MASTER SET

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SOME REMARKS ON CORRELATIONS BETWEEN CONFLICT AND INSTABILITY

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A study of the stability of classical systems under the influence of two conflicting single-well, single-variable potentials is presented. The control parameters are a so-called conflict parameter c , the distance between the two conflicting attractors, and an asymmetry parameter a which determines the relative strength of the two attractors. For convex potentials, the influence of conflict on the system stability depends on the sign of the third derivative of the conflicting potential functions, resulting in two categories called U- and V-type potentials. The introduction of a concave branch (bias) in convex conflicting potentials qualitatively changes the system behavior, by generating bifurcation and discontinuity. The cusp catastrophe can be redefined as the conflict between two such biased single-well potentials. The main interest of the new representation potentially lies in the evocative power of the control parameters a and c . The mapping of (a, c) on (u, v) , the traditional control parameters of the cusp catastrophe, is given. The cusp develops above a certain critical value of the conflict parameter c .

INDEX TERMS: Conflict, competition, frustration, single-well potential, bifurcation, catastrophe theory, cusp, stability, instability

1. INTRODUCTION

A traditional description of systems behavior, intermediate between phenomenological and local interaction descriptions, involves potential or energy functions. The system evolves towards an equilibrium state which is represented by a minimum of a suitably defined potential function $\mathcal{P}$ (gradient dynamics). The system state is supposed to be totally defined by n state variables $(x_1, x_2, \dots, x_n)$. A clear distinction should be made between these state variables and so-called control parameters $(a_1, a_2, \dots, a_m)$ which represent the various interactions of the system with its environment and control the potential shape. The potential $\mathcal{P}$ is a function of both state variables and control parameters $\mathcal{P}(a_1, \dots, a_m, x_1, \dots, x_n)$, the equilibrium states of the system being defined by the sets of x_j values such as $\partial\mathcal{P}/\partial x_j = 0$ for all x_j . The potential minima are attractors of the system dynamics, and the portion of state space within which any state evolves toward the attractor defines the attractor basin.

The response of the system to a small excitation ε will depend on the shape of the potential function near the equilibrium point. Several typical situations can occur, as illustrated on Figure 1 on a one-variable potential. If the system is in a deep, narrow minimum of the potential function, a small excitation ε will not move the system very much away from its equilibrium position (Fig. 1a). If it is located in a large, flat potential well, the same small excitation ε will generate a much bigger perturbation of the system around its equilibrium

position (Fig. 1b). Finally, if the system is trapped in a local minimum with a very small potential barrier which can be overcome by the excitation ε , it will be allowed to escape from its equilibrium position toward another one, thus experiencing an abrupt (catastrophic), irreversible move (Fig. 1c).

An interesting question is: *how is the potential shape modified when control parameters vary?*, since the stability of the system depends on the answer. In general, the potential shape results from the superposition (balance) of a number of elementary potentials, characteristic of the interactions between the system and different parts of its environment. Each specific, individual interaction determines its own potential that will be called *component potential*, while the resulting global interaction will define a so-called *overall potential*, summation of all the component potentials, and consequence of the competition between these potentials. In the following, it will be assumed that each component potential is single-well, or more precisely, that it is characterized by a single, punctual attractor. This study has been limited to simple situations concerning one-dimensional state space only, where the overall potential results from the contribution of (and conflict between) two single-variable component potentials.

The aim of this work is to try to establish, in such simple cases, some correlations between the stability of a system and the degree of conflict between the two underlying interactions. In this study, the term "conflict" is being used in a very specific sense, which is defined in Section II, together with the different parameters used through the paper. The analysis and results presented in the subsequent sections pertain only to that sense of the term "conflict", and cannot necessarily be extrapolated to any other. Those situations where the conflict is solved in a smooth, continuous way, are analyzed in Section III, where a classification of convex potentials, whose competition leads to two opposite behaviors with respect to stability, is presented. Section IV deals with those cases where conflict generates discontinuities in the system state, and for which a relationship with Catastrophe Theory is attempted. The cusp catastrophe is described as resulting from the conflict between two single-well potentials, and a study of the instability of the cusp around its bifurcation set is presented. Some conclusions are drawn in Section V.

Figure 1 Influence of a small excitation ε on the stability of a system (whose state is represented by a full circle) originally in stable equilibrium in a potential well, for different potential shapes: deep and narrow potential (a), flatter potential (b), local minimum with small barrier (c). Ordinate represents potential energy and abscissa the 1D state space. The total length of the horizontal segments measures the destabilization parameter d_ε .

2. DEFINITIONS

All throughout this article, differentiation marks will concern the state variable x only.

2.1. Destabilization

To estimate the stability of a given system, it is useful to have a measure of the fluctuation of the system caused by a given excitation, that is, of the flatness of the potential around the equilibrium position. For this purpose, a *destabilization parameter* d_ε is defined as the portion of the state space spanned by the system when it is submitted to an excitation ε . It might be considered as a rough estimate of the intrinsic instability of the system. In a 1D state space, this scalar quantity d_ε represents a segment length, as illustrated by the horizontal segments in the examples of Figure 1. The larger this scalar is, the less stable the system (the less resistant to perturbations).

2.2. Control Parameters

2.2.1. Conflict parameter c As a common sense definition, a conflicting situation will be said to occur between the two component potentials if their attractors (minima) do not coincide. The distance c between the two attractors in the state space will be used as a measure of the importance of the conflict between the two component potentials. In the following, any mention to the term "conflict" will refer to this definition. Since the two component potentials are single-well, in any case, the overall minimum (minima) will be located somewhere in between the two component minima.

2.2.2. Strength parameter a In this paper, for convenience, the conflicting potentials p_1 and p_2 will be considered as belonging to the same generic type p , and symmetrical, which is often the case, for instance when the system interacts with two environmental elements of the same kind. The two component potentials will thus be written $p_1(x) = k_1 p(x)$ and $p_2(x) = k_2 p(-x)$, where the minus sign indicates the symmetry, and k_1 and k_2 are positive constants. These two constants control the strength of the elementary interactions, since they determine the deepness of the component potentials: the smaller k is, the flatter the potential and the weaker the corresponding attractor. Since only two attractors are considered, one strength parameter is enough to determine their relative strength: it will be called a (for asymmetry), such that $k_1 = (1 + a)/2$ and $k_2 = (1 - a)/2$. For $a = 0$, p_1 and p_2 have equal strength since $k_1 = k_2 = \frac{1}{2}$, while $a = \pm 1$ represent limit cases where one of the potentials is flat ($k_j = 0$) and the other has full strength ($k_j = 1$).

2.2.3. General formulation The influence of the two control parameters a and c on the relative shape and position of the two component potentials is schematized on figure 2.

In summary, an overall potential function with one state variable x and two control parameters a and c will be considered. For convenience, and without any loss of generality, the minimum of the generic potential p will be taken as the origin ($p(0) = p'(0) = 0$), and the

Figure 2 Influence of the two control parameters a and c on the relative shape and position of the two component potentials (full and dotted lines). The purpose of this article is to study the resulting distortion of the overall potential in some simple and specific cases.

two component potentials will be considered as symmetrically shifted away ($\pm c/2$) from the origin. Therefore, the overall potential function $\mathcal{P}$ can be written:

$$\mathcal{P}(a, c, x) = \frac{1+a}{2} p\left(\frac{c}{2} + x\right) + \frac{1-a}{2} p\left(\frac{c}{2} - x\right) \quad (1)$$

with a between -1 and $+1$, c positive, and p a single-well potential function with $p(0) = p'(0) = 0$.

The purpose of this study is to evaluate the influence of the control parameters a and c on the overall potential shape, that is, the relationship $d_e = f(a, c)$ for a given generic potential p and a given excitation ε .

3. CONFLICT WITHIN CONTINUITY

In this section, only cases where the generic potential p is convex in its whole definition range will be considered, or at least where the state region of interest (around the overall minimum) is determined by the overlap of the two convex branches of the component potentials. In such a region $\mathcal{P}''$ is positive, since k_1 and k_2 are positive as well as p'' . This means that the overall potential is convex, with one minimum only whatever the values of a and c , which will determine a unique equilibrium position for the system. Since the minimum is unique, there is no possibility for the system to jump from one equilibrium position to another one by varying a control parameter; the system state varies continuously by smoothly varying a control parameter: convexity generates continuity.

3.1. Influence of the Conflict Parameter c

Considering the influence of conflict only and before any mathematical consideration, two qualitatively different types of behavior are expected in the simplest cases: either the overall minimum deepens when c increases, or it flattens. This will depend on the shape of the component potentials. Intuitively, increasing conflict between two flat-minimum potential wells (U-shape) will tend to deepen the minimum of the overall potential, whereas flattening will be observed for increasing conflict between two narrow-minimum potentials (V-shape). A general, mathematical, definition of so-called U-type and V-type potentials will now be given.

The overall potential $\mathcal{P}$ is considered a function of one state variable x and one control parameter only, the conflict parameter c ($a = 0$). The generic potential $p(x)$ is assumed to be at least of class C^3 , with possibly a discontinuity of the third derivative at the origin. From the hypotheses, it follows:

$$\begin{aligned}\mathcal{P}(0, c, x) &= \frac{1}{2}p\left(\frac{c}{2} + x\right) + \frac{1}{2}p\left(\frac{c}{2} - x\right) && \text{(from (1))} \\ p(0) = p'(0) &= 0 && \text{(origin)} \\ p''(x) &\geq 0 && \text{(convexity)}\end{aligned}$$

The problem is to determine how $\mathcal{P}$ is modified around its minimum when c is increased.

3.1.1. Ground-state Since $\mathcal{P}''(0, c, x) = \frac{1}{2}p''\left(\frac{c}{2} + x\right) + \frac{1}{2}p''\left(\frac{c}{2} - x\right) \geq 0$, the overall potential remains convex, as seen earlier. Consequently, $\mathcal{P}'(0, c, x)$ is an increasing function of x , with $\mathcal{P}'(0, c, 0) = \frac{1}{2}p'\left(\frac{c}{2}\right) - \frac{1}{2}p'\left(\frac{c}{2}\right) = 0$, that is $\mathcal{P}'(0, c, x)$ is negative

for $x < 0$ and positive for $x > 0$. Thus $x = 0$ corresponds to the (absolute) minimum of $\mathcal{P}$, and $\mathcal{P}$ is symmetrical relative to $x = 0$ whatever the value of c , since

$$\mathcal{P}(0, c, -x) = \frac{1}{2}p\left(\frac{c}{2} - x\right) + \frac{1}{2}p\left(\frac{c}{2} + x\right) = \mathcal{P}(0, c, x).$$

The equilibrium position is at the origin and remains stable whatever the importance of the conflict c .

Besides, $\mathcal{P}(0, c, 0) = p(c/2)$ with $c > 0$. As $p(c/2)$ increases with c positive, the minimum of the overall potential $\mathcal{P}$ is an increasing function of the conflict parameter c : shifting away the two component potentials p_1 and p_2 increases the ground state energy of the system. This is typically what is observed in frustrated magnetic systems (sinusoidal potential wells) where this increase has been used to quantify frustration [Lacorre, 1987].

3.1.2. Excitations The problem is now what will be the behavior of such systems experiencing conflict when they are excited? What matters is the shape of the potential well $\mathcal{P}(0, c, x)$ close to its minimum $x = 0$, and the way it evolves when the conflict c varies. Taylor's series expansion of the component potentials can be used to determine it.

$$\begin{aligned}
 p\left(\frac{c}{2} + x\right) &= p\left(\frac{c}{2}\right) + x p'\left(\frac{c}{2}\right) + \frac{x^2}{2!} p''\left(\frac{c}{2}\right) + \frac{x^3}{3!} p'''\left(\frac{c}{2}\right) + \frac{x^4}{4!} p^{(4)}\left(\frac{c}{2}\right) + \dots \\
 p\left(\frac{c}{2} - x\right) &= p\left(\frac{c}{2}\right) - x p'\left(\frac{c}{2}\right) + \frac{x^2}{2!} p''\left(\frac{c}{2}\right) - \frac{x^3}{3!} p'''\left(\frac{c}{2}\right) + \frac{x^4}{4!} p^{(4)}\left(\frac{c}{2}\right) - \dots
 \end{aligned}$$

Hence:

$$\mathcal{P}(0, c, x) = \frac{1}{2} p\left(\frac{c}{2} + x\right) + \frac{1}{2} p\left(\frac{c}{2} - x\right) = p\left(\frac{c}{2}\right) + 0 + \frac{x^2}{2!} p''\left(\frac{c}{2}\right) + 0 + \frac{x^4}{4!} p^{(4)}\left(\frac{c}{2}\right) + 0 + \dots$$

If now only small perturbations of the system are considered, which will not move it very far away from its equilibrium position, such that the expansion can be limited to the second order, one gets:

$$\mathcal{P}(0, c, x) = p\left(\frac{c}{2}\right) + \frac{x^2}{2} p''\left(\frac{c}{2}\right) + \mathcal{O}(x^4)$$

The positive second order coefficient $p''(c/2)$ is important, since it determines the shape of the overall potential: it can be seen that this shape depends on the conflict parameter c . If the system is submitted to a small excitation ε , it will span the state space between $+x_e$ and $-x_e$, with x_e such that $\varepsilon = \mathcal{P}(0, c, x_e) - \mathcal{P}(0, c, 0) = \mathcal{P}(0, c, x_e) - p\left(\frac{c}{2}\right) \approx \frac{x_e^2}{2} p''\left(\frac{c}{2}\right)$.

Therefore, the expression of the destabilization parameter $d_e (= 2|x_e|)$ as a function of ε and of the conflict parameter c is:

$$d_e \approx 2 \sqrt{\frac{2\varepsilon}{p''\left(\frac{c}{2}\right)}}$$

For small excitations, the importance of the perturbation as a function of the conflict c depends on the variation of the second derivative of p , that is on the sign of the third derivative:

- if $p'''(c > 0)$ is positive, the perturbation decreases when conflict increases,
- if $p'''(c > 0)$ is negative, the perturbation increases when conflict increases.

With reference to the intuitive reasoning of the beginning of section 3.1, we will thereafter call “U-type potential” a convex potential with positive third derivative for positive variable, and “V-type potential” a convex potential with negative third derivative.

In more complex cases where the third derivative changes sign, the behavior will vary depending on the degree of conflict.

3.2. Some Case Studies

Table I gives a list of potentials, some of them commonly used in physics and chemistry, which have been classified according to their type, U or V. An increasing conflict between two U-type, *resp.* V-type potentials will result in a deepening, *resp.* flattening of the resulting overall potential. An illustration is given in figure 3, which represents the variation of the destabilization parameter d_e with the conflict parameter c for the potentials of Table 1.

Table 1 Some convex interaction potentials classified according to their type

Potential	$p(x)$	$c_{lim}^{\dagger}$	$p'''(b=c/2)^{\ddagger}$	$\text{sign}(p''')^{\ddagger}$	Type
secant	$(\cos x)^{-1} - 1$	π	$(\sin b)(5 + \sin^2 b)/\cos^4 b$	+	U
catenary	$\frac{(e^x + e^{-x})}{2} - 1$	$+\infty$	$\frac{(e^b - e^{-b})}{2}$	+	U
LJ _{rep.} [§]	$(x - 2^{1/6})^{-12}$ $- (x - 2^{1/6})^{-6} + 1/4$	$2^{7/6}$	$168(b - 2^{1/6})^{-15}$ $(2(b - 2^{1/6})^6 - 13)$	+	U
power	$ x ^q, q > 2$	$+\infty$	$q(q-1)(q-2)b^{q-3}$	+	U
parabola	x^2	$+\infty$	0	0	U/V
power	$ x ^q, 1 < q < 2$	$+\infty$	$q(q-1)(q-2)b^{q-3}$	-	V
LJ _{att.} [§]	$(x + 2^{1/6})^{-12}$ $- (x + 2^{1/6})^{-6} + 1/4$	$2((26/7)^{1/6} - 2^{1/6})$	$168(b + 2^{1/6})^{-15}$ $(2(b + 2^{1/6})^6 - 13)$	-	V
hyperbola	$(1 + x^2)^{1/2} - 1$	$+\infty$	$-3b(1 + b^2)^{-5/2}$	-	V
cosine	$1 - \cos x$	π	$-\sin b$	-	V
Morse	$1 - e^{-x^2}$	$\sqrt{2}$	$-4b(3 - 2b^2)e^{-b^2}$	-	V

[†]The quantity c_{lim} represents the maximal allowed value of c in order either to remain within the definition range of the potential or to be consistent with the convexity hypothesis.

[‡]Expressions and signs for p''' are given for $c > 0$.

[§]LJ means Lennard-Jones; this potential is considered either through its repulsive (rep.) or its attractive (att.) branch.

Power potentials $p(x) = |x|^q$ ($q > 1$) are U-type or V-type potentials depending on the value of q : U-type for $q > 2$ and V-type for $1 < q < 2$. A particular case, intermediate between U-type and V-type potentials, is that of a parabolic potential x^2 with a null third derivative: with such a potential, the magnitude of the perturbation does not depend on the importance of the conflict. As an example, for a given excitation, the oscillation of a mass between two springs does not depend on the stretching or compression of the springs (within their elastic limit). In this case, the shape of the potential is fully determined by the system insensitivity to the degree of conflict.

Let us now examine more closely two potentials of physical significance.

3.2.1. Sinusoidal potential For this periodic potential $p(x) = 1 - \cos x$, only one period is considered. The limit $c_{lim} = \pi$ corresponds to the convexity condition (the two previous points are also valid for the secant potential).

An interesting point with the sinusoidal potential is that the whole Taylor series with no limit to the second order parameter can be considered, since.

$$\begin{aligned} \mathcal{P}(0, c, x) - p\left(\frac{c}{2}\right) &= \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots\right) \cos\left(\frac{c}{2}\right) \\ &= (1 - \cos x) \cos\left(\frac{c}{2}\right) \end{aligned}$$

Since $1 - \cos x$ is always positive whatever the value of x , and $\cos(c/2)$ decreases when c increases, the overall potential flattens when the conflict increases. The interesting point is

Destabilization curves for convex potentials

Figure 3 Evolution with the degree of conflict c of the destabilization d_e for a system under conflicting influence of the U-type and V-type potentials of Table I. In order to allow a direct comparison, the applied excitation is different in each case and has been calculated in such a way that the resulting perturbation without conflict is the same for all examples ($d = 0.1$).

that the previous remark is valid whatever the value of x . The consequence is that a sinusoidal potential can be considered as V-type even for large perturbation.

Sinusoidal potentials are of interest in several physics fields. In magnetism for instance, the Heisenberg hamiltonian for the coupling of two interacting spins S_i and S_j writes $-J_{ij}S_iS_j$, where J_{ij} is the coupling constant. In the classical approximation, S_i and S_j are vectors so that the coupling energy is $e_{ij} = -J_{ij}|S_i||S_j|\cos\theta_{ij}$, where θ_{ij} is the angle between spins S_i and S_j . That is, spin S_i under the influence of its interaction with spin S_j , experiences a sinusoidal potential. A conflicting situation may occur when a spin S interacts with two

spins S_1 and S_2 . Such a situation is called *magnetic frustration*. It occurs, for instance, in triangular cycles (platelets) of spins in antiferromagnetic interaction (see Fig. 4).

Spin S , the system under study, is supposed to interact with spins S_1 and S_2 through two antiferromagnetic (AF, negative) interactions of same strength J . A conflicting situation for S occurs when S_1 and S_2 are not parallel, for instance if S_1 and S_2 interact through an AF interaction J' such that $J' < 0.5J < 0$. Spin S experiences an overall potential which is the summation of two conflicting interaction potentials with S_1 and S_2 . The conflict parameter c is the angle between spins S_1 and S_2 , which depends on the ratio J'/J through the relationship $c = 2\text{Arccos}\left(\frac{J}{2J'}\right)$. A complete interpretation, in terms of so-called *constraint functions* [Lacorre, 1987], of the magnetic systems experiencing frustration has been proposed [Lacorre, 1988]. According to this interpretation, the site constraint function of spin S is, in this case, $f_s = -\frac{J}{2J'}$, and the site constraint angle is $\theta_s = \text{Arccos}(-f_s)$.

Straightforwardly, it can be seen that the site constraint angle θ_s is nothing else but half the conflict parameter c . The constraint function is another measure of the degree of conflict, as the conflict parameter c (they vary the same way), but it is more convenient to handle since it varies between -1 and $+1$, thus providing an absolute scale of frustration. For more details, see [Lacorre, 1987] and [Lacorre, 1988].

As evidenced in Table 1 and illustrated in Figure 4, increasing the conflict (i.e., the frustration) decreases the stability of the system (i.e., flattens the overall potential). This shows that, for a given excitation, a frustrated magnetic system will experience larger thermal ($\varepsilon = kT$) and quantum fluctuations, for instance, than a non-frustrated system. As a matter of fact (see for instance [Oguchi, *et al.*, 1985], [Férey, *et al.*, 1986], [McLenaghan and Sherrington, 1985]), it is well known that frustrated magnetic systems have lower ordering temperatures and larger spin reduction (zero-point motion). Strong evidence for a correlation between these features and the constraint function are presented elsewhere [Lacorre, 1988].

The important point to stress here is that such behaviors as lower ordering temperature or larger zero-point motion do not result from the presence of frustration only, but rather are a direct consequence of the interaction potential shape, which is sinusoidal, that is, V-type: had the magnetic interaction potential been U-type (secant, for instance, or hyperbolic cosine), frustration would have caused exactly the opposite effects (higher ordering temperature, smaller spin reduction).

3.2.2. Lennard-Jones potential The Lennard-Jones potential is not convex in the whole definition range, but in this section, the study will be limited to the convex branch of the potential. This type of potential is usually used to describe a Van der Waals interaction between two atoms (see for instance [Kittel, 1968], p. 85). For instance, one can consider a central atom in Van der Waals interaction with two identical atoms in symmetrical positions, one at each side (see Fig. 5). If the three atoms are allowed to move freely, they will adopt a non-conflicting position (Fig. 5a), on which the two extreme atoms are considered fixed (Gedankenexperiment). At a given temperature, the central atom will be submitted to a thermal agitation of a certain amplitude. If the extreme atoms are now moved slightly toward each other (compression) and left fixed at these new positions, at the same temperature as previously, the central atom will have a smaller amplitude of thermal agitation

non-conflicting orientations

Figure 4 A frustrated triangular platelet of spins in antiferromagnetic interaction as an illustration of a conflicting situation between two sinusoidal potentials. The studied system is the central spin S . The conflict parameter c is controlled by the strength of interaction J' . As c increases (upper to lower), the overall potential flattens such that for a given, either thermal, or quantum excitation, the spin fluctuation increases.

On the right side: component potentials (dotted lines) and overall potential (full lines). The abscissa units are radian.

non-conflicting positions

Figure 5 An atom (for instance Argon) in between two similar atoms as an illustration of a conflicting situation between two Lennard-Jones potentials. The studied system is the central atom. The shape of the component (dotted lines) and overall potentials (full lines) and the amplitude of fluctuation for a given excitation are shown in the following situations:

- a) no conflict,
- b) compression on the extreme atoms ($c < 0$),
- c) separation of the extreme atoms, with $0 < c < c_{lim}$,
- d) separation of the extreme atoms, with $c > c_{lim}$.

(Fig. 5b). Conversely, if the atoms are moved slightly away from each other, the thermal agitation of the central atom will be larger (Fig. 5c).

Such a behavior originates from the shape of the Lennard-Jones potential, of the type $(x + 2^{1/6})^{-12} - (x + 2^{1/6})^{-6} + \frac{1}{4}$ (see Fig. 6a). This potential is defined for

$x > -2^{1/6} \approx -1.122$; it is convex for $-2^{1/6} < x < \left(\frac{26}{7}\right)^{1/6} - 2^{1/6} \approx 0.122$ and concave for

$x > \left(\frac{26}{7}\right)^{1/6} - 2^{1/6}$. Within the convex part, a distinction has to be made depending on the sign of x : if x is negative (repulsive branch), the potential is U-type, while it is V-type if x is positive (attractive branch). Such a difference in the convexity type of the Lennard-Jones potential is responsible for the opposite influence of conflict on the stability of the central atom upon compression and repulsion conditions. Moving the extreme atoms away might cause the concave branch of the potentials to overlap (Fig. 5d) and lead to a qualitatively different behavior, which will be examined in section 4.1.

Figure 6 a) Characteristics of the Lennard-Jones potential. b) Evolution $d_e = f(c)$ of the destabilization with conflict between two Lennard-Jones potentials for three values of excitation ϵ . The dashed line corresponds to the "mathematical" bifurcation. Note that the "delay" described in the text can be made as small as desired by reducing the excitation.

In this example, conflict leads to two opposite behaviors with respect to the stability of the system depending on what branch of the potential defines the equilibrium states. Had the potential shape not been known, the simple qualitative observation of the specific behavior of the system under conflict would have shown that the underlying interaction potential is part U-type/part V-type.

3.3. Influence of Control Parameters c and a

The second control parameter a , which tunes the relative strength of the conflicting interactions, is now allowed to vary. For instance in the previous examples, the two antiferromagnetic interactions will not have the same strength ($J_1 \neq J_2$ instead of J) or the two extreme atoms in Van der Waals interaction will not be the same.

According to expression (1), the overall potential $\mathcal{P}$ can now be written:

$$\mathcal{P}(a, c, x) = \frac{1+a}{2} p\left(\frac{c}{2} + x\right) + \frac{1-a}{2} p\left(\frac{c}{2} - x\right) \text{ with } -1 \leq a \leq 1.$$

In general, the overall minimum does not coincide with the origin $x = 0$.

In the (a, c) control plane, the line $a = 0$ represents the one-control-parameter case studied above, while the lines $a = 1$ and $a = -1$ correspond to the extreme cases where one interaction is of zero strength (flat potential, see Fig. 2) with constant equal instability whatever the value of c . When the generic potential p is symmetrical ($p(x) = p(-x)$), the shape of the overall potential is the same for $a = \pm 1$ as for the non-conflicting case $c = 0$ ($p(X)$, with $X = x \pm \frac{c}{2}$). In this case, the destabilization is the same on the three segments of the control plane (a, c) : $a = 1$, $a = -1$ and $c = 0$.

Above the control plane, the destabilization parameter d_e defines a surface hereafter called "destabilization surface" (In some respects, the "constraint surface" defined in [Lacorre, 1988] for magnetic phase diagrams is equivalent to a destabilization surface for magnetic systems). On this surface, in the symmetrical case ($p(x) = p(-x)$), the segments $a = 1$, $a = -1$ and $c = 0$ belong to a plane with constant destabilization (reference plane). The surface develops below or above this plane, depending whether the potential is U-type or V-type, respectively.

Figure 7 gives the destabilization surfaces of representative examples of U-type and V-type potentials chosen among those studied earlier.

4. CONFLICT GENERATING DISCONTINUITY

We will now explore those cases where conflict induces abrupt changes in the system state. Since conflict between convex potential wells generate only a smooth change in the system state, it is necessary to introduce a concave branch in the component potentials.

4.1. Introduction of a Concave Branch in a Convex Potential Well

In paragraph 3.2.2, the Lennard-Jones potential $(x + 2^{1/6})^{-12} - (x + 2^{1/6})^{-6} + \frac{1}{4}$ has been used as an example of mixed U-type/V-type convex potential, when considered close to its minimum (see Fig. 6a). However, further apart from its minimum (typically for

Figure 7 Destabilization surfaces $d_e(a, c)$ for some representative examples of U-type (upper row) and V-type (lower row) convex potentials.

$x > x_l = \left(\frac{26}{7}\right)^{\frac{1}{6}} - 2^{\frac{1}{6}} \approx 0.122$, the potential becomes concave. Let us now allow the two concave branches of the conflicting potentials to overlap, by shifting the two extreme atoms away from each other with a conflict parameter c greater than the limit $c_{lim} = 2x_l \approx 0.244$. The overall potential has now two symmetrical minima on each side of the origin (see Fig. 5d). Thus the equilibrium position of the central atom is not unique anymore, neither intermediate between the two extreme atoms, but there are two equilibrium positions shifted towards each of the atoms: the system has experienced bifurcation at the critical value of $c = c_{lim}$. Figure 6b shows the evolution of the destabilization parameter d_e as a function of the conflict c for the Lennard-Jones potential (with c considered negative for compression). An increase of the instability with the conflict up to the bifurcation point (maximum instability) is observed, then a decrease of the instability. Such an evolution of the stability of conflicting systems experiencing bifurcation is probably very general, as will be seen later (a similar behaviour is also observed in the Morse potential, for instance).

Another general characteristic of this kind of system is the existence of a certain delay between the (mathematical) appearance of the bifurcation in the overall potential and its effective "detection" by the system, due to the magnitude of the excitation it experiences. In principle, such a delay can be made as small as desired by reducing the excitation (see Fig. 6b).

It seems clear, from this example, that the triggering factor for the appearance of a bifurcation is the presence of a concave part in the conflicting potential wells. Such a point will be illustrated with another example.

As seen in the previous section, the potential $p(x) = x^4$ is U-type (see Table 1). The slightly more complex potential well $x^4 + x$ can be expressed, after appropriate shifts in order to get the minimum of the well at the origin, as $p(x) = (x - m)^4 + x - m^4$, with $m = \left(\frac{1}{4}\right)^{\frac{1}{3}} \approx 0.630$. Such a potential (see Fig. 8a) is convex on the whole definition range, since $p''(x) = 12(x - m)^2$ is positive whatever the value of x . This potential is supposed to conflict in its $x > 0$ branch with its symmetrical counterpart $p(-x) = (x + m)^4 - x - m^4$.

Figure 8 Introduction of a concave branch (bias) in a convex potential: Characteristics of the potentials (a) and curves $d_{\varepsilon} = f(c)$ (b). The dashed vertical segment indicates "mathematical" bifurcation for biased potential $x^4 - x^2 + x$.

The third derivative $p'''(\frac{c}{2} > 0)$ is equal to $12(c - 2m)$, which is negative provided that $c < 2m$ (V-type potential). If the degree of conflict is not too large, such that $c \leq 2m$, the system becomes more and more unstable when the conflict increases (note that the introduction of a lower degree term resulted in a change in the potential type, from U-type in the pure x^4 potential to V-type in the $x^4 + x$ potential).

It is possible to introduce a concave segment in the previous potential by subtracting an x^2 term in order to get a potential of the type $x^4 - x^2 + x$ (Fig. 8a). After an appropriate shift, $p(x) = (x - m)^4 - (x - m)^2 + x - m^4 + m^2$ has its minimum (equilibrium position) at the origin if $m = \left(\frac{1}{8} + \frac{\sqrt{57}}{72}\right)^{\frac{1}{3}} + \left(\frac{1}{8} - \frac{\sqrt{57}}{72}\right)^{\frac{1}{3}} \sim 0.885$. Such a potential is convex in the whole

definition range, except in a concave segment $m - \frac{1}{\sqrt{6}} < x < m + \frac{1}{\sqrt{6}}$. As in the previous case, the third derivative $p'''(c > 0)$ is negative provided that $c < 2m \approx 1.769$. This potential is considered to conflict with its symmetrical counterpart in the same conditions as previously. The plot on Figure 8b gives the evolution of the destabilization as a function of the intensity of the conflict for the three potential wells of Figure 8a.

Comparison with the pure convex case leads to a straightforward conclusion: the introduction of a concave segment in the convex potential has instigated the appearance of a bifurcation, for a conflict parameter $c = 2m - \sqrt{\frac{2}{3}} \approx 0.953$ (dashed vertical segment in Fig. 8b). As for the Lennard-Jones potential (Fig. 6b), the destabilization increases on increasing conflict, reaching its maximum at the bifurcation or close to it, then the appearance of two minima stabilizes the equilibrium.

If now the second control parameter a is introduced, according to (1), the overall potential becomes:

$$\mathcal{P}(a, c, x) = \frac{1+a}{2} p\left(\frac{c}{2} + x\right) + \frac{1-a}{2} p\left(\frac{c}{2} - x\right),$$

with

$$p(x) = (x - m)^4 - (x - m)^2 + x - m^4 + m^2, \text{ and } m = \left(\frac{1}{8} + \frac{\sqrt{57}}{72}\right)^{\frac{1}{3}} + \left(\frac{1}{8} - \frac{\sqrt{57}}{72}\right)^{\frac{1}{3}}$$

Not paying much attention to the constant term, which is not important for the present study, one gets:

$$\begin{aligned} \mathcal{P}(a, c, x) = & x^4 + 2a(c - 2m)x^3 + \left(\frac{3}{2}(c - 2m)^2 - 1\right)x^2 \\ & + \left(\frac{1}{2}(c - 2m)^3 - c + 2m + 1\right)ax + cst. \end{aligned}$$

An appropriate parameter change of x to $x + a\left(m - \frac{c}{2}\right)$ simplifies the previous expression to:

$$\mathcal{P}(a, c, x) = x^4 + \left(\frac{3}{2}(1 - a^2)(2m - c)^2 - 1\right)x^2 + ((1 - a^2)(2m - c)^3 + 1)ax + cst,$$

that is

$$\mathcal{P}(a, c, x) = x^4 + ux^2 + vx + cst,$$

with

$$u = \frac{3}{2}(1 - a^2)(2m - c)^2 - 1,$$

and

$$v = ((1 - a^2)(2m - c)^3 + 1)a.$$

The expression $x^4 + ux^2 + vx$ is well known to be that of the cusp in Catastrophe theory. What has been done is to express the cusp catastrophe as the conflict between two single-well potentials.

4.2. The Cusp as Conflict between Two Single-Well Potentials

In his outstanding book "Stabilité Structurale et Morphogénèse" René Thom [1972] showed that, in a four-dimensional space, there were only seven possible most simple and qualitatively different configurations of discontinuities in state variables, generated by a continuous variation of some control parameters governing the shape of a smooth potential. Among these configurations, called elementary catastrophes, the cusp or Riemann-Hugoniot catastrophe is probably the most famous since it has been used to describe the discontinuities observed in a variety of bimodal phenomena ranging from optics (caustics), to mechanics (collapse of elastic structures), thermodynamics (phase transitions), embryology (cellular differentiation), ethology (aggression in dogs), psychology (bistable perception), psychiatry (anorexia), political science (decision making, revolutions), linguistics (archetypal morphologies of verbs), semiology (semiotic square), among others (for applications to Catastrophe Theory, see for instance [Hilton, 1976], [Zeeman, 1977], [Poston and Stewart, 1978], [Saunders, 1980], [Gilmore, 1981] and references therein). It is a qualitative mathematical model, independent from any substrate, which allows an extraordinarily wide application range, from physics to metaphysics, from "hard" to "soft" science.

4.2.1. The traditional representation of the cusp The cusp catastrophe is generated by the universal unfolding of the singularity x^4 . The unfolding $x^4 + ux^2 + vx$ reveals the degenerate nature of the critical point of x^4 which is lifted by the perturbation introduced by the added terms. The control parameters u and v define the shape of the potential and determine the equilibrium positions, for which the state variable x is such that $4x^3 + 2ux + v = 0$. The equilibrium surface of the cusp is defined by the set $x(u, v)$ such that x is a real solution of the previous equation: its general shape (see Fig. 9a) gave its name to the cusp. The potential has one extremum (minimum), except within the cusp region $8u^3 + 27v^2 < 0$, where it has three extrema (two minima separated by a maximum). In the cusp region, the minima defining stable equilibrium positions correspond to the upper and lower sheets

of the cusp, while the maximum (unstable equilibrium) is represented by the (usually unreachable) intermediate sheet.

When the parameters (u, v) controlling the system are varied continuously, the state of the system evolves according to the position of the representative point on the equilibrium surface. In the major part of the control plane, a smooth variation of the control parame-

Figure 9 The cusp catastrophe:

a) the equilibrium surface above the control plane with the cusp bifurcation set.

b) distortion of the potential across the cusp region ($u = \text{negative constant}$, v varies). The full dot represents the equilibrium position within the perfect delay convention. The total length of the horizontal segments represents the destabilization parameter d_e , the system being perturbed by an excitation ϵ . Note the discontinuity anticipation (left arrow) discussed in the text (imperfect delay convention).

ters implies a continuous variation of the system state, except on the bifurcation set defined by the curves $8u^3 + 27v^2 = 0$, where a discontinuous variation is observed: the representative point on the equilibrium surface jumps either from the upper sheet to the lower one, or from the lower to the upper. Another specific characteristics of the model is clearly visible: a back and forth motion on the control plane does not necessary bring the system back to its original state (hysteresis across the cusp). In other words, the state of the system is not defined by the control parameters u and v alone, but also by the history of the motion. This is true only across the cusp, inside what will be called the *cusp region* ($u < 0$), but not in what is sometimes referred to as the *neutral* or *normal region* ($u > 0$), where complete reversibility and smooth behaviour are observed.

Figure 9b illustrates the distortion of the overall potential while crossing the cusp region (v is varied, u being fixed negative). In the previous paragraph, the convention adopted concerning the system state is called "perfect delay convention": the system remains in a potential well up to the point where this well disappears, while crossing the second bifurcation branch; the system then experiences a state discontinuity. Another convention is also used for some applications, the Maxwell convention: the system remains in a potential well as long as the minimum is global. Therefore, the state discontinuity occurs earlier than in the perfect delay convention, and it is reversible. An illustration of this phenomenon is given by phase transitions, for instance. In the stability study of the current article, the perfect delay convention cannot strictly be applied since the system is perturbed by a (small) excitation ε : this excitation allows it to escape from a local minimum before the bifurcation branch is effectively crossed. Such an anticipation has the same origin as the "delay" evoked in Section IV-1. The situation is thus intermediate between the perfect delay and Maxwell conventions, sometimes called "imperfect delay convention". It is illustrated in Figure 9b.

4.2.2. The new representation of the cusp Traditionally, the cusp is interpreted as reflecting the conflict between two attractors defined by the double-well of the generic potential, the strength of an attractor depending on the deepness of the corresponding well. However, no individual existence is assigned to each attractor (or "pregnancy" in Thom's terminology [Thom, 1981], [Thom, 1988], [Thom, 1990] p. 53). What is shown in Section IV-1 is that it is possible to decompose the overall double-well potential in two conflicting single-well potentials, representative of the two competing pregnancies, the relative strength of which is defined by the control parameter a , and the degree of conflict between pregnancies by the control parameter c .

The relationship between parameters a and c and the traditional control parameters u and v are given in Section IV-1. The mapping of (a, c) on (u, v) is illustrated in Figure 10, with a between -1 and $+1$ and c between 0 and $2m$ taken as the maximum possible conflict. It can be seen that the area in the (u, v) plane covered by the mapping overlaps all four quadrants, including a part of the so-called neutral region ($u > 0$) which therefore, according to this description, already bears traces of some conflict. On the left part of the mapping, the two points A and B represent the pure archetypal shape of the two conflicting attractors, of the two individual pregnancies. Any potential represented by a point inside the mapping region can be generated by a given degree of conflict (parameter c) between these two archetypal potentials with a given relative strength (parameter a).

The two generic symmetrical points can be chosen anywhere within the region $8u^3 + 27v^2 > 0$ with $u < 0$, arbitrarily close to or far from the bifurcation set. Let $(\alpha, \pm \beta)$ be the

Figure 10 New representation of the cusp: mapping of (a, c) on (u, v) with $\alpha = -1$ and $\beta = 1$. Point A and its symmetrical counterpart B represent the archetypal conflicting pregnancies. Any point within the mapping region can be expressed as the conflict c between A and B with relative strength a . Curves within the mapping region are iso- a or iso- c lines (iso- c lines join points A and B while iso- a lines do not).

coordinates of A and B in the (u, v) plane, with $8\alpha^3 + 27\beta^2 > 0$ and $\alpha < 0$. In the previous example, $\alpha = -1$ and $\beta = 1$. In the general case, the overall potential can be written:

$$\mathcal{P}(a, c, x) = \frac{1+a}{2} p\left(\frac{c}{2} + x\right) + \frac{1-a}{2} p\left(\frac{c}{2} - x\right)$$

with

$$p(x) = (x - m)^4 + \alpha(x - m)^2 + \beta x - m^4 - \alpha m^2,$$

m depending on α and β (typically such that $4m^3 + 2\alpha m - \beta = 0$ since, by definition, $p'(0) = 0$). The relationships between (a, c) and (u, v) are, in the general case:

$$u = \frac{3}{2}(1 - a^2)(2m(\alpha, \beta) - c)^2 + \alpha,$$

$$v = ((1 - a^2)(2m(\alpha, \beta) - c)^3 + \beta)a.$$

The mapping region is delimited by the curve $c = 0$ and the segment $c = 2m(\alpha, \beta)$. An interesting feature is that this region can be split in two domains (see figure 11a), with qualitatively different regimes:

- a low-conflict domain $0 < c \leq 2m - \sqrt{\frac{2}{3}|\alpha|}$, within which a smooth variation in the relative strength of the competing attractors always changes the system state smoothly;

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Figure 11 The mapping region of the cusp as built up from two domains, a low-conflict (light gray) and a strong conflict (dark gray) domains:
a) within the traditional (u, v) reference frame,
b) within the new (a, c) reference frame. Conflicting pregnancies A and B are now represented by the facing upper and lower segments.

- a strong-conflict domain $2m - \sqrt{\frac{2}{3}}|\alpha| < c < 2m$ in which a smooth variation in the attractor strengths can generate an abrupt jump (a discontinuity) in the system state.

The two domains are also clearly seen in Figure 11b, where the bifurcation set is plotted inside the (a, c) reference frame.

The conclusion is that, when competing potentials exhibit a concave branch, a strong enough conflict can generate bifurcation and discontinuities in the system state.

4.2.3. Some comments Since Catastrophe Theory generates pure mathematical models, independent from any substrate, and expressed in their simplest possible formulation, a change of variable is often necessary in order to get expressions as a function of the control parameters specific to a given application. For instance, for a certain type of applications, E.C. Zeeman [1977] uses a set of coordinates simply deduced by a rotation ($u - v, -u - v$) from the original set. Different in nature is the new formulation proposed in the above section, since it concerns the essence of the overall potential itself, and involves its decomposition in two individual components, each representing one of the two conflicting influences.

Despite its relative complexity in reference to the extreme simplicity of the original formulation, the proposed representation of the cusp might appear to be useful since the new control parameters (a, c) have a very clear significance which could expectedly simplify interpretation in some applications. For instance, the fundamental difference between crossing a branch of the bifurcation set parallel to u and parallel to v might not appear straightforward by a mere examination of the overall potential distortion (see Fig. 12), whereas it is within the new formulation (see Fig. 13). A simple example, borrowed from R. Thom, concerning his application of Catastrophe Theory to linguistics will serve as an illustration. According to Thom [1970], verbs can be interpreted as specific trajectories within the universal unfolding of catastrophes, and he has numbered up to 16 different archetypal morphologies describing space-time processes. For instance, crossing the bifurcation set of the cusp catastrophe parallel to the u -axis ($v = \text{cst}$) or to the v -axis ($u = \text{negative cst}$) can be interpreted as the linguistic morphologies of the verbs "to capture" and "to become", respectively. Such an interpretation is straightforward within the new representation just by considering the evolution of the component potentials (Fig. 13).

Sections $u = \text{cst}$ and $v = \text{cst}$ are almost exactly equivalent to sections $c = \text{cst}$ and $a = \text{cst}$, respectively, in the new representation¹ (compare paths ① and ② on Fig. 12a and Fig. 13a). Figure 13b gives the evolution of the component potentials A and B, as c or a are varied across the bifurcation set. A decrease of c (the distance in the state space between A and B) while a (their relative strength) remains constant is very evocative of a union, or the capturing of B, the weakest, by A, the strongest (Fig. 13b path ①). Now, when a (the relative strength) varies while c (the distance) remains constant, the evolution of the component potentials during the variation, as well as its interpretation, appear to be very different from the previous one. For instance, when a is equal to 1, A has its full, archetypal shape while B does not exist (flat potential). As a decreases, B appears and deepens, increasing

¹ In fact, the equivalence would be exact if the archetypal points A and B were rejected to infinite. Thom [1970] pointed out that linear sections are not the only possible ones, and even in some cases, incurvation is necessary.

Figure 12 a) The two main sections across the cusp bifurcation set, in the traditional representation (path ①: $v = \text{cst}$; path ②: $u = \text{cst}$)

b) Potential distortion along path ① ($v = 0.3$; from left to right: $u = -1, -0.86, -0.72, -0.58, -0.44$) and path ② ($u = -0.72$; from left to right: $v = 0.1, 0.2, 0.3, 0.4, 0.5$)

Figure 13 Same as figure 12, within the cusp new representation (with $\alpha = -1$ and $\beta = 1$):

a) The two main sections (path ①: $a = \text{cst}$; path ②: $c = \text{cst}$)

b) Variation of the relative position and strength of the component potentials along path ① ($a = 0.2$; from left to right: $c = 1.76, 1.28, 0.84, 0.4, 0$) and path ② ($c = 1.28$; from left to right: $a = 0.98, 0.80, 0, -0.80, -0.98$)

Path ① represents the verb "to capture" and path ② the verb "to become".

in strength while A weakens, flattens, till it eventually disappears while B reaches its full, archetypal shape (Fig. 13b path ②). During the process, the distance between the two attractors remains the same, while the "substance" of A is transferred to B. It provides a clear and very suggestive illustration of the verb "to become": A becomes B.

Comparatively, the interpretation and fundamental difference between the two meanings seems less evident in the traditional cusp representation (see Fig. 12b).

It appears that the new representation also accounts fairly well for more complex biological processes [Lacorre, 1995].

4.3. Stability on the Cusp

In the same way as for the case of convex potentials treated in the previous article, the destabilization on the cusp will now be studied as a function of control parameters u and v , or equivalently a and c . As previously, the destabilization will be measured by the portion of state space spanned by the system under the influence of a given, constant excitation. Although low-conflict and neutral regions do not coincide exactly (see figure 11a), both are characterized by single-well overall potentials and they will be studied jointly; the same for the strong-conflict and cusp regions, which share the same characteristics.

4.3.1. Low-conflict (or neutral) region Figure 14a and b give the destabilization surface of the low-conflict ((a,c) coordinates) and neutral ((u,v) coordinates) regions, respectively. Figure 14a is thus directly comparable to Figure 7 (V-type) since in both cases only convex parts of the potentials are concerned: note the strong similitude. The general trend is an increase of destabilization with conflict, due to the V-type character of component potentials around the minimum, as previously emphasized.

An interesting feature is to be noticed: the continuous increase of instability when moving from any point in the neutral region (Fig. 14b) towards the origin ($u = 0, v = 0$), so that this point, corresponding to the organizing centre of the cusp, exhibits the highest instability in the whole neutral region. Thus emphasis is put on the fact that the neutral region, usually considered of little interest in regard to the richness of the cusp region, already bears hints of the discontinuities which characterize the cusp bifurcation set, through this increase of instability. As a matter of fact, the neutral region does not reflect a true steady regime, but rather it conceals a certain, although low, amount of underlying conflict (Fig. 14a), resulting in this "smooth" destabilization.

Such conflict, when increased further, will give birth to the cusp.

4.3.2. Strong-conflict (or cusp) region When conflict is such that the concave parts of the two competing potentials overlap (that is when c is greater than $2m - \sqrt{\frac{2}{3}}|\alpha|$) or equivalently when u becomes negative, bifurcation occurs for small values of a or v , in the same way as already described in Section 4.1 for Lennard-Jones potentials. Within this region, when the degree of conflict is held constant and the relative strength of the competing attractors is smoothly varied (a varying from -1 to $+1$) or equivalently when u is held fixed and v varies continuously from $-\infty$ to $+\infty$, an abrupt jump (discontinuity) in the system state is observed at the crossing of the bifurcation set. The amplitude of the jump will be usually much larger than the spanning amplitude of the excited state in a convex potential well, especially if the excitation is small. It means that the destabilization parameter d_e will exhibit a huge increase just at, or just before (see Section IV-2.1 and Fig. 9b) the crossing of the bifurcation set (of a fold of the cusp), as illustrated in Figure 15.

Figure 14 Destabilization surface ($\epsilon = 0.005$) above the control plane of the cusp within the new (left) and traditional (right) representations: for the low-conflict and neutral region only: a) $\log d_\epsilon(a, c)$ and b) $\log d_\epsilon(u, v)$; including strong-conflict and cusp region: c) $\log d_\epsilon(a, c)$ and d) $\log d_\epsilon(u, v)$.

Figure 15 shows the evolution of destabilization due to a small excitation when v is varied, for a negative and a positive value of u . In the cusp region (when u is negative) or, equivalently, in the strong conflict region, the following behavior is observed when v (respectively a) is varied across the bifurcation set:

- before the discontinuity: the destabilization progressively increases as the representative point approaches the bifurcation set;
- at the fold, or close to it: there is an abrupt increase of instability due to the jump of the system state from the vanishing local minimum to the global minimum. Just after

Figure 15 The destabilization $d_e(v)$ for a negative and a positive values of u .

this irreversible jump, the system instability strongly decreases, and becomes even smaller than just before the jump;

- after the discontinuity: the destabilization progressively decreases as the representative point moves away from the bifurcation set.

In summary, the discontinuity (abrupt jump) occurs at a maximum of instability, and its effect is to release the instability. In Figure 15, this situation is compared with that occurring in the neutral (or low-conflict) region, where instability is released without discontinuity in the system state. As already mentioned, another difference is the non-reversibility (hysteresis) observed in the strong-conflict region, while there is reversibility for the system state in the low-conflict region.

Figures 14c-d give a view of the destabilization surface, including the neutral or low-conflict region. In the cusp or strong-conflict region and starting from the organizing centre, a double crest of instability develops above the bifurcation set. When the control point crosses the bifurcation set, the point representative of the system state on the destabilization surface first passes through the first branch of the double crest, then climbs up the other branch (see arrows in Fig. 15).

It can be seen in the strong-conflict region, that the maximum instability for a given conflict value continues increasing with conflict, as in the low-conflict region, but now, first it does not occur at equal attractor strengths ($a \neq 0$) and second, it corresponds to a discontinuity, unlike in the low-conflict region. The increase of maximum instability with strengthening conflict is due to the increasing gap between the competing attractors. The minimum instability (maximum stability) occurs when one of the attractors has maximum strength ($a = \pm 1$), the other being inexistent (flat potential). It is straightforward that a conflict with a

null-strength attractor (which in fact is no longer an attractor) does not change the stability of the system, and could merely be considered as a non-conflicting situation.

A final remark concerns the incidence of the catastrophe (discontinuity) on the global stability of the system, since it (the catastrophe) seems to concentrate the major part of instability: the system appears to be much more stable in the strong-conflict region before and after the catastrophe, than it is in the low-conflict region before and after the maximum of instability (see figure 15).

4.3.3. The Maxwell convention If the Maxwell convention is used (the system always seeks the global minimum) instead of the (im)perfect delay convention, the equilibrium surface has the shape presented on figure 16-upper part. This surface resembles the P-V-T phase diagram of fluids at the liquid-gas transition. In fact, it is well known that the Van der Waals equation of state can be expressed in the same form as that of the cusp equilibrium surface, with the volume or density as state variable, the control parameters u and v being simply related to the pressure and temperature. In the Maxwell convention, unlike in the (im)perfect delay convention, the transition is reversible (no hysteresis); it occurs at $v = 0$ (u negative), or $a = 0$ ($c > 2m - \sqrt{\frac{2}{3}|\alpha|}$).

The destabilization surface of the cusp in the Maxwell convention is shown on Figure 16-lower part. The portion corresponding to the neutral (or low-conflict) region is the same as that in the imperfect delay convention. Emphasis is put on the fact that the critical point ($u = v = 0$, or $a = 0$, $c = 2m - \sqrt{\frac{2}{3}|\alpha|}$) corresponds to the maximum instability in the neutral (low-conflict) region.

Figure 16 The cusp in the Maxwell convention: equilibrium surface above destabilization surface ($\varepsilon = 0.005$)

Within the strong conflict region, the maximum instability takes place on the critical line (remember that strong fluctuations usually occur at phase transitions). On the critical line, instability increases with increasing conflict. When going across the transition line, the instability first increases, then the transition occurs with maximum instability, then the instability decreases symmetrically.

The other remarks presented in the previous paragraph are still valid with the Maxwell convention.

5. SUMMARY AND CONCLUSION

In this paper, a description of the stability of systems under the influence of two conflicting single-well and single-variable attractor basins has been attempted, with the distance between attractors taken as the conflict parameter. Their stability depends on the shape of the overall attractor basin resulting from the conflict between the two component attractors. A conflict between two convex potentials is always solved in a smooth, continuous way, since the summation of two convex functions always gives a convex function. However, the stability of the system varies with the degree of conflict, in a way which depends on the sign of the third derivative of the conflicting potential functions: two categories of single-well convex potentials have thus been defined in Section 3, U-type and V-type potentials.

The only way to get a qualitatively different behavior, with discontinuity in the system state and bifurcation is to "bias" the conflicting attractor basins by introducing a concave segment within the otherwise convex potentials. This has been developed on a few examples in Section 4. One of the main results of this paper is that the cusp catastrophe, from the Elementary Catastrophe Theory, can be described as the conflict between two "concavely biased" convex potentials. Such a concave bias is just another, slightly different formulation of the unfolding of singularity x^4 : one might say that the unfolding is encoded within the generic potential. The important point here does not concern the exact quantitative value of the concave bias (the point representative of the archetypal conflicting attractors can be chosen almost anywhere in a wide region of the (u,v) control plane), but rather the qualitative information that such a bias of the single-well conflicting potentials is absolutely necessary (the u coordinates of the representative points *have to* be negative) in order to generate the cusp through the conflict.

Another important conclusion of the present description is that the cusp develops above a certain critical degree of conflict (depending on the exact nature of the bias) below which the conflict is solved in a smooth, continuous way. However, in this low-conflict region, traces of the increasing conflict can be found in the increasing instability of the system which, as it reaches its maximum value corresponding to the organizing centre of the catastrophe, generates the cusp.

Concerning the new representation of the cusp, despite the relative apparent complexity of the formulation and besides the fact that it gives a coherent pattern within the present study, its main advantage potentially resides in providing more meaning than the purely mathematical parameters u and v : the notions of strength (parameter a) and separation (parameter c) might appear more appealing to clarify concepts. For instance, they suggest an alternate detailed description [Lacorre, 1995] of the "predation loop" introduced by Thom [1972]. More generally, one can wonder whether a similar type of representation

could be used for the other, more complex catastrophes implying conflict between more than two attractors.

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