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Abstracts

Adelic quadratic spaces

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(joint work with Gaël Rémond)

We examine the links between linear and quadratic equations through the search of algebraic solutions of small heights.

0.1. The starting point is a theorem by Cassels (1955, [1]) and Davenport (1957, [3]) which asserts that if $q: \mathbb{Q}^n \rightarrow \mathbb{Q}$ is a non-zero isotropic quadratic form with integral coefficients $(a_{i,j})_{i,j}$ then there exists a vector $x = (x_1, \dots, x_n) \in \mathbb{Z}^n \setminus \{0\}$ such that $q(x) = 0$ and

$$\sum_{i=1}^n x_i^2 \leq \left(2\gamma_{n-1}^2 \sum_{i,j} a_{i,j}^2 \right)^{(n-1)/2}$$

(γ_{n-1} is the Hermite constant). Our aim is to give a generalization of this statement in the context of rigid adelic spaces (introduced in the preceding talk by Gaël Rémond).

Let K be an algebraic extension of $\mathbb{Q}$ and n be a positive integer. We denote by $c_K(n)$ the supremum over all rigid adelic spaces E over K of the real numbers

$$\inf \{ H_E(x)^n H(E)^{-1} ; x \in E \setminus \{0\} \}$$

($H_E(x)$ and $H(E)$ are the heights of x and E with respect to the metrics on E). According to [6], the field K is called a *Siegel field* if $c_K(n) < +\infty$ for all $n \geq 1$. We have $c_{\mathbb{Q}}(n) = \gamma_n^{n/2}$,

$$c_K(n) \leq \left(n |\Delta_{K/\mathbb{Q}}|^{1/[K:\mathbb{Q}]} \right)^{n/2}$$

if K is a number field of absolute discriminant $\Delta_{K/\mathbb{Q}}$ and

$$c_{\overline{\mathbb{Q}}}(n) = \exp \left\{ \frac{n}{2} \left(\frac{1}{2} + \dots + \frac{1}{n} \right) \right\}$$

(see [6]).

An *adelic quadratic space* (E, q) over K is a rigid adelic space E/K endowed with a quadratic form $q: E \rightarrow K$. In this framework, several problems can be raised (here, small = of small height):

- 1) Existence of a small isotropic vector,
- 2) Existence of a small maximal totally isotropic subspace,
- 3) Existence of a basis of E composed of small isotropic vectors.

There exist between 25 and 30 articles in the literature dealing with these questions (essentially when K a number field or $\overline{\mathbb{Q}}$). A common divisor to these works is the notion of Siegel's lemma. We shall provide solutions to these three problems, which are optimal with respect to the height of q .

0.2. The following statement gives an answer to the problem 2.

Theorem 1. *Assume q is isotropic. Then, for all $\varepsilon > 0$, there exists a maximal totally isotropic subspace F of E of dimension $d \geq 1$ and height*

$$H(F) \leq (1 + \varepsilon)c_K(n - d) (2H(q))^{(n-d)/2} H(E).$$

Here $H(q)$ is the height of q built from local operators norms (see [7]). For instance, in the context of Cassels and Davenport Theorem, one can prove that $H(q) \leq (\sum_{i,j} a_{i,j}^2)^{1/2}$. Theorem 1 generalizes and improves theorems by Schlickewei (1985, $K = \mathbb{Q}$, [9]), Vaaler (1987, K number field, [10]) and Fukshansky (2008, $K = \overline{\mathbb{Q}}$, [5]). Using a Siegel's lemma in such a subspace F , we obtain an answer to Problem 1:

Quadratic Siegel's lemma. *If q is isotropic then, for all $\varepsilon > 0$, there exists $x \in E \setminus \{0\}$ such that $q(x) = 0$ and*

$$H_E(x) \leq (1 + \varepsilon) \left(c_K(n) (2H(q))^{(n-d)/2} H(E) \right)^{1/d}.$$

The proof of Theorem 1 follows from an estimate of the height of a suitable q -orthogonal symmetric of an almost minimal height subspace F (chosen among maximal totally isotropic subspaces of E) and from a Siegel's lemma used with the quotient E/F . To be interesting, Theorem 1 must be applied in a Siegel field ($c_K(n - d) < \infty$). But the converse is true: it can be also proved that to be a Siegel field is a necessary condition when a quadratic Siegel's lemma exists (take $q(x) = \ell(x)^2$ with $\ell: E \rightarrow K$ a linear form and use [6, § 4.8]).

0.3. Now, let us tackle the problem of a small isotropic basis of an adelic quadratic space (E, q) over a Siegel field K . Assume that there exists a nondegenerate isotropic vector in E . It is well known then that there exists a basis $(e_1, \dots, e_n)$ of E such that $q(e_i) = 0$ for all $1 \leq i \leq n$. Our goal is to have also the heights of e_i 's *small*. An obvious approach rests on an induction process, choosing $e_i \in E \setminus K.e_1 \oplus \dots \oplus K.e_{i-1}$ with small height and $q(e_i) = 0$. That leads us to the following variant of the quadratic Siegel's lemma:

- 1a) Let I be an ideal of the ring of polynomials of E and denote by $Z(I)$ the set of zeros $\{x \in E; \forall P \in I, P(x) = 0\}$. How to bound

$$\inf \{H_E(x); q(x) = 0 \text{ and } x \notin Z(I)\} ?$$

(Quadratic Siegel's lemma avoiding an algebraic set.)

To simplify, we state our result only for the standard adelic space $E = K^n$.

Theorem 2. *Let $q: K^n \rightarrow K$ be a quadratic form and let I be an ideal of $K[X_1, \dots, X_n]$ generated by polynomials of (total) degree $\leq M$. Assume (i) $q \neq 0$ and (ii) $\exists x \notin Z(I); q(x) = 0$. Then there exists a constant $c(n, K) \geq 1$, which depends only on n and K , such that the vector x in condition (ii) can also be chosen with height*

$$H_{K^n}(x) \leq c(n, K) M^3 H(q)^{(n-d+1)/2}$$

where d is the dimension of maximal totally isotropic subspaces of (K^n, q) .

The constant $c(n, K)$ can be made fully explicit (see [7, § 7]). This statement generalizes and improves previous results by Masser (1998, $K = \mathbb{Q}$, $Z(I)$ hyperplane, [8]), Fukshansky (2004, K number field, $Z(I)$ union of hyperplanes, [4]) and Chan, Fukshansky & Henshaw (2014, [2]). Moreover the exponent $(n - d + 1)/2$ of $H(q)$ is best possible: take $E = \mathbb{Q}^n$, $a, d \geq 1$ integers, $Z(I) = \{x_d = 0\}$ and

$$q(x) = 2x_{d+1}x_d - a^2x_d^2 - (x_{d+2} - ax_{d+1})^2 - \cdots - (x_n - ax_{n-1})^2.$$

We have $H(q) = O_{a \rightarrow +\infty}(a^2)$ and if x is isotropic then $|x_n| \geq a^{n-d+1}|x_d|/4$. The proof of Theorem 2 relies on an avoiding Siegel's lemma and a geometric lemma. From Theorem 2 can easily be deduced a small-height isotropic basis of E .

Complete proofs and further results are given in [7].

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