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A Note on Graphs of Dichromatic Number 2

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Abstract
Neumann-Lara and Škrekovski conjectured that every planar digraph is 2-colourable. We show that this conjecture is equivalent to the more general statement that all oriented $K_5$-minor-free graphs are 2-colourable.

1 Introduction
Digraphs and graphs considered here are loopless, and without parallel or anti-parallel arcs. A directed edge starting in $u$ and ending in $v$ is denoted by $(u,v)$, $u$ is called its tail while $v$ is its head. In a digraph $D$, a vertex set $X \subseteq V(D)$ is called acyclic if the induced subdigraph $D[X]$ is acyclic. An acyclic colouring of $D$ with $k$ colours is a mapping $c : V(D) \to [k]$ such that $c^{-1}([i])$ is acyclic for all $i \in [k]$. The dichromatic number $\vec{\chi}(D)$ is defined as the minimal $k \geq 1$ for which such a colouring exists. For an undirected graph $G$, the dichromatic number $\vec{\chi}(G)$ is defined as the maximum dichromatic number an orientation of $G$ can have.

This notion has been introduced in 1982 by Neumann-Lara [NL82], was rediscovered by Mohar [Moh83], and since then has received further attention, see [AH15, MW16, ACH+16, HM17, LM17, BHKL18, HLTW19] for some recent results.

In analogy to the famous Four-Colour-Theorem, the following intriguing conjecture was proposed by Neumann-Lara [NL85] and independently by Škrekovski [BFJ+04].

Conjecture 1. If $G$ is a planar graph, then $\vec{\chi}(G) \leq 2$.

The strongest partial result obtained so far is due to Li and Mohar who showed the following:

Theorem 1 ([LM17]). Every planar digraph without directed triangles admits an acyclic 2-colouring.

The purpose of this note is to show the following:

Theorem 2. The following statements are equivalent:

• Every planar graph $G$ has $\vec{\chi}(G) \leq 2$.

• Every $K_5$-minor-free graph $G$ fulfils $\vec{\chi}(G) \leq 2$. Moreover, any orientation of $G$ admits an acyclic 2-colouring without monochromatic triangles.

This strengthening is similar to the situation for undirected graph colourings, where it is known that all $K_5$-minor-free graphs are 4-colourable [AH77, AHK77, Wag37].

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2 2-Colours of Planar Digraphs

In the following, we will use the term planar triangulation when we mean a maximal planar graph on at least three vertices. It is well-known that the latter (up to the choice of the outer face and reflections) admit combinatorially unique crossing-free embeddings in the plane or on the sphere, in which every face is bounded by a triangle (from now on called facial triangles). A frequent tool in our proof will be the following Lemma, which has already been used in [LM17].

Lemma 3 ([LM17]). Let $D_1$ and $D_2$ be digraphs which intersect in a tournament. Suppose that $c_1 : V(D_1) \to [k], c_2 : V(D_2) \to [k]$ are acyclic colourings such that $c_1|_{V(D_1) \cap V(D_2)} = c_2|_{V(D_1) \cap V(D_2)}$. Then the common extension of $c_1$ and $c_2$ to $V(D_1) \cup V(D_2)$ defines an acyclic $k$-colouring of $D_1 \cup D_2$.

In this section we prepare the proof of Theorem 2 with some strengthen but equivalent formulations of Neumann-Lara’s Conjecture.

Proposition 1. The following statements are equivalent:

(i) Neumann-Lara’s Conjecture, i.e., every planar digraph has an acyclic 2-colouring.

(ii) Every oriented planar triangulation admits an acyclic 2-colouring without monochromatic facial triangles.

(iii) For any planar triangulation $T$, any facial triangle $a_1a_2a_3$ in $T$, and any non-monochromatic pre-colouring $p : \{a_1, a_2, a_3\} \to \{1, 2\}$, every orientation $\vec{T}$ of $T$ admits an acyclic 2-colouring $c : V(\vec{T}) \to \{1, 2\}$ without monochromatic facial triangles such that $c(a_i) = p(a_i), i \in \{1, 2, 3\}$.

(iv) For any planar triangulation $T$, any triangle $a_1a_2a_3$ in $T$, and any non-monochromatic pre-colouring $p : \{a_1, a_2, a_3\} \to \{1, 2\}$, every orientation $\vec{T}$ of $T$ admits an acyclic 2-colouring $c : V(\vec{T}) \to \{1, 2\}$ without monochromatic triangles such that $c(a_i) = p(a_i), i \in \{1, 2, 3\}$.

![Figure 1: The octahedron-orientation $O_6$.](image)

Proof. $(i) \implies (ii)$: Suppose that every planar digraph is 2-colourable and let $\vec{T}$ be an arbitrary orientation of a planar triangulation $T$. Looking at the orientation $O_6$ of the octahedron graph depicted in Figure 1, it is easily observed that in any acyclic 2-colouring the boundary of the outer face cannot be monochromatic. Now consider a crossing-free spherical embedding of $\vec{T}$. For every facial triangle in this embedding whose orientation is transitive, we take a copy of $O_6$ and glue this copy into the face in such a way that the outer three edges of $O_6$ are identified with the three edges of the facial triangle (to make the orientations of the identified edges compatible,
it might be necessary to reflect and rotate the embedding of $O_6$ shown in Figure 1. This creates a crossing-free embedding of a planar oriented triangulation $T^\Delta$. By assumption, $T^\Delta$ admits an acyclic 2-colouring. This colouring restricted to the vertices of the subdigraph $T$ clearly is still valid. Furthermore, no triangle in $T$ can be monochromatic: This follows by definition if the triangle forms a directed cycle. If the orientation is transitive, by definition of $T^\Delta$, a monochromatic colouring would contradict the fact that $O_6$ has no acyclic 2-colouring with the outer three vertices being coloured the same.

$(ii) \implies (iii)$: Suppose that $(ii)$ holds. Let $\bar{T}$ be an orientation of a planar triangulation $T$ and let $a_1a_2a_3$ be the vertices of a facial triangle $t$ of $T$, equipped with a non-monochromatic pre-colouring $p : \{a_1, a_2, a_3\} \rightarrow \{1, 2\}$.

**Case 1: $t$ is not directed in $\bar{T}$.** By relabelling, we may assume the transitive orientation $(a_1, a_2), (a_1, a_3), (a_2, a_3) \in E(\bar{T})$. Now consider a plane embedding of $\bar{T}$ in which $a_1a_2a_3$ forms the bounding triangle of the outer face and where $a_1, a_2, a_3$ appear in clockwise order. We now define a new oriented planar triangulation $\bar{T}^*$ as follows: We consider the embedding and orientation of the $K_4$ as shown in Figure 2 in which the outer face has a clockwise direction and the central vertex is a source. Into each of the three inner faces of this embedding, we can now glue a copy of $\bar{T}$ with the described embedding in such a way that all orientations of identified edges agree. The vertex $a_1$ from each copy now is identified with the central vertex, which we call $x$. This oriented planar triangulation $\bar{T}^*$ according to assumption admits an acyclic 2-colouring $c^* : V(\bar{T}^*) \rightarrow \{1, 2\}$ without monochromatic facial triangles. By relabelling the colours, we may assume that $c^*(x) = 1$. Because the outer triangle is not monochromatic, there have to be edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ on the outer triangle such that $c^*(u_1) = c^*(v_2) = 1, c^*(u_2) =$
c^*(v_1) = 2. To both c_1, c_2 we have a corresponding copy of \( \vec{T} \), and the 2-colourings c_1, c_2 of \( \vec{T} \) which c^* induces on these copies are still valid acyclic colourings. Furthermore, there are no monochromatic facial triangles in \( \vec{T} \) with respect to c_1, c_2: Every bounded facial triangle of \( \vec{T} \) in the corresponding copy also forms a bounded facial triangle of \( \vec{T}^* \), while the outer triangles of the copies contain c_1 respectively c_2 and are thus not monochromatic under c_1 respectively c_2. From the way we glued the three copies of \( \vec{T} \) we conclude that c_1(a_1) = c^*(x) = 1, c_1(a_2) = c^*(u_1) = 1, c_1(a_3) = c^*(v_1) = 2 and c_2(a_1) = c^*(x) = 1, c_2(a_2) = c^*(u_2) = 2, c_2(a_3) = c^*(v_2) = 1.

Now consider the oriented planar triangulation \( \vec{T} \) which is obtained from \( \vec{T} \) by reversing the orientations of all edges. Let us rename the vertices of \( t \) according to \( a_1' := a_3, a_3' := a_1, a_2' := a_2 \). We have \((a_1', a_2'), (a_1', a_3'), (a_2', a_3') \in E(\vec{T})\). We can therefore apply the same arguments as above to \( \vec{T} \) with the transitive labeling \( a_1' a_2' a_3' \) of the triangle \( t \). Hence, after flipping all colours from 1 to 2 and 2 to 1 we have found extending colourings \( p \).

Finally, because the acyclic 2-colourings of \( \vec{T} \) and \( \vec{T} \) coincide, we conclude that c_1, c_2, c_1', c_2' : \( V(\vec{T}) \rightarrow \{1, 2\} \) of \( \vec{T} \) without monochromatic facial triangles such that c_1'(a_1') = c_2'(a_2') = 1, c_1'(a_3') = 2 and c_2'(a_3') = 1, c_2'(a_2') = 2, c_2'(a_3') = 1.

Case 2: \( t \) is directed in \( \vec{T} \). After relabelling (and possibly exchanging the colours 1 and 2), we may suppose that \((a_1, a_2), (a_2, a_3), (a_3, a_1) \in E(\vec{T})\) and \( p(a_1) = 1, p(a_2) = 1, p(a_3) = 2 \). Consider the orientation \( \vec{T} \) obtained from \( \vec{T} \) by reversing the edge \( e = (a_3, a_1) \). By the first case, we know that \( \vec{T} \) admits an acyclic 2-colouring \( c_e \) without monochromatic facial triangles which extends \( p \). Because the endpoints of \( e \) receive different colours, it follows directly that \( c_e \) also defines an acyclic 2-colouring of \( \vec{T} \) with the required properties, and the claim follows also in this case.

(iii) \( \Rightarrow \) (iv): We prove the statement by induction on the number of vertices. In the base case, where \( T \) is an oriented triangle, the statement clearly holds true. Now let \( n \geq 4 \), and assume that the statement holds for all triangulations with less than \( n \) vertices. Let \( \vec{T} \) be an arbitrary orientation of some planar triangulation \( T \) with \( n \) vertices. If \( T \) is 4-connected, then the only triangles in \( T \) are the facial triangles and therefore the claim follows from (iii). Therefore we may suppose that \( T \) is not 4-connected, i.e., there exists a separating triangle \( x_1, x_2, x_3 \) in \( T \). Consider some plane crossing-free embedding of \( T \). Here, \( x_1, x_2, x_3 \) separates the vertices in its interior \( V_{\text{in}} \subseteq V(T) \) from those in its exterior \( V_{\text{out}} \subseteq V(T) \). Let \( \vec{T}_{\text{out}} := \vec{T}[V_{\text{out}} \cup \{x_1, x_2, x_3\}] \) and \( \vec{T}_{\text{in}} := \vec{T}[V_{\text{in}} \cup \{x_1, x_2, x_3\}] \). Both form oriented planar triangulations on less than \( n \) vertices. To prove that \( \vec{T} \) satisfies the inductive claim, let \( t = a_1a_2a_3 \) be a given triangle in \( T \) equipped with a non-monochromatic pre-colouring \( p : \{a_1, a_2, a_3\} \rightarrow \{1, 2\} \). We must either have \(\{a_1, a_2, a_3\} \subseteq V_{\text{out}} \cup \{x_1, x_2, x_3\}\) or \(\{a_1, a_2, a_3\} \subseteq V_{\text{in}} \cup \{x_1, x_2, x_3\}\). Assume that we are in the first case, the second case is completely analogous. Then, by the induction hypothesis, there exists an acyclic colouring \( c_{\text{out}} : V(\vec{T}_{\text{out}}) \rightarrow \{1, 2\} \) without monochromatic triangles such that \( c_{\text{out}}(a_i) = p(a_i), i \in \{1, 2, 3\} \). The restriction of \( c_{\text{out}} \) to \( x_1, x_2, x_3 \) now defines a non-monochromatic pre-colouring for \( \vec{T}_{\text{in}} \), and it follows from the induction hypothesis that there exists an acyclic 2-colouring \( c_{\text{in}} \) of \( \vec{T}_{\text{in}} \) without monochromatic triangles which agrees with \( c_{\text{out}} \) on \( V(\vec{T}_{\text{out}}) \cap V(\vec{T}_{\text{in}}) = \{x_1, x_2, x_3\} \). By Lemma 3, the common extension of \( c_{\text{out}}, c_{\text{in}} \) to \( V(\vec{T}) \)
now defines an acyclic 2-colouring of $\vec{T}$, extending the given pre-colouring $p$ of $t$ and without monochromatic triangles. This verifies the inductive claim.

$(iv) \Rightarrow (i)$: This follows since every planar graph is a subgraph of a planar triangulation.

Because any edge in a planar triangulation lies on a triangle, we directly obtain the following.

**Corollary 4.** Under the assumption of Neumann-Lara’s Conjecture, every orientation of a planar graph admits an acyclic 2-colouring without monochromatic triangles which can be chosen to extend any given pre-colouring of an edge or any non-monochromatic pre-colouring of a triangle.

### 3 $K_5$-Minor-Free Graphs

Given a pair $G_1, G_2$ of undirected graphs such that $V(G_1) \cap V(G_2)$ forms a clique of size $i$ in both $G_1$ and $G_2$, and such that $|V(G_1)|, |V(G_2)| > i$, the graph $G$ with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$ is called the *proper $i$-sum* of $G_1$ and $G_2$. A graph obtained from $G$ by deleting some (possibly all or none) of the edges in $E(G_1) \cap E(G_2)$ is said to be an *$i$-sum* of $G_1$ and $G_2$. The central tool in proving Theorem 2 is the following classical result due to Wagner. By $V_8$ we denote the so-called Wagner graph, that is the graph obtained from $C_8$ by joining any two diagonally opposite vertices by an edge.

**Theorem 5 ([Wag37]).** A simple graph is $K_5$-minor-free if and only if it can be obtained from planar graphs and copies of $V_8$ by means of repeated $i$-sums with $i \in \{0, 1, 2, 3\}$.

$V_8$ is triangle-free and admits an acyclic 2-colouring for any orientation. Moreover, it can be easily checked that such a colouring can be chosen to extend any given pre-colouring of two adjacent vertices. We are now in the position to prove Theorem 2.

**Proof of Theorem 2.** Assume that Neumann-Lara’s Conjecture holds true. We have to prove that every oriented $K_5$-minor-free graph admits an acyclic 2-colouring without monochromatic triangles.

Any orientation of a $K_5$-minor-free graph admits an acyclic 2-colouring without monochromatic triangles which can be chosen to extend any given pre-colouring of an edge or any non-monochromatic pre-colouring of a triangle.

Assume towards a contradiction that there exists a $K_5$-minor-free graph $G$ which does not satisfy this claim and choose $G$ minimal with respect to the number of vertices, and among all such graphs maximal with respect to the number of edges.

By Corollary 4 we know that the claim is fulfilled by all planar graphs and by $V_8$, and so it follows from Theorem 5 that $G$ is the $i$-sum of two $K_5$-minor-free graphs $G_1, G_2$ with fewer vertices, where $0 \leq i \leq 3$. By the minimality assumption, we therefore know that $G_1, G_2$ satisfy the claim. Clearly, every super-graph of $G$ does not satisfy the assertion as well. Therefore, by the assumed edge-maximality, $G$ must in fact be the proper $i$-sum of $G_1$ and $G_2$.

Now choose some orientation $\vec{G}$ of $G$ for which the above claim fails. Denote by $\vec{G}_1, \vec{G}_2$ the induced orientations on the subgraphs $G_1, G_2$.

Let $e = uv \in E(G) = E(G_1) \cup E(G_2)$ be an arbitrary edge with a given pre-colouring $p : \{u, v\} \rightarrow \{1, 2\}$. W.l.o.g. assume that $e \in E(G_1)$. Let $c_1 : V(G_1) \rightarrow \{1, 2\}$ be an acyclic 2-colouring of $\vec{G}_1$ without monochromatic triangles which extends $p$. The clique $C = V(G_1) \cap V(G_2)$ in $G_2$ is either empty, a single vertex, and edge or a triangle. In each case, the restriction $p' = c_1|C$ (if non-empty) can be considered as a pre-colouring of a vertex, an edge or a triangle in $G_2$ with two colours. In the case where $C$ is a triangle, by the choice of $c_1$, we furthermore know that
the pre-colouring $p$ is not monochromatic. We therefore conclude that in any case, there is an acyclic 2-colouring $c_2 : V(G_2) \to \{1, 2\}$ of $G_2$ without monochromatic triangles which extends $p'$. Therefore $c_1$ and $c_2$ agree on $V(G_1) \cap V(G_2) = C$ and it follows from Lemma 3 that the common extension of $c_1, c_2$ to $V(G)$ defines an acyclic 2-colouring of $\vec{G}$ which extends $p$. Because every triangle of $G$ is fully contained in $G_1$ or $G_2$, we also have that there are no monochromatic triangles under $c$.

Similarly, for any triangle $t = a_1a_2a_3$ in $G$ equipped with a non-monochromatic pre-colouring $p : \{a_1, a_2, a_3\} \to \{1, 2\}$, we may assume w.l.o.g. that $t$ is fully contained in $G_1$. Again, we find a pair of acyclic 2-colourings $c_1, c_2$ of $G_1, G_2$ such that $c_1$ extends $p$, $c_2$ coincides with $c_1$ on the clique $V(G_1) \cap V(G_2)$, and there are no monochromatic triangles in $G_j$ under $c_j$ for $j = 1, 2$.

Finally, the common extension of $c_1, c_2$ to $V(G)$ by Lemma 3 defines an acyclic 2-colouring of $\vec{G}$ with the desired properties.

From this we conclude that $\vec{G}$ admits an extending acyclic 2-colouring without monochromatic triangles for any pre-colouring of an edge and for any non-monochromatic pre-colouring of a triangle. This is a contradiction to our choice of $\vec{G}$. This shows that the initial assumption was false and concludes the proof of the Theorem. 

\section{Conclusion}

A natural question that comes out from the discussion in this paper is the following.

\textbf{Question 1.} \textit{What is the largest minor-closed class $G_2$ of undirected graphs with dichromatic number at most 2?}

While $\chi(K_6) = 2$, it is known that $\chi(K_7) = 3$. Therefore, $G_2$ is a subclass of the $K_7$-minor-free graphs. However, $G_2$ seems to be a lot smaller than this class. In fact, there are $K_{3,3}$-minor-free graphs with dichromatic number greater than 2, see Figure 3 for a simple example.

![Figure 3: An oriented $K_{3,3}$-minor-free graph without an acyclic 2-colouring.](image)
References


