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Abstract

The first goal of this article is to identify, for different defaultable claims, the fundamental processes which uniquely determine the pre-default price and therefore require to be modelled. The main message to the reader is that although the use of the default intensity or hazard process is ubiquitous, it may not uniquely characterise the price of some defaultable claims. The second goal is to better consolidate the reduced form approach with the structural approach, by extending the reduced form approach to allow for default times which can occur at stopping times and do not satisfy the immersion property.

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Introduction

In this article, we discuss and extend in a self-contained manner the reduced form approach for credit risk modelling. The current work is motivated by the observation that, in the literature of credit risk modelling, there are two main streams of approach:

1. The default time is assumed to exist and satisfy some probabilistic conditions. This is done in the recent works of Gehmlich and Schmidt [17], Fontana and Schmidt [11] and Jiao and Li [23].
2. The default time is constructed from some processes which are given in advance. For example through the Cox construction in the reduced form approach (presented in Section 1.4) and hitting times in the structural approach.

In the first approach, the question of whether one can construct a random time with the assumed probabilistic conditions are rarely asked and answered. While in the second approach there appears to be a lack of general consensus on what is the fundamental process or object that requires modelling. More specifically, in the reduced form models, many works start by either modelling the default intensity, the survival process or the conditional density of default with little to no justification for their choice. On the other hand, in the structural approach, the default time is modelled using the exit time of some process which in principle should reflect the default mechanism. However in practice, it is not clear what is the most appropriate way to model the default mechanism.

Additionally, it is also widely agreed that the reduced form models and structural models do not mix well with each other. This is mostly because in the classic reduced form models, due to the continuity assumption placed on the default intensity process, the default event is not allowed to happen at stopping times (avoidance property). On the other hand, although in the structural approach the default events are allowed to happen at stopping times or first hitting times, but unlike the reduced form approach, it suffers from the fact that the default intensity process or the compensator becomes extremely difficult to compute for complex models of the underlying default mechanism. It also does not take into account that the moment of default can potentially happen at other stopping times.

To this end, the first goal of this article is to identify, for different defaultable claims, the fundamental processes which uniquely determine the pre-default price (and therefore require modelling). Our main message to the reader is that although the use of the default intensity or hazard process is ubiquitous, it may not uniquely characterise the price of some defaultable claims. As original contribution, we revisit the formula of Duffie et al. [10] and the work of Coculescu and Nikeghabli [6] to make precise the role of the hazard process and the assumptions under which the default intensity or the hazard process is sufficient in determining the pre-default price. Finally, we explain why the modelling of the survival process or the conditional distribution of default appears to be the more natural choice in certain situations.

To better consolidate the reduced form approach with the structural approach, the second goal of this work is to extend the reduced form approach by providing constructions of default times which can occur at stopping times and do not satisfy the immersion property[1]. That is, given a financial market with information modelled by a given filtration $\mathbb{F}$, we extend the Cox-construction in order to construct default times which can occur at a family of $\mathbb{F}$-stopping times, each with a positive probability. This extends the models considered in Gehmlich and Schmidt [17], Fontana and Schmidt [11], where although the hazard process can be discontinuous, however it can only be discontinuous at predictable stopping times. To the best of our knowledge, given a family of $\mathbb{F}$-stopping times, it is not clear (until now) how to construct a random time or default time whose graph is contained in that family of $\mathbb{F}$-stopping times and does not satisfy the immersion property. From a practical point of view, this is an important extension since it has long been argued that, in certain applications, default times should not be allowed to happen at predictable stopping times as default events in principle should happen without warning.

[1] We recall that $\mathbb{F}$ is immersed in $\mathbb{G}$, where $\mathbb{F} \subset \mathbb{G}$ if any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. In the literature, this is also called hypothesis (H)

The structure of this work is as follows. We first introduce in Section 1 the necessary tools and background from the general theory of processes, stochastic calculus for optional processes and the standard Cox-construction of a random time.

In Section 2, we take a bottom up approach to identify the key assumptions on the default time and the key processes which uniquely determine the pre-default price of a defaultable claim. In particular, we revisit and extend the results of Coculescu and Nikeghbali [38] in Corollary 2.5 and Corollary 2.14 and identify the exact assumptions on the default time (and the claim) such that the hazard process will uniquely determine the pre-default price. In addition, we revisit and extend results on the multiplicative decomposition of the Azéma supermartingale in Proposition 2.3 and Proposition 2.9. The main message of this section is that, in general, the hazard process is not sufficient in determining the pre-default price and depending on the form of the claim it could be more natural to model the conditional distribution of the default.

In Section 3 we take a top-down approach and present several methods to construct random times which satisfy the desired properties which were previously identified in Section 2. From a mathematical point of view, the existence of a random time (which can be constructed on an extended probability space) having the requested properties is a main concern. As our main original contribution, we present in subsection 3.3.1 a construction of *the optional multiplicative system* associated with a positive optional supermartingale. We then extend the previous works of Jeanblanc and Song [20, 21], Li and Rutkowski [29] on the construction of random times with a given dual predictable projection and illustrate, in subsection 3.3.2, how the optional multiplicative system can be used to construct a random time with a given dual optional projection. From a modelling perspective, this allows one to construct default times for which the default event is allowed to happen at a pre-determined family of stopping times. Finally, we also extend the $\natural$-model or the natural-model of Jeanblanc and Song [21] and Song [40] to the optional setting.

1 Tools, Notations and Terminologies

We work on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with a filtration $\mathbb{F}$ satisfying the usual conditions. Therefore all martingales are assumed to be right continuous. Given an integrable process X, we adopt the standard notation ${}^{o,\mathbb{F}}X$ and ${}^{p,\mathbb{F}}X$ for the optional and predictable projection of a process X and $V^{o,\mathbb{F}}$ and $V^{p,\mathbb{F}}$ for the dual optional and dual predictable projection of a finite variation process V. For a càdlàg process X, we denote by $\Delta X_t = X_t - X_{t-}$ its jump at time t.

Given a random time τ (a positive $\mathcal{G}$-measurable random variable), we set $A_t = \mathbb{1}_{\{\tau \le t\}}$ and call this process the default process. In the rest of the article, we shall assume that the random time τ is finite. We denote by $\mathbb{G}$ the smallest filtration containing $\mathbb{F}$ satisfying the usual conditions such that τ is a stopping time in $\mathbb{G}$. The filtration $\mathbb{G}$ is called the progressive enlargement of $\mathbb{F}$ with τ. We recall that, for any $\mathbb{G}$-predictable process Y there exists an $\mathbb{F}$-predictable process y such that $Y_t \mathbb{1}_{\{t \le \tau\}} = y_t \mathbb{1}_{\{t \le \tau\}}$ (see Jeulin [22, lemma 4.4]). The process y is called an $\mathbb{F}$-predictable reduction of Y. If Y is a $\mathbb{G}$-optional process, there exists an $\mathbb{F}$-optional process y such that $Y_t \mathbb{1}_{\{t<\tau\}} = y_t \mathbb{1}_{\{t<\tau\}}$. The process y is called an $\mathbb{F}$-optional reduction of Y, or the predefault value of Y. In the rest of the paper, if not mentioned, the filtration $\mathbb{F}$ is always taken to be the base filtration and in order to reduce notation, whenever there is no confusion, we will not explicitly write the dependence on the filtration $\mathbb{F}$ when writing the projections and the dual projections.

The main tool used in this study is the stochastic calculus for optional semimartingales developed by Gal'čuk [13] [14] [15] or the $\underline{\underline{\mathsf{A}}}$-semimartingale developed by Lenglart [27]. For more details on the general theory for stochastic processes we refer to He et al. [12], for results from the theory of enlargement of filtrations to Jeulin [22]. The reader can also refer to Aksamit and Jeanblanc [2] for a modern exposition, in the english language, of the theory of enlargement of filtration.

1.1 Stochastic calculus for optional semimartingales

In this work we adopt the notation and framework of Gal'cŭk in [13, 14]. However we stress that we do not use the full power of the calculus for optional semimartingales as the filtration $\mathbb{F}$ is assumed to satisfy the usual conditions. We present below some useful results from the stochastic calculus for optional semimartingales specialised to our setting.

In the case where the filtration $\mathbb{F}$ satisfies the usual conditions any $\mathbb{F}$-optional semimartingale X is of the form $X = X_0 + M + A$, where M is a (càdlàg) local martingale and A is a làglàd process of finite variation. For a given optional semimartingale X we denote its right and left jumps by $\Delta^+ X = X_+ - X$ and $\Delta X = X - X_-$. For a finite variation làglàd process A, we can decompose A into $A = A^c + A^d + A^g$ where $A^d = \sum \Delta A_s$, $A^g = \sum \Delta^+ A_s$ and A^c is continuous, and we introduce the right-continuous part of A as $A^r := A^c + A^d$. For any optional semimartingale $X = X_0 + M + A$, we can write $X = X_0 + X^r + X^g$ where $X^r := M + A^r$ and $X^g := A^g$ are called the right-continuous and left-continuous part of X respectively. We denote by $\mathscr{E}(X)$ the optional stochastic exponential of X, defined as the solution of

$$Y_t = 1 + \int_{(0,t]} Y_{s-} dX^r_s + \int_{[0,t)} Y_s dX^g_{s+}$$

which satisfies

$$\mathscr{E}_t(X) = 1 + \int_{(0,t]} \mathscr{E}_{s-}(X) dX^r_s + \int_{[0,t)} \mathscr{E}_s(X) dX^g_{s+}$$

or, in a closed form

$$\mathscr{E}_t(X) = \mathscr{E}_0(X) e^{X_t - \frac{1}{2}\langle X^c, X^c\rangle_t} \prod_{0<s\le t} (1 + \Delta X_s) e^{-\Delta X_s} \prod_{0\le s<t} (1 + \Delta^+ X_s) e^{-\Delta^+ X_s} .$$

The optional stochastic logarithm of a strictly positive optional semimartingale X satisfies

$$\mathscr{L}_t(X) = \int_{(0,t]} \frac{1}{X_{s-}} dX^r_s + \int_{[0,t)} \frac{1}{X_s} dX^g_{s+}.$$

Given two optional semimartingales X and Y, the quadratic variation of X and Y is defined by

$$[X, Y]_t = \langle X^c, Y^c\rangle_t + \sum_{0<s\le t} \Delta X^r_s \, \Delta Y^r_s + \sum_{0\le s<t} \Delta^+ X^g_s \, \Delta^+ Y^g_s$$

and similarly to the càdlàg case, the following formula is satisfied

$$\mathscr{E}(X + Y + [X, Y]) = \mathscr{E}(X)\mathscr{E}(Y). \tag{1.1}$$

For a strictly positive optional semimartingale X, it can be shown that $X_0 \mathscr{E}(\mathscr{L}(X)) = X$. In the case where the process X is a càdlàg semimartingale the optional stochastic exponential and the optional logarithm, both reduce to the standard càdlàg stochastic exponential and logarithm. Therefore we shall not introduce two sets of notations.

Note that given a supermartingale which is not càdlàg, i.e., an optional supermartingale, the usual Doob-Meyer decomposition can not be applied. However, we can use the Mertens decomposition (see Gal'cŭk [14] and Mertens [32, Theorem T3]) which we recall below.

Definition 1.1 *An $\mathbb{F}$-adapted làglàd process X is said to be $\mathbb{F}$-strongly predictable if X is $\mathbb{F}$-predictable and X_+ is $\mathbb{F}$-optional.*

Theorem 1.2 (Doob-Meyer-Mertens-Gal'cŭk decomposition, [14] [32]) *An optional supermartingale X admits a decomposition $X = M - A$, where M is a (local) uniformly integrable martingale and A is an increasing strongly predictable (locally) integrable process with $A_0 = 0$ if and only if X belongs to the class-(D) (class-(DL)). This decomposition is unique to within indistinguishability.*

1.2 The Azéma (optional) supermartingale

Given a random time τ and the increasing process $A = \mathbb{1}_{[\![\tau,\infty[\![}$, we introduce:

1. The $\mathbb{F}$-dual predictable (resp. optional) projection of A denoted by $A^{p,\mathbb{F}}$ (resp. $A^{o,\mathbb{F}}$) or simply A^p (resp. A^o) when there is no confusion about the filtration.
2. The Azéma supermartingale $Z = 1 - {}^oA$ and the Azéma optional supermartingale $\widetilde{Z}$ associated to τ, that is,
$$Z_t = \mathbb{P}(\tau > t \,|\, \mathcal{F}_t) \qquad \text{and} \qquad \widetilde{Z}_t = \mathbb{P}(\tau \geq t \,|\, \mathcal{F}_t) = Z_t - \Delta A^{o,\mathbb{F}}_t.$$
3. The $\mathbb{F}$-conditional cumulative distribution of τ defined by $F_t(u) := \mathbb{P}(\tau \leq u \,|\, \mathcal{F}_t)$ and the $\mathbb{F}$-conditional survival distribution of τ given by $G_t(u) := 1 - F_t(u) = \mathbb{P}(\tau > u \,|\, \mathcal{F}_t)$.

It is well known that the processes Z_- and $\widetilde{Z}$ do not vanish on the set $[\![0,\tau]\!]$. Furthermore, the process Z is càdlàg and $\widetilde{Z}$ is làdlàg with the properties that $\widetilde{Z}_+ = Z_+ = Z$ and $\widetilde{Z}_- = Z_-$. Also the conditional cumulative distribution and the conditional survival distribution are clearly valued in $[0,1]$ and from tower property, for $u \geq t$, one has $F_t(u) = \mathbb{E}[1 - Z_u \,|\, \mathcal{F}_t]$. Moreover, for any fixed u, the process $F(u)$ is an $\mathbb{F}$-martingale and, for any fixed t, the map $u \to F_t(u)$ is non-decreasing.

We present below the additive decomposition of the supermartingales Z and $\widetilde{Z}$. It is well known that $Z + A^o$ and $Z + A^p$ are uniformly integrable martingales (see Corollary 5.31, page 152 of He et al. [12]) and we denote by m and M, the $\mathbb{F}$-martingales

$$m := Z + A^o \tag{1.2}$$
$$M := Z + A^p. \tag{1.3}$$

From the above, the Doob-Meyer decomposition of Z is given by

$$Z = M - A^p\,. \tag{1.4}$$

The Doob-Meyer-Mertens-Gal'cŭk decomposition of the optional supermartingale $\widetilde{Z}$ is given by

$$\widetilde{Z} = m - A^o_-\,. \tag{1.5}$$

Note that $\widetilde{Z}$ is an optional process as it is the difference between the càdlàg process m and the predictable process A^o_-.

Remark 1.3 We stress that mathematically, the knowledge Z allows one to uniquely identify A^p and vice versa. This is because in principle A^p can be obtained from the Doob-Meyer decomposition of Z and conversely if the dual optional projection A^p is known, then the Azéma supermartingale Z can be, in principle, obtained by computing ${}^o(A^p_\infty - A^p)$. Similarly the knowledge of $\widetilde{Z}$ allows us to uniquely identify A^o and vice versa. This is because in principle A^o_- (and thus A^o) can be obtained from the Doob-Meyer-Mertens-Gal'cŭk decomposition of $\widetilde{Z}$ and conversely if the dual optional projection A^o is know then the Azéma optional supermartingale $\widetilde{Z}$ can be obtained by computing ${}^o(A^o_\infty - A^o_-)$.

Finally, from the Azéma optional supermartingale $\widetilde{Z}$, one can compute the Azéma supermartingale Z by computing $\widetilde{Z}_+$. However it is not evident that $\widetilde{Z}$ can be obtained from Z. The main message we hope to convey here is that the dual optional projection A^o is the process which contains the most information.

Definition 1.4 *We say that τ avoids $\mathbb{F}$-stopping times ($\mathbb{F}$-predictable stopping times) if, for any finite $\mathbb{F}$-stopping time ($\mathbb{F}$-predictable stopping time) ϑ, one has $\mathbb{P}(\tau = \vartheta) = 0$.*

This is important to consider default times which do not enjoy the above avoidance property [17, 23]. In particular, from a financial modelling perspective this means that the default is allowed to happen at $\mathbb{F}$-stopping times and therefore allowing the modeller to introduce features from structural models of default. We recall some facts related with the avoidance property. If τ is a random time, then it avoids $\mathbb{F}$-stopping times if and only if A^o is continuous [2, Th. 1.43]. In that case, $A^o = A^p$. On the other hand, if A^p is continuous, then τ avoids all $\mathbb{F}$-predictable stopping times and the only $\mathbb{F}$-stopping times which are not avoided by τ are the jumps times of A^o.

For example, suppose $\mathbb{F}$ is a Poisson filtration and we take $\tau = T_1$ where T_1 is the first jump of the Poisson process N (so that $\mathbb{F} = \mathbb{G}$). In this case, one has $A^p_t = \lambda(t \wedge T_1)$ which is continuous and $A^o_t = \mathbb{1}_{\{t<T_1\}}$[2]. It is clear that τ does not avoid the $\mathbb{F}$-stopping time T_1 (which is not predictable).

1.3 The compensator and the hazard process

The compensator and the hazard process are ubiquitous in the credit risk literature. However their role is not always clear and we shall later see that, in general, the knowledge of the hazard process does not uniquely characterise the Azéma supermartingale and hence the price of defaultable claims. In order to be precise, we present below a self contained section on the notion of compensator and hazard process.

Let $\mathbb{K}$ be a filtration satisfying the usual conditions on $(\Omega, \mathcal{G}, \mathbb{P})$ and let τ be a $\mathbb{K}$-stopping time. The $\mathbb{K}$-compensator of A is defined to be the unique $\mathbb{K}$-predictable non-decreasing process $\Gamma^{\mathbb{K}}$ such that $\Gamma^{\mathbb{K}}_0 = 0$ and $A - \Gamma^{\mathbb{K}}$ is a $\mathbb{K}$-martingale. It is clear that $\Gamma^{\mathbb{K}}$ is stopped at time τ, i.e., $\Gamma^{\mathbb{K}}_t = \Gamma^{\mathbb{K}}_{t\wedge\tau}$, and depends on the choice of $\mathbb{K}$ (as well of the probability and, if needed, we shall write $\Gamma^{\mathbb{K},\mathbb{P}}$). If $\Gamma^{\mathbb{K}}$ is continuous, the $\mathbb{K}$-stopping time τ is a $\mathbb{K}$-totally inaccessible stopping time, and if $\Gamma^{\mathbb{K}}$ is absolutely continuous w.r.t. the Lebesgue measure then its derivative is called the $\mathbb{K}$-intensity rate.

Given a random time τ, we recall the form of the $\mathbb{G}$-compensator of the $\mathbb{G}$-adapted non-decreasing process A (see [2, Prop. 2.15]).

Lemma 1.5 (Jeulin-Yor formula) *The $\mathbb{G}$-compensator of A is the process $\Gamma^{\mathbb{G}}$ given by*

$$\Gamma^{\mathbb{G}}_t := \int_{(0,t]} \mathbb{1}_{\{\tau \geq u\}} Z^{-1}_{u-} dA^p_u,$$

we shall also refer to $\Gamma^{\mathbb{G}}$ as the compensator of τ.

Note that the process $\Gamma^{\mathbb{G}}$ is well defined (we use that $\mathbb{E}(\Gamma^{\mathbb{G}}_\tau) = \mathbb{E}(\int_0^\infty dA^p_s)$, see Jeulin [22], page 64). It is not hard to see that an $\mathbb{F}$-predictable reduction of the compensator $\Gamma^{\mathbb{G}}$ is given by

$$\Gamma_t = \int_{(0,t]} Z^{-1}_{u-} dA^p_u,$$

which we define to be the hazard process.

Definition 1.6 *The process Γ is called the $\mathbb{F}$-hazard process or simply the hazard process*

We note that although the process Z might vanish, the process Γ, as it is non-decreasing, is always well defined, except that it can take the value infinity. It is also important to stress that the $\mathbb{F}$-predictable reduction of $\Gamma^{\mathbb{G}}$ is not unique. One can also consider for any $a > 0$ the process

$$\int_{(0,\cdot]} (Z_{s-} + a\mathbb{1}_{\{Z_{s-}=0\}})^{-1} dA^p_s$$

or the process $\int_{(0,\cdot]} \mathbb{1}_{\{Z_{u-}>0\}} Z^{-1}_{u-} dA^p_u$ which are both $\mathbb{F}$-predictable reduction of $\Gamma^{\mathbb{G}}$.

[2] The equality holds from Theorem 5.30 in [12] and the fact that $\mathbb{P}(T_1 < \infty) = 1$

Example 1.7 In the case where τ is an $\mathbb{F}$-stopping time, then $\mathbb{G} = \mathbb{F}$, and $Z = 1 - A$, $\widetilde{Z} = 1 - A_-$ and $A^o = A$. The equality $A^p = A$ holds if and only if τ is an $\mathbb{F}$-predictable stopping time and in that case $\Gamma^{\mathbb{G}} = A$.

(i) If τ is a stopping time in a Brownian filtration, then since the optional σ-algebra is equal to the predictable σ-algebra, we have $\Gamma^{\mathbb{G}} = A$.

(ii) If $\mathbb{F}$ is a Poisson filtration generated by N with intensity rate λ, and $\tau = T_1$, the first jump time of N, then $\Gamma^{\mathbb{G}}_t = \int_0^{T_1 \wedge t} \lambda ds = \lambda(T_1 \wedge t)$ and $\Gamma_t = \lambda t$. In this case $\widetilde{Z}_t$ is equal to $1 - A^o_{t-}$ which is left continuous and not càdlàg.

For convenience of the reader, we recall the following key lemma [2, Lemma 2.9, Corollary 2.10]

Lemma 1.8 *(i) Let X be a $\mathcal{G}$-measurable integrable r.v. Then, for any $t \geq 0$,*

$$\mathbb{E}[X|\mathcal{G}_t]\mathbb{1}_{\{\tau>t\}} = \mathbb{1}_{\{\tau>t\}} \frac{1}{Z_t} \mathbb{E}[X\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t] \, .$$

(ii) Let h be a bounded $\mathbb{F}$-predictable process. Then, using the fact that τ if finite (hence $Z_\infty = 0$)

$$\mathbb{E}[h_\tau|\mathcal{G}_t] = h_\tau \mathbb{1}_{\{\tau \leq t\}} + \mathbb{1}_{\{\tau>t\}} \frac{1}{Z_t} \mathbb{E}\Big[\int_t^\infty h_u dA^p_u \Big| \mathcal{F}_t \Big] \, . \tag{1.6}$$

(iii) Let h be a bounded $\mathbb{F}$-optional process. Then,

$$\mathbb{E}[h_\tau|\mathcal{G}_t] = h_\tau \mathbb{1}_{\{\tau \leq t\}} + \mathbb{1}_{\{\tau>t\}} \frac{1}{Z_t} \mathbb{E}\Big[\int_t^\infty h_u dA^o_u \Big| \mathcal{F}_t \Big] \, . \tag{1.7}$$

1.4 The Cox construction

The most commonly used method, in the credit risk literature, to construct a default time is the Cox-construction which enables one to construct a random time which admits a given compensator or hazard process. In general, one needs to enlarge the given probability space to do so: Indeed, suppose that $\mathbb{F}$ is a Brownian filtration and $\Gamma^{\mathbb{F}}$ an $\mathbb{F}$-adapted continuous non-decreasing process, it is not possible to construct an $\mathbb{F}$-stopping time τ with compensator equal to $\Gamma^{\mathbb{F}}_{t\wedge\tau}, t \geq 0$, since such a stopping time would be totally inaccessible stopping time, and it is well known that all stopping times in a Brownian filtration are predictable.

Given a non-decreasing $\mathbb{F}$-adapted process X, the Cox-construction models the default time as the first time a non-decreasing process X crosses a barrier U which is independent of $\mathcal{F}_\infty$. We first consider the case where $X = X^r$ is right continuous. In this case, we set

$$\tau = \inf\{s \, : X^r_s \geq U\}$$

where U is a uniform random variable (constructed on an extended space)[3] taking values in $[0, 1]$ and is independent from $\mathcal{F}_\infty$. In this case, the following equality holds

$$\{\tau \leq t\} = \{U \leq X^r_t\}. \tag{1.8}$$

From this equality and the fact that U is independent from $\mathcal{F}_\infty$, we can deduce that $Z = 1 - X$ and the cumulative conditional distribution of τ is given by

$$\mathbb{P}(\tau \leq u \,|\, \mathcal{F}_t) = \mathbb{E}[X^r_u \,|\, \mathcal{F}_t], \quad \forall u, t \in \mathbb{R}_+.$$

In particular, $\mathbb{P}(\tau \leq u|\mathcal{F}_t) = \mathbb{P}(\tau \leq u|\mathcal{F}_u) = X^r_u = 1 - Z_u$ for $t \geq u$.

[3] Here we find it more convenient to use a uniform random variable rather than an exponential random variable, but this is not essential.

In the case where $X = X^g$ is left continuous, the previous set equality (1.8) is no more valid and we define τ as

$$\tau = \inf\{s \,:\, X^g_s > U\}$$

and the set equality $\{\tau < t\} = \{U < X^g_t\}$ leads to $\widetilde{Z}_t = \mathbb{P}(\tau \geq t \,|\, \mathcal{F}_t) = 1 - X^g_t$. By taking the limit from the right, we have $Z = \widetilde{Z}_+ = 1 - X^g_+$.

In both of the above cases, the filtration $\mathbb{F}$ is immersed in $\mathbb{F} \vee \sigma(U)$, hence in $\mathbb{G}$, and the Azéma supermartingale Z is non-increasing (see [2, Chapter 3]). Any random time τ constructed using the above method will be called a Cox-time.

In the case where $X = X^r$ is right-continuous and $\mathbb{F}$-predictable, then the Azéma supermartingale Z of the constructed random time τ is the predictable non-increasing process $Z = 1 - X$ and from uniqueness of the Doob-Meyer decomposition of Z, we have $A^p = 1 - Z = X$. Therefore the $\mathbb{G}$-compensator of A is

$$\Gamma^{\mathbb{G}}_t = \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} dA^p_s = -\mathscr{L}_{t\wedge\tau}(1 - X)$$

and it admits $\Gamma = -\mathscr{L}(1 - X)$ as the $\mathbb{F}$-hazard process.

The above shows that if one wants to model directly the hazard process Γ, which is a right-continuous $\mathbb{F}$-predictable non-decreasing process, then one can take $X = 1 - \mathscr{E}(-\Gamma)$ in the Cox-construction. In the literature, it is common to assume that the hazard Γ is continuous and consider $X = 1 - e^{-\Gamma}$. In this setting, the constructed random time τ is then a $\mathbb{G}$-totally inaccessible stopping time and $A^o = A^p = 1 - Z$. We point out that in the classical intensity setup [33], the default intensity process is modelled using a non-negative $\mathbb{F}$-adapted process γ and we take $\Gamma_t = \int_{(0,t]} \gamma_s ds$.

In the case where X is right continuous and not predictable then, although from the Cox-construction the equality $Z = 1 - X$ is still valid, however it is not always true that $A^p = X$. This is because X is optional and, in this case, the dual predictable projection A^p is equal to X^p. In this special case, we can identify X with the dual optional projection A^o. This is due to the fact that any Cox-time is a pseudo-stopping time (see Definition 2.11), which implies that $Z = 1 - A^o$ and therefore $X = A^o$ (see for example [2, Theorem 3.35]).

Remark 1.9 We want to again stress that the ability to identify the dual optional (predictable) projection A^o (A^p) with the model input X in the case where X is non-continuous is crucial for the purpose of modelling default times. The reason is that it allows us to model the behaviour of the dual optional (predictable) projection A^o (A^p) through the modelling of X. This allows one to model the probability that the constructed default time is equal to a pre-determined family of (predictable) stopping times, i.e., the jump times of X.

2 Credit Risk Pricing - Bottom Up Approach

In this section, we take a bottom up view and suppose that we are given a default time τ. The aim is to show that depending on the structure of the defaultable claim and the assumptions placed on the default time, different processes are required to be modelled in order to uniquely determine the price a default claim. In the following, we suppose that $\mathbb{P}$ is the pricing measure and the interest rate is zero. Otherwise, the assumptions and the definition of the characteristics of τ have to be considered under the pricing measure.

2.1 Claims of Type I

Given a finite time horizon $[0,T]$, we say that a defaultable financial contract is of type I if the payoff is given by

$$C(T,\tau) = Y_T \mathbb{1}_{\{\tau > T\}} + C_\tau \mathbb{1}_{\{\tau \leq T\}}$$

where τ, Y_T and C are given in advance, and we suppose that $Y_T \in \mathcal{F}_T$ and C is either an $\mathbb{F}$-predictable or an $\mathbb{F}$-optional process.

The default payment C is $\mathbb{F}$-predictable: Let us suppose that C is bounded and $\mathbb{F}$-predictable, then from direct computation using the key lemma, we see that

$$\mathbb{E}[C(T,\tau)\,|\,\mathcal{G}_t] = Z_t^{-1}\mathbb{E}[Y_T Z_T + \int_{(t,T]} C_u dA_u^p \,|\, \mathcal{F}_t]\mathbb{1}_{\{\tau > t\}} + C_\tau \mathbb{1}_{\{\tau \leq t\}}$$

and the pre-default value of the claim is given by

$$\pi_t = Z_t^{-1}\mathbb{E}[Y_T Z_T + \int_{(t,T]} C_u dA_u^p \,|\, \mathcal{F}_t] = Z_t^{-1}\mathbb{E}[Y_T Z_T - \int_{(t,T]} C_u dZ_u \,|\, \mathcal{F}_t].$$

From the above, we see that the pre-default value is uniquely determined by the Azéma supermartingale Z or equivalently by the dual predictable projection A^p.

The default payment C is $\mathbb{F}$-optional: In the case where C is bounded and $\mathbb{F}$-optional, then one has to replace A^p with A^o to obtain

$$\pi_t = Z_t^{-1}\mathbb{E}[Y_T Z_T + \int_{(t,T]} C_u dA_u^o \,|\, \mathcal{F}_t].$$

From the above, we see that in the case where the default payment is optional, the pre-default price is uniquely determined by the quantities A^o and Z. In this situation, we note that it is sufficient to know A^o, since the Azéma supermartingale Z can be, in principle, retrieved by computing $Z = {}^o(A^o_\infty - A^o)$. However, unlike the case where the defaultable claim C is an $\mathbb{F}$-predictable process, it is not sufficient to model the Azéma supermartingale Z. This is because the process A^o cannot be uniquely determined from Z (via the Doob-Meyer decomposition).

In the following, we adopt the following notation for the pre-default price of type I claim,

$$\pi_t = Z_t^{-1}\mathbb{E}[Y_T Z_T + \int_{(t,T]} C_u^{(j)} dA_u^j \,|\, \mathcal{F}_t] \quad j \in \{o,p\}, \tag{2.1}$$

where process $C^{(j)}$ is a predictable (optional) claim if $j = p$ ($j = o$).

In the literature of credit risk modelling, the hazard process or the intensity process is ubiquitous. This is mainly due to the fact that in the classic reduced form approach to credit risk modelling, given some non-decreasing continuous *hazard process* $\Gamma_t = \int_0^t \gamma_u du$ where γ is the *intensity process*, by using the Cox construction one can construct a random time τ such that $Z = A^p = A^o = e^{-\Gamma}$ and τ satisfies the immersion property. In this setting, we clearly see that for Cox-times the pre-default price π_t given above in (2.1) is uniquely determined by the hazard process Γ.

However for an arbitrary random time or default time, the Azéma supermartingale Z is in general not continuously decreasing and $Z \neq A^p \neq A^o \neq e^{-\Gamma}$. In fact a long standing question is under what assumptions, on the default time τ, can the pricing formula for type I claims be uniquely characterized by the hazard process or the intensity process. This question was explored in the work of Coculescu and Nikeghbali [6] under continuity assumption and, in the following, we extend their work and identify the most general conditions under which the hazard process uniquely determines the pre-default price of type I claims.

2.1.1 The predictable case

In order to be precise, we recall, from Definition 1.6, that the hazard process is an $\mathbb{F}$-predictable process Γ such that the $\mathbb{G}$-compensator of τ is given by Γ^τ. To proceed we introduce the following class of random times which is inspired by the definition of pseudo-stopping times.

Definition 2.1 (Strict pseudo-stopping time) *A finite random time τ is called an $\mathbb{F}$-strict pseudo-stopping time if for all uniformly integrable $\mathbb{F}$-martingale Y, we have $\mathbb{E}[Y_{\tau-}] = \mathbb{E}[Y_0]$.*

The class of strict pseudo-stopping time has already appeared in the literature as a special case of the recently introduced *invariance time* in Crépey and Song [7]. In short, an invariance time is a random time for which there exists a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that τ is a strict pseudo-stopping time under $\mathbb{Q}$.

Proposition 2.2 *Given a finite random time τ, the following conditions are equivalent*
(i) The random time τ is a strict pseudo-stopping time.
(ii) The Azéma supermartingale Z of τ is non-increasing and predictable or equivalently ${}^oA = A^p$.
(iii) For all uniformly integrable $\mathbb{F}$-martingale Y, the process $Y^{\tau-}$ is a uniformly integrable $\mathbb{G}$-martingale where $Y_t^{\tau-} = Y_t \mathbb{1}_{\{t<\tau\}} + Y_{\tau-}\mathbb{1}_{\{\tau\le t\}}$.

Proof: To see that *(i)* is equivalent to *(ii)*, we first suppose that τ is a strict pseudo-stopping time

$$\begin{aligned}\mathbb{E}[Y_{\tau-}] &= \mathbb{E}[\int_{[0,\infty)} Y_{u-} dA_u^p] \\ &= \mathbb{E}[\int_{[0,\infty)} {}^p(Y_\infty)_u dA_u^p] \\ &= \mathbb{E}[Y_\infty A_\infty^p] = \mathbb{E}[Y_0]\end{aligned}$$

which implies that $A_\infty^p = 1$. It follows that the Doob-Meyer decomposition of Z is given by $Z = 1 - A^p$, since $Z_t = \mathbb{E}[A_\infty^p - A_t^p | \mathcal{F}_t]$. Conversely, the equality ${}^oA = A^p$ and the finiteness of τ implies that $A_\infty^p = 1$, and from $\mathbb{E}[Y_{\tau-}] = \mathbb{E}[Y_\infty] = \mathbb{E}[Y_0]$ we see that τ is a strict pseudo-stopping time.

To see that *(i)* and *(iii)* are equivalent, we first suppose that Y is a uniformly integrable $\mathbb{F}$-martingale. For every $\mathbb{G}$-stopping time ν, from page 186 of Dellacherie et al. [8], we know there exists an $\mathbb{F}$-stopping time σ such that $\tau \wedge \nu = \tau \wedge \sigma$. Therefore, from the fact that τ is a strict pseudo-stopping time we have

$$\mathbb{E}[Y_\nu^{\tau-}] = \mathbb{E}[Y_{\tau-}^\sigma] = \mathbb{E}[Y_0]$$

which shows that Y is a uniformly integrable $\mathbb{G}$-martingale by Theorem 1.42 [19]. To show the converse, it is sufficient to consider $Y^{\tau-}$ at τ and apply the optional sampling theorem in $\mathbb{G}$.

In the following, we study the relationship between the hazard process Γ and the Azéma supermartingale Z. We present a result on the multiplicative decomposition of a strictly positive supermartingale, which can be found on page 138 of Jacod and Shiryaev [19], but here we provide a specialized proof for the Azéma supermartingale, which is more insightful for our purposes.

Proposition 2.3 *Suppose that Z is strictly positive then*

$$Z = \mathscr{E}(-\Gamma)\mathscr{E}(N)$$

where N is a local martingale given by $N_t = \int_{]0,t]} ({}^pZ_u)^{-1} dM_u$, the martingale M being defined in (1.3) and Γ being the hazard process.

Proof: Starting from the $\mathbb{F}$-hazard process Γ, we write

$$\begin{aligned}\Gamma_t = \int_{(0,t]} Z_{u-}^{-1} dA_u^p &= \int_{(0,t]} Z_{u-}^{-1} dA_u^p - \int_{(0,t]} Z_{u-}^{-1} dM_u + \int_{(0,t]} Z_{u-}^{-1} dM_u \\ &= -\mathscr{L}_t(Z) + \int_{(0,t]} Z_{u-}^{-1} dM_u \\ &= -\mathscr{L}_t(Z) + N_t - \int_{(0,t]} \frac{\Delta A_u^p}{Z_{u-}(Z_{u-} - \Delta A_u^p)} dM_u \\ &= -\mathscr{L}_t(Z) + N_t - [\Gamma, N]_t\end{aligned}$$

where $N_t := \int_{]0,t]} (Z_- - \Delta A^p)_u^{-1} dM_u$. The last equality above follows from Yœurp's lemma (see Proposition 1.16 [2]) and we point out that $Z_- - \Delta A^p = {}^pZ$ so that $N_t = \int_{]0,t]} ({}^pZ_u)^{-1} dM_u$. Then by taking the stochastic exponential of both sides above, we obtain from (1.1) the equality

$$Z = Z_0 \mathscr{E}(-\Gamma + N + [-\Gamma, N]) = Z_0 \mathscr{E}(-\Gamma)\mathscr{E}(N).$$

which concludes the proof.

Corollary 2.4 *A random time τ is a finite strict pseudo-stopping time if and only if its Azéma supermartingale takes the form $Z = Z_0 \mathscr{E}(-\Gamma)$.*

Corollary 2.5 *If τ is a finite strict pseudo-stopping time then the pre-default value for a financial product of Type I with a payoff $C^{(p)}$ is uniquely obtained in terms of the hazard process. That is*

$$\pi_t = \mathscr{E}_t(-\Gamma)^{-1} \mathbb{E}[Y_T \mathscr{E}_T(-\Gamma) + \int_{(t,T]} C_u^{(p)} d\mathscr{E}_u(-\Gamma) \,|\, \mathcal{F}_t].$$

Remark 2.6 Here it is important to note that if τ is not a strict pseudo-stopping time then the pre-default price π is only uniquely characterized by the Azéma supermartingale Z. Although the hazard process is ubiquitous in the literature, however one can construct many random times with the same hazard process but different Azéma supermartingales. Such examples are given in subsection 2.1.3.

2.1.2 The optional case

To study claims which are optional, we introduce below the optional hazard process. The optional hazard or more specifically the optional Jeulin-Yor formula given in Lemma 2.7 have also appeared in the recent working paper of Choulli et al. [5] where the martingale representation theorem in the filtration $\mathbb{G}$ was studied. The shift from the hazard process to the optional hazard process allows one to model random times which can happen at $\mathbb{F}$-stopping times.

Lemma 2.7 (Optional Jeulin-Yor formula) *Given a random time τ, the $\mathbb{G}$-adapted process*

$$\mathbb{1}_{\{\tau \le t\}} - \int_{(0,t]} \mathbb{1}_{\{\tau \ge u\}} \widetilde{Z}_u^{-1} dA_u^o$$

is a $\mathbb{G}$-martingale.

Proof: To prove this decomposition, viewed as an optional Jeulin-Yor formula, note that for $s \le t$,

$$\mathbb{E}[\mathbb{1}_{\{\tau \le t\}} \,|\, \mathcal{G}_s] = \mathbb{1}_{\{\tau \le s\}} + \mathbb{E}[\mathbb{1}_{\{s < \tau \le t\}} \,|\, \mathcal{G}_s]$$

For any $G_s \in \mathcal{G}_s$, there exists $F_s \in \mathcal{F}_s$ such that $F_s 1\!\!1_{\{\tau>s\}} = G_s 1\!\!1_{\{\tau>s\}}$. Then we have

$$\begin{aligned}\mathbb{E}[1\!\!1_{G_s} 1\!\!1_{\{s\le\tau<t\}}] &= \mathbb{E}[1\!\!1_{F_s}\int 1\!\!1_{\{s<u\le t\}} dA^o_u] \\ &= \mathbb{E}[1\!\!1_{F_s}\int 1\!\!1_{\{s<u\le t\}} 1\!\!1_{\{\tau\ge u\}} \widetilde{Z}_u^{-1} dA^o_u] \\ &= \mathbb{E}[1\!\!1_{G_s}\int_{]s,t]} 1\!\!1_{\{\tau\ge u\}} \widetilde{Z}_u^{-1} dA^o_u]\end{aligned}$$

which concludes the proof.

Definition 2.8 *An $\mathbb{F}$-optional hazard process of τ is the process $\widetilde{\Gamma}$ given by $\widetilde{\Gamma}_t := \int_{[0,t]} \widetilde{Z}_u^{-1} dA^o_u$.*

To understand the relationship between the $\mathbb{F}$-optional hazard process $\widetilde{\Gamma}$ and the Azéma optional supermartingale $\widetilde{Z}$, we consider $\widetilde{\Gamma}_-$ and suppose that $\widetilde{Z}\widetilde{Z}_- > 0$. The optional multiplicative decomposition of the Azéma supermartingale Z has previously been considered in Kardaras [25].

Proposition 2.9 *Suppose $\widetilde{Z}$ is strictly positive then*

$$\widetilde{Z} = \mathscr{E}(-\widetilde{\Gamma}_-)\mathscr{E}(\widetilde{N}).$$

where $\widetilde{N}_t := \int_{]0,t]} \widetilde{Z}_{u-}^{-1} dm_u$ with m defined in (1.2).

Proof: Given the $\mathbb{F}$-optional hazard process $\widetilde{\Gamma}$, we write

$$\begin{aligned}\widetilde{\Gamma}_{t-} &= \int_{[0,t[} \widetilde{Z}_u^{-1} dA^o_u - \int_{]0,t]} \widetilde{Z}_{u-}^{-1} dm_u + \int_{]0,t]} Z_{u-}^{-1} dm_u \\ &= \int_{]0,t]} \widetilde{Z}_{u-}^{-1} dA^{o,c}_u - \int_{]0,t]} \widetilde{Z}_{u-}^{-1} dm_u + \int_{[0,t[} \widetilde{Z}_u^{-1} dA^{o,g}_{u+} + \int_{]0,t]} Z_{u-}^{-1} dm_u \\ &= -\mathscr{L}_t(\widetilde{Z}) + \widetilde{N}_t\end{aligned}$$

where $\mathscr{L}(\widetilde{Z})$ is the optional stochastic logarithm of $\widetilde{Z}$ and $\widetilde{N}_t := \int_{]0,t]} \widetilde{Z}_{u-}^{-1} dm_u$. By taking the optional stochastic exponential of both sides above, we obtain from (1.1)

$$\begin{aligned}\widetilde{Z} &= \mathscr{E}(-\widetilde{\Gamma}_- + \widetilde{N}) \\ &= \mathscr{E}(-\widetilde{\Gamma}_-)\mathscr{E}(\widetilde{N})\end{aligned}$$

where the second equality holds since $[-\widetilde{\Gamma}_-, \widetilde{N}] = 0$ as $\Delta\widetilde{\Gamma}_- = 0$ and $\Delta^+\widetilde{N} = 0$.

Remark 2.10 Note that $\mathscr{E}(\widetilde{N})$ is càdlàg, $\mathscr{E}(-\widetilde{\Gamma}_-)$ is càglàd and therefore $\widetilde{Z}$ is not càdlàg.

Definition 2.11 (Pseudo-stopping time) *A random time τ is called an $\mathbb{F}$-pseudo-stopping time if for all uniformly integrable $\mathbb{F}$-martingale Y, we have $\mathbb{E}[Y_\tau] = \mathbb{E}[Y_0]$.*

Theorem 2.12 *(i) A finite random time τ is a pseudo-stopping time if and only if its Azéma's optional supermartingale can be expressed as*

$$\widetilde{Z} = \mathscr{E}(-\widetilde{\Gamma}_-).$$

(ii) Let τ be a finite honest time, then the Azéma optional supermartingale can be expressed as

$$\widetilde{Z} = \mathscr{E}(-\widetilde{\Gamma}_-)\widetilde{M}$$

where $\widetilde{M}$ is a local martingale.

Note that the above equalities are valid even in the case when $\widetilde{Z}$ is allowed to vanish.

Proof: (i) Suppose that τ is a pseudo-stopping time then from Theorem 1 of Nikeghbali and Yor [37] or Aksamit and Li [1] we know that $m = 1$ and thus $\widetilde{N} = 1$. The representation of $\widetilde{Z}$ is then obtained by applying Proposition 2.9 to $\widetilde{Z} + \epsilon$ for $\epsilon > 0$, and taking the limit as $\epsilon \downarrow 0$ using the monotone convergence theorem. For the converse, by Theorem 1 of Aksamit and Li [1], if the Azéma optional supermartingale $\widetilde{Z}$ is left continuous and decreasing then τ is a pseudo-stopping time.

(ii) The validity of the multiplicative decomposition on the whole real line for honest times is rather involved. Therefore we refer the readers to Lemma 2.1 in Li [28].

Remark 2.13 Theorem 2.12 (i) extends Theorem 3.8 of Coculescu and Nikeghbali [6] and gives complete characterization of pseudo-stopping times in terms of the associated $\mathbb{F}$-optional hazard process. From Proposition 2.9 and Theorem 2.12, we see that the knowledge of the $\mathbb{F}$-optional hazard process is insufficient in retrieving the associated Azéma optional supermartingale (and thus the Azéma supermartingale) and is sufficient only when the default time is a pseudo-stopping time.

Corollary 2.14 *If τ is a finite pseudo-stopping time then the pre-default value for a financial product of Type I with a payoff $C^{(o)}$ is uniquely characterised by the optional hazard process, that is*

$$\pi_t = \mathscr{E}_{t+}(-\widetilde{\Gamma}_-)^{-1}\mathbb{E}[Y_T\mathscr{E}_{T+}(-\widetilde{\Gamma}_-) + \int_{(t,T]} C^{(o)}_u d\mathscr{E}_{u+}(-\widetilde{\Gamma}_-) \,|\, \mathcal{F}_t].$$

Finally, we revisit Proposition 3.3 in Coculescu and Nikeghbali [6], where it is stated that if Z is continuous then the random time τ avoids all $\mathbb{F}$-stopping times. Unfortunately, the proof is not fully correct, to illustrate this we give a constructive counter example in Example 3.13.

Proposition 2.15 *Let τ be a finite random time such that its associated Azéma's supermartingale Z is continuous, then τ avoids all predictable stopping times.*

Proof: If Z is continuous then $\Delta M = \Delta A^p$. However since M is a martingale, the size of a jump at predictable time is zero, then for any predictable stopping times T, $\Delta A^p_T = \mathbb{P}(\tau = T \,|\, \mathcal{F}_{T-}) = 0$, which implies that $\mathbb{P}(\tau = T) = 0$.

2.1.3 The hazard process and the Azéma supermartingale

From Corollary 2.4 and Theorem 2.12, we note that the knowledge of the (optional) hazard process does not uniquely characterize the Azéma (optional) supermartingale unless the random time is a strict pseudo-stopping time (pseudo-stopping time). In other words, one can have two different random times with the same (optional) hazard process, but different Azéma (optional) supermartingale.

Example 2.16 For any honest time in a Brownian filtration which avoids stopping times, one can compute its hazard process Γ and $\mathscr{E}(-\Gamma)$ which is finite on the whole real line. Taking $X = 1-\mathscr{E}(-\Gamma)$ in the Cox construction, one obtains a random time (on an extended space) such that the hypothesis (H) holds and its Azéma supermartingale Z is equal to the predictable non-increasing process $\mathscr{E}(-\Gamma)$. The corresponding $\mathbb{F}$-hazard process is given by Γ.

This example gives two random times, one being the honest time and the other a Cox-time. They have the same hazard process, but different Azéma's supermartingale (the Azéma supermartingale of the honest time is not decreasing).

Example 2.17 Constructive example. Suppose one first models the hazard process Γ as an absolutely continuous process with derivative γ, in a Brownian filtration. Then, it is shown in [31] that, for any $\mathbb{F}$-adapted bounded b, the solution of

$$dY_t = -\gamma_t Y_t dt + b_t Y_t(1 - Y_t)dB_t$$

is a supermartingale valued in $[0, 1]$ with multiplicative decomposition $Y = Ne^{-\Gamma}$, hence Γ is the hazard process of the associated τ.

2.2 Claims of Type II

Given a finite time horizon $[0,T]$, we say that a defaultable financial contract is of type II if it the terminal payoff is given by

$$Y_T\mathbb{1}_{\{\tau>T\}}+C_\tau(\zeta)\mathbb{1}_{\{\tau\le T\}}$$

where $Y_T\in\mathcal{F}_T$ and for any y, the process $C(y)$ is $\mathbb{F}$-optional, $\zeta\in\mathcal{F}_T$ so that $C_\tau(\zeta)\in\mathcal{G}_T$ and the terminal payoff is $\mathcal{G}_T$-measurable. Then the pre-default price of a claim of type II is given by

$$\begin{aligned}\pi_t\mathbb{1}_{\{\tau>t\}}&=\mathbb{E}[Y_T\mathbb{1}_{\{\tau>T\}}+C_\tau(\zeta)\mathbb{1}_{\{\tau\le T\}}\,|\,\mathcal{G}_t]\mathbb{1}_{\{\tau>t\}}\\&=\mathbb{E}[Y_T\mathbb{1}_{\{\tau>T\}}|\mathcal{G}_t]\mathbb{1}_{\{\tau>t\}}+\mathbb{E}[C_\tau(\zeta)\mathbb{1}_{\{t<\tau\le T\}}\,|\,\mathcal{G}_t]\\&=Z_t^{-1}\mathbb{E}[Y_TZ_T+C_\tau(\zeta)\mathbb{1}_{\{t<\tau\le T\}}\,|\,\mathcal{F}_t]\mathbb{1}_{\{\tau>t\}}\end{aligned}$$

where the last equality follows from the key lemma. In order to further compute the pre-default value, i.e., $Z_t^{-1}\mathbb{E}[Y_TZ_T+C_\tau(\zeta)\mathbb{1}_{\{t<\tau\le T\}}\,|\,\mathcal{F}_t]$, it is evident that one requires the knowledge of the $\mathcal{F}_T$-conditional cumulative distribution of τ denoted by $F_T(u)$. If we suppose that the $\mathcal{F}_T$-conditional distribution is known, then the pre-default price of the claim is given by

$$\pi_t=Z_t^{-1}\mathbb{E}[Y_TZ_T+\int_{(t,T]}C_u(\zeta)\,F_T(du)\,|\,\mathcal{F}_t]. \tag{2.2}$$

From the above, we see that the Azéma (optional) supermartingale and the dual projections are insufficient in determining the pre-default price and one requires the knowledge of the $\mathcal{F}_T$-conditional distribution $F_T(u)$ for $u\le T$.

In view of this, we collect below some simplifying assumptions on the $\mathcal{F}_T$-conditional distribution of τ which have appeared in the literature.

Hypothesis (H): If the random time τ satisfies the hypothesis (H) or the immersion property, then $F_t(u)=F_u(u)=1-Z_u=1-\mathscr{E}_u(-\Gamma)$.

$$\pi_t=Z_t^{-1}\mathbb{E}[Y_TZ_T-\int_{(t,T]}C_u(\zeta)dZ_u\,|\,\mathcal{F}_t]. \tag{2.3}$$

In this case the pre-default price is uniquely determined by the hazard process Γ.

Hypothesis (HP): The random time τ satisfies the hypothesis (HP) if there exists a random field $(C_{u,t})_{u\le t}$, increasing in u and decreasing and adapted in t such that $F_t(u)=C_{u,t}F_t$ for all $u\le t$, where $F_t=F_t(t)$. In this case, the pre-default price is then given by

$$\pi_t=Z_t^{-1}\mathbb{E}[Y_TZ_T+F_T\int_{(t,T]}C_u(\zeta)\,dC_{u,T}\,|\,\mathcal{F}_t]. \tag{2.4}$$

For modelling purposes, one often consider the case where $C_{u,t}$ is *completely separable* in u and t, that is $\mathcal{F}_t$-conditional distribution takes the form $F_t(u)=C_uM_t$, where M is a martingale and C is a non-decreasing adapted process.

The random field $(C_{u,t})_{u\le t}$ is also called the multiplicative system associated with the submartingale F (see Definition 3.8) which we will study in more detail in subsection 3.3.1 (we refer also to Meyer [35] and the references within). For more details and examples in applications we refer to Gapeev et al. [16], Jeanblanc and Song [20] and Li and Rutkowski [29]. In the special case where τ is an honest time then it is well known that, for all $t\ge 0$, there exists an $\mathcal{F}_t$-measurable random variable τ_t such that for $u\le t$, we have the equality $\mathbb{P}(\tau\le u\,|\,\mathcal{F}_t)=\mathbb{1}_{\{\tau_t\le u\}}\mathbb{P}(\tau\le t\,|\,\mathcal{F}_t)$ and the random field $C_{u,t}$ is given by $\mathbb{1}_{\{\tau_t\le u\}}$.

Jacod's Hypothesis [Jacod's absolute continuity hypothesis]: There exists a non-negative $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $(\omega, t, u) \to p_t(\omega, u)$ càdlàg in t such that
(1) for every u, the process $(p_t(u), t \geq 0)$ is a non-negative $\mathbb{F}$-martingale,
(2) denoting by η the law of τ, for every $t \geq 0$, the measure $p_t(u)\eta(du)$ equals $\mathbb{P}(\tau \in du \,|\, \mathcal{F}_t)$, in other words, for any Borel (bounded) function f

$$\mathbb{E}[f(\tau)|\mathcal{F}_t] = \int_{\mathbb{R}_+} f(u)p_t(u)\eta(du)\,.$$

In particular $Z_t = \int_{]t,\infty[} p_t(u)\eta(du)$. Such a family is called the conditional density and there are not so many explicit examples of random times where this density exists, except those of Chaleyat Maurel and Jeulin [3] and Mansuy and Yor [34, p.34]. Note that if τ is a honest time then it satisfies Jacod's hypothesis if and only if it takes only countable values (see [2, Remark 5.31b]).

The equivalence Jacod's hypothesis is the same as the above except that the conditional density is strictly positive.

Extended Jacod's Hypothesis: There exists a non-negative $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $(\omega, t, u) \to p_t(\omega, u)$ càdlàg in t and a $\mathbb{F}$-adapted non-decreasing process D such that for $u \leq t$

$$F_t(u) = \int_{[0,u]} p_t(s)dD_s.$$

where D is an $\mathbb{F}$-adapted increasing process and for every u, the process $(p_t(u), t \geq u)$ is a $\mathbb{F}$-martingale. For details and examples, we refer to Li and Rutkowski [30] and Song [42] and [23] for a particular case.

2.2.1 The Azéma supermartingale and the conditional law

To conclude the first part of the article, we point out that unless the default time τ satisfies the hypothesis (H) then the Azéma (optional) supermartingale does not uniquely characterise the pre-default price of a claim of type II. Mathematically, one can construct infinitely many random times with the same Azéma (optional) supermartingale but each with a different $\mathbb{F}$-conditional distribution.

Example 2.18 We examine the extension of the William's path decomposition example studied in Nikeghabli and Yor [37]. Let $\mathbb{F}$ be a Brownian filtration and L be an honest time, with Azéma supermartingale Z, such that $\underline{Z}_t := \inf_{u \leq t} Z_u$ is continuous. We consider the pseudo-stopping time (see [37]) given by

$$\rho = \sup\{t < L : Z_t = \underline{Z}_t\}.$$

The Azéma supermartingale of ρ is given by $\underline{Z}$, and the $\mathbb{F}$-conditional distribution of ρ can be easily computed and is given by

$$G_t(u) := \mathbb{P}(\rho > u \,|\, \mathcal{F}_t) = Z_t \mathbb{1}_{\{T_u < t\}} + \underline{Z}_t\, \mathbb{1}_{\{T_u \geq t\}} + (1 - Z_t)\mathbb{1}_{\{T_u < L_t \leq t\}} \qquad \forall t \geq u,$$

where L_t is an $\mathcal{F}_t$-measurable random variable such that for $t \geq 0$ satisfies $L\mathbb{1}_{\{L \leq t\}} = L_t \mathbb{1}_{\{L \leq t\}}$ and $T_u = \inf\{t > u : Z_t \leq \underline{Z}_t\}$.

On the other hand, one can apply the Cox-construction described in subsection 1.4 with $X = 1 - \underline{Z}$ and construct a random time τ^{Cox} with a non-increasing Azéma's supermartingale $Z^{\text{Cox}} = \underline{Z}$. Therefore ρ and τ^{Cox} have the same Azéma supermartingale. However for $t > u$,

$$\mathbb{P}(\tau^{\text{Cox}} > u \,|\, \mathcal{F}_t) = \mathbb{P}(\tau^{\text{Cox}} > u \,|\, \mathcal{F}_u) = Z_u^{\text{Cox}} = \underline{Z}_u$$

whereas, for $t > T_u$, $G_t(u)$ is different from $\underline{Z}_t$.

3 Credit Risk Pricing - Top Down Approach

In this part of the article, having derived the formula for the pre-default price of claims of type I and type II, we proceed to take a top down view. In the following, we do not assume that a default time is given, instead, we consider as model inputs the objects which uniquely determines the pre-default price and show that it is possible to construct a random time which is consistent with these model inputs. To remind the reader, we summary below some results from section 2.

Claim of type I: We recall that the pre-default price of type I claim is given by

$$\pi_t = Z_t^{-1}\mathbb{E}[Y_T Z_T + \int_{(t,T]} C_u^{(j)} dA_u^j \,|\, \mathcal{F}_t] \quad j \in \{o, p\}, \tag{3.1}$$

where process $C^{(j)}$ is a predictable (optional) claim if $j = p$ ($j = o$).

Predictable claim: The pre-default price is uniquely determined by the Azéma supermartingale Z or equivalently the dual predictable projection process A^p.

Optional claim: The pre-default price is uniquely determined by the Azéma supermartingale Z and the dual optional projection process A^o. In fact the pre-default price is uniquely determined by A^o, since the Azéma supermartingale Z can be computed by setting $Z = {}^o(A_\infty - A^o)$.

Question: Given a non-decreasing optional process B such that $\mathbb{E}(B_\infty - B_{t-}|\mathcal{F}_t)$ takes values in $[0, 1]$, does there exist a random time τ (on a possibly extended space) such that $A^o = B$ and does not satisfy the hypothesis (H).

Remark 3.1 In view of the uniqueness of the Doob-Meyer (Doob-Meyer-Mertens-Gal'cŭk) decomposition, the existence of a random time with a given dual predictable (optional) projection is equivalent to the existence of a random time with a given Azéma (optional) supermartingale.

Claims of Type II: We recall that the pre-default price of a claim of type II is given by

$$\pi_t = Z_t^{-1}\mathbb{E}[Y_T Z_T + \int_{(t,T]} C_u(\zeta)\, F_T(du) \,|\, \mathcal{F}_t]. \tag{3.2}$$

The pre-default price is thus uniquely determined by the $\mathcal{F}_T$-conditional distribution $(F_T(u))_{u\leq T}$ and it should be considered as model inputs. Note that the Azéma supermartingale $(Z_u)_{u\leq T}$ can be retrieve from $\mathcal{F}_T$-conditional distribution $(F_T(u))_{u\leq T}$ by taking conditional expectations.

Question: Given a random field $(F_t(u), u, t \in \mathbb{R}_+)$, valued in $[0, 1]$ which is increasing in u and, for fixed u, is a martingale in t, does there exist a random time τ such that $\mathbb{P}(\tau \leq u \,|\, \mathcal{F}_t) = F_t(u)$.

Definition 3.2 *A predictable (optional) non-decreasing process B is said to be a valid dual predictable (optional) projection if $X := 1 - {}^o(B_\infty - B)$ takes value in $[0, 1]$.*

Definition 3.3
(i) A supermartingale is said to be a valid Azéma's supermartingale if it takes values in $[0, 1]$.
(ii) An optional supermartingale is said to be a valid Azéma's optional supermartingale if it takes values in $[0, 1]$ and the unique strongly predictable no-decreasing process in the Doob-Meyer-Mertens-Gal'cŭk decomposition is left continuous.

Definition 3.4 *A random field $(F_t(u), u, t \in \mathbb{R}_+)$ is called a valid $\mathbb{F}$-conditional distribution if it takes values in $[0, 1]$, is right continuous and non-decreasing in u and is such that, for a fixed u, $(F_t(u))_{t\geq 0}$ is an $\mathbb{F}$-martingale .*

3.1 The extended Cox construction

As shown in the previous subsection, Cox-times satisfy the hypothesis (H) and in that setting one can easily identify the dual optional projection of the constructed time. However, to the best of our knowledge, it is still a open question as to whether one can construct a random time τ with a given dual optional projection but does not satisfy hypothesis (H).

Given an $\mathbb{F}$-conditional distribution $(F_t(u), u, t \in \mathbb{R}_+)$ and a random variable U which is uniformly distributed on $[0, 1]$ and independent from $\mathcal{F}_\infty$, we consider an extension of the Cox-construction. That is we construct a random time τ by setting

$$\tau := \inf \{u : F_\infty(u-) > U\}.$$

From the above definition we can deduce the set equality $\{\tau < t\} = \{U < F_\infty(t-)\}$ and therefore

$$\mathbb{P}(\tau < u \,|\, \mathcal{F}_\infty) = F_\infty(u-) \quad \text{and} \quad \mathbb{P}(\tau \leq u \,|\, \mathcal{F}_\infty) = F_\infty(u).$$

By taking the $\mathcal{F}_u$-conditional expectation of both hand sides in the above, we retrieve the Azéma optional supermartingale and the Azéma supermartingale. That is[4]

$$\mathbb{P}(\tau < u \,|\, \mathcal{F}_u) = F_u(u-) \quad \text{and} \quad \mathbb{P}(\tau \leq u \,|\, \mathcal{F}_u) = F_u(u). \tag{3.3}$$

The above shows that given a valid $\mathbb{F}$-conditional distribution $(F_t(u), u, t \in \mathbb{R}_+)$, it is always possible to construct a random time τ such that $\mathbb{P}(\tau \leq u \,|\, \mathcal{F}_t) = F_t(u)$ and $\mathbb{P}(\tau < u \,|\, \mathcal{F}_t) = F_t(u-)$.

Definition 3.5 *Given a valid Azéma supermartingale Y (resp. Azéma optional supermartingale $\widetilde{Y}$), an $\mathbb{F}$-conditional distribution $(F_t(u), u, t \in \mathbb{R}_+)$ is said to be consistent with Y (resp. $\widetilde{Y}$) if $1 - F_u(u) = Y_u$ (resp. $1 - F_u(u-) = \widetilde{Y}_u$).*

Remark 3.6 To this end, in view of (3.3), the uniqueness of the Doob-Meyer decomposition and Theorem 5.30 in [12], in order to construct a random time with a given in advanced dual predictable projection B, it is sufficient to construct an $\mathbb{F}$-conditional distribution $(F_t(u), u, t \in \mathbb{R}_+)$ which is consistent with the Azéma supermartingale ${}^o(B_\infty - B)$.

Similarly, in view of (3.3), the uniqueness of the Doob-Meyer-Mertens-Gal'cŭk decomposition and Theorem 5.30 in [12], we see that in order to construct a random time with a given in advanced dual optional projection B, it is sufficient to construct an $\mathbb{F}$-conditional distribution $(F_t(u), u, t \in \mathbb{R}_+)$ which is consistent with the Azéma optional supermartingale ${}^o(B_\infty - B_-)$.

3.2 Random times with given dual predictable projection

From Remark 3.6, we see that in order to construct a random time with given in advanced dual predictable projection B, it is sufficient to construct a valid $\mathbb{F}$-conditional distribution which is consistent with the Azéma supermartingale $Y = {}^o(B_\infty - B)$. The construction of a valid $\mathbb{F}$-conditional distribution which is consistent with a given in advanced valid Azéma supermartingale was first considered in Gapeev et al. [16] and was later extended in Jeanblanc and Song [20] [21], Li and Rutkowski [29, 30] and Song [40, 41]. Without going into all the details, we will survey the literature and present below some existing methodologies.

In Gapeev et al. [16] and later Jeanblanc and Song [20] [21], the authors have considered the case where the Azéma supermartingale is of the form $Z = \mu e^{-\Lambda}$ where μ is a local martingale and Z, μ, Λ are continuous. In this case a closed form of a valid conditional $\mathbb{F}$-conditional distribution which is consistent with given in advanced Azéma supermartingale Z is obtained and is given by

$$F_t(u) = (1 - Z_t) \exp\left\{ - \int_u^t \frac{Z_s}{1 - Z_s} d\Lambda_s \right\}. \tag{3.4}$$

[4]By monotone convergence theorem it is clear that $F_t(u-)$ is +a martingale for $t \geq u$.

In this setting, the $\mathbb{F}$-conditional distribution $F_t(u)$ is differentiable w.r.t. u and the dynamics for $t > u$ is given by

$$\begin{aligned} dF_t(u) &= -\frac{F_t(u)}{1-Z_t} dM_t, \quad t \geq u \\ F_u(u) &= 1 - Z_u \end{aligned}$$

where M is the martingale part of the Doob-Meyer decomposition of Z. From this, one can construct a random time τ on an extended probability space and a probability $\mathbb{Q}$ such that $\forall t$, $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$ and $\mathbb{Q}(\tau \leq u \,|\, \mathcal{F}_t) = F_t(u)$. Obviously, for $t \leq u$ one has $F_t(u) = \mathbb{E}(F_u(u)|\mathcal{F}_t)$, hence we pay attention only to the family for $t \geq u$.

In Jeanblanc and Song [21], under the assumptions that $Z > 0$ is continuous and $1 - Z > 0$, a non-linear extension of the above model was considered. That is for any continuous local martingale Y and any bounded Lipschitz function f with $f(0) = 0$, a conditional distribution $M_t(u)$ which is consistent with $1-Z$ can be constructed as a solution of the following stochastic differential equation

$$\begin{aligned} dF_t(u) &= -F_t(u) \left(\frac{1}{(1-Z_t)} dM_t + f(M_t(u) - 1 + Z_t) dY_t \right), \quad t \geq u \\ F_u(u) &= 1 - Z_u \,. \end{aligned}$$

On the other hand, in Li and Rutkowski [29], for a valid Azéma supermartingale Z, it was shown that an $\mathbb{F}$-conditional distribution can be constructed using the *predictable multiplicative system* associated with $1 - Z$ introduced in Meyer [35]. In particular, if $1 - Z > 0$ then the constructed $\mathbb{F}$-conditional distribution satisfies the following system of differential equations: For any $u \geq 0$,

$$\begin{aligned} dF_t(u) &= -\frac{F_{t-}(u)}{(1 - {}^pZ_t)} dM_t \quad t \geq u \\ F_u(u) &= 1 - Z_u \end{aligned}$$

and it can be shown that

$$F_t(u) = (1 - Z_t) \exp\Big(-\int_{(u,t]} \frac{dA^{p,c}_s}{1 - {}^pZ_s}\Big) \prod_{u<s\leq t} \Big(1 - \frac{\Delta A^p_s}{1 - {}^pZ_s}\Big)$$

where $A^{p,c}$ is the continuous part of A^p. We point out here that random times which are constructed using multiplicative systems satisfy the hypothesis (HP).

Finally, in Song [40], inspired again by Meyer [35] and Yœurp and Meyer [43], the following non-linear system of stochastic differential equations (referred to as the $\natural$-equation or the natural equation) was considered under the assumptions that $ZZ_- > 0$ and $1 - {}^pZ > 0$,

$$\begin{aligned} dX^{u,x}_t &= -\frac{X^{u,x}_{t-}}{(1 - {}^pZ_t)} dM_t + \sigma(t, X^{u,x}_{t-}) dY_t, \qquad t \geq u \\ X^{u,x}_u &= x \,. \end{aligned} \tag{3.5}$$

where Y is an $\mathbb{F}$-local martingale. It was shown in [40] that if the pair (σ, Y) satisfies the following hypothesis 3.7 then, (σ, Y) is referred to as a $\natural$-pair, and the family of solutions $X^{u,1-Z_u}_t$ for $t \geq u$ can be considered as a model for an $\mathbb{F}$-conditional distribution which is consistent with the given Azéma supermartingale Z.

Hypothesis 3.7 *For every $u \geq 0$, on the interval $[u, \infty[$, we assume that the pair (σ, Y) satisfies (i) for given initial point x, the process $\sigma(\cdot, X^{u,x}_-)(1 - {}^pZ - X^{u,x}_-)^{-1} \mathbb{1}_{\{1 - {}^pZ_- - X^{u,x}_- \neq 0\}}$ is integrable with respect to Y and*

$$-(1 - {}^pZ)^{-1} \Delta M - \sigma(\cdot, X^{u,x}_-)(1 - {}^pZ - X^{u,x}_-)^{-1} \mathbb{1}_{\{1 - {}^pZ - X^{u,x}_- \neq 0\}} \Delta Y > -1,$$

(ii) the process $-\sigma(\cdot, X^{u,x})(X^{u,x}_-)^{-1}\mathbb{1}_{\{-X^{u,x}_-\neq 0\}}$ *is integrable with respect to* Y *and*

$$-(1-{}^pZ)^{-1}\Delta M + \sigma(\cdot, X^{u,x}_-)(X^{u,x}_-)^{-1}\mathbb{1}_{\{-X^{u,x}_-\neq 0\}}\Delta Y > -1,$$

(iii) given another $\mathbb{F}$*-adapted process* X' *taking values in* $[0,1]$*, the process* $(\sigma(\cdot, X'_-)-\sigma(\cdot, X^{u,x}_-))(X'-X_-)^{-1}\mathbb{1}_{\{X'-X^{u,x}_-\neq 0\}}$ *is integrable with respect to* Y *and*

$$-(1-{}^pZ)^{-1}\Delta M - (\sigma(\cdot, X'_-)-\sigma(\cdot, X^{u,x}_-))(X'_- - X^{u,x}_-)^{-1}\mathbb{1}_{\{X'_- - X^{u,x}_-\neq 0\}}\Delta Y > -1.$$

To conclude, we point out that it was shown in Theorem 3.7 of [40] that the set of ♮-pairs or natural-pairs is non-empty.

3.3 Random times with given dual optional projection

From Remark 3.6, we see that to construct a random time with given in advanced dual optional projection B, it is sufficient to construct a valid $\mathbb{F}$-conditional distribution which is consistent with the Azéma optional supermartingale $\widetilde{Y} = {}^o(B_\infty - B_-)$.

3.3.1 Optional multiplicative systems

To construct a random time which does not satisfy hypothesis (H) and has a given dual optional projection, we introduce below results on the optional multiplicative system associated with a positive optional submartingale.

The notion of a predictable multiplicative system was first introduced in Meyer [35] and later extended to the optional case in Li and Rutkowski [29]. However, in [29], the existence of an optional multiplicative system was only established for the case of the Azéma submartingale, that is we need a given in advance random time τ. In the following, given a positive optional submartingale X, we show the existence of an optional multiplicative system associated with X_+.

Recall that any positive optional submartingale X and its right continuous modification X_+ are of class-(D), therefore $X_\infty = \lim_{t\uparrow\infty} X_{t+}$ exists and is in L^1. For the reader's convenience, we recall below the definition of a multiplicative system.

Definition 3.8 *A multiplicative system is a positive random field* $(C_{u,t})_{u,t\in[0,\infty]}$ *satisfying the following conditions:*
(i) for all $u \le s \le t$ *the equality* $C_{u,s}C_{s,t} = C_{u,t}$ *holds; moreover,* $C_{u,t} = 1$ *for* $u \ge t$*,*
(ii) for any fixed $u \in \mathbb{R}_+$*, the process* $(C_{u,t})_{t\in[0,\infty]}$ *is adapted and non-increasing,*
(iii) for any fixed $t \in \mathbb{R}_+$*, the process* $(C_{u,t})_{u\in[0,\infty]}$ *is right-continuous and non-decreasing.*
A multiplicative system is called predictable (resp. optional) when for each u*, the process* $(C_{u,t})_{t\in[0,\infty]}$ *is predictable (resp. optional).*

Definition 3.9 *Suppose that* X *is a positive optional submartingale, we say that* $(C_{u,t})_{u,t\in[0,\infty]}$ *is a multiplicative system associated with* X_+ *if, in addition to conditions (i)-(iii) of Definition 3.8, we have, for all* $t \in [0,\infty]$*,*

$$\mathbb{E}[C_{t,\infty}X_\infty|\mathcal{F}_t] = X_{t+}. \tag{3.6}$$

The main tool used to obtain the existence of an optional multiplicative system associated with a submartingale is the Doob-Meyer-Mertens-Gal'čuk decomposition. Given any positive optional submartingale X, we can write $X = X_0 + M^X + A^X$, where M^X is an uniformly integrable martingale and A^X is a strongly predictable process of integrable variation and we can decompose A^X into $A^{X,r} + A^{X,g}$, where $A^{X,r}$ is càdlàg and $A^{X,g}$ is càglàd.

Lemma 3.10 *Let X be a strictly positive optional submartingale, bounded below by a strictly positive constant. Let the random field $\bar{C}_{u,t}$ be defined by $\bar{C}_{u,t} = 1$ for all $u \geq t$ and satisfy the equation*

$$\bar{C}_{u,t} = 1 - \int_{]u,t]} \bar{C}_{u,s-}({}^pX_s)^{-1}\, dA^{X,r}_s - \int_{[u,t[} \bar{C}_{u,s}(X_{s+})^{-1}\, dA^{X,g}_{s+}, \quad \forall\, t \geq u.$$

Then, for any u, the process $(\bar{Q}_{u,t} := \bar{C}_{u,t}X_t)_{t\in[u,\infty]}$ is a positive uniformly integrable martingale and it satisfies

$$d\bar{Q}_{u,t} = -\bar{C}_{u,t-}\, dM^X_t \tag{3.7}$$

and $C_{u,t} := \bar{C}_{u+,t+}$ is a multiplicative system associated with X_+.

Proof: We will first show that, for any fixed u, the process $(\bar{C}_{u,t})_{t\in[0,\infty]}$ is positive and bounded by one. To this end, it suffices to observe that it is a non-increasing process with $\bar{C}_{u,t} = 1$ for $t \leq u$ and from (3.10), the left jump satisfies $\bar{C}_{u,t} = \bar{C}_{u,t-}(1 - ({}^pX)^{-1}_t \Delta A^{X,r}_t)$ for $t > u$. Since $X_{t-} = ({}^pX)_t - \Delta A^{X,r}_t$, we obtain

$$0 < ({}^pX)_{t-}X^{-1}_{t-} = 1 - ({}^pX)^{-1}_{t-}\Delta A^{X,r}_t \leq 1,$$

while the right jump satisfies $\bar{C}_{u,t+} = \bar{C}_{u,t}(1 - X^{-1}_{t+}\Delta A^{X,g}_{t+})$ for $t \geq u$. Since $X_t = X_{t+} - \Delta A^{X,g}_t$, we obtain similarly

$$0 < X_{t+}X^{-1}_t = 1 - X^{-1}_{t+}\Delta A^{X,g}_{t+} \leq 1.$$

Thus we conclude that the process $(\bar{C}_{u,t})_{t\in[u,\infty]}$ is positive and bounded by one. Therefore, for any u, the process $(\bar{Q}_{u,t} = \bar{C}_{u,t}X_t)_{t\in[u,\infty]}$ is positive. Next, we show that, for any u the process $(\bar{Q}_{u,t})_{t\in[u,\infty]}$ is a uniformly integrable martingale. To this end, from the Itô formula we obtain

$$d\bar{Q}_{u,t} = -\bar{C}_{u,t-}\, dM^X_t$$

which is a uniformly martingale since $\bar{C}_{u,t}$ is bounded by one and M^X is uniformly integrable. For ease of notation we set

$$Y_t := 1 - \int_{(0,t]} ({}^pX_s)^{-1}\, dA^{X,r}_s - \int_{[0,t)} (X_{s+})^{-1}\, dA^{X,g}_{s+}.$$

From the form of the stochastic exponential, we have

$$\bar{C}_{u,t} = \exp\left\{Y_t - Y_u - \frac{1}{2}\int_{(u,t]} d\left\langle Y^c, Y^c\right\rangle_s\right\} \prod_{u<s\leq t}(1+\Delta Y_s)e^{-\Delta Y_s} \prod_{u\leq s<t}(1+\Delta^+ Y_s)e^{-\Delta^+ Y_s}.$$

From the above, it is clear that $\bar{C}_{u,t} = \bar{C}_{0,t}\bar{C}^{-1}_{0,u}$ and $\bar{C}_{u,u} = 1$.

Strictly speaking the random field $\bar{C}_{u,t}$ is not a multiplicative system associated with X since it is not right continuous in u. Therefore we need to regularise the random field by considering for all $u \leq t$ and $\epsilon > 0$

$$\bar{C}^{-1}_{0,u+\epsilon}\bar{C}_{0,t+\epsilon}X_{t+\epsilon} = X_{u+\epsilon} + \bar{C}^{-1}_{0,u+\epsilon}\int_{(0,t+\epsilon]} \bar{C}_{0,s-}dM^X_s - \bar{C}^{-1}_{0,u+\epsilon}\int_{(0,u+\epsilon]} \bar{C}_{0,s-}dM^X_s.$$

The right continuity of the stochastic integral gives for all $t \geq u$

$$\begin{aligned}\bar{C}^{-1}_{0,u+}\bar{C}_{0,t+}X_{t+} &= X_{u+} - \bar{C}^{-1}_{0,u+}\int_{(0,t]} \bar{C}_{0,s-}dM^X_s + \bar{C}^{-1}_{0,u+}\int_{(0,u]} \bar{C}_{0,s-}dM^X_s \\ &= X_{u+} - \int_{(u,t]} \bar{C}^{-1}_{0,u+}\bar{C}_{0,s-}dM^X_s.\end{aligned}$$

We note that for $s > u$, the term $\bar{C}_{u+,s-} = \bar{C}^{-1}_{0,u+}\bar{C}_{0,s-}$ is positive and bounded by one. Therefore $C_{u,t} := \bar{C}^{-1}_{0,u+}\bar{C}_{0,t+}$ is a multiplicative system associated with the submartingale X_+.

Corollary 3.11 *For any optional submartingale X, there exists an optional multiplicative system associated with X_+.*

Proof: We set $C_{u,t} := \lim_{\epsilon\downarrow 0} C^\epsilon_{u,t}$ where $C^\epsilon_{u,t}$ is defined in Lemma 3.10 for $X^\epsilon = X + \epsilon$. It is obvious that the family $C^\epsilon_{u,t}$ is non-increasing in t and non-decreasing in u. Moreover,

$$\mathbb{E}_{\mathbb{P}}\left[X_\infty C_{u,\infty} \,\middle|\, \mathcal{F}_u\right] = \mathbb{E}_{\mathbb{P}}\left[\lim_{\epsilon\downarrow 0}(X_\infty + \epsilon)C^\epsilon_{u,\infty} \,\middle|\, \mathcal{F}_u\right] = \lim_{\epsilon\downarrow 0}\mathbb{E}_{\mathbb{P}}\left[X^\epsilon_\infty C^\epsilon_{u,\infty} \,\middle|\, \mathcal{F}_u\right]$$
$$= \lim_{\epsilon\downarrow 0}(X_{u+} + \epsilon) = X_{u+}$$

where we used the monotone convergence theorem and the third equality follows from Lemma 3.10.

Remark 3.12 The significance of the above results is of two folds. Firstly, the existence of the optional multiplicative system $C_{u,t}$ associated with X_+ generalises the Madan-Roynette-Yor formula (see for example [39]). That is the Madan-Roynette-Yor formula or the equality

$$\mathbb{E}[\mathbb{1}_{\{\tau_t<u\}}(K - S_t)^+|\mathcal{F}_u] = (K - S_u)^+ \tag{3.8}$$

where $K > 0$ is the strike and S is the stock (a continuous uniformly integrable martingale), can be retrieved as a special case of (3.6) as soon as one sets[5] $X_\infty = X_t := (K - S_t)^+$ and $C_{u,\infty} = C_{u,t} = \mathbb{1}_{\{\tau_t<u\}}$ where $\tau_t = \sup\{s \le t : X_s = 0\}$.

In general equation (3.6) extends the equality $\mathbb{E}[\mathbb{1}_{\{\tau_\infty<u\}}X_\infty|\mathcal{F}_u] = X_u$ (resp. $\mathbb{E}[\mathbb{1}_{\{\tau_\infty<u\}}X_\infty|\mathcal{F}_u] = X_{u+}$) where X is a (resp. optional) semimartingale of class-(Σ) and $\tau = \tau_\infty = \sup\{s : X_s = 0\}$. For more details, we refer the reader to Cheridito et al. [4], Li [28] and the references within.

Secondly, given a valid Azéma optional supermartingale X, the random field $\bar{C}_{u,t}$ constructed in Lemma 3.10 can be used to construct a $\mathbb{F}$-conditional distribution which is consistent with X.

3.3.2 Application to construction of random time

Given an adapted integrable càdlàg non-decreasing process B such that $X_t := 1 - \mathbb{E}[B_\infty - B_{t-} \,|\, \mathcal{F}_t]$ is positive, we present a method to construct a random time τ such that the dual optional projection of $A := \mathbb{1}_{[\![\tau,\infty[\![}$ is given by B. As in the Cox-construction of random times, we suppose that there exists an uniformly distributed random variable U on $[0, 1]$ which is independent of $\mathcal{F}_\infty$. We note that since B_- is left continuous we have $B_- = B^c + B^g$.

Let $X^\epsilon := X + \epsilon$, one can construct a random field $\bar{C}^\epsilon_{u,t}$ as done in (3.10) and by applying monotone convergence theorem as done in Corollary 3.11 to show that there exists a random field $\bar{C}_{u,t}$ such that $X_t\bar{C}_{u,t}$ is a martingale for $u \le t$. Furthermore for a fixed t, $\bar{C}_{u,t}$ is left continuous in u, since $\bar{C}^\epsilon_{u,t}$ is left continuous in u and this property is preserved in the limit.

The important observation is that $F_t(u) = X_t\bar{C}_{u+,t}$ is an $\mathbb{F}$-conditional distribution which is consistent with the valid Azéma optional supermartingale $1 - X = {}^o(B_\infty - B_-)$. Therefore we can define $\tau := \inf\{u \ge 0 : X_\infty\bar{C}_{u,\infty} > U\}$ and for every $t \ge 0$, we have $\{\tau < t\} = \{X_\infty\bar{C}_{t,\infty} > U\}$. From (3.7) and independence of U, we have

$$\mathbb{P}(\tau < t \,|\, \mathcal{F}_t) = \mathbb{E}[X_\infty\bar{C}_{t,\infty} > U \,|\, \mathcal{F}_t] = \mathbb{E}[X_\infty\bar{C}_{t,\infty}|\, \mathcal{F}_t] = X_t.$$

We deduce from the uniqueness of the Doob-Meyer-Mertens-Gal'cŭk decomposition, $B_- = (A^o)_-$ and since both B and A^o are càdlàg (or one can apply Theorem 5.30 in [12]) we obtain $B = A^o$.

We now provide a counter example to Proposition 3.3 in Coculescu and Nikeghbali [6].

[5] We assume that all processes are stopped at t.

Example 3.13 Let K be some continuous non-decreasing process which takes value zero at time zero and $\lim_{t\to\infty} K_t = \infty$, We consider the non-decreasing continuous process

$$Y_t := \lambda(t \wedge T_1) + K_t$$

where T_1 be the first jump of a Poisson process N with parameter λ. For any continuous differentiable cumulative distribution function $F : \mathbb{R}_+ \to [0,1]$ with $F(0) = 0$ and bounded derivative $f = F' > 0$,

$$\begin{aligned}
F(Y_t) &= \lambda \int_{(0,t]} f(Y_s) d(s \wedge T_1) + \int_{(0,t]} f(Y_s) dK_s \\
&= \left\{ - \int_0^t f(Y_s) d\left(N_s - \lambda s\right)^{T_1} \right\} + \left\{ \int_0^t f(Y_s) dN_s^{T_1} \right\} + \left\{ \int_0^t f(Y_s) dK_s \right\} \\
&=: \mu_t^d + \zeta_t^d + \zeta_t^c
\end{aligned}$$

where μ^d is a purely discontinuous martingale, ζ^d a purely discontinuous non-decreasing process and ζ^c a non-decreasing continuous process. The process $F(Y)$ is a continuous submartingale valued in $[0,1]$ such that $F(Y_0) = 0$ and $F(Y_\infty) = 1$.

To construct a random time τ whose Azéma supermartingale is equal to $1 - F(Y)$, we consider the optional submartingale

$$\widetilde{F} = \mu^d + \zeta_-^d + \zeta^c \,.$$

By using Lemma 3.10 with the optional submartingale $\widetilde{F}$, we can construct a random time τ which has the property that $\mathbb{P}(\tau < t \,|\, \mathcal{F}_t) = \widetilde{F}$ and $\mathbb{P}(\tau \le t \,|\, \mathcal{F}_t) = F(Y_t)$. The constructed random time τ has a continuous Azéma supermartingale but does not avoid the first jump time of the Poisson process N since the dual optional projection of τ is given by

$$A_t^o = \zeta_t^d + \zeta_t^c = \left\{ \int_{(0,t]} f(Y_s) dN_s^{T_1} \right\} + \left\{ \int_{(0,t]} f(Y_s) dK_s \right\}.$$

To this end we point out that it is not difficult to include a continuous martingale component in F.

Moreover, the constructed time τ is an example of a random time where a non-increasing Azéma supermartingale does not imply that the random time is a pseudo-stopping time. It is also an example of a strict pseudo-stopping time which is not a pseudo-stopping time and do not satisfy the immersion hypothesis.

3.3.3 Extension of the natural-model

From (3.7), for every $u \ge 0$, we see that if $(1 - \widetilde{Z}_-)(1 - \widetilde{Z}) > 0$, then the dynamics of the constructed $\mathbb{F}$-conditional distribution satisfies the following linear differential equation

$$\begin{aligned}
dF_t(u-) &= -\frac{F_{t-}(u-)}{1 - \widetilde{Z}_{t-}} dm_t, \qquad t \ge u \\
F_u(u-) &= 1 - \widetilde{Z}_u.
\end{aligned}$$

In the following, by mimcing the predictable case presented in Song [40] we present an optional extension of the $\natural$-model or natural model. That is for a given $\mathbb{F}$-local martingale Y and real valued Borel function σ, we consider a non-linear extension of the above system of equations given by

$$\begin{aligned}
dX_t^{u,x} &= -X_{t-}^{u,x}(1 - \widetilde{Z}_{t-})^{-1} dm_t + \sigma(t, X_{t-}^{u,x}) dY_t, \qquad t \ge u \\
X_u^{u,x} &= x \,.
\end{aligned} \tag{3.9}$$

where x is an $\mathcal{F}_u$-measurable random variable.

Then given the appropriate conditions on the pair (σ, Y), the solution to the above system of equations with initial condition $1-\widetilde{Z}$ (after regularisation in u) should provide a model of the $\mathbb{F}$-conditional distribution distribution which is consistent with $1-\widetilde{Z}$. That is $\sup_{s<u} X_t^{s,1-\widetilde{Z}_s}$ should provide a model for $F_t(u-)=\mathbb{P}(\tau<u \,|\, \mathcal{F}_t)$, which satisfies the property that $F_u(u-)=1-\widetilde{Z}_u=\sup_{s<u} X_u^{s,1-\widetilde{Z}_s}$.

Lemma 3.14 *Given an $\mathbb{F}$-local martingale N such that $\Delta N>-1$ and an $\mathbb{F}$-adapted non-decreasing process V, the non-negative solution to the following làglàd linear stochastic differential equation*

$$Y_t^{u,x}=x+\int_{]u,t]} Y_{s-}^{u,x} dN_s+\int_{[u,t[} dV_s, \qquad t\geq u \tag{3.10}$$

is given by

$$Y_t^{u,x}=\mathscr{E}_{u,t}(N)\Big(x+\int_{[u,t[}[\mathscr{E}_{u,s}(N)]^{-1}dV_s\Big), \qquad t\geq u$$

where $\mathscr{E}_{u,t}(N):=\mathscr{E}_t(N)[\mathscr{E}_u(N)]^{-1}$.

Proof: For fixed $u\geq 0$, it is sufficient to apply integration by parts formula to $\mathscr{E}_{u,t}(N)K_{u,t}$ where $K_{u,t}=x+\int_{[u,t[}[\mathscr{E}_{u,s}(N)]^{-1}dV_s$. By noticing that $K_{u,t}=K_{u,t-}=K_{u,t}^g$, we obtain

$$\begin{aligned}\mathscr{E}_{u,t}(N)K_t &= x+\int_{[u,t[}\mathscr{E}_{u,s}(N)dK_{u,s+}^g+\int_{]u,t]}K_{u,s-}\mathscr{E}_{u,s-}(N)dN_s\\ &= x+\int_{[u,t[}dV_s+\int_{]u,t]}\mathscr{E}_{u,s-}(N)K_{u,s-}dN_s.\end{aligned}$$

By standard results on stochastic exponentials, the solution is non-negative if $\Delta N>-1$.

For simplicity, let us set $d\widetilde{m}_s=-(1-\widetilde{Z}_{s-})^{-1}dm_s$ then we have

$$1-\widetilde{Z}_t=\int_{(0,t]}(1-\widetilde{Z}_{s-})d\widetilde{m}_s+A_{t-}^o \tag{3.11}$$

which satisfies equation (3.10). To this end, we present below similar hypothesis on the pair (σ, Y) as those given in hypothesis 3.7 or on page 8 of Song [40]. The difference here is that we consider $(1-\widetilde{Z}_-)$ and $d\widetilde{m}_t=(1-\widetilde{Z}_{t-})^{-1}dm_t$ instead of $(1-{}^pZ)$ and $d\widetilde{M}_t=(1-{}^pZ_t)^{-1}dM_t$ respectively.

Hypothesis 3.15 *For every $u\geq 0$, on the interval $[u,\infty[$, we assume that the pair (σ, Y) satisfies (i) for given initial point x, the process $\sigma(\cdot, X_-^{u,x})(1-\widetilde{Z}_- - X_-^{u,x})^{-1}\mathbb{1}_{\{1-\widetilde{Z}_- - X_-^{u,x}\neq 0\}}$ is integrable with respect to Y and*

$$\Delta\widetilde{m}-\sigma(\cdot, X^{u,x})(1-\widetilde{Z}_- - X_-^{u,x})^{-1}\mathbb{1}_{\{1-\widetilde{Z}_- - X_-^{u,x}\neq 0\}}\Delta Y>-1,$$

(ii) the process $-\sigma(\cdot, X^{u,x})(X_-^{u,x})^{-1}\mathbb{1}_{\{-X_-^{u,x}\neq 0\}}$ is integrable with respect to Y and

$$\Delta\widetilde{m}+\sigma(\cdot, X_-^{u,x})(X_-^{u,x})^{-1}\mathbb{1}_{\{-X_-^{u,x}\neq 0\}}\Delta Y>-1,$$

(iii) given another $\mathbb{F}$-adapted process X' taking values in $[0,1]$, the process $-(\sigma(\cdot, X'_-)-\sigma(\cdot, X_-^{u,x}))(X'-X_-)^{-1}\mathbb{1}_{\{X'-X_-^{u,x}\neq 0\}}$ is integrable with respect to Y and

$$\Delta\widetilde{m}-(\sigma(\cdot, X'_-)-\sigma(\cdot, X_-^{u,x}))(X'_- - X_-^{u,x})^{-1}\mathbb{1}_{\{X'_- - X_-^{u,x}\neq 0\}}\Delta Y>-1.$$

Lemma 3.16 *Suppose that the pair (σ, Y) satisfies Hypothesis 3.15. Then,*
(i) if $x \le 1 - \widetilde{Z}_u$, one has $X_t^{u,x} \le 1 - \widetilde{Z}_t$ for all $t \in [u, \infty[$,
(ii) if $x > 0$, one has $X_t^{u,x} > 0$ for all $t \in [u, \infty[$,
(iii) if $x \le y$, one has $X_t^{u,x} \le X_t^{u,y}$ for all $t \in [u, \infty[$.

Proof: By taking the difference between (3.11) and (3.5) we obtain

$$\begin{aligned} 1 - \widetilde{Z}_t - X_t^{u,x} &= 1 - \widetilde{Z}_u - x + \int_{]u,t]} (1 - \widetilde{Z}_{s-} - X_{s-}^{u,x}) \left[d\widetilde{m}_s + \frac{\mathbb{1}_{\{1 - \widetilde{Z}_{s-} - X_{s-}^{u,x} \neq 0\}} \sigma(s, X_{s-}^{u,x})}{(1 - \widetilde{Z}_{s-} - X_{s-}^{u,x})} dY_s \right] + A_{t-}^o \\ &= 1 - \widetilde{Z}_u - x + \int_{]u,t]} (1 - \widetilde{Z}_{s-} - X_{s-}^{u,x}) \, d\widetilde{N}_s + A_{t-}^o. \end{aligned}$$

The result then follows from Lemma 3.14 and Hypthesis 3.15 (i). Similarly (ii) follows from Lemma 3.14 and Hypthesis 3.15 (ii), and finally (iii) follows from Lemma 3.14 and Hypothesis 3.15 (iii).

Theorem 3.17 *Suppose that the pair (σ, Y) satisfies Hypothesis 3.15 and that $X_t^{u,x}$ is a family of solutions to the system of stochastic differential equations (3.9). Let us set $M_t^u := X_t^{u, 1 - \widetilde{Z}_u}$, then the family*

$$F_t(u-) := \begin{cases} 1 - \widetilde{Z}_u & t = u \\ \sup_{s<u} M_t^s \wedge (1 - \widetilde{Z}_t), & t \in]u, \infty[\end{cases}$$

is an $\mathbb{F}$-conditional distribution which is consistent with $1 - \widetilde{Z}$.

Proof: For all $\mathbb{F}$-stopping times T such that $0 < u \le T \le \infty$, we have

$$\begin{aligned} 0 \le \mathbb{E}[M_T^u - \sup_{s<u} M_T^s \wedge (1 - \widetilde{Z}_T)] &= \mathbb{E}[M_T^u - \sup_{s<u} M_T^s] \\ &= \mathbb{E}[1 - \widetilde{Z}_u] - \sup_{s<u} \mathbb{E}[1 - \widetilde{Z}_s] \\ &= A_{u-}^o - A_{u-}^o = 0. \end{aligned}$$

The above computation implies that M^u and $\sup_{s<u} M^s$ are indistinguishable on $[u, \infty[$ and therefore the map $[0, t] \ni u \to M_t(u)$ is left continuous and non-decreasing function.

To this end, we can proof an analogy of Theorem 3.7 in Song [40] which gives sufficient conditions for Hypothesis 3.15 to be satisfied. However we do not do it here, and as an example, one can take Y to be a continuous $\mathbb{F}$-local martingale, $\sigma(t, x) = x \cdot f(1 - \widetilde{Z}_{t-} - x)$ where f is bounded, Lipschitz continuous and $f(0) = 0$. Then Hypothesis 3.15 will be satisfied.

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