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Schur-positivity via products of grid classes

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Abstract. Characterizing sets of permutations whose associated quasisymmetric function is symmetric and Schur-positive is a long-standing problem in algebraic combinatorics. In this paper we present a general method to construct Schur-positive sets and multisets, based on geometric grid classes and the product operation. Our approach produces many new instances of Schur-positive sets, and provides a broad framework that explains the existence of known such sets that until now were sporadic cases.

Résumé. La caractérisation des ensembles de permutations dont la fonction quasisymmetrique associée est symétrique et Schur-positive est un problème de longue date dans la combinatoire algébrique. Dans cet article, nous présentons une méthode générale pour construire des ensembles et multiensembles Schur-positifs, basée sur des *grid* classes géométriques et l'opération de multiplication. Notre approche produit beaucoup de nouveaux cas d'ensembles Schur-positifs, et elle fournit un cadre général qui explique l'existence de tels ensembles qui jusqu'à maintenant étaient des cas sporadiques.

Keywords. Schur-positivity, descent, symmetric group, grid class, Kronecker product, arc permutation, quasi-symmetric function

1 Introduction

Given any subset A of the symmetric group $\mathcal{S}_n$, define the quasi-symmetric function

$$\mathcal{Q}(A) := \sum_{\pi \in A} F_{n,\mathrm{Des}(\pi)},$$

where $\mathrm{Des}(\pi) := \{i : \pi(i) > \pi(i+1)\}$ is the descent set of π and $F_{n,\mathrm{Des}(\pi)}$ is Gessel's *fundamental quasi-symmetric function* defined by

$$F_{n,D}(\mathbf{x}) := \sum_{\substack{i_1 \le i_2 \le \ldots \le i_n \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

The following long-standing problem was first posed in [10].

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Problem 1.1 *For which subsets of permutations $A \subseteq \mathcal{S}_n$ is $\mathcal{Q}(A)$ symmetric?*

A symmetric function is called *Schur-positive* if all coefficients in its expansion in the Schur basis are nonnegative. The problem of determining whether a given symmetric function is Schur-positive is a major problem in contemporary algebraic combinatorics [18].

By analogy, a set (or, more generally, a multiset) A of permutations in $\mathcal{S}_n$ is called *Schur-positive* if $\mathcal{Q}(A)$ is symmetric and Schur-positive. Classical examples of Schur-positive sets of permutations include inverse descent classes and Knuth classes [9], conjugacy classes [10, Theorem 5.5] and permutations of fixed inversion number [3, Prop. 9.5].

An exotic example of a Schur-positive set, different from all the above ones, was recently found: the set of arc permutations, which may be characterized as those avoiding the patterns $\{\sigma \in \mathcal{S}_4 : |\sigma(1)-\sigma(2)| = 2\}$. A bijective proof of its Schur-positivity is given in [6]. Inspired by this example, Woo and Sagan raised the problem of finding other Schur-positive pattern-avoiding sets [13]. Our goal in this paper is to provide a conceptual approach that explains the existing results and produces new examples of Schur-positive pattern-avoiding sets of permutations. An important tool in our approach will be to consider products of geometric grid classes.

A *geometric grid class* (introduced by Albert et al. [4]) consists of those permutations that can be drawn on a specified set of line segments of slope ± 1, whose locations and slopes are determined by the positions of the corresponding non-zero entries in a matrix M with entries in $\{0, 1, -1\}$. We use the term grid to refer to this set of line segments.

Let $\mathcal{G}_n(M)$ be the set of permutations in $\mathcal{S}_n$ that can be obtained by placing n dots on the grid in such a way that there are no two dots on the same vertical or horizontal line, labeling the dots with $1, 2, \dots, n$ by increasing y-coordinate, and then reading them by increasing x-coordinate.

Let $\mathcal{G}(M) = \bigcup_{n\geq 0} \mathcal{G}_n(M)$. We call $\mathcal{G}(M)$ a *geometric grid class*, or simply a *grid class* for short (all grid classes that appear in this paper are geometric grid classes). Since removing dots from the drawing of a permutation on a grid yields drawings of the permutations that it contains, it is clear that every geometric grid class is closed under pattern containment, and so it is characterized by its set of minimal forbidden patterns, which is always finite, as shown in [4].

Example 1.2 Left-unimodal permutations*, defined as those for which every prefix forms an interval in $\mathbb{Z}$, are those in the grid class*

$$\mathcal{L} := \mathcal{G}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

A drawing of the permutation 4532617 *on this grid is shown on the left of Figure 1.*

In general, the product of Schur-positive subsets of $\mathcal{S}_n$ does not give a Schur-positive multiset or set. We are interested in finding families of subsets whose product is Schur-positive, either as a multiset or as a set.

Let $[n] := \{1, 2, \dots, n\}$. The *descent set* of a permutation $\pi \in \mathcal{S}_n$ is defined by $\mathrm{Des}(\pi) := \{i : \pi(i) > \pi(i+1)\}$. For each $J \subseteq \{1, \dots, n-1\}$, define the *descent class* $D_{n,J} := \{\pi \in \mathcal{S}_n : \mathrm{Des}(\pi) = J\}$, and its inverse $D^{-1}_{n,J} := \{\pi^{-1} : \pi \in D_{n,J}\}$. Our first main result about products of Schur-positive sets is Theorem 4.5, where we prove that for every Schur-positive set $\mathcal{B} \subseteq \mathcal{S}_n$ and every $J \subseteq [n-1]$, the multiset product $\mathcal{B}D^{-1}_{n,J}$ is Schur-positive. Even though the proof details are not included in this extended abstract due to lack of space, they appear in the full version of this paper [8], where we provide a representation-theoretic proof that involves Solomon's descent representations, Kronecker products of symmetric functions and Stanley's shuffling theorem.

If instead of considering multiset products we are interested in the underlying sets being Schur-positive, we have a more restricted theorem (Theorem 5.7): for every $J \subseteq [n-2]$, the set product $D_{n-1,J}^{-1} C_n$ is Schur-positive, where $C_n = \langle c \rangle = \{c^k : 0 \le k < n\}$ is the cyclic subgroup generated by the n-cycle $c = (1, 2, \dots, n)$, and $D_{n-1,J}^{-1}$ is interpreted as a subset of $\mathcal{S}_n$ by identifying $\mathcal{S}_{n-1}$ as the set of the permutations in $\mathcal{S}_n$ that fix n. The proof combines a descent-set-preserving bijection together with sieve methods and representation theoretic arguments.

Schur-positivity of various set and multiset products of grid classes follows from the above two results. In Section 5 we discuss applications to vertical and horizontal rotations of grids, of which arc permutations are a special case, and thus we obtain a short proof of their Schur-positivity. As another application, we prove Schur-positivity of certain multiple-column grid classes, including the grid $\mathcal{G}(M_k)$ obtained by vertical rotation of a one-column grid all of whose slopes are positive (see the drawing of the grid $\mathcal{G}(M_3)$ in Fig. 2).

The paper concludes with a list of some open problems, questions, conjectures, and ideas for further work in Section 6.

2 Preliminaries

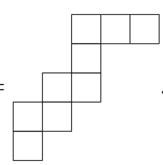
2.1 Zigzag tableaux and classes of permutations

For a skew shape λ/μ, let $\mathrm{SYT}(\lambda/\mu)$ be the set of standard Young tableaux of shape λ/μ. We use the English notation in which row indices increase from top to bottom. For $T \in \mathrm{SYT}(\lambda/\mu)$, define its descent set by $\mathrm{Des}(T) := \{i : i+1 \text{ lies southwest of } i \text{ in } T\}$.

A *zigzag shape* is a path-connected skew shape that does not contain a 2×2 square. For example, every hook is a zigzag shape. There is a natural bijection between the set of all subsets of $[n-1]$ and the set of all zigzag shapes of size n, where the size is defined to be the number of cells.

Definition 2.1 *Given a subset $J \subseteq [n-1]$, let $Z_{n,J}$ be the zigzag shape with n cells labeled $1, \dots, n$ increasing in the northeast direction, where cell $i+1$ is immediately above cell i if $i \in J$, and immediately to the right of cell i otherwise.*

For example, if $n = 9$ and $J = \{1, 3, 5, 6\}$, then

$Z_{n,J} =$  .

Consider the map from the set of standard Young tableaux (SYT for short) of all zigzag shapes of size n to permutations in $\mathcal{S}_n$ defined by listing the entries of the SYT, starting from the southwest corner and moving along the shape. The restriction of this map to the set $\mathrm{SYT}(Z_{n,J})$ of tableaux of a fixed zigzag shape $Z_{n,J}$ is a bijection to permutations in $\mathcal{S}_n$ with descent set J. This bijection has the property that the descent set of the SYT becomes the descent set of the inverse of the associated permutation.

The well-known Robinson–Schensted correspondence maps each permutation $\pi \in \mathcal{S}_n$ to a pair (P_π, Q_π) of standard Young tableaux of the same shape $\lambda \vdash n$. The *Knuth class* corresponding to a standard Young tableau T of size n is the set of permutations $\pi \in \mathcal{S}_n$ such that $P_\pi = T$. It is known that inverse descent classes are disjoint unions of Knuth classes.

For sets A and B of permutations on disjoint finite sets of letters, denote by $A \sqcup\!\sqcup B$ the set of all shuffles of a permutation in A with a permutation in B. For example, if $A = \{12, 21\}$ and $B = \{43\}$, then $A \sqcup\!\sqcup B = \{1243, 1423, 1432, 4123, 4132, 4312, 2143, 2413, 2431, 4213, 4231, 4321\}$. Shuffles will play an important role in Section 4.

For partitions $\mu \vdash k$ and $\nu \vdash n-k$, let (μ, ν) be the skew Young diagram obtained by placing Young diagrams of shape μ and ν so that the northeast (NE) corner of the Young diagram of shape μ coincides with the southwest (SW) corner of the Young diagram of shape ν.

The following result is well known, and some generalizations of it can be found in [5, 12].

Theorem 2.2 *Let A be a Knuth class of shape μ on the letters $1, \ldots, k$ and B a Knuth class of shape ν on the letters $k+1, \ldots, n$ then $A \sqcup\!\sqcup B$ is a union of Knuth classes, and the distribution of* Des *is the same over $A \sqcup\!\sqcup B$ and over* $\mathrm{SYT}(\mu, \nu)$.

The following result, due to Stanley, will be used in Section 4. For bijective proofs, see [11, 15].

Proposition 2.3 ([16, Ex. 3.161]) *Given two permutations σ and τ of disjoint sets of integers, the distribution of the descent set over all shuffles of σ and τ depends only on* $\mathrm{Des}(\sigma)$ *and* $\mathrm{Des}(\tau)$.

2.2 Quasi-symmetric functions and Schur-positivity

The *Frobenius image* of an $\mathcal{S}_n$-character $\chi = \sum_{\lambda \vdash n} c_\lambda \chi^\lambda$ is the symmetric function $\mathrm{ch}(\chi) := \sum_{\lambda \vdash n} c_\lambda s_\lambda$.

The quasi-symmetric function $\mathcal{Q}(A)$ from Section 1 may be naturally extended to multisets of permutations. We say that a (multi)set of permutations $\mathcal{B}$ in $\mathcal{S}_n$ is *Schur-positive for a complex $\mathcal{S}_n$-representation ρ* if

$$\mathcal{Q}(\mathcal{B}) = \mathrm{ch}(\chi^\rho).$$

It was recently shown that $\mathcal{B}$ is Schur positive for ρ if and only if χ^ρ may be evaluated by a certain $\{-1, 0, 1\}$-weighted enumeration of $\mathcal{B}$ [3, Theorem 1.5] [1, Theorem 3.2].

The following variation of Problem 1.1 was proposed in [3].

Problem 2.4 *For which subsets of permutations $A \subseteq \mathcal{S}_n$ is $\mathcal{Q}(A)$ Schur-positive?*

Classical examples of Schur-positive sets are listed in Section 1. An example of a Schur-positive set that does not fall in any of these cases is the set of arc permutations in $\mathcal{S}_n$.

Definition 2.5 *A permutation $\pi \in \mathcal{S}_n$ is an* arc permutation *if, for every $1 \le j \le n$, the first j letters in π form an interval in $\mathbb{Z}_n$ (where the letter n is identified with zero). Denote by $\mathcal{A}_n$ the set of arc permutations in $\mathcal{S}_n$.*

For example, $12543 \in \mathcal{A}_5$, but $125436 \notin \mathcal{A}_6$, since $\{1, 2, 5\}$ is an interval in $\mathbb{Z}_5$ but not in $\mathbb{Z}_6$.

Arc permutations were introduced in the study of flip graphs of polygon triangulations [2]. Some combinatorial properties of these permutations, including their description as a union of grid classes and their descent set distribution are studied in [6]. In particular, it follows from [6, Theorem 5] that $\mathcal{A}_n$ is a Schur-positive set. One of the goals of this paper is to explain this result by providing a general recipe for constructing Schur-positive subsets of $\mathcal{S}_n$.

3 Simple examples of Schur-positive grid classes

3.1 One-column grid classes

Grid classes whose matrix M consists of one column are particularly interesting because they are unions of inverse descent classes, and thus Schur-positive. Without loss of generality, we may assume that M is a $\{1,-1\}$-matrix. For $\mathbf{v} \in \{1,-1\}^k$ let $\mathcal{G}^{\mathbf{v}}$ denote the one-column grid class where the entries of $\mathbf{v}$ are read from bottom to top. For convenience, we only write the signs of the entries of $\mathbf{v}$. For example, $\mathcal{G}^{-+} = \mathcal{L}$ is the class of left-unimodal permutations, and $\mathcal{G}^{++}$ is the class of shuffles of two increasing sequences. In general, we denote by $\mathcal{G}^{+^k}$ the class of shuffles of k increasing sequences.

It is easy to see that $\mathcal{G}^{+^k}$ consists of those permutations with $\mathrm{des}(\pi^{-1}) \le k-1$, and similarly for $\mathcal{G}^{-^k}$.

Proposition 3.1 1. *$\mathcal{Q}(\mathcal{G}^{+^k})$ is the multiplicity-free sum of symmetric functions of zigzags of height at most k. Additionally, $\mathcal{Q}(\mathcal{G}_n^{+^k}) = \sum_{\lambda \vdash n} |\{P \in \mathrm{SYT}(\lambda) : |\mathrm{Des}(P)| \le k-1\}|\, s_\lambda$.*

2. *$\mathcal{Q}(\mathcal{G}^{-^k})$ is the multiplicity-free sum of symmetric functions of zigzags of width at most k. Additionally, $\mathcal{Q}(\mathcal{G}_n^{-^k}) = \sum_{\lambda \vdash n} |\{P \in \mathrm{SYT}(\lambda) : |\mathrm{Des}(P)| \ge n-k\}|\, s_\lambda$.*

We obtain a particularly simple expression when $k=2$:

$$\mathcal{Q}(\mathcal{G}_n^{+^2}) = s_n + \sum_{a=1}^{\lfloor \frac{n}{2} \rfloor} (n-2a+1) s_{n-a,a}.$$

A similar argument also yields

$$\mathcal{Q}(\mathcal{L}_n) = \sum_{k=0}^{n-1} s_{n-k,1^k}.$$

General one-column grid classes can be expressed as unions of inverse descent classes as follows.

Proposition 3.2 *For every $\mathbf{v} \in \{+,-\}^k$,*

$$\mathcal{G}_n^{\mathbf{v}} = \mathcal{S}_n \setminus \bigsqcup_{\substack{\mathbf{u} \in \{+,-\}^{n-1} \\ \mathbf{v} \le \mathbf{u}}} D^{-1}_{n,J_{\mathbf{u}}} = \bigsqcup_{\substack{\mathbf{u} \in \{+,-\}^{n-1} \\ \mathbf{v} \not\le \mathbf{u}}} D^{-1}_{n,J_{\mathbf{u}}},$$

where $\mathbf{v} \le \mathbf{u}$ denotes that $\mathbf{v}$ is a subsequence of $\mathbf{u}$, and $J_{\mathbf{u}} := \{i : u_i = +\}$. Consequently, $\mathcal{G}_n^{\mathbf{v}}$ is a Schur-positive set.

Example. We have

$$\mathcal{G}_5^{-++} = \mathcal{S}_5 \setminus \left(D^{-1}_{5,\{2,3\}} \sqcup D^{-1}_{5,\{2,4\}} \sqcup D^{-1}_{5,\{3,4\}} \sqcup D^{-1}_{5,\{1,3,4\}} \sqcup D^{-1}_{5,\{2,3,4\}} \right).$$

Corollary 3.3 *For every $\mathbf{v} \in \{+,-\}^k$,*

$$\mathcal{Q}(\mathcal{G}_n^{\mathbf{v}}) = \sum_{\substack{\mathbf{u} \in \{+,-\}^{n-1} \\ \mathbf{v} \not\le \mathbf{u}}} s_{Z_{n,J_u}}$$

Recall from [14] *the Solomon descent subalgebra* of $\mathbb{C}[\mathcal{S}_n]$, which is spanned by the elements $d_{n,J} := \sum_{\pi \in D_{n,J}^{-1}} \pi \in \mathbb{C}[\mathcal{S}_n]$. For $\mathbf{v} \in \{+,-\}^k$ and $n > k$, let $g_{n,\mathbf{v}} := \sum_{\pi \in \mathcal{G}_n^{\mathbf{v}}} \pi \in \mathbb{C}[\mathcal{S}_n]$. Letting $k = n-1$, we get the following immediate consequence of Proposition 3.2.

Corollary 3.4 *The set*

$$\{g_{n,\mathbf{v}} :\ \mathbf{v} \in \{+,-\}^{n-1}\}$$

forms a basis for the Solomon descent subalgebra in $\mathbb{C}[\mathcal{S}_n]$.

3.2 Colayered permutations

Another family of Schur-positive grid classes consists of the so-called colayered permutations.

Definition 3.5 *The* k-colayered grid *class* $\mathcal{Y}^k$ *is determined by the* $k \times k$ *identity matrix:*

$$\mathcal{Y}^k = \mathcal{G}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Let $\mathcal{Y} = \bigcup_{k \geq 1} \mathcal{Y}^k$ *be the set of colayered permutations with an arbitrary number of layers. Let* $\mathcal{Y}_n^k = \mathcal{Y}^k \cap \mathcal{S}_n$ *and* $\mathcal{Y}_n = \mathcal{Y} \cap \mathcal{S}_n$.

Fig. 1 shows the 2- and 3-colayered grids.

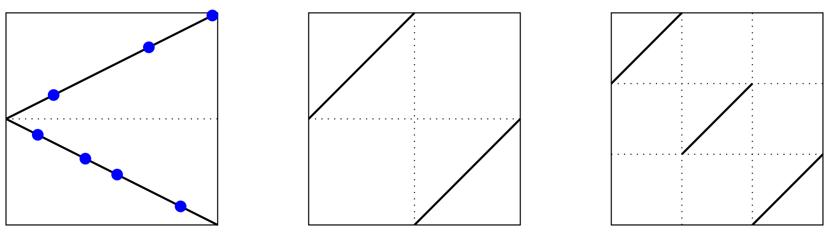


Fig. 1: A drawing of the permutation 4532617 on the grid for $\mathcal{L}$ (left), and the grids for $\mathcal{Y}^2$ (center) and $\mathcal{Y}^3$ (right).

Example. We have that $\mathcal{Y}_5^1 = \{12345\}$ and $\mathcal{Y}_5^2 = \{12345, 51234, 45123, 34512, 23451\}$.

Proposition 3.6 *1. For every* $1 \leq k \leq n$,

$$\mathcal{Q}(\mathcal{Y}_n^k \setminus \mathcal{Y}_n^{k-1}) = s_{n-k+1,1^{k-1}}.$$

2. $\mathcal{Q}(\mathcal{Y}_n)$ *is the Frobenius image of the character of the exterior algebra* $\wedge V$, *where* V *is the* n-*dimensional natural representation space of* $\mathcal{S}_n$.

Corollary 3.7 *For every* $1 \leq k \leq n$, $\mathcal{Y}_n^k$ *is a Schur-positive set and*

$$\mathcal{Q}(\mathcal{Y}_n^k) = s_n + s_{n-1,1} + s_{n-2,1,1} + \cdots + s_{n-k+1,1^{k-1}}.$$

4 Products of Schur-positive sets

Given two subsets $A, B \subseteq \mathcal{S}_n$, define its product AB to be the multiset of all permutations obtained as $\pi\sigma$ where $\pi \in A$ and $\sigma \in B$. To denote the underlying set, without multiplicities, we write $\{AB\}$. We call AB the multiset product and $\{AB\}$ the set product of A and B.

We are interested in pairs of Schur-positive subsets $A, B \subseteq \mathcal{S}_n$ whose product is Schur-positive, either as a multiset or as a set. This section contains one of our main results, Theorem 4.5.

4.1 Basic examples

In general, the product of Schur-positive subsets of $\mathcal{S}_n$ does not give a Schur-positive multiset or set. For example, the subsets $A = \{2134, 3412, 1243\}$ and $B = \{2143, 3412\}$ in $\mathcal{S}_4$ are Schur-positive, but $AB = \{AB\}$ is not.

Lemma 4.1 *The set product of two subsets consisting of all permutations of fixed Coxeter length is Schur-positive.*

Note, however, that the multiset product of subsets of permutations of fixed Coxeter length is not necessarily Schur-positive. For example, the multisets $\{\pi \in \mathcal{S}_4 : \ell(\pi) = 1\}^2$ and $\{\pi \in \mathcal{S}_5 : \ell(\pi) = 2\}^2$ are not Schur-positive, where ℓ denotes the Coxeter length. The next lemma shows that conjugacy classes behave better with respect to products.

Lemma 4.2 *Multiset and set products of conjugacy classes in $\mathcal{S}_n$ are Schur-positive.*

Conjugacy classes span the center of the group algebra $\mathbb{C}[\mathcal{S}_n]$. Inverse descent classes span the descent subalgebra [14], from where the next result follows.

Proposition 4.3 *Multiset and set products of inverse descent classes in $\mathcal{S}_n$ are Schur-positive.*

Theorem 4.5 is a significant strengthening of Proposition 4.3 in the multiset case. An important tool is the following result about shuffles, whose proof uses Theorem 2.2 and Prop. 2.3.

Lemma 4.4 *Fix a set partition $U \sqcup V = [n]$ with $|U| = k$. Let A and B be Schur-positive sets of the symmetric groups on U and V, respectively. Then $A \sqcup\!\sqcup B$ is a Schur-positive set of $\mathcal{S}_n$, and*

$$\mathcal{Q}(A \sqcup\!\sqcup B) = \mathcal{Q}(A)\mathcal{Q}(B). \tag{1}$$

In other words, if A is a Schur-positive set for the $\mathcal{S}_k$-representation ϕ and B is a Schur-positive set for the $\mathcal{S}_{n-k}$-representation ψ, then $A \sqcup\!\sqcup B$ is a Schur-positive set for the induced representation $(\phi \otimes \psi) \uparrow_{\mathcal{S}_k \times \mathcal{S}_{n-k}}^{\mathcal{S}_n}$.

4.2 Right multiplication by an inverse descent class

We use the notation $\{j_1, \ldots, j_t\}_<$ to indicate that the elements of the set satisfy $j_1 < j_2 < \cdots < j_t$. For $J = \{j_1, \ldots, j_t\}_< \subseteq [n-1]$, let $\mathcal{S}_{\bar{J}}$ denote the Young subgroup $\mathcal{S}_{j_1} \times \mathcal{S}_{j_2 - j_1} \times \cdots \times \mathcal{S}_{n-j_t}$, and let

$$R_{n,J} := \{\pi \in \mathcal{S}_n : \operatorname{Des}(\pi) \subseteq J\} = \bigsqcup_{I \subseteq J} D_{n,I}. \tag{2}$$

The *Kronecker product* of two symmetric functions $f, g \in \Lambda^n$ is defined by

$$f * g := \sum_{\mu,\ \nu,\ \lambda \vdash n} \langle f, s_\mu \rangle \langle g, s_\nu \rangle c_{\mu,\nu,\lambda} s_\lambda,$$

where $c_{\mu,\nu,\lambda} := \langle \chi^\mu \otimes \chi^\nu, \chi^\lambda \rangle$.

Theorem 4.5 *Let $\mathcal{B} \subseteq \mathcal{S}_n$ be a Schur-positive set for the $\mathcal{S}_n$-representation ρ. Then for every $J \subseteq [n-1]$, the following hold.*

1. *The multiset*
$$\mathcal{B} R_{n,J}^{-1}$$
is a Schur-positive multiset of $\mathcal{S}_n$ for $\rho \downarrow_{\mathcal{S}_{\bar{J}}} \uparrow^{\mathcal{S}_n}$.

2. *The multiset*
$$\mathcal{B} D_{n,J}^{-1}$$
is a Schur-positive multiset of $\mathcal{S}_n$ for $\rho \otimes S^{Z_{n,J}}$, where $Z_{n,J}$ is the zigzag shape as in Definition 2.1, and $S^{Z_{n,J}}$ is the corresponding Specht module. Equivalently,
$$\mathcal{Q}(\mathcal{B} D_{n,J}^{-1}) = \mathcal{Q}(\mathcal{B}) * \mathcal{Q}(D_{n,J}^{-1}),$$
where $*$ denotes the Kronecker product.

5 Vertical and horizontal rotations

In this section we explore some applications of Theorem 4.5 with a geometric flavor.

5.1 Vertical rotations

If we choose the Schur-positive set $\mathcal{B}$ to be a colayered grid class, which is Schur-positive by Corollary 3.7, a consequence of Theorem 4.5 is that $\mathcal{Y}_n^k D_{n,J}^{-1}$ is a Schur-positive multiset of $\mathcal{S}_n$ for every $k \geq 1$ and every $J \subseteq [n-1]$.

Note that $C_n = \mathcal{Y}_n^2$. For $A \subseteq \mathcal{S}_n$, the product $C_n A$ (resp. AC_n) is the multiset of vertical (resp. horizontal) rotations of the elements of A. The following is a consequence of Theorem 4.5.

Corollary 5.1 *For every $J \subseteq [n-1]$, the multiset $C_n D_{n,J}^{-1}$ of vertical rotations of an inverse descent class is a Schur-positive multiset for $S^{Z_{n,J}} \downarrow_{\mathcal{S}_{n-1}} \uparrow^{\mathcal{S}_n}$.*

By Proposition 3.2, every one-column grid class $\mathcal{G}_n^{\mathbf{v}}$ can be expressed as a disjoint union of inverse grid classes. Thus, we also get the following.

Corollary 5.2 *For every one-column grid class $\mathcal{G}^{\mathbf{v}}$ and every $k \geq 1$,*

$$\mathcal{Y}_n^k \mathcal{G}_n^{\mathbf{v}}$$

is a Schur-positive multiset of $\mathcal{S}_n$.

Taking $k = 2$, Corollary 5.2 implies that the multiset of vertical rotations $C_n \mathcal{G}_n^{\mathbf{v}}$ is Schur-positive. In general, we do not know if the underlying set is always Schur-positive (see Conjecture 6.1).

For an arbitrary one-column grid class $\mathcal{G}^{\mathbf{v}}$, the set $\{C_n \mathcal{G}^{\mathbf{v}}\}$ of its vertical rotations may not be a grid class, but it is a union of grid classes. For example, taking the class $\mathcal{L} = \mathcal{G}^{-+}$ of left-unimodal permutations, we get that $\{C_n \mathcal{L}_n\} = \mathcal{A}_n$, the set of arc permutations. It is shown in [6] that arc permutations consist of the union of two grid classes. Our approach provides a simple proof of the following result, which is a reformulation of [6, Theorem 5].

Proposition 5.3 *$\mathcal{A}_n$ is a Schur-positive set, and*

$$\mathcal{Q}(\mathcal{A}_n) = s_n + s_{1^n} + \sum_{k=2}^{n-2} s_{n-k,2,1^{k-2}} + 2\sum_{k=1}^{n-2} s_{n-k,1^k}. \tag{3}$$

For the particular case of $\mathbf{v} = +^k$, Corollary 5.2 implies that $C_n \mathcal{G}_n^{+^k}$ is a Schur-positive multiset. The underlying set $\{C_n \mathcal{G}_n^{+^k}\}$ is the grid class $\mathcal{G}_n(M_k)$, where M_k is the $2 \times 2k$ matrix whose odd rows are $(1, 0)$ and whose even rows are $(0, 1)$. The grid $\mathcal{G}(M_3)$ is drawn in Fig. 2. The main result in this section is that $\mathcal{G}_n(M_k)$ is a Schur-positive set. Let us start with the case $k = 2$.

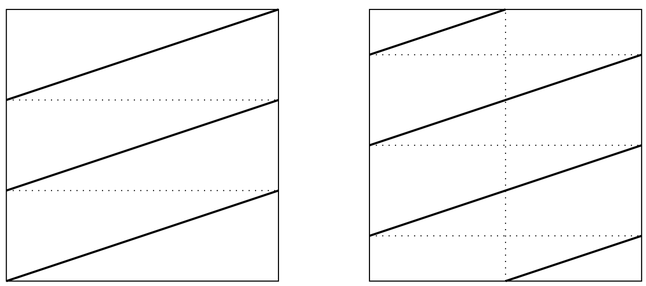


Fig. 2: The grids $\mathcal{G}^{+^3}$ (left) and $\mathcal{G}(M_3)$ (right).

Proposition 5.4 *$\mathcal{G}_n(M_2)$ is a Schur-positive set, and*

$$\mathcal{Q}(\mathcal{G}_n(M_2)) = s_n + (n-1)s_{n-1,1} + \sum_{a=2}^{\lfloor \frac{n}{2} \rfloor - 1} (2n - 4a + 2)s_{n-a,a} + \sum_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} (n - 2a)s_{n-a-1,a,1} + \begin{cases} 2s_{\frac{n}{2},\frac{n}{2}} & \textit{for even } n \geq 4, \\ 4s_{\frac{n+1}{2},\frac{n-1}{2}} & \textit{for odd } n \geq 5. \end{cases}$$

Using vertical rotations we can prove (see [8] for details) that, for every $k \geq 1$, $\mathcal{Q}(\mathcal{G}_n(M_k))$ is symmetric. To prove Schur-positivity of this grid we consider horizontal rotations next.

5.2 Horizontal rotations

In this subsection we identify $\mathcal{S}_{n-1}$ with the subset of $\mathcal{S}_n$ consisting of those permutations π with $\pi(n) = n$. In particular, subsets of $\mathcal{S}_{n-1}$ such as $D_{n-1,J}$ and $R_{n-1,J}$ are considered as subsets of $\mathcal{S}_n$ as well. We obtain some results about horizontal rotations of these sets using bijective techniques.

For $J = \{j_1, \dots, j_t\}_< \subseteq [n-2]$, let $L_{n,J}$ be the skew shape of size n consisting of disconnected horizontal strips of sizes $j_1, j_2 - j_1, \dots, n-1-j_t, 1$ from left to right, each strip touching the next one in one vertex. Let T be a SYT of shape $L_{n,J}$ and let T_1 the entry in its upper-right box.

Example. The skew SYT

$$T = \begin{array}{ccccc} & & & & \boxed{4} \\ & \boxed{1} & \boxed{3} & \boxed{5} & \\ \boxed{2} & & & & \end{array}$$

has shape $L_{5,\{1\}}$ and $T_1 = 4$.

Lemma 5.5 *For every $J \subseteq [n-2]$, $I \subseteq [n-1]$, and $1 \le k \le n$,*

$$|\{\pi \in R^{-1}_{n-1,J} C_n : \pi^{-1}(n) = k, \ \mathrm{Des}(\pi) = I\}| = |\{T \in \mathrm{SYT}(L_{n,J}) : T_1 = k, \ \mathrm{Des}(T) = I\}|.$$

Corollary 5.6 *For every $J \subseteq [n-2]$, $R^{-1}_{n-1,J} C_n$ is a Schur-positive set for $1_{S_{\bar{J}}} \uparrow^{\mathcal{S}_n}$.*

The above result is the key ingredient in the proof of the following theorem.

Theorem 5.7 *For every $J \subseteq [n-2]$, $D^{-1}_{n-1,J} C_n$ is a Schur-positive set for $S^{Z_{n-1,J}} \uparrow^{\mathcal{S}_n}$.*

This theorem can now be used to prove that $\mathcal{G}_n(M_k)$ is a Schur-positive set, by first showing that this grid is a horizontal rotation of a grid in $\mathcal{S}_{n-1}$, namely $\mathcal{G}_n(M_k) = \mathcal{G}^{+^k}_{n-1} C_n$.

Corollary 5.8 *For every $k \ge 1$, $\mathcal{Q}(\mathcal{G}_n(M_k))$ is symmetric and Schur-positive.*

The next corollary is an enumerative consequence of Theorem 5.7. It is not hard to show that $\mathcal{A}_n = C_n \mathcal{L}_{n-1}$ but $\mathcal{A}_n \ne \mathcal{L}_{n-1} C_n$ (in fact, the set $\bigcup_n \mathcal{L}_{n-1} C_n$ is not closed under pattern containment). Nevertheless, the following equidistribution phenomenon holds, where we use the notation $\mathbf{x}^D := \prod_{i \in D} x_i$.

Corollary 5.9

$$\sum_{\pi \in \mathcal{L}_{n-1} C_n} \mathbf{x}^{\mathrm{Des}(\pi)} = \sum_{\pi \in \mathcal{A}_n} \mathbf{x}^{\mathrm{Des}(\pi)}.$$

We end this section by stating a generalization of Theorem 5.7, which will be proved in a forthcoming paper [7].

Theorem 5.10 *If $\mathcal{B} \subseteq \mathcal{S}_{n-1}$ is a Schur-positive set for the $\mathcal{S}_{n-1}$-representation ρ, then $\mathcal{B} C_n$ is a Schur-positive set for $\rho \uparrow^{\mathcal{S}_n}$.*

It should be noted that an analogous statement for vertical rotation does not hold. For example, $\{2143, 2413\}$ is a Knuth class in $\mathcal{S}_4$, thus Schur-positive, but $C_5\{2143, 2413\}$ is not Schur-positive.

6 Final remarks and open problems

We have shown in Corollary 5.8 that vertically rotated one-column grid classes are Schur-positive when all slopes have the same sign, that is, $\{C_n \mathcal{G}^{\mathbf{v}}_n\}$ is Schur-positive when $\mathbf{v} = +^k$ or $\mathbf{v} = -^k$ (the latter case follows by symmetry). By Proposition 5.3, this phenomenon also holds when $\mathbf{v} = -+$ or $\mathbf{v} = +-$. Computer experiments suggest that the following more general statement is true.

Conjecture 6.1 *For every one-column grid class $\mathcal{G}^{\mathbf{v}}$, the set $\{C_n \mathcal{G}^{\mathbf{v}}_n\}$ is a Schur-positive set.*

The following conjecture suggests a far-reaching generalization of Corollary 5.9.

Conjecture 6.2 *For every $J \subseteq [n-2]$,*

$$\mathcal{Q}(C_n D_{n-1,J}^{-1}) = \mathcal{Q}(D_{n-1,J}^{-1} C_n),$$

thus, by Theorem 5.7, $C_n D_{n-1,J}^{-1}$ is a Schur-positive set for $S^{Z_{n-1,J}} \uparrow^{\mathcal{S}_n}$.

Note that both $C_n D_{n-1,J}^{-1}$ are $D_{n-1,J}^{-1} C_n$ are sets (that is, elements have multiplicity one). Conjecture 6.2 no longer holds when C_n is replaced by a general Schur-positive set $\mathcal{B} \subset \mathcal{S}_n$.

Regarding multiset products of Schur-positive sets and inverse descent classes, computer experiments support the following conjecture.

Conjecture 6.3 *Let $\mathcal{B} \subset \mathcal{S}_n$ be Schur-positive. Then, for every $J \subseteq [n-1]$,*

$$\mathcal{Q}(D_{n,J}^{-1} \mathcal{B}) = \mathcal{Q}(\mathcal{B} D_{n,J}^{-1}).$$

In particular, by Theorem 4.5, the multiset $D_{n,J}^{-1} \mathcal{B}$ is Schur-positive.

Note that when $\mathcal{B} = C_n$, the conjecture involves horizontal and vertical rotations of an inverse descent class.

Another natural geometric operation on grids consists of stacking one grid on top of another. To avoid ambiguity, we will only consider the case where one of the stacked grids has a single column.

Definition 6.4 (The stacking operation) *For a grid class $\mathcal{H}$ and $\mathbf{v} \in \{+,-\}^k$, let $\mathcal{G}^{\mathbf{v},\mathcal{H}}$ ($\mathcal{G}^{\mathcal{H},\mathbf{v}}$) be the grid class obtained by placing the grid for $\mathcal{G}^{\mathbf{v}}$ below (atop) the one for $\mathcal{H}$.*

Fig. 3 shows an example of the stacking operation.

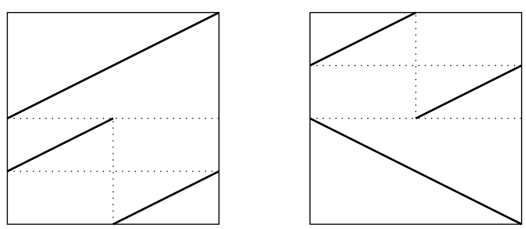


Fig. 3: The grids for $\mathcal{G}^{\mathcal{Y}^2,(1)}$ (left) and $\mathcal{G}^{(-1),\mathcal{Y}^2}$ (right).

Proposition 6.5 $\mathcal{G}^{\mathcal{Y}^2,(1)}$ *and* $\mathcal{G}^{(-1),\mathcal{Y}^2}$ *are Schur-positive.*

Question 6.6 *Let $\mathcal{H}$ be a Schur-positive grid and let $\mathbf{v} \in \{+,-\}^k$. Is the stacked grid $\mathcal{G}^{\mathbf{v},\mathcal{H}}$ necessarily Schur-positive?*

Computer experiments hint for an affirmative answer.

We conclude with a natural question regarding restriction of Schur-positive grids.

Question 6.7 *Let $\mathcal{G}$ be a grid class. Does $\mathcal{Q}(\mathcal{G}_n)$ being Schur-positive imply that $\mathcal{Q}(\mathcal{G}_{n-1})$ is Schur-positive?*

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