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Nicolae Cîndea* Andaluzia Matei† Sorin Micu‡ Constantin Niţă§

Abstract

We consider a contact model with power-law friction in the antiplane context. Our study focuses on the boundary optimal control, paying attention on existence and uniqueness results, on optimality conditions as well as on a computation method. The computation technique is based on linearisation combined with a fixed point method and the saddle point theory.

Keywords: nonlinear boundary value problem, antiplane model, power-law friction, optimal control, optimality conditions, fixed point, approximation
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1 Introduction

In this paper we consider a model describing the antiplane shear deformation of an elastic, isotropic, nonhomogeneous cylindrical body, in frictional contact on a part of the boundary with a rigid foundation. If we refer the cylinder to a cartesian coordinate system $Ox_1x_2x_3$ such that its generators are parallel with the axis Ox_3, the cross section of the body is a bounded connected open set $\Omega \subset Ox_1x_2$. The boundary Γ of Ω is Lipschitz continuous and partitioned in three measurable parts Γ_1, Γ_2 and Γ_3 of positive measure. From the mathematical point of view the model consists of the following nonlinear boundary value problem

$$\begin{aligned}
\operatorname{div}(\mu(x)\nabla u(x)) + f_0(x) = 0 \qquad & \text{in } \Omega, && (1)\\
u(x) = 0 \qquad & \text{on } \Gamma_1, && (2)\\
\mu(x)\partial_\nu u(x) = f_2(x) \qquad & \text{on } \Gamma_2, && (3)\\
\mu(x)\partial_\nu u(x) = -g(x)\,|u(x)|^{r-1}\,u(x) \qquad & \text{on } \Gamma_3. && (4)
\end{aligned}$$

The unknown of the problem is a function $u = u(x_1, x_2) : \overline{\Omega} \to \mathbb{R}$ that represents the third component of the displacement vector which in the antiplane model has the particular form $(0, 0, u(x_1, x_2))$. The function $\mu = \mu(x_1, x_2) : \overline{\Omega} \to \mathbb{R}$ is a coefficient of the material, the functions $f_0 = f_0(x_1, x_2) : \Omega \to \mathbb{R}, f_2 = f_2(x_1, x_2) : \Gamma_2 \to \mathbb{R}$ are related to the density of the body forces and the density of the surface traction, respectively and $g : \Gamma_3 \to \mathbb{R}_+$ is the coefficient of friction. The vector $\nu = \nu(x_1, x_2) = (\nu_1(x_1, x_2), \nu_2(x_1, x_2))$ represents the outward unit normal vector to the boundary and $\partial_\nu u = \nabla u \cdot \nu$. The boundary condition (4)

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on Γ_3 depends on the positive parameter r and it is called in the literature *power-law friction*. Formally, Coulomb law with the friction coefficient $g = g(x)$ (also known as friction Tresca law) is obtained from (4) in the limit $r \to 0$. For this reason, our study will be mainly focused on the case $0 < r < 1$ but the case $r \geq 1$ will be also addressed. For details on antiplane contact models and their mathematical analysis we refer, e.g., to [7].

Our aim is to study an optimal control problem which consists in minimizing the distance between u and a given target u_d by acting with a control force f_2 only on the part Γ_2 of the boundary. At the same time, we want to keep as small as possible the L^2-norm of the control f_2. Consequently, our objective will be to minimize the functional $J : L^2(\Gamma_2) \to \mathbb{R}$ defined by

$$J(f_2) = \frac{\alpha}{2}\, \|\nabla(u - u_d)\|^2_{L^2(\Omega)} + \frac{\beta}{2}\, \|f_2\|^2_{L^2(\Gamma_2)}\,, \tag{5}$$

where α and β are two positive real numbers and u is the solution of (1)-(4). Thus, we are dealing with a control problem in which we want to keep the deformation of the body as close as possible to a reference target by acting on a small part of the boundary with a minimal cost.

To study this problem we consider an equivalent formulation consisting in minimizing the bifunctional

$$L(u, f_2) = \frac{\alpha}{2}\, \|\nabla(u - u_d)\|^2_{L^2(\Omega)} + \frac{\beta}{2}\, \|f_2\|^2_{L^2(\Gamma_2)}\,,$$

defined on the set $\mathcal{V}_{ad}$ of pairs $[u, f_2]$ verifying (1)-(4). A minimizer of L will be called optimal pair and its second component optimal control. By using the direct method in the calculus of variations, we justify the existence of at least one optimal pair. Notice that we cannot use the classical convex minimization results because the admissible set $\mathcal{V}_{ad}$ is not convex. Studying the case $0 < r < 1$, we are able to write an optimality condition only for a regularized problem by means of a parameter ρ. Convergence results as ρ tends to 0 are proved for the state problem as well as for the control problem. On the other hand, for the case $r \geq 1$, an optimality condition is written for the original problem, without any regularization. Then we focus on the computation of the optimal control. To this end in view, we linearize the problem and, by means of the saddle point theory, we characterize the corresponding optimal control. We go back to the original nonlinear problem by using a fixed point technique. In each of the cases $r \in (0, 1)$ and $r \geq 1$ we show that, under appropriate smallness assumptions on the data, it is possible to define a contraction map and to obtain an optimal pair from its unique fixed point. In the case $r \geq 1$ for the original problem and $r \in (0, 1)$ for the regularized one, this technique allows to prove the uniqueness of the optimal control, a result which is not so common in this non convex context.

Our results represent a contribution to the optimal control theory of contact problems; see also, e.g., [3, 9, 10] for other recent results on the existence of optimal controls for problems governed by variational and hemivariational inequalities.

The paper has the following structure. In Section 2 we describe the functional setting and we introduce the state and the optimal control problems. Section 3 is devoted to the optimality conditions, treating separately the problem in the case $r \geq 1$ and the regularized version of it when $0 < r < 1$. In Section 4 we present convergence results of the regularized problem to the original one. In Section 5 we study the fixed point method approach to our optimal control problem and in Section 6 we present some numerical experiments based on it.

2 The optimal control problem

Firstly, let us describe the functional setting and fix the appropriate hypothesis allowing to study problem (1)-(4). We assume that

$$f_0 \in L^2(\Omega), \quad f_2 \in L^2(\Gamma_2), \tag{6}$$

$$\mu \in L^\infty(\Omega) \text{ and there exists } \mu^* > 0 \text{ such that } \mu(x) \geq \mu^* \text{ a.e. } x \in \Omega, \tag{7}$$

$$g \in L^\infty(\Gamma_3) \text{ and } g(x) \geq 0 \text{ a.e. } x \in \Gamma_3, \tag{8}$$

and we introduce the following Hilbert space

$$V = \{v \in H^1(\Omega) : \gamma v = 0 \text{ a.e. on } \Gamma_1\},$$

where γ denotes the Sobolev trace operator. We consider the inner product on V, defined by

$$(\cdot,\cdot)_V : V \times V \to \mathbb{R}, \quad (u,v)_V = \int_\Omega \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x,$$

and denote by $\|\cdot\|_V$ the associated norm. We recall that, according to, e. g., [8, Theorem 2.21], for each $s \geq 1$, the trace operator $\gamma : H^1(\Omega) \to L^s(\Gamma)$ is linear, continuous and compact. So, there exists $c_0 = c_0(s) > 0$ such that

$$\|\gamma v\|_{L^s(\Gamma)} \leq c_0 \|v\|_V \qquad (v \in V). \tag{9}$$

In the sequel, although c_0 in (9) depends on the exponent s of the space $L^s(\Gamma)$, to simplify the writing, we shall replace it by an absolute upper bound c_0 independent of s. This is justified by the fact that we apply this inequality only for a finite number of values s. Moreover, we recall the following Poincaré type inequality

$$\|v\|_{L^2(\Omega)} \leq c_P \|v\|_V \qquad (v \in V), \tag{10}$$

where c_P is a positive constant depending on Ω and Γ_1.

Let $r > 0$. We define the operator $A : V \to V$ and the functional $j : V \to \mathbb{R}$ by the equalities

$$(Au, v)_V = \int_\Omega \mu(x) \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x \qquad (u, v \in V), \tag{11}$$

$$j(v) = \frac{1}{r+1} \int_{\Gamma_3} g(x)\,|\gamma v(x)|^{r+1}\,\mathrm{d}\Gamma \qquad (v \in V), \tag{12}$$

and note that by assumptions (7)-(8) and since $\Omega \subseteq \mathbb{R}^2$ the integrals in (11) and (12) are well defined. We consider the following *state problem* representing the weak formulation of (1)-(4):

$$\left.\begin{aligned} &\textit{Let } f_2 \in L^2(\Gamma_2) \textit{ (called control). Find } u \in V \textit{ such that} \\ &(Au, v-u)_V + j(v) - j(u) \geq \int_\Omega f_0(x)(v(x) - u(x))\,\mathrm{d}x \\ &\qquad + \int_{\Gamma_2} f_2(x)(\gamma v(x) - \gamma u(x))\,\mathrm{d}\Gamma \qquad (v \in V). \end{aligned}\right\} \tag{SP}$$

According to [7, Theorem 3.1], for every control $f_2 \in L^2(\Gamma_2)$, problem (SP) has a unique solution $u = u(f_2) \in V$ verifying

$$\|u\|_V \leq \frac{1}{\mu^*} \left(\|f_0\|_{L^2(\Omega)} + c_0 \|f_2\|_{L^2(\Gamma_2)} \right), \tag{13}$$

where μ^* is the constant in (7) and c_0 is the constant in (9).

Now, let us introduce the optimal control problem we want to study. Let α, $\beta > 0$ be two positive constants and we define the following functional

$$L : V \times L^2(\Gamma_2) \to \mathbb{R}, \quad L(u, f_2) = \frac{\alpha}{2} \|u - u_d\|_V^2 + \frac{\beta}{2} \|f_2\|^2_{L^2(\Gamma_2)}. \tag{14}$$

Furthermore, we denote

$$\mathcal{V}_{\rm ad} = \left\{ [u, f_2] \;\middle|\; [u, f_2] \in V \times L^2(\Gamma_2), \text{ such that (SP) is verified} \right\},$$

and we introduce the following *optimal control problem:*

$$\textit{Find } [u^*, f_2^*] \in \mathcal{V}_{ad} \textit{ such that } L(u^*, f_2^*) = \min_{[u,f_2] \in \mathcal{V}_{\rm ad}} L(u, f_2). \tag{OCP}$$

Notice that, problem (OCP) is equivalent to the minimization of the functional J introduced in (5). Unlike J, the functional L given by (14) is convex. However, the minimization domain $\mathcal{V}_{\rm ad}$ is not convex. By using the same type of arguments as in [6, Theorem 3.7] we can deduce the following result.

Theorem 1. *Let $r > 0$ and assume that* (6)*,* (7) *and* (8) *hold. Then,* (OCP) *has at least one solution* $[u^*, f_2^*]$.

A solution of (OCP) will be called an *optimal pair.* The second component of the optimal pair is called an *optimal control.*

Remark 1. *Under the previous hypothesis we cannot ensure the uniqueness of the optimal pair. In Sections 5.2 and 5.3 we shall give some uniqueness results under more restrictive conditions (see Theorems 7 and 9 below).*

Remark 2. *When the power r in* (1)-(4) *tends to zero we obtain in the limit that the boundary condition* (4) *is replaced by the Tresca friction law*

$$|\mu(x)\partial_\nu u(x)| \leq g(x), \quad \mu(x)\partial_\nu u(x) = -g(x)\frac{u(x)}{|u(x)|} \textit{ if } u(x) \neq 0 \textit{ on } \Gamma_3. \tag{15}$$

Therefore, we can see (4) *as a regularized version of* (15)*. Finally, let us notice that a boundary control problem for a model with Tresca friction law has been studied in [6].*

3 Optimality conditions

In this section we study the optimality conditions corresponding to our minimization problem (OCP). We shall analyze separately the cases $r \geq 1$ and $r \in (0, 1)$. In the former case, the regularity properties of the functional j allows to deduce an optimality condition using a classical approach due to J.-L. Lions (see, for instance [5] and also [1, Lemma 3.11, p. 1127]). When $r \in (0, 1)$ we cannot use directly this argument and we need to introduce a regularized version of the problem depending on a small parameter ρ. In the sequel, $D^k G$ will denote the $k-$th derivative of the function G.

3.1 The case $r \geq 1$

The following result gives the optimality condition in this case.

Theorem 2. *Let $r \geq 1$. Any optimal control f_2^* of problem* (OCP) *verifies*

$$f_2^* = -\frac{1}{\beta}\gamma\left(\eta(f_2^*)\right), \tag{16}$$

where, for each $f_2 \in L^2(\Gamma_2)$, $\eta(f_2)$ is the unique solution of equation

$$\alpha(u - u_d, w)_V = \left(\eta(f_2), Aw + D^2 j(u) w\right)_V \qquad (w \in V), \tag{17}$$

and $u = u(f_2)$ is the solution of (SP).

Proof. Let us define $F : V \times L^2(\Gamma_2) \to V$,

$$F(u, f_2) = Au + D^1 j(u) - y(f_0) - y(f_2), \qquad (u \in V, f_2 \in L^2(\Gamma_2)), \tag{18}$$

where $y(f_0)$ and $y(f_2)$ are two elements from V given by

$$(y(f_0), v)_V = \int_\Omega f_0(x) v(x)\,\mathrm{d}x \qquad (v \in V), \tag{19}$$

$$(y(f_2), v)_V = \int_{\Gamma_2} f_2(x) \gamma v(x)\,\mathrm{d}\Gamma \qquad (v \in V). \tag{20}$$

Notice that $F(u, f_2) = 0$ is equivalent to the fact that u is a solution of (SP).

According to Lemma 3.11 from [1], we obtain that the derivative of the functional J defined by (5) is given by

$$\begin{aligned}(D^1 J(f_2), \xi)_{L^2(\Gamma_2)} =&(\partial_2 L(u(f_2), f_2), \xi)_{L^2(\Gamma_2)} \\ &- (\eta(f_2), \partial_2 F(u(f_2), f_2)\xi)_V \qquad (\xi \in L^2(\Gamma_2)),\end{aligned} \tag{21}$$

if $\partial_1 F(u(f_2), f_2)$ is a homeomorphism in V and $\eta(f_2) \in V$ verifies (17). Since $\partial_1 F(u, f_2)v = Av + D^2 j(u)v$, the homeomorphism property of $\partial_1 F(u(f_2), f_2)$ follows from the fact that, according to Lax Milgram's lemma, for each $h \in V$, there exists a unique $v^* \in V$ such that

$$(Av^*, w)_V + (D^2 j(u(f_2))v^*, w)_V = (h, w)_V \qquad (w \in V). \tag{22}$$

From (21) we deduce that

$$(D^1 J(f_2), \xi)_{L^2(\Gamma_2)} = \beta(f_2, \xi)_{L^2(\Gamma_2)} + (\eta(f_2), y(\xi))_V \qquad (\xi \in L^2(\Gamma_2)), \tag{23}$$

where $y(\xi)$ is defined by (20). Since f_2^* is a minimizer of J, from (23) we obtain the following optimality condition

$$\beta(f_2^*, \xi)_{L^2(\Gamma_2)} + (\eta(f_2^*), y(\xi))_V = 0 \qquad (\xi \in L^2(\Gamma_2)), \tag{24}$$

where $\eta(f_2^*)$ is the unique solution of (17) with $f_2 = f_2^*$. By taking into account (20), relation (24) is equivalent to (16) which concludes the proof. $\square$

Remark 3. *It is easy to see that, for each $u \in V$, the second derivative of j in u is given by*

$$\left(D^2 j(u)w, v\right)_V = r \int_{\Gamma_3} g(x) \left|\gamma u(x)\right|^{r-1} \gamma w(x) \gamma v(x)\,\mathrm{d}\Gamma \qquad (v, w \in V). \tag{25}$$

Notice that, if $r \in (0, 1)$, the right hand side expression in (25) *is not well-defined. This indicates that j given by* (12) *is not sufficiently smooth to derive it two times and no optimality condition can be obtained. In the next section we regularize the function j and we deduce an optimality condition for the corresponding problem.*

3.2 The case $r \in (0, 1)$

Let us fix $\rho > 0$. We introduce the following regularized version of the state problem (SP):

$$\left.\begin{aligned} &\textit{Let } f_2 \in L^2(\Gamma_2) \textit{ (called control). Find } u_\rho \in V \textit{ such that} \\ &(Au_\rho, v - u_\rho)_V + j_\rho(v) - j_\rho(u_\rho) \geq \int_\Omega f_0(x)(v(x) - u_\rho(x))\,\mathrm{d}x \\ &\qquad + \int_{\Gamma_2} f_2(x)(\gamma v(x) - \gamma u_\rho(x))\,\mathrm{d}\Gamma \qquad (v \in V), \end{aligned}\right\} \tag{SP$^\rho$}$$

where the functional $j_\rho : V \to \mathbb{R}$ is defined as follows

$$j_\rho(v) = \frac{1}{r+1} \int_{\Gamma_3} g(x) \left(\sqrt{|\gamma v(x)|^{2r+2} + \rho^2} - \rho \right) \mathrm{d}\Gamma \qquad (v \in V). \tag{26}$$

As in the case of problem (SP), we can deduce that, for every $f_2 \in L^2(\Gamma_2)$, the problem (SP$^\rho$) has a unique solution $u_\rho = u_\rho(f_2) \in V$, verifying (13).

By defining the new admissible set,

$$\mathcal{V}_{\mathrm{ad}}^{\rho} = \left\{ [u, f_2] \;\middle|\; [u, f_2] \in V \times L^2(\Gamma_2), \text{ such that } (\mathrm{SP}^\rho) \text{ is verified} \right\},$$

and by using the functional L from (14), we introduce a new *optimal control problem*:

$$\textit{Find } [u_\rho^*, f_{2,\rho}^*] \in \mathcal{V}_{\mathrm{ad}}^{\rho} \textit{ such that } L(u_\rho^*, f_{2,\rho}^*) = \min_{[u,f_2] \in \mathcal{V}_{\mathrm{ad}}^{\rho}} L(u, f_2). \tag{OCP$^\rho$}$$

As in the case of Theorem 1, the next result can be proved following [6, Theorem 3.7].

Theorem 3. *Assume that* (6)*,* (7) *and* (8) *hold. Then,* (OCP$^\rho$) *has at least one solution* $[u_\rho^*, f_{2,\rho}^*]$.

In this last part of this section we obtain an optimality condition for the problem (OCP$^\rho$). We have the following result.

Theorem 4. *Let* $\rho > 0$ *and* $r \in (0, 1)$*. Any optimal control* $f_{2,\rho}^*$ *of problem* (OCP$^\rho$) *verifies*

$$f_{2,\rho}^* = -\frac{1}{\beta} \gamma \left(\eta(f_{2,\rho}^*) \right), \tag{27}$$

where, for each $f_2 \in L^2(\Gamma_2)$*,* $\eta(f_2)$ *is the unique solution of equation*

$$\alpha(u_\rho - u_d, w)_V = \left(\eta(f_2), Aw + D^2 j_\rho(u_\rho) w \right)_V \qquad (w \in V), \tag{28}$$

and, for all $v \in V$

$$\left(D^2 j_\rho(u_\rho) w, v \right)_V = \int_{\Gamma_3} g(x) \frac{|u_\rho(x)|^{2r} \left[r \, |u_\rho(x)|^{2r+2} + (2r+1)\rho^2 \right]}{\left(|u_\rho(x)|^{2r+2} + \rho^2 \right)^{3/2}} w(x) v(x) \, \mathrm{d}\Gamma,$$

$u_\rho = u_\rho(f_2)$ *being the solution of* (SP$^\rho$).

Proof. It is similar to the proof of Theorem 2 and we omit it. □

Remark 4. *The replacement of the functional* j *from* (SP) *by its regularized version* j_ρ *in* (SP$^\rho$) *has enabled us to deduce the optimality condition* (27)-(28) *for the corresponding minimization problem* (OCP$^\rho$)*. Indeed,* j_ρ *is a convex, lower semi-continuous and two times Gâteaux differentiable functional. The question that an optimality condition can be obtained for the initial minimization problem* (OCP) *in the case* $r \in (0, 1)$ *remains open.*

4 Convergence properties

In the first part of this section we prove the convergence of the unique solution of problem (SP$^\rho$) to the solution of (SP).

Proposition 1. *Let $r \in (0,1)$, $f_0 \in L^2(\Omega)$ and $f_2 \in L^2(\Gamma_2)$ be given and let u be the corresponding solution of problem* (SP). *For each $\rho > 0$, let u_ρ be the solution of problem* (SP$^\rho$). *Then $u_\rho \to u$ in V as $\rho \to 0$.*

Proof. Since g verifies (8), the following inequality holds for all $v \in V$,

$$\int_{\Gamma_3} g(x) \left(\rho + |\gamma v(x)|^{r+1} - \sqrt{|\gamma v(x)|^{2r+2} + \rho^2} \right) \mathrm{d}\Gamma \leq \rho \int_{\Gamma_3} g(x)\, \mathrm{d}\Gamma. \tag{29}$$

On the other hand, by definition of the functionals j_ρ and j we obtain

$$|j_\rho(v) - j(v)| \leq \frac{1}{r+1} \int_{\Gamma_3} g(x) \left| \sqrt{|\gamma v(x)|^{2r+2} + \rho^2} - \rho - |\gamma v(x)|^{r+1} \right| \mathrm{d}\Gamma.$$

From the last inequality and (29) we are led to

$$|j_\rho(v) - j(v)| \leq \rho \int_{\Gamma_3} g(x)\, \mathrm{d}\Gamma \qquad (v \in V). \tag{30}$$

The conclusion follows from [7, Theorem 3.6]. □

Next, we prove that a solution of the minimization problem (OCP) can be obtained as limit of solutions $[u_\rho^*, f_{2,\rho}^*]$ to the regularized minimization problems (OCP$^\rho$) when ρ tends to zero.

Theorem 5. *For each $\rho > 0$, let $[u_\rho^*, f_{2,\rho}^*]$ be a solution of problem* (OCP$^\rho$). *Then, there exists a subsequence of the family $\left([u_\rho^*, f_{2,\rho}^*]\right)_{\rho>0}$, denoted in the same way, and a solution $[u^*, f_2^*]$ of problem* (OCP) *such that*

$$u_\rho^* \to u^* \text{ in } V \text{ and } f_{2,\rho}^* \rightharpoonup f_2^* \text{ in } L^2(\Gamma_2) \text{ as } \rho \to 0. \tag{31}$$

Proof. Let u_ρ^0 be the unique solution of (SP$^\rho$) with $f_2 = 0$. From (13) we deduce that

$$L(u_\rho^*, f_{2,\rho}^*) \leq L(u_\rho^0, 0) \leq \alpha \left(\frac{\|f_0\|_{L^2(\Omega)}^2}{(\mu^*)^2} + \|u_d\|_V^2 \right).$$

Thus, we deduce that $\left([u_\rho^*, f_{2,\rho}^*]\right)_{\rho>0}$ is a bounded sequence in $V \times L^2(\Gamma_2)$. Consequently, there exists $[u^*, f_2^*] \in V \times L^2(\Gamma_2)$ such that, passing eventually to a subsequence, but keeping the notation to simplify the writing, we have

$$u_\rho^* \rightharpoonup u^* \text{ in } V \text{ and } f_{2,\rho}^* \rightharpoonup f_2^* \text{ in } L^2(\Gamma_2) \text{ as } \rho \to 0.$$

In fact,

$$u_\rho^* \to u^* \text{ in } V \text{ as } \rho \to 0. \tag{32}$$

Indeed, since the operator A is strongly monotone, by (SP$^\rho$), we have

$$\begin{aligned} \mu^* \left\| u_\rho^* - u^* \right\|_V^2 &\leq (Au^*, u^* - u_\rho^*)_V + (Au_\rho^*, u_\rho^* - u^*)_V \\ &\leq (Au^*, u^* - u_\rho^*)_V + j_\rho(u^*) - j_\rho(u_\rho^*) \\ &\quad - (f_0, u^* - u_\rho^*)_{L^2(\Omega)} - (f_{2,\rho}^*, \gamma u^* - \gamma u_\rho^*)_{L^2(\Gamma_2)}. \end{aligned}$$

Now, (32) follows immediately from the above inequality if we prove that

$$\lim_{\rho\to 0}(j_\rho(u^*) - j_\rho(u^*_\rho)) = 0. \tag{33}$$

To prove (33) we remark that

$$\left|j_\rho(u^*) - j_\rho(u^*_\rho)\right| \le \frac{\|g\|_\infty}{r+1}\int_{\Gamma_3}\left|\left|\gamma u^*(x)\right|^{r+1} - \left|\gamma u^*_\rho(x)\right|^{r+1}\right| \,\mathrm{d}\Gamma.$$

From the above inequality, since $u^*_\rho \rightharpoonup u^*$ as ρ tends to zero, we deduce from [8, Theorem 2.21] (with $s = r + 1$) that (33) holds true.

On the other hand we have that $[u^*, f^*_2] \in \mathcal{V}_{\mathrm{ad}}$. Indeed, since $[u^*_\rho, f^*_{2,\rho}] \in \mathcal{V}^\rho_{\mathrm{ad}}$, $u^*_\rho \to u^*$ in V and $f^*_{2,\rho} \rightharpoonup f^*_2$ in $L^2(\Gamma_2)$ as ρ tends to zero, by passing to the limit in (SP$^\rho$) we deduce that $[u^*, f^*_2]$ verifies (SP).

Let $[\widehat{u}, \widehat{f}_2] \in \mathcal{V}_{\mathrm{ad}}$ be a solution of (OCP). For each $\rho > 0$, let u_ρ be the unique solution of the problem (SP$^\rho$) with $f_2 = \widehat{f}_2$. Obviously, $[u_\rho, \widehat{f}_2] \in \mathcal{V}^\rho_{\mathrm{ad}}$ for each $\rho > 0$. From Proposition 1 we deduce that the sequence $(u_\rho)_{\rho>0}$ converges to $\widehat{u}$ in V as $\rho \to 0$. Since the functional L is convex and continuous, we have

$$L(u^*, f^*_2) \le \liminf_{\rho\to 0} L(u^*_\rho, f^*_{2,\rho}). \tag{34}$$

Moreover, since $[u^*_\rho, f^*_{2,\rho}]$ is a solution of (OCP$^\rho$), it follows that

$$\limsup_{\rho\to 0} L(u^*_\rho, f^*_{2,\rho}) \le \limsup_{\rho\to 0} L(u_\rho, \widehat{f}_2) = L(\widehat{u}, \widehat{f}_2), \tag{35}$$

and, consequently,

$$L(u^*, f^*_2) \le L(\widehat{u}, \widehat{f}_2). \tag{36}$$

On the other hand, since $[\widehat{u}, \widehat{f}_2]$ is a solution of (OCP), we have that

$$L(\widehat{u}, \widehat{f}_2) \le L(u^*, f^*_2). \tag{37}$$

Thus, from (36)-(37) we deduce that $L(\widehat{u}, \widehat{f}_2) = L(u^*, f^*_2)$, and the proof of the theorem is complete. $\square$

Remark 5. *Theorem 5 shows that the regularized problem* (OCP$^\rho$)*, for which we dispose of the optimality condition* (27)-(28)*, may be used to approximate a solution of* (OCP)*.*

5 A fixed point method

5.1 An auxiliary linear problem

Given $z \in L^2(\Gamma_3)$, we consider the linear problem

$$\begin{cases} \mathrm{div}(\mu(x)\nabla u(x)) + f_0(x) = 0 & (x \in \Omega) \\ u(x) = 0 & (x \in \Gamma_1) \\ \mu(x)\partial_\nu u(x) = f_2(x) & (x \in \Gamma_2) \\ \mu(x)\partial_\nu u(x) = z(x) & (x \in \Gamma_3). \end{cases} \tag{38}$$

It is easy to see that for every $f_0 \in L^2(\Omega)$ and $f_2 \in L^2(\Gamma_2)$ problem (38) has a unique weak solution $u \in V$. For every $z \in L^2(\Gamma_3)$, we consider the following optimal control problem:

$$\textit{Find } [u^*, f^*_2] \in \vartheta_{\mathrm{ad}} \textit{ such that } L(u^*, f^*_2) = \inf_{[u,f_2]\in\vartheta_{\mathrm{ad}}} L(u, f_2), \tag{39}$$

where L is given by (14) and the admissible set ϑ_{ad} is defined as follows

$$\vartheta_{\text{ad}} = \left\{[u, f_2] \in V \times L^2(\Gamma_2) \text{ such that (38) is verified in a weak sense}\right\}.$$

Notice that (38) is similar to (1)-(4), with the nonlinear boundary condition on Γ_3 replaced by a simpler, non homogeneous one. Moreover, unlike (OCP), the minimization problem (39) is convex and much easier to study.

Since (38) is a linear problem, we can decompose its solutions as a sum of two functions $\overline{v}$ and w, the solutions of (38) with $f_2 = 0$ and (38) with $z = f_0 = 0$, respectively. If $w_d = \overline{v} - u_d$, the optimal control problem (39) becomes equivalent to the following one:

$$\textit{Find } [w^*, f_2^*] \in \mathcal{W}_{\text{ad}} \textit{ such that } L_{\mathcal{W}}(w^*, f_2^*) = \inf_{[w,f_2]\in\mathcal{W}_{\text{ad}}} L_{\mathcal{W}}(w, f_2), \qquad (\text{OCP}^{\text{lin}})$$

where $\mathcal{W}_{\text{ad}}$ is the linear space of all pairs $[w, f_2] \in V \times L^2(\Gamma_2)$ verifying the variational formulation of (38) with $z = f_0 = 0$ and $L_{\mathcal{W}}$ is defined by

$$L_{\mathcal{W}}(w, f_2) = \frac{\alpha}{2}\|w + w_d\|_V^2 + \frac{\beta}{2}\|f_2\|^2_{L^2(\Gamma_2)} \qquad ([w, f_2] \in \mathcal{W}_{\text{ad}}). \tag{40}$$

Notice that $L_{\mathcal{W}}$ is a strictly convex functional in $\mathcal{W}_{\text{ad}}$. By using the saddle point theory we can give an optimality condition for the minimization problem (OCP^{lin}).

Proposition 2. *There exists a unique solution* $[w^*, f_2^*, \lambda^*] \in V \times L^2(\Gamma_2) \times V$ *of the following variational problem*

$$\left\{\begin{array}{lcl} a\left([w^*, f_2^*], [\widetilde{w}, \widetilde{f_2}]\right) + b\left([\widetilde{w}, \widetilde{f_2}], \lambda^*\right) & = & \ell\left([\widetilde{w}, \widetilde{f_2}]\right) \\ b\left([w^*, f_2^*], \widetilde{\lambda}\right) & = & 0, \end{array}\right. \tag{41}$$

for any $[\widetilde{w}, \widetilde{f_2}] \in V \times L^2(\Gamma_2)$ *and* $\widetilde{\lambda} \in V$*, where*

$$a : [V \times L^2(\Gamma_2)] \times [V \times L^2(\Gamma_2)] \to \mathbb{R},$$

$$a\left([w, f_2], [\widetilde{w}, \widetilde{f_2}]\right) = \alpha\,(w, \widetilde{w})_V + \beta(f_2, \widetilde{f_2})_{L^2(\Gamma_2)}, \tag{42}$$

$$b : [V \times L^2(\Gamma_2)] \times V \to \mathbb{R},$$

$$b\left([w, f_2], \widetilde{\lambda}\right) = \left(f_2, \gamma\widetilde{\lambda}\right)_{L^2(\Gamma_2)} - \left(Aw, \widetilde{\lambda}\right)_V, \tag{43}$$

$$\ell : V \times L^2(\Gamma_2) \to \mathbb{R}, \quad \ell\,([w, f_2]) = -\alpha\,(w_d, w)_V\,. \tag{44}$$

Moreover, $[w^*, f_2^*]$ *is the unique solution of the constrained minimization problem* (OCP^{lin}).

Proof. Firstly, we show that the variational problem (41) has a unique solution. We remark that the bilinear form a is continuous and coercive, the bilinear form b is continuous and the linear functional ℓ is continuous. Moreover, the following *inf-sup* property is verified for some constant $\varrho > 0$:

$$\inf_{\lambda\in V} \sup_{[w,f_2]\in V\times L^2(\Gamma_2)} \frac{b\,([w, f_2], \lambda)}{\|[w, f_2]\|_{V\times L^2(\Gamma_2)}\|\lambda\|_V} \geq \varrho. \tag{45}$$

Indeed, to prove (45), for each $\lambda \in V$, let $w_\lambda \in V$ be the unique solution of the variational problem

$$b\,([w_\lambda, \gamma\lambda], \varphi) = (\lambda, \varphi)_V \qquad (\varphi \in V). \tag{46}$$

From (46) and (43) we deduce that $b\left([w_\lambda, \gamma\lambda], \lambda\right) = \|\lambda\|_V^2$ and $\|w_\lambda\|_V \leq \frac{c_0^2+1}{\mu^*}\|\lambda\|_V$, where μ^* and c_0 are the constants in (7) and (9), respectively. The last two relations imply that, for each $\lambda \in V$, we have

$$\frac{b\left([w_\lambda, \gamma\lambda], \lambda\right)}{\|[w_\lambda, \gamma\lambda]\|_{V\times L^2(\Gamma_2)}\|\lambda\|_V} \geq \frac{\mu^*}{\sqrt{(c_0^2+1)^2 + (\mu^* c_0)^2}}.$$

The last inequality implies that (45) is verified with $\varrho = \frac{\mu^*}{\sqrt{(c_0^2+1)^2+(\mu^* c_0)^2}}$.

We conclude that the variational problem (41) has a unique solution $[w^*, f_2^*, \lambda^*] \in V \times L^2(\Gamma_2) \times V$ (see, for instance, [2, Theorem 4.2.3]).

Notice that, the second relation in (41) implies that $[w^*, f_2^*] \in \mathcal{W}_{\mathrm{ad}}$. Moreover, the first relation in (41) ensures that $[w^*, f_2^*]$ is a critical point of the functional $L_{\mathcal{W}}(\,\cdot\,,\,\cdot\,) + b\left([\,\cdot\,,\,\cdot\,], \lambda^*\right)$ defined in $V \times L^2(\Gamma_2)$. Since this functional is convex, it follows $[w^*, f_2^*]$ is a minimizer. Hence, $[w^*, f_2^*]$ is a solution of the constrained minimization problem ($\mathrm{OCP}^{\mathrm{lin}}$). The strict convexity of $L_{\mathcal{W}}$ yields the uniqueness of the minimizer and the proof is complete. $\square$

Remark 6. *The solution $[w^*, f_2^*, \lambda^*] \in V \times L^2(\Gamma_2) \times V$ of* (41) *is in fact a saddle point for the functional $L_{\mathcal{W}} + b$.*

Now we define the operator $\mathcal{C} : V \times L^2(\Omega) \times L^2(\Gamma_3) \to L^2(\Gamma_2)$ as follows

$$\mathcal{C}(u_d, f_0, z) = f_2^*, \tag{47}$$

where f_2^* is given by Proposition 2.

The following result gives some useful estimates for the operator $\mathcal{C}$.

Proposition 3. *If $\mathcal{C}$ is the operator defined by* (47)*, then:*

- *There exists $M_1 > 0$ such that, for any $z \in L^2(\Gamma_3)$:*

$$\|\mathcal{C}(u_d, f_0, z)\|_{L^2(\Gamma_2)} \leq M_1 \left(\|u_d\|_V + \|f_0\|_{L^2(\Omega)} + \|z\|_{L^2(\Gamma_3)}\right). \tag{48}$$

- *There exists $M_2 > 0$ such that, for any z_1, $z_2 \in L^2(\Gamma_3)$:*

$$\|\mathcal{C}(u_d, f_0, z_1) - \mathcal{C}(u_d, f_0, z_2)\|_{L^2(\Gamma_2)} \leq M_2\|z_1 - z_2\|_{L^2(\Gamma_3)}. \tag{49}$$

- *There exists $M_3 > 0$ such that, for any z_1, $z_2 \in L^2(\Gamma_3)$:*

$$\|u_1 - u_2\|_V \leq M_3\|z_1 - z_2\|_{L^2(\Gamma_3)}, \tag{50}$$

  where u_i is the weak solution of the linear elliptic problem (38) *with $z = z_i$ and $f_2 = \mathcal{C}(u_d, f_0, z_i)$, $i \in \{1, 2\}$.*

Proof. Since $[w^*, f_2^*]$ is the minimizer of ($\mathrm{OCP}^{\mathrm{lin}}$), we have that

$$\frac{\beta}{2}\|f_2^*\|_{L^2(\Gamma_2)}^2 \leq L_{\mathcal{W}}(w^*, f_2^*) \leq L_{\mathcal{W}}(0,0) = \frac{\alpha}{2}\|w_d\|_V^2,$$

from which we deduce that

$$\|f_2^*\|_{L^2(\Gamma_2)} \leq \sqrt{\frac{\alpha}{\beta}}\left(\|\overline{v}\|_V + \|u_d\|_V\right). \tag{51}$$

Since $\overline{v}$ verifies (38) with $f_2 = 0$, we deduce that

$$\|\overline{v}\|_V \leq \frac{\max\{c_0,\, c_P\}}{\mu^*}\left(\|f_0\|_{L^2(\Omega)} + \|z\|_{L^2(\Gamma_3)}\right), \tag{52}$$

where c_0 and c_P are the constants from (9) and (10), respectively. From (51)-(52) we deduce that (48) holds with $M_1 = \sqrt{\frac{\alpha}{\beta}} \max\left\{\frac{\max\{c_0, c_P\}}{\mu^*}, 1\right\}$.

Let us now pass to prove (49). Let $\mathcal{C}(u_d, f_0, z_i) = f_{2,i}^*$, $i = 1, 2$. From the optimality condition in (41), we deduce that

$$a\left([\theta^*, \tau^*], [\widetilde{w}, \widetilde{f_2}]\right) + b\left([\widetilde{w}, \widetilde{f_2}], \lambda_1^* - \lambda_2^*\right) = -\alpha\left(\overline{v}(z_1) - \overline{v}(z_2), \widetilde{w}\right)_V,$$

where $\theta^* = w_1^* - w_2^*$, $\tau^* = f_{2,1}^* - f_{2,2}^*$ and $\overline{v}(z)$ is the weak solution of (38) with $f_2 = 0$.

Now, if we chose $[\widetilde{w}, \widetilde{f_2}] = [\theta^*, \tau^*]$ and we take into account the second relation in (41), we obtain that

$$a\left([\theta^*, \tau^*], [\theta^*, \tau^*]\right) = -\alpha\left(\overline{v}(z_1) - \overline{v}(z_2), \theta^*\right)_V. \tag{53}$$

Relation (53), combined with the definition of a, the fact that $\overline{v}(z_1) - \overline{v}(z_2)$ verifies (38) with $f_2 = f_0 = 0$ and estimate (52), implies that

$$\|\theta^*\|_V \leq \frac{\max\{c_0,\, c_P\}}{\mu^*}\|z_1 - z_2\|_{L^2(\Gamma_3)}, \tag{54}$$

and

$$\beta\|\tau^*\|^2_{L^2(\Gamma_2)} \leq \alpha\left(\frac{\max\{c_0,\, c_P\}}{\mu^*}\right)^2 \|z_1 - z_2\|^2_{L^2(\Gamma_3)}, \tag{55}$$

The last relation gives us precisely (49) with $M_2 = \sqrt{\frac{\alpha}{\beta}}\frac{\max\{c_0, c_P\}}{\mu^*}$.

To prove (50) we remark that

$$\int_{\Gamma_2}\left(\mathcal{C}(u_d, f_0, z_1)(x) - \mathcal{C}(u_d, f_0, z_2)(x)\right)(u_1 - u_2)(x)\,\mathrm{d}\Gamma$$
$$+ \int_{\Gamma_3}(z_1(x) - z_2(x))(u_1 - u_2)(x)\,\mathrm{d}\Gamma - \int_\Omega \mu(x)|\nabla(u_1 - u_2)(x)|^2\,\mathrm{d}x = 0.$$

Consequently, we obtain that

$$\|u_1 - u_2\|_V \leq \frac{c_0}{\mu^*}\left(\|z_1 - z_2\|_{L^2(\Gamma_3)} + \|\mathcal{C}(u_d, f_0, z_1) - \mathcal{C}(u_d, f_0, z_2)\|_{L^2(\Gamma_2)}\right).$$

From the last estimation, combined with (49), we infer that (50) holds with $M_3 = \frac{c_0}{\mu^*}(1 + M_2)$ and the proof of the proposition is complete. □

Finally, we have the following lemma which will be used in the next two sections.

Lemma 1. *If $\rho > 0$ and $r \in [0, 1]$ then, for any $a, b \in \mathbb{R}$,*

$$\left|\frac{|a|^{2r}a}{\sqrt{|a|^{2r+2} + \rho^2}} - \frac{|b|^{2r}b}{\sqrt{|b|^{2r+2} + \rho^2}}\right| \leq (3r + 1)\left(\frac{1}{\rho}\right)^{\frac{1-r}{1+r}}|a - b|. \tag{56}$$

On the other hand, if $r > 1$ then, for any $a, b \in \mathbb{R}$,

$$\left||a|^{r-1}a - |b|^{r-1}b\right| \leq r\,|a - b|\,(|a| + |b|)^{r-1}. \tag{57}$$

Proof. For inequality (56), we use Mean Value Theorem applied to the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = |t|^{2r}t/\sqrt{|t|^{2r+2} + \rho^2}$, and take into account that $\sup_{t \in \mathbb{R}}|h'(t)| \leq (3r + 1)\rho^{\frac{r-1}{1+r}}$.

Concerning (57), the same idea may be used by considering $h(t) = |t|^{r-1}t$. □

5.2 The case $r \in (0,1)$

This section is devoted to present and to study a fixed point method for the approximation of the optimal controls of problem (OCP$^\rho$). Let $r \in (0,1)$ and $\rho > 0$ be two real numbers. Now, given $g \in L^\infty(\Gamma_3)$, we define the operator $\mathcal{N} : L^2(\Gamma_3) \to L^2(\Gamma_3)$ by

$$\mathcal{N}(z)(x) = -g(x)\frac{|\gamma(u)(x)|^{2r}\,\gamma(u)(x)}{\sqrt{|\gamma(u)(x)|^{2r+2} + \rho^2}} \qquad (z \in L^2(\Gamma_3), x \in \Gamma_3), \tag{58}$$

where u is the solution of (38) associated to z and $f_2 = \mathcal{C}(u_d, f_0, z)$. Clearly, $\mathcal{N}$ is well defined.

Remark 7. *If z is a fixed point of the operator $\mathcal{N}$, then u is a solution of the nonlinear problem* (SP$^\rho$) *and $f_2^* = \mathcal{C}(u_d, f_0, z)$ gives an optimal control for the nonlinear minimization problem* (OCP$^\rho$).

The remaining part of this section is devoted to present some sufficient conditions for the existence of a fixed point of $\mathcal{N}$. The most important properties of the map $\mathcal{N}$ are the following.

Theorem 6. *Let $\rho > 0$ be given. There exists a positive constant δ such that the operator $\mathcal{N}$ defined by* (58) *is a contraction in $L^2(\Gamma_3)$ for every $g \in L^\infty(\Gamma_3)$ satisfying*

$$\left(\frac{1}{\rho}\right)^{\frac{1-r}{1+r}} \|g\|_{L^\infty(\Gamma_3)} < \delta. \tag{59}$$

Thus $\mathcal{N}$ has a unique fixed point $z_\rho^* \in L^2(\Gamma_3)$. Furthermore, $[u_\rho^*, f_{2,\rho}^*]$ is a solution of (OCP$^\rho$), *where $f_{2,\rho}^* = \mathcal{C}(u_d, f_0, z_\rho^*)$ and u_ρ^* is the solution of* (SP$^\rho$) *with $f_2 = f_{2,\rho}^*$.*

Proof. For each $z_1,\ z_2 \in L^2(\Gamma_3)$ the following estimate holds

$$\begin{aligned}&\|\mathcal{N}(z_1) - \mathcal{N}(z_2)\|^2_{L^2(\Gamma_3)}\\ &\quad\le \|g\|^2_{L^\infty(\Gamma_3)} \int_{\Gamma_3} \left| \frac{|\gamma(u_1)(x)|^{2r}\,\gamma(u_1)(x)}{\sqrt{|\gamma(u_1)(x)|^{2r+2} + \rho^2}} - \frac{|\gamma(u_2)(x)|^{2r}\,\gamma(u_2)(x)}{\sqrt{|\gamma(u_2)(x)|^{2r+2} + \rho^2}} \right|^2 \mathrm{d}\Gamma,\end{aligned}$$

where u_i is the weak solution of the linear elliptic problem (38) with $z = z_i$ and $f_2 = \mathcal{C}(u_d, f_0, z_i)$, $i \in \{1, 2\}$. By applying (56) from Lemma 1, we deduce that

$$\|\mathcal{N}(z_1) - \mathcal{N}(z_2)\|_{L^2(\Gamma_3)} \le \frac{3r+1}{\rho^{\frac{1-r}{1+r}}} \|g\|_{L^\infty(\Gamma_3)} \|\gamma(u_1) - \gamma(u_2)\|_{L^2(\Gamma_3)}. \tag{60}$$

From the trace theorem and (50) we get

$$\|\mathcal{N}(z_1) - \mathcal{N}(z_2)\|_{L^2(\Gamma_3)} \le c_0 M_3 \frac{3r+1}{\rho^{\frac{1-r}{1+r}}} \|g\|_{L^\infty(\Gamma_3)} \|z_1 - z_2\|_{L^2(\Gamma_3)}. \tag{61}$$

Thus, for $\left(\frac{1}{\rho}\right)^{\frac{1-r}{1+r}} \|g\|_{L^\infty(\Gamma_3)}$ small enough, $\mathcal{N}$ is a contraction on $L^2(\Gamma_3)$.

The last part of the theorem follows from the definition of the operator $\mathcal{C}$ in (47) and Remark 7. $\square$

In Section 6 we shall approximate the control $\mathcal{C}(u_d, f_0, z_\rho^*)$ by using the classical fixed point iteration method. Let us finish this section by proving that, in the hypothesis of Theorem 6, the optimal control is unique.

Theorem 7. *Suppose that condition* (59) *holds. Then the optimization problem* (OCP$^\rho$) *has exactly one optimal pair* $[u_\rho^*, f_{2,\rho}^*]$.

Proof. Let $[\widetilde{u}_\rho^*, \widetilde{f}_{2,\rho}^*]$ and $[u_\rho^*, f_{2,\rho}^*]$ be two optimal pairs. Since $[\widetilde{u}_\rho^*, \widetilde{f}_{2,\rho}^*]$ and $[u_\rho^*, f_{2,\rho}^*] \in \mathcal{V}_{\mathrm{ad}}^\rho$ it follows that if $f_{2,\rho}^* = \widetilde{f}_{2,\rho}^*$ then $u_\rho^* = \widetilde{u}_\rho^*$.

Suppose that $f_{2,\rho}^* \neq \widetilde{f}_{2,\rho}^*$. Let z_ρ^* and $\widetilde{z}_\rho^*$ be given by

$$z_\rho^* = -\frac{g\left|\gamma(u_\rho^*)\right|^{2r}\gamma(u_\rho^*)}{\sqrt{\left|\gamma(u_\rho^*)\right|^{2r+2}+\rho^2}}, \quad \widetilde{z}_\rho^* = -\frac{g\left|\gamma(\widetilde{u}_\rho^*)\right|^{2r}\gamma(\widetilde{u}_\rho^*)}{\sqrt{\left|\gamma(\widetilde{u}_\rho^*)\right|^{2r+2}+\rho^2}}.$$

We recall that $f_{2,\rho}^* = \mathcal{C}(u_d, f_0, z_\rho^*)$ and $\widetilde{f}_{2,\rho}^* = \mathcal{C}(u_d, f_0, \widetilde{z}_\rho^*)$. Arguing by contradiction and taking into account (49), we conclude that $z_\rho^* \neq \widetilde{z}_\rho^*$.

On the other hand, from definition (58) of the operator $\mathcal{N}$, we have that z_ρ^* and $\widetilde{z}_\rho^*$ are both fixed points of $\mathcal{N}$. Therefore, $\mathcal{N}$ has two different fixed points which is a contradiction.

Hence $f_{2,\rho}^* = \widetilde{f}_{2,\rho}^*$ and the proof is complete. □

5.3 The case $r \geq 1$

In this case we can use a similar fixed point method to study the existence of the optimal controls for problem (OCP). Let $B(0,R)$ denote the ball in $L^2(\Gamma_3)$ centered in 0 and of radius R. Given $g \in L^\infty(\Gamma_3)$, we define the operator $\mathcal{N} : B(0,R) \to B(0,R)$ by

$$\mathcal{N}(z)(x) = -g(x)\left|\gamma(u)(x)\right|^{r-1}\gamma(u)(x) \qquad (z \in B(0,R), x \in \Gamma_3), \tag{62}$$

where u is the solution of (38) with $f_2 = \mathcal{C}(u_d, f_0, z)$ given by (47).

The following result gives the most important properties of the map $\mathcal{N}$.

Theorem 8. *Let $R > 0$. There exists $\delta = \delta(R) > 0$ such that $\mathcal{N}$ is a contraction on $B(0,R)$ if*

$$\|g\|_{L^\infty(\Gamma_3)} < \delta. \tag{63}$$

Thus $\mathcal{N}$ has a unique fixed point $z^* \in B(0,R)$. Furthermore, $[u^*, f_2^*]$ is a solution of (OCP), *where $f_2^* = \mathcal{C}(u_d, f_0, z^*)$ and u^* is the solution of* (SP) *with $f_2 = f_2^*$.*

Proof. Since the case $r = 1$ is obvious, we consider that $r > 1$. Firstly, we show that $\mathcal{N}$ is well defined on $B(0,R)$. From (48), it is easy to prove that

$$\|u\|_V \leq \frac{\max\{c_0,\, c_P\}}{\mu^*}(M_1+1)\left(\|f_0\|_{L^2(\Omega)} + \|u_d\|_V + \|z\|_{L^2(\Gamma_3)}\right). \tag{64}$$

Next, by using (9), we have that

$$\begin{aligned}\|\mathcal{N}(z)\|_{L^2(\Gamma_3)}^2 &= \int_{\Gamma_3}|g(x)|^2\left|\gamma(u)(x)\right|^{2r}d\Gamma \\ &\leq \|g\|_{L^\infty(\Gamma_3)}^2\|\gamma(u)\|_{L^{2r}(\Gamma_3)}^{2r} \leq c_0^{2r}\|g\|_{L^\infty(\Gamma_3)}^2\|u\|_V^{2r}.\end{aligned}$$

Hence, $\mathcal{N}(B(0,R)) \subset B(0,R)$ if

$$\|g\|_{L^\infty(\Gamma_3)} < \frac{R}{\left(c_0\frac{\max\{c_0,\, c_P\}}{\mu^*}(M_1+1)\right)^r\left(\|f_0\|_{L^2(\Omega)} + \|u_d\|_V + R\right)^r}. \tag{65}$$

Now, we show that $\mathcal{N}$ is a contraction on $B(0,R)$. Let z_1, $z_2 \in B(0,R)$. From (57) in Lemma 1 we deduce that

$$\begin{aligned}
&\|\mathcal{N}(z_1)-\mathcal{N}(z_2)\|^2_{L^2(\Gamma_3)} \leq \\
&\leq r^2\|g\|^2_{L^\infty(\Gamma_3)}\int_{\Gamma_3}|\gamma(u_1)(x)-\gamma(u_2)(x)|^2\left(|\gamma(u_1)(x)|+|\gamma(u_2)(x)|\right)^{2(r-1)}d\Gamma \\
&\leq r^2\|g\|^2_{L^\infty(\Gamma_3)}\|\gamma(u_1)-\gamma(u_2)\|^2_{L^p}\left(\|\gamma(u_1)\|_{L^{\frac{2(r-1)p}{p-2}}}+\|\gamma(u_2)\|_{L^{\frac{2(r-1)p}{p-2}}}\right)^{2(r-1)},
\end{aligned}$$

where $p=\frac{2}{3-2r}$ if $r\in(1,\frac{3}{2})$ and $p=3$ if $r\geq\frac{3}{2}$. From the last estimate, (9), (50) and (64) we obtain that

$$\begin{aligned}
&\|\mathcal{N}(z_1)-\mathcal{N}(z_2)\|^2_{L^2(\Gamma_3)} \leq r^2c_0^{2r}\|g\|^2_{L^\infty(\Gamma_3)}\|u_1-u_2\|^2_V\left(\|u_1\|_V+\|u_2\|_V\right)^{2(r-1)} \\
&\leq r^2c_0^{2r}M_3^2\left(2\frac{\max\{c_0,\,c_P\}}{\mu^*}(M_1+1)\right)^{2(r-1)}\|g\|^2_{L^\infty(\Gamma_3)}\|z_1-z_2\|^2_{L^2(\Gamma_3)} \\
&\left(\|z_1\|_{L^2(\Gamma_3)}+\|z_2\|_{L^2(\Gamma_3)}+\|u_d\|_V+\|f_0\|_{L^2(\Omega)}\right)^{2(r-1)}.
\end{aligned}$$

Thus, if g verifies (63) and δ is sufficiently small, the application $\mathcal{N}$ is a contraction on $B(0,R)$. The last part of the theorem follows from the definition of the operator $\mathcal{C}$ in (47) and the fact that, if z is a fixed point of the operator $\mathcal{N}$, then u is a solution of the nonlinear problem (SP). □

Remark 8. *In the case $r>1$ the arguments used in the proof of Theorem 8 show that $\mathcal{N}$ is a contraction on $B(0,R)$ even if (63) is not verified, by assuming that R, $\|u_d\|_V$ and $\|f_0\|_{L^2(\Omega)}$ are small enough. Indeed, given any $g\in L^\infty(\Gamma_3)$, we have that, for sufficiently small R, $\|u_d\|_V$ and $\|f_0\|_{L^2(\Omega)}$, both relation (65) and the contraction property of $\mathcal{N}$ are verified.*

As in Theorem 7 for the case $r\in(0,1)$, we can show that problem (OCP) has a unique solution under appropriate hypothesis.

Theorem 9. *Under the hypothesis of Theorem 8, the optimization problem (OCP) with $r\geq 1$ has exactly one optimal pair $[u^*, f_2^*]$.*

6 Numerical results

In this section we use the characterization of the optimal pair in Theorems 6 and 8 to propose a numerical approximation scheme. Indeed, under appropriate conditions (59) and (63), respectively, given $z^0\in L^2(\Gamma_3)$, the sequence of successive approximations $(z_n)_{n\geq 0}$ defined by $z^{n+1}=\mathcal{N}(z^n)$ for every $n\geq 0$, converges in $L^2(\Gamma_3)$ to the fixed point z^* of $\mathcal{N}$ which gives the optimal pair. We propose the following algorithm:

At each step of the Algorithm 1, an optimal control $\mathcal{C}(u_d, f_0, z^n)$ for (OCP$^{\rm lin}$) is computed by (41). From a numerical point of view it is convenient to consider a minimization problem with respect to a pair $[w, f_2]\in V\times H^1(\Omega)$. This permits to avoid the simultaneous manipulation of distributed and boundary terms. More precisely, for a given parameter $\zeta>0$ we solve, instead of (OCP$^{\rm lin}$), the following minimization problem:

$$\textit{Find } [w^*, f_2^*]\in\mathcal{W}^\zeta_{\rm ad} \textit{ such that } L^\zeta_{\mathcal{W}}(w^*, f_2^*)=\inf_{[w,f_2]\in\mathcal{W}^\zeta_{\rm ad}} L^\zeta_{\mathcal{W}}(w, f_2), \qquad (\mathrm{OCP}^{lin}_\zeta)$$

let $u_d \in V$, $f_0 \in L^2(\Omega)$, $g \in L^\infty(\Gamma_3)$, $\epsilon > 0$, $n_{\max} \in \mathbb{N}^*$;
let $z^0 \in L^2(\Gamma_3)$;
let $f_2^0 = \mathcal{C}(u_d, f_0, z^0)$;
let $n = 0$;
do
  let $z^{n+1} = \mathcal{N}(z^n)$;
  let $n = n + 1$;
  let $f_2^n = \mathcal{C}(u_d, f_0, z^n)$;
  let $E^n = \|z^{n+1} - z^n\|_{L^2(\Gamma_3)}$;
until $(E^n \leq \epsilon)$ **or** $(n > n_{\max})$.

Algorithm 1: A fixed point algorithm to compute the optimal control.

with $\mathcal{W}_{\rm ad}^\zeta = \big\{[w, f_2] \in V \times H^1(\Omega) \text{ such that } [w, \gamma f_2] \in \mathcal{W}_{\rm ad}\big\}$ and

$$L_{\mathcal{W}}^\zeta(w, f_2) = \frac{\alpha}{2}\|w + w_d\|_V^2 + \frac{\beta}{2}\|\gamma f_2\|_{L^2(\Gamma_2)}^2 + \frac{\zeta}{2}\|f_2\|_{H^1(\Omega)}^2.$$

We notice that all the results from Section 5 can be easily adapted to this new functional framework. In practice, we numerically approach the unique solution of the following mixed formulation:

$$\left\{\begin{array}{lcl} a^\zeta\left([w, f_2], [\widetilde{w}, \widetilde{f_2}]\right) + b^\zeta\left([\widetilde{w}, \widetilde{f_2}], \lambda\right) & = & \ell^\zeta\left([\widetilde{w}, \widetilde{f_2}]\right) \\ b^\zeta\left([w, f_2], \widetilde{\lambda}\right) & = & 0, \end{array}\right. \tag{66}$$

for any $[\widetilde{w}, \widetilde{f_2}] \in V \times H^1(\Omega)$ and $\widetilde{\lambda} \in V$. For every $[w, f_2]$, $[\widetilde{w}, \widetilde{f_2}]$ in $V \times H^1$ and $\lambda \in V$, the bilinear form a^ζ is given by

$$a^\zeta\left([w, f_2], [\widetilde{w}, \widetilde{f_2}]\right) = a\left([w, \gamma f_2], [\widetilde{w}, \gamma\widetilde{f_2}]\right) + \zeta(f_2, \widetilde{f_2})_{H^1(\Omega)},$$

and $b^\zeta([w, f_2], \lambda) = b([w, \gamma f_2], \lambda)$, $\ell^\zeta([w, f_2]) = \ell([w, \gamma f_2])$. Remark that the term $\zeta(f_2, \widetilde{f_2})_V$ appearing in a^ζ could be seen as a stabilization term. To approximate the solutions of (66), we consider a triangulation $\mathcal{T}_h$ of Ω, where h is the largest diameter of triangles forming $\mathcal{T}_h$. For every $h > 0$, we define the finite dimensional spaces $V_h \subset V$ and $W_h \subset H^1(\Omega)$ as the Lagrange P_1 finite elements spaces: $V_h = \{\varphi \in V \mid \varphi|_T \text{ is an affine function, for every } T \in \mathcal{T}_h\}$ and $W_h = \{\varphi \in H^1(\Omega) \mid \varphi|_T \text{ is an affine function, for every } T \in \mathcal{T}_h\}$. The corresponding discrete version of (66) is the following finite dimensional system: find $[w^h, f_2^h, \lambda^h] \in V_h \times W_h \times V_h$ solution to

$$\left\{\begin{array}{lcl} a^\zeta\left([w^h, f_2^h], [\widetilde{w}^h, \widetilde{f}_2^h]\right) + b^\zeta\left([\widetilde{w}^h, \widetilde{f}_2^h], \lambda^h\right) & = & \ell^\zeta\left([\widetilde{w}^h, \widetilde{f}_2^h]\right) \\ b^\zeta\left([w^h, f_2^h], \widetilde{\lambda}^h\right) & = & 0, \end{array}\right. \tag{67}$$

for any $[\widetilde{w}^h, \widetilde{f}_2^h] \in V_h \times W_h$ and $\widetilde{\lambda}^h \in V_h$. If the inf-sup condition

$$\inf_{\lambda^h \in V_h} \sup_{[w^h, f_2^h] \in V_h \times W_h} \frac{b^\zeta\left([w^h, f_2^h], \lambda^h\right)}{\|[w^h, f_2^h]\|_{V_h \times W_h} \|\lambda^h\|_{V_h}} \geq \varrho_h > 0, \tag{68}$$

holds uniformly with respect to h, then for every h there exists an unique solution of (67), and the sequence of solutions converges to the solution of (66) when $h \to 0$.

In all the numerical experiments the domain Ω is either the disk $\Omega_{\rm d} \subset \mathbb{R}^2$ centered in the origin and of radius 1, with its boundary formed by the three arcs of circle represented in Figure 1 (left), either the unit square $\Omega_{\rm s} = (0,1) \times (0,1)$ with the boundary $\Gamma = \bigcup_{i=1} \Gamma_i$ as depicted in Figure 2 and $\Gamma_3 = \Gamma_3^{\rm l} \cup \Gamma_3^{\rm r}$.

We set $\mu \equiv 1$ in Ω, which corresponds to a homogeneous material, and f_0 and g are two functions to be chosen later. In order to numerically validate the optimal control strategy proposed in Section 5 and used in Algorithm 1, we compute firstly a reference solution to the nonlinear problem (SP^ρ) for a given value of f_2, which, for the seek of clarity, we denote f_2^{ex}, on the boundary Γ_2. The reference solution u_{ex} of (SP^ρ) was approached by numerically solving its associated variational formulation on a very fine mesh of the domain Ω (mesh $\mathcal{T}_{\mathrm{d},h}^{\mathrm{ref}}$ and mesh $\mathcal{T}_{\mathrm{s},h}^{\mathrm{ref}}$ described in Table 1). Then we choose $u_d = u_{\mathrm{ex}}$ and we apply Algorithm 1 on other five coarser meshes $(\mathcal{T}_{\mathrm{d},h}^i)_{1\leq i\leq 5}$ and $(\mathcal{T}_{\mathrm{s},h}^i)_{1\leq i\leq 5}$. For every triangulation $\mathcal{T}_h^i$, the notation h stands for the maximum diameter of the triangles composing the triangulation. On each of these meshes we compute an optimal pair $[u_h^*, f_{2,h}^*]$ which aims to approach $[u_d, f_2^{\mathrm{ex}}]$. In Table 1 we list the number of points and elements associated to these meshes and in Figure 1 and Figure 2 we display the domain Ω and the coarsest mesh we consider for our study.

$\Omega = \Omega_{\mathrm{d}}$	$\mathcal{T}_{\mathrm{d},h}^1$	$\mathcal{T}_{\mathrm{d},h}^2$	$\mathcal{T}_{\mathrm{d},h}^3$	$\mathcal{T}_{\mathrm{d},h}^4$	$\mathcal{T}_{\mathrm{d},h}^5$	$\mathcal{T}_{\mathrm{d},h}^{\mathrm{ref}}$
$\sharp$ points	592	2 349	8 984	35 878	143 489	344 148
$\sharp$ triangles	1 102	4 536	17 646	71 114	285 396	686 294
$\sharp$ segments in Γ	80	160	320	640	1280	2 000
$\Omega = \Omega_{\mathrm{s}}$	$\mathcal{T}_{\mathrm{s},h}^1$	$\mathcal{T}_{\mathrm{s},h}^2$	$\mathcal{T}_{\mathrm{s},h}^3$	$\mathcal{T}_{\mathrm{s},h}^4$	$\mathcal{T}_{\mathrm{s},h}^5$	$\mathcal{T}_{\mathrm{s},h}^{\mathrm{ref}}$
$\sharp$ points	441	1 681	6 561	25 921	103 041	251 001
$\sharp$ triangles	800	3 200	12 800	51 200	204 800	500 000
$\sharp$ segments in Γ	80	160	320	640	1280	2 000

Table 1: Description of six meshes of the unit disk Ω_{d} and of the square Ω_{s}, respectively.

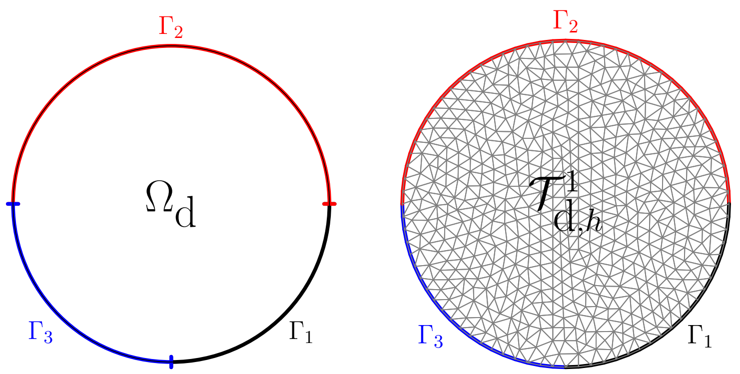

Figure 1: Unit disk Ω_{d} with the partition of its boundary (left) and the associated mesh $\mathcal{T}_{\mathrm{d},h}^1$ (right).

Algorithm 1 was implemented in `FreeFem++` [4]. The choice of parameters α, β and ζ appearing in the definition of the functional $L_{\mathcal{W}}^{\zeta}$ is, in general, a difficult issue. In our case, since we want that the reconstructed solution u^* to be as close as possible to the target u_d, the coefficient α should be much larger than β. More precisely, if it is not specified otherwise, for the numerical experiments which follow, we choose $\alpha = 10^8$. The parameter β is chosen equal to 1 and $\zeta = \beta h$. Remark that when $h \to 0$ the parameter ζ goes to zero and, therefore, the solution of ($\mathrm{OCP}_{\zeta}^{lin}$) is close to the solution of ($\mathrm{OCP}^{\mathrm{lin}}$).

In what follows we present several numerical experiment for one or both domains Ω_{d} and Ω_s and different choice of parameters.

Numerical experiment 1 ($r = 0.5$): For this numerical experiment we choose $\Omega = \Omega_{\mathrm{d}}$ and let $f_0 \equiv 0$ in Ω, $f_2^{\mathrm{ex}} \equiv 0$ on Γ_2 and $g \equiv 1$ on Γ_3. For this choice of parameters, the

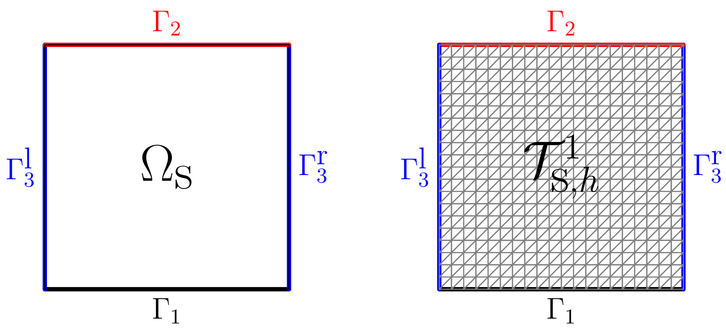

Figure 2: Unit square Ω_{s} with the partition of its boundary (left) and the associated mesh $\mathcal{T}_{\mathrm{s},h}^1$ (right).

solution of problem $(\mathrm{OCP}_{\zeta}^{lin})$ is evidently the pair $[0,0]$ and, therefore, the solution of (OCP^{ρ}) is $[w^*, f_2^*] = [0,0]$. For an initial guess of $z_0 = 10$, the norms of solutions $[u_h^*, f_{2,h}^*]$ computed by the Algorithm 1 are close to zero ($< 10^{-15}$) for values of $\rho > 0$ large enough. In Table 2 we gather the number of iterations needed for the convergence in Algorithm 1 for the mesh $\mathcal{T}_{\mathrm{d},h}^2$ and several values of ρ. For all the numerical simulations we choose the parameter ϵ appearing in the stopping criteria of Algorithm 1 equal to 10^{-6}. We remark that the number of iterations needed for convergence rapidly grows when ρ becomes very small. Thus, for values of ρ close to zero we lose the convergence. This is due to the fact that if ρ is small the condition (59) is no more fulfilled.

ρ	10	1	0.25	0.15	0.14	0.13
$\sharp$ iterations	6	9	15	37	212	Non-convergence

Table 2: Number of iterations needed for convergence for mesh $\mathcal{T}_h^2$ and different values of ρ for the Experiment 1.

Numerical experiment 2 ($r = 0.5$): A less trivial example is the following one: $f_0(x,y) = \sqrt{x^2+y^2}$ for $(x,y) \in \Omega$, $f_2^{\mathrm{ex}}(x,y) = x$ for $(x,y) \in \Gamma_2$ and $g \equiv 1$ on Γ_3. The results obtained in this case, for $\rho = 10$ and Ω being Ω_{d} and Ω_{s}, respectively, are summarized in Table 3.

$\Omega = \Omega_{\mathrm{d}}$	$\mathcal{T}_{\mathrm{d},h}^1$	$\mathcal{T}_{\mathrm{d},h}^2$	$\mathcal{T}_{\mathrm{d},h}^3$	$\mathcal{T}_{\mathrm{d},h}^4$	$\mathcal{T}_{\mathrm{d},h}^5$
$\Vert u_h^* - u_{\mathrm{ex}}\Vert_V$	0.27	0.17	0.12	0.071	0.026
$\Vert f_{2,h}^* - f_2^{\mathrm{ex}}\Vert_{L^2(\Gamma_2)}$	0.048	0.036	0.066	0.071	0.086
$\sharp$ iterations	5	5	5	5	5
$\Omega = \Omega_{\mathrm{s}}$	$\mathcal{T}_{\mathrm{s},h}^1$	$\mathcal{T}_{\mathrm{s},h}^2$	$\mathcal{T}_{\mathrm{s},h}^3$	$\mathcal{T}_{\mathrm{s},h}^4$	$\mathcal{T}_{\mathrm{s},h}^5$
$\Vert u_h^* - u_{\mathrm{ex}}\Vert_V$	0.014	0.0067	0.003	0.0012	0.00032
$\Vert f_{2,h}^* - f_2^{\mathrm{ex}}\Vert_{L^2(\Gamma_2)}$	0.000087	0.000073	0.00025	0.00032	0.00025
$\sharp$ iterations	5	5	5	5	5

Table 3: Reconstruction errors and number of iterations needed for convergence for meshes $\mathcal{T}_{\mathrm{d},h}^i$ and $\mathcal{T}_{\mathrm{s},h}^i$ of domains Ω_{d} and Ω_{s}, respectively and for (f_0, f_2^{ex}) given in the Experiment 2.

The results for Experiment 2 obtained for $\Omega = \Omega_{\mathrm{s}}$ are illustrated in Figure 3. More exactly, at left we display the optimal solution u_h^* computed on the mesh $\mathcal{T}_{\mathrm{s},h}^3$ and, at right, we plot the optimal controls $f_{2,h}^*$ obtained for different meshes. Since Γ_2 can be represented as the graph of continuous function defined for $x \in (-1,1)$, we can represent the optimal controls as functions of x. Considering different values of ρ, we observe again that the number of iterations needed for the convergence explodes when $\rho \to 0$, for small values of ρ the converge being lost.

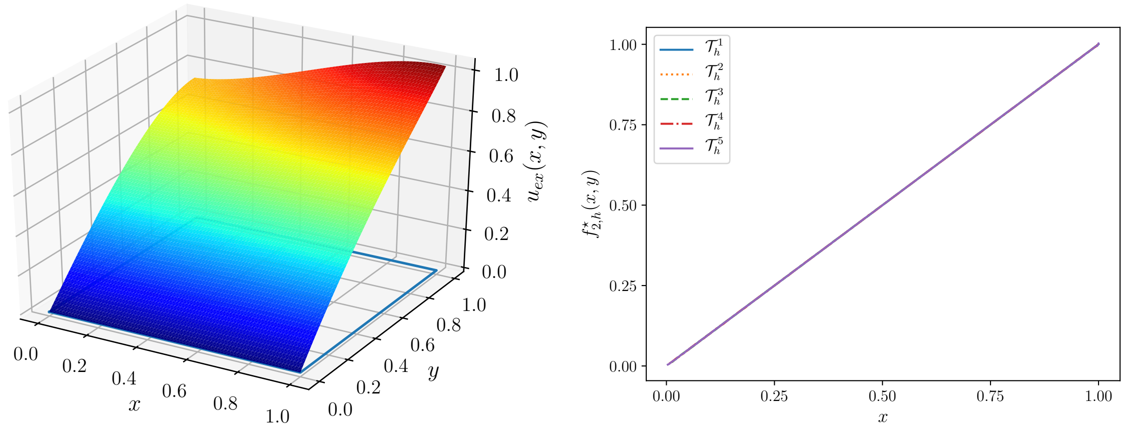

Figure 3: Optimal solution u_h^* (left) on $\mathcal{T}^3_{\mathrm{s},h}$ and the optimal control $f^*_{2,h}$ computed on different meshes for $\rho = 10$ and data from Experiment 2.

Numerical experiment 3 ($r = 2$): Finally we consider an example with the parameter $r = 2$. We choose here the same data as for Experiment 2. In conformity to the results in Section 5.3, we choose $\rho = 0$ and therefore the operator $\mathcal{N}$ is given by (62). As observed in Table 4 and in oposition to the Experiments 1 and 2, the number of iterations needed to the convergence remains reasonable for $\rho = 0$. This is in complete agreement to the results in Section 5.3. As expected, the value of $\|g\|_{L^\infty(\Gamma_3)}$ affects the convergence of the fixed point algorithm: e.g., for $g \equiv 2$ the number of iterations needed for convergence on the mesh $\mathcal{T}_h^1$ is 165 and for $g \equiv 3$ the convergence is lost.

	$\mathcal{T}_h^1$	$\mathcal{T}_h^2$	$\mathcal{T}_h^3$	$\mathcal{T}_h^4$	$\mathcal{T}_h^5$
$\|u_h^* - u_{\mathrm{ex}}\|_V$	0.014	0.0067	0.003	0.0012	0.00033
$\|f^*_{2,h} - f_2^{\mathrm{ex}}\|_{L^2(\Gamma_2)}$	0.00038	0.00011	0.00023	0.00033	0.00028
$\sharp$ iterations	10	11	11	11	20

Table 4: Reconstruction errors and number of iterations needed for convergence for meshes $\mathcal{T}_h^i$ with $i \in \{1, 2, 3, 4, 5\}$ for $r = 2$, (f_0, f_2^{ex}) given in the Experiment 3 and $\rho = 0$.

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References

[1] A. Amassad, D. Chenais and C. Fabre, Optimal control of an elastic contact problem involving Tresca friction law, *Nonlinear Anal.* 48 (2002) 1107-1135.

[2] D. Boffi, F. Brezzi and M. Fortin, *Mixed finite element methods and applications*, Springer Ser. Comput. Math. 44 (2013).

[3] H. Hechaichi and M. Sofonea, Analysis and Control of a Nonlinear Boundary Value Problem, *Nonlinear Anal. Model. Control* 22(6) (2017) 841-860.

[4] F. Hecht, New development in freefem++, *J. Numer. Math.* 20 (2012) 251-265.

[5] J.-L. Lions, *Contrôle Optimale Des Systèmes Gouvernés Par Des Équations Aux Dérivées Partielles*, Dunod, Paris, (1968).

[6] A. Matei and S. Micu, Boundary optimal control for nonlinear antiplane problems, *Nonlinear Anal.* 74 (5) (2011) 1641-1652.

[7] A. Matei and M. Sofonea, *Variational inequalities with applications. A study of antiplane frictional contact problems*, Adv. Mech. Math. 18 Springer New York (2009).

[8] S. Migorski, A. Ochal and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, Adv. Mech. Math. 26 Springer (2013).

[9] M. Sofonea, Convergence Results and Optimal Control for a Class of Hemivariational Inequalities, *SIAM J. Math. Anal.* 50 (2018) 4066-4086.

[10] M. Sofonea, Optimal control of variational-hemivariational inequalities, *Appl. Math. Optim.* 79 (3) (2019) 621-646.

7 Appendix

Lemma 2. *Given f_0, $f_2 := f_2^{ex}$ and g, let u_{ex} be the unique solution of* (SP$^\rho$)*. If we choose $u_d = u_{ex}$ in definition* (5) *of the functional J, then the following estimates hold*

$$\begin{aligned}\left\|\nabla(u_\rho^* - u_{ex})\right\|_{L^2(\Omega)} &\leq \sqrt{\frac{\beta}{\alpha}}\,\|f_2^{ex}\|_{L^2(\Gamma_2)}\,,\\ \left\|f_{2,\rho}^* - f_2^{ex}\right\|_{L^2(\Gamma_2)} &\leq c_0\sqrt{\frac{\beta}{\alpha}}\,\|f_2^{ex}\|_{L^2(\Gamma_2)}\,.\end{aligned} \tag{69}$$

Consequently, $[u_{ex}, f_2^{ex}]$ represents a good approximation of the optimal pair $[u_\rho^*, f_{2,\rho}^*]$ in $V \times L^2(\Gamma_2)$, if the quantity $\sqrt{\frac{\beta}{\alpha}}$ is small enough.

Proof. For $f_2 = f_2^{\mathrm{ex}}$, we have that

$$J(f_2) = \frac{\alpha}{2}\,\|\nabla(u_{\mathrm{ex}} - u_{\mathrm{ex}})\|^2_{L^2(\Omega)} + \frac{\beta}{2}\,\|f_2\|^2_{L^2(\Gamma_2)} = \frac{\beta}{2}\,\|f_2\|^2_{L^2(\Gamma_2)}\,. \tag{70}$$

Under hypothesis (59), there exists a unique optimal pair $[u_\rho^*, f_{2,\rho}^*]$ and

$$J(f_{2,\rho}^*) = \frac{\alpha}{2}\left\|\nabla(u_\rho^* - u_{\mathrm{ex}})\right\|^2_{L^2(\Omega)} + \frac{\beta}{2}\left\|f_{2,\rho}^*\right\|^2_{L^2(\Gamma_2)} \leq J(f_2^{\mathrm{ex}}) = \frac{\beta}{2}\,\|f_2^{\mathrm{ex}}\|^2_{L^2(\Gamma_2)}\,.$$

The last relation, combined with (70) and (9), implies that (69) holds and the proof is complete. □

Let us evaluate the constant ϱ_h appearing in the inf-sup condition (67). We consider the following hypotheses:

$$\begin{aligned}&(Aw^h, w^h)_{V_h} \geq \mu_h^*\left\|w^h\right\|^2_{V_h}\,,\\ &(Aw^h, v^h)_{V_h} \leq M_h\left\|w^h\right\|_{V_h}\left\|v^h\right\|_{V_h}\,,\\ &\left\|\gamma w^h\right\|_{L_h^2(\Gamma_2)} \leq c_0^h\left\|w^h\right\|_{V_h}\,,\\ &(w^h, v^h)_{V_h} \leq \left\|w^h\right\|_{V_h}\left\|v^h\right\|_{V_h}\,,\\ &(\gamma w^h, \gamma v^h)_{L_h^2(\Gamma_2)} \leq \left\|\gamma w^h\right\|_{L_h^2(\Gamma_2)}\left\|\gamma v^h\right\|_{L_h^2(\Gamma_2)}\,.\end{aligned}$$

In the above hypotheses we can prove inequality (68). Indeed, to prove (68), for each $\lambda^h \in V_h$, let $w^h_{\lambda^h} \in V_h$ be the unique solution of the variational problem

$$b\left([w^h_{\lambda^h}, \gamma\lambda^h], \varphi^h\right) = \left(\lambda^h, \varphi^h\right)_{V_h} \qquad (\varphi^h \in V_h). \tag{71}$$

From (71) and (43) we deduce that $b\left([w^h_{\lambda^h}, \gamma\lambda^h], \lambda^h\right) = \|\lambda^h\|^2_{V_h}$ and $\|w^h_{\lambda^h}\|_{V_h} \leq \frac{(c_0^h)^2+1}{\mu_h^*}\|\lambda^h\|_{V_h}$. The last two relations imply that, for each $\lambda^h \in V_h$, we have

$$\frac{b\left([w^h_{\lambda^h}, \gamma\lambda^h], \lambda^h\right)}{\|[w^h_{\lambda^h}, \gamma\lambda^h]\|_{V_h \times L_h^2(\Gamma_2)}\|\lambda^h\|_{V_h}} \geq \frac{\mu_h^*}{\sqrt{\left((c_0^h)^2 + 1\right)^2 + (\mu_h^* c_0^h)^2}}.$$

The last inequality implies that (68) is verified with

$$\varrho^h = \frac{\mu_h^*}{\sqrt{\left(\left(c_0^h\right)^2 + 1\right)^2 + \left(\mu_h^* c_0^h\right)^2}}. \tag{72}$$

Finally, let us study the behavior of the minimal pairs of the functional $L_{\mathcal{W}}^{\zeta}$ when ζ tends to zero. We have the following result.

Theorem 10. *For each $\zeta > 0$, let $[w_\zeta^*, f_{2,\zeta}^*]$ be the solution of problem* (OCP$_\zeta^{lin}$)*. Then, there exists a subsequence of the family $\left([w_\zeta^*, f_{2,\zeta}^*]\right)_{\zeta>0}$, denoted in the same way, and the solution $[w^*, f_2^*]$ of problem* (OCP$^{\mathrm{lin}}$) *such that*

$$w_\zeta^* \to w^* \text{ in } V \text{ and } \gamma f_{2,\zeta}^* \rightharpoonup f_2^* \text{ in } L^2(\Gamma_2) \text{ as } \zeta \to 0. \tag{73}$$

Proof. Let $[w^0, 0] \in \mathcal{W}_{\mathrm{ad}}^{\zeta}$. It follows that $\left\|w^0\right\|_V^2 = 0$ and

$$L_{\mathcal{W}}^{\zeta}(w_\zeta^*, f_{2,\zeta}^*) \le L_{\mathcal{W}}^{\zeta}(w^0, 0) \le \alpha \left\|w_d\right\|_V^2.$$

Since $w_d = \overline{v} - u_d$ and taking into account (52), we deduce that $\left([w_\zeta^*, \gamma f_{2,\zeta}^*]\right)_{\zeta>0}$ is a bounded sequence in $V \times L^2(\Gamma_2)$. Consequently, there exists $[w^*, f_2^*] \in V \times L^2(\Gamma_2)$ such that, passing eventually to a subsequence, but keeping the notation to simplify the writing, we have

$$w_\zeta^* \rightharpoonup w^* \text{ in } V \text{ and } \gamma f_{2,\zeta}^* \rightharpoonup f_2^* \text{ in } L^2(\Gamma_2) \text{ as } \zeta \to 0.$$

In fact,

$$w_\zeta^* \to w^* \text{ in } V \text{ as } \zeta \to 0. \tag{74}$$

Indeed, since $[w_\zeta^*, f_{2,\zeta}^*] \in \mathcal{W}_{\mathrm{ad}}^{\zeta}$ we deduce that its verifies the variational formulation of (38) with $z = f_0 = 0$ given by

$$(Aw_\zeta^*, v)_V - \int_{\Gamma_2} \gamma f_{2,\zeta}^*(x) \gamma v(x)\, \mathrm{d}\Gamma = 0 \qquad (v \in V). \tag{75}$$

Since the operator A is strongly monotone, by using (75), we have

$$\begin{aligned} \mu^* \left\|w_\zeta^* - w^*\right\|_V^2 &\le (Aw^*, w^* - w_\zeta^*)_V + (Aw_\zeta^*, w_\zeta^* - w^*)_V \\ &= (Aw^*, w^* - w_\zeta^*)_V + \int_{\Gamma_2} \gamma f_{2,\zeta}^*(x) \gamma (w_\zeta^* - w^*)(x)\, \mathrm{d}\Gamma. \end{aligned}$$

Hence, (74) follows immediately if we passing to the limit when ζ tends to zero in the above inequality.

On the other hand, since $[w_\zeta^*, f_{2,\zeta}^*] \in \mathcal{W}_{\mathrm{ad}}^{\zeta}$, $w_\zeta^* \to w^*$ in V and $\gamma f_{2,\zeta}^* \rightharpoonup f_2^*$ in $L^2(\Gamma_2)$ as ζ tends to zero, by passing to the limit in (75) we deduce that $[w^*, f_2^*] \in \mathcal{W}_{\mathrm{ad}}$.

Let $[\widehat{w}, \widehat{f_2}] \in \mathcal{W}_{\mathrm{ad}}$ be the solution of (OCP$^{\mathrm{lin}}$). For each $\widetilde{f_2} \in H^1(\Omega)$ with the property that $\gamma \widetilde{f_2} = \widehat{f_2}$, we have that

$$L_{\mathcal{W}}(w^*, f_2^*) \le \liminf_{\zeta \to 0} L_{\mathcal{W}}^{\zeta}(w_\zeta^*, f_{2,\zeta}^*) \le \limsup_{\zeta \to 0} L_{\mathcal{W}}^{\zeta}(\widehat{w}, \widetilde{f_2}) = L_{\mathcal{W}}(\widehat{w}, \widehat{f_2}). \tag{76}$$

On the other hand, since $[\widehat{w}, \widehat{f_2}]$ is the solution of (OCP$^{\text{lin}}$), we have that

$$L_{\mathcal{W}}(\widehat{w}, \widehat{f_2}) \leq L_{\mathcal{W}}(w^*, f_2^*). \tag{77}$$

Thus, from (76) and (77) we deduce that $L_{\mathcal{W}}(\widehat{w}, \widehat{f_2}) = L_{\mathcal{W}}(w^*, f_2^*)$, and the proof of the theorem is complete. $\square$

Let α, β, $\zeta > 0$ be three positive constants and we define the following functional $L^\zeta : V \times H^1(\Omega) \to \mathbb{R}$,

$$L^\zeta(u, f_2) = \frac{\alpha}{2} \|u - u_d\|_V^2 + \frac{\beta}{2} \|\gamma f_2\|_{L^2(\Gamma_2)}^2 + \frac{\zeta}{2} \|f_2\|_{H^1(\Omega)}^2. \tag{78}$$

We introduce the following *optimal control problem:*

$$\textit{Find } [u_\zeta^*, \gamma f_{2,\zeta}^*] \in \mathcal{V}_{ad} \textit{ such that } L^\zeta(u_\zeta^*, f_{2,\zeta}^*) = \min_{[u, \gamma f_2] \in \mathcal{V}_{\text{ad}}} L^\zeta(u, f_2). \tag{OCP$^\zeta$}$$

By using the same type of arguments as in Theorem 1 we obtain that (OCP$^\zeta$) has at least one solution $[u_\zeta^*, \gamma f_{2,\zeta}^*]$. Next, we have the following result.

Theorem 11. *For each $\zeta > 0$, let $[u_\zeta^*, \gamma f_{2,\zeta}^*]$ be a solution of problem* (OCP$^\zeta$)*. Then, there exists a subsequence of the family $\left([u_\zeta^*, \gamma f_{2,\zeta}^*]\right)_{\zeta > 0}$, denoted in the same way, and a solution $[u^*, f_2^*]$ of problem* (OCP) *such that*

$$u_\zeta^* \to u^* \textit{ in } V \textit{ and } \gamma f_{2,\zeta}^* \rightharpoonup f_2^* \textit{ in } L^2(\Gamma_2) \textit{ as } \zeta \to 0. \tag{79}$$

Proof. As in the proof of Theorem 5, there exists $[u^*, f_2^*] \in V \times L^2(\Gamma_2)$ such that, passing eventually to a subsequence, but keeping the notation to simplify the writing, we have

$$u_\zeta^* \rightharpoonup u^* \text{ in } V \text{ and } \gamma f_{2,\zeta}^* \rightharpoonup f_2^* \text{ in } L^2(\Gamma_2) \text{ as } \zeta \to 0.$$

Moreover, following the idea in the proof of Theorem 5, we can prove that

$$u_\zeta^* \to u^* \text{ in } V \text{ as } \zeta \to 0.$$

On the other hand we have that $[u^*, f_2^*] \in \mathcal{V}_{\text{ad}}$. Indeed, since $[u_\zeta^*, \gamma f_{2,\zeta}^*] \in \mathcal{V}_{\text{ad}}$, $u_\zeta^* \to u^*$ in V and $\gamma f_{2,\zeta}^* \rightharpoonup f_2^*$ in $L^2(\Gamma_2)$ as ζ tends to zero, by passing to the limit in (SP) we deduce that $[u^*, f_2^*]$ verifies (SP).

Let $[\widehat{u}, \widehat{f_2}] \in \mathcal{V}_{\text{ad}}$ be a solution of (OCP). For each $\widetilde{f_2} \in H^1(\Omega)$ with the property that $\gamma \widetilde{f_2} = \widehat{f_2}$, we have that

$$L(u^*, f_2^*) \leq \liminf_{\zeta \to 0} L^\zeta(u_\zeta^*, f_{2,\zeta}^*) \leq \limsup_{\zeta \to 0} L^\zeta(\widehat{u}, \widetilde{f_2}) = L(\widehat{u}, \widehat{f_2}). \tag{80}$$

On the other hand, since $[\widehat{u}, \widehat{f_2}]$ is a solution of (OCP), we have that

$$L(\widehat{u}, \widehat{f_2}) \leq L(u^*, f_2^*). \tag{81}$$

Thus, from (80) and (81) we deduce that $L(\widehat{u}, \widehat{f_2}) = L(u^*, f_2^*)$, and the proof of the theorem is complete. $\square$