive, on the contrary the same student may use both of them in the solution process. In our opinion there must be a connection between them, however a thorough study of these relationships goes beyond the goals of this paper.

Our main hypothesis is that *in the process of solving-open ended problems students seem to be more generally inclined to consider the drawings they produce together with their history.*

That means that one may interpret a drawing which accompanied a certain activity as incorporating the activity itself. As a consequence a drawing interpreted with its history might turn out unusable to support other activities.

This hypothesis and the analysis, previously carried out, of the elements reflecting the figural - conceptual balance in a drawing, represent the key point of our study. As a matter of fact they provide the key to interpret the particular behaviour highlighted in § 2 and concerning the specific moment of conjecture formulation.

Our hypothesis suggests that the answer to the question posed in § 2 may be found taking into account the *history* of the produced drawings.

5 Giacomo's protocol (11th grade, scientific high school)

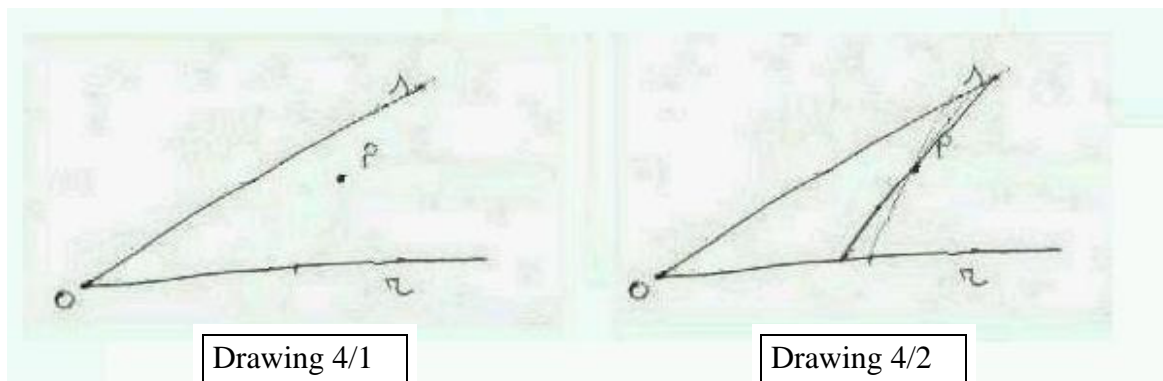
In the following we shall report an excerpt from Giacomo's protocol relative to the activities of identification and formulation of the conjecture. We shall analyse this excerpt in the light of the previously carried out considerations.

In reporting Giacomo's protocol we will mark in "times" what has been said during the interview and in "italics" what Giacomo did.

The original drawings were scanned and processed by means of computer in order to reproduce exactly, on the basis of the analysis of the videotapes, the ways in which they appeared at each moment of the problem solving session. Each drawing is labeled by a couple of numbers. The former characterizes the drawing with respect to all the others produced during the interview, the latter indicates the "steps" of the drawing production and of its successive modifications.

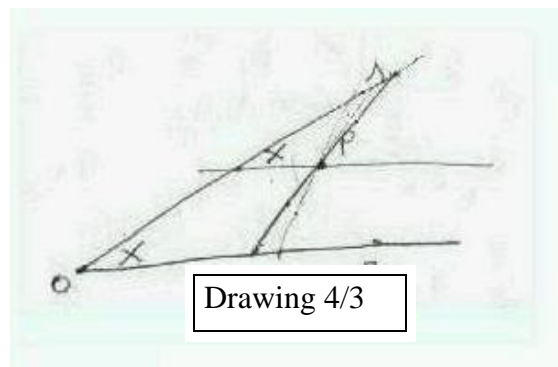
Phase 1: identifying a conjecture

10. **Int:** what does "determine a line segment ..." mean?
11. **Gia:** finding a point ... a line segment which passes through P and has just its ends on r and s and such that P is its midpoint ... identifying it, finding it
12. *He draws the angle rOs and the point P (drawing 4/1); he only just sketches some segments passing through P and traces one of them (drawing 4/2)*



This is the fourth drawing Giacomo has made searching for the correct conjecture. The previous drawings, which we cannot show here, share characteristics with this one, for example amplitude of the angle, orientation of its sides and position of point P on the inside. Giacomo evidently produced from the very beginning drawings which seemed to be sufficiently *satisfactory* (§ 3.1).

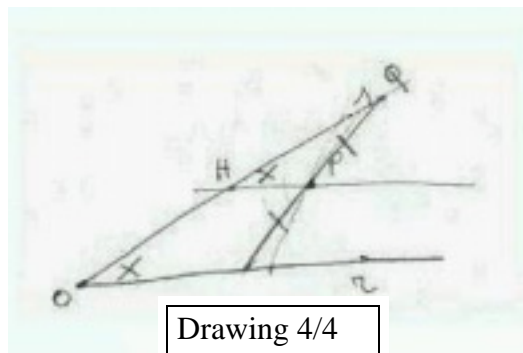
His first intervention on this drawing consists in tracing some segments passing through P (item 12.). Taking into account that he attended a computer-based course in geometry in which microworld Cabri was used, we can suppose that he is trying to reproduce on paper the dynamism typical of the exploration Cabri allows to perform on the screen of the computer.



13. **Gia:** one ... one can make the line parallel to r pass through P - *he draws the parallel line for P (drawing 4/3) - at this point .. - he marks the corresponding angles*

After having drawn the segment to be determined Giacomo traces the straight line parallel to r passing through P ; this is not the first time he has drawn it, he had previously considered two straight lines parallel to r and s without drawing the unknown segment. Here he seems to be trying to combine the strategy of the parallels with that of drawing the unknown segment.

14. **Gia:** here, practically ... if I draw the segment through P - *he stresses the traced segment - which should be practically divided in half, I should find that, owing to the similitude ... this one is half of this one (he marks the two halves of the segment)* and therefore even the ratio of the other two sides, that is ... - *he writes the letters Q and H (drawing 4/4) - OQ and OH must be 2*

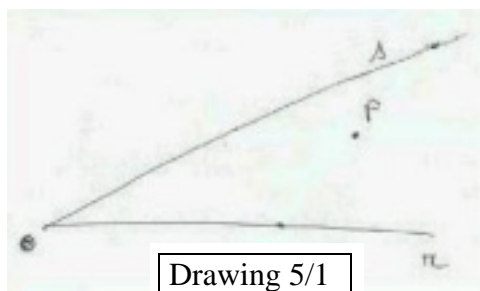


Drawing 4/4

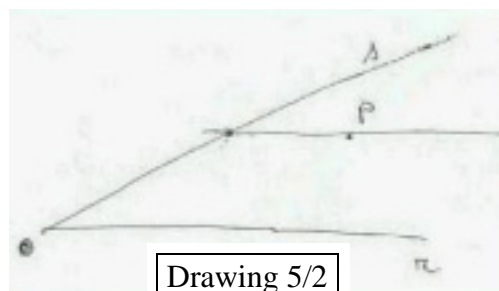
In the two previous items (13. and 14.) Giacomo identified the conjecture clearly; the introduction of labels reflects the awareness on Giacomo's part that the drawn elements are the ingredients able to solve the problem. The use of the conditional mood does not express uncertainty about the correctness of the conjecture, Giacomo seems immediately positive about, or of the inferences; rather it could indicate the awareness that reasoning is carried out starting from the unknown segment instead of starting from the data of the problem.

In order to successfully conclude his problem solving session Giacomo has to explicitly formulate its conjecture and prove it. The drawing he has just produced (drawing 4/4) seems *satisfactory* (§ 2), moreover it appears rather precise and orderly, it is almost completely devoid of elements unnecessary to formulate the conjecture. One could expect that Giacomo formulated its conjecture referring to such drawing but as we will see this did not happen.

Phase 2: formulating a conjecture



Drawing 5/1

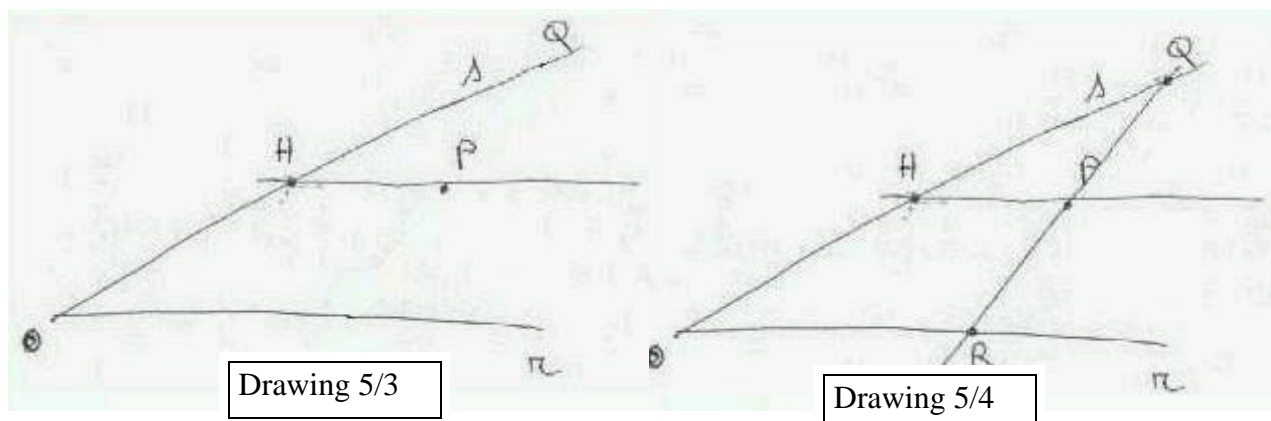


Drawing 5/2

14. **Gia:** then, in order to find the segment one can proceed as follows: - *he draws the angle and point P writing the letters (drawing 5/1)* - given point P, one draws the line parallel to r (*he draws it (drawing 5/2)*) - after which (*he marks with H the intersection point between s and the parallel to r*) he doubles it ... I would obtain OH, let us consider a segment of equal length - *he marks Q on s (drawing 5/3)* - and one draws the line ... it has its ends - *he traces the straight line QP and marks with R the intersection between the line and r (drawing 5/4)* - on the arms of the angle and P as its midpoint

Notwithstanding drawing 4/4 could appear a suitable support for concluding the problem solving session, while formulating his conjecture Giacomo produces a new drawing.

Why does Giacomo decide to produce a new drawing?



Drawing 4 has been produced during the search of a solution and *its history reflects this activity*: the unknown segment was the first to be drawn because it was assumed as known (drawing 4/2), whilst the parallel line was traced afterwards (drawing 4/3).

In order to formulate the conjecture, Giacomo needs to restore the identified relations (item 14.), this time starting from the data of the problem (i.e. the angle rOs and point P). Thus, the need to produce a new drawing seems to be linked to the need of restoring the correct logical order of the conjecture. But according to our hypothesis, Giacomo interprets drawing 4/4 together with its *history* as representing the search for a conjecture and therefore it cannot adequately support the process of fomulation of the conjecture itself.

Let us note that Giacomo does not only produce an ordinary new drawing, but the drawing he produces is practically identical to the previous one. Observing the steps relative to the two drawings (drawing 4 and drawing 5) we notice that the first two (drawings 4/1 and 5/1) as well as the last two (drawings

4/4 and 5/4) are practically equal. What is completely different is their history, as we expected in the light of our previous analysis.

In order to explain the highlighted similarities we have to consider that formulating a conjecture is a rather demanding activity. It requires the greatest attention on the part of the student. However, even producing a new drawing may be a demanding activity: as previously discussed (§ 3.1) any drawing has to be *satisfactory*. Drawing 4/1 was already satisfactory, producing a similar drawing allows Giacomo to be engaged only in the formulation of the conjecture, avoiding the need to control the correctness, generality and good gestalt of the new drawing (§ 3.2). The fact that even the two conclusive frames (drawing 4/4 and 5/4) are equal may represent a sort of check of the correctness of the formulation of the conjecture; their perfect correspondence assures that no mistakes have been made.

In conclusion Giacomo's behaviour seems to synthetise and harmonise two distinct needs: on the one hand, having a drawing adequately supporting the activity of conjecture formulation; on the other hand, being able to draw the greatest attention on this activity.

6 Conclusions

Giacomo's protocol may be considered paradigmatic, similar examples could be provided, but the brevity of this paper does not allow to do that.

Two elements of fundamental importance emerge from the analysis of the protocol:

- when producing a drawing the student tries to reconcile and harmonize needs of both conceptual and figural nature, respecting the criteria of correctness, generality and good form;
- a drawing produced by the student is interpreted together with its history. The lines by which it is composed have been drawn in particular moments and in a certain order with respect to a specific activity of the problem solving process. The drawing with its history represents this activity.

Our previous research (Maracci 1998) already highlighted students' difficulties in managing drawings in problem solving. The considerations presented in the previous discussion of the experimental data allow us to propose a hypothesis which contributes to a better specification of these difficulties. Managing drawings in the solution of open-ended problems is an activity to be performed at two distinct levels:

- *achronic level*: a drawing has to be a satisfactory representation;
- *diachronic level*: a drawing has to be considered according to its becoming over time, i.e. together with its development.

The abilities of controlling and managing drawings at both levels seem to be fundamental in the problem solving activity; as a consequence we think that the development of such competencies has to become an explicit educational goal. In order to reach such goal specific activities remain to be identified and included in school practice; in particular, students must be stimulated to operate on the two levels both separately and simultaneously.

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ANALYSIS OF EYE-MOVEMENTS DURING FUNCTIONAL VERSUS PREDICATIVE PROBLEM SOLVING

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Abstract: *The theory of functional/predicative thinking, once developed to describe differences in students' behaviour while solving programming tasks, is applied to analyse eye-movements while solving tasks of visual pattern completion (QuaDiPF-tasks, Schwank 1998/2000). Predicative thinking requires that in order to meaningfully complete the pattern the subject has to get involved with the logic of the static structure of the pattern, functional thinking requires to get involved in a dynamic reading of the logic of the pattern. The QuaDiPF-tasks proved to be useful in other experiments to predict typical functional or predicative behaviour of the subjects. The eye-movement-study is a second approach after an EEG-study to use not only qualitative methods for the classification of problem solving behaviour, but also quantitative ones.*

1. Qualitative investigations of functional/predicative thinking

In the very first days of the theory of predicative versus functional thinking (Schwank 1986, 1993-1995) we used the nonverbal intelligence test APM (Advanced Progressive Matrices) from Raven (1965) to balance our different groups of subjects spread over such different countries as Germany, Indonesia and China (Marpaung 1986, Xu 1994). The focus of interest was on differences in students' behaviour while solving some programming problems using different types of microworlds. The label “*predicative*” was used to characterize a problem solving behaviour highly orientated towards and sensible for features, relations and judgements, whereas the label “*functional*” was used to characterize a problem solving behaviour highly orientated towards and sensible for courses, modes of actions and effects. The experiments were run in the form of constructive teaching experiments (Cobb et al. 1983), hence, qualitative analysis methods were to apply. Although one could have expected strong cultural influences on the cognitive processes involved when solving our programming tasks, the results show that knowledge about the national relationship cannot help to describe or to predict the individual types of problem solving behaviour of our subjects. Instead, the distinction of functional and predicative thinking was very helpful. The results show that students tend to have a strong preference for either of the two thinking styles, which shows up most clearly in cases when students are quite intelligent and do reach the limits of their intellectual power when challenged by specific tasks.

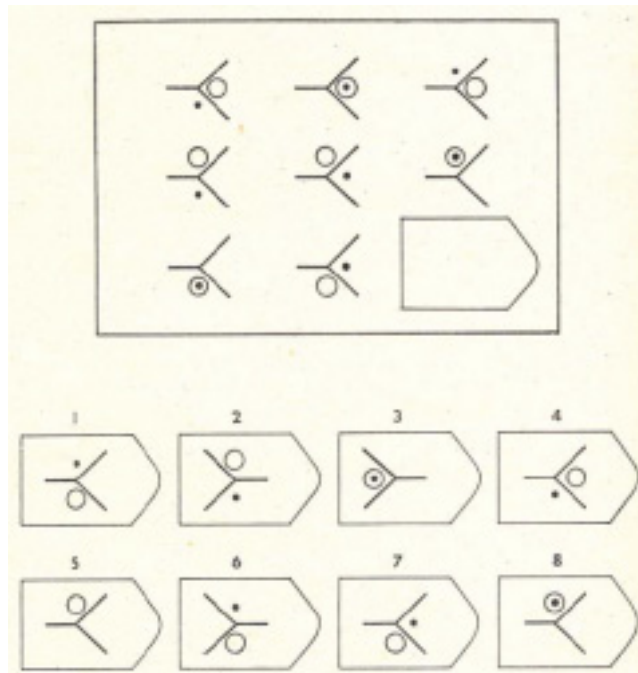


Fig. 1: Typical Raven-task (APM II-14)

goes as follows: Each figure consists of three objects: an upsilon, a point and a circle. The upsilon is the same in each figure. In each row, the circle is at the same place. In each column, the point is at the same place.

The interesting point is that there does also exist a functional strategy to solve this task: Perceiving the pattern a process is invented which produces the last element in a row or column. In each row, the point moves around, and in each column, the circle moves around. The object around which the movement takes place remains stable.

The moment we realized these two different approaches of inductive thinking (Schwank 1996) while solving matrices tasks, we had the idea for a new tool to test subjects in how far they are able to think in a predicative way and how far they are able to think in a functional way.

We invented new tasks, which differ considerably in their level of pure functional and pure predicative difficulty (QuaDiPF-Test; Schwank 1998/2000). As we are interested in the decomposition of thinking processes, we eliminated the possibility to start visually matching procedures by not offering a sample choice of 8 solutions as in the original APM-test. Instead of such a kind of choice the subjects are asked to draw their solution figure and to argue why their figure fits the pattern. The sessions are videotaped and the reasons given by the subjects are qualitatively analysed.

In several experiments, we use QuaDiPF-tasks as well as other tasks like term rewriting tasks (Cohors-Fresenborg 2001; Striethorst, in press) or

Only later, we began wondering why students, who were remarkable for their good functional behaviour in our programming tasks, had the same good results in the APM-test as students, who were remarkable for their good predicative behaviour. At first glance, it seems that solving APM-tasks requires getting involved with the logic of the static structure of the pattern and therefore, one has to recognize the regular recurrence of certain features of the elements (e.g. shape: square, circle ...; size: small, big, ..., position: left, right, ...). One of the possible predicative analyses of the APM II task No. 14 (Fig. 1)

programming tasks (Xu 1994; Armbrust, in press). The results of the QuaDiPF-test allow a good prediction whether a strong functional or a strong predicative behaviour of the subjects is to be expected in cases of the other problem solving tasks.

2. Quantitative investigations of functional/predicative thinking

EEG-Studies

Compared with the complex tasks used in our experiments about programming behaviour or term rewriting, the QuaDiPF-tasks are very simple - actually in a very specific sense. The only visible activity the subject has to carry out while thinking about a QuaDiPF-task is to move his/her eyes. Due to this fact and because of the simplicity of presenting these tasks for mental exercises suddenly EEG-experiments on functional and predicative thinking became possible. So far, we have been running two of such experiments together with Jan Born and members of his research group at the Medical University of Lübeck, the first of them has been completely evaluated (Mölle et al. 2000, Schwank 1999). In this study, the EEG was recorded in 22 young men while solving QuaDiPF-tasks. The results are that the EEG complexity during predicative thinking decreased in comparison to functional thinking and mental relaxation, with this reduction being most pronounced over the right and parietal cortex; a reduction in dimensional complexity during functional thinking as compared to mental relaxation which was concentrated over the left central cortex, although significant, was less clear.

Eye-Movements-Studies



Fig. 2: Student wearing an EyeLink headband solving QuaDiPF-tasks

Having known for long about the possibilities of eye-movement-studies (Galley 2001), we were looking for some partners who would like to co-operate with us. Carpenter et al. (1990) made an attempt to analyse the manner how subjects solve the Raven Matrices using eye-tracking methods. In summary they stated: *“The processes that distinguish among individuals are primarily the ability to induce abstract relations ...”* (p. 404) and the only pure functional Raven task they excluded from analysis: *“Problem was not classifiable by our taxonomy”* (p. 431). The question is of course,

what distinguishes functional problem solving strategies from predicative ones. Finally, we met Franz Mechsner, Max Planck Institute for Psychological Research (MPIPR), Munich, who agreed to run a joint experiment in one of their

laboratories and we started with about 20 subjects in summer 1999. The screen resolution in this first pilot study was not the best (640x480). As partners in the new course of study “Cognitive Science” at the University of Osnabrück, we got the possibility to use the new equipment for eye-movement studies, e.g. now we can work with a screen resolution of 1024x768. In a second pilot study, we tested 12 subjects in Osnabrück using an EyeLink system from SensoMotoric Instruments (for technical details see <http://www.smi.de>). In the sessions, which are videotaped, the subjects are confronted with the tasks on a screen, they wear a headband (Fig. 2) with two ultra-miniature high-speed cameras in order to record their eye-movements (saccades, fixations). After having solved a task, the subjects are asked to draw their solution figure and to argue why it fits the pattern well.



Fig. 3a: Entire eye-movements solving the task
Duration: 01:11,344

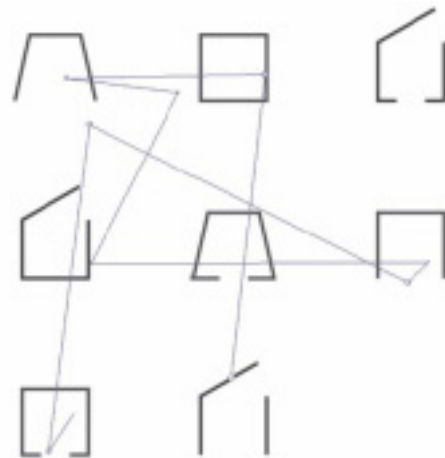


Fig. 3b: Start orientation
Time: 00:02,310 / Duration: 00:01,155

Figure 3a shows the eye-movements of a predicative solution. The data is scanned regarding interesting sequences of eye-movements: which parts of the pattern are looked at in which sequences. Fig. 3b-3e show such sequences. Time and duration are given in [min]:[sec], [msec]. The self-explaining predicative argumentation of the subject is given below. [E1-E8 refer to the single elements of the figures reading from left to right, starting in the first row.]

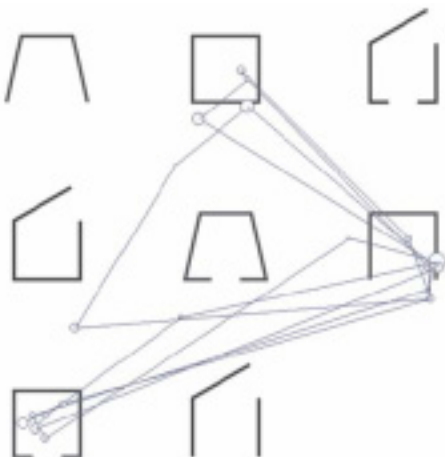


Fig. 3c: Realising the set of “squares”
Time: 00:22,642 / Duration: 00:05,166 492

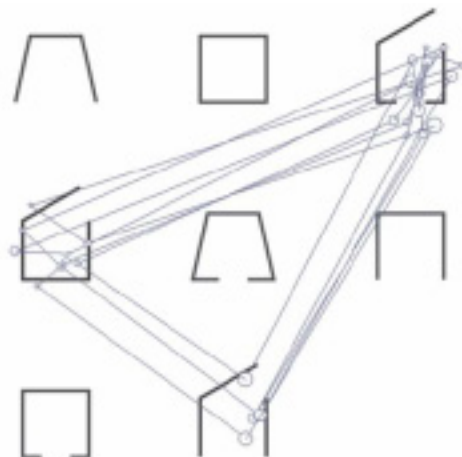


Fig. 3d: Realising the set of “houses”
Time: 00:28,541 / Duration: 00:07,445

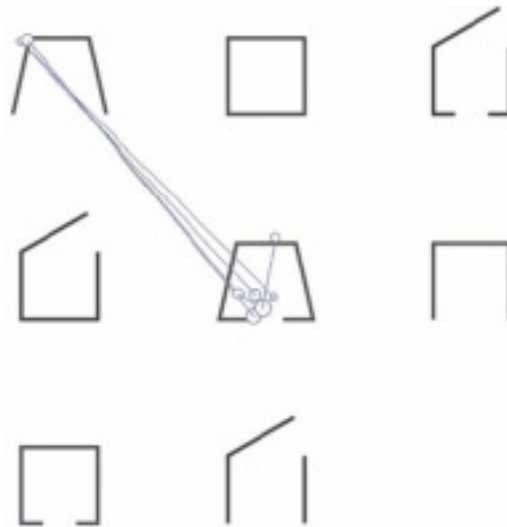


Fig. 3e: Realising the set of “trapezia”
Time: 01:02,880 / Duration: 00:02,336

Predicative Problem Solving - Example

S[subject]: (*Finished the drawing*) So. Well, it's not very nice in shape. It is supposed to look like (*S points to E[lement]1*) this one closed. Hm, well, I'll give the following reasons for that, hm, there are three different shapes each time. (*S points to the elements in the first row*). Once it has (*S points to E1*) - they have no bottom line each time. Once with (*S points to E2*) a half full, hm, empty, yes, half a bottom line and once with none at all. This can be seen in there. Once full (*S points to E2*), nothing at all (*S points to E6*) and once half (*S points to E7*). Once here (*S points to E3*) hm half, once full (*S point to E4*) and once none at all (*S points to E8*). And here this is missing (*S points to E1*), then, once none at all (*S points to E5*), once half and then it has got (*S points to E9*) to be in full. I would say so.

V[ersuchs]P[erson]: So. Ja, also formschön ist es nicht. Das soll so aussehen, wie (*VP zeigt auf E1*) das hier zugeschlossen. Ähm, das begründe ich jetzt so, ähm, hier sind jeweils drei verschiedene Formen. (*VP zeigt die Elemente der ersten Zeile*) Einmal hat das (*VP zeigt auf E1*) haben die jeweils einen leeren Boden. Einmal einen (*VP zeigt auf E2*) halb voll äh leeren ja einen halben Boden und einmal überhaupt keinen. Das kann man hier dran sehen. Einmal voll (*VP zeigt auf E2*), gar nichts (*VP zeigt auf E6*) und einmal halb (*VP zeigt auf E7*). Einmal hier (*VP zeigt auf E3*) äh halb, einmal voll (*VP zeigt auf E4*) und einmal gar nicht (*VP zeigt auf E8*). Und hier fehlt das (*VP zeigt auf E1*) dann einmal gar nicht (*VP zeigt auf E5*), einmal halb und dann muss (*VP zeigt auf E9*) einmal voll sein. Würd' ich mal sagen.

Functional Problem Solving - Example

S[subject]: Hm. Left and right, so they move inside as a circle and then outside.

E[xperimenter]: Hm.

S: This repeats every time. (*S points to the left and right side of the “square” in E1 and then moves along the curves in E2 and E3*).

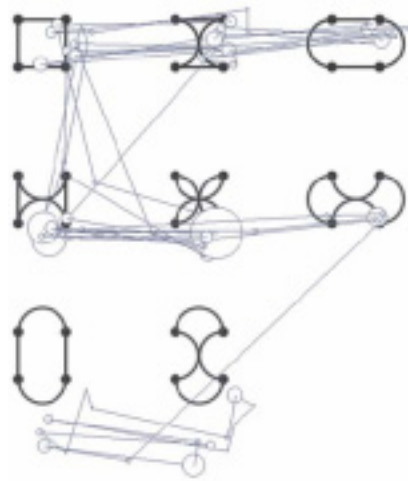


Fig. 4: Entire eye-movements
Duration: 00:23,236
The calibration at the bottom
of the screen wasn't very
successful.

ich nicht drauf geachtet. Ja ist vielleicht das ... ein Moment. - Ja ist das selbe praktisch, ne. Erst nach innen, dann nach außen.

E: Is there also an explanation for the columns?
This was an explanation for the rows.

S: Here? Or what? (*S moves along the columns, one after the other*)

E: Yes.

S: I did not consider them. Yes, this might be ... one moment. - Yes, it's the same, basically. First to the inside, then to the outside.

V[ersuchs]P[erson]: Äh. Links und rechts, die gehen also als Kreis nach innen und dann nach außen. (V[ersuchs]L[eiter]: Ähm.) VP: Jedesmal wiederholt sich das. (VP zeigt auf die linke und rechte Seite des "Quadrates" in E1 und fährt dann die Bögen in E2 und E3 entlang) VL: Gibt's noch ne Erklärung in den Spalten? Das war ja in den Zeilen erklärt. VP: Hier? Oder was? (VP durchfährt nacheinander die Spalten) VL: Ja. VP: Da habe

This functional solution - the figures result from their predecessors by moving - is interesting for several reasons. After short reflection, a fairly well-founded solution is given. Thereby, the subject used only information he checked with his eyes in the rows as it is obvious from his eye-movements-data (Fig. 4). The experimenter's demand for further explanation shows the subject being aware of that.

In case of our EEG-studies, we had to decide that the subjects give the reasons for their solutions only after they created them mentally because any physical movement would have made the EEG-data unsuitable for further analyses. It was never clear of which status the explanations, given in retrospect, were. What is their relationship to the thoughts of the subjects at the moment they were developing their solutions? It might be that the verbal explanations are more or less nothing but nice sentences which the experimenter would like to hear, and thus produced for this purpose. E.g. in a particularly unfavourable case, the subject could have thought functionally, but argued predicatively. Even though it is fascinating to investigate brain mechanisms more directly via EEG-methods, insight is increased by investigating eye-movements as seen here.

The eyes' scanning of the QuaDiPF-tasks, controlled by the brain, at least shows the process of attempts to orient oneself in the task, where the gaze gets caught, which parts of the task - in which sequences - are preferred compared to those which are disregarded.

3. Outlook

So far, we know that we find traces of eye-movements, which fit perfectly to the argumentations of the subjects. It seems difficult (or even impossible) to cheat with the eyes, when using the eyes as an essential tool during thinking processes. When we were learning to use the technical equipment for the investigation of eye-movements, several members of our research group checked the usage of the EyeLink headband and tried to simulate the solving of known QuaDiPF-tasks. The reproduction of a solution or the attempt to simulate predicative/functional thinking (which requires guarding the thoughts) is accompanied by quite different eye-movements than those during actual thinking processes creating a new solution without specific constraints.

Dependent on the type of a QuaDiPF-task, the analyses of the eye-movements show whether a solution was produced in a predicative or (probably) in a functional way. In one type of predicative QuaDiPF-tasks, it is even possible to distinguish between two possibilities of predicative problem solving. In Fig. 3 the triangles along which the eyes move, are typical for the predicative approach “creating sets taking into account typical features”.

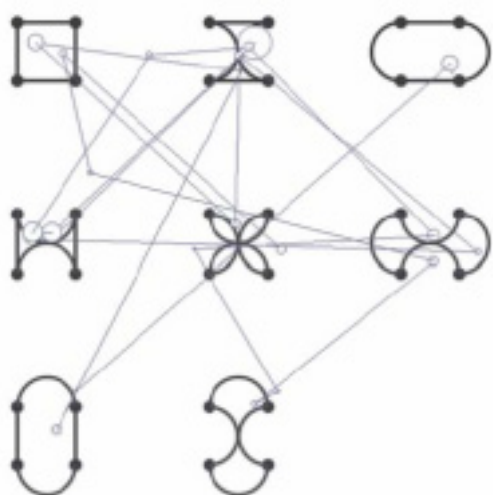


Fig. 5: Searching for pairs
Time: 00:08,953 / Duration: 00:06,355

Another predicative approach is to break down the elements in their components (bottom, walls, top) and to check the regularities of their occurrences in the rows and columns: in each row an open bottom occurs once, so does a half-open bottom and closed bottom; furthermore straight walls occur twice, bevelled walls once; finally they are closed tops twice and a sloping tops once, the same pattern fits in the columns. The eye-movements from this kind of predicative solution don't proceed along such triangles as presented in Fig. 3, instead they proceed along the rows and the columns.

Eye-movements like the one given in Fig. 4 confront us with problems. We find similar eye-movements in cases of the following suitable predicative procedure: Analogous to the way described above, the elements are broken down in their components (bottom, walls, top), then it is very easy to see that in each row, the bottoms and the tops remain the same, the same holds for the walls in the columns. Unfortunately, we are only sure that it is predicative behaviour we are dealing with, if a subject builds sets, which can nicely be seen in the eye-movements (Fig. 5): solving the given task, a subject could argue that there are always pairs of elements, the “square” is missing his partner, the figure in the middle is just the centre point. In fact, so

far, the first analyses of about 30 subjects show that the set-building is one of the most striking characteristics in the eye-movements of predicative behaving subjects. In case of the QuaDiPF-task given in Fig. 5, the situation is not satisfying; because the predicative set-building approach is of lower quality than the predicative decomposition approach. Hence, such tasks are not suitable for distinguishing quite good predicative problem solvers from quite good functional problem solvers exclusively by means of the analyses of eye-movements.

In further experiments, we will use an another type of functional QuaDiPF-tasks (Fig. 6), for which in the predicative approach only a set-building strategy helps, but not a decomposition one. In addition, we will pay more attention to such QuaDiPF-tasks for which the degree of difficulty differs depending on whether one tries to find a predicative or a functional solution. E.g. the QuaDiPF-task given in Fig. 7 is rather difficult, but it is easier to find a solution based on functional arguments than to find a similarly good solution based on predicative arguments. Again, here, the set-building approach is only second class.

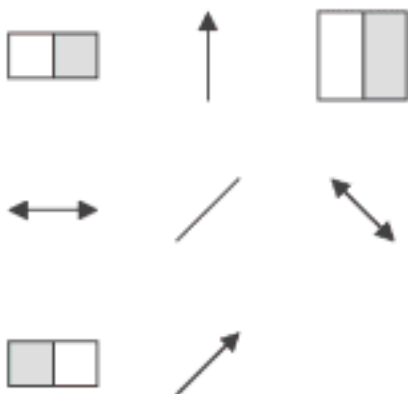


Fig. 6: QuaDiPF Set C - 17

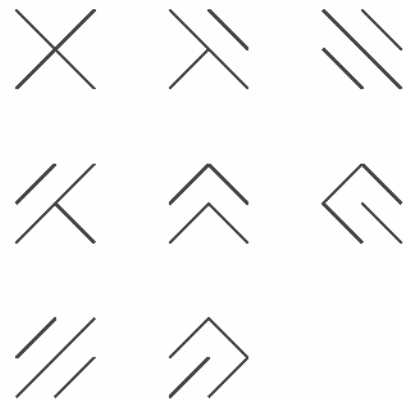


Fig. 7: QuaDiPF Set C - 13

We expect that - as in the past - only in rare cases we will find predicative problem solvers, who read the rows and columns and thereby elaborate a good predicative solution. These subjects will stand out from the others because of the short times they need to solve pure, difficult predicative QuaDiPF-tasks like the one given in Fig. 3. And, of course, we will develop more QuaDiPF-tasks, which fulfil the conditions of the meaningful measurement of eye-movements.

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And last, but not least, we would like to thank our student researchers and our doctoral students. In the ongoing experiment on eye-movements, mainly Silke Brinkschmidt and Stephan Armbrust (both Institute for Cognitive Mathematics, University of Osnabrück) have been involved. In the position of a guest student researcher, Silke Brinkschmidt tested the subjects at the MPIPR, Stephan Armbrust has provided the programme to start the analysis of the data on eye-movements and run the second pilot study at Osnabrück.

WORKING GROUP 6

Assessment and curriculum

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INTRODUCTION TO WG6 ASSESSMENT AND CURRICULUM

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The study of assessment and curriculum as a theme of research in mathematics education directs the attention to some immediate connections between research and practice. With an emphasis on large scale assessment and national curricula, group 6 primarily assumed a researcher's perspective of institutional phenomena - what is "officially" going on in the different European nations, and how are assessment methods and curricula evolving at the present time? It included a focus on comparisons between nations, but it also brought out the national traditions, which may be very stable and partly resistant to international influences.

The presentations in the group provided a basis for discussions that went beyond the particular national instances. What kinds of ambitions can be identified? How does one tackle the difficulties involved in aligning curriculum and assessment? In which ways does the "official" picture misrepresent what really goes on in the classrooms?

Some reflection will show that "school mathematics", as evidenced in assessment methods and curricula, is dynamic and that the new aspects are highly dependent on other changes in culture and on the available technology.

ELEMENTS ON EVOLUTION OF OFFICIAL CURRICULUM IN FRANCE THE CASE OF INEQUALITIES IN THE "COLLÈGE" LEVEL

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Abstract: *Our communication is a part of a more general study of the evolution of the curriculum in France in Twentieth century, seen from the thematic of inequalities. We will present here just some elements of the evolution of the official curriculum as it appears in the Official Syllabus in the "Collège" level (11 to 14). We focus on the following questions: are there inequalities in the curriculum? If any, what kind of inequalities? And in what curricular context do they "live"?*

Résumé: *Notre communication fait partie d'un travail plus général sur l'étude de l'évolution du curriculum en France au XXème siècle en prenant comme point d'attaque les inéquations. Ici nous allons nous restreindre à la présentation de quelques éléments de l'évolution du curriculum officiel tel qu'il se présente dans les programmes et les Instructions Officiels au niveau du Collège en nous intéressant aux questions suivantes : les inéquations sont-elles présentes dans le curriculum ? Si oui, quels sont les types d'inéquations ? Quels sont les contextes curriculaires dans lesquels ces types d'inéquations vivent ?*

Our communication is a part of a more general study of the evolution of the curriculum in France in the Twentieth century, seen from the thematic of inequalities. We will only present here some elements of the evolution of the official curriculum as it appears in the Official Syllabus in the "Collège" level (11 to 14). Our theoretical frame is the ecological approach initiated in France by Yves Chevallard. Two levels of analysis are thus considered. The first level is composed with "habitats" and "niches", that is to say the places and functions of the studied objects. The second level is focused on "praxéologies" (Chevallard 1999) that is to say tasks, techniques used to achieve these tasks, technologies understood as the discourse explaining and justifying techniques,

¹ Translated by Jean-Philippe Drouhard, équipe CESAME

and theories defined as technologies applied to technologies. We will study the curriculum looking for an articulation between these two levels of analysis. Here we focus on the following questions: are there inequalities in the curriculum? If any, what kind of inequalities? And in what curricular context do they "live"?

1. Presence of inequalities in the official syllabus

The object "Inequality" is present in the secondary level teaching during the whole twentieth century. Generally speaking, it appears at least in the "collège" syllabus. We can find it:

- In the 1902-1905 reform, in the year "3rd Classical A and B" and "3rd Modern" (14 ys.) as: *"first degree numerical in equations and inequalities with one unknown"*, where the word "inequality" refer also to numerical inequalities;
- In the 1947 reform, in the year "3rd B" (form without Latin) as: *"first degree numerical equations and inequalities with one unknown. Graphical Interpretation"*;
- In the 1960 reform, in the year "3rd" as: *"equations and inequalities; position of the problem; signification, in these problems, of the signs =, >, <. First degree numerical equations and inequalities with one unknown, with numerical coefficients. Graphical interpretation"*;
- In the 1971 syllabus, this object appears in the year "4th" (13 ys.) under the following forms: *"upon numerical examples, first degree numerical equations and inequalities with one unknown" (4th)*, and *"examples leading to one or two first degree equations and inequalities with one or two unknowns, with numerical coefficients. Graphical representation of the solutions of first degree equations or inequalities with two unknowns."* (3rd)
- In the 1977 syllabus too, inequalities appear both in the year "4th" under the following forms: *"numerical examples of first degree equations and inequalities with one unknown" (4th)*, and *"First degree equations and inequalities with two unknowns, with numerical coefficients. Solving of an equation, an inequality, a system of two equations; graphical solving of a system of equations or inequalities. Examples of various first degree problems "* (3rd)
- In the 1985 syllabus, this object appears in the year 4th: *"Solving of problems leading to first degree equations, inequalities with one unknown" (4th)*, and in 3rd: *"First degree equations and inequalities. Graphical solving methods of first degree equations and inequalities, with numerical coefficients. Solving methods of a system of first degree*

equations or inequalities with two unknowns, with numerical coefficients. Examples of various problems coming down to first degree "(3rd).

- In the present syllabus (1996-99), the object "inequality" appears in the year 3rd as: "*First degree inequality with one unknown.*", and "*Solving of first degree problems, or problems coming down to first degree*". The students' abilities being "*to solve a first degree inequality with one unknown with numerical coefficients. To represent the solutions on a graduated straight line*" and "*to put into equations and to solve a problem leading to a first degree equation, inequality or system of two equations*".

Inequalities are therefore constantly present in the secondary school, except in the 1925 reform where they are introduced only from the year 2nd (15 ys.). The kind of inequality is constant, too: mostly first degree inequalities with one unknown, except in 1971, 1977 and 1985 where appear also first degree inequalities with two unknowns, related to graphical representations and systems (1977 and 1985), related also to the "graphical solving methods" (1985).

Some syllabuses are about graphical representation of the solutions of an inequality (1947, 1960, 1971), others about explicit graphic solving method (1977, 1985). The latter is also about "*Solving methods of a system of first degree equations or inequalities*" without further details about these methods. Moreover the 1996-99 syllabus distinguishes to solve (an equation, an inequality or a system of two equations) from to solve graphically systems of equation and inequalities, which supposes that solving a systems of inequalities will be graphical only. Other precision is also given, for instance the "numerical coefficients" (1960, 1971, 1977 and 1985) and the problem solving (1977, 1985 and 1996).

Equations are constantly close to inequalities and, except in the 1902 and 1996 syllabuses, the two words appear together in the same sentences. We will develop this point below.

Another observation: in French there are two different words, "inégalité" (inequality) and "inéquation" ("inequation") (except in the 1902 syllabus). Carlo Bourlet writes, in his book "Elementary Algebra": "just as one distinguishes equalities in two species, identities and equations, one may distinguish amongst inequalities, those which occur for any value of the letters, and those which occur only if some letters, called unknown, are given particular values. We call the latter "conditional inequalities" (p. 160). In a footnote, the same author writes: "one uses also, sometimes, the words "unidentities" and "inequations" but as these expressions are rather unusual we preferred not to use them.

2. Context of inequalities in the syllabuses

In order to precise the context of inequalities in the various syllabuses we will consider three broad stages. First stage, until the "new math" reform (1902 - 1970), second stage, the "new math" until the 1977(counter-)reform (1970-1977), third, the present one, mainly from 1985.

2.1. *First stage*

During the first stage inequalities are a part of "algebra" and the syllabuses of years 4th and 3rd are divided in two or three parts amongst "arithmetic", "algebra" and "geometry". In the 1902-1905 syllabus, inequalities come after operations on positive numbers, monomials, polynomials, and equations (first degree equations with one unknown, solving two equations with two unknown, systems of equations with more than two unknown). The problems of setting into equations come then, but there is no explicit reference to inequalities. Afterwards come the variations of the expression $ax + b$ together with its graphical representations; eventually come the second degree equations.

In 1947 equations, polynomials and monomials are also very close to inequalities but other objects come beforehand: locating a point in the plane, variables and functions from usual magnitudes, linear functions and graphical representations. First degree equations with an unknown come before inequalities and problems on equations come after inequalities.

In the 1960 syllabus we find again the same objects in the same order. However there is a difference: it is no more about functions given from usual graphical magnitude but instead, simply about functions. This program brings another "little change", since in the year 4th inequalities are introduced with the comparison of negative numbers.

In brief, during this period the context of inequalities is essentially made of equations. Inequalities are "almost" as equations, they follow the study of the equations and their techniques, (algebraic technique) just taking into account the sign of the a when both members of an inequality are multiplied by a . Inequalities are a part of "classical" algebra, of which the main topic are equations.

"Small" changes operate on this organisation, which will turn into more radical changes in the second stage. These changes are firstly the appearance of the notion of function and of the graphical representation before the study of

equations and inequalities (1947 and 1960), then the appearance of the ordered structure of real numbers (although not explicit).

2.2. *Second stage*

This second stage is characterised by the dramatic changes of the so-called "new maths" reform. With respect to inequalities, the changes were "prepared" beforehand as we saw above. The classical organisation in the "collège" in "arithmetic", "algebra" and "geometry" is replaced by another organisation. For instance, in the year of 4th, there are four parts: Relations, Decimal numbers and approach to real numbers, Geometry of the straight line, Plane geometry. In the year of 3rd, there are three parts: Real numbers, algebraic computation and numerical functions; Euclidean plane; Euclidean plane geometry. Inequalities appear in year 4th in relationship with the fact that \mathbb{R} is a totally ordered field; in year 3rd in relationship with the total ordered and the notion of interval. In this year, first degree inequalities with two unknowns appear, and also the graphical representation of the solution of this kind of inequality. Equations remain close to inequalities, and both are depending on numerical functions. This notion of function becomes then a core notion of the curricular organisation (just as number structures) and forms the basis for the study of equations and inequalities (at least for their definitions, even if it is not in the techniques that are taught and used).

The 1977 reform takes a step back but it is not possible to come back to the same point as before the "new math" reform. The organisation of the year 4th is in two parts: Numerical computation; Plane geometry; in year 3rd we come back to a traditional organisation in two parts (Algebra, Geometry). The organisation related to inequalities is no longer the traditional one, however: in year 4th the first degree inequalities are associated with the order relation, and the first degree inequalities with two unknowns appear in year 3rd in the algebraic part. It is to note that the graphical solving of a system of inequalities is explicitly noted and that problems do not refer explicitly to equations nor to inequalities.

In brief, the context of inequalities is, at this stage, essentially structural and functional. Structural because the inequality is associated with the order relation (in 1971, students are taught the demonstration that \mathbb{R} is a totally ordered field!) and functional since this object is depending upon the notion of function and upon its graphical representation. Inequalities enter here the field of "modern" algebra (study of numerical structures, in particular order structures) and, by means of functions, as a tool for calculus (this aspect is more evident in the French "Lycée", 15-17 ys.). As a consequence of this relationship, we may

note that in the syllabuses, appear first degree inequalities with one or two unknowns and systems of inequalities, of which the graphical solving is explicitly organised. Inequalities, although keeping a privileged association with equations, become more autonomous, in particular in the field of calculus.

2.3. *Third stage*

In the third stage, since 1985, all the years of the "collège" are organised in three parts: Geometrical works, Numerical works, Data organisation and processing, functions. Inequalities appear in the 'Numerical works' part in the year 4th (1985), and associated with the order relation in the year 3rd (1996). The solving of problems leading to inequalities is expected in 4th and in 3rd (1985 and 1996). The graphical solving method of a system of first degree inequalities is explicitly expected in 1985; however this type of inequality and the systems of inequalities vanished in the present syllabus.

In brief, in this stage we observe a movement towards a structural (numerical) context and to the equations context by restraining to a unique type of inequalities, whereas systems of first degree equations with two unknown remain present.

Tentative conclusion

The study of the various curricular contexts related to inequalities must carry on with the study of the types of problems and of the various techniques (see Assude, 2000), which are their conditions of possibility. Moreover, these curricular studies must be completed by the study of other kind of context (historical, social, epistemological, among others). For this, works of historians of teaching is essential.

The study of a curriculum may be done from various points of view. We took here the point of view of the institutions, applied to the official syllabus. This point of view, however, must be completed by others, in particular by the actual practices, by the analysis of school books, but also, as we already said, focusing on the three following kinds of context: social and economical, philosophical, ideological and cultural, and epistemological and content-based. A multidisciplinary approach is obviously one of the conditions for the success of such an enterprise.

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ABOUT THE REFORM OF MATHEMATICS EXAMINATION IN HUNGARY¹

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Abstract: *In the article the authors present some results of an empirical study on the final examination in mathematics of the previous four years. The results show that the former developmental procedure of the tests does not produce test items what satisfy the requirements of objectivity, validity and reliability. The distributions of the test achievements in different years varies very much. A new development procedure is under development.*

The past decade is the decade of the reforms in education all over the world. The great and fast social, economical and technological changes press the educational policy makers to think over what is the new role of schools. It is also question now, what is the social need, what to teach, how to teach in the new era, how to make education more effective and how to ensure the correct measure of the efficiency of education and the success of individual students. (Niss, 1993)

Hungary also takes part in this long lasting procedure what includes - among others - the restructuring of the school system, the development of curriculum, the reform of the textbook “industry” and the examination reform. (Curriculum, 2000; Lukács, 2001)

During this reform we have the opportunity to study the past and present examination procedures theoretical and empirical ways. (Tompá-a, 1999; Tompa-b, 1999)

In this article we would like to describe some aspects of the Mathematics examination reform in Hungary on the basis of an empirical study.

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1. Some characteristics of the Hungarian mathematical examination system

Mathematics in the Hungarian education system always played a great role by tradition. One of the indicators of this fact is that the maturity examination procedure includes compulsory mathematics exam. One or another form of mathematics exam is compulsory for each pupils at the end of all types of secondary school. (Regularly at the age of 18-19.) We summarise the mathematical examination system in Table 1.

Type of the examination	Centrally developed matriculation exam		Centrally developed combined matriculation and entrance exam	
	For “academic” high school	For vocational-oriented high school	For technical and mathematical oriented higher education	For economy oriented higher education
Version				
No. of items	6 open problems + 1 theorem to be proved	6 open problems + 1 theorem to be proved	8 open problems	8 open problems
Time	180 min.	180 min.	240 min.	240 min.
Score	Maximum 80	Maximum 80	Maximum 100	Maximum 100
Evaluators	Regular teacher	Regular teacher	External + regular teacher	External + regular teacher
Form	(*) Selection of known problems	(*) Selection of known problems	On printed sheets	On printed sheets

Table 1: Summary of the existing Mathematics matriculation exams in Hungary

According to the table we can differentiate 2 main types of matriculation exams in Mathematics:

- a) Those pupils who want to finish their Mathematics education after high school it is possible to take a less difficult examination either in “academic” or in vocational oriented high schools. This exam is centrally designed by a national institute and has to be taken in the school, so it is called the “school exam”. This exam is regularly compiled by a committee of experts. The 6 problems and the theorem to be proved are selected from a collection of 4000 problems of a specific textbook (exercises book). This book is well known (Gimes, 1992) and is used by students during their 4-years of studies. The content of the exam (the 6 problems and the theorem) is announced via mass media (National Radio and National Television) on the examination day.
- b) Those pupils who want to enter a higher education institution (which requires entrance examination in Mathematics) there are the two types of combined exams, according to the field of interest. These exams fulfill

two purposes. They serve as matriculation exam and entrance exam to the higher education institution as well. They are more difficult than the school exams. These sorts of exams are also developed by team of experts and the problems are given to the pupils on sheets in the venue of the exam. The entrance examination takes place one or another higher education institution depending on where the pupils applied to.

Both types of examination are developed by team of experts. The teams consist of mathematicians and mathematics educators with higher education and high school teaching background. The teams make very careful selections according to their best knowledge, and the experiences gained in the previous years examinations. In this procedure the problems are not pre-tested, and we do not have any data about them in advance. It happens sometimes that an exam more difficult in one year than another.

2. Some data about the past years' examination

In order to get more information about the validity, objectivity and reliability of the present Mathematics examination we collected data on the school exam in year 1995-1998 from the "academic" high schools. It is to know that even if the development and the launch of the exam is central the schools are not obliged to report their results to any central place, and keeping the pupils' work is compulsory only for two years after the exam was taken.

2.1 Examples of the examination problems

In this article there is no place to show all problems of the exams in the studied period. However let us demonstrate the nature and the quality of the problems with Exam 97 and Exam 98. The intention is to have almost the similar characteristics of the examination year to year (number of problems, topics, validity, difficulty of the problems etc.) There are only problems covered by the national core curriculum and the theorem to be proved is also familiar to the pupils. So the Mathematics examination is supposed to measure the acquired knowledge.

School Mathematics Examination - 1997

1. (1214.) Does the area change if we lengthen one of the sides of a square by $\frac{1}{5}$ times than its original length, and shorten the other side by an equal amount? If it does, determine the percentage change.

2. (1548.) Solve the following inequality in the set of real numbers:
 $\log_1(3x-2) > 0$
3. (2385.) The radii of the base and the upper circle of a truncated cone are 'R' and 'r'. A plane parallel to the base divides the superficies of the cone into two parts having equal areas. What will be the radius of the circle intersected by the above plane from the truncated cone?
4. (3054.) Solve the following equation in the set of real numbers:
 $\sqrt{3} \sin 2x = 2 \sin^2 x + 1$
5. (3196.) The vectors from the origin to two adjacent vertices of a square are **a**(5; -2) and **b**(-4; 4). Give the co-ordinates of the vectors to the other vertices of the square.
6. (4051.) How many positive divisors are there for 2700?
7. Proof of a theorem (Meeting points of the altitudes of the triangle).

School Mathematics Examination - 1998

1. (1068.) Solve the following equation in the set of natural numbers:
 $\log(x+1) + \log(x-1) = \log 8 + \log(x-2)$.
2. (2066.) The length of one of the parallel sides of a trapezoid is 4.8 cm, and the length of the remaining sides is 3.2 cm. What is the area of the trapezoid? What are the angles of the trapezoid?
3. (3385.) Find the point of the x-axis at which the line segment having endpoints A(0; -3) and B(6; 5) subtends a right angle.
4. (2394.) The length of the sides of the base of a square pyramid is 8 cm, and the slant height of the side faces is 12 cm. What will be the radius of the sphere which is tangent to all the faces of the pyramid?
5. (861.) Solve the following equation in the set of non-negative numbers:
 $|4-x^2| = 2$.
6. (4036.) How many 4-digit numbers can be made out of the digits 1, 3, 5, 7 and 9, if each digit may appear only once? How many of these begin with 13? How many numbers are there which has first digit as 1 and at the same time, last digit being 3?
7. Proof of a theorem (The leg-theorem).

(The numbers in parentheses refer to the numbers of the problems in the specific exercises book.)

2.2 About participants of the study

The design of the sample was representative for the population of age 18-19 attending academic high schools. (There were only some schools that could not present the data, but the sample was big enough, so it did not mean bias problem.) A number of assistants went to see the schools and collected the results about the examinations. To give the data it was voluntary. Table 2 shows the number of examinees taking part in the study.

Year	1995	1996	1997	1998
No. of papers	1514	3740	4262	2423

Table 2: Number of participants in the study

We have also collected not only the final scores of the examination but 10-15 percent of the complete examination sheets has been copied (without the pupils' identity) in order to analyse them deeper later.

2.3 Some results of the study

Our intention is to make deep analysis of the Mathematics examination based on the collected data, but here we would like to demonstrate only some general statements. Table 3 and 4 show the average of pupils' achievements on each problem, the theorem and the whole test in each year. So we can compare the achievements in the tests of the different years.

Year	P1	P2	P3	P4	P5	P6	Theorem	Total
1995	72,9	52,3	45,4	33,6	75,6	68,3	43,7	53,6
1996	93,9	71,5	43,7	10,0	31,5	42,5	53,5	47,2
1997	78,6	78,6	16,2	29,2	33,3	49,0	54,5	54,5
1998	81,6	83,3	42,7	39,1	54,2	60,7	60,7	58,2

Table 3: Pupils' achievements on the different problems in percentages (Problems are in the order of their appearance in the test)

This table shows that there are great differences in the total achievements year after year, and the difficulty of the problems (test items) differs very much. The first two problems supposed to be the easy items, and the range of their difficulties is 52,3-93,9. (See Table 3.) Rather great differences are what we can realise.

The question is can it be due to the different populations?

However it is more interesting what Table 4 shows that is the great differences in difficulties of the items belonging to the same topic.

Year	To1	To2	To3	To4	To5	To6	Theorem	Total
1995	72,9	75,6	68,3	45,4	33,6	52,3	43,7	53,6
1996	93,9	43,7	42,5	71,5	10	31,5	53,5	47,2
1997	78,6	29,2	49	78,6	16,2	33,3	54,5	54,5
1998	54,2	81,6	60,7	83,3	39,1	42,7	60,7	58,2

Table 4: Pupils' achievements on the different problems in percentages (The order is according to related topics)

The observation of the great differences is easier if we see the graph made on the basis of the data in Table 4.

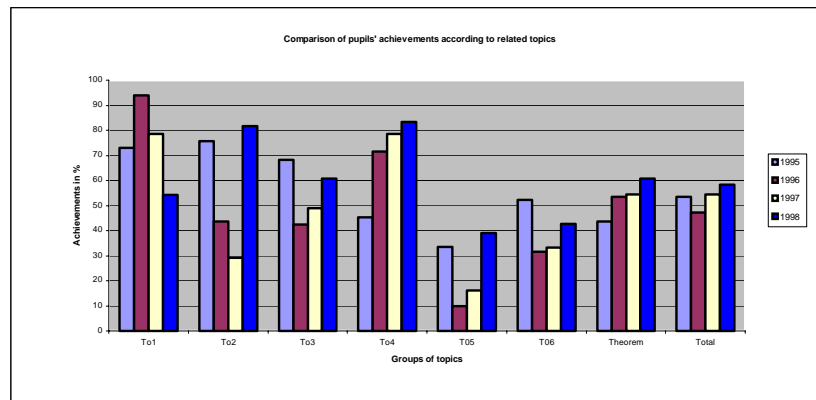


Figure 1: Comparison of pupils' achievements in the matriculation exams year 1995-1998

The topic list is the following:

To 1: Algebra1 – percentages, linear and second order equation

To 2: Algebra2 – logarithmic, trigonometric equations and functions

To 3: Number theory

To 4: Plane geometry, coordinate geometry, geometric calculation

To 5: Location, solid geometry

To 6: Vectors, trigonometry

Theorem: Proof of a theorem from the plane geometry

The question is the same: Can it be due to the different populations?

2.4 The distribution of pupils achievements

In the following figures we show the distribution of the pupils' achievements on the whole tests in the four years.

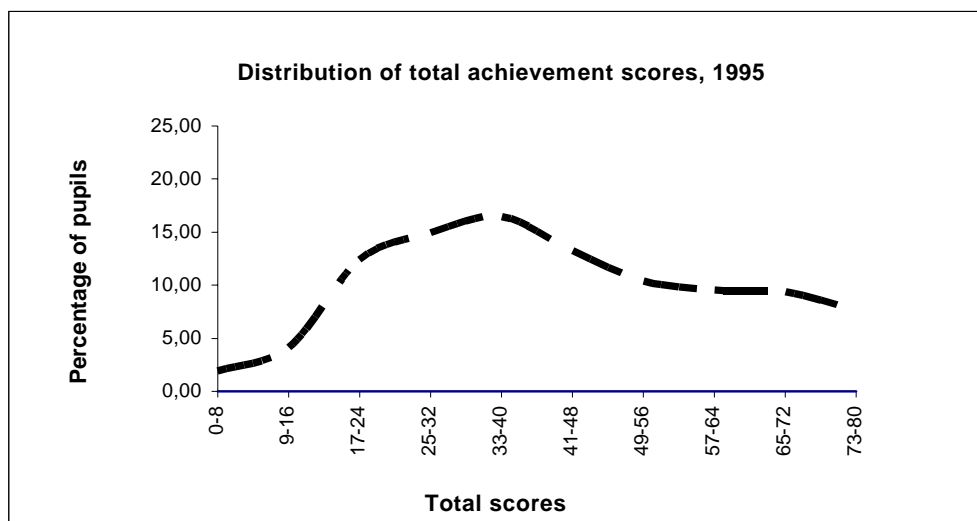


Figure 2: The distribution of total achievements' scores – year 1995

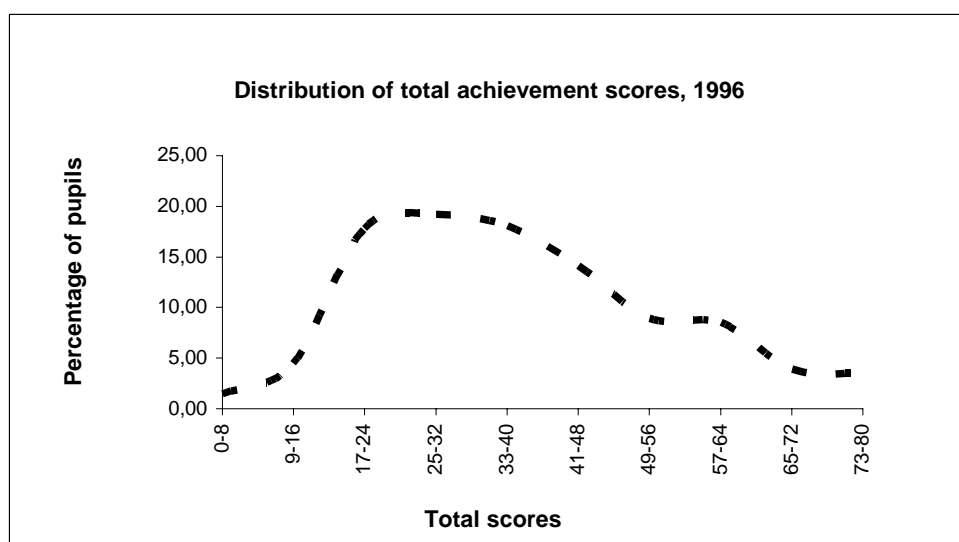


Figure3: The distribution of total achievements' scores – year 1996

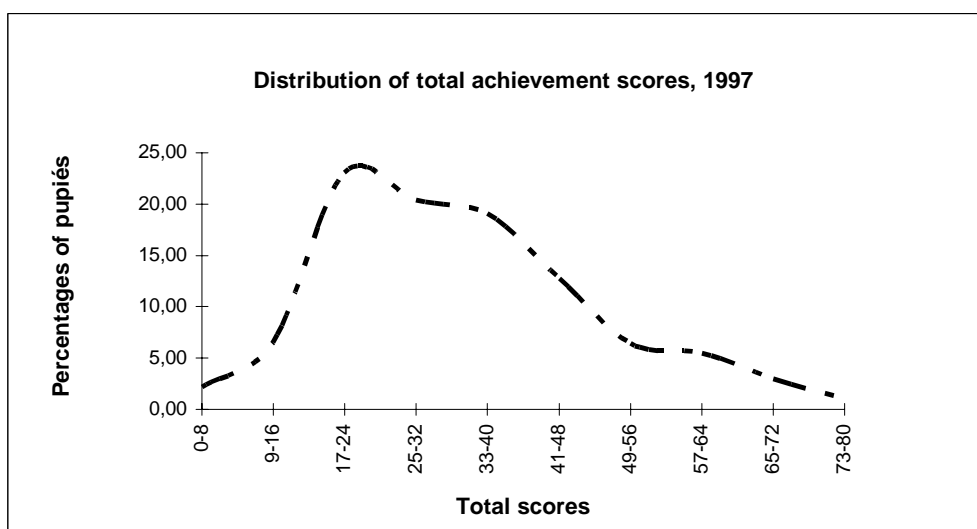


Figure 4: The distribution of total achievements' scores – year 1997

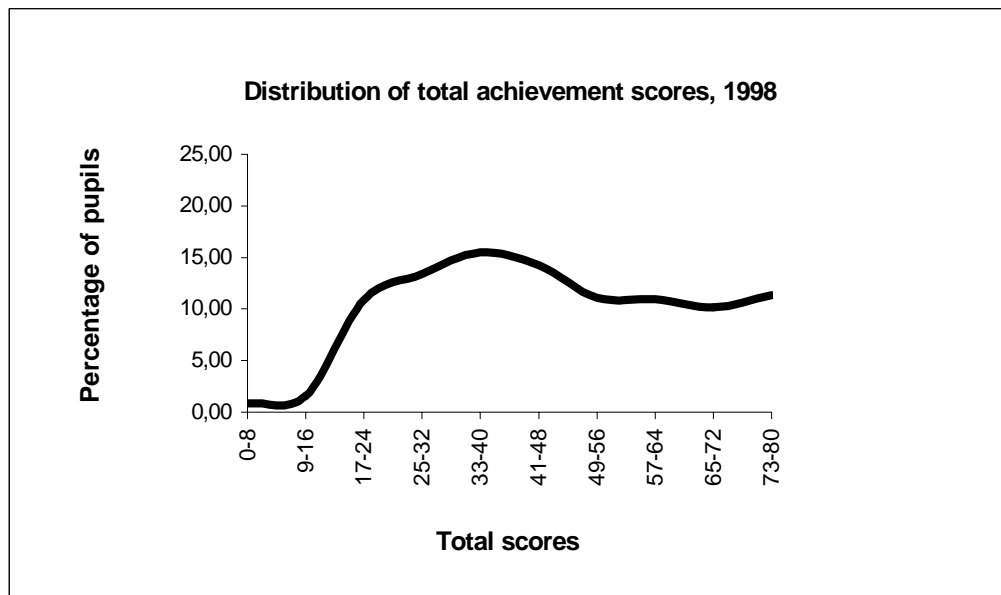


Figure 5: The distribution of total achievements' scores – year 1998

These four figures show *very different pictures of achievement distribution* of the roughly the same four populations.

It is not believable that the populations one after the other differ that much. It rather indicates that there has to do something about the reliability, validity and objectivity of the Mathematics examination. And we are working on now.

3. Some reasons and trends of the change

In this short article there is no possibility to describe a detailed program of the examination reform, but we can summarise the main reasons and the trends of the changes. They are the followings:

- The examination system is rather old (more then 20 years)
- New curriculum includes new topics, it implies changes in examination
- The tests need better characteristics according to the educational measurement theory
- Better validity (harmonising requirements and test items)
- Better reliability
- Equality in the difficulty year to year
- Comparability

So our main reform activity is focusing now on the building of examination item bank. That means to develop a lot of new and old types of

problems, to pre-test them and on the basis of empirical data to parametrize them.

At the end of the developmental and research period we have to present a new examination model by 2003.

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MATHEMATICS TEXTBOOKS AND STUDENTS' ACHIEVEMENT IN THE 7TH GRADE: WHAT IS THE EFFECT OF USING DIFFERENT TEXTBOOKS

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Abstract: *Finland participated in the TIMSS 1999 in 1998-2000, and the results of this study are analysed at the moment both nationally and internationally. Because of the nature of the Finnish curriculum, which was decentralised in the mid 90's, a textbook analysis on three most popular textbooks in Grade 7 was done to find out more about the nature of the intended and implemented curriculum in this grade level. The implemented curriculum was also explored by using the teacher questionnaire data of TIMSS 1999. In this paper results of these analyses are presented, as well as some connections between curricular characteristics and students' achievement. The used textbook does not seem to have an effect on the overall results in the achievement test. However, in the analyses of smaller groups of items or individual items in the test, the textbook seems to matter, and it should not be forgotten.*

Introduction

Finland participated in the TIMSS 1999 study (Third International Mathematics and Science Study Repeat) in 1998-2000. The main interests of TIMSS 1999 were to get information about mathematics and science achievement, and the factors influencing students' achievement in different educational systems. The target population of TIMSS 1999 was the 13-years old children, which in Finland meant the 7th Grade students.

The aim of my study was to find out the effect of using different mathematics textbooks in Grade 7. In what measure does the use of different textbooks explain the variation within students' achievement?

Theoretical Background

The theoretical background of this study was based on the conceptual framework for TIMSS, which was already used in the Second International

Mathematics Study. The curriculum is seen to be formed of three embedded levels: the intended, the implemented, and the attained curriculum. These have certain contexts, where they work, and each factor effecting education can be situated in some of these contexts. (Kangasniemi 1989, Robitaille et al. 1993)

The intended curriculum is the mathematics content as defined at the national or the educational system level. In Finland this is outlined in the Framework Curriculum for the Comprehensive School 1994 published by the National Board of Education. It includes general goals for the education in Finland for Grades 1 to 9. Based on these general aims the individual schools are responsible to prepare their own curricula. As a consequence there may be great variation in the school level curricula.

The implemented curriculum is the mathematics content made available to students. In a way it is the intended curriculum interpreted by teachers or other organisers of instruction. Because of this interpretation it is not identical to the intended curriculum, but it is however influenced by the intended curriculum.

The attained curriculum means the outcomes of education. In mathematics it means the concepts, processes, and attitudes learned during instruction. This is influenced by the intended and implemented curriculum.

Textbooks

In the three level curriculum model outlined above textbooks can be related to both the intended and the implemented curriculum. This is why they can be seen as an intermediate level in the model and they are termed the potentially implemented curriculum (Schmidt et al. 1997). Depending on the educational system the role of the textbooks differ considerably and they can correspond more to either of the two levels related to them.

On the basis of earlier studies in Finland the textbooks are very closely related both to the intended and the implemented curriculum. However the correspondence to the intended curriculum may have changed after the curriculum reform in 1994, which resulted in the Framework Curriculum mentioned earlier. In this connection the centralised control of the textbooks was finished. The teachers' use of textbooks is explored in this study, but based on earlier research in this field it is very regular (Korhonen 1994, Kupari 1993).

Context of the Study

In order to better understand the ideas of this study, it is reasonable to give a short description about the Finnish school system and mathematics teaching.

Finnish School

The time the data of this research was gathered the Finnish comprehensive school included two parts: the lower level from grade 1 up to 6 and the upper level from grade 7 up to 9. Nowadays the "border" between the lower and upper level has been abolished at least in the legislation. The starting age of school is usually the year the child gets 7 years old, sometimes the school can be started a year earlier.

Curriculum Reform in the 1990's

During the 1990's the Finnish school system left the centralised control and the schools were given more possibilities for decision making. The National Board of Education released the Framework Curriculum for the Comprehensive School in the year 1994. It gives guidelines to the local administrators, who are then supposed to prepare their own school curricula. For example the Framework for the Comprehensive School does not specify, when various subjects are to be taught or what are the upper limits of lessons used to the subjects. It just gives the minimum number of lessons that must be taught during the upper and lower level in different subjects. It also states the general educational and subject-related goals of the instruction.

Mathematics Teaching

The approach of mathematics in the upper and lower levels are quite different. In the lower level mathematics teaching is based on a spiral approach: the same main contents are covered each year, but some new ideas are brought up every year.

In the upper level mathematics teaching is usually organised course-based. In practise this means that there are contents which are taught thoroughly just once during the upper level (e.g. percentage). The schools may themselves decide, when different content areas are taught, and often the order of the content areas is the same as in the used mathematics textbook.

Research Questions

The research questions of the study were the following:

1. What are the content areas to be found in the Finnish mathematics textbooks in Grade 7 according to the TIMSS framework?
2. What contents have been taught in Grade 7 announced by the teachers?
3. Is there any connection between the potentially implemented and implemented curriculum discussed in questions 1 and 2 and the attained curriculum described by the students' achievement in the TIMSS 1999 test?

Description of Data

The TIMSS 1999 sample in Finland consisted of 171 schools, 3290 students and 641 teachers. Of the schools 151 were Finnish speaking, and 20 Swedish speaking, and of the teachers 175 taught mathematics. The data gathering took place in April 1999.

Textbook Analysis

Information of the mathematics and science textbooks the sample schools used in Grade 7 in the school year 1998-99 was collected. Responses were received from 134 schools. This study is based on the data from 104 schools, which used the following textbooks: Kolmio (K, 47 schools), Plussa (P, 37 schools), and Matematiikan maailma (MM, 20 schools).

These three textbooks were analysed in this research. The textbook analysis was done using mainly the same method as in TIMSS 1995 (Bianchi et al. 1998, Isager 1996, McKnight 1992, McKnight et al. 1992, Schmidt et al. 1997). The textbooks were divided into analysis units and then the units into small blocks. These blocks were given codes describing content and performance according to the mathematics frameworks used in both TIMSS 1995 and 1999. Each of the blocks could have more than one code in both aspects: for instance decimal fractions and measurement was a usual combination in the textbooks. The results are presented with percentages, which express the proportion of the textbook devoted to the content area in question. The percentages are not additive because of the possibility to give multiple codes to a single block.

The reliability of the textbook analysis was estimated by using two reliability coders. There were some differences between the main coder and the other coders in given codes but in most cases the differences could be explained by not using multiple codes in a similar way or by the overlap of the categories. As a consequence, the results presented here can be considered fairly reliable.

The results of the study are reported using the categories of the TIMSS 1995 mathematics frameworks, and some of it is presented in table 1.

Table 1. Main content categories and some subcategories of the TIMSS 1995 mathematics frameworks

1. Numbers
 - 1.1. Whole Numbers
 - 1.2. Fractions and Decimals
 - 1.3. Integer, Rational, and Real Numbers
 - 1.4. Other Numbers and Number Concepts
 - 1.5. Estimation and Number Sense Concepts

2. Measurement
3. Geometry: Position, Visualisation, and Shape
4. Geometry: Symmetry, Congruence, and Similarity
5. Proportionality
6. Functions, Relations, and Equations
7. Data Representation, Probability, and Statistics
8. Elementary Analysis
9. Validation and Structure
10. Other Content, e.g. History of Mathematics and Applications of Mathematics.

Questionnaires and Achievement Test in TIMSS 1999

Other sources of information in this study were the achievement tests and different questionnaires in TIMSS 1999. There were background questionnaires to teachers, students and schools. This study uses the data from the teacher questionnaire considering the use of textbooks and the contents taught to students during the school year 1998-99. Also the achievement test data is used.

Results

1. What are the content areas to be found in the Finnish grade 7 mathematics textbooks according to the TIMSS framework?

The results presented here are based on the analysis of the three most used Finnish textbooks in grade 7. The content categories presented here have been chosen considering their relevance to questions number 2 and 3.

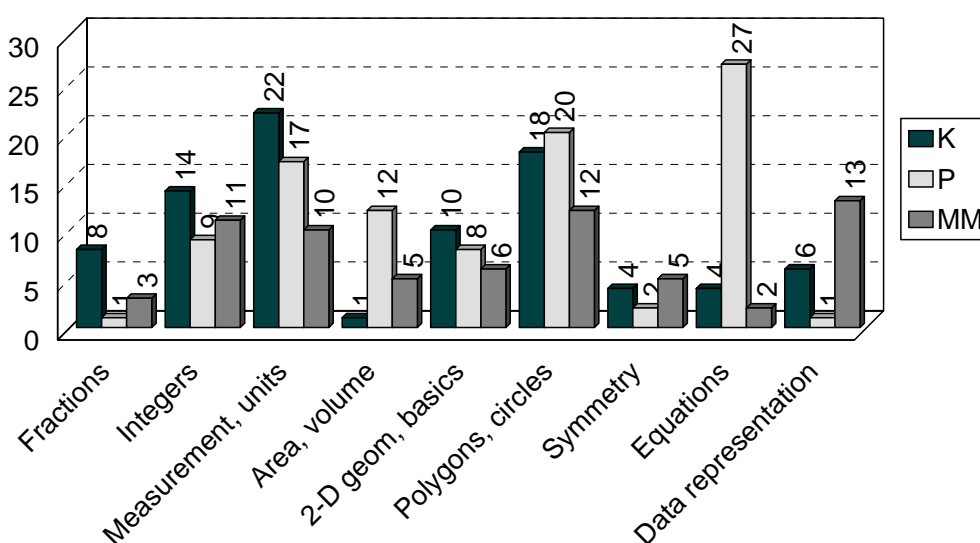


Figure 1. Proportion of pages (%) with given content codes.

There were some shared content areas emphasised in all of the three textbooks: integers (negative numbers) and two-dimensional geometry considering both basics and polygons and circles. Bigger differences among textbooks occurred in case of equations, fractions, computing area and volume, and data representation. The textbook P differed distinctly from the others with the great proportion of Equations, which accounted for 27% of the textbook. As an interesting detail division of fractions was presented only in one of the textbooks. It is interesting to see students' achievements in these content areas, because most of them are usually not thoroughly taught in the lower level.

2. What contents have been taught in Grade 7 announced by the teachers?

The teachers were asked in what extent they had taught content areas included in the content frameworks during the last school year. The scale used was “Taught earlier”, “1-5 lessons”, “Over 5 lessons”, “Not yet taught”, “I don’t know”. In the analysis of these answers the categories “1-5 lessons” and “Over 5 lessons” were combined to one category “Taught this year”. The results were consistent with the results from the textbook analysis, and some of them are presented in Table 2.

Table 2. Content areas taught according to the responses of the mathematics teachers.

Content category	K % taught	P % taught	MM % taught
Fractions, computations	87	31	36
Decimal fractions, computations	85	53	80
Percentages	12	8	69
Negative numbers	88	92	86
Measurement, units	65	63	78
Perimeter and area of combined shapes	22	50	76
Volume of solids	10	56	45
Cartesian coordinates	87	93	86
2-D geometry	87	69	100
Congruence/similarity (also symmetry)	31	11	90
Solving equations	19	94	16
Data representation	65	57	72
Simple probabilities	2	0	66

In general teachers using the textbook K had taught number concepts more often than the others. Exceptions in this pattern were percentages and negative numbers (integers). The latter was covered in all of the books and also by almost all teachers. The former was covered only in the textbook MM, and teachers using this book had usually taught the content. The teachers using textbooks K and P had maybe used some other materials in the teaching of this

content. In the case of textbook K, the structure of the book can explain the results. It consists of a theory book covering all materials to be taught in the upper level, and separate exercise books for each Grade.

The same explanation is appropriate at least concerning the results of solving equations and simple probabilities. The most surprising result were found in the category congruence, similarity and symmetry. The textbook analysis results were quite equal but the teachers using textbook MM had taught this content distinctly more often than others. However, most of the results were very consistent with the contents of the textbooks displayed in Figure 1.

3. Relationships between instructional contents and student achievement

Before considering results of the achievement test, it is reasonable to present results about teachers' usage of mathematics textbooks. Teachers were asked whether they use a textbook and what proportion of their teaching is based on a textbook. According to the answers 99% of the students were taught by teachers using textbooks. Furthermore 85% of the students were taught by teachers, who used textbooks in over a half of the lessons, and only 2% of the students used a textbook in less than a quarter of their lessons. These results are a good reason to analyse the textbooks, when looking for possible explanations to variation in students' achievement. The results showed quite high use of textbooks, when compared with international results from TIMSS 1995 (Foxman 1999).

In general there were no significant differences in the achievement between students using different textbooks. In TIMSS 1999 the items were grouped in five main categories: *fractions and number sense, measurement, geometry, algebra and data representation*. The average scores for the overall mathematics achievement and each of the categories above were very similar between students using different textbooks. However, there could be seen some differences, when the items were grouped according to content subcategories.

Table 3. Average percent correct calculated in content subcategories.

Content subcategory	K	P	MM
Fractions, computations (8 items)	52	48	46
Division of fractions (1 Item)	38	5	0
Equations, algebraic expressions (9 items)	40	55	39
Perimeter and area (7 items)	38	47	46
Volume (1 item)	51	57	66
Congruence, similarity, and symmetry (10 items)	50	50	52
Data representation (12 items)	78	76	81
Simple probabilities (7 items)	67	71	75

In table 3 computations with fractions gives a quite good picture of the results. The percentages were quite close to each other, even though the textbook analysis and teachers' answers would have let us assume, that the users of the textbook K should have done better than the others. This was the case, but the difference was very small. However there is a very simple possible explanation for this: the computations with fractions has been already taught during the lower level. But there is one exception, namely the division of fractions, and the users of textbook K did much better than the others in the item of this subcategory.

The results for congruence, similarity, and symmetry were a bit surprising. Based on the teacher questionnaire results one could have assumed that the users of textbook MM would have had higher scores than the other two groups. However, this is not the case and the results actually showed the same similarity as the textbook analysis. A closer look at the items might give some explanation to these results. Anyway, the results of the other categories were quite consistent with the results from the textbook analysis and teacher questionnaire, though the differences in achievement among different textbook users were small.

Conclusions and continuation of the study

Although the textbooks analysed covered different contents in Grade 7, the differences in students' achievement were small. This implies, that Finnish students have a very equal educational background in mathematics, when they come from the lower level to the upper level. This was the result also in a recent national mathematics assessment study in 6th Grade (Niemi 2001). Some statistically significant differences among different textbook users were found, but they were very small. On the other hand there seems to appear some differences on the upper level depending on the textbook used at the school and the content assessed. In the overall results of TIMSS 1999 there were no differences among different textbook users and this is a good result remembering the goal of equality in the Finnish school system. The result considering overall scores has been similar in national mathematics assessment studies in the 9th Grade (Korhonen 1994, 2001). Anyhow, when looking at smaller groups of items or individual items the textbook seems to have an effect on the results. This was also found in the mathematics assessment of Grade 9 done in year 2000 (Korhonen 2001). This indicates, that when doing item level analyses, the textbook must be taken into account as a school level variable: looking at the results of the division of fractions item justifies this.

In the next phase of the study different content and item groups will be analysed. One problem to be solved is to determine the statistical significance of the differences in achievement. The textbook analysis has been expanded to cover also the textbooks used in Grades 5 and 6. The results of these analysis will be studied further and hopefully these results will give more information and better explanations concerning the results of TIMSS 1999.

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WORKING GROUP 7

**The role of metaphors and images in the learning
and understanding of mathematics**

Group leaders:

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Núria Gorgorió

INTRODUCTION TO WG7

WORKING TOGETHER ON METAPHORS AND IMAGES

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The participants in the working group soon agreed that the goal of the sessions would be to share and compare the various standpoints and research problems. We were not trying to achieve an agreement on the various concepts and constructs we were dealing with, or on the different possible focuses of attention or theoretical approaches. Our aim was to create understanding and search for common points. Too much agreement would have been suspicious, especially taking into account the wide range of issues that could be tackled on a working group labelled "The Role of Metaphors and Images in the Learning and Understanding of Mathematics".

As regards mathematical subjects, a wide range of domains were addressed by the members of the group: elementary arithmetic, algebra, geometry, analysis ... From a didactical standpoint as well, various theoretical frames were represented: some referred mostly to Duval, some to Nunez and/or Lakoff, some to Fischbein ... and some to several of these. This double "classification" shows that many -if not all- mathematical domains (and not only geometry) are concerned with metaphors and images, and that this theme has many strong theoretical references. *A contrario*, this shows that some work is still needed to provide the researchers with a more general frame, which could be helpful in any area of mathematics (and perhaps even beyond them).

However, despite the differences in interests and standpoints, we all considered that the usual ways of teaching mathematics nowadays have negative effects on the developing and using of images by students. Therefore, we concluded that some changes should take place in the teaching of mathematics, in order to help students make a more effective use of them. Therefore, the discussions around the research works presented within the working group aimed at knowing more about the role of images and metaphors in the learning and teaching processes of mathematics in order to have models to try to improve the situation. It seemed clear to the group that if any changes are to be produced, then teachers, among others, could be agents for these. But then, the question arose whether or not we can teach prospective or in-service teachers to teach their students how and when to use metaphors and images?

We all seemed to accept that students spontaneously produce their own images and metaphors, and that a long-term intention of teaching is to have the students functioning as mathematicians do. Images and metaphors were recognised as being extremely useful in mathematics, not only for communication but also for conceptualisation. Therefore, during the discussion of the different research approaches, experimental designs and research methods, many issues arose about how to conduct the students to do the same as mathematicians do: what are the metaphors and images that should be promoted by teachers? How do the models used by the teachers help or disturb the students? How can teachers recognise and collect their students' initial thinking mode? Is the active imitation of the teacher's models helpful in the students' construction of their own images and metaphors? Even if everyday metaphors can be very rich, is intuition linked to everyday life always helpful for the formal thinking that we want to promote? Or can it be an obstacle? The potentiality of the source domain relies on the possibility for inferences, but how can we come to know the spontaneous source domain of our students?

It seemed obvious to the group that more research is needed, focusing not only on partial aspects but also on trying to find a more global interpretation of the issues, from an open-minded theoretical standpoint, trying to build on the different theoretical constructs already existing, without introducing new terms that only introduce small differences and not merely on doing 'theoretical tourism'.

To conclude, we would like to say that we were very pleased to work in this group; its size (10 members) was well adapted to its way of functioning, since there was substantial time to discuss the paper of every participant, and we recommend it for future congresses of ERME. This work resulted in changes in the papers that are presented here below, and we hope that it improved their quality. We also hope that the other members took as much pleasure as we did in working together.

USING « GEOMETRICAL INTUITION » TO LEARN LINEAR ALGEBRA

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Abstract: *University teachers often declare that using « geometrical intuition » would help students in their learning of linear algebra. My work aims at investigating such an affirmation. I will first present a definition of « geometrical intuition », relying upon the work by Fischbein about intuition in mathematics, and especially the use of models. I submitted a questionnaire to post-graduate students, in order to draw out the different aspects of geometrical intuition and the various kinds of models they use in their practice of linear algebra. The use of a geometrical or a figural model can be associated with a good understanding of linear algebra, but it can also raise specific difficulties.*

Introduction

Many teachers mention geometry as helpful for learning and working in linear algebra. In a recent report about teaching geometry, the mathematical teaching reflection commission writes:

“The link between linear algebra and geometry seems to be more interesting in the opposite way, because usual geometry in dimension 2 or 3 provides an intuitive basis to work when the dimension is greater than 3, or even infinite...” (C.R.E.M. 2000)

This quotation raises several questions. What is the “usual geometry in dimension 2 and 3”? Is it the geometry now taught in secondary school, or Euclidean geometry (with the historical meaning of that expression)? What is exactly an “intuitive basis”? Do students use such a basis when they work in linear algebra, and does it really improve their understanding and practices? These are the questions I address and try to answer in my doctoral dissertation (Chartier 2000). I will first present the theoretical framework I used in order to make the meaning of the expression “geometrical intuition” precise, and to study it in the practices of students. I used Fischbein’s work: indeed it allowed me to present a definition of geometrical intuition, and to describe potential uses of it in the practices of students. I proposed a questionnaire to post-graduate students, in order to observe such uses of geometrical intuition. I will present here a short review of the questionnaire and of the analysis of the answers.

1. Geometrical intuition

1.1. *Intuition in mathematics : the work of Fischbein*

According to Fischbein (1987), intuition offers behaviourally meaningful representations, allowing the reasoning activity to rely upon apparently certain conceptions. An important factor of intuition is the use of models. Fischbein defines a model as follows: a system B represents a model of system A if, on the basis of a certain isomorphism, a description or solution produced in terms of A may be reflected consistently in terms of B and vice-versa. If a notion cannot be represented intuitively, one tends to produce (deliberately or unconsciously) a model which can replace the notion in the reasoning process. Fischbein distinguishes several kinds of models; I will briefly present here the ones I used for my study.

Abstract and intuitive models

These are two opposite kinds of models. Some mathematical relations are abstract models for concrete realities. For example, a 3-dimensional Euclidean space can intervene as an abstract model for the physical space. On the contrary, an intuitive model can be perceived like a concrete reality; for example, drawings of arrows can constitute an intuitive model in vectorial plane geometry (in this paper, the term “drawing” denotes a material representation).

Analogical and paradigmatic models

These are two opposite kinds of intuitive models. In the case of analogical models, the model and the original belong to two distinct conceptual systems. In the case of a paradigmatic model, the model is a subclass of the original.

Intra-mathematical and extra-mathematical models

An analogical model can be either intra-mathematical, or extra-mathematical. For example, a drawing is an extra-mathematical model (in that particular case, I will use the expression “figural model”); a geometry can be used as an intra-mathematical, analogical model for linear algebra.

In my work, “geometry” will denote a mathematical theory which is an abstract model for physical space, and hence has a special connection with reality. A geometrical model is a model stemming from a certain type of geometry.

A geometrical model is then directly related to physical space, and apparently offers certain conceptions to the reasoning activity (it can therefore be considered as intuitive). It can inspire solving strategies and confirm meaningfulness of solutions: but it can also smuggle uncontrolled components into the reasoning process, especially because it is associated with a figural

model. Different geometrical models can be used in different parts of linear algebra; these models can be either analogical (if they stem from elementary Euclidean geometry, independent of linear algebra) or paradigmatic (if they stem from a geometry relying on linear algebra).

“Geometrical intuition” denotes here the use of a geometrical or figural model.

Studying the use of figural models led me to refer also to Fischbein’s theory concerning figural concepts. According to Fischbein, a figural concept is “a mental entity which is controlled by a concept, but which preserves its spatiality” (Fischbein 1993). Fischbein uses figural concepts in geometry; here, I will mention figural models rather than figural concepts; indeed in linear algebra, there is no certitude about the possibility of fusion between the conceptual and the figural aspects.

1.2. *Students practices in linear algebra and geometrical intuition*

Several geometries can be involved in students practices, and therefore provide different models for general linear algebra, with the associated figural models.

- *Secondary school geometry*: all my research takes place in France, where geometry is a very important topic in the secondary school curriculum. Since 1986, linear algebra is not presented at secondary school in France; but some notions like basis, projections and symmetries are still encountered in vectorial plane and space geometry. It is then possible that some students establish links between secondary school geometry and linear algebra. Moreover, a questionnaire proposed to university teachers, that I will not present here, showed that they are still strongly influenced by the choices made during the “modern mathematics” reform. Thus, some teachers praise a structural approach of linear algebra, with almost no drawings. Students, who followed such a course, if they are looking for an apparently concrete support, may find it in secondary school geometry.
- *University geometry*: some universities propose a geometry course preceding the introduction of linear algebra (in France, it corresponds also to a position discussed during the modern mathematics reform, and opposed to the structural approach). Various contents are presented, depending on the university: analytical geometry, axiomatic affine geometry ... These courses are intended to help students understanding of linear algebra; they are a source of geometrical models in linear algebra, deliberately proposed by teachers.

Models issued from these geometries are analogical models; a paradigmatic model can also intervene if, for example, students use linear algebra in \mathbb{R}^2 and \mathbb{R}^3 as a model for general linear algebra. Linear algebra itself is a model for polynomials, or sequences... The relation “being a model for” is transitive; thus, the figural models used, for example, in linear algebra in \mathbb{R}^2 and \mathbb{R}^3 can also be models for polynomials.

Each use of these models could be investigated for itself. The use of drawings, for example, requires specific didactical research, where various activities of the students can be examined: the heuristic use of drawings, to understand the course, or to solve linear algebra problems; the production of drawings when the students are asked to draw ... Some of the results I will present here are related to different uses of drawings; but I will not give here details about the heuristic function of drawings, because I present a short review of a work aiming at specifying global structures describing the use of geometrical and figural models by students.

2. Testing students practices

2.1. *The questionnaire*

The aim of a questionnaire for the students is to provide answers to the two following questions:

- What kind of geometrical or figural models do students use in their linear algebra practices? I will try to observe if students use secondary school geometry notions and results in linear algebra, and whether it helps them or rather creates difficulties. I will investigate the use of drawings, especially in critical cases. Do students illustrate linear algebra situations with drawings, even in dimension four and higher? Do they accept the representation of polynomials as vectors?
- Are some kinds of geometrical models associated with a good understanding of linear algebra, while other are correlated with misconceptions and difficulties?

I have chosen to interview post-graduate students, because they have already learned the main notions of linear algebra, including the theory of Euclidean spaces. I gave them a questionnaire and then met them individually for further questions. The students I asked are in the University of Rennes, in France; they did not follow geometry courses at the university. 47 students agreed to fill in the questionnaire and answer the interview.

I can not examine here the whole questionnaire. I will briefly present two questions, referring to two different aspects of “geometrical intuition”.

2.2. Symmetries

I proposed the following exercise in the questionnaire:

Give a 4x4 matrix representing a symmetry.

I have chosen the notion of symmetry, because it is first encountered in geometry courses, even in primary school; it is then studied again in linear algebra, and Euclidean spaces.

That exercise is not a usual one: matrices of symmetries are generally required when the symmetry is a reflection, in a Euclidean space of dimension 2 or 3, during the study of the orthogonal group. The choice of the dimension 4 prevents the students from staying in a geometrical context; however, they can use some geometrical properties of the symmetries to answer. They can for example associate a drawing to a symmetry in dimension 2 or 3, representing the image of a suitable basis, and generalise it in dimension 4.

They can also refer only to structural properties, using for example the characterisation of the matrix M of a symmetry by $M^2=I$ (I being the identity matrix) or simply the fact that, in a suitable basis, the matrix of a symmetry is diagonal with coefficients equal to 1 or -1 on the diagonal.

During the interview following the written questionnaire, students were asked for details of their investigation process: which properties of the symmetry did they use; did they use a drawing? (This is the only case of “heuristic” use of drawing I will consider here.) I also asked the students, if they were able to produce a drawing to illustrate such a symmetry. I supposed that the fact the dimension was 4 would prevent some of them to produce a drawing.

I used the following criteria for the analysis of the answers:

- *Correct answer*: I consider the answer as correct as soon as a matrix of a symmetry has been given, even with no explanation or justification.
- *Central symmetry*: some students present the matrix they propose as representing a central symmetry. In that case, the matrix proposed was $(-I)$, the diagonal matrix whose diagonal coefficients are (-1) . The notion of central symmetry is not encountered at university; these students use reminders of secondary school in a non-appropriate context.
- *Linear algebra symmetry*: when, in their argumentation, students use linear algebra properties of the symmetries, talking, for instance, of the kernel of $(s-Id)$, or of supplementary spaces, or of the square of a symmetry.
- *Affine drawing*: students representing a symmetry acting on points.
- *Vectorial drawing*: students representing a symmetry acting on vectors.

- *Coordinates*: students filling in the matrix using the analytical expression of the symmetry.
- *Basis* : students filling in the matrix using the image of a basis.

The following table gives the results concerning these different points, for the 47 students:

	Central sym.	Linear algebra sym.	Affine drawing	Vectorial drawing	Coord.	Basis
Central symmetry	16	1	3	1	9	4
Linear algebra symmetry	1	16	1	10	3	12
Affine drawing	3	1	9	0	7	1
Vectorial drawing	1	10	0	11	1	9
Coordinates	9	3	7	1	16	0
Basis	4	12	1	9	0	17

Most of the students gave a matrix of a symmetry. But a third of them presented it as a central symmetry; only another third used linear algebra properties of symmetries. Less than a half proposed a drawing (even after I asked for one); most of the others argued that the dimension being four, it was not possible to draw. Half the drawings proposed were affine, and half vectorial. Only four students declared, during the interview, that they used a drawing to solve the exercise.

I studied the correlation between the different items; it led me to distinguish two groups of students, gathering two third of the ones who proposed a suitable matrix.

In the first group (13 students), students use linear algebra properties relative to the notion of symmetry; they use the image of a basis to find the matrix. They do not all give a drawing; but all their drawings are vectorial.

In the second group (12 students), students talk of “central symmetry”; they use coordinates to find the coefficients of the matrix, and the drawings they give are essentially affine. These students refer to the notion of symmetry they learned at secondary school; their use of coordinates allow them to find the matrix, although they refer to a symmetry acting on points.

2.3. *Representing polynomials*

The question I present here belongs to the interview. I asked the students to produce, if possible, a drawing to illustrate notions or properties of linear algebra. I proposed for example the following property:

P, Q, R are the polynomials defined by $P(X) = X$, $Q(X) = X^2$ and $R(X) = X^3$. $\{P, Q, R\}$ is linearly independent.

My aim was to observe if the notion of a set of independent vectors is associated with a drawing, even if the corresponding vector space does not stem from geometry. I assumed that the students had already constructed some figural concept in relation with the notion of basis; since that notion, and the corresponding drawing, are already encountered in secondary school. But $\{X, X^2, X^3\}$ is not often considered as a basis. Interpreting a set of independent vectors as a basis requires the implicit use of a span subspace, which is unfamiliar to many students.

I have chosen polynomials, rather than functions, in order to prevent confusion with vectorial representation and graphs of functions.

Less than half of the students give a drawing here. Some students drew the graph of the corresponding functions; finally, 28 students (among 47) proposed no drawing, or an unsuitable drawing. 18 students propose a suitable drawing, with vectors. Two third of them evoke the notion of basis. I observed the difficulty I mentioned above, about the span subspace.

Some students say: “It would have been easier with $\{1, X, X^2\}$, because it is a basis”. These students refer to the space $\mathbb{R}_2[X]$; some others say that the situation takes place in $\mathbb{R}_3[X]$, thus in particular in dimension 4, so they are not able to draw. It seems that the notion of a set of independent vectors is not strongly associated with a drawing, while the link with the same drawing, for the notion of basis, is strong enough for the students to use it, even to represent polynomials.

3. Conclusion

Comparing the results of the different parts of the questionnaire led me to distinguish three attitudes among the students.

Students using affine drawings in linear algebra

These students use an affine figural model, unsuitable for linear algebra. That model probably stems from secondary school geometry; because as I

mentioned it above, these students did not have affine geometry courses at the university. They also evoke secondary school properties in their answers.

Perhaps no other figural model has been proposed to them since they entered the university, if they only encountered teachers praising a structural approach to linear algebra. They clearly encounter difficulties in linear algebra (all of them failed at least at two of the three linear algebra questions I proposed).

Students using vectorial drawings in linear algebra

These students seem to frequently use vectorial drawings in their practice of linear algebra. Around a half of these students seem to have good competence in linear algebra, while the other half seem to encounter difficulties. So the use of vectorial drawings, more convenient than affine drawings to illustrate linear algebra situations, does not seem to be associated with a good understanding of linear algebra.

Students using no or few drawings, and succeeding in linear algebra

Some students seem to use almost no drawing, while they seem to have a good understanding in linear algebra. They sometimes mention using a mental representation, but mostly when they do not remember a property; they do not seem to find it useful otherwise. The existence of that group suggests that a purely algebraic approach to linear algebra is possible, at least for some students.

Using geometry does not provide a way of avoiding all the difficulties related to the learning of linear algebra. One of the advantages of linear algebra is that it unifies several mathematical domains; the historical part of my work (Chartier 2000) shows that the wish to unify these domains has been a determining factor in the genesis of linear algebra. In France, the transposition process that took place during the “modern mathematics” reform, and led to the introduction of linear algebra in the secondary school curriculum, accentuated the link between geometry and linear algebra. A geometrical model can be used to introduce linear algebra, but it must be associated with others, as polynomials or sequences for instance; linear algebra can not be presented as a mere generalisation of plane or space affine geometry.

However, the use of a figural, vectorial model not only in geometry but also in vector spaces of functions or polynomials, and in general linear algebra, can be useful for the students. Some of them succeed without it, while some students use a vectorial figural model and fail anyway. But the use of an affine figural model is clearly associated with difficulties in linear algebra. A vectorial model, proposed by the teacher, could at least prevent the students from using an affine model. It may also help them to understand the unifying aspect of linear algebra.

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THE TRANSITION FROM ALGEBRA TO ANALYSIS: THE USE OF METAPHORS IN A GRAPHIC CALCULATOR ENVIRONMENT

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Abstract: *This paper focuses on the introduction of the concept of limit at secondary school level. Drawing on research on the role of metaphors, movement and perception in the construction of mathematical concepts, we developed an approach to the concept of limit which considers perceptive elements as a starting point. The students are required to construct and manipulate representations of functions in a graphic calculator environment (namely, the TI89). The transition from the register of graphical representations to the symbolic one takes place in two ways: through the re-elaboration of the language used by the students to describe their graphic manipulation of the functions and through the introduction of symbols which will support the algebraic manipulation to be done later on.*

Introduction

The work presented in this paper lies within the field of research on the didactics of analysis, at secondary school level. A number of studies, centred on the teaching and learning of analysis, have pointed out difficulties and obstacles emerging when the students face this mathematical field. M. Artigue (1991, 1998) classifies these difficulties into three main categories:

- ⟨ difficulties due to the complexity of the elementary objects of the field, such as real numbers, sequences and functions;
- ⟨ difficulties due to the conceptualisation and formalisation of the concept of limit, the basic concept of the field;
- ⟨ difficulties due to the specific features of analysis and to the gap between algebraic and analytic thinking modes.

She also stresses that, up to now, research in that area has mainly dealt with the first two categories of difficulties, with much of the work about the notion of function (Dubinsky & Harel, 1992), limit (Cornu, 1991; Sierpinska, 1985), derivatives and areas (Schneider, 1991). From a theoretical point of view, research relied on different frames: the duality between concept definition and

concept images (Tall & Vinner), the duality between processes and objects (Sfard, 1992) and the ensuing notion of procept (Tall, 1994), the notion of epistemological obstacle (Cornu, Sierpiska, Schneider). Progressively, as evidenced in (Tall, 1996), research has included also technological issues, with the development or use of computer languages favouring the encapsulation of processes into objects, the interplay between the different semiotic registers used in analysis, and, more recently, the analysis of potential and pitfalls of computer algebraic systems.

Up to now, research has been less sensitive to the third category of difficulties mentioned above. M. Legrand (1993) has pointed out some “rupture” between algebraic and analytic thinking modes, focusing on the necessary changes in the meaning of equality (to be considered not only as a sign for equivalence as this was the case in algebra, but mainly as a sign for “infinite proximity”) and on the change in reasoning modes (especially from reasoning by equivalence to reasoning by sufficient conditions).

Our research project mainly deals with this dimension. We study the entrance in the field of analysis and the transition towards new thinking modes, of students who have already gained some familiarity with algebraic thinking modes. These are characterised by important changes, beyond those mentioned above:

- < the dominant role played by infinite processes with respect to finite processes;
- < the transition from global points of view on functional objects to local points of view;
- < the complexification of algebraic practices linked to the necessary differentiation between orders of magnitude in computations.

Beyond the theoretical frames which have traditionally been attached to research on the didactics of analysis, in this study we are especially sensitive to what can be offered by cognitive approaches which stress the role played by metaphors, perception and movement in human cognition (Longo, 1997, 2000; Lakoff & Núñez, 2000; Sfard, 1994). In the next paragraph, we briefly present these approaches and the way they have influenced our research work.

1. Some basic elements of our research project

1.1 Construction of sense in mathematics

In his research, Longo (1997, 2000) presents a cognitive approach to the problem of the Foundations of Mathematics, introducing elements of analysis

outside the formalistic approach, which is usually taken in this field of study. He assumes that simple mathematical objects are cognitive facts and that mathematical problems and constructions are expressions of thinking rules, which also have a cognitive nature, as for example: comparing, classifying, making analogies. We draw on his work as concerns the importance he gives to:

1. the interactions with the real world (perceptions, spatial relationships, movement) in the process of construction of meaning for new mathematics concepts;
2. To the reciprocal and constant influences between the perceptive (phenomenological) level and the mathematical level (at which concepts are formalised) in doing mathematics.

1.2 Metaphorical structure of mathematics

Besides Longo's research, Lakoff and Núñez's work shows an analysis of the structure of mathematics that is strongly based on the human mind structure ("mind-based mathematics"). The study of the cognitive structures which seem to support mathematical reasoning (as for example metaphors) is the key element of this research.

Metaphors¹ become an important basis for understanding some mathematical concepts. The operational aspect, which distinguishes the definition of metaphor previously presented, proves a key element with respect to our research. This seems to be a constant element of mathematical thinking. For example we can consider some aspects of the analysis of the concept of limit, based on the following definition: "suppose that, as the variable x gets closer and closer to the point a from either side, $f(x)$ gets correspondingly closer and closer to a unique value L . We define L to be 'the Limit of $f(x)$ as x approaches a '". The metaphorical character of this definition is highlighted by the use of the words "approach" and "closer and closer": they are connected with the everyday understanding and experience of motion through space.

2. Considerations and hypothesis

As regards our study of the transition from algebra to analysis, we remark that the treatment of functions can involve global and local dynamic aspects that are

¹ Metaphor is an important kind of conceptual mappings (the concepts are organised through vast networks of these mappings), "which project the inferential structure of a *source domain* onto a *target domain*" (Lakoff & Núñez, 2000)

respectively related to algebraic and analytic work. For example, a global dynamic aspect is given by the transformation of the representative curve function through known geometrical transformations. This change corresponds also to a transformation of the algebraic formula. Another example regards the tangent (Castela, 1995): the global perception of the tangent to a curve (a straight line that intersects the curve in two coincident points but that does not cross it anywhere) is algebraic and geometric. These dynamic aspects (and metaphors) seem to be different if we are concerned with a local context. Drawing on the previously mentioned research, the first hypothesis is that the basis for the introduction of the notion of limit relates to perceptual aspects, that involve a transition from a global to a local point of view. We choose the graphic register as a starting point because the dynamic aspects (with the metaphors which can be defined) seem to be more useful in this case rather than in the algebraic register.

The second hypothesis is that it is possible to introduce some kind of dynamics into the treatment of graphical representations of functions through the Zoom-controls of graphical calculators in order to foster the passage to a local point of view. Using these Zoom-controls, we meet a phenomenon, identified as the phenomenon of “microstraightness”, which can be expressed as follows: “a graph enlarged around a point seems to become a line”. We want to study the cognitive and educational potentialities of the zoom and of the related metaphor of “microstraightness”. Our hypothesis is that this graphical manipulation allows the discovery of a phenomenon that requires the introduction of a new operator, i.e. the limit, into the construction of its mathematical model. The passage to the algebraic register becomes necessary to overcome the limits of the graphic register². In the construction of the model, the symbols that are introduced acquire meaning on the basis of the previous graphical work and the computations bring into play orders of magnitude.

3. The classroom experiment

The theoretical elements we have previously highlighted were used to develop “didactic engineering” (Brousseau, 1998) in order to introduce mathematical activity which seems to constitute the basis for analytic concepts such as concept of limit and derivative. The experiment was implemented in a fourth year classroom of a ‘Liceo Scientifico PNI’³ (12th grade) in May 2000 and it consisted of 6 sessions. Every student was given a TI-89 graphic-symbolic

² Were we are not able to distinguish between the tangent line and the other straight lines, which mix up with the curve around the chosen point only through the Zoom-controls.

³The students have 5 mathematics classes per week

calculator to keep for all experiments. The eighteen students were divided into six groups; each group was given one calculator. Each session was organised in two parts: group activities first and then collective work orchestrated by the teacher. The teacher and myself managed all the sessions, lasting one hour and forty minutes each. The observation of the sessions was organised in the following way: a video camera filmed the whole classroom, both during the group activities and during the collective debates. Five university students observed five groups filling a grid, constructed by us on the basis of our *a priori* analysis. The *a posteriori* analysis is based on all the data collected (videos and fieldnotes) and on the *a priori* analysis.

In the following we describe the first two sessions pointing out the elements highlighted above.

3.1 The first session: Some elements of analysis and the students' work

Goal: to lead the students to encounter the “microstraightness” phenomenon.

Activity. Two different tasks are given to different groups in the classroom. The exploration of representative graphs of some functions around fixed points, through the Zoom-controls of the calculators, is the main task (we propose six functions, some of which are differentiable everywhere and others which have singularity points). At the beginning the exploration is guided, as the number of zooms to execute is suggested; then the students are asked to use the zoom more freely. The change from the calculator environment to paper and pencil is determined by us. The students are asked to draw on paper what they see in the initial window ZoomStd (the dimensions of standard window are $[-10, 10] \times [-10, 10]$) (first step), then what they see after two zooms (second step) and finally what they see at the end of exploration (third step). In this way, the whole exploration is in front of the students' eyes and provides a record of their work. A global point of view on the whole function shows its variation, extremes and infinite branches. The three pencil and paper representations each student makes have the purpose of introducing the phenomenon of “microstraightness”: the given curves are different but at the end of the exploration almost all of them are locally represented as lines. One function isn't differentiable at chosen point. These elements show that the phenomenon they encountered has a local character. In the collective phase, that follows the group activity, the teacher gathers the results of students' observations, underlying the property previously pointed out and discussing with the students a name for it (which ends up to be “linear zooming”).

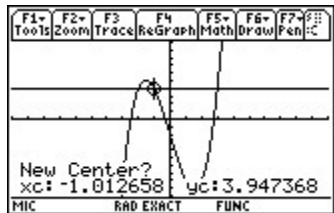
Some variables of the situation: the choice of the functions to explore and of the values of x , at which the functions are to be calculated, allows the students to highlight the local feature of the phenomenon. The ZoomStd was set

at the beginning in order to provide the same starting conditions for all the students. After doing many zooms, the axes are no longer visible, therefore a horizontal line⁴ is introduced, as well as another reference for the y-coordinates of the point around which the increases are made.

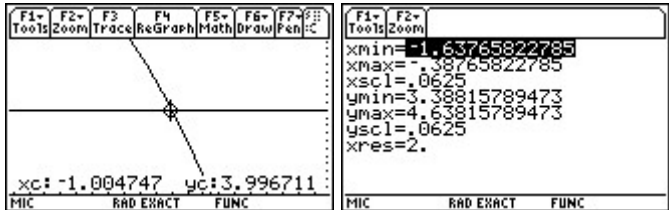
We now present some elements of analysis, taken from the work of a group of students during the first lesson.

Task: “Explorations of $y_1(x) = x^3 - 7x - 2$ around the given point $(-1;4)$ ”.
Students: CF, GL, CA. Teacher: T

First step. The ZoomStd representation shows all the typical graphical features that students are used to see when they work with graphs, since the beginning of their work with function: the co-ordinates system, the grid units.

STUDENTS' WORK	ANALYSIS
<i>The students perform exploration work. They represent function on the standard window; the plotted line corresponds to the constant function y_6. The chosen point is marked by the cursor</i>	 <p style="text-align: center;"><i>Fig. 1</i></p>

Second Step. The exploration of the graph around the chosen point begins and this allows the transition from a global to a local point of view. This takes place by choosing the increases with the calculator. At each step, everything that does not appear on the display-screen must be ignored.

<p><i>After reading the task again, the students are considering $y_1(x)$</i></p> <p>23. CA: “Do two zooms...do ZoomIn!”</p> <p><i>After two subsequent ZoomIns⁵ around the given point, we obtain the following fig 2 is obtained⁶</i></p>	<p>The guided process stresses the transition to a local point of view which needs totally new gestures for the students: it is a new dynamic aspect “of entering (going into) the graph”.</p>  <p style="text-align: center;"><i>Fig. 2</i></p>
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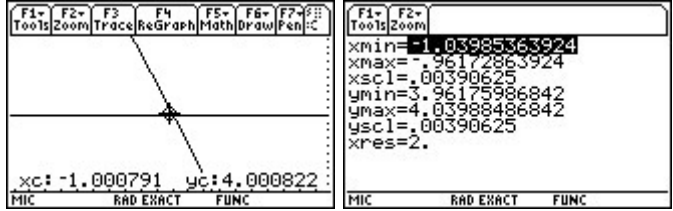
⁴ whose equation is $y_6(x)=4$, which equals the values taken by the different functions

⁵ With the command ZoomIn, the new window is homothetic to the previous one

⁶ the dimensions of this window are displayed next to it

24. GL: “Draw the graphical representation we obtained” <i>CF draws the representation of a line.</i>	CF’s drawing depends on his interpretation of the screen; but another student makes an objection. At first the perceptive image of a line prevails (#25), then the memory of the curve they started from leads the students to say that it cannot be a line.
25. CA: “what’s this line? But it isn’t a line!”	
26. GL: “Yes, it seems to become a line; it looks as if it becomes more and more similar to a line! It approaches to become a line”	Then, the exploration goes on with another ZoomIn, which leads GL, who has used the calculator since the beginning of the activity, to suggest that the graph becomes a line (#26).

Third step. The students are not told when they have to stop the exploration. Our hypothesis is that they will stop when some kind of stable state seems to be reached for the first function, and that they will try to reproduce the same kind of phenomenon when dealing with the other examples.

<i>Three other ZoomIns follow the observation #26. Exploration stops at fig3, which is then reproduced on paper by CF.</i>	CF needs to improve the precision for the coordinates of the fixed point on the calculator. 
	<i>Fig. 3</i>

<i>The students analyse the second function</i>	
31. CF: “Enlarge it” <i>addressed to Angelo, who does two ZoomIns.</i>	There the effect of enlargements begins to be clear.
32. GL: “Come on! It’s a line!” (...)	The use of zooms does not show any difficulties; the transition to a local point of view is indeed more difficult. The two students show different reactions to the image they see on the screen. CF, who had been dealing with the representations on paper, finds the problem of not having any reference system (#39), when he needed to draw the screen on their paper sheet. GL, who had seen a line in the exploration of the previous example, identifies a line again. Such a representation shows the transition to a local point of view (#38).
38. GL: “yeeeeaaahh, it becomes a line, come on! ...”	
39. CF: “there are no graphic references”	

<i>The fifth function is linear and it is represented on the standard window (ZoomStd).</i> 41. GL: “Well, ... it already was a line, so it remains a line!” (...)	GL clearly shows he constructed the image of a line and the step of recognising the graph as a line constitutes the standard way of stopping the exploration (#41).
<i>They work with a function, which is not differentiable at the chosen point.</i> 46. GL: “But, that will never become a line ... excuse me, what was the point?” 47. CA: “Go back and repeat”	The discovery of straight line is their criterion for stopping exploration. If it is not obtained, there is a mistake in process. The exploration begins (#47).

3.2 The second session: Some elements of analysis and the students' work

Goal: to go beyond this first perceptive approach of local linearity and to help students to build a mathematical model for this phenomenon.

Activity. We created a unique group activity, which required finding the equation of a line “to which the graph around a given point seems to approach” for a quadratic function, because parabolas are familiar objects to Italian students in grade 12. We hypothesise that this knowledge will help the mathematisation process we aim at. More precisely, students are asked to test the microstraightness of the parabola in the neighbourhood of the given point, and then to find the equation of the line they have obtained on the screen of their calculator (this can be easily achieved by using the Trace command in order to obtain the co-ordinates of a second point of this line). It is expected that different equations will be obtained, with close coefficients of course, leading to some interesting discussions about the way the microstraightness phenomenon can be mathematised. Mathematisation will be then helped by the use of the ZoomOut command, which will restore a more global vision with a parabola and lines in a tangential position with respect to it. An algebraic computation will then allow the students to find the equation of the tangent and compare it with the equations previously obtained, linking the tangent yet known with the microstraightness phenomenon and giving to the tangent a new status of limit object.

We present a part of the collective discussion, in which the break secant/tangent and the necessity to specify the “idea of points closer and closer” are evident. Students: GA, PM, CF, DAL. Teacher: T

<p>16. T: “$y=x^2$, is it ok? I will enlarge it. This is the equation of a line. They seem very close. If I re-explode it, what do I find?” (...) “If you observe from far off, what do you find?”</p> <p>17. GA: “it should seem a tangent”</p> <p>18. T: “... you say it should be a tangent”. <i>He draws the tangent at the graph at the given point on the blackboard (...)</i></p> <p>25. PM: “but, it can’t be the tangent, because there we have taken two points of the graph ... if we have taken two points, it will be a secant line”</p> <p>26. CF: “The unique known given point, which we have, is the one provided by the text, we have found the other point in a different way”. (...)</p>	<p>In the group work phase, CF and CA interpreted the line they got on the screen as the tangent line to the curve and calculated its equation algebraically. The beginning of the classroom discussion shows that other students considered the line they got after some enlargements as the tangent line to the parabola at the given point.</p> <p>According to our hypothesis, the conflict between the conception of tangent line and the representation of straight lines, obtained during group activity, comes out.</p> <p>CF explains his solution process</p>
<p>54. T: “If I want to find the tangent line...?” (...)</p> <p>55. DAL: “I should make the two points closer and closer” (<i>gesture with his hands</i>)</p> <p><i>The teacher fosters the transition from the graphics to a numerical point of view.</i></p> <p><i>The first step is that of writing the ratio of increments on the blackboard for the points $(2.5;f(2.5))$ and $(2.5+h; f(2.5+h))$.</i></p>	<p>The teacher goes on with discussing a way of determining the tangent line in order to solve the conflict.</p> <p>It seems that DAL continues the enlargement game which started with the calculator: he considers the nearness of the points of the graph as a spatial nearness which may get to a contact.</p> <p>This transition is made through a translation into numerical or symbolic elements of what was said around the points on the graph.</p> <p>The teacher fostered the operationalisation of the images obtained after exploration.</p>

123. PM: "If I want the tangent, I take the same point 2.5"	The conflict is transferred to h; this transfer encourages some calculations.
124. T: "Aaahhh, let's take it, let's take 2.5. Above we have 6.25 ... what's the result for the ratio of increment, PM?"	
125. PM: "Undefined"	
126. Teacher: "We have zero over zero. Then, it isn't right." (...)	The parameter h is contextualised. Its value different from zero has sense in this situation, not only algebraically.
136. DAL: "I consider a point approaching more and more ..."	The experience with the calculator formed the basis and support for the discussion. Interventions #136 contain an aspect of what has been identified as the "microstraightness metaphor", which is then translated in the calculation with h. The parameter h has a fixed order of magnitude, based on the situation in which it was introduced.
138. AI: "get a point nearer"	
140. DAL: 2,500001	
165. T: "(...) what Mathematics made? They considered plus something very small."	

4. Conclusions

This paper shows an attempt to use new theoretical constructs, referring to metaphors and dynamic aspects, for the study of the introduction of analysis in the classroom. In this part we will outline some preliminary conclusions coming from the on-going analysis of the data collected up to now. Different approaches (Artigue et. al., 1998) to the concept of limit with calculators have been presented, our approach is different from these. Our analysis seems to show that the "zoom approach" and the "microstraightness" metaphor may be helpful in order to address the transition from global to local points of view. Moreover, the microstraightness phenomenon appears as a striking one to the students, motivating mathematical attention. However, the mathematisation of this phenomenon is not so easy. The constructed situation helps to question the initial perceptive evidence: "it becomes a line", but generates an evident cognitive conflict with previous conceptions of the tangent object which are of an algebraic-geometrical nature, and associated with other images and perceptions. Of course, such conflicts cannot be avoided as they are constitutive of the transition towards analysis. They cannot be resolved at a perceptive and metaphorical level and the data collected show the role that the game between the algebraic and the graphic registers plays. They also tend to show that this effect cannot be managed in an autonomous way by the group of students, but the way the teacher organises and pilots the discussion, which is fostered by the situation, is essential. The conversion from the graphic register to the algebraic

one was possible thanks to the introduction of the symbol “h” at the end of the second session.

The study of the potentialities of the use of metaphors in the teaching and learning of mathematics is just beginning. Still many issues are to be investigated, as for example: do different metaphors have different results on learning processes and/or teaching processes (and, if yes, how and why)? Is it possible to construct metaphors for the teaching and learning of some mathematical concepts?

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TEACHERS'USE OF SEMIOTIC REGISTERS

From teachers' use of semiotic registers ... to their effects on how arithmetical problems are solved

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Keywords: *semiotic registers, arithmetical problems, primary school, mathematical textbooks*

Abstract: *In order to examine whether the form of data-presentation may influence success in solving a given problem, we are interested in the teachers' use of semiotic registers. We are particularly concerned to see if 8 or 9 year old children are confronted in their classrooms with different semiotic registers and if the teachers are aware of this confrontation in the way they work with their pupils.*

Introduction

In our research (1) which investigates the effects of solving a given problem, concerning data-presentation (which includes text, table, graph, diagram, text mixed with drawings not related to the data, and a variety of registers), we worked on the results of a questionnaire which we presented to Primary school-teachers. Eighty-one teachers of three districts belonging to two different Local Education Authorities (France) and 1081 pupils were concerned.

This paper presents the form and the results of the teacher's questionnaire which was articulated in four parts:

- the pupil tools
- solving problems at school
- the teacher's lesson preparation
- the classroom

We have located our study principally within the theoretical framework developed by R. Duval (2). The learning of mathematics requires cognitive activities which in natural language particularly requires the use of a variety of semiotic systems. Besides the transformations of representations in the interior of a given semiotic system, it is necessary to consider the cognitive activity of

the conversion of representations from one semiotic register to another, together with their articulation. According to R. Duval, these essential activities of conversion and articulation become involved in the activity of resolution of problems at the centre of the didactic situation created by the teacher.

1. Pupils tools

Our first question was : « *When they have to solve arithmetical problems, do pupils have to read varied registers of semiotic representation?* » So, we asked if the mathematical text books really contain a variety of representations and we also asked if the 8 or 9 year old pupils have, and also use mathematical workbooks in their classrooms.

In order to investigate these questions, we examined different mathematical text books which were edited by different French editors between 1960 to 1970 and between 1995 to 1999.

We used the classification proposed by Raymond Duval (3) who distinguishes two types of situations:

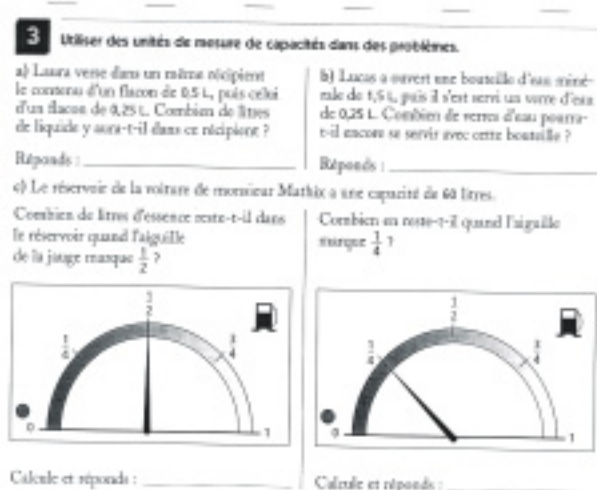
- the situations where the representations are produced by one and only one system (for example : development of a discourse, development of an algebraic calculus ; transformation of a geometrical figure ...). In this case, the production is made just as if each representation was self-sufficient for the part of the step in which it is produced.
- the situations where the representations are produced in order to be placed in parallel or to be associated with other representations produced by another system (for example : a key for a caricature ; the hypothesis for a geometrical figure ; a drawing just near a descriptive or a narrative text). In this case, the production is made just as if, for some representations, an auxiliary representation was necessary.

	analysed exercices	Terms presented in natural language in situation of « self-sufficient production »	Terms presented in natural language Not in situation of « self-sufficient production »
In the mathematical textbooks published from 1960 to 1970			
number	72	65	7
percentage		90,3%	9,7%
In the mathematical textbooks published since 1995			
number	41	13	28
percentage		31,7%	68,3%

Table 1

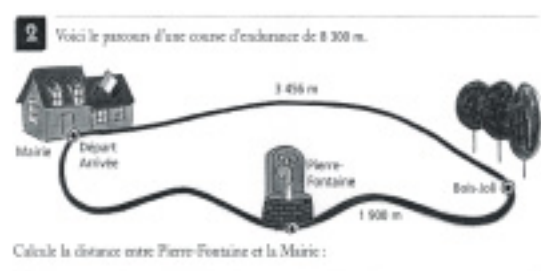
For the representations, we distinguished different functions. We adopted the classification proposed by R. DUVAL (4) who has found seven functions about the auxiliary representations. In our analysis of the 113 exercises, we noticed principally four functions:

- *further information*: it means that there are informations which are not present in the principal representation (see problem 2),
- *illustration*
- *explanatory interpretation* which means that the auxiliary representation gives informations which are already present in the principal representation
- *heuristic interpretation* which means that the auxiliary representation offers possibilities of processing which are very different of these of principal representation-register.



Problem 1

extracts: CHAMPEYRACHE G, *Maths élem CE2*, Belin, Paris, 1997



Problem 2

Functions of auxiliary representations

		Terms presented in natural language Not in situation of « self-sufficient production »				
	Analysed exercises	Total Number Percentage	Further information Number Percentage	Illustration Number Percentage	Explanatory interpretation Number Percentage	Heuristic interpretation Number Percentage
Math. books 1960 to 1970	72	7 (9,7%)	3 (43,0%)	0 (0,0%)	3 (43,0%)	1 (14,0%)
Math. books 1995 to 1999	41	28 (68,3%)	19 (49,0%)	8 (20,5%)	8 (20,5%)	4 (10,0%)

Table 2

We noticed that the actual textbooks contain a variety of representations (see table 1). But are these textbooks really present in the classrooms and do pupils use them when they have to solve problems?


So when comparing the different functions of representations in books between 1960 and 1970 and between 1995 and 1999 (see table 2), we can ask whether through their mathematical books, 8 or 9 year old pupils can effectually be confronted with a variety of semiotic representations, which was not the case before. They also may be confronted with the conversion (5) of representations where they need to change from one register to another. This example is taken from a textbook of 1995:

Marie possède 35F, Nadège 54F, Laure 15F et Hélène 22F.
A la crêperie, elles commandent toutes une crêpe et une boisson.
• Que reste-t-il à chacune après cette dépense ?

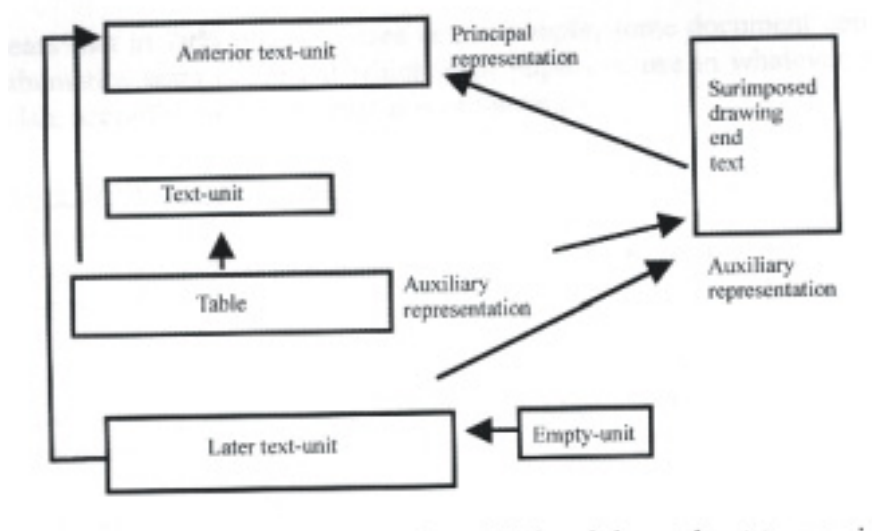
Complète le tableau.

	Marie	Nadège	Laure	Hélène
somme possédée				
somme dépensée				
somme restante				

Hélène veut acheter en plus une glace.
Combien doit-elle emprunter à ses camarades ?



Problem 3 (extract: CORRIEU, L, dir, *Vivre les mathématiques CE2*, A. Colin, Paris, 1995)



According to R. Duval (5), in any mathematical activity, at least two registers of representations are simultaneously used and you have the possibility to change a register when you want. Mathematical comprehension thus assumes the coordination of at least two semiotic registers of representations.

But are the pupils actually confronted with this variety of representations?

The following questions were given to the teachers:

Does each pupil in your class in the 8/9 age group have a maths text book?

a maths workbook? _____

If so please give the title, editor and collection. _____

If not, is there at least one book for two pupils? _____

please give the title, editor, collection _____

Number of mathematical-books per pupil (8 or 9 years old)

At least onebook or working file per pupil	79%
No book or working file per pupil	21%

Table 3

It appears that in 79% (see table 3) of the classes in our sample, some document consisting of mathematics texts is present which each pupil can use in whatever manner they wish, according to the instructions of the teacher.

2. Solving problems at school

The following questions about the frequency of exercises were given to the teachers:

• Number of problems

On average, how often do you set arithmetical problems to your pupils in the 8/9 age group?

once a day ☐ *number or problems set* _____

once a week ☐ *number of problems set* _____

once a fortnight ☐ *number of problems set* _____

Other response

Frequency of arithmetical problem resolution (8 or 9 year old pupil)

Once a day	10,0%
Three times a week	1,2%
Twice a week	3,7%
Once a week	76,2%
Once a fortnight	6,3%
Variable	2,5%

Table 4

• Approach to the problem

Please tick the box corresponding to the approach that you most frequently adopt.

As a general rule, when you set a problem to your pupils in the 8/9 age group

- *do you read the problem to your pupils* ☐
- *do you ask one of your pupils to read the problem to the class* ☐
- *do you ask your pupils to read the problem silently* ☐
- in this case* : *do you then read out the problem to the class* ☐
- : *do you then ask a pupil to read out the problem to the class* ☐
- : *the problem is not read out in class.* ☐

In more than 3 classes out of 4 the pupils are confronted with a weekly activity of resolution, but this happens daily only in one class in ten (see table 4). This results in the resolution of problems becoming an infrequent activity if we quantify the time which a pupil spends in the activity in a school year in this didactic situation.

• Correction of the problem

As a general rule where problem solving is concerned you usually opt for:

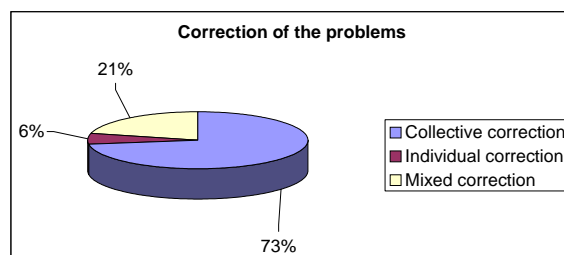
- *individual correction* ☐
- *collective correction* ☐

If you usually use collective correction:

You use as a basis for this correction:

- *the correct solution of one of your pupils* ☐
- *several correct solutions of pupils presented in different ways* ☐
- *the comparison of at least one correct solution and one erroneous solution.* ☐
- *you prefer giving the correct solution yourself immediately* ☐
- *you include systematically during the correction yet another solution that has not been given by your pupils.* ☐

Again, in 3 classes out of 4, the correction phase is managed by whole class participation, as is the French custom. In this activity of collective correction more than 80% of the teachers start with a comparison of the solutions proposed by the pupils (see table 5). Some (48%) rely on a comparison between an exact answer and an erroneous one, while others (37%) use a comparison between many correct answers. Clearly, in this case, some teachers utilise representations involving errors since they recognise the status and the role in learning of erroneous representations. We could also interpret these facts as a tentative and implicit taking into account by the teachers of the variability of semiotic registers of representations and expressions produced by the pupils in the resolution of the problems.



Graphic 1

Start point: <u>correct answer by a pupil</u>	6,3%
Start point: <u>comparison of a number of correct answers</u> , using different methods.	36,7%
Start point: confrontation of at least one correct answer with at least one incorrect answer.	48,1%
The teacher immediately proposes their own solution.	0%
The teacher systematically introduces a different form of solving than the one(s) proposed by the pupils during the collective correction.	2,6%
No response	6,3%

Table 5

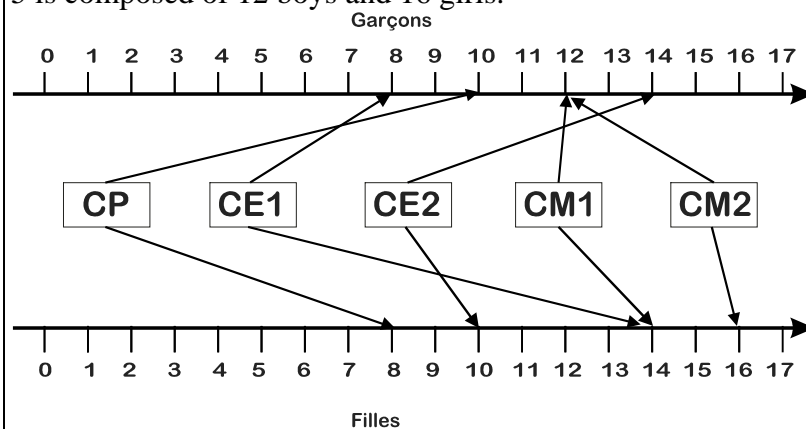
A teacher who systematically introduces a different form of solving explained that this moment is very important in her learning, because she introduces different forms of representation and makes it possible for the pupils to progress from one register to another. For example, move from a table to a text (or natural language), in order to solve and to explain the resolution.

This is an example of a problem during the correction phase with a number of different registers:

Problem 1

Here are the numbers of pupils in the Marie Curie Primary school:

In Primary 1 there are 10 boys and 8 girls. In Primary 2 there are 8 boys and 14 girls. In primary 3 there are 14 boys and 10 girls. In primary 4 there are 12 boys and 14 girls. Primary 5 is composed of 12 boys and 16 girls.



classrooms	CP	CE1	CE2	CM1	CM2
boys	10	8	14	12	12
girls	8	14	10	14	16

- a) What is the total number of pupils attending this school?
b) How many boys are there in this school?

3. Lesson Preparation

We also investigated the tools used by the teachers in preparing their tasks and we analysed several elements related to classroom practice using the following questions:

•Teachers' resources

*Do you use a specific book for the preparation of you maths lessons ? _____
If so please give the title, editor, collection, master's copy, pupil's copy)*

If not what other method of preparation do you use? _____

•Problem solving

As a general rule, when preparing your lessons involving problems, for pupils in the 8/9 age group :

- a) do you prepare a written solution to the problem in advance ? ☐
d) do you usually calculate the reply mentally ? ☐
c) do you put down onto paper, all the possible ways of solving the problem ? ☐

If you have replied "yes " to question c) , please explain what, for you, is the purpose of this method of preparation and in what way you use it in class with your pupils.

The table below shows that the modality “mental resolution” becomes dominant before the “written form of the solution”. We might interpret this result as explaining the fact that most of the teachers spend most of the time in a situation which does not address the explanation by reference to a semiotic register of representation and expression.

	yes	no	No response
The teacher systematically solves the problem in <u>writing beforehand</u> .	21,2%	75%	3,8%
The teacher usually solves the problem mentally.	53,7%	42,5%	3,8%
The teacher systematically explores <u>in writing</u> a number of ways of solving the problem.	29,6%	66,6%	3,8%

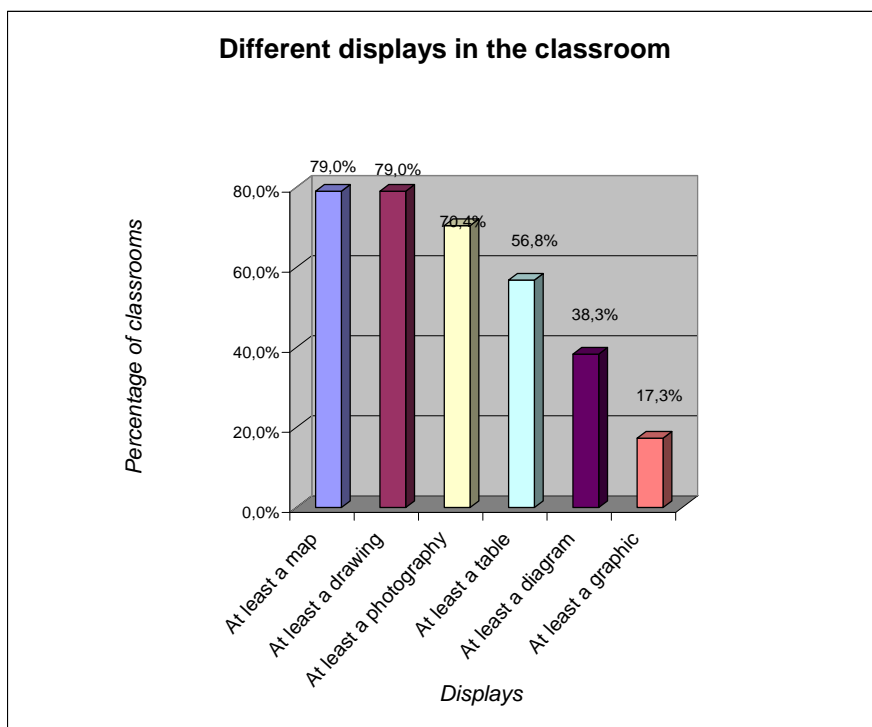
Table 6

4. Display in the classroom

The kinds of displays used by teachers in the classroom were investigated as follows:

Please give in the space provided the number of elements displayed on the class-room walls, visible for the pupils, at the precise moment of filling in this form :

- tables / grids _____
- diagrams _____
- maps _____
- graphs _____
- children's drawings, reproductions of artists' works _____
- photos _____
- others (indicate the nature) _____



Graphic 2

These results show that in the classes in this sample the teachers use some display such as types of table, scheme which increases the variety of semiotic registers utilised in the resolution of mathematical problems. But we noticed that only 17% of the classrooms use display such as graphics. However we do not have here any information about the relation between these two elements: the use of displays and solving problems.

Conclusion

The objet of our study is to locate the use of semiotic registers of representation and expression in the teaching-learning situations proposed by mathematics teachers in French primary schools. The results that we have obtained come from responses to a questionnaire given to a sample of 81 teachers training a total of 1081 pupils of age 8 to 9 years old, in classes of CE2 (3rd year in French primary school).

It emerged that 79% of the classes (see table 3) use mathematical books or working files and in these actual mathematical books or working files, 68% of the exercises which we analysed are presented with a variety of representations (see table 1). The pupils thus have to learn to deal with the problems when some of the essential information is not in the written text. Only 10% of the 81 teachers proposed a daily review and solution during school time. The style of correction could suggest that the variability of semiotic registers of representation and expression is taken into account, but we noticed that only 2,6% of the teachers systematically introduces a different form of solving than the one(s) proposed by the pupils during the collective correction (see table 5). We also noticed that 57% of the classrooms used at least a table in the display but that only 17% use at least a graph (see graphic 2). However, the written practice of problem resolution by the teacher at the time of the preparation of the didactic sequences was relatively neglected in favour of a “mental resolution” which does not provide the opportunity for comparison and use of explicit semiotic registers. A more careful analysis of the data will ultimately allow other richer aspects of the use of these semiotic registers to emerge.

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VERBALIZATION AS A MEDIATOR BETWEEN FIGURAL AND THEORETICAL ASPECTS

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Abstract: *The study reported in this paper concerns the dialectic relationship between the figural register and the natural language register when students try to solve plane geometry problems. I will present a theoretical framework and some preliminary results concerning the following problem: how and to what extent does natural language act as a mediator and a control tool between the “operational handling” (Duval) of the drawing and the theoretical reference (in our case, Euclidian geometry)?*

1. The theoretical framework and the research hypotheses

In geometry, since we deal with theoretical objects and their representations, we need to state what we mean by “drawing”, “figure” and “geometric object”. Even if in the literature it is not usual to consider these elements separately, after Fischbein, Parzysz and Laborde’s definitions I think it becomes necessary. So, I considered the definitions given by Parzysz (1988) and by Laborde and Capponi (1994).

Parzysz suggests that “*the FIGURE is the geometrical object which is described by the text defining it*” and “*The figure is most often REPRESENTED*” (Parzysz pg. 80). Parzysz calls “drawing” the illustration of a figure.

Referring to Parzysz’ elaboration, Laborde and Capponi propose the following definition: “*Drawing can be considered as a signifier of a theoretical reference (an object of a geometric theory, like Euclidean Geometry or Projective Geometry). A geometric figure involves the joining of a given reference to all of its drawings: it can be defined as the set of all couples which have the reference as the first term, while the second term belongs to the universe of all possible drawings of the reference*”.

Referring to the abovementioned elaborations, from now on I will consider “*geometric object*” the *object of a geometric theory related to a definition*. The “*description*” will be the *verbal presentation of the “geometric object”* (i.e. the text of the definition). By “*drawing*” I then mean *one of the different graphical expressions of the definition itself*.

I can now define the “figure” (F) as the set of couples made up by the geometrical object (O) and one among the drawings (d_i) that are material representations of that geometrical object (O):

$$F = \{(O, d_1), (O, d_2), (O, d_3), \dots (O, d_i)\}.$$

In this way the theoretical aspect is linked to the graphic one and a kind of bridge is established between them.

The differences (and relations) between drawing, figure and geometrical object play a very important role in handling the drawing when trying to solve a plane geometry problem. Therefore, we adopted the “operational handling” of a drawing considered in Duval's theory (Duval, 1994): operational handling of drawing (“*appréhension opératoire*”¹) involves an immediate perception of the drawing and its different variations (“*mereologiques*”, optical or of position)².

Our research concentrates mainly on analysing the influence of natural language on the relationship between the operational handling of drawings and the theoretical reference to which it is related. We define “theoretical reference” in a given geometric theory as theorems and definitions of that theory, which are related to the figure by the student who is solving the problem. Since solving a plane geometry problem involves reciprocal relationships between drawing and theory, we note a two-way relationship between the handling of the drawing and the choosing of a particular theoretical reference: choosing a particular theoretical reference leads to the operational handling of the drawing and vice versa the drawing operational handling can suggest how to choose a particular theoretical reference.

Carrying on this idea the following general hypothesis can be formulated:

H) The relationship described above, between the drawing and the theoretical reference, is guided and controlled by the natural language.

Testing this general hypothesis is not an easy task. A preliminary study is needed in order to identify the functions of the natural language in geometrical problem solving and specify what guide and control function mean. In order to perform this investigation it is necessary to develop an appropriate Research methodology.

¹ « L’appréhension opératoire est l’appréhension d’une figure en ses différents modifications ».

² A variation is called “*mereologique*” when it divides the drawing into parts; it is designated as *optical* if it is an enlargement or a reduction of the drawing ; it is called *positional* when the figure background changes position.

A preliminary investigation about these issues is reported in this paper.

I will elaborate a research methodology suitable to investigate the function of the natural language in geometrical problem solving. Moreover, I will present preliminary results intended to support general hypothesis H).

1.1 Research methodology

In order to analyse the function of language I considered a situation of communication between two students who were solving the problem. Communication is necessary to address to an interlocutor, but, at the same time, just because we address ourselves to an interlocutor, language can support a conceptual evolution.

The research's hypothesis immediately pose a problem concerning research methodology: How can we access the students' solving process? Basically, I tried to elaborate a methodology that should be suitable for tackling this problem. Such a problem depends on the fact that language plays two different roles: it is a tool for the researcher (as a "revealer" of students' processes) and, at the same time, it is a tool for students, because they use it to solve the problem (and our inquiry concerns its role in the solving process). We elaborated a model for analysing protocols, based on the use of language as a revealer, which allowed us to point out the role of language as a problem-solving tool for students. The model is based on the assumption that solving processes are mainly expressed through two registers: the linguistic and the figural. Then, the model distinguishes between two strategies: one developing from the figure drawn after reading the text of the problem, and the other developing from the question posed in the text or from the sub-questions obtained by transforming that question.

We named these strategies "*drawing strategy*" and "*discourse strategy*".

The "*drawing strategy*" involves handling the drawing (in Duval's sense of operational handling of drawing), or its perceptive apprehension (cf. Duval, 1994) in order to construct a "work environment" by means of a list of information

The "*discourse strategy*" consists of a structured sequence of questions, starting from the question of the text; or from some key words taken from the text, by talking with a schoolmate, or with a teacher; or from a key configuration isolated in the drawing.

So, the “discourse strategy” is closely linked to the text of the problem but, on the contrary, the “drawing strategy” is not strictly related to the text.

Both strategies may intervene in the same student’s solution.

- *Aim of strategies*

If the “discourse strategy” consists of a structured sequence of questions, its aim is a structured sequence of answers. On the contrary, the aim of the “drawing strategy” is collecting information starting from the drawing or by acting on the drawing itself. So, the change of aim in the procedure is the key element that reveals the intention to go on to another solving strategy. The “discourse strategy” usually is a part of a deductive strategy, in which the aim is to prove something. On the contrary, the aim of the “drawing strategy” isn’t proving (indeed this strategy is used to create a set of information that constitutes the working environment).

- *Criteria for distinguishing between the two strategies*

We now try to provide some criteria that are useful for recognising a “drawing strategy”. In detail, language makes it possible to recognise this strategy when we can detect:

- Words that refer to perception, such as “*you can see that...*”
- Words and adverbs indicating space, such “*here, there,*” besides demonstrative adjective or pronouns, such “*this (one), that (one), ...*”, accompanied by gestures;
- The present tense recurring frequently
- A descriptive rather than deductive discourse, without any connection linking the information in the list.
- Unjustified inferences: they carry the formal shape of ordinary inferences “*since we know that..., then it follows necessarily that...*” But the term “*necessarily*” introduces perceptive evidence and takes the place of “*then, since...*” The following is an example containing some of the above-mentioned inferences, made at the drawing level.

123. Taina: “because, since we have OD diagonal, I go on tracing the OD line, then we have the parallel, no, the perpendicular, which is AE, since it is a circle, since we know that OA, OD and OE are circle radii and that AO is equal to AD and that OE is also equal to AD, then necessarily DE is equal too”.

We now try to provide some rules that are useful for recognising a “discourse strategy”:

- The variation in the use of verb forms and tenses. For instance a sentence like “*we should be able to demonstrate that*” points out an attempt to get out of the solving procedure, in order to provide a plan of it.
- The complexity of sentences: coordination between several complete propositions.
- The “final”³ structure of a sentence as “*to have...it is necessary that...*” This structure allows us to determine the theoretical reference that guided the answers to the questions.
- The presence of “key words” such as “*perpendicular*” or “*isosceles triangle, height, medians*”. These words play a key role for the subject, which refers back to a concept belonging to his/her knowledge system. Therefore, these words are a kind of bridge between the subject's knowledge and the text of the problem or the discussion with some schoolmates. Let's take a look at the word “parallelogram”, for example: it reminds the subject of the quadrilateral figure, then the student will relate it to all the theorems and properties defining it which are part of his/her knowledge system, thus becoming capable of handling the drawing.
 - The presence of “*Key configurations*”, which is recognised and isolated by the subject in the drawing.
 - The deductive structure of discourse (the premises of a deductive step is the conclusion of the preceding one)

2. Experimental situation and early research results

As pointed out above, the object of our research is the functions of language in the link between the operational handling of a drawing and the theoretical reference to which it is related. So, our aim is to analyse the students' oral and written texts using the models of the “drawing strategy” and the “discourse strategy” in order to point out the function of the natural language. As an early result of our analysis we were able to identify various behaviours which we called “action models”, in the students’ solving processes. Such models put some specific functions of the natural language into evidence. A short description of the first experiment performed is presented, followed by the early results.

³ This term indicates that the subject begins to search for the “cause” starting from the “effect” (consequence). The action focuses on a search of the theoretical reference to reach the “effect” that, in the specific case of our pre-experimentation, is the rhombus.

2.1 The experimental situation

We performed a preliminary experiment involving an Italian Scientific Upper School and French Grade X students (14-15 year old). They worked in pairs, trying to solve a plane geometry problem involving geometrical objects already studied by the students in the middle school. It is necessary to underline that there is a substantial difference between the syllabi about geometry teaching in France and in Italy. Briefly, we can say that France is teaching the geometry of the transformations while Italy is teaching the Euclidean geometry.

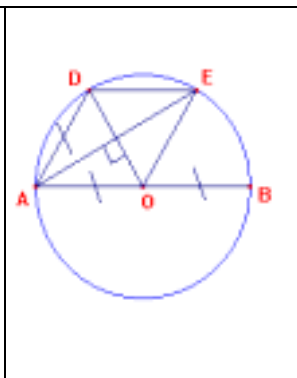
Audio and video-recordings as well as students' written texts were collected.

Task

Given a circle C ; its centre O ; its diameter AB ; D is a point on this circle, so that $AD = AO$.

The perpendicular to DO through A meets the circle C again in point E .

Prove that $OADE$ is a rhombus.



During the experimentation different versions of this task were proposed, for instance, without drawing or with different data. We are presenting now only the above version.

We can consider the necessity to represent a mathematical object and several semiotic systems from the duality of cognitive modes: images and language, to the symbolic algebraic writing notations. Now, to represent the geometrical object we consider the linguistic register and the figural register as semiotic systems. Within a semiotic system we have the representations of geometry objects as compositions of signs. So that for any geometrical object we can have different representations produced by different semiotic systems. However, this variety of semiotic systems raises coordination problems. The code activity represents a coordination system between the conceptual aspects and the figural aspects. For this reasons the problem given by the form of drawing and statement forces students to coordinate two registers: the figural and the linguistic ones. As Duval said “the conceptual understanding is possible when such a coordination for mathematical objects are not confused with content of representation” (Duval, 2000)

2.2 Early research results

We identified several "verbal action models" implemented by pupils: some of them will be described.

It is within the context of these models that we are trying to determine the functions of natural language as a problem-solving tool.

Action models in the "drawing strategy"

Among the results obtained by analysing the students' "drawing strategy", there is one action model involving the creation of a list of information. This list can be analysed in two different ways: as a simple list of information or as a list where some information can be connected to the drawing interpretation or to an inference. Let's use a French student's work as an example of simple list (Taina worked with Sophie - see later for additional excerpts).

40. Taina: diagonals AE and OD cut each other in their middle point, making a right angle, and AO is equal to EO, EO is equal to AD.

The information in the list is, obviously:

- 1) Diagonals AE and OD intersect each other in their middle point.
- 2) They both make a right angle
- 3) AO is equal to EO
- 4) EO is equal to AD.

As we can see, the information in the list is not related to each other.

How and where do students get the information (theoretical references, geometrical relations, properties, etc.) for making their own list? We already said that the information in a list can be collected from the drawing, through operational handling or through the perception of it, but it can also be collected through implicit or explicit inferences. It is necessary to underline that these inferences are not linked together by a deductive development of the discourse, because they are tools to add new information to the list. Here are two examples of second type list: Taina - Sophie and Gaelle - Camille:

– Explicit inference:

59. Sophie: Look! AO is a radius of the circle and EO is a radius of the circle too (*this information comes out from the drawing interpretation field*⁴)
60. Taina: then, AO is equal to EO too
61. Sophie: and AO is equal to AD too
62. Taina: so, and AO is equal to AD, so AD is equal to OE

⁴ The drawing interpretation field it was defined by Laborde as the set of spatial drawing's properties which are related to the geometrical properties of the object. (Laborde and Capponi, 1994, pp. 171 – 172)

– Implicit inference:

36. Gaëlle: maybe, look! this one is symmetrical to this one (*OA and DE*) then it is the same.

The above inference is implicit, since it comes out from the drawing perception and not from the transition towards a deductive procedure, even if the connector “then” occurs. We can see some revealing signs of the “drawing strategy” in the action model of the list:

- The verb “*to look at*”, related to a perception of the figure (interventions 59 and 36)
- Deictics such as “*this one*” are present. They take the place of the object name as, for instance, in intervention (36) where the segment AO and the segment DE are not named but indicated by gestures and by the deictic “this one”
- The recurring present tense
- A descriptive discourse is present in (36) with the adjectives “symmetrical” and “the same” which come from the drawing perception. At the same, in (40) “*diagonals AE and OD cut each other in their middle point*” is a descriptive discourse deriving from the drawing perception.

Experience shows that students try to handle the list whenever it becomes too long to be managed. Such handling involves some operations geared to modify the list. These operations include: putting the information in the correct order, picking up useful information and leaving useless information off the list through inference, adding some information to the list. For instance, the following dialogue is an example of how information is deleted from the list:

<p>26 O: But ...wait, ...look: this one is equal to that one ($AD = AO$)</p> <p>27 D: NO, but... look: OD is equal to OE, which is equal to OA, because they are radii of the circle, three radii...then... since this one is equal to that one ($OD = OE$) and OE is equal to AO, then DE is equal to OE</p>	<p>Intervention 26/27</p> <p>List:</p> <p>C₁: $AD = AO$</p> <p>C₂: OE radius</p> <p>C₃: AO radius</p> <p>By inference we can get to the information</p> <p>C₄: $OE = AO$</p>
<p>31 D: Since in the text they say that AD is equal to OA, and that OE is a radius, then it (OE) is equal to AO, because AO is a radius. Then OE is equal to OA, which is equal to AD.</p>	<p>Intervention 31</p> <p>Handled list (useless information C₂, C₃ kept out of the list through inference):</p> <p>C₄: $OE = AO$</p> <p>C₁: $AO = AD$</p>

In order to point out the functions played by language in the drawing strategy and its handling in the relationship between the operational handling and the theoretical reference, let's see the results of analysing the student's oral and written work:

- 1) The ordering function of language (language as an “organiser”): there is no order in the information carried by drawing, because it is global and two-dimensional (the operational handling of the drawing, doesn't give any ordered information). On the contrary, language is straight and sequential and because of these qualities the information must come out in order.
- 2) The selective memory function of language (language as a selective memory tool), which makes it possible to select only the useful part of the information given in the drawing. The drawing has everything, but there's even too much! Therefore we need to select the information and thus build a system to keep such data in mind. (For instance, deleting an useless information: see the interventions 26(31))
- 3) The function of control in handling the drawing: the presence of language in handling the figure is a tool for controlling the entire operation. (For an example, see the interventions 36(46))

Action models in the “Discourse strategy”

Based on the results obtained by analysing the students' output related to the “discourse strategy”, two action models were identified: one related to the discourse procedure starting from the question of the problem, and the other related to the procedure started by key words (or by key configurations). The latter will be described in detail in this report while the former will not be discussed at this time.

The *key words* act like a kind of label and carry out two functions: they let students recall a particular theoretical reference and they can refer to a linking concept, by which it is possible to switch to another theoretical field, leaving the given one (such a situation is not described in this report).

The action model that refers to *key words* that can be used to recall the theoretical reference needed for the solution. In this action the word is associated to the concept⁵, that is the word recalls the concept starting from a

⁵I adopted the Vergnaud's definition of concept. The concept is composed by three elements:

- The situations which gives signification to the concept (in our case, the problem)
- The invariants by which the “schemes” of action act (in our case the geometrical properties)
- The linguistic and non-linguistic forms by which the concept is symbolically represented (in our case the drawing, the definition...)

figural aspect up to a link with a theoretical aspect, or starting from the theoretical aspect up to a link with a figural aspect. Usually, the associative operations are started by pronouncing a word or by reading it. The concept allows us to consider particular geometrical objects. In this sense, based on the definition of Parallelogram, we can consider two equal segments which are also parallel segments. This defines a set of information that must be found again in the drawing by handling it: we need to identify two opposite and parallel segments.

We can identify a “Key configuration” in the same way. This configuration acts as a sort of label to recall a particular concept obviously starting from the figural aspect.

Here is an example of how *key words* work:

36. Gaelle: maybe we can prove... well, look at it! This one is symmetrical to the other one (AO and DE), so it is the same.
37. Camille: and then?
38. Gaelle: and then we should be able to prove that (*this is a meta discourse*) it is parallel to that one there (*AO parallel to DE*).
39. Camille: yes, but what we have to say is that this one is the middle point (*the diagonals intersection*).
- ...it is the middle point of this one and of that one (*DO and AE*)...wait! AO is equal to AD... and *what if we could prove that (meta discourse)* triangle DAO is isosceles? Because, you know, it is important with reference to this one (*DO*).
42. Gaelle: yes, because it is the height.
43. Camille: yes, it is the height.
44. Gaelle: Yes, it is also the median ...Yeees!!! it is the median!!!
45. Camille: and this means that it is an isosceles triangle because the height is equal to the median... AE is perpendicular to OD and AH is the height in the triangle ADO (*H intersection of the two diagonals*).
46. Gaelle: then AE cuts OD in the middle.

The sequence of these pieces of information (isosceles triangle, height and medians, as underlined in the text) is the standard sequence of the properties by which an isosceles triangle is described in France. After naming the medians, Gaelle realizes that it is connected to "middle point"; then she relates this word to the theoretical reference, the isosceles triangle, and then she goes on to the discourse procedure.

We can see some revealing signs of the “discourse strategy” in the “key word” action model:

- The variation in the use of verb forms and tenses: for example in intervention (38) “...*we should be able to prove that*”
- The complexity of sentences by the coordination between several complete propositions: for example in interventions (39) “*what if we could prove that...*” or (45) “*this means that...because...and...*”

- The deductive structure of the discourse: for example in interventions (45-46) “*this means that it is an isosceles triangle...then AE cuts OD in the middle*”
- The presence of “key words”: “*isosceles triangle, height, median*”

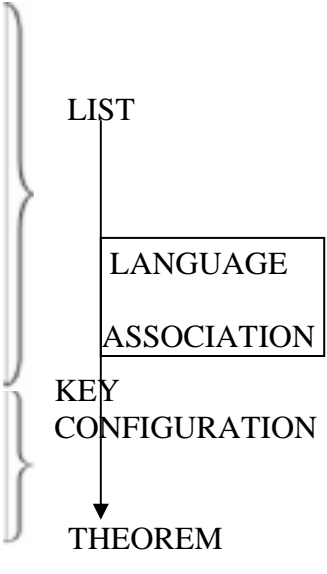
We notice that the meta-discourse usually reveals the transition to a new strategy (see the above-mentioned interventions no. 38 and no. 41) and it plays a function of control on the solving procedure.

The dialogue reported above is an example showing quite clearly how a change of aim is decisive in transforming the descriptive structure of the discourse into a deductive one, to go on to the discourse strategy. The language functions support the change of aim, so the language association function supports the passage from a list of information to a theoretical reference, as we can see in the following paragraph.

An example of the passage from a “drawing strategy” to a “discourse strategy”

We will describe a language protocol extract in which it is possible to identify the passage from the handling manipulation of the drawing to the choosing of a particular theoretical reference. The handling manipulation of the drawing produces a list of information, while the evocation of a particular theoretical reference is possible on the base of a “key configuration” action model. The language function, which supports the evocation of a particular theoretical reference starting from an action model of a list and a “key configuration”, is the language association function. Briefly, this language function allows the association of a Key configuration (key word) to a concept.

Here is an example of how the *language association function* works:

<p>1 E: hypothesis: AO is a radius of the circle, OE is a radius of the circle, and AD is equal to AO. H is the intersection point between DO and AE. AHO is a rectangle triangle but...AOD is an isosceles triangle, not an equilateral triangle, so the triangle AHO is a particular triangle with angles of 30°, 60° and 90°.</p> <p>2 A: hypothesis: all of these triangles (<i>AHO, AHD, DHE, OHE</i>) are equal among themselves because AD is equal to AO, then AO is equal to OE because they are radii and ED is equal to the radii because we have demonstrated it.... and then?</p> <p>3 E: <u>DH and HO are equal and AH is equal to HE too</u> so hence, ... there is a <u>theorem about the diagonals of the parallelogram which cuts them in their middle point</u>. Is it so?</p> <p>4 A: Yes, “if the diagonals of a quadrilateral cut each other in their middle point, then it is a parallelogram”</p> <p>5 E: Ok, and if a parallelogram has four equal sides, then it is a rhombus!</p>	 <pre> graph TD LIST --> LA[LANGUAGE ASSOCIATION] LA --> KC[KEY CONFIGURATION] KC --> THEOREM </pre>
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The interventions (1) and (2) are examples of the list of information, on the contrary, the interventions (3), (4) and (5) concern a theoretical reference (“*if the diagonals of a quadrilateral cut each other in their middle point, then it is a parallelogram*”) in a deductive structure of the discourse. We can recognize an operational handling of the drawing in the sub-configuration of rectangle triangles and an inference by which the information “*AOD equilateral triangle*” is obtained.

In intervention (3) the theorem (theoretical reference) is evoked. How do students evoke the theorem starting from the list?

I think that the key element, which works as a bridge, is the configuration of diagonals, which cut each other in their middle point. The configuration is a sort of label, which makes possible to evocate a theoretical aspect (the theorem) of the concept linked to the parallelogram. For this reasons this label is a “key configuration”. The function of language here is that of an association function, which makes possible to link the figural aspect of the parallelogram to the theoretical aspect (mentioned theorem).

3. Conclusion

Up to now, results seem to show that natural language plays a really important role of mediation between the handling of the drawing and the theoretical reference in plane geometry problem solving. By using the set of criteria elaborated to recognize students' strategies (the drawing strategy and the discourse strategy), I was able to identify different functions that the natural language seems to have assumed in students' problem solving. In particular, I was able to partly describe the function of mediation of natural language and give a list of some other functions. Among these: the association function, the guide function, the planning function and the control function.

It is within the context of action models that we are trying to determine and to put into evidence the functions of natural language as a problem-solving tool.

The first experiment I carried out suggests that further steps in the research project should be made at a micro-analysis level concerning the identification of functions of the natural language by considering students' "action models" related to their macro-strategies. This analysis should make it possible to find the appropriate area where the teacher can intervene in students' problem-solving activities.

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FROM ICONS TO SYMBOLS: REFLECTIONS ON THE HISTORICAL DEVELOPMENT OF THE LANGUAGE OF ALGEBRA

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***Abstract:** This paper examines the ways in which the language of algebra has developed through the problems arising from iconic representation in elementary arithmetic and geometry. The ability of a language to express, explain and integrate new ideas is discussed, some examples are drawn from historical material, and the idea of “semiotic condensation” is proposed as a way of thinking about teaching and research.*

1. The Languages of Mathematics

Algebra is a powerful and expressive medium which lives in the space between arithmetic and geometry, abstracting the essences of arithmetic and geometric objects and providing a means of transformation from one kind of reality to the other. It hides the complexity of the processes which have developed over thousands of years and which now enable us to formulate sophisticated mathematical models to aid the solution of a wide range of problems. This facility of modelling has enabled different cultures to cope with increasingly complex human problems; to be able to classify problem situations and abstract from them essential aspects which lead to methods for their solution. In doing this we have invented written representations of objects, operations, relations and procedures, enabling the development of abstract concepts which take us far from the physical world and into the intellectual world of pure mathematics. The development of algebraic language in its many forms, is a vital and fundamental part of human endeavour.

In order to examine the development of algebraic language, I propose a working definition of language as “an infinitely developable symbol system”. The term “symbol system” is inclusive of context, discourse, and genre and refers to general forms of text. Thus I concur with Pimm (1995) that mathematics is a specialised language with its own contexts, metaphors, symbol systems and purposes. Furthermore, I aim to show that there is not just one “mathematical language” but a number of different languages which have arisen at different times, each having its own special purpose and its own particular

rules and domain of application. Mathematical language in its broadest sense therefore involves a number of symbol systems, each arising from attempts to overcome particular problematic situations. Identifying these different languages can help to reveal situations where the variation from one language to another causes difficulties for students. According to Kvasz (2000) as a mathematical language develops, it has the following aspects, to a greater or lesser degree:

- a) *Logical Power* concerns the ability of the language to develop general procedures and proofs and make it possible to create a concept of reality independent of the language itself, against which results can be verified. *Logical Boundaries* occur when this is not possible. For example, the language of elementary arithmetic does not contain variables, so all problems have to be formulated with concrete numbers and tested against real situations. There is no independent, general procedure whereby calculations can be tested.
- b) *Expressive Power* is the ability to express concepts, formulate problems and provide models arising from a given problem situation. *Expressive boundaries* happen where the language is unable to express certain problematic situations. To cope with new problems, elementary arithmetic (of positive integers) has been gradually extended to include fractions, negative, non rational, and complex numbers.
- c) *Explanatory Power* concerns the ability to explain whether or not a problem can be solved by referring to the logical structure of the language or to the ontological basis of the concepts. Hence the full explanation of why and how “imaginary” numbers “worked” had to wait until the concept of complex numbers was developed and integrated into the current algebraic language.
- d) *Integrative Power*: as a language develops in response to the growing number of problems it meets in different contexts, it gradually evolves the ability to create universal analytic methods and so to include a wide range of problems.

2. 2. Icons, Indexes and Symbols

Recent interpretation of the available archaeological evidence suggests that over time we have created icons, used indexes and developed symbols which first replace and later become the objects of thought (Deacon 1997). In our case mathematical objects are represented by icons, indexes and symbols which we use as tools to develop processes whereby we describe and manipulate the world. The distinction between icons, indexes and symbols is a subtle one. On one level, an icon can be taken to represent the object itself. For example, a passport photograph is iconic of the person it represents. The interpretive

process that generates iconic reference is what we call recognition. The word “re-cognition” means thinking about something again, and “re-presentation” is to present something again. Iconic relationships are the most basic means by which things can be “re-presented”, and hence “re-cognised”.

The next level of sophistication is indexical. The function of indexes are to associate one thing with another. Numerals on a scale can indicate temperature, pressure, consumption of fuel, etc., and we learn to associate the smell of smoke with fire, and so on. Our indexical competence is constructed from a set of relationships between icons, and indexical interpretation is accomplished by using iconic relationships to interpret new stimuli.

Symbols¹ denote objects, operations, relations, and other situations, but the nature of symbols means that they also have connotations (Barthes 1973). For symbols there is some form of social connection; a convention, a tacit agreement or an explicit code which makes the link between what we see and its deeper meaning. For example, the algebraic expression $ax^2 + bx + c$, essentially denotes a series of instructions to take some numbers and operate on them according to particular arithmetical rules. However, the connotations involved indicate that the individual elements have different meanings; “ x ” stands for a variable, usually a real number; “ a ”, “ b ” and “ c ” are parameters; the “ $+$ ” sign means addition in the field of real numbers; the whole expression is a quadratic form which can be re-presented as a parabola; the curve is one of the conic sections, and so on. We often claim that algebraic and graphical representations denote the “same” mathematical object; however, this kind of symbolic identity only arrives with the greater sophistication of written text and the gradual accretion of the accompanying conventions. The situation is even more confusing for students when they meet the expression $(x - a)(x - b)$ which denotes the same object, but is a change of connotation because it focusses on different algebraic properties of the object. Algebraic transformations are invariant with respect to denotation of the symbolic expression they act upon because they can change the connotation but not the denotation of the symbolic expression itself. The converse is not true; if two different expressions have the same denotation, they are not always reducible to each other by algebraic transformations. For example the two equations $x^2 - 4 = 0$ and $x^2 + 5 = 0$ both denote zero, but have very different connotations.

While denotation concerns the first, immediate level of meaning, connotation, the second level, is much more complex. This second level of meaning Barthes calls the “myth”. Any analysis of the second level meanings involves an explanation of the semiotic process whereby the myths have been established, and of course different people may see different meanings, and hence interpret different myths. Failure to appreciate the semiotic content of symbols leads to problems when students try to make a transformation from one

form to another (Duval 2000). Since the symbol system has been developed historically, it is important for students and teachers to have some appreciation of what this involves and how we came by the symbols we use today².

3. Mathematics in History

The history of algebra has been popularly divided into three periods: “Rhetorical” where problems are expressed in words, and solutions given as instructions, as in Egyptian and Babylonian mathematics; “Syncopated” where some symbols are beginning to be used as in the early Renaissance; and “Symbolic” where algebra appears in the time of Viete and Descartes, rather like we write it today. These three stages defined a progression from some kind of “primitive” state to a more sophisticated way of dealing with problems, but even a brief investigation into historical detail will show that this interpretation is much too simplistic.

Mathematics, like any other human enterprise develops in “communities of practice” (Wenger 1998, Adler, 2000), and within these communities we find an evolution of shared meanings. Newcomers are initiated into the practices which are passed on from one generation to another. Not all of these meanings or variations of practice are written down, but are shared as “tacit knowledge” by the community (Polanyi 1964). However, not all people share the same meanings and so variations occur where new problems can be seen from different points of view.

Illustrating the development of mathematics by reference to iconic, indexical and symbolic representations has its problems. The narrative form of this paper means that conventions of writing arrange ideas in a certain order, which then implies that there is some kind of progression from one idea to another. While this may be true in one sense, it is important to realise that cognitive development is holistic, and while there may be some necessary stages, rarely is it the case that this development can be seen as a series of distinct phases where those involved pass through and leave behind one phase before reaching the next.

The problem is that no objects in themselves are icons, indices or symbols. They are interpreted as such according to what is produced in response to someone perceiving them. This means that the differences between iconic, indexical or symbolic relationships derive from regarding things either with respect to their form, their correlations with other things, or their involvement in systems of conventional relationships or practices. We continually act in the flow of meaning which encompasses all of these modes. The use we make of

these ideas depends on our purpose, and so the definition and exemplification of these terms in mathematics (as also in everyday language) is inherently ambiguous.

4. The Invention of Icons

In the Ancient Near East, iconic representation originated in Neolithic times. Small stones were used to represent objects for the purpose of calculation. These gave way to clay models representing the objects, animals, or products which were the subject of the calculations. There now exists a considerable amount of evidence (Friberg 1984, Nissen et. al. 1993) to indicate that the first kind of writing invented was not for any literary or religious purpose, but as numerical symbols. The principal motivation for the invention of writing was for the purpose of economic administration which included the development of number systems and ways of calculation. As writing developed, it took over a wide variety of representations which depended on the ways in which objects were counted, and the qualitative significance of the objects themselves within the society.

So the first mathematical language was an arithmetic based on iconic representations, and aspects of this can be seen in “Pythagorean” arithmetic. Pythagorean representations consist of patterns of dots for odd and even numbers, triangular and square numbers, etc., which provided visual justification for the establishment of basic arithmetical relationships. This kind of iconic representation led to the belief that all objects were composed of a large but finite number of elements. In this respect, the iconic representation of integers failed to solve the problem of representation of nonrational numbers, so this language failed in its expressive power because it was unable to express exactly the situation represented by the diagonal of the square. Fowler (1997) remarks that the translation of the Greek term for these numbers ought to be “inexpressible” (not “incommensurable”).

Another kind of problem solving was motivated by ritual practices. Siedenberg (1962) describes how the famous problem of doubling the size of the altar originated in the Vedic rituals of ancient India. In the geometry of classical Greece, we have a sophisticated system of iconic representation of variations of this problem in the transformation of areas as developed in Euclid Book II.

Knorr (1986) and Netz (1999) have examined the tradition of solving geometric problems in ancient Greek mathematics, and claim that *geometric icons have ontological primacy* since they were the first means of re-presenting a problem situation. The ways we talk about diagrams and instruct people to

draw them embody a number of verbal conventions, and so diagrams are a referent for the text. The truth value of an assertion lies in the relation between the assertion and the diagram, and the identity of the object rests with the visual faculty. For example, for the straight line AB , the relation $AB = BA$ is “obvious” as can easily be seen from a diagram.

So in the geometrical icon there exists an inherent kind of generality where the naming of objects and their relations is based on their *visual* status, and the identity of geometrical objects is confirmed by visual and not symbolic means. Today we rely on “capturing the beast with a nest of verbal formulation” (Netz 1999 p. 38) which is exactly the process described in Lakatos (1976) where the ontological primacy of the physical object referred to by the term “polyhedron” is challenged, and strings of symbols are created to replace it.

The important role of the diagram in relation to the accompanying assertions in the text can be seen in many examples in the development of algebra from Al Khowarizmi to Cardano and beyond. We can also see this ontological primacy of the diagram in the development of Projective Geometry which arose from the practical art of Perspective and where the early proof methods relied upon “projection and section”.

5. Classification, Systematisation and the Development of Algorithms

Very gradually, as well as the representations of the things, we have descriptions of what can be done to or with the things. These descriptions are initially in terms of “action metaphors” describing the operations on the objects, and are *indexical* because they indicate what is to be enacted. As the level of sophistication grew, the procedural character of many of the calculations was gradually recognised, and so methods which were seen to be similar evolved into more general techniques which became applicable outside their original domain. However, it was not possible to use this iconic language of elementary arithmetic to express unity or order. In the early stages, the classification of problems was based on their practical use in a specific context.

The conjunction of the classification of problems by context and the gradual use of a more universal system of calculation led to the development of a systematic approach which can be found in texts of the early second millennium BC. The usual instructions to the student were “we do it like this...”. But it was still the case that the language of elementary arithmetic did not make it possible to create a concept of reality independent of the language itself, against which results could be verified. Two different methods applied to the same problem could come up with different results, but this was avoided by

everybody using the same method. For example, a method given for measuring the area of a trapezoidal field, (take the average of the two short sides and multiply by the length of the field) is correct for rectangles, but wrong in other cases.

An efficient notation is not always necessary for the development of algorithms. In Babylonian mathematics the instructions for finding the sides of a rectangle given the sum and the product of two adjacent sides appears in many general histories of mathematics. The purpose of these algorithms for scribal practice is now well documented (Robson 1996) and the exercises were carefully arranged to give integer solutions. Negative solutions and other problematic situations were not shown, and the inability to do some calculations within the arithmetic system of positive integers and fractions eventually led to development of a new language which could express these problems more clearly. For example, the tablet which shows a calculation of the diagonal of a square, while not giving the arithmetic algorithm explicitly, suggests a realisation of an infinite process which had to be approximated for practical purposes. (Resnikoff and Wells, 1973). Much later, problems of this kind were expressed in a geometrical language which was still iconic, but enabled a distinction to be made between number and magnitude.

6. Representation of Objects, Operations and Relations

Notation began as iconic representations, which were gradually decontextualised and became “the thing”, the unknown. Basic arithmetic operations and relations like addition, subtraction and equality, were initially represented by verbal expressions or their abbreviations, and as the arithmetical and writing techniques developed, these representations evolved to cover a wider and more sophisticated range of operations and relations. We can regard the development of notation as a process which combines the iconic representation of objects and the indexical representation of operations and relations. The notation becomes more efficient as the signs themselves become decontextualised from their qualitative origins and this allows abstraction and generalisation to occur. In this way, the rules for symbol manipulation become clearer (to the initiates) and this provides the potential for the development of new concepts. In contrast, as this happens, it makes participation in the meanings more difficult for any 'outsider'.

However, the verbal expressions for relations were not all superseded by written signs. In the classical geometrical tradition, the diagram is the referent for the text, and this persisted until the change from geometry to algebra as a basis for mathematical reasoning was begun in the mid nineteenth century with the 'arithmetisation of analysis'³. Relations in geometry are initially defined in a

visual sense; same, similar, perpendicular, parallel, congruent, etc., but again became more complex. As the concepts and techniques develop these relations come to be expressed in text which begins to capture the idea but which still uses the diagram as the referent.

7. Symbolism and the Reification of Concepts

Symbols have a semiotic history and are the result of the development of systems of representation which move from iconic through indexical to symbolic modes. The major period for the development of algebraic symbolism was from the latter part of the fifteenth century to about the middle of the seventeenth century and is marked by a number of important contributions beginning about the time of Pacioli's "*Summa de Arithmetica...*" (1492) and continuing to Descartes and Fermat in the 1630s whose most significant contribution was to *combine the two languages of algebra and geometry into the new language of analytic geometry*. Descartes declared that his programme was to develop a system for the easier solution of geometrical problems, and the graphs were an important referent for the written algebra. Furthermore, two significant problems had arisen in the use of both the language of elementary arithmetic and elementary geometry; these were the status of negative and imaginary numbers. However, by the 14th century the use of double entry bookkeeping began to give negative quantities a numerical "reality", so that the manipulation of negative numbers had a referent in real world experience.

In the early stages, the solutions to problems were written in prose and numerals with no consistent general notation to express operations, relations and other ideas and the proliferation of symbols in the sixteenth century to cope with the elaboration of problems became a serious obstacle to understanding.⁴ However, during this period the development of printing began to modify the way in which algebraic symbols were written since many of the complicated symbols were gradually replaced by choices from the easily available alphabet, and others (like the square root sign) quickly became 'standardised'.

This raises an interesting question as to whether, when a particular notation is introduced, the original concept could really be conveyed exactly by the new notation. For example, according to Vergnaud's (1990) theory of conceptual fields, a concept is a system which includes semiotic representations, so when you introduce a language, (or a notation) you change the concept. The system of representation chosen changes our apprehension of the original iconic representation, bringing other images of connections and contexts to mind. This can often help in broadening our concept images (Tall and Vinner 1981) but it can lead to false interpretations as shown in Hoyrup's (1994) discussion on the

varieties of discourse in 'subscientific mathematics' in the development of algebraic processes from arithmetic problems in the period up to about the ninth century AD.

Cardano (1545) advocated a general theoretical basis for the solution of equations. He used geometric demonstrations to justify his algebraic solutions for quadratic and cubic equations but was clearly uneasy since he could not supply any geometric justification for quartic solutions, nor could he find a way of expressing the new numbers in geometric terms. His application of the quadratic algorithm and the problem of the “irreducible case” for cubic and quartic equations led him to the discovery of new entities which he insisted were numbers to be “imagined”. *Thus he began to extend the domain of application of algebra from integers and fractions to negative and the new “imaginary” numbers.*

The turning point in the development of the Language of Algebra is generally recognised to be the work of Vieta who in his *In Artem Analyticam Isagoge* of 1591 demonstrated how the consistent use of letters of the alphabet could make the operations which went into the building of the terms in an expression more visible. In this work he generalised algebraic calculation and freed himself almost entirely from geometric representation. His claim that the key to solving equations is to know how they are built up in the first place was an extremely powerful principle which gave rise to new methods and new ways of conceiving the problems. After Vieta both Descartes and Newton gave algebraic solutions for geometric problems and geometric solutions for algebraic problems, and mathematicians were beginning to use the new symbolic entities and operate with them as if they were “real” quantities. However there was still a strong belief in the ontological primacy of the geometrical diagram.

8. Abstraction of General Theories

In *La Geometrie* (1637) Descartes produced a weak version of the fundamental theorem of algebra, proposing that equations would have as many roots as their highest power, but since imaginary numbers had no geometric construction and corresponded to no definite quantity, they were not included. Girard had already stated in his *L'invention Nouvelle en L'Algebre* (1629) the general principle of the fundamental theorem where every equation in x^n has n roots, but the mathematical community was not ready to make this leap into unknown territory. Finally, in his *Treatise of Algebra* (1685) Wallis showed that algebraists could not solve the irreducible case unless they were prepared to admit “imaginaries” as quantities which could be manipulated according to certain rules.

Towards the end of the eighteenth century the work of Lagrange, Vandermonde and Ruffini began to produce a generalised theory of algebraic equations. The essential activity of “re-cognising” patterns, regularities and similarities in the role of concepts and procedures was the key to this gradual process of abstraction and generalisation. In the evolution of mathematical ideas, we see that at certain stages where particular problems arise, new icons, indexes and aspects of the symbol system are employed in order to overcome the difficulties. These developments of the symbolic systems helped in their turn to develop new aspects of the languages of mathematics to enable them to become more integrative, expressive and explanatory, and gradually shift attention away from purely iconic representation into the abstract symbolic world we now inhabit as mathematicians.

It is clear that these transitions from icon through index to symbol did not occur at one time, nor is it true to say that we now do mathematics exclusively in a symbolic mode. It is also clear that we have not banished the icon altogether, for it is likely that the ontological primacy still rests with the objects we can see, or that we create on our page, or on our computer screen.

9. What lessons can we learn for the classroom?

In the process of understanding, the most attractive and immediate ways of gaining access to an idea are through iconic representations. In the early stages of learning, icons have ontological primacy and are the first way in which we begin to access deeper meanings. We use all kinds of icons with young children to represent objects, demonstrate properties, relations, and so on. This reliance on the icon never entirely leaves us; and it may be one of the most dominant aspects of the way we learn. In teaching mathematics we use many signs which re-present objects and ideas. These indexes are used to focus attention on particular relations or aspects that are significant in the mathematical sense. The “semiotic condensation” of icons and indexes into symbols is the most powerful aspect of mathematisation.

We can find many examples of this in our teaching of mathematics at all levels. Duval's work on the difficulties students of high school and university have in translating from one representation to another is indicative of the deep rooted problems students have in understanding the subtle meanings in different notations (Duval 2000). Work done by Vassakos (2001) on the problems students in the Greek Lyceum have in understanding the Dirichlet - Bourbaki definition of function deconstructs a series of aspects necessary for the understanding of this concept which rely heavily on iconic representations.

These, and other similar results indicate that serious attention should be paid to the ways in which symbols are introduced much earlier in the school system.

The motivation for the development of mathematical language was the need to tackle more sophisticated problems which arose from internal pressures concerning the nature of the mathematical problems themselves. However, the problems for which the symbols were originally developed are not now of any relevance to pupils. We seem not to be aware of these facts in our presentation of mathematics and expect pupils to be able to understand easily the semiotic significance of the different representations and symbol systems we employ. The way in which we might see an algebraic expression and a graph as “equivalent”, and the way in which we gain insights into the properties of these objects by movement from one kind of representation to another is a very important part of our mathematical awareness. The ongoing work of the 'Arithmetic to Algebra' project shows how, given interesting problem contexts and a sensitive teaching approach, young pupils can develop and manipulate meaningful mathematical symbolism of their own, and come to some understanding (albeit implicit) of the process of building a symbol system⁵ and becoming confident in the use of mathematical languages (Malara and Navarra 2001).

Notes

1. Here I make no distinction between the terms 'sign' and 'symbol' (compare Radford 2001 p. 239 and his note 2). Later, I use the term 'notation' to mean the particular choice of alphanumeric or agreed conventional mark made on the paper. A collection of marks becomes a symbolic re-presentation when it is interpreted in a particular way.
2. This is not necessarily a reference to history of mathematics in the usual sense. Vygotsky (1978 Ch. 8) talks about the development of written speech, and similarly, the development of mathematical writing is also embedded in a meaningful historical socio-cultural context.
3. However, Nunez and Lakoff (1998) raise serious questions about the popular belief in the banishment of intuition in Weierstrass' formulation of the limiting process.
4. For example Rudolff (*Die Coss* 1525) and Stifel (*Arithmetica Integra* 1544) did considerable work on equations with powers and non-rational numbers and created a great variety of symbols to express the numbers and relations between them. However, the learning of new symbols is difficult - it is easy to get confused if they are too elaborate and not clearly related to each other, and the audience already has a fairly efficient orthographic system in the Roman alphabet lettering in which the problems could be posed anyway, so

why go to the bother of learning something new when there is already a simpler system available? The payoff in terms of intellectual effort in learning a new system must be seen to be worthwhile.

5. This longitudinal project, with teachers and children in Italian Middle Schools in Belluno, aims at introducing algebraic symbolism by using a number of pedagogical devices to develop contexts and discourses where children are encouraged to work collectively on solving problems and communicate their solutions by developing notations for the objects and processes employed. In many respects this project is similar to that described by Radford (2001) whose work I have already cited.

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POSTERS

SAMPLE OF PEDAGOGICAL COMMUNICATION IN MATHEMATICS LESSON

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Keywords: *communication of students, change in character of communication, contentual aspect of communication, formal aspect of communication*

Abstract: This poster presentation deals with changes in the character of communication between students in the course of gaining pieces of knowledge.

We search for and study phenomena which, in our opinion, describe the character of students' communication. We focus our attention on the following two aspects:

- contentual (semantic) - this aspect includes especially searching for and finding an idea, deciding for a strategy; discussion about the found strategy; explanation, argumentation, abstraction; following the theme, holding attention; ...
- formal - here we observe mainly voice intensity; wording; frequency and length of pauses; affective reaction; ...

By means of the mentioned phenomena we try to discover differences between communication (a) before learning a topic and gaining new pieces of knowledge, (b) after familiarising oneself with the topic and new pieces of knowledge.

We will show the discovered differences on a sample of students' communication in the course of learning "Pythagorean theorem".

Acknowledgments: The research was supported by the Research Project GACR 406/99/D080.

THE EFFECTS OF TYPE OF SUPPORT ON CHILDREN'S THINKING WHEN TACKLING MATHEMATICAL INVESTIGATIONS

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Keywords: *image schema, cognitive objects, natural numbers, preschool*

Abstract: This study is stimulated by an interest in mathematical investigations and in their value in maths education. The aim of the study is to consider the

effects of support provided within a task on children's performance when carrying out investigations and in turn to improve the question writing process in this domain. Such support should improve access for children of all abilities, to enable them to show evidence of positive achievement and be rewarded for doing so. The idea of providing support in a written investigation is not intended to make the activity easier, but rather to make the activity more valid, prompting pupils, thus enabling them to spend more valuable time on the task. Evidently, this support should not offer a complete solution, but rather assist pupils in shaping a response. A set of parallel investigations was designed to explore differences in pupils' procedural and strategic working in response to the support provided. Types of support considered were worked and un-worked examples, suggestions for recording systems and 'clues'. A pilot study was completed in December 2000 and the full trial will be completed with 500 ten and eleven year olds in March 2001. Pilot scripts were marked against criteria developed in terms of theories of problem solving and focus on recording systems, strategies used and the influence of the support given, as well as the overall success of the solutions. The study of children's written scripts was complemented by clinical interviews allowing their conceptual as well as procedural and strategic knowledge to be explored. This poster will set out background theories and beliefs regarding investigations, the methodology, some of the results to date along with the possible implications of the results of the project.

THEORY AND PRACTICE OF TEACHING FROM PRE-SERVICE AND IN-SERVICE TEACHER EDUCATION - PHENOMENA OF IN-SERVICE PRACTICE TRAINING

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Keywords: *didactic thinking, phenomena of in-service training*

Abstract: The aim of in-service training of teachers is first of all to further develop their didactic thinking. In line with this main aim there is a series of partial objectives such as drawing teachers into participating in research, publishing the research results as well as new items, including information about math competitions, solution of topical questions as for instance evaluation, quality of text books and so on.

Summarisation and analysis of 10 years experience collected in 530 classes of in-service training in 10 districts of the Czech Republic – aimed at teachers of 1st degree of fundamental schools. There is no doubt that teachers of this category ought to undergo in-service training in math didactics. In-service training is been blocked by **a certain number of phenomena**.

Some of them will be illustrated (examples of pupils work and photo-documentation) by the poster the function of which is to provoke discussion and exchange of information.

IMPROVING TEACHERS' BELIEFS ABOUT MATHEMATICAL EDUCATION

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Keywords: *Approaches to teaching, constructivism, teacher's beliefs*

Abstract: In the Czech Republic changes in education are being prepared. These changes originate in:

- the national community needs (in 1999 these were expressed in the government document Conception of education and development of the educational system in the Czech Republic),
- from the postulates formulated in the scope of unifying school systems in Europe as an educational need for 21st century.

Our research projects proved that:

- in the Czech Republic the present mathematical (and not only mathematical) education is aimed mainly towards students' performance but does not develop their abilities and potencies accordingly, transmission and instructive approaches prevail,
- the basis of expected changes of the (mathematics) education nature is the change of the teacher's beliefs.

Experimentally, we check activities supporting the development and cultivation of teacher's beliefs about the nature of mathematical education:

Two target groups in our experiment:

A. students, future teachers of mathematics:

- In most cases we diagnosed a strong fixation to the prototype of a teacher of instructive or transmission type,

- the basic activity – guided confrontation of students with educational reality coming from the constructivist nature of mathematics education, the stress being put on the creation of the space for influencing the individual conception of teaching profession by individual students.
- B. practising teachers interested in further education (further education for practising teachers is not obligatory in the Czech Republic at the moment)
- were in most cases diagnosed as teachers of transmission or instructive type,
 - basic activity – active participation in workshops with follow-up activities for concrete teaching, teachers are guided to grasp and describe precisely didactic problems, to search independently their alternative solutions using corresponding theoretical knowledge and self-reflection of both running thinking processes and realised activities.

Our experiments and some of their outcomes will be presented on our poster.

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EULER'S THEOREM

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Keywords: *Platon's solids, Euler's theorem, polyhedrons*

Abstract: Suggestion how to introduce polyhedrons to primary school pupils: The well known solids- cube, pyramid etc., vertex, edge, face, net (surface), cube volume, shapes of solids (convex and non-convex solids), Platon's solids,

number of vertexes (v), edges (e), faces (f), Euler's theorem $f + v - e = 2$. There are also included results of research, which was aimed to realization of the above-described method. It continues in my previous work in this area, which was supported by grants of Ministry of Education of Czech Republic and grants of Palacký University.

REASONING PROCESS AND PROCESS OF CONTROL IN ALGEBRA

RECENT THEORETICAL AND EXPERIMENTAL TRENDS IN THE PROJECT "CESAME"

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Reasoning process and process of control

Writing in algebra is not (in general) just to think in its mind and then to take a pencil and write something in a sheet of paper. In elementary algebra, reasoning is made *with* signs, *on* signs, *by* signs. In the international research group CESAME we try to address this point, amongst others. In order to understand better the role of written signs in the algebraic reasoning, we were led to two related questions:

1. To what extent do students and teachers share a common opinion about the role and use of written signs?
2. What are the processes of control involved in reasoning and using of written symbols in elementary algebra?

We conceived the late question as a concrete starting point for the study. As a matter of fact, the subject's reasoning process is not easy to observe directly. Particularly in algebra, the "syntactical" reasoning (i. e. the subject's reasoning process on written symbols) is very often implicit or hidden by the use of unquestioned algorithms. This is why we chose to study the subject's processes of control, which are intimately related to the syntactical reasoning but are (slightly) easier to observe directly.

In this Poster we will not address the first question but rather we will present a preliminary experimental study aimed to collect clues on the students' processes of control when achieving a task involving some algebraic reasoning.

The circumference problem¹

NAME:

We ask you to work alone, to write your draft computations in this sheet of paper, to use ink pen, neither to use eraser nor white correcting fluid. All your ideas are interesting, please write them down in details.

Problem:

A Student² has to solve the following system:

$$\begin{cases} x^2 + xy + y^2 = 25 \\ xy = 0 \end{cases}$$

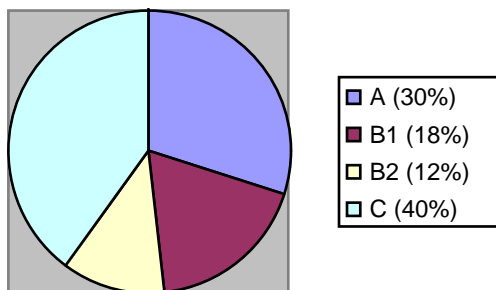
He says that the solution is the set of points such as:

$$x^2 + y^2 = 25$$

Indicate whether his solution is correct or not, giving mathematical explanations of how he could proceed.

Synthesis of the Productions of the Students from Buenos Aires (Argentina)

Percentages are of a population of College students, very beginners in the "CBC" (Common Basic Cycle) of the University of Buenos Aires.



- A. Give a correct answer (via a counterexample). They explain why Juan made a mistake, a. s. o.: 30%.

¹ We show the problem as it appeared to the students

² "Juan" in the Argentinian version

B. Say that Juan is right, but in a defective way (30%). They form two categories:

B1) Those who say that Juan done well but "did not take into account that just one of the two (x or y) must be null, he just take into account xy as a number", and stop here. These students do not succeed to integrate this into a system, to reformulate it as a *set* of solutions, set that has to preserve all the *information* and in particular the denotation³. However, their writings are correct (60% of the 30%).

B2) Those who say that Juan done well but in an incomplete way "because he did not know which (x or y) was 0" (40% of the 30%).

C. Those who produce a contradictory information but do not appear to perceive it, since they let it their writings as they are (40%)

Example of this kind of production

"Juan done well, he made a substitution: as $xy = 0$ he could substitute it in the equation $x^2 + xy + y^2 = 25$ and leaves $x^2 + y^2 = 25$.

Then, as $xy = 0$, therefore $x = 0$ or $y = 0$. In the first case we deduce that $y^2 = 25$ therefore $y = \pm 5$, and in the second $x^2 = 25$ therefore $x = \pm 5$.

In general they conclude indicating the four points, but do not see the relationship with the first answer.

Here it is interesting to note that the control may come:

- From a reasoning on the conservation of all solutions
- From consideration of the solution set, at least with a change of register (Duval, 1995) and "seeing" a circumference into $x^2 + y^2 = 25$!

Further Steps

As a second step we made interviews of students (in France and in Argentina) in order to better understand and more in details their various reasoning processes and the associated processes of control. The analysis of these interviews is a part of the ongoing study.

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SPACE IMAGINATION ON THE CUBE

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Keywords: *Space imagination, cube and motion, walking on a cube, understanding the direction ahead-to the back, type of cube model, tilting a die, regularity in tilting.*

Abstract: The poster presents some results of the space imagination research in the environment “cube and motion” carried out on 8-to-14-year old students.

The following two types of tasks were used: “Walking on a cube” and “Tilting a die”. Some phenomena are important for the solving strategy in both types of tasks. These are for example the use of movement and difficulties in understanding the direction ahead-to the back. Other phenomena are important only in one of the above mentioned tasks: The type of imagined cube model plays a role in the first task, regularity in tilting in the second one. By using these exercises a difference between boys’ and girls’ space imagination and between younger and older students’ space imagination was investigated.

GENERATING KNOWLEDGE AND MEANING TO TEACH MATHEMATICS¹

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Keywords: *Professional knowledge; Pré-service/In-service teacher education; learning difficulties in mathematics; situated knowledge*

Abstract: This work is inscribed in the scope of the investigations on the teaching and learning of concrete mathematical notions and on the idea of situated knowledge as sources for the generation of pedagogical content knowledge by the teacher. It refers to a research² that has been conducted by university professors, teachers and students of elementary school teachers' training courses in secondary level and aims at: 1) identifying and analyzing the difficulties to learn mathematics presented by students of elementary school; 2) planning and developing activities based on the reflections about the difficulties found and their implications for qualifying teachers; 3) promoting an articulation between theory and practice based on the social interaction among the subjects involved. The results of the research have to do with the type of professional knowledge produced; with the relationship between theory and practice in the qualification of teachers and with the reduction of the high levels of difficulty in mathematics learning. The investigative action has made it possible to achieve the following results: 1) the development of professional knowledge of teachers and future teachers in three relevant domains: concepts related to mathematical contents, problem-solving processes, inter-relationship among concepts, history of concepts ... (mathematical knowledge); knowledge on teaching procedures, planning of situations to approach concepts...(pedagogical knowledge); how students learn and think about specific mathematical content, what difficulties they present, what mistakes they make...(knowledge of the learners' cognition in mathematics); 2) the conduction of an proposal for initial and continuing education; 3) the improvement of learning in mathematics.

¹ Work deriving from project *Mathematical education and the necessary qualification for investigators in Mathematical Education*, developed at the Department of Mathematics Didactics, University of Seville. Financial support Sao Paulo State Agency for Research Support (FAPESP) since 09/2000 up to 06/2001.

² *Improvement of teachers' qualification in elementary school teachers' training courses (secondary school): the difficulties in learning mathematics by students in the 1st to the 4th grades as a reference for reflection*. UNESP/Brazil. Supported by FAPESP since 1998. Researchers in charge: Leny Rodrigues M. Teixeira and Vinicio de M. Santos.

STUDY OF ANTHROPO-DIDACTIC FUNCTIONS AND COGNITIVE EFFECTS OF INTERACTIONS IN THREE CONTRASTED TEACHING CONTEXTS

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Keywords: *didactical interactions - theory of didactical situations - anthropo-didactical approach*

Abstract: Which didactical functions are assumed by interactions between the teacher and pupils in mathematical teaching?

The research concerned 7 classes in elementary school (142 pupils). Each teacher did two lessons on the resolution of TTT problems (Vergnaud). There had an interval of 10 days between the lessons, and they were preceded and followed by a pre-test and a post-test comprising 22 problems involving only two numbers (lower than 10).

These interactions are in the present case approached as adaptations of modes of the teachers' action to two types of subservience: the first case, defined within anthropological limits, allows identification of a certain number of non-didactical conditions (the teachers' pedagogical conceptions, for instance); the second case, strictly didactical, permits the identification of the objective conditions of teaching. This two-way approach (anthropo-didactical) shows that the different forms of interaction can be explained as a result of the adequacy (or inadequacy) of these two types of subserviences; therefore, a unique interactive form gives evidence of very different didactical intentions and generates varying cognitive effects according to the different didactical contexts.

THE ATOMIC ANALYSIS OF THE CONCEPTUAL FIELDS: SIMILARITY (A CASE STUDY)

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Keywords: *development, understanding, similarity, solving scheme*

Abstract: Filip, a junior-secondary student, was subjected to a series of tests and observations. At the end of 1996 he has taken part in the experiment in which he

has been taught how to understand the statement: “figures have the same shape”. Three years later I have met him again. During the investigation I tried to describe actual level of his competence on the using an informal idea of similarity. The poster shows the results of the investigation.

SEMIOTIC REPRESENTATIONS IN THE PROCESS OF CONSTRUCTION OF MATHEMATICAL CONCEPT

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Keywords: *semiotic representations, construction of mathematical concept, articulation of semiotic representations, structuralist methods*

Abstract: Understanding the role of the semiotic system of representations will help us to comprehend how students construct mathematical concept. For example similarity in mathematic is one of those concepts formulated at a quite later age. Semiotic representations and their mutual transformation play a substantial role in creating the concept of similarity. A number of studies (Duval, Hitt) show that variability of semiotic representations during the genesis of the concept is an essential part of the accurate understanding of the concept. We try to identify the signs and sign representation used by Czech pupils of elementary school during different phases of the similarity concept. It shows the possibilities of mutual coherent articulation of these semiotic representations.

THE ROLE OF IMAGE SCHEMATA IN THE DEVELOPMENT OF NEW COGNITIVE OBJECTS

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Keywords: *image schema, cognitive objects, natural numbers, preschool*

Abstract: Our current work bases on the theories of Sfard, Dubinsky and Malle regarding the development of new cognitive objects with respect to the development of natural numbers. These theories propose that new cognitive concepts are developed by actions with concrete material objects, which uncover

relations between these objects. By disregarding the concrete material objects, *action schemata* and *relation schemata* are developed, coordinated and condensed to build new cognitive objects. In these theories, the existence of *image schemata*, a schematic structure representing the most important features of an object, an action or a relation in a pictorial, very stylistic manner, is only suggested. Nevertheless image schemata are believed to be very important in the development of elementary mathematical objects. The purpose of the work is twofold: First, to clarify and exactly describe the concept of image schemata and its role in developing new objects, and second, to present a research study examining the existence of image schemata with the result that such an existence is very likely.

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