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HIDDEN REGULAR VARIATION FOR POINT PROCESSES AND THE SINGLE/MULTIPLE LARGE POINT HEURISTIC

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We consider regular variation for marked point processes with independent heavy-tailed marks and prove a single large point heuristic: the limit measure is concentrated on the cone of point measures with one single point. We then investigate successive hidden regular variation removing the cone of point measures with at most k points, $k \geq 1$, and prove a multiple large point phenomenon: the limit measure is concentrated on the cone of point measures with $k + 1$ points. We show how these results imply hidden regular variation in Skorokhod space of the associated risk process, in connection with the single/multiple large point heuristic from [Rhee et al. \(2019\)](#). Finally, we provide an application to risk theory in a reinsurance model where the k largest claims are covered and we study the asymptotic behavior of the residual risk.

1. Introduction. Regular variation is a fundamental concept in the analysis of rare event probabilities for heavy-tailed models that was widely popularized by [Resnick \(2007, 2008\)](#) and finds natural applications in risk theory ([Embrechts et al., 1997](#); [Asmussen and Albrecher, 2010](#); [Mikosch, 2010](#); [Hult and Lindskog, 2011](#)).

Regular variation was first considered on the finite-dimensional space \mathbb{R}^d and formulated in terms of vague convergence on the compactified space $E = [-\infty, +\infty]^d$. In such a context, a random element $X \in \mathbb{R}^d$ is said to be regularly varying if there exists a positive sequence $a_n \rightarrow \infty$ such that

$$n\mathbb{P}(a_n^{-1}X \in \cdot) \xrightarrow{v} \mu(\cdot), \quad \text{as } n \rightarrow \infty,$$

where \xrightarrow{v} stands for vague convergence on $E \setminus \{0\}$. If the limit measure μ concentrates only on the axes $F = \cup_{i=1}^d (\{0\}^{i-1} \times \mathbb{R} \times \{0\}^{d-i})$, we say that X is asymptotically independent. One can then wonder if some dependence among the components appears in a different regime. The concept of hidden regular variation was introduced for this purpose in [Resnick \(2002\)](#) and formulated in terms of vague convergence on the space $E \setminus F$ of the form $n\mathbb{P}(\tilde{a}_n^{-1}X \in \cdot) \xrightarrow{v} \tilde{\mu}(\cdot)$. We refer to [Heffernan and Resnick \(2005\)](#), [Maulik and Resnick \(2004\)](#), [Mitra and Resnick \(2011\)](#) for further developments on finite-dimensional hidden regular variation.

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To go beyond finite-dimensional spaces, M_0 -convergence on a general metric space was introduced by [Hult and Lindskog \(2006\)](#). This notion avoids the compactification procedure and replaces the compactly supported continuous test functions from vague convergence by bounded continuous functions with support bounded away from the origin. This results in an elegant theory that provides a convenient framework for regular variation in infinite dimensional spaces. Infinite dimensional regular variation theory includes the analysis of heavy-tailed stochastic processes ([Hult and Lindskog, 2005, 2007](#); [Hult et al., 2005](#)) or times series ([Basrak and Segers, 2009](#); [Dombry et al., 2018](#)).

In order to consider hidden regular variation in a function space, the theory of M -convergence was further extended in [Lindskog et al. \(2014\)](#) removing cones larger than the origin. The main example provided there is the infinite dimensional space \mathbb{R}_+^∞ with application to hidden regular variation of Lévy processes in Skorokhod space $\mathbb{D}([0, 1], \mathbb{R})$. [Rhee et al. \(2019\)](#) provide further insight into the hidden regular variation for regularly varying Lévy processes and random walks in connection with sample path large deviations. They propose a single/multiple large jump heuristic where the limit measure in (hidden) regular variation is supported by the cone of functions with one single (multiple) large jumps.

In this paper, we provide the first detailed analysis of (hidden) regular variation properties for point processes. Point processes are an important tool in applied probability and stochastic modelling and are widely used in risk theory. The Lévy processes mentioned above can be seen as functionals of Poisson point processes, so that one can expect to deduce the regular variation of the former from those of the latter *via* a continuous mapping theorem. For these reasons, we believe regular variation at the level of point process is an important and fundamental conceptual tool. A first result stating the regular variation of Poisson point processes with regularly varying intensity measure appears in [Dombry et al. \(2018\)](#) with a single large point heuristic. We propose here an analysis beyond the Poisson case and consider independently marked point processes (with heavy-tailed mark distribution) and successive hidden regular variation with different orders. We derive a general criterion for (hidden) regular variation in terms of pointwise Laplace functional and apply it to several models of increasing complexity: marked Poisson point processes, independently marked point processes and triangular arrays of independently marked point processes. For all these models, a similar structure for successive hidden regular variation is discovered and a single/multiple jump heuristic is proved.

The structure of the paper is the following. In Section 2, we set the necessary background on measure spaces and regular variation following the lines of [Lindskog et al. \(2014\)](#). We also settle the point process framework and provide in Theorem 2.5 a characterization of hidden regular variation of point processes in terms of convergence of their Laplace functionals. Section 3 states the main results of this paper on the successive hidden regular variation for independently marked point processes. In Section 4, we use the continuous mapping theorem to derive the hidden regular variation properties in Skorokhod space of the risk processes associated to a marked point process. An application to a reinsurance problem is also discussed with the asymptotic analysis of the residual risk after reinsurance of the largest claims. All the proofs are gathered in Sections 5, 6 and 7.

Notations and shortcuts: In the following, $[a]$ denotes the integer part of a , $(a)_+ = \max(a, 0)$ the positive part of a and $a \wedge b$ the minimum between a and b , for $a, b \in \mathbb{R}$. The indicator function of the set A is denoted by $\mathbb{1}_A$. For a set A , $\text{int}A$, $\text{cl}A$ and ∂A are respectively the interior, closure and boundary of A . The equivalence of two real sequences $u_n \sim v_n$ means that $u_n/v_n \rightarrow 1$ as $n \rightarrow \infty$; the notations $u_n = o(v_n)$ and $u_n = O(v_n)$ mean respectively

that $u_n/v_n \rightarrow 0$ and that u_n/v_n remains bounded as $n \rightarrow \infty$. We denote by $\varepsilon_x(\cdot)$ the Dirac measure at x and by \otimes the tensor product of measures.

2. Background on measure spaces, regular variation and point processes. We set up in this section the mathematical background necessary for this paper. We start by defining the framework and notations for measure spaces and then turn to regular variation and hidden regular variation, following the lines of [Lindskog et al. \(2014\)](#), Sections 2 and 3. We finally present some background on point processes.

2.1. Background on measure spaces. We denote by (E, d_E) a complete separable metric space endowed with its Borel σ -algebra $\mathcal{B}(E)$ generated by the open balls $B_{x,r}^E = \{x' \in E : d_E(x, x') < r\}$, $x \in E$, $r > 0$. When there is no confusion, we omit the superscript E and write simply $B_{x,r}$. The r -neighborhood of a subset $A \subset E$ is the open set A^r of points that are at distance less than r from A , that is $A^r = \cup_{x \in A} B_{x,r}$. We say that a subset B is bounded away from A if $B \cap A^r = \emptyset$ for some $r > 0$.

The space of bounded continuous real-valued functions on E is denoted by $\mathcal{C}_b(E)$. The set of finite Borel measures on E is denoted by $\mathbb{M}_b(E)$. A sequence of measures $(\mu_n)_{n \geq 1}$ is said to converge weakly to μ in $\mathbb{M}_b(E)$, noted $\mu_n \xrightarrow{\mathbb{M}_b(E)} \mu$, if $\int f d\mu_n \rightarrow \int f d\mu$ for all $f \in \mathcal{C}_b(E)$. The Prohorov distance on $\mathbb{M}_b(E)$ defined by

$$d_{\mathbb{M}_b(E)}(\mu_1, \mu_2) = \inf_{\varepsilon > 0} \{\mu_1(A) \leq \mu_2(A^\varepsilon) + \varepsilon \text{ for all } A \in \mathcal{B}(E)\}$$

metrizes weak convergence and, equipped with this metric, $\mathbb{M}_b(E)$ is a complete separable metric space; see [Kallenberg \(2017\)](#).

For $F \subset E$ a closed subset, we denote by $\mathbb{M}(E \setminus F)$ the set of Borel measures μ on $E \setminus F$ that assign finite mass on sets bounded away from F , that is, such that $\mu(E \setminus F^r) < \infty$ for all $r > 0$. In the following proposition, corresponding to [Lindskog et al. \(2014, Theorem 2.2\)](#), we provide equivalent characterizations of convergence in $\mathbb{M}(E \setminus F)$.

PROPOSITION 2.1 (Convergence in $\mathbb{M}(E \setminus F)$). *Let $\mu_n, \mu \in \mathbb{M}(E \setminus F)$. The convergence $\mu_n \rightarrow \mu$ in $\mathbb{M}(E \setminus F)$, noted $\mu_n \xrightarrow{\mathbb{M}(E \setminus F)} \mu$, is defined by the following equivalent properties:*

i) for all $f \in \mathcal{C}_b(E)$ with support bounded away from F ,

$$\int_E f d\mu_n \rightarrow \int_E f d\mu \quad \text{as } n \rightarrow \infty;$$

ii) for all $A \in \mathcal{B}(E)$ bounded away from F ,

$$\mu(\text{int}A) \leq \liminf_{n \rightarrow \infty} \mu_n(A) \leq \limsup_{n \rightarrow \infty} \mu_n(A) \leq \mu(\text{cl}A);$$

iii) there exists a sequence $r_i \downarrow 0$ such that $\mu_n^{r_i} \rightarrow \mu^{r_i}$ in $\mathbb{M}_b(E \setminus F^{r_i})$ as $n \rightarrow \infty$, for each $i \geq 1$, where $\mu_n^{r_i}$ (resp. μ^{r_i}) denotes the restriction of μ_n (resp. μ) to $E \setminus F^{r_i}$.

This notion of convergence is metrized by the distance

$$(2.1) \quad \rho(\mu_1, \mu_2) = \int_0^\infty \{\rho_r(\mu_1^r, \mu_2^r) \wedge 1\} e^{-r} dr$$

where μ^r denotes the restriction of μ to $E \setminus F^r$ and ρ_r the Prohorov metric on $\mathbb{M}_b(E \setminus F^r)$. Furthermore, $\mathbb{M}(E \setminus F)$ endowed with the distance ρ is a complete separable metric space ([Lindskog et al., 2014, Theorem 2.3](#)).

2.2. *Background on regular variation.* Regular variation intrinsically involves the notion of scaling and cones. A scaling on a complete and separable metric space is a multiplication by positive reals, that is a continuous mapping $(0, \infty) \times E \rightarrow E$ satisfying

$$\begin{aligned} 1x &= x, \\ u_1(u_2x) &= (u_1u_2)x, \quad \text{for all } u_1, u_2 > 0. \end{aligned}$$

Equivalently, a scaling is a continuous group action of $(0, \infty)$ on E .

A cone is a Borel set $F \subset E$ that is stable under the group action, that is $x \in F$ implies $ux \in F$ for all $u > 0$. In the following, we assume that F is a closed cone such that

$$d(x, F) < d(ux, F), \quad \text{for all } u > 1, x \in E \setminus F,$$

where $d(x, F) = \inf\{d(x, y) : y \in F\}$ denotes the distance to the cone F .

DEFINITION 2.2 (Regular variation).

- A measure $\nu \in \mathbb{M}(E \setminus F)$ is said to be regularly varying if there exists a positive sequence $a_n \rightarrow \infty$ and a non-null measure $\mu \in \mathbb{M}(E \setminus F)$ such that

$$n\nu(a_n \cdot) \longrightarrow \mu(\cdot) \quad \text{in } \mathbb{M}(E \setminus F).$$

When such a convergence holds, we write $\nu \in \text{RV}(E \setminus F, \{a_n\}, \mu)$.

- An E -valued random element X defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is said to be regularly varying on $E \setminus F$ if there exists a positive sequence $a_n \rightarrow \infty$ and a non-null measure $\mu \in \mathbb{M}(E \setminus F)$ such that

$$n\mathbb{P}(a_n^{-1}X \in \cdot) \longrightarrow \mu(\cdot) \quad \text{in } \mathbb{M}(E \setminus F).$$

When such a convergence holds, we write $X \in \text{RV}(E \setminus F, \{a_n\}, \mu)$.

Here, by abuse of notation, $n\mathbb{P}(a_n^{-1}X \in \cdot)$ is seen as the restriction to $E \setminus F$ of the rescaled distribution of X ; similarly we say that a measure $\nu \in \mathbb{M}_b(E)$ is regularly varying on $E \setminus F$ if its restriction to $E \setminus F$ is regularly varying.

There are many equivalent formulations of regular variation, another important one being the convergence of

$$b_n\nu(n \cdot) \longrightarrow \mu(\cdot) \quad \text{or} \quad b_n\mathbb{P}(n^{-1}X \in \cdot) \longrightarrow \mu(\cdot) \quad \text{in } \mathbb{M}(E \setminus F)$$

for some other positive sequence b_n , related to a_n by $a_{[b_n]} \sim b_{[a_n]} \sim n$. Also the convergence along integers $n \rightarrow \infty$ can be reinforced into convergence along a real variable $x \rightarrow \infty$. We refer to [Lindskog et al. \(2014\)](#), Definition 3.2 and Theorem 3.1 for exhaustive statements.

An important consequence of regular variation is the existence of a regular variation index $\alpha > 0$ such that μ is homogeneous of order $-\alpha < 0$, that is

$$\mu(u \cdot) = u^{-\alpha} \mu(\cdot) \quad \text{for all } u > 0.$$

Then, (a_n) and (b_n) are regularly varying sequences at infinity with index $1/\alpha$ and α respectively, that is

$$\lim_{n \rightarrow \infty} \frac{a_{[nu]}}{a_n} = u^{1/\alpha} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{b_{[nu]}}{b_n} = u^\alpha, \quad \text{for all } u > 0.$$

Importantly, regular variation not only gives a rate of convergence for rare events probabilities of the type $\mathbb{P}(X \in xA)$ for large $x > 0$ but also provides the typical behavior of X given the rare event $X \in xA$. This is formulated in terms of a conditional limit theorem, as in the following proposition.

PROPOSITION 2.3. *Let $X \in \text{RV}(E \setminus F, \{a_n\}, \mu)$ and $A \in \mathcal{B}(E)$ be bounded away from F such that $\mu(A) > 0$ and $\mu(\partial A) = 0$. Then, as $x \rightarrow \infty$,*

$$\mathbb{P}(x^{-1}X \in \cdot \mid X \in xA) \longrightarrow \mu_A(\cdot) := \frac{\mu(A \cap \cdot)}{\mu(A)} \quad \text{in } \mathbb{M}_b(E).$$

The next proposition will be crucial in our proof of Theorems 4.1 and 4.3 and is of independent interest. It states a regular variation criterion similar to the well known second converging together theorem for weak convergence (see Billingsley (1968) Theorem 4.2 or Resnick (2007) Theorem 3.5) and is especially useful when combined with truncation arguments.

PROPOSITION 2.4. *Let E be a complete separable metric space and consider E -valued random variables X and $X_{n,m}$, $n, m \geq 1$. Let $F \subset E$ be a closed cone. Assume that there is a positive sequence (a_n) and $k \geq 1$ such that:*

- i) for each $m \geq 1$, $n^k \mathbb{P}(a_n^{-1}X_{n,m} \in \cdot) \longrightarrow \mu_m(\cdot)$ in $\mathbb{M}(E \setminus F)$ as $n \rightarrow \infty$;*
- ii) $\mu_m \longrightarrow \mu$ in $\mathbb{M}(E \setminus F)$ as $m \rightarrow \infty$;*
- iii) for all $\varepsilon > 0$, $r > 0$*

$$\begin{aligned} \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} n^k \mathbb{P}(d(a_n^{-1}X_{n,m}, a_n^{-1}X) > \varepsilon, d(a_n^{-1}X_{n,m}, F) > r) &= 0 \\ \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} n^k \mathbb{P}(d(a_n^{-1}X_{n,m}, a_n^{-1}X) > \varepsilon, d(a_n^{-1}X, F) > r) &= 0. \end{aligned}$$

Then, $n^k \mathbb{P}(a_n^{-1}X_n \in \cdot) \longrightarrow \mu(\cdot)$ in $\mathbb{M}(E \setminus F)$ as $n \rightarrow \infty$.

We finally provide some intuition on successive hidden regular variation. Most often, regular variation is used when $F = F_0 = \{0\}$ is reduced to a single point 0, called the origin of E and satisfying $u0 = 0$ for all $u > 0$. Then, for an E -valued random variable X , X/n converges in distribution to 0 as $n \rightarrow \infty$. This limit theorem is quite uninformative since it simply states that $\mathbb{P}(X/n \in A) \rightarrow 0$ for all Borel sets A bounded away from 0. It is hence sensible to rescale these probabilities and consider the convergence $b_n \mathbb{P}(X/n \in \cdot) \rightarrow \mu_0(\cdot)$ in $\mathbb{M}(E \setminus F_0)$. This implies roughly $\mathbb{P}(X/n \in A) \sim \mu_0(A)/b_n$ and is much more informative, provided $\mu_0(A) > 0$. When $\mu_0(A) = 0$, it is natural to look for a higher order scaling $b_n^{(1)}$ such that $b_n^{(1)}/b_n \rightarrow \infty$. The support of the homogeneous measure μ_0 is a closed cone F_1 such that $\mu_0(A) = 0$ for all A bounded away from F_1 . This leads us to consider the convergence $b_n^{(1)} \mathbb{P}(X/n \in \cdot) \rightarrow \mu_1(\cdot)$ in $\mathbb{M}(E \setminus F_1)$. This procedure can be repeated and we may obtain successive hidden regular variation of the form

$$(2.2) \quad b_n^{(k)} \mathbb{P}(n^{-1}X \in \cdot) \rightarrow \mu_k(\cdot) \quad \text{in } \mathbb{M}(E \setminus F_k), \quad k \geq 0,$$

where $(F_k)_{k \geq 1}$ are increasing cones, $\mu_k \in \mathbb{M}(E \setminus F_k)$ are measures with disjoint supports and $b_n^{(k)} > 0$ are rate functions such that $\lim_{n \rightarrow \infty} b_n^{(k)}/b_n^{(k-1)} = \infty$.

These successive regular variation results can also be formulated in terms of large deviations as in [Rhee et al. \(2019\)](#), Theorem 3.2. For a Borel set $A \subset E \setminus \{0\}$, define

$$\mathcal{K}(A) = \max\{k \geq 0 : A \cap F_k = \emptyset\} \quad \text{and} \quad \mathcal{I}(A) = \mu_{\mathcal{K}(A)}(A).$$

The successive regular variation from Equation (2.2) is equivalent to the large deviations

$$\mathcal{I}(\text{int}A) \leq \liminf_{n \rightarrow \infty} b_n^{(\mathcal{K}(A))} \mathbb{P}(n^{-1}X \in A) \leq \limsup_{n \rightarrow \infty} b_n^{(\mathcal{K}(A))} \mathbb{P}(n^{-1}X \in A) \leq \mathcal{I}(\text{cl}A),$$

for all Borel set $A \subset E \setminus \{0\}$ such that $\mathcal{K}(A)$ is finite with A bounded away from $F_{\mathcal{K}(A)}$. In the following, we use mostly the terminology of regular variation instead of large deviations.

2.3. Background on point processes. We refer to [Daley and Vere-Jones \(2003, 2008\)](#) and [Snyder and Miller \(1991\)](#) for a complete review of point process theory.

Given a complete separable metric space E and a closed subset $F \subset E$, we have seen that the measure space $\mathbb{M}(E \setminus F)$ endowed with the metric ρ is a complete and separable metric space. The subset $\mathcal{N}(E \setminus F)$ of \mathbb{N} -valued measures is the subset of point measures of the form $\pi = \sum_{i \in I} \varepsilon_{x_i}$, where the x_i 's are in $E \setminus F$ and I is a countable index set. The condition that π is finite on subsets that are bounded away from F implies that the family $(x_i)_{i \in I}$ is at most countable and any accumulation point must belong to F . The mapping $x \mapsto \varepsilon_x$ defines an isometric embedding $E \setminus F \rightarrow \mathcal{N}(E \setminus F)$.

[Kallenberg \(2017\)](#) shows that $\mathcal{N}(E \setminus F)$ is a closed subset of $\mathbb{M}(E \setminus F)$ and hence a complete separable metric space. A point process in $E \setminus F$ is a random variable with values in $\mathcal{N}(E \setminus F)$, that is a measurable application from some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ into $\mathcal{N}(E \setminus F)$ endowed with its σ -algebra. When E is equipped with a scaling function, a natural scaling induced on $\mathcal{N}(E \setminus F)$ is

$$u\pi = \sum_{i \in I} \varepsilon_{ux_i}, \quad u > 0, \quad \pi = \sum_{i \in I} \varepsilon_{x_i} \in \mathcal{N}(E \setminus F).$$

That is, the multiplication acts on each point of the point measure. Using this structure, one can define regularly varying point processes. [Dombry et al. \(2018\)](#) prove that if Π is a Poisson point process on $E \setminus \{0\}$ with regularly varying intensity measure $\nu \in \text{RV}(E \setminus \{0\}, \{a_n\}, \mu)$, then Π is regularly varying in $\mathcal{N}(E \setminus \{0\})$ with sequence (a_n) and limit measure μ^* defined as the image of μ under $x \mapsto \varepsilon_x$, that is

$$\mu^*(B) = \int_E \mathbb{1}_{\{\varepsilon_x \in B\}} \mu(dx), \quad B \in \mathcal{B}(\mathcal{N}(E \setminus \{0\})).$$

In this paper, we extend this result in several ways: we obtain successive hidden regular variation not only for Poisson point processes with $F = \{0\}$, but for general independently marked point process with regularly varying mark distribution.

Our results are based on a criterion for regular variation in $\mathcal{N}(E \setminus F)$ extending Theorem A.1 in [Dombry et al. \(2018\)](#). For brevity, we note here $\mathcal{N} = \mathcal{N}(E \setminus F)$. For $k \geq 0$, we consider the closed cone $\mathcal{N}_k \subset \mathcal{N}$ of point measures with at most k points that is

$$\mathcal{N}_k := \left\{ \pi = \sum_{i=1}^p \varepsilon_{x_i}; 0 \leq p \leq k; x_1, \dots, x_p \in E \setminus F \right\}.$$

When $k = 0$, $\mathcal{N}_0 = \{0\}$ is reduced to the null measure. For μ^* a Borel measure on \mathcal{N} , we denote by \mathcal{B}_{μ^*} the class of Borel sets $A \in \mathcal{B}(E)$ that are bounded away from F and such that $\mu^*(\pi(\partial A) > 0) = 0$. The following theorem provides a criterion for convergence in $\mathbb{M}(\mathcal{N} \setminus \mathcal{N}_k)$ in terms of finite-dimensional distributions and Laplace functional.

THEOREM 2.5 (Convergence in $\mathbb{M}(\mathcal{N} \setminus \mathcal{N}_k)$).

Let $k \geq 0$ and $\mu^*, \mu_1^*, \mu_2^*, \dots \in \mathbb{M}(\mathcal{N} \setminus \mathcal{N}_k)$. The following statements are equivalent, where convergences are meant as $n \rightarrow \infty$:

(i) $\mu_n^* \rightarrow \mu^*$ in $\mathbb{M}(\mathcal{N} \setminus \mathcal{N}_k)$;

(ii) for all $p \geq 1$, $A_1, \dots, A_p \in \mathcal{B}_{\mu^*}$ and $(m_1, \dots, m_p) \in \mathbb{N}^p$ such that $\sum_{i=1}^p m_i \geq k + 1$,

$$\mu_n^*(\pi(A_i) = m_i, 1 \leq i \leq p) \rightarrow \mu^*(\pi(A_i) = m_i, 1 \leq i \leq p);$$

(iii) there exists a decreasing sequence $r_i \downarrow 0$ such that

$$\int_{\mathcal{N}} e^{-\pi(f)} \mathbb{1}_{\{\pi(E \setminus F^{r_i}) \geq k+1\}} \mu_n^*(d\pi) \rightarrow \int_{\mathcal{N}} e^{-\pi(f)} \mathbb{1}_{\{\pi(E \setminus F^{r_i}) \geq k+1\}} \mu^*(d\pi),$$

for all bounded Lipschitz functions $f : E \rightarrow [0, \infty)$ vanishing on F^{r_i} and where $\pi(f) = \int_E f(x) \pi(dx)$.

As a consequence of a standard approximation argument, when (iii) holds for bounded Lipschitz test functions $f : E \rightarrow [0, \infty)$ vanishing on F^{r_i} , it holds also for all $f \in \mathcal{C}_b(E)$ vanishing on F^{r_i} . Furthermore, when $k = 0$, we retrieve exactly Theorem A.1 in [Dombry et al. \(2018\)](#) since (iii) is then equivalent to

$$(2.3) \quad \int_{\mathcal{N}} (1 - e^{-\pi(f)}) \mu_n^*(d\pi) \rightarrow \int_{\mathcal{N}} (1 - e^{-\pi(f)}) \mu^*(d\pi),$$

because the contribution of the event $\{\pi(E \setminus F^{r_i}) = 0\}$ in the integral vanishes since f is supported by $E \setminus F^{r_i}$.

3. Regular variation for marked point processes. The simplest example from risk theory we want to consider is the Poisson point process Π on $E = [0, T] \times [0, \infty)$ with fixed $T > 0$ and intensity $\lambda(dt)\nu(dx)$, where λ is a finite measure on $[0, T]$ and ν a probability measure on $(0, \infty)$. Then Π has finitely many points almost surely and can be represented as

$$\Pi = \sum_{i=1}^N \varepsilon_{(T_i, X_i)}.$$

Each point (T_i, X_i) represents a claim occurring at time T_i with size $X_i > 0$. The random variable N denotes the total number of claims up to time T which is here Poisson distributed with mean $\lambda([0, T])$. The arrival times $T_1 \leq \dots \leq T_N$ form a Poisson point process with intensity λ on $[0, T]$. The claim sizes X_1, \dots, X_N distributed as ν are independent of the claim number and arrival times. We consider regular variation of the Poisson point process Π when the claim size distribution ν is regularly varying.

For the purpose of generality, we consider the more abstract and general framework where $E = \mathcal{T} \times \mathcal{X}$ is the cartesian product of two complete and separable metric spaces $(\mathcal{T}, d_{\mathcal{T}})$ and $(\mathcal{X}, d_{\mathcal{X}})$. We think of \mathcal{T} as the time component and \mathcal{X} as the space component. We equip E with the distance

$$d_E(z, z') = d_{\mathcal{T}}(t, t') + d_{\mathcal{X}}(x, x')$$

for $z = (t, x), z' = (t', x') \in E = \mathcal{T} \times \mathcal{X}$. The induced scaling on E is defined by $u \cdot (t, x) = (t, ux)$ for $u > 0, t \in \mathcal{T}, x \in \mathcal{X}$. That is the scaling operates on the space component only. We also

assume that \mathcal{X} possesses an origin noted $0_{\mathcal{X}}$, or simply 0 when no confusion is possible. The subset $F = \mathcal{T} \times \{0\} \subset E$ is a cone representing the time axis. Note that the r -neighborhood of F is simply $F^r = \mathcal{T} \times B_{0,r}^{\mathcal{X}}$.

We develop a regular variation theory in $\mathcal{N} = \mathcal{N}(E \setminus F)$ for independently marked point processes of the form

$$(3.1) \quad \Pi = \sum_{i=1}^N \varepsilon_{(T_i, X_i)}$$

where

$$(3.2) \quad \Psi = \sum_{i=1}^N \varepsilon_{T_i}$$

is a finite point process on \mathcal{T} representing the claim arrivals and, independently, X_1, X_2, \dots are i.i.d. random variables on $\mathcal{X} \setminus \{0\}$ with regularly varying distribution ν . We consider three particular situations, namely, marked Poisson point processes, independently marked point processes, and triangular arrays of independently marked point processes.

3.1. Regular variation for marked Poisson point processes. We first focus on the simple situation of a marked Poisson point process, that is the base point process Ψ in Equation (3.2) is a Poisson point process on \mathcal{T} with finite intensity measure $\lambda \in \mathbb{M}_b(\mathcal{T})$. Then Π is a Poisson point process with product intensity measure $\lambda \otimes \nu$. Recall the notations $E = \mathcal{T} \times \mathcal{X}$, $F = \mathcal{T} \times \{0\}$ and $\mathcal{N} = \mathcal{N}(E \setminus F)$ the space of \mathcal{N} -valued point measures on $E \setminus F$. For $k \geq 0$, $\mathcal{N}_k \subset \mathcal{N}$ denotes the closed cone of point measures with at most k points.

THEOREM 3.1 (RV for marked Poisson point processes).

Assume $\lambda \in \mathbb{M}_b(\mathcal{T})$, $\nu \in RV(\mathcal{X} \setminus \{0\}, (a_n), \mu)$ and consider Π a Poisson point process on $E \setminus F$ with intensity $\lambda(dt)\nu(dx)$. Then, for $k \geq 0$,

$$(3.3) \quad n^{k+1} \mathbb{P}(a_n^{-1} \Pi \in \cdot) \longrightarrow \mu_{k+1}^*(\cdot) \quad \text{in } \mathbb{M}(\mathcal{N} \setminus \mathcal{N}_k),$$

where the limit measure μ_{k+1}^* is non-null and given by

$$(3.4) \quad \mu_{k+1}^*(B) = \frac{1}{(k+1)!} \int_{E^{k+1}} \mathbb{1}_{\{\sum_{i=1}^{k+1} \varepsilon_{(t_i, x_i)} \in B\}} \otimes_{i=1}^{k+1} \lambda(dt_i) \mu(dx_i), \quad B \in \mathcal{B}(\mathcal{N} \setminus \mathcal{N}_k).$$

Equivalently, Equation (3.3) can be rephrased in terms of regular variation as

$$\Pi \in \text{RV}(\mathcal{N} \setminus \mathcal{N}_k, \{a_{\lfloor n^{1/(k+1)} \rfloor}\}, \mu_{k+1}^*)$$

or

$$b_n^{k+1} \mathbb{P}(n^{-1} \Pi \in \cdot) \longrightarrow \mu_{k+1}^*(\cdot) \quad \text{in } \mathbb{M}(\mathcal{N} \setminus \mathcal{N}_k)$$

where b_n is such that $b_n \nu(n \cdot) \rightarrow \mu(\cdot)$ in $\mathbb{M}(\mathcal{X} \setminus \{0\})$. If α denotes the regular variation index of ν , then Π is regularly varying in $\mathcal{N} \setminus \mathcal{N}_k$ with index $(k+1)\alpha$. The limit measure μ_{k+1}^* is the image of $(\lambda \otimes \mu)^{k+1}$ under the mapping

$$(z_1, \dots, z_{k+1}) \in E^{k+1} \longmapsto \sum_{i=1}^{k+1} \varepsilon_{z_i} \in \mathcal{N}$$

and is concentrated on the cone of point measures with exactly $k + 1$ points.

Theorem 3.1 provides successive hidden regular variation as discussed in Section 2.2 and can be interpreted as a single/multiple large point heuristic, see Rhee et al. (2019). For $k = 0$, the regular variation writes $b_n \mathbb{P}(\Pi/n \in \cdot) \rightarrow \mu_1^*(\cdot)$ in $\mathbb{M}(\mathcal{N} \setminus \{0\})$. This is a single large jump heuristic since the limit measure is supported by the cone \mathcal{N}_1 of point measures with at most one point. Removing this cone, we obtain hidden regular variation with index 2α in $\mathbb{M}(\mathcal{N} \setminus \mathcal{N}_1)$ and the convergence $b_n^2 \mathbb{P}(\Pi/n \in \cdot) \rightarrow \mu_2^*(\cdot)$. This is a multiple large jump heuristic with two large points and μ_2^* is supported by the cone \mathcal{N}_2 . Removing the cone \mathcal{N}_2 , we obtain hidden regular variation with index 3α in $\mathbb{M}(\mathcal{N} \setminus \mathcal{N}_2)$ and so forth.

3.2. *Independently marked point processes.* The results on regular variation for marked Poisson point processes are extended to general independently marked point processes as defined in Equation (3.1). For $k \geq 1$, the k -th factorial moment measure of the base point process $\Psi = \sum_{i=1}^N \varepsilon_{T_i}$ is defined by

$$M_k(A) = \mathbb{E} \left[\Psi^{(k)}(A) \right], \quad A \in \mathcal{B}(\mathcal{X}^k),$$

where

$$\Psi^{(k)} = \sum_{1 \leq i_1 \neq \dots \neq i_k \leq N} \varepsilon_{(T_{i_1}, \dots, T_{i_k})}$$

is the k -th factorial power of Ψ . The k -th factorial moment measure M_k is finite if and only if N has a finite moment of order k , see Daley and Vere-Jones (2003) Chapter 5.2 for more details on these notions.

THEOREM 3.2 (RV for independently marked point processes).

Consider the independently marked point process Π defined by Equation (3.1). Assume that $\nu \in RV(\mathcal{X} \setminus \{0\}, (a_n), \mu)$ and that, for $k \geq 0$, the base point process Ψ has a finite and non-null $(k + 1)$ -th factorial moment measure M_{k+1} . Then,

$$(3.5) \quad n^{k+1} \mathbb{P}(a_n^{-1} \Pi \in \cdot) \rightarrow \mu_{k+1}^*(\cdot) \quad \text{in } \mathbb{M}(\mathcal{N} \setminus \mathcal{N}_k),$$

with non-null limit measure defined, for $B \in \mathcal{B}(\mathcal{N} \setminus \mathcal{N}_k)$, by

$$(3.6) \quad \mu_{k+1}^*(B) = \frac{1}{(k+1)!} \int_{E^{k+1}} \mathbf{1}_{\{\sum_{i=1}^{k+1} \varepsilon_{(t_i, x_i)} \in B\}} M_{k+1}(dt_1, \dots, dt_{k+1}) \otimes_{i=1}^{k+1} \mu(dx_i).$$

Theorem 3.2 is indeed a generalization of Theorem 3.1: for a Poisson point process Ψ with finite intensity measure λ , the k -th factorial moment measure is finite for all $k \geq 1$ and equal to $M_k = \lambda^{\otimes k}$, so that Equations (3.4) and (3.6) agree.

3.3. *Marked point processes based on triangular arrays.* This section is motivated by the following simple situation. When $E = [0, T] \times [0, \infty)$, consider i.i.d. claim sizes X_i and a finite deterministic number of claims m arising at discrete times $T_i = Ti/m$, $i = 1, \dots, m$. We are interested in the asymptotic regime when $m = m_n \rightarrow \infty$ and the regular variations in this regime. A singular feature of the arrival times is that the empirical distribution $m_n^{-1} \sum_{i=1}^{m_n} \varepsilon_{iT/m_n}$ converges weakly as $n \rightarrow \infty$ to the uniform distribution on $[0, T]$.

The general abstract setting is the following: on $E = \mathcal{T} \times \mathcal{X}$, we consider a sequence of independently marked point processes

$$(3.7) \quad \Pi_n = \sum_{i=1}^{m_n} \varepsilon_{(T_i^n, X_i)}, \quad n \geq 1,$$

where the number of points $m_n \rightarrow \infty$ is deterministic, the arrival times are given by a triangular array of \mathcal{T} -valued random variables $\{T_i^n, n \geq 1, 1 \leq i \leq m_n\}$ and, independently, the marks X_i are i.i.d. with distribution ν on $\mathcal{X} \setminus \{0\}$. We assume no independence in the triangular array, but we suppose the convergence in probability of the empirical distribution to a probability measure λ , that is

$$(3.8) \quad m_n^{-1} \Psi_n := m_n^{-1} \sum_{i=1}^{m_n} \varepsilon_{T_i^n} \xrightarrow{\mathbb{M}_b(\mathcal{T})} \lambda \quad \text{in probability as } n \rightarrow \infty.$$

The main difference from the previous Sections 3.1 and 3.2 is that the number of points m_n tends to infinity as $n \rightarrow \infty$, whereas it was previously of fixed (random) size N . We prove in the following theorem that similar regular variation results still hold, but with different rates.

THEOREM 3.3 (RV for sequences of marked point processes based on triangular arrays).

Consider the sequence of independently marked point processes Π_n , $n \geq 1$, defined by Equation (3.7). Assume $\nu \in RV(\mathcal{X} \setminus \{0\}, (a_n), \mu)$ and Equation (3.8) holds. Then, for $k \geq 0$,

$$(3.9) \quad n^{k+1} \mathbb{P}(a_n^{-1} \Pi_n \in \cdot) \longrightarrow \mu_{k+1}^*(\cdot) \quad \text{in } \mathbb{M}(\mathcal{N} \setminus \mathcal{N}_k),$$

with non-null limit measure μ_{k+1}^ as in Theorem 3.1 Equation (3.4).*

In the following examples, we apply Theorem 3.9 interchanging the roles of the sequences (m_n) and (n) in order to compare the result with classical large deviation principles.

EXAMPLE 3.4. Consider a probability measure $\lambda \in \mathbb{M}_b(\mathcal{T})$ and the binomial point process

$$\Pi_n = \sum_{i=1}^n \varepsilon_{(T_i, X_i)}$$

with the (T_i, X_i) , $1 \leq i \leq n$ iid with distribution $\lambda \otimes \nu$. By the law of large numbers, Equation (3.8) holds interchanging the roles of the sequences (m_n) and (n) and with $T_i^n = T_i$. If $\nu \in RV(\mathcal{X} \setminus \{0\}, (a_n), \mu)$, Theorem 3.3 provides the convergence

$$(3.10) \quad m_n^{k+1} \mathbb{P}(a_n^{-1} \Pi_n \in \cdot) \longrightarrow \mu_{k+1}^*(\cdot) \quad \text{in } \mathbb{M}(\mathcal{N} \setminus \mathcal{N}_k), \quad k \geq 0,$$

for any sequence $m_n \rightarrow \infty$ as $n \rightarrow \infty$, with μ_{k+1}^* as in Equation (3.4).

EXAMPLE 3.5. A typical situation where Theorem 3.3 applies is when $\mathcal{T} = [0, T]$ and $T_i^n = iT/n$, $1 \leq i \leq m_n = n$. Then Equation (3.8) holds with λ the uniform distribution on $[0, T]$. If $\nu \in RV(\mathcal{X} \setminus \{0\}, (a_n), \mu)$, it is well-known that $\Pi_n = \sum_{i=1}^n \varepsilon_{(iT/n, X_i)}$ suitably rescaled converges in distribution to a Poisson point process (Resnick, 2007, Theorem 6.3). More precisely, $a_n^{-1} \Pi_n \xrightarrow{d} \Pi$ in \mathcal{N} , with Π a Poisson point process with intensity $\lambda \otimes \mu$. Theorem 3.3 considers the large deviation regime and states the regular variation properties of Π_n as in Equation (3.10).

EXAMPLE 3.6. A slightly more complex situation is based on a renewal sequence $(T_i)_{i \geq 1}$ on $\mathcal{T} = [0, \infty)$. We assume the inter-arrival times $T_i - T_{i-1}$ to be independent and identically distributed with finite mean $\tau > 0$. By the renewal Theorem (see, e.g., Mikosch (2010), Theorem 2.2.4), the number of arrivals up to time t , noted $N(t)$, satisfies $N(t)/t \rightarrow 1/\tau$ almost surely as $t \rightarrow \infty$. Consider $T_i^n = T_i/n$, $1 \leq i \leq m_n$. It would be natural to consider $m_n = N(nT)$ for some $T > 0$ but Theorem 3.3 does not cover the case of a random number m_n . Instead, we consider $m_n = nT/\tau \sim \mathbb{E}[N(nT)]$. Interchanging again the roles (m_n) and (n) , we define $\Pi_n = \sum_{i=1}^{nT/\tau} \varepsilon_{(T_i/n, X_i)}$ and the renewal theorem implies that $\sum_{i=1}^{nT/\tau} \varepsilon_{T_i/n}$ converges to the uniform distribution on $[0, T]$. If $\nu \in \text{RV}(\mathcal{X} \setminus \{0\}, \{a_n\}, \mu)$, Theorem 3.3 provides the convergence

$$(m_n)^{k+1} \mathbb{P}(a_{nT/\tau m_n}^{-1} \Pi_n \in \cdot) \longrightarrow \mu_{k+1}^*(\cdot) \quad \text{in } \mathbb{M}(\mathcal{N} \setminus \mathcal{N}_k),$$

for any sequence $m_n \rightarrow \infty$ as $n \rightarrow \infty$.

4. Applications.

4.1. *Regular variation of risk processes in Skorokhod space.* We focus on risk processes that are based on marked point processes on $\mathcal{T} \times \mathcal{X} = [0, T] \times [0, \infty)$. Recall that the marked point process

$$\Pi = \sum_{i=1}^N \varepsilon_{(T_i, X_i)}$$

represents the situation where N claims arise on $[0, T]$ at times $0 \leq T_1 \leq \dots \leq T_N \leq T$ and with sizes $X_1, \dots, X_N > 0$. We focus here on the associated risk process defined as

$$(4.1) \quad R(t) = \sum_{i=1}^N X_i \mathbb{1}_{\{T_i \leq t\}}, \quad t \in [0, T].$$

For an insurance company, it models the evolution over time of the total claim amount. When the claim arrival times $T_1 \leq \dots \leq T_N$ form an homogeneous Poisson point process and the claim sizes are i.i.d., $(R(t))_{0 \leq t \leq T}$ is a compound Poisson process. More generally, we consider the case when Π is an independently marked point process as in Section 3.2. The risk process R is a pure jump process and can be seen as a random element of the Skorokhod space $\mathbb{D} = \mathbb{D}([0, T], \mathbb{R})$ of càd-làg functions. Recall that endowed with the Skorokhod metric, \mathbb{D} is a complete separable metric space, see Billingsley (1968, Chapter 3). We define $\mathbb{D}_k \subset \mathbb{D}$ the closed cone of càd-làg functions with at most k discontinuity points (or jumps) on $[0, T]$.

In the next theorem, we derive the (hidden) regular variation properties of the risk process R on \mathbb{D} from the regular variation properties of Π on \mathcal{N} . The statement is very similar to Theorem 3.2 and is in fact derived from it using a continuous mapping theorem together with technical truncation arguments relying on Proposition 2.4.

THEOREM 4.1. *Let $\mathcal{T} \times \mathcal{X} = [0, T] \times [0, \infty)$ and Π be an independently marked point process on $\mathcal{T} \times \mathcal{X}$ as defined in (3.1). Assume $\nu \in \text{RV}([0, \infty) \setminus \{0\}, \{a_n\}, \mu)$ with $\mu(dx) = \alpha x^{-\alpha-1} dx$ for some $\alpha > 0$. Let $k \geq 0$ and assume Ψ has a finite and non-null $(k+1)$ -th factorial moment measure noted M_{k+1} . Then*

$$(4.2) \quad n^{k+1} \mathbb{P}(a_n^{-1} R \in \cdot) \longrightarrow \mu_{k+1}^\#(\cdot) \quad \text{in } \mathbb{M}(\mathbb{D} \setminus \mathbb{D}_k),$$

with non-null limit measure defined, for $B \in \mathcal{B}(\mathbb{D} \setminus \mathbb{D}_k)$, by

$$(4.3) \quad \mu_{k+1}^\#(B) = \int_{E^{k+1}} \mathbb{1}_{\left\{(\sum_{i=1}^{k+1} x_i \mathbb{1}_{\{t_i \leq u\}})_{0 \leq u \leq T} \in B\right\}} M_{k+1}(dt_1, \dots, dt_{k+1}) \otimes_{i=1}^{k+1} \mu(dx_i).$$

The limit measure $\mu_{k+1}^\#$ is the image measure of μ_{k+1}^* defined in Equation (3.6) under the mapping

$$\pi = \sum_{i \in I} \varepsilon_{(t_i, x_i)} \in \mathcal{N} \mapsto \left(\sum_{i \in I} x_i \mathbb{1}_{\{t_i \leq u\}} \right)_{0 \leq u \leq T} \in \mathbb{D}$$

with I a countable index set. In the case $k = 0$, \mathbb{D}_0 is the space of continuous functions on $[0, T]$ and Theorem 4.1 provides hidden regular variation in $\mathbb{D} \setminus \mathbb{D}_0$ with a single large jump heuristic: in this regime, the rescaled distribution of the risk process R converges in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_0)$ to $\mu_1^\#$ which is concentrated on the cone of càd-làg functions with exactly one jump. For $k \geq 1$, we obtain successive hidden regular variation and multiple large jump heuristics: removing the cone \mathbb{D}_k , we obtain a limit measure $\mu_{k+1}^\#$ concentrated on the cone of càd-làg functions with $k + 1$ jumps. This is closely related to the results by Rhee et al. (2019).

REMARK 4.2. The cone \mathbb{D}_k is in fact larger than the exact support of $\mu_k^\#$. Indeed, in view of Equation (4.3), the support of $\mu_k^\#$ is the cone of pure jump process with exactly k jumps. Let us denote by \mathbb{J}_k , $k \geq 0$, the cone consisting of pure jump functions with at most k jumps. In comparison, \mathbb{D}_k is the cone of càd-làg functions with at most k jumps and is strictly larger than \mathbb{J}_k . When considering successive hidden regular variations as discussed in the end of Section 2.2, we should rather consider the increasing sequence of cones \mathbb{J}_k , $k \geq 0$, instead of the sequence \mathbb{D}_k , $k \geq 0$. Such results are stronger but harder to establish. In Corollary 4.8 below, we cover the triangular array case in connection with an application to reinsurance.

The next theorem considers risk processes build on triangular arrays. Let Π_n , $n \geq 1$, be the sequence of marked point processes defined by Equation (3.7). When $\alpha > 1$, it is as usual necessary to center the risk process and we consider

$$\tilde{R}_n(t) = \sum_{i=1}^{m_n} (X_i - c) \mathbb{1}_{\{T_i^n \leq t\}}, \quad t \in [0, T], n \geq 1,$$

where $c = \mathbb{E}[X]$ if X has a finite expectation and $c = 0$ otherwise.

THEOREM 4.3. Let $\mathcal{T} \times \mathcal{X} = [0, T] \times [0, \infty)$ and Π_n , $n \geq 1$, be the sequence of marked point processes defined by Equation (3.7). Assume Ψ_n satisfies assumption (3.8) and the regular variation condition $\nu \in \text{RV}([0, \infty) \setminus \{0\}, \{a_n\}, \mu)$ with $\mu(dx) = \alpha x^{-\alpha-1} dx$ for some $\alpha > 0$. Depending on the value of $\alpha > 0$, we assume furthermore:

- $\mathbb{E}[X_1] < \infty$ if $\alpha = 1$;
- $\mathbb{E}[X_1^2] < \infty$ if $\alpha = 2$;
- $a_{nm_n}^2/m_n \rightarrow \infty$ as $n \rightarrow \infty$ if $\alpha \geq 2$.

Then, for $k \geq 0$,

$$(4.4) \quad n^{k+1} \mathbb{P}(a_{nm_n}^{-1} \tilde{R}_n \in \cdot) \longrightarrow \mu_{k+1}^\#(\cdot) \quad \text{in } \mathbb{M}(\mathbb{D} \setminus \mathbb{D}_k),$$

with $\mu_{k+1}^\#$ given by Equation (4.3).

EXAMPLE 4.4. We consider the setting of [Rhee et al. \(2019\)](#). Let $(X_n(t))_{0 \leq t \leq 1}$ be a Lévy process with Lévy measure ν and jump part

$$J_n(t) = \int_{x>1} xN([0, nt] \times dx)$$

where N is the Poisson random measure with measure $Leb \otimes \nu$ on $[0, n] \times (0, \infty)$ and Leb denotes the Lebesgue measure. Then $J_n(t)$ is a compound Poisson process and its centered version \tilde{J}_n is tail equivalent to the centered version of the Lévy process X_n ; see Proposition 6.1 of [Rhee et al. \(2019\)](#). The process \tilde{J}_n is close to \tilde{R}_n when approximating the number of jumps $N([0, n] \times [1, \infty))$ by its expectation n , interchanging the roles of m_n and n and considering T_i^n as in Example 3.6 for $T = \tau = 1$. Theorem 4.3 yields

$$m_n^{k+1} \mathbb{P}(a_{nm_n}^{-1} \tilde{R}_n \in \cdot) \longrightarrow \mu_{k+1}^\#(\cdot) \quad \text{in } \mathbb{M}(\mathbb{D} \setminus \mathbb{D}_k).$$

This is the one-sided large deviation principles for centered Lévy processes of [Rhee et al. \(2019\)](#) considering (m_n) so that $a_{nm_n} \sim n$, i.e. $m_n \sim (n\nu(n, \infty))^{-1} \rightarrow \infty$ as $n \rightarrow \infty$. It shows that X_n , \tilde{J}_n and \tilde{R}_n satisfies the same one-sided large deviation principles.

REMARK 4.5. The integrability conditions for $\alpha = 1$ or $\alpha = 2$ could be dropped thanks to extra classical but technical arguments. For the sake of simplicity, we focus on the integrable case only. Because the sequence (a_n) is regularly varying with index $1/\alpha$, the condition $a_{nm_n}^2/m_n \rightarrow \infty$ is satisfied as soon as $\alpha > 2$ and $m_n = o(n^p)$ for some $p < 2/(\alpha - 2)$. This ensures that the growth $m_n \rightarrow \infty$ is not too fast.

REMARK 4.6. When $\alpha \geq 1$, the centering is not necessary under the extra assumption $a_{nm_n}/m_n \rightarrow \infty$. That is the risk process $R_n(t) = \sum_{i=1}^{m_n} X_i \mathbb{1}_{\{T_i^n \leq t\}}$ satisfies

$$n^{k+1} \mathbb{P}(a_{nm_n}^{-1} R_n \in \cdot) \longrightarrow \mu_{k+1}^\#(\cdot) \quad \text{in } \mathbb{M}(\mathbb{D} \setminus \mathbb{D}_k).$$

This is immediately derived from Equation (4.4) because the magnitude of the centering term is bounded by $m_n/a_{nm_n} \rightarrow 0$. Note that the condition $a_{nm_n}/m_n \rightarrow \infty$ holds as soon as $\alpha > 1$ and $m_n = o(n^p)$ for some $p < 1/(\alpha - 1)$.

4.2. *Reinsurance of the largest claims.* We provide in this section an application of the preceding results to risk theory with the study of a reinsurance model. The reader may find an exhaustive review of risk theory and the mathematical issues that it raises in [Asmussen and Albrecher \(2010\)](#) and [Mikosch \(2010\)](#). We focus on a reinsurance treaty of extreme value type called the *largest claims reinsurance*. Assume that at the time the contract is underwritten, say $t = 0$, the reinsurance company guarantees that it will cover the k largest claims over the period $[0, T]$. For a risk process of the form (4.1), the contract covers the risk

$$R_k^+ = \sum_{i=1}^k X_{N+1-i:N},$$

where $X_{1:N} \leq \dots \leq X_{N:N}$ denote the order statistics of X_1, \dots, X_N . The value R_k^+ is known only at time T and, during the contract life, the covered risk evolves as

$$R_k^+(t) = \sum_{i=1}^k X_{N(t)+1-i:N(t)}, \quad t \in [0, T],$$

where $N(t) = \sum_{i=1}^N \mathbf{1}_{\{T_i \leq t\}}$ is the number of claims up to time t and $X_{1:N(t)} \leq \dots \leq X_{N(t):N(t)}$ the order statistics of $X_1, \dots, X_{N(t)}$. We use here the convention $X_{N(t)+1-i:N(t)} = 0$ if $i > N(t)$. The subscriber of the contract needs to assess its residual risk

$$R_k^- = \sum_{i=1}^{N-k} X_{i:N}$$

that is not covered by the reinsurance treaty and its evolution over time is

$$R_k^-(t) = \sum_{i=1}^{N(t)-k} X_{i:N(t)}, \quad t \in [0, T].$$

Our results state the regular variation properties of the residual risk R_k^- . The following proposition is crucial in our approach. When $k = 0$, it states that the sum of independent heavy-tailed random variables has the same tail behavior as their maximum. For non-ordered random variables, the result has been shown in [Tillier and Wintenberger \(2018\)](#) Proposition 7 under similar moment conditions. To our best knowledge, the more general statement with random number of terms N and arbitrary order statistic $k \geq 0$ is new.

PROPOSITION 4.7. *Let $(X_i)_{i \geq 1}$ be i.i.d. non-negative random variables with cumulative distribution function F , assumed heavy-tailed with index $\alpha > 0$. Independently, let N be random variable with values in \mathbb{N} . Assume $\mathbb{E}[N^{k+1}] < \infty$, and, if $\alpha \geq 1$, assume furthermore that $\mathbb{E}[N^p] < \infty$ for some $p > (k+1)\alpha$. Then,*

$$\mathbb{P} \left(\sum_{i=1}^{N-k} X_{i:N} > x \right) \sim \mathbb{P}(X_{N-k:N} > x) \sim \frac{\mathbb{E}[N^{k+1}]}{(k+1)!} (1 - F(x))^{k+1}, \quad \text{as } x \rightarrow \infty,$$

with $\mathbb{E}[N^{k+1}] = \mathbb{E}[N!/(N-k-1)!]$ the $(k+1)$ -th factorial moment of N .

Interestingly, we have the relationships

$$(4.5) \quad X_{N-k:N} = 2d(R, \mathbb{D}_k) \quad \text{and} \quad \sum_{i=1}^{N-k} X_{i:N} = d(R, \mathbb{J}_k),$$

where d is the Skorokhod metric, \mathbb{D}_k is the cone of càd-làg functions with at most k jumps (see [Lemma 6.1](#) below) and \mathbb{J}_k is the cone of piecewise constant càd-làg functions with at most k jumps. [Proposition 4.7](#) states that, for large x , the events $\{d(R, \mathbb{D}_k) \geq x/2\}$ and $\{d(R, \mathbb{J}_k) \geq x/2\}$ are asymptotically equivalent (note that one inclusion always holds). As a consequence, we obtain the following corollary.

COROLLARY 4.8. *Assume that the assumptions of [Theorem 4.1](#) are satisfied. When $\alpha \geq 1$, assume furthermore that $\mathbb{E}[N^p] < \infty$ for some $p > (k+1)\alpha$. Then, the conclusion of [Theorem 4.1](#) holds with $\mathbb{D} \setminus \mathbb{D}_k$ replaced by $\mathbb{D} \setminus \mathbb{J}_k$.*

In terms of successive hidden regular variations, it is sensible to work with \mathbb{J}_k instead of \mathbb{D}_k because the support of $\mu_k^\#$ is exactly \mathbb{J}_k . From a technical point of view, the results are stronger and more difficult to establish in $\mathbb{D} \setminus \mathbb{J}_k$ because the distance $d(R, \mathbb{J}_k)$ involves a sum of order statistics so that extra integrability conditions are required to control the sum.

Going back to our original problem of largest claims in reinsurance, we deduce from [Corollary 4.8](#) the following results.

PROPOSITION 4.9. *Assume that the assumptions of Theorem 4.1 are satisfied. When $\alpha \geq 1$, assume furthermore that $\mathbb{E}[N^p] < \infty$ for some $p > (k+1)\alpha$. Then:*

i) (regular variation) *The residual risk satisfies*

$$\mathbb{P}(R_k^- > x) \sim \frac{\mathbb{E}[N^{k+1}]}{(k+1)!} (1 - F(x))^{k+1}, \quad \text{as } x \rightarrow \infty.$$

ii) (conditional limit theorem) *The typical behavior of the risk process given a large residual risk is given by*

$$\mathbb{P}(x^{-1}R \in \cdot \mid R_k^- > x) \xrightarrow{d} \mathbb{P}\left(\left(\sum_{i=1}^{k+1} Z_i \mathbf{1}_{\{S_i \leq t\}}\right)_{t \in [0, T]} \in \cdot\right), \quad \text{as } x \rightarrow \infty,$$

where \xrightarrow{d} stands for weak convergence in Skorokhod space \mathbb{D} , Z_1, \dots, Z_{k+1} are i.i.d. with standard α -Pareto distribution and, independently, (S_1, \dots, S_{k+1}) has distribution $M_{k+1}(ds_1, \dots, ds_{k+1})/M_{k+1}([0, T]^{k+1})$.

iii) (residual risk monitoring) *Assume that M_{k+2} is finite and non-null and that both M_{k+1} and M_{k+2} are continuous measures. If $\alpha \geq 1$, assume furthermore that $\mathbb{E}[N^p] < \infty$ for some $p > (k+2)\alpha$. Then, for all $0 < t_0 < t_1 < T$ and $u > 1$,*

$$(4.6) \quad \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \lim_{x \rightarrow \infty} (1 - F(x))^{-1} \mathbb{P}(R_k^-(t_1) > ux \mid x < R_k^-(t_0) < (1 + \varepsilon)x) \\ &= \frac{M_{k+2}([0, t_0]^{k+1} \times (t_0, t_1])}{M_{k+1}([0, t_0]^{k+1})} \left((u-1)^{-(k+1)\alpha} + ((u-1)^{-\alpha} - 1)_+ \right). \end{aligned}$$

The third item allows to monitor the residual risk during the contract lifetime and assesses the risk of a larger loss at time $t_1 > t_0$ given that the loss is approximately x at time t_0 . The regular variation of the conditional probability with respect to the null event $R_k^-(t_0) = x$ does not follow from Corollary 4.8 explaining why we introduce the approximate conditioning with $\varepsilon \rightarrow 0$.

EXAMPLE 4.10. In the simple case of a compound Poisson process when the claims form an homogeneous Poisson point process with intensity $\lambda > 0$, the results from Proposition 4.9 simplify as follows. The residual risk is regularly varying such that

$$\mathbb{P}(R_k^- > x) \sim \frac{\lambda^{k+1}}{(k+1)!} (1 - F(x))^{k+1} \quad \text{as } x \rightarrow \infty.$$

The limiting conditional risk process in item ii) corresponds to $k+1$ independent claim arrivals with occurrence times S_1, \dots, S_{k+1} uniform on $[0, T]$ and independent magnitudes Z_1, \dots, Z_{k+1} with standard α -Pareto distribution. Finally, the limit in Equation (4.6) equals

$$\lambda(t_1 - t_0) \left((u-1)^{-(k+1)\alpha} + ((u-1)^{-\alpha} - 1)_+ \right).$$

5. Proofs related to Section 3.

5.1. *Proof of Theorem 3.1.* For the proof of Theorem 3.1, we need the following lemma that characterizes the convergence of the tensor product of measures in $\mathbb{M}(E \setminus F)$. We recall that $E = \mathcal{T} \times \mathcal{X}$ and $F = \mathcal{T} \times \{0\}$.

LEMMA 5.1. *Assume $\nu_n \rightarrow \mu$ in $\mathbb{M}(\mathcal{X} \setminus \{0\})$ and $\lambda \in \mathbb{M}_b(\mathcal{T})$. Then for $k \geq 1$, we have*

$$(\lambda \otimes \nu_n)^{\otimes k} \rightarrow (\lambda \otimes \mu)^{\otimes k} \quad \text{in } \mathbb{M}(E^k \setminus \cup_{i=1}^k (E^{i-1} \times F \times E^{k-i})).$$

PROOF OF LEMMA 5.1. Let $f : E \rightarrow \mathbb{R}$ be a continuous bounded function vanishing on $F^r = \mathcal{T} \times B_{0,r}$ for some $r > 0$. Possibly replacing $r > 0$ by a smaller value, we can assume $\nu_n(B_{0,r}^c) \rightarrow \mu(B_{0,r}^c)$ (see Theorem 2.2 (i) in [Hult and Lindskog \(2006\)](#) that asserts that this convergence holds for all but countably many $r > 0$). By Fubini-Tonelli theorem,

$$\int_{E \setminus F} f(t, x) \lambda(dt) \nu_n(dx) = \int_{\mathcal{T}} \left(\int_{\mathcal{X} \setminus \{0\}} f(t, x) \nu_n(dx) \right) \lambda(dt).$$

For all $t \in \mathcal{T}$, the function $x \mapsto f(t, x)$ is bounded continuous and vanishes on $B_{0,r}$ so that the convergence $\nu_n \rightarrow \mu$ in $\mathbb{M}(\mathcal{X} \setminus \{0\})$ implies from Proposition 2.1 that

$$\int_{\mathcal{X} \setminus \{0\}} f(t, x) \nu_n(dx) \rightarrow \int_{\mathcal{X} \setminus \{0\}} f(t, x) \mu(dx) \quad \text{as } n \rightarrow \infty.$$

Furthermore, since f is bounded, say by $M > 0$, and since $\nu_n(B_{0,r}^c) \rightarrow \mu(B_{0,r}^c)$ is also bounded, say by $L > 0$, we have

$$\left| \int_{\mathcal{X} \setminus \{0\}} f(t, x) \nu_n(dx) \right| \leq M \nu_n(B_{0,r}^c) \leq ML, \quad t \in \mathcal{T}.$$

Lebesgue convergence theorem finally entails

$$\int_{E \setminus F} f(t, x) \lambda(dt) \nu_n(dx) \rightarrow \int_{E \setminus F} f(t, x) \lambda(dt) \mu(dx),$$

proving the convergence $\lambda \otimes \nu_n \rightarrow \lambda \otimes \mu$ in $\mathbb{M}(E \setminus F)$. A direct application of [Rhee et al. \(2019, Lemma 2.2\)](#) ensures that, for $k \geq 1$,

$$(5.1) \quad (\lambda \otimes \nu_n)^{\otimes k} \longrightarrow (\lambda \otimes \mu)^{\otimes k} \quad \text{in } \mathbb{M}\left(E^k \setminus \cup_{i=1}^k (E^{i-1} \times F \times E^{k-i})\right).$$

□

We are now ready to prove Theorem 3.1. For the sake of clarity, we begin with the proof of regular variation in the case $k = 0$, before considering hidden regular variation in the case $k \geq 1$.

PROOF OF THEOREM 3.1, CASE $k = 0$. The proof uses similar arguments than in Theorem 3.3 in [Dombry et al. \(2018\)](#). We have to show that

$$n\mathbb{P}(a_n^{-1}\Pi \in \cdot) \longrightarrow \mu_1^*(\cdot) \quad \text{in } \mathbb{M}(\mathcal{N} \setminus \{0\}).$$

According to Theorem 2.5 Equation (2.3), the convergence holds if and only if

$$(5.2) \quad \lim_{n \rightarrow \infty} n\mathbb{E} \left[1 - e^{-(a_n^{-1}\Pi)(f)} \right] = \int_{\mathcal{N}} \left(1 - e^{-\pi(f)} \right) \mu_1^*(d\pi)$$

for all $f : E \rightarrow [0, \infty)$ bounded continuous with support bounded away from F . By definition of μ_1^* ,

$$\int_{\mathcal{N}} \left(1 - e^{-\pi(f)} \right) \mu_1^*(d\pi) = \int_{\mathcal{T} \times \mathcal{X}} \left(1 - e^{-f(t,x)} \right) \lambda(dt) \mu(dx).$$

Besides, since Π is a Poisson point process,

$$\begin{aligned} n\mathbb{E} \left[1 - e^{-(a_n^{-1}\Pi)(f)} \right] &= n \left(1 - \mathbb{E} \left[e^{-\int_E f(a_n^{-1}z)\Pi(dz)} \right] \right) \\ &= n \left(1 - \exp \left[\int_{\mathcal{T} \times \mathcal{X}} \left(e^{-f(t, a_n^{-1}x)} - 1 \right) \lambda(dt) \nu(dx) \right] \right) \\ &= n \left(1 - \exp \left[-\frac{1}{n} \int_{\mathcal{T} \times \mathcal{X}} \left(1 - e^{-f(t,y)} \right) \lambda(dt) \nu_n(dy) \right] \right) \end{aligned}$$

with $\nu_n = n\nu(a_n \cdot)$. As the function $1 - e^{-f}$ is bounded continuous with support bounded away from F and as $\lambda \otimes \nu_n \rightarrow \lambda \otimes \mu$ in $\mathbb{M}(E \setminus F)$ as $n \rightarrow \infty$ (see Lemma 5.1), we have

$$\int_{\mathcal{T} \times \mathcal{X}} \left(1 - e^{-f(t,y)} \right) \lambda(dt) \nu_n(dy) \longrightarrow \int_{\mathcal{T} \times \mathcal{X}} \left(1 - e^{-f(t,y)} \right) \lambda(dt) \mu(dy), \quad \text{as } n \rightarrow \infty,$$

and hence

$$\begin{aligned} n\mathbb{E} \left[1 - e^{-\int_{E \setminus F} f(a_n^{-1}z)\Pi(dz)} \right] &= n \left(1 - \exp \left[-\frac{1}{n} \int_{\mathcal{T} \times \mathcal{X}} \left(1 - e^{-f(t,y)} \right) \lambda(dt) \nu_n(dy) \right] \right) \\ &\longrightarrow \int_{\mathcal{T} \times \mathcal{X}} \left(1 - e^{-f(t,x)} \right) \lambda(dt) \mu(dx). \end{aligned}$$

This proves Equation (5.2) and concludes the proof of Theorem 3.1, case $k = 0$. \square

PROOF OF THEOREM 3.1, CASE $k \geq 1$. Let $r > 0$ be fixed such that $\mu(\partial B_{0,r}^c) = 0$ and let $f : \mathcal{T} \times \mathcal{X} \mapsto [0, \infty)$ be a bounded continuous function vanishing on $F^r = \mathcal{T} \times B_{0,r}$. For $k \geq 1$, we will prove that

$$\begin{aligned} &n^{k+1} \mathbb{E} \left[e^{-\sum_{i=1}^N f(T_i, a_n^{-1}X_i)} \mathbf{1}_{\{\sum_{i=1}^N \varepsilon_{a_n^{-1}X_i}(B_{0,r}^c) \geq k+1\}} \right] \\ &\longrightarrow \frac{1}{(k+1)!} \int_{E^{k+1}} e^{-\sum_{i=1}^{k+1} f(t_i, x_i)} \otimes_{i=1}^{k+1} \mathbf{1}_{\{x_i \in B_{0,r}^c\}} \lambda(dt_i) \mu(dx_i) \\ (5.3) \quad &= \int_{\mathcal{N}} e^{-\pi(f)} \mathbf{1}_{\{\pi(B_{0,r}^c) \geq k+1\}} \mu_{k+1}^*(d\pi). \end{aligned}$$

In view of Theorem 2.5 (iii), this implies the convergence (3.3).

Since $\sum_{i=1}^N \varepsilon_{X_i}$ is a Poisson random measure with intensity $\lambda(\mathcal{T})\nu(dx)$, we have

$$\begin{aligned}
& n^{k+1} \mathbb{P} \left(\sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) \geq k+1 \right) \\
&= n^{k+1} e^{-\lambda(\mathcal{T})\nu(a_n B_{0,r}^c)} \sum_{j=k+1}^{\infty} \frac{(\lambda(\mathcal{T})\nu(a_n B_{0,r}^c))^j}{j!} \\
(5.4) \quad &= \frac{(n\lambda(\mathcal{T})\nu(a_n B_{0,r}^c))^{k+1}}{(k+1)!} e^{-\lambda(\mathcal{T})\nu(a_n B_{0,r}^c)} \sum_{j=0}^{\infty} \frac{(k+1)!}{(k+1+j)!} (\lambda(\mathcal{T})\nu(a_n B_{0,r}^c))^j
\end{aligned}$$

and

$$(5.5) \quad n^{k+1} \mathbb{P} \left(\sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) = k+1 \right) = \frac{(n\lambda(\mathcal{T})\nu(a_n B_{0,r}^c))^{k+1}}{(k+1)!} e^{-\lambda(\mathcal{T})\nu(a_n B_{0,r}^c)}.$$

Regular variation $\nu \in \text{RV}(\mathcal{X} \setminus \{0\}, \{a_n\}, \mu)$ implies $n\nu(a_n B_{0,r}^c) \rightarrow \mu(B_{0,r}^c) > 0$ as $n \rightarrow \infty$. Then, combining with Equations (5.4) and (5.5), we deduce

$$n^{k+1} \mathbb{P} \left(\sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) \geq k+1 \right) \sim n^{k+1} \mathbb{P} \left(\sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) = k+1 \right) \rightarrow \frac{c^{k+1}}{(k+1)!}$$

with $c = \lambda(\mathcal{T})\mu(B_{0,r}^c)$. As a consequence, the left-hand side of Equation (5.3) satisfies

$$\begin{aligned}
& n^{k+1} \mathbb{E} \left[e^{-\sum_{i=1}^N f(T_i, a_n^{-1} X_i)} \mathbb{1}_{\{\sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) \geq k+1\}} \right] \\
& \sim \frac{c^{k+1}}{(k+1)!} \frac{\mathbb{E} \left[e^{-\sum_{i=1}^N f(T_i, a_n^{-1} X_i)} \mathbb{1}_{\{\sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) \geq k+1\}} \right]}{\mathbb{P} \left(\sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) \geq k+1 \right)} \\
(5.6) \quad & \sim \frac{c^{k+1}}{(k+1)!} \mathbb{E} \left[e^{-\sum_{i=1}^N f(T_i, a_n^{-1} X_i)} \mid \sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) = k+1 \right].
\end{aligned}$$

Because f vanishes on $\mathcal{T} \times B_{0,r}$, the sum $\sum_{i=1}^N f(T_i, a_n^{-1} X_i)$ depends only on the points of Π in $\mathcal{T} \times a_n B_{0,r}^c$. The independence property of Poisson point processes ensures that, given Π has $k+1$ points in $\mathcal{T} \times a_n B_{0,r}^c$, those points are independent and uniformly distributed with distribution $c_n^{-1} \mathbb{1}_{\{x \in a_n B_{0,r}^c\}} \lambda(dt) \nu(dx)$ for $c_n = \lambda(\mathcal{T})\nu(a_n B_{0,r}^c)$ (see [Kingman \(1993\)](#), Chapter 2.4). We deduce

$$\begin{aligned}
& \mathbb{E} \left[e^{-\sum_{i=1}^N f(T_i, a_n^{-1} X_i)} \mid \sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) = k+1 \right] \\
&= \int_{E^{k+1}} e^{-\sum_{i=1}^{k+1} f(t_i, a_n^{-1} x_i)} \otimes_{i=1}^{k+1} c_n^{-1} \mathbb{1}_{\{x_i \in a_n B_{0,r}^c\}} \lambda(dt_i) \nu(dx_i).
\end{aligned}$$

Introduce $\nu_n(\cdot) = n\nu(a_n\cdot)$. Since, $c_n \sim cn^{-1}$, from Lemma 5.1, we deduce

$$\begin{aligned} & \mathbb{E}\left[e^{-\sum_{i=1}^N f(t_i, a_n^{-1}X_i)} \mid \sum_{i=1}^N \varepsilon_{a_n^{-1}X_i}(B_{0,r}^c) = k+1\right] \\ &= \frac{1}{n^{k+1}c_n^{k+1}} \int_{E^{k+1}} e^{-\sum_{i=1}^{k+1} f(t_i, x_i)} \otimes_{i=1}^{k+1} \mathbb{1}_{\{x_i \in B_{0,r}^c\}} \lambda(dt_i) \nu_n(dx_i) \\ &\rightarrow \frac{1}{c^{k+1}} \int_{E^{k+1}} e^{-\sum_{i=1}^{k+1} f(t_i, x_i)} \otimes_{i=1}^{k+1} \mathbb{1}_{\{x_i \in B_{0,r}^c\}} \lambda(dt_i) \mu(dx_i). \end{aligned}$$

This together with Equation (5.6) implies Equation (5.3), completing the proof of Theorem 3.1 in the case $k \geq 1$. \square

5.2. *Proof of Theorem 3.2.* We begin with three preliminary lemmas that will be useful for the proof of Theorem 3.2.

LEMMA 5.2. *For $n \geq 1$ and $p \geq 0$, let S be a random variable with binomial distribution with parameter (n, p) . Then, for $k = 1, \dots, n$,*

$$\mathbb{P}(S \geq k) \leq \binom{n}{k} p^k.$$

PROOF. Let X_1, \dots, X_n be independent Bernoulli random variables with parameter p . Then $S = \sum_{i=1}^n X_i$ has a binomial distribution with parameter (n, p) . Furthermore, $S \geq k$ if and only if there is a subset $I \subset \{1, \dots, n\}$ with cardinality k such that $X_i = 1$ whenever $i \in I$. The union bound implies

$$\mathbb{P}(S \geq k) \leq \sum_{i \in I} \mathbb{P}(X_i = 1, i \in I).$$

The lemma follows because there are $\binom{n}{k}$ terms all equal to p^k . \square

LEMMA 5.3. *Assume $\nu \in \text{RV}(\mathcal{X} \setminus \{0\}, \{a_n\}, \mu)$ and let $f : \mathcal{T} \times \mathcal{X} \rightarrow [0, \infty)$ be a bounded continuous function with support bounded away from the axis $\mathcal{T} \times \{0\}$. Then, as $n \rightarrow \infty$,*

$$(5.7) \quad (L_n f)(t) := n\mathbb{E}[1 - e^{-f(t, X_i/a_n)}] \rightarrow \int_{\mathcal{X}} (1 - e^{-f(t, x)}) \mu(dx) =: (L f)(t) \quad \text{for all } t \in \mathcal{T}$$

and furthermore the functions $L_n f$, $n \geq 1$, are uniformly bounded on \mathcal{T} .

PROOF. For all fixed $t \in \mathcal{T}$, the function $x \in \mathcal{X} \mapsto 1 - e^{-f(t, x)}$ is bounded continuous and vanishes on a neighborhood of 0 so that the convergence $n\nu(a_n\cdot) \rightarrow \mu(\cdot)$ in $\mathbb{M}(\mathcal{X} \setminus \{0\})$ implies Equation (5.7). Furthermore, since f is nonnegative bounded and with support bounded away from $\mathcal{T} \times \{0\}$, we have

$$0 \leq f(t, x) \leq M \mathbb{1}_{\{d_{\mathcal{X}}(0, x) \geq r\}} \quad \text{for some } M \geq 0 \text{ and } r > 0,$$

whence

$$(L_n f)(t) \leq (1 - e^{-M}) n\nu(a_n B_{0,r}^c) \rightarrow (1 - e^{-M}) \mu(B_{0,r}^c)$$

is uniformly bounded on \mathcal{T} . \square

LEMMA 5.4. Assume $\Psi = \sum_{i=1}^N \varepsilon_{T_i}$ has a finite intensity measure $\lambda \in \mathbb{M}_b(\mathcal{T})$ and define its Laplace functional as

$$L_\Psi(g) = \mathbb{E}[e^{-\Psi(g)}] = \mathbb{E} \left[\prod_{i=1}^N e^{-g(T_i)} \right], \quad \text{for } g : \mathcal{T} \rightarrow [0, \infty) \text{ measurable.}$$

Let $(g_n)_{n \geq 1}$ and g be nonnegative bounded functions on \mathcal{T} such that

$$\lim_{n \rightarrow \infty} g_n(t) = g(t), \quad \text{for all } t \in \mathcal{T},$$

and assume $(g_n)_{n \geq 1}$ is uniformly bounded on \mathcal{T} . Then, as $n \rightarrow \infty$,

$$n [1 - L_\Psi(n^{-1}g_n)] \longrightarrow \int_{\mathcal{T}} g(t)\lambda(dt).$$

PROOF. By definition of the Laplace functional L_Ψ , we have

$$n [1 - L_\Psi(n^{-1}g_n)] = \mathbb{E} \left[n \left(1 - e^{-n^{-1}\Psi(g_n)} \right) \right]$$

where

$$n \left(1 - e^{-n^{-1}\Psi(g_n)} \right) \longrightarrow \Psi(g), \quad \text{almost surely as } n \rightarrow \infty.$$

Furthermore, the inequality $1 - e^{-x} \leq x$, $x \geq 0$, together with the uniform bound $g_n(t) \leq M$, $n \geq 1$, imply the domination condition

$$n \left(1 - e^{-n^{-1}\Psi(g_n)} \right) \leq \Psi(g_n) \leq M\Psi(1),$$

where $\Psi(1) = N$ satisfies $\mathbb{E}[N] = \lambda(\mathcal{T}) < \infty$. We deduce, thanks to Lebesgue convergence Theorem,

$$n [1 - L_\Psi(n^{-1}g_n)] \longrightarrow \mathbb{E}[\Psi(g)].$$

Campbell's Theorem states the equality $\mathbb{E}[\Psi(g)] = \int_{\mathcal{T}} g(t)\lambda(dt)$ which concludes the proof. \square

PROOF OF THEOREM 3.2, CASE $k = 0$. Similarly as in the proof of Theorem 3.1, we need to prove that

$$(5.8) \quad n\mathbb{E} \left[1 - e^{-\int_E f(a_n^{-1}z)\Pi(dz)} \right] \rightarrow \int_E \left(1 - e^{-f(t,x)} \right) \lambda(dt)\mu(dx), \quad \text{as } n \rightarrow \infty,$$

for all bounded continuous functions $f : E \rightarrow [0, \infty)$ with support bounded away from F . Conditioning with respect to the base point process Ψ , the left-hand side of Equation (5.8) is rewritten as

$$\begin{aligned} n\mathbb{E} \left[1 - \exp \left(- \int_E f(a_n^{-1}z)\Pi(dz) \right) \right] &= n\mathbb{E} \left[\mathbb{E} \left[1 - \exp \left(- \sum_{i=1}^N f(T_i, X_i) \right) \middle| \Psi \right] \right] \\ &= n \left\{ 1 - \mathbb{E} \left[\mathbb{E} \left[\prod_{i=1}^N e^{-f(T_i, X_i)} \middle| \Psi \right] \right] \right\} \\ &= n \left\{ 1 - \mathbb{E} \left[\prod_{i=1}^N \left(1 - \frac{1}{n} (L_n f)(T_i) \right) \right] \right\}. \end{aligned}$$

In the last equality, we use the conditional independence of the X_i given Ψ and the definition of $L_n f$ in Equation (5.7). The Laplace functional of the base point process Ψ defined in Lemma 5.4 satisfies, for $h : \mathcal{T} \rightarrow (0, 1]$ measurable,

$$\mathbb{E} \left[\prod_{i=1}^N h(T_i) \right] = \mathbb{E} \left[\exp \left(- \sum_{i=1}^N -\ln h(T_i) \right) \right] = L_\Psi(-\ln h).$$

This gives

$$n\mathbb{E} \left[1 - \exp \left(- \int_E f(a_n^{-1}z)\Pi(dz) \right) \right] = n \left[1 - L_\Psi \left(-\ln \left(1 - \frac{1}{n} L_n f \right) \right) \right].$$

We define

$$g_n(t) = -n \ln \left(1 - \frac{1}{n} (L_n f)(t) \right) \quad \text{and} \quad g(t) = (Lf)(t).$$

Lemma 5.3 implies that $g_n(t) \rightarrow g(t)$, as $n \rightarrow \infty$, and is uniformly bounded on \mathcal{T} . Lemma 5.4 then entails

$$\begin{aligned} n\mathbb{E} \left[1 - \exp \left(- \int_E f(a_n^{-1}z)\Pi(dz) \right) \right] &= n \left[1 - L_\Psi(n^{-1}g_n) \right] \\ &\rightarrow \int_{\mathcal{T}} g(t)\lambda(dt) = \int_E \left(1 - e^{-f(t,x)} \right) \lambda(dt)\mu(dx). \end{aligned}$$

This proves Equation (5.8) and Theorem 3.2 in the case $k = 0$. \square

PROOF OF THEOREM 3.2, CASE $k \geq 1$. The proof is an adaptation of the proof of Theorem 3.1, case $k \geq 1$. We fix $r > 0$. Conditioning upon the events $\{N = j\}$, $j \geq 1$, and using the independence of the marks, we have

$$\begin{aligned} n^{k+1}\mathbb{P} \left(\sum_{i=1}^N \varepsilon_{a_n^{-1}X_i}(B_{0,r}^c) \geq k+1 \right) &= n^{k+1} \sum_{j=k+1}^{\infty} \mathbb{P}(N = j) \mathbb{P} \left(\sum_{i=1}^j \varepsilon_{a_n^{-1}X_i}(B_{0,r}^c) \geq k+1 \right) \\ (5.9) \qquad \qquad \qquad &= \sum_{j=k+1}^{\infty} \mathbb{P}(N = j) n^{k+1} \mathbb{P}(S_{j,n} \geq k+1) \end{aligned}$$

with $S_{j,n}$ a random variable with binomial distribution with parameter $(j, \nu(a_n B_{0,r}^c))$. Regular variation implies $\nu(a_n B_{0,r}^c) \sim n^{-1}\mu(B_{0,r}^c)$, as $n \rightarrow \infty$, so that the binomial probability distribution has asymptotic

$$(5.10) \quad n^{k+1}\mathbb{P}(S_{j,n} \geq k+1) \sim n^{k+1}\mathbb{P}(S_{j,n} = k+1) \sim \binom{j}{k+1} \mu(B_{0,r}^c)^{k+1}.$$

In order to apply dominated convergence and plug-in the equivalent (5.10) in Equation (5.9), we use the upper bound from Lemma 5.2 yielding

$$n^{k+1}\mathbb{P}(S_{j,n} \geq k+1) \leq \binom{j}{k+1} n^{k+1} \nu(a_n B_{0,r}^c)^{k+1}.$$

The convergence $n\nu(a_n B_{0,r}^c) \rightarrow \mu(B_{0,r}^c)$ implies the bound $n\nu(a_n B_{0,r}^c) \leq M$ for some $M > 0$. We deduce the uniform bound

$$n^{k+1}\mathbb{P}(S_{j,n} \geq k+1) \leq M^{k+1} \binom{j}{k+1} \leq \frac{M^{k+1}}{(k+1)!} j^{k+1}.$$

Since Ψ has a finite factorial moment measure of order $k + 1$, N has a finite moment of order $k + 1$ and

$$\sum_{j=k+1}^{\infty} \mathbb{P}(N = j) \frac{M^{k+1}}{(k+1)!} j^{k+1} \leq \frac{M^{k+1}}{(k+1)!} \mathbb{E}[N^{k+1}] < \infty.$$

Equations (5.9) and (5.10) together with dominated convergence imply

$$\begin{aligned} n^{k+1} \mathbb{P}\left(\sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) \geq k+1\right) &\sim n^{k+1} \mathbb{P}\left(\sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) = k+1\right) \\ &\rightarrow c := \mu(B_{0,r}^c)^{k+1} \sum_{j \geq k+1} \binom{j}{k+1} \mathbb{P}(N = j). \end{aligned}$$

The limit c is positive as soon as $\mathbb{P}(N \geq k+1) > 0$ which is ensured by the condition M_{k+1} non-null. We deduce

$$\begin{aligned} n^{k+1} \mathbb{E}\left[e^{-\sum_{i=1}^N f(T_i, a_n^{-1} X_i)} \mathbb{1}_{\{\sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) \geq k+1\}}\right] \\ \sim n^{k+1} \mathbb{E}\left[e^{-\sum_{i=1}^N f(T_i, a_n^{-1} X_i)} \mathbb{1}_{\{\sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) = k+1\}}\right]. \end{aligned}$$

Next, we condition on $\{N = j\}$ and introduce the symmetric probability π_j on \mathcal{T}^j giving the position of the points of Ψ conditionally on $\{N = j\}$, see (Daley and Vere-Jones, 2003, Section 5.3). Noting $p_j = \mathbb{P}(N = j)$, we get

$$\begin{aligned} \mathbb{E}\left[e^{-\sum_{i=1}^N f(T_i, a_n^{-1} X_i)} \mathbb{1}_{\{\sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) = k+1\}}\right] \\ = \sum_{j \geq k+1} \int_{E^j} e^{-\sum_{i=1}^j f(t_i, a_n^{-1} x_i)} \mathbb{1}_{\{\sum_{i=1}^j \varepsilon_{x_i}(a_n B_{0,r}^c) = k+1\}} p_j \pi_j(dt_1, \dots, dt_j) \nu(dx_1) \dots \nu(dx_j). \end{aligned}$$

Consider the j -th term for fixed $j \geq k + 1$. The event $\sum_{i=1}^j \varepsilon_{x_i}(a_n B_{0,r}^c) = k + 1$ can be decomposed into a disjoint union of $\binom{j}{k+1}$ disjoint events indexed by a subset $I \subset \{1, \dots, j\}$ of size $k + 1$ such that $x_i \in a_n B_{0,r}^c$ if $i \in I$ and $x_i \in a_n B_{0,r}$ if $i \notin I$. By symmetry, each events yields an equal contribution and we consider $I = \{1, \dots, k + 1\}$. Then $f(t_i, a_n^{-1} x_i) = 0$ for $i > k + 1$ and we can integrate out the corresponding x_i 's, yielding the contribution

$$\nu(a_n B_{0,r})^{j-k-1} \int_{\mathcal{T}^j \times (a_n B_{0,r}^c)^{k+1}} e^{-\sum_{i=1}^{k+1} f(t_i, a_n^{-1} x_i)} p_j \pi_j(dt_1, \dots, dt_j) \nu(dx_1) \dots \nu(dx_{k+1}).$$

We then multiply by $\binom{j}{k+1}$, introduce the j th Janossi measure $J_j = j! p_j \pi_j$ of Ψ (see Daley and Vere-Jones (2003) Chapter 5.3) and sum over $j \geq k + 1$ to get

$$\begin{aligned} \mathbb{E}\left[e^{-\sum_{i=1}^N f(T_i, a_n^{-1} X_i)} \mathbb{1}_{\{\sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) = k+1\}}\right] \\ = \sum_{j \geq k+1} \frac{\nu(a_n B_{0,r})^{j-k-1}}{(k+1)!(j-k-1)!} \dots \\ \dots \int_{\mathcal{T}^j \times (a_n B_{0,r}^c)^{k+1}} e^{-\sum_{i=1}^{k+1} f(t_i, a_n^{-1} x_i)} J_j(dt_1, \dots, dt_j) \nu(dx_1) \dots \nu(dx_{k+1}). \end{aligned}$$

Note that $\nu(a_n B_{0,r}) \rightarrow 1$ and $n\nu(a_n \cdot) \rightarrow \mu(\cdot)$ as $n \rightarrow \infty$. Then, multiplying by n^{k+1} , from Lemma 5.1, we have

$$\begin{aligned} & n^{k+1} \mathbb{E} \left[e^{-\sum_{i=1}^N f(T_i, a_n^{-1} X_i)} \mathbb{1}_{\{\sum_{i=1}^N \varepsilon_{a_n^{-1} X_i}(B_{0,r}^c) = k+1\}} \right] \\ & \sim \frac{1}{(k+1)!} \sum_{j \geq k+1} \int_{\mathcal{T}^j \times (B_{0,r}^c)^{k+1}} e^{-\sum_{i=1}^{k+1} f(t_i, x_i)} \frac{J_j(dt_1, \dots, dt_j)}{(j-k-1)!} \mu(dx_1) \dots \mu(dx_{k+1}) \\ & = \frac{1}{(k+1)!} \int_{\mathcal{T}^j \times (B_{0,r}^c)^{k+1}} e^{-\sum_{i=1}^{k+1} f(t_i, x_i)} M_{k+1}(dt_1, \dots, dt_j) \mu(dx_1) \dots \mu(dx_{k+1}) \\ & = \int_{\mathcal{N}} e^{-\pi(f)} \mathbb{1}_{\{\pi(F_r^c) \geq k+1\}} \mu_{k+1}^*(d\pi). \end{aligned}$$

In view of Theorem 2.5, the convergence (3.5) in $\mathbb{M}(\mathcal{N} \setminus \mathcal{N}_k)$ follows. Note that in the third line above, we use Fubini-Tonelli Theorem to exchange summation and integration as well as the following identity linking Janossy measures and factorial moment measures (Daley and Vere-Jones (2003) Theorem 5.4.II)

$$M_{k+1}(B) = \sum_{m \geq 0} \frac{J_{k+1+m}(B \times \mathcal{T}^m)}{m!}, \quad B \in \mathcal{B}(\mathcal{T}^{k+1}).$$

This concludes the proof. \square

5.3. *Proof of Theorem 3.3.* The next lemma is a variant of Lemma 5.3 that will be useful for the proof of Theorem 3.3.

LEMMA 5.5. *Under the same notations and assumptions as in Lemma 5.3, assume furthermore that $f : \mathcal{T} \times \mathcal{X} \rightarrow [0, \infty)$ is Lipschitz continuous and \mathcal{T} is locally compact. Then the convergence $L_n f \rightarrow Lf$ in Equation (5.7) is uniform on \mathcal{T} .*

PROOF. Lemma 5.3 states the pointwise convergence $(L_n f)(t) \rightarrow (Lf)(t)$ for all $t \in \mathcal{T}$. Under the assumption that f is K -Lipschitz, the family of functions $\{L_n f, n \geq 1\}$ is equicontinuous, since for $t, t' \in \mathcal{T}$, we have

$$\begin{aligned} |(L_n f)(t) - (L_n f)(t')| & \leq n \int_{B_{0,r}^c} |e^{-f(t,x/a_n)} - e^{-f(t',x/a_n)}| \nu(dx) \\ & \leq n\nu(a_n B_{0,r}^c) M K d_{\mathcal{T}}(t, t'), \end{aligned}$$

where f is bounded by M with support included in $\mathcal{T} \times B_{0,r}^c$ and $n\nu(a_n B_{0,r}^c) \rightarrow \mu(B_{0,r}^c)$ is bounded. Pointwise convergence together with equicontinuity implies uniform convergence thanks to Arzela-Ascoli theorem applied on \mathcal{T} locally compact. \square

PROOF OF THEOREM 3.3, CASE $k = 0$. The proof uses arguments and notations from the proof of Theorem 3.2. We need to prove that

$$(5.11) \quad n\mathbb{E} \left[1 - e^{-\int_E f(a_n^{-1} z) \Pi(dz)} \right] \rightarrow \int_{\mathcal{T} \times \mathcal{X}} \left(1 - e^{-f(t,x)} \right) \lambda(dt) \mu(dx), \quad \text{as } n \rightarrow \infty.$$

On the first hand, we have

$$\begin{aligned}
n\mathbb{E} \left[1 - e^{-\int_E f(a_{nm_n}^{-1}z)\pi(dz)} \right] &= n \left[1 - \mathbb{E} \left[\prod_{i=1}^{m_n} \left(1 - \frac{1}{nm_n} (L_{nm_n}f)(T_i^n) \right) \right] \right] \\
&= n \left[1 - \mathbb{E} \left[\exp \left(\frac{1}{m_n} \sum_{i=1}^{m_n} m_n \log \left(1 - \frac{1}{nm_n} (L_{nm_n}f)(T_i^n) \right) \right) \right] \right] \\
(5.12) \quad &= n \left[1 - \mathbb{E} \left[\exp \left(-\frac{1}{nm_n} \sum_{i=1}^{m_n} (L_{nm_n}f)(T_i^n) + o\left(\frac{1}{n}\right) \right) \right] \right].
\end{aligned}$$

In the last equality, we use the fact that $L_{nm_n}f(t)$ is uniformly bounded on \mathcal{T} (see Lemma 5.3) so that the $o(1/n)$ terms in the Taylor expansions of the logarithm are uniform. We have also $(L_{nm_n}f)(t) \rightarrow Lf(t)$ uniformly on \mathcal{T} from Lemma 5.5 that ensures

$$\left| \frac{1}{m_n} \sum_{i=1}^{m_n} ((L_{nm_n}f)(T_i^n) - Lf(T_i^n)) \right| \leq \sup_{t \in \mathcal{T}} |(L_{nm_n}f)(t) - Lf(t)| \rightarrow 0.$$

Using (3.8) we apply Portmanteau theorem on Lf bounded continuous in order to obtain

$$\frac{1}{m_n} \sum_{i=1}^{m_n} (Lf)(T_i^n) = \int (Lf)(t) m_n^{-1} \Psi_n(dt) \rightarrow \int (Lf)\lambda(dt)$$

in probability. Combining the two results we obtain the convergence in probability

$$\frac{1}{m_n} \sum_{i=1}^{m_n} (L_{nm_n}f)(T_i^n) \rightarrow \int (Lf)(t)\lambda(dt), \quad t \rightarrow \infty.$$

Going back to Equation (5.12),

$$\begin{aligned}
&n \left[1 - \mathbb{E} \left[\exp \left(-\frac{1}{nm_n} \sum_{i=1}^{m_n} (L_{nm_n}f)(T_i^n) + o\left(\frac{1}{n}\right) \right) \right] \right] \\
&= n \left[1 - \mathbb{E} \left[1 - \frac{1}{nm_n} \sum_{i=1}^{m_n} (L_{nm_n}f)(T_i^n) + o\left(\frac{1}{n}\right) \right] \right] \\
&= \mathbb{E} \left[\frac{1}{m_n} \sum_{i=1}^{m_n} (L_{nm_n}f)(T_i^n) + o(1) \right] \\
&\rightarrow \int Lf(t)\lambda(dt),
\end{aligned}$$

where the last step follows from the fact that the term is bracket converges in probability and is uniformly bounded. Since $\int Lf(t)\lambda(dt) = \int_{\mathcal{T} \times \mathcal{X}} (1 - e^{-f(t,x)}) \lambda(dt)\mu(dx)$, this shows Equation (5.11) and concludes the proof. \square

The next two lemmas are used in the proof of Theorem 3.3, case $k \geq 1$

LEMMA 5.6. Assume \mathcal{T} locally compact and let $k \geq 0$. For $f : \mathcal{T} \times \mathcal{X} \rightarrow [0, \infty)$ Lipschitz continuous vanishing on $\mathcal{T} \times B_{0,r}$, the sequence of functions $L_{k+1,n}f$ defined on \mathcal{T}^{k+1} by

$$(L_{k+1,n}f)(t_1, \dots, t_{k+1}) = \int_{(B_{0,r}^c)^{k+1}} e^{-\sum_{1 \leq i \leq k+1} f(t_i, x_i)} \nu_n(dx_1) \cdots \nu_n(dx_{k+1}), \quad n \geq 1,$$

converges uniformly to $L_{k+1}f$ defined by

$$(L_{k+1}f)(t_1, \dots, t_{k+1}) = \int_{(B_{0,r}^c)^{k+1}} e^{-\sum_{1 \leq i \leq k+1} f(t_i, x_i)} \mu(dx_1) \cdots \mu(dx_{k+1}).$$

Furthermore the sequence $L_{k+1,n}f$ is uniformly bounded.

PROOF. When $k = 0$, Lemma 5.6 reduces to Lemma 5.5. The proof of the case $k \geq 1$ follows the same lines and is omitted for the sake of brevity. \square

LEMMA 5.7. Under condition (3.8), the normalized factorial moment measure of order $k + 1$ of Ψ_n satisfies

$$(5.13) \quad m_n^{-(k+1)} M_{k+1}^{\Psi_n} \longrightarrow \lambda^{\otimes(k+1)} \quad \text{in } \mathbb{M}_b(\mathcal{T}^{k+1}) \text{ as } n \rightarrow \infty.$$

PROOF. Condition (3.8) implies the convergence in probability, as $n \rightarrow \infty$,

$$m_n^{-1} \Psi_n(f) = m_n^{-1} \sum_{i=1}^{m_n} f(T_i^n) \longrightarrow \lambda(f) = \int_{\mathcal{T}} f d\lambda$$

for all bounded continuous $f : \mathcal{T} \rightarrow \mathbb{R}$. Considering the tensor product $f = f_1 \otimes \cdots \otimes f_{k+1}$ with bounded continuous functions $f_i : \mathcal{T} \rightarrow \mathbb{R}$, $i = 1, \dots, k + 1$, this implies the convergence in probability

$$m_n^{-(k+1)} \Psi_n^{\otimes(k+1)}(f) = \prod_{\ell=1}^{k+1} m_n^{-1} \sum_{i=1}^{m_n} f_\ell(T_i^n) \longrightarrow \prod_{\ell=1}^{k+1} \int_{\mathcal{T}} f_\ell d\lambda = \int_{\mathcal{T}^{k+1}} f d\lambda^{k+1}.$$

Since f is bounded, the convergence in probability implies the convergence of expectations

$$(5.14) \quad \mathbb{E} \left(m_n^{-(k+1)} \Psi_n^{\otimes(k+1)}(f) \right) \longrightarrow \int_{\mathcal{T}^{k+1}} f d\lambda^{k+1}.$$

On the other hand, the left-hand side of (5.14) can be rewritten as

$$\begin{aligned} \mathbb{E} \left[m_n^{-(k+1)} \Psi_n^{\otimes(k+1)}(f) \right] &= m_n^{-(k+1)} \mathbb{E} \left[\sum_{1 \leq i_1, \dots, i_{k+1} \leq m_n} f(T_{i_1}^n, \dots, T_{i_{k+1}}^n) \right] \\ &= m_n^{-(k+1)} \left(\mathbb{E} \left[\sum_{1 \leq i_1 \neq \dots \neq i_{k+1} \leq m_n} f(T_{i_1}^n, \dots, T_{i_{k+1}}^n) \right] + \text{remainder} \right) \\ (5.15) \quad &= m_n^{-(k+1)} M_{k+1}^{\Psi_n}(f) + O(m_n^{-1}). \end{aligned}$$

On the second line, the remainder collects all the terms with multi-index (i_1, \dots, i_{k+1}) having at least two equal components. The number of such terms is asymptotically equivalent to

$\binom{k+1}{2}m_n^k$ and each term is bounded so that the remainder is $O(m_n^k)$, yielding Equation (5.15). Together with Equation (5.14), this implies

$$m_n^{-(k+1)}M_{k+1}^{\Psi_n}(f) \longrightarrow \int_{\mathcal{T}^{k+1}} f d\lambda^{\otimes(k+1)} \quad \text{as } n \rightarrow \infty.$$

This holds for all test functions $f = f_1 \otimes \cdots \otimes f_{k+1}$, which is enough to prove the announced convergence (5.13). \square

PROOF OF THEOREM 3.3, CASE $k \geq 1$. We fix $r > 0$ such that $\mu(\partial B_{0,r}^c) = 0$ and a bounded continuous function $f : \mathcal{T} \times \mathcal{X} \mapsto [0, \infty)$ vanishing on $F^r = \mathcal{T} \times B_{0,r}$. For $k \geq 1$, we need to prove that

$$\begin{aligned} & n^{k+1} \mathbb{E} \left[e^{-\sum_{i=1}^{m_n} f(t_i, a_{nm_n}^{-1} X_i)} \mathbb{1}_{\{\sum_{i=1}^{m_n} \varepsilon_{a_{nm_n}^{-1} X_i}(B_{0,r}^c) \geq k+1\}} \right] \\ \rightarrow & \frac{1}{(k+1)!} \int_{E^{k+1}} e^{-\sum_{i=1}^{k+1} f(t_i, x_i)} \otimes_{i=1}^{k+1} \mathbb{1}_{\{x_i \in B_{0,r}^c\}} \lambda(dt_i) \mu(dx_i) \\ (5.16) \quad & = \int_{\mathcal{N}} e^{-\pi(f)} \mathbb{1}_{\{\pi(B_{0,r}^c) \geq k+1\}} \mu_{k+1}^*(d\pi). \end{aligned}$$

The proof follows the proof of Theorem 3.2. Due to independent marking, we have

$$n^{k+1} \mathbb{P} \left(\sum_{i=1}^{m_n} \varepsilon_{a_{nm_n}^{-1} X_i}(B_{0,r}^c) \geq k+1 \right) = n^{k+1} \mathbb{P}(S_{m_n} \geq k+1)$$

where S_{m_n} is a random variable with binomial distribution $(m_n, \nu(a_{nm_n} B_{0,r}^c))$. Regular variation implies $\nu(a_{nm_n} B_{0,r}^c) \sim (nm_n)^{-1} \mu(B_{0,r}^c)$, as $n \rightarrow \infty$, so that the binomial probability distribution satisfies

$$\begin{aligned} & n^{k+1} \mathbb{P}(S_{m_n} \geq k+1) \sim n^{k+1} \mathbb{P}(S_{m_n} = k+1) \\ & \sim n^{k+1} \binom{m_n}{k+1} (nm_n)^{k+1} \mu(B_{0,r}^c)^{k+1} \\ (5.17) \quad & \sim \frac{\mu(B_{0,r}^c)^{k+1}}{(k+1)!}, \quad n \rightarrow \infty. \end{aligned}$$

We deduce, as $n \rightarrow \infty$,

$$\begin{aligned} & n^{k+1} \mathbb{E} \left[e^{-\sum_{i=1}^{m_n} f(T_i^n, a_n^{-1} X_i)} \mathbb{1}_{\{\sum_{i=1}^{m_n} \varepsilon_{a_{nm_n}^{-1} X_i}(B_{0,r}^c) \geq k+1\}} \right] \\ & \sim n^{k+1} \mathbb{E} \left[e^{-\sum_{i=1}^{m_n} f(T_i^n, a_{nm_n}^{-1} X_i)} \mathbb{1}_{\{\sum_{i=1}^{m_n} \varepsilon_{a_{nm_n}^{-1} X_i}(B_{0,r}^c) = k+1\}} \right]. \end{aligned}$$

The event $\left\{ \sum_{i=1}^{m_n} \varepsilon_{a_{nm_n}^{-1} X_i}(B_{0,r}^c) = k+1 \right\}$ can be decomposed into an union of $\binom{m_n}{k+1}$ disjoint events indexed by a subset $I \subset \{1, \dots, m_n\}$ of size $k+1$ such that $x_i \in a_{nm_n} B_{0,r}^c$ if $i \in I$ and $x_i \in a_{nm_n} B_{0,r}$ if $i \notin I$. We obtain

$$\begin{aligned} & n^{k+1} \mathbb{E} \left[e^{-\sum_{i=1}^{m_n} f(T_i^n, a_{nm_n}^{-1} X_i)} \mathbb{1}_{\{\sum_{i=1}^{m_n} \varepsilon_{a_{nm_n}^{-1} X_i}(B_{0,r}^c) \geq k+1\}} \right] \\ & = n^{k+1} \nu(a_{nm_n} B_{0,r})^{m_n - k - 1} \sum_{I \subset \{1, \dots, m_n\}} \mathbb{E} \left[e^{-\sum_{i \in I} f(T_i^n, a_{nm_n}^{-1} X_i)} \mathbb{1}_{\{X_i \in a_{nm_n} B_{0,r}^c, i \in I\}} \right]. \end{aligned}$$

Conditionally on $(T_i^n)_{1 \leq i \leq m_n}$, we have

$$\begin{aligned} & (nm_n)^{k+1} \mathbb{E} \left[e^{-\sum_{i \in I} f(T_i^n, a_{nm_n}^{-1} X_i)} \mathbb{1}_{\{X_i \in a_{nm_n} B_{0,r}^c, i \in I\}} \mid (T_i^n)_{1 \leq i \leq m_n} \right] \\ &= (nm_n)^{k+1} \int_{(a_{nm_n} B_{0,r}^c)^{k+1}} e^{-\sum_{i \in I} f(T_i^n, a_{nm_n}^{-1} x_i)} \nu(dx_{i_1}) \cdots \nu(dx_{i_{k+1}}) \\ &= \int_{(B_{0,r}^c)^{k+1}} e^{-\sum_{i \in I} f(T_i^n, x_i)} \nu_{nm_n}(dx_{i_1}) \cdots \nu_{nm_n}(dx_{i_{k+1}}). \end{aligned}$$

As $n \rightarrow \infty$ we obtain the equivalence

$$\begin{aligned} & n^{k+1} \mathbb{E} \left[e^{-\sum_{i=1}^{m_n} f(T_i^n, a_{nm_n}^{-1} X_i)} \mathbb{1}_{\{\sum_{i=1}^{m_n} \varepsilon_{a_{nm_n} X_i}^{-1}(B_{0,r}^c) \geq k+1\}} \right] \\ & \sim \frac{1}{m_n^{k+1}} \frac{1}{(k+1)!} \sum_{1 \leq i_1 \neq \dots \neq i_{k+1} \leq m_n} \mathbb{E}[(L_{nm_n, k+1} f)(T_{i_1}^n, \dots, T_{i_{k+1}}^n)] \\ & \sim \frac{1}{(k+1)!} \int (L_{nm_n, k+1} f)(t) m_n^{-(k+1)} M_{k+1}^{\Psi_n}(dt). \end{aligned}$$

By Lemma 5.7, $m_n^{-(k+1)} M_{k+1}^{\Psi_n} \rightarrow \lambda^{\otimes(k+1)}$ in $\mathbb{M}_b(\mathcal{T}^{k+1})$. The desired result follows thanks to dominated convergence. \square

6. Proofs related to Section 4.

6.1. *Proofs related to Section 4.1.* The following lemma gives an explicit expression for the distance in the Skorokhod space $\mathbb{D} = \mathbb{D}([0, T], \mathbb{R})$ to the cone \mathbb{D}_k of functions with at most k discontinuity points. We first introduce some notations. A càd-làg function $x \in \mathbb{D}$ has at most countably many discontinuity points $(t_i)_{i \in I}$ with size $|x(t_i) - x(t_i^-)|$ and for every $\varepsilon > 0$, the number of jumps with size larger than ε is finite - for a discontinuity point t_i , $i \in I$, the notation $x(t_i^-)$ stands for the left limit of x at t_i . Reordering the sequence of jump sizes $|x(t_i) - x(t_i^-)|$, $i \in I$, we define the non-negative sequence

$$\Delta_1(x) \geq \Delta_2(x) \geq \Delta_3(x) \geq \dots$$

That is $\Delta_1(x)$ is the largest jump (in absolute value), $\Delta_2(x)$ the second largest jump, etc. If x has finitely many jumps, say $k \geq 0$, we set $\Delta_m(x) = 0$ for $m \geq k+1$. It follows that $x \in \mathbb{D}_k$ if and only if $\Delta_{k+1}(x) = 0$.

LEMMA 6.1. *For all $k \geq 0$ and $x \in \mathbb{D}$, $d(x, \mathbb{D}_k) = \frac{1}{2} \Delta_{k+1}(x)$.*

PROOF. We first consider the case $k = 0$ when \mathbb{D}_0 is the space of continuous functions. For $x \in \mathbb{D}$ and $y \in \mathbb{D}_0$, we prove that $d(x, y) \geq \frac{1}{2} \Delta_1(x)$. Indeed, since y is continuous, $d(x, y)$ is equal to the uniform distance $\|x - y\|_\infty$, with $\|\cdot\|_\infty$ the uniform norm for bounded functions on $[0, T]$. Considering $t_1 \in [0, T]$ such that $\Delta_1(x) = |x(t_1) - x(t_1^-)|$, we get

$$d(x, y) = \|x - y\|_\infty \geq \max(|x(t_1) - y(t_1)|, |x(t_1^-) - y(t_1^-)|) \geq \frac{1}{2} \Delta_1(x).$$

Since $y \in \mathbb{D}_0$ is arbitrary, we deduce $d(x, \mathbb{D}_0) \geq \frac{1}{2} \Delta_1(x)$.

For the reverse inequality, it is enough to exhibit a sequence of continuous functions y_n such that

$$(6.1) \quad d(x, y_n) = \|x - y_n\|_\infty \rightarrow \frac{1}{2}\Delta_1(x).$$

A simple construction is *via* convolution: define $y_n = x * f_n$ where $*$ is the convolution operator and f_n is the density of the uniform distribution on $[-1/n, 1/n]$ (one should more precisely first extend the definition of x by letting $x(t) = x(0)$ for $t \leq 0$ and $x(t) = x(T)$ for $t \geq T$). Then, Equation (6.1) is satisfied, proving $d(x, \mathbb{D}_0) \leq \frac{1}{2}\Delta_1(x)$.

We next consider the case $k \geq 1$. Any function $y \in \mathbb{D}_k$ can be decomposed as $y = j + c$ where j is a pure jump function with at most k jumps and c is a continuous function. It follows

$$d(x, y) = d(x, j + c) = d(x - j, c) = \frac{1}{2}\Delta_1(x - j),$$

where the last equality relies on the case $k = 0$. Since j has at most k jumps, the functions x and $x - j$ share the same discontinuity points except at most k of them. This implies $\Delta_1(x - j) \geq \Delta_{k+1}(x)$ with equality if j kills the k largest jumps of x . Hence

$$d(x, y) = \frac{1}{2}\Delta_1(x - j) \geq \frac{1}{2}\Delta_{k+1}(x)$$

and, taking the infimum for $y \in \mathbb{D}_k$, $d(x, \mathbb{D}_k) \geq \frac{1}{2}\Delta_{k+1}(x)$. The reverse inequality is proven taking $y = c + j$ with j killing exactly the k largest jumps of x so that

$$d(x, \mathbb{D}_k) \leq d(x, y) = \frac{1}{2}\Delta_1(x - j) = \frac{1}{2}\Delta_{k+1}(x).$$

□

PROOF OF THEOREM 4.1. Consider the rescaled risk process

$$R_n^0(t) = \sum_{i=1}^N a_n^{-1} X_i \mathbf{1}_{\{T_i \leq t\}}, \quad t \in [0, T],$$

and, for $\delta > 0$, the truncated rescaled risk process

$$(6.2) \quad R_n^\delta(t) = \sum_{i=1}^N a_n^{-1} X_i \mathbf{1}_{\{a_n^{-1} X_i > \delta\}} \mathbf{1}_{\{T_i \leq t\}}, \quad t \in [0, T].$$

The proof involves the following three steps, corresponding to conditions *i*) – *iii*) of Proposition 2.4:

- 1) Using the continuous mapping theorem (Lindskog et al., 2014, Theorem 2.3), we show that $n^{k+1}\mathbb{P}(R_n^\delta \in \cdot) \rightarrow \hat{\mu}_{k+1}^\delta(\cdot)$ in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_k)$, with limit measure

$$\mu_{k+1}^{\#\delta}(B) = \int_{E^{k+1}} \mathbf{1}_{\{\sum_{i=1}^{k+1} z_{(t_i, x_i)} \in B\}} M_{k+1}(dt_1, \dots, dt_{k+1}) \otimes_{i=1}^{k+1} \mathbf{1}_{\{x_i > \delta\}} \mu(dx_i),$$

where $z_{(t,x)} = (x \mathbf{1}_{\{t \leq u\}})_{0 \leq u \leq T} \in \mathbb{D}$.

- 2) We prove that $\mu_{k+1}^{\#\delta} \rightarrow \mu_{k+1}^\#$ in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_k)$ as $\delta \rightarrow 0$.

3) We prove that R_n^δ and R_n^0 satisfy, for any $\varepsilon, r > 0$,

$$(6.3) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} n^{k+1} \mathbb{P}(d(R_n^0, R_n^\delta) > \varepsilon, d(R_n^0, \mathbb{D}_k) > r) = 0$$

where d denotes the Skorokhod metric on \mathbb{D} .

Both conditions iii) of Proposition 2.4 hold under the only condition (6.3) because $d(R_n^\delta, \mathbb{D}_k) \leq d(R_n^0, \mathbb{D}_k)$. Then the result $n^{k+1} \mathbb{P}(a_n^{-1} R \in \cdot) \rightarrow \mu_{k+1}^\#(\cdot)$ in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_k)$ follows from Proposition 2.4 with $E = \mathbb{D}$, $F = \mathbb{D}_k$, $X = R = a_n R_n^0$, $X_{n,m} = a_n R_n^\delta$ and $m = [1/\delta]$.

Step 1. For $\delta > 0$, we have $R_n^\delta = T_\delta(a_n^{-1} \Pi)$ with $T_\delta : \mathcal{N} \rightarrow \mathbb{D}$ the measurable mapping defined by

$$T_\delta : \pi = \sum_{i=1}^N \varepsilon_{(t_i, x_i)} \mapsto \sum_{i=1}^N \mathbb{1}_{\{x_i > \delta\}} z_{(t_i, x_i)}.$$

Theorem 3.2 together with the continuous mapping theorem (Lindskog et al., 2014, Theorem 2.3) imply

$$n^{k+1} \mathbb{P}(R_n^\delta \in \cdot) = n^{k+1} \mathbb{P}(T_\delta(a_n^{-1} \Pi) \in \cdot) \xrightarrow{\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_k)} \mu_{k+1}^* \circ T_\delta^{-1}(\cdot) = \mu_{k+1}^{\#\delta}(\cdot).$$

It remains to check that the conditions for the continuous mapping theorem are satisfied. First, note that $T_\delta^{-1}(\mathbb{D}_k) \subset \mathcal{N}_k$ and that $T_\delta^{-1}(B)$ is bounded away from \mathcal{N}_k for all $B \in \mathcal{B}(\mathbb{D})$ bounded away from \mathbb{D}_k . Besides, T_δ is continuous at every point π such that $\pi([0, T] \times \{\delta\}) = 0$, which can be proved with similar arguments as in the proof of Lemma 3.2 in Eyi-Minko and Dombry (2016). It is easily seen that μ_{k+1}^* has no mass on $\{\pi([0, T] \times \{\delta\}) \neq 0\}$ so that the discontinuity set of T_δ has vanishing μ_{k+1}^* -measure.

Step 2. It is a straightforward application of the monotone convergence Theorem since the indicator function $\mathbb{1}_{\{x_i > \delta\}}$ converges monotonically to $\mathbb{1}_{\{x_i > 0\}}$ as $\delta \downarrow 0$.

Step 3. The Skorokhod distance between the risk process R_n^0 and its truncated version R_n^δ is upper bounded by

$$d(R_n^\delta, R_n^0) \leq \|R_n^0 - R_n^\delta\|_\infty = \sum_{i=1}^N a_n^{-1} X_i \mathbb{1}_{\{a_n^{-1} X_i \leq \delta\}} \leq \delta N.$$

On the other hand, Lemma 6.1 implies

$$\{d(R_n^0, \mathbb{D}_k) > r\} = \{\Delta_{k+1}(R_n^0) > 2r\} = \{(a_n^{-1} \Pi)([0, T] \times (2r, \infty)) \geq k+1\}.$$

We deduce

$$\mathbb{P}\left(d(R_n^\delta, R_n^0) > \varepsilon, d(R_n^0, \mathbb{D}_k) > r\right) \leq \mathbb{P}\left(\delta N > \varepsilon, (a_n^{-1} \Pi)([0, T] \times (2r, \infty)) \geq k+1\right).$$

Denote $S_{j,n}$ a random variable with binomial distribution with parameter $(j, \nu(2a_n r, \infty))$. Conditioning on $N = j$ and applying Lemma 5.2, we get the upper bound, for $\delta > 0$ small enough so that $\varepsilon/\delta > k+1$,

$$\sum_{j=\varepsilon/\delta}^{\infty} \mathbb{P}(S_{j,n} \geq k+1) \mathbb{P}(N = j) \leq \sum_{j=\varepsilon/\delta}^{\infty} \binom{j}{k+1} \nu(2a_n r, \infty)^{k+1} \mathbb{P}(N = j).$$

The convergence $\nu_n(2r, \infty) \rightarrow \mu(2r, \infty)$ implies the bound $\nu_n(2r, \infty) \leq M$ for some $M > 0$, whence the uniform bound

$$\begin{aligned} n^{k+1} \mathbb{P}\left(d(R_n^\delta, R_n^0) > \varepsilon, d(R_n^0, \mathbb{D}_k) > r\right) &\leq \sum_{j=\varepsilon/\delta}^{\infty} \binom{j}{k+1} j^{k+1} \nu_n(2r, \infty)^{k+1} \mathbb{P}(N = j) \\ &\leq \sum_{j=\varepsilon/\delta}^{\infty} \mathbb{P}(N = j) \frac{M^{k+1}}{(k+1)!} j^{k+1} \\ &\leq \frac{M^{k+1}}{(k+1)!} \mathbb{E}[N^{k+1} \mathbf{1}_{\{N \geq \varepsilon/\delta\}}]. \end{aligned}$$

The assumption that Ψ has a finite factorial measure M_{k+1} implies $\mathbb{E}[N^{k+1}] < \infty$ so that the right-hand side converges to 0 as $\delta \rightarrow 0$, proving Equation (6.3). \square

PROOF OF THEOREM 4.3. The proof is very similar to the proof of Theorem 4.1 with R_n^0 and R_n^δ replaced respectively by

$$\tilde{R}_n^0(t) = \sum_{i=1}^{m_n} a_{nm_n}^{-1} (X_i - c) \mathbf{1}_{\{T_i^n \leq t\}}, \quad t \in [0, T],$$

and

$$R_n^\delta(t) = \sum_{i=1}^{m_n} a_{nm_n}^{-1} X_i \mathbf{1}_{\{a_{nm_n}^{-1} X_i > \delta\}} \mathbf{1}_{\{T_i^n \leq t\}}, \quad t \in [0, T].$$

Steps 1) and 2) are proved exactly in the same way but the proof of Step 3) is more involved in the case of a triangular array. We now prove

$$(6.4) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} n^{k+1} \mathbb{P}(d(\tilde{R}_n^0, R_n^\delta) > \varepsilon, d(\tilde{R}_n^0, \mathbb{D}_k) > r) = 0.$$

Recall that $d(\tilde{R}_n^0, \mathbb{D}_k) = \frac{1}{2} \Delta_{k+1}(\tilde{R}_n^0)$ and note that $\Delta_{k+1}(\tilde{R}_n^0) \leq \Delta_{k+1}(R_n^0) \vee (a_{nm_n}^{-1} c)$. Moreover, the Skorokhod distance between the risk process \tilde{R}_n^0 and R_n^δ is upper bounded by

$$d(R_n^\delta, \tilde{R}_n^0) \leq \|R_n^\delta - \tilde{R}_n^0\|_\infty = \max_{1 \leq t \leq m_n} \left| \sum_{i=1}^t a_{nm_n}^{-1} (X_i \mathbf{1}_{\{a_{nm_n}^{-1} X_i \leq \delta\}} - c) \right|.$$

We deduce

$$\begin{aligned} &n^{k+1} \mathbb{P}\left(d(R_n^\delta, \tilde{R}_n^0) > \varepsilon, d(\tilde{R}_n^0, \mathbb{D}_k) > r\right) \\ &\leq n^{k+1} \mathbb{P}\left(\max_{1 \leq t \leq m_n} \left| \sum_{i=1}^t a_{nm_n}^{-1} (X_i \mathbf{1}_{\{a_{nm_n}^{-1} X_i \leq \delta\}} - c) \right| > \varepsilon, \Delta_{k+1}(R_n^0) \vee (a_{nm_n}^{-1} c) > 2r\right) \\ &= n^{k+1} \mathbb{P}\left(\max_{1 \leq t \leq m_n} \left| \sum_{i=1}^t a_{nm_n}^{-1} (X_i \mathbf{1}_{\{a_{nm_n}^{-1} X_i \leq \delta\}} - c) \right| > \varepsilon, \Delta_{k+1}(R_n^0) > 2r\right). \end{aligned}$$

where the last equality holds for n large enough so that $a_{nm_n}^{-1} c \leq 2r$. Hence, Equation (6.4) is equivalent to

$$(6.5) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} n^{k+1} p_n^\delta = 0$$

with

$$p_n^\delta = \mathbb{P}\left(\max_{1 \leq t \leq m_n} \left| \sum_{i=1}^t a_{nm_n}^{-1} (X_i \mathbb{1}_{\{a_{nm_n}^{-1} X_i \leq \delta\}} - c) \right| > \varepsilon, \Delta_{k+1}(R_n^0) > 2r\right).$$

Note that $\Delta_{k+1}(R_n^0) = a_{nm_n}^{-1} X_{m_n-k:m_n}$. Introduce independent random variables U_1, \dots, U_{m_n} with uniform distribution on $[0, 1]$ and their order statistics $U_{1:m_n} \leq \dots \leq U_{m_n:m_n}$ and denote F^\leftarrow the quantile function. By the inversion method, (X_1, \dots, X_{m_n}) has the same distribution as $(F^\leftarrow(U_1), \dots, F^\leftarrow(U_{m_n}))$. Possibly changing the underlying probability space, we assume without loss of generality that $X_i = F^\leftarrow(U_i)$ and similarly $X_{i:m_n} = F^\leftarrow(U_{i:m_n})$, $1 \leq i \leq m_n$. The conditioning event $\Delta_{k+1}(R_n^0) > 2r$ is then equal to $U_{m_n-k:m_n} > F(2ra_{nm_n})$ and we have

$$p_n^\delta = \mathbb{P}\left(\max_{1 \leq t \leq m_n} \left| \sum_{i=1}^t a_{nm_n}^{-1} (F^\leftarrow(U_i) \mathbb{1}_{\{F^\leftarrow(U_i) \leq a_{nm_n} \delta\}} - c) \right| > \varepsilon, U_{m_n-k:m_n} > F(2ra_{nm_n})\right)$$

For $\delta \leq 2r$, $F^\leftarrow(U_{m_n-k:m_n}) > 2ra_{nm_n}$ implies $F^\leftarrow(U_{m_n+1-i:m_n}) > a_{nm_n} \delta$ for $i = 1, \dots, k+1$ and the terms corresponding to the $k+1$ largest order statistics are equal to $ca_{nm_n}^{-1}$. Since there are at most $k+1$ of them, their contribution is at most $(k+1)ca_{nm_n}^{-1} \rightarrow 0$ and is smaller than $\varepsilon/2$ for large n so that the contributions of the terms corresponding to the $m_n - k - 1$ smallest order statistics must be larger than $\varepsilon/2$. Denote by $\sigma(i)$ the rank of observation i , i.e. $U_{\sigma(i):m_n} = U_i$, $1 \leq i \leq m_n$. There is a unique permutation σ' of $\{1, \dots, m_n - k - 1\}$ such that the $m_n - k - 1$ smallest order statistics appear in the same order in the sequences $(U_i)_{1 \leq i \leq m_n} = (U_{\sigma(i):m_n})_{1 \leq i \leq m_n}$ and $(U_{\sigma'(i):m_n})_{1 \leq i \leq m_n - k - 1}$. We obtain that p_n^δ is bounded from above by

$$\mathbb{P}\left(\max_{1 \leq t \leq m_n - k - 1} \left| \sum_{i=1}^t a_{nm_n}^{-1} (F^\leftarrow(U_{\sigma'(i):m_n}) \mathbb{1}_{\{F^\leftarrow(U_{\sigma'(i):m_n}) \leq a_{nm_n} \delta\}} - c) \right| > \varepsilon/2, U_{m_n-k:m_n} > u_n\right)$$

with $u_n = F(2ra_{nm_n})$. Conditionally on $U_{m_n-k:m_n} = u$, the vector $(U_{i:m_n})_{1 \leq i \leq m_n - k - 1}$ has the same distribution as the order statistic on an independent uniform sample on $[0, u]$ with size $m_n - k - 1$. By exchangeability of (U_1, \dots, U_{m_n}) , the distribution of σ is independent of the order statistics $(U_{i:m_n})_{1 \leq i \leq m_n}$ and uniform on the set of permutations of $\{1, \dots, m_n\}$. It follows that the permutation σ' over $\{1, \dots, m_n - k - 1\}$ is uniform and independent of $(U_{i:m_n})_{1 \leq i \leq m_n - k - 1}$ and hence that $(V_i)_{1 \leq i \leq m_n - k - 1} = (U_{\sigma'(i):m_n - k - 1}/u)_{1 \leq i \leq m_n - k - 1}$ has independent components uniform on $[0, 1]$. We deduce

$$\begin{aligned} & \mathbb{P}\left(\max_{1 \leq t \leq m_n - k - 1} \left| \sum_{i=1}^t a_{nm_n}^{-1} (F^\leftarrow(U_{\sigma'(i):m_n}) \mathbb{1}_{\{F^\leftarrow(U_{\sigma'(i):m_n}) \leq a_{nm_n} \delta\}} - c) \right| > \varepsilon/2 \mid U_{m_n-k:m_n} = u\right) \\ &= \mathbb{P}\left(\max_{1 \leq t \leq m_n - k - 1} \left| \sum_{i=1}^t a_{nm_n}^{-1} (F^\leftarrow(uV_i) \mathbb{1}_{\{F^\leftarrow(uV_i) \leq a_{nm_n} \delta\}} - c) \right| > \varepsilon/2\right) \\ &\leq 3 \max_{1 \leq t \leq m_n - k} \mathbb{P}\left(\left| \sum_{i=1}^t a_{nm_n}^{-1} (F^\leftarrow(uV_i) \mathbb{1}_{\{F^\leftarrow(uV_i) \leq a_{nm_n} \delta\}} - c) \right| > \varepsilon/6\right) \end{aligned}$$

where the last line follows from Etemadi's inequality ([Etemadi, 1985](#)). Integrating with respect to $U_{m_n-k:m_n} > F(2ra_{nm_n})$, we obtain the upper bound

$$p_n^\delta \leq 3\mathbb{E}\left[\pi_n^\delta(U_{m_n-k:m_n}) \mathbb{1}_{\{U_{m_n-k:m_n} > F(2ra_{nm_n})\}}\right]$$

with

$$\pi_n^\delta(u) = \max_{1 \leq t \leq m_n - k} \mathbb{P} \left(\left| \sum_{i=1}^t a_{nm_n}^{-1} (F^{\leftarrow}(uV_i) \mathbb{1}_{\{F^{\leftarrow}(uV_i) \leq a_{nm_n} \delta\}} - c) \right| > \varepsilon/6 \right).$$

In the following, we provide upper bounds for $\pi_n^\delta(u)$ and prove that

$$(6.6) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} n^{k+1} \mathbb{E} \left[\pi_n^\delta(U_{m_n - k : m_n}) \mathbb{1}_{\{U_{m_n - k : m_n} > F(2ra_{nm_n})\}} \right] = 0$$

which implies Equation (6.5). We classically have to distinguish four different cases. In each case, we will use the following Lemma.

LEMMA 6.2. *Let X be a non-negative regularly varying random variable with index $\alpha > 0$ and survival function $\bar{F} = 1 - F$. Then, for any $p > \alpha$, we have*

$$\mathbb{E} \left[\left(\frac{X}{x} \right)^p \mathbb{1}_{\{X \leq x\}} \right] \sim \frac{\alpha}{p - \alpha} \bar{F}(x), \quad x \rightarrow \infty.$$

PROOF. The proof follows from Karamata's theorem; see for instance (Bingham et al., 1989, Theorem 1.6.4), Equation (1.6.3). \square

Proof of Equation (6.6) in the case $\alpha < 1$.

Using Markov inequality, we provide an upper bound for $\pi_n^\delta(u)$ as follows: for any $1 \leq t \leq m_n - k$ and $\varepsilon > 0$, we have

$$\begin{aligned} & \mathbb{P} \left(\left| \sum_{i=1}^t a_{nm_n}^{-1} (F^{\leftarrow}(uV_i) \mathbb{1}_{\{F^{\leftarrow}(uV_i) \leq a_{nm_n} \delta\}} - c) \right| > \varepsilon/6 \right) \\ & \leq 6\varepsilon^{-1} \sum_{i=1}^t \mathbb{E} \left[a_{nm_n}^{-1} F^{\leftarrow}(uV_i) \mathbb{1}_{\{F^{\leftarrow}(uV_i) \leq a_{nm_n} \delta\}} \right] \\ & \leq 6\varepsilon^{-1} m_n \mathbb{E} \left[a_{nm_n}^{-1} F^{\leftarrow}(uV_1) \mathbb{1}_{\{F^{\leftarrow}(uV_1) \leq a_{nm_n} \delta\}} \right]. \end{aligned}$$

We use the fact that there are at most m_n summands as well as the fact that the V_i 's are iid.

Using the change of variable $x = F^{\leftarrow}(uv)$, we have

$$\mathbb{E} \left[F^{\leftarrow}(uV_1) \mathbb{1}_{\{F^{\leftarrow}(uV_1) \leq a_{nm_n} \delta\}} \right] = \int_0^{F^{\leftarrow}(a_{nm_n} \delta)/u} F^{\leftarrow}(uv) dv = u^{-1} \int_0^{a_{nm_n} \delta} x F(dx).$$

We recognize the expression $u^{-1} \mathbb{E}[X_1 \mathbb{1}_{X_1 \leq a_{nm_n} \delta}]$ on which we apply Lemma 6.2 with $p = 1 > \alpha$. We deduce that

$$\pi_n^\delta(u) \leq \frac{6}{\varepsilon u} m_n \delta \mathbb{E} \left[\frac{X_1}{\delta a_{nm_n}} \mathbb{1}_{\{X_1 \leq a_{nm_n} \delta\}} \right] \sim \frac{6}{\varepsilon u} m_n \delta \frac{\alpha}{1 - \alpha} \bar{F}(a_{nm_n} \delta) = o(1),$$

as $m_n \bar{F}(a_{nm_n} \delta) \sim \delta^{-\alpha}/n \rightarrow 0$ as $n \rightarrow \infty$.

Proof of Equation (6.6) in the case $\mathbb{E}[X_1] < \infty$.

When the variable X_1 is integrable, there is a common previous step consisting in centering

the partial sums in Equation (6.6). Let us denote $c_{n,\delta}(u) = \mathbb{E}[F^{\leftarrow}(uV_1)\mathbb{1}_{\{F^{\leftarrow}(uV_1) \leq a_{nm_n}\delta\}}]$ the centering term. Using $c = \mathbb{E}[X_1]$ and $u > F(2ra_{nm_n})$, we obtain

$$\begin{aligned} c_{n,\delta}(u) - c &= u^{-1} \int_0^{F(a_{nm_n}\delta)} F^{\leftarrow}(v)dv - \int_0^1 F^{\leftarrow}(v)dv \\ &= u^{-1} \int_{F(a_{nm_n}\delta)}^1 F^{\leftarrow}(v)dv + (u^{-1} - 1) \int_0^1 F^{\leftarrow}(v)dv. \end{aligned}$$

Using that for n large enough we have $u^{-1} < F(2ra_{nm_n})^{-1} \leq 2$ so that

$$|c_{n,\delta}(u) - c| \leq 2(\mathbb{E}[X_1\mathbb{1}_{X_1 > a_{nm_n}\delta}] + \bar{F}(a_{nm_n}\delta)\mathbb{E}[X_0]) = O(a_{nm_n}\bar{F}(a_{nm_n})), \quad n \rightarrow \infty,$$

by an application of Karamata's theorem. Thus $m_n a_{nm_n}^{-1} |c_{n,\delta} - c| = O(n^{-1})$ uniformly over $u > F(2ra_{nm_n})$ as $n \rightarrow \infty$ and Equation (6.5) is implied by

$$(6.7) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} n^{k+1} \mathbb{E} \left[\tilde{\pi}_n^\delta(U_{m_n-k:m_n}) \mathbb{1}_{\{U_{m_n-k:m_n} > F(2ra_{nm_n})\}} \right] = 0$$

with

$$\tilde{\pi}_n^\delta(u) = \max_{1 \leq t \leq m_n - k} \mathbb{P} \left(\left| \sum_{i=1}^t a_{nm_n}^{-1} (F^{\leftarrow}(uV_i) \mathbb{1}_{\{F^{\leftarrow}(uV_i) \leq a_{nm_n}\delta\}}) - c_{n,\delta}(u) \right| > \varepsilon/7 \right).$$

Proof of Equation (6.7) in the case $1 \leq \alpha < 2$ with $\mathbb{E}[X_1] < \infty$.

Applying Markov inequality of order $p = 2$, we obtain for any $1 \leq t \leq m_n - k$

$$\begin{aligned} &\mathbb{P} \left(\left| \sum_{i=1}^t a_{nm_n}^{-1} (F^{\leftarrow}(uV_i) \mathbb{1}_{\{F^{\leftarrow}(uV_i) \leq a_{nm_n}\delta\}}) - c_{n,\delta}(u) \right| > \varepsilon/7 \right) \\ &\leq 49\varepsilon^{-2} a_{nm_n}^{-2} \mathbb{E} \left[\left(\sum_{i=1}^t (F^{\leftarrow}(uV_i) \mathbb{1}_{\{F^{\leftarrow}(uV_i) \leq a_{nm_n}\delta\}}) - c_{n,\delta}(u) \right)^2 \right] \\ &\leq 49\varepsilon^{-2} a_{nm_n}^{-2} m_n \text{Var} \left(F^{\leftarrow}(uV_1) \mathbb{1}_{\{F^{\leftarrow}(uV_1) \leq a_{nm_n}\delta\}} \right). \end{aligned}$$

Using the change of variable $x = F^{\leftarrow}(uv)$ and Lemma 6.2 with $p = 2 > \alpha$, we have

$$\begin{aligned} \text{Var} \left(F^{\leftarrow}(uV_1) \mathbb{1}_{\{F^{\leftarrow}(uV_1) \leq a_{nm_n}\delta\}} \right) &\leq \int_0^{F(a_{nm_n}\delta)/u} F^{\leftarrow}(uv)^2 dv = u^{-1} \mathbb{E}[X_1^2 \mathbb{1}_{\{X_1 \leq a_{nm_n}\delta\}}] \\ &= O(a_{nm_n}^2 \bar{F}(a_{nm_n})). \end{aligned}$$

We finally get

$$\tilde{\pi}_n^\delta(u) = O(m_n \bar{F}(a_{nm_n})) = O(n^{-1}) = o(1), \quad \text{as } n \rightarrow \infty.$$

Proof of Equation (6.7) in the case $\alpha \geq 2$ with $\text{Var}(X_1) < \infty$.

From the Fuk-Nagaev inequality, see (Petrov, 1995) Equation (2.79) page 78, for any $p > \alpha$,

we have

$$\begin{aligned}
& \mathbb{P}\left(\sum_{i=1}^t a_{nm_n}^{-1} (F^{\leftarrow}(uV_i) \mathbb{1}_{\{F^{\leftarrow}(uV_i) \leq a_{nm_n} \delta\}}) - c > \varepsilon/7\right) \\
& \leq \left(\frac{7(p+2)}{p}\right)^p m_n \mathbb{E}\left[\left|a_{nm_n}^{-1} (F^{\leftarrow}(uV_1) \mathbb{1}_{\{F^{\leftarrow}(uV_1) \leq a_{nm_n} \delta\}})\right|^p\right] \\
& \quad + \exp\left(-\frac{2}{e^p (p+2)^2} \frac{\varepsilon^2}{49m_n \text{Var}(a_{nm_n}^{-1} F^{\leftarrow}(uV_1) \mathbb{1}_{\{F^{\leftarrow}(uV_1) \leq a_{nm_n} \delta\}})}\right) \\
& \leq c_p^1 \varepsilon^{-p} m_n a_{nm_n}^{-p} \mathbb{E}\left[F^{\leftarrow}(uV_1)^p \mathbb{1}_{\{F^{\leftarrow}(uV_1) \leq a_{nm_n} \delta\}}\right] + \exp\left(-c_p^2 \frac{\varepsilon^2 a_{nm_n}^2}{m_n \text{Var}(F^{\leftarrow}(uV_1))}\right) \\
& = A_1 + A_2
\end{aligned}$$

for some constants $c_p^1, c_p^2 > 0$ depending only of the order p . Lemma 6.2 again implies for some $p > \alpha \geq 2$

$$A_1 \sim c_p^1 \varepsilon^{-p} \delta^p m_n \frac{\alpha}{p - \alpha} \bar{F}(a_{nm_n} \delta) = o(1)$$

since $\delta^p m_n \bar{F}(a_{nm_n} \delta) \sim \delta^{p-\alpha}/n \rightarrow 0$ as $n \rightarrow \infty$ for $p > \alpha$. It remains to deal with the second term A_2 . Since $\alpha \geq 2$, under the condition $a_{nm_n}^2/m_n \rightarrow \infty$ then $A_2 \rightarrow 0$, which ends the proof of the case $\alpha \geq 2$ with $\text{Var}(X_1) < \infty$. This proves Equations (6.7) and (6.5) and concludes the proof of Theorem 4.3. \square

6.2. Proofs related to Section 4.2.

PROOF OF PROPOSITION 4.7. The proof involves three steps. We start by proving regular variation of $X_{N-k:N}$ and then we show that the tail distribution of $\sum_{i=1}^{N-k-1} X_{i:N}$ is negligible regarding the tail of $X_{N-k:N}$. In Step 3, we finally show

$$(6.8) \quad \mathbb{P}\left(\sum_{i=1}^{N-k} X_{i:N} > x\right) \underset{x \rightarrow \infty}{\sim} \mathbb{P}(X_{N-k:N} > x).$$

Recall the notations: F^{\leftarrow} denotes the quantile function and $U_{1,n} \leq \dots \leq U_{n,n}$ the order statistics of a iid sample U_1, \dots, U_n uniformly distributed on $[0, 1]$.

Step 1. For $x > 0$, we have

$$\mathbb{P}(X_{N-k:N} > x) = \sum_{n=0}^{\infty} \mathbb{P}(X_{N-k:n} > x) \mathbb{P}(N = n) = \sum_{n=0}^{\infty} \mathbb{P}(S_{n,x} > k) \mathbb{P}(N = n)$$

where $S_{n,x} \sim \mathcal{B}(n, 1 - F(x))$ is a Binomial random variable. Since $\mathbb{P}(S_{n,x} \geq k+1) \underset{x \rightarrow \infty}{\sim} \mathbb{P}(S_{n,x} = k+1)$ and

$$\frac{\mathbb{P}(S_{n,x} = k+1)}{(1 - F(x))^{k+1}} \leq \binom{n}{k+1} = \frac{n^{[k+1]}}{(k+1)!}, \quad x > 0$$

Lebesgue's theorem yields

$$\sum_{n=0}^{\infty} \frac{\mathbb{P}(S_{n,x} > x)}{(1 - F(x))^{k+1}} \mathbb{P}(N = n) \xrightarrow{x \rightarrow \infty} \sum_{n=0}^{\infty} \frac{n^{[k+1]}}{(k+1)!} \mathbb{P}(N = n) = \frac{1}{(k+1)!} \mathbb{E}[N^{[k+1]}].$$

Then,

$$(6.9) \quad \lim_{x \rightarrow \infty} \frac{\mathbb{P}(X_{N-k:N} > x)}{(1 - F(x))^{k+1}} = \frac{1}{(k+1)!} \mathbb{E}[N^{[k+1]}] < \infty$$

under the assumption $\mathbb{E}[N^{k+1}] < \infty$. Since X is regularly varying with index α , Equation (6.9) proves that $X_{N-k:N}$ is regularly varying with index $(k+1)\alpha$.

Step 2. For $x > 0$, we have

$$\mathbb{P}\left(\sum_{i=1}^{N-k-1} X_{i:N} > x\right) = \sum_{n=k}^{\infty} \mathbb{E}\left[\mathbb{P}\left(\sum_{i=1}^{n-k-1} F^{\leftarrow}(U_{i:n}) \geq x \mid U_{n-k:n}\right)\right] \mathbb{P}(N = n).$$

By independence and Markov inequality to the order p , we obtain

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{n-k-1} F^{\leftarrow}(U_{i:n}) \geq x \mid U_{n-k:n} = u\right) &\leq x^{-p} \mathbb{E}\left[\left(\sum_{i=1}^{n-k-1} F^{\leftarrow}(U_{i:n})\right)^p \mid U_{n-k:n} = u\right] \\ &= x^{-p} \mathbb{E}\left[\left(\sum_{i=1}^{n-k-1} F^{\leftarrow}(uU_i)\right)^p\right] \\ &\leq x^{-p} n^p \int_0^1 F^{\leftarrow}(uv)^p dv. \end{aligned}$$

We have for $u \geq 0$

$$\int_0^1 F^{\leftarrow}(uv)^p dv \leq M F^{\leftarrow}(u)^p (1 - u).$$

Integrating with respect to u , it follows that

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{n-k-1} X_{i:n} > x\right) &\leq \mathbb{E}\left[\mathbb{P}\left(\sum_{i=1}^{n-k-1} F^{\leftarrow}(U_{i:n}) \geq x \mid U_{n-k:n}\right)\right] \\ &\leq x^{-p} n^p M \mathbb{E}[F^{\leftarrow}(U_{n-k:n})^p (1 - U_{n-k:n})] \\ &\leq x^{-p} n^p M \mathbb{E}[X_{n-k:n}^p (1 - F(X_{n-k:n}))] \end{aligned}$$

as $F(F^{\leftarrow}(u)) \leq u$ for all $u > 0$. From Step 1, $X_{n-k:n}$ is regularly varying with index $(k+1)\alpha$. Since X is regularly varying with index α , then $\mathbb{E}[X_{n-k:n}^p (1 - F(X_{n-k:n}))] = c < \infty$ for $p < k\alpha$. Integrating now with respect to n , for $p > (k+1)\alpha$ we have

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{N-k-1} X_{i:N} > x\right) &\leq \sum_{n=k}^{\infty} x^{-p} n^p M \mathbb{E}[X_{n-k:n}^p (1 - F(X_{n-k:n}))] \mathbb{P}(N = n) \\ &\leq x^{-p} M c \sum_{n=k+2}^{\infty} n^p \mathbb{P}(N = n) \\ &= o((1 - F(x))^{k+1}) \end{aligned}$$

under the assumption $\mathbb{E}[N^p] < \infty$ for $p > (k+1)\alpha$. This proves $\mathbb{P}\left(\sum_{i=1}^{N-k-1} X_{i:n} > x\right) = o(\mathbb{P}(X_{N-k:N} > x))$ and concludes Step 2.

Step 3. Obviously we have $\mathbb{P}\left(\sum_{i=1}^{N-k} X_{i:N} > x\right) \geq \mathbb{P}(X_{N-k:N} > x)$ for $x > 0$, it remains to prove that

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}\left(\sum_{i=1}^{N-k} X_{i:N} > x\right)}{\mathbb{P}(X_{N-k:N} > x)} \leq 1.$$

For any $\varepsilon > 0$ and $x \geq 0$ it holds

$$\begin{aligned} & \mathbb{P}\left(\sum_{i=1}^{N-k} X_{i:N} > x\right) \\ &= \mathbb{P}\left(\sum_{i=1}^{N-k} X_{i:N} > x, \sum_{i=1}^{N-k-1} X_{i:N} > \varepsilon x\right) + \mathbb{P}\left(\sum_{i=1}^{N-k} X_{i:N} > x, \sum_{i=1}^{N-k-1} X_{i:N} \leq \varepsilon x\right) \\ &\leq \mathbb{P}\left(\sum_{i=1}^{N-k-1} X_{i:N} > \varepsilon x\right) + \mathbb{P}(X_{N-k:N} > (1-\varepsilon)x). \end{aligned}$$

On the one hand, we have

$$(6.10) \quad \frac{\mathbb{P}\left(\sum_{i=1}^{N-k-1} X_{i:N} > \varepsilon x\right)}{\mathbb{P}(X_{N-k:N} > x)} = \frac{\mathbb{P}\left(\sum_{i=1}^{N-k-1} X_{i:N} > \varepsilon x\right)}{\mathbb{P}(X_{N-k:N} > \varepsilon x)} \cdot \frac{\mathbb{P}(X_{N-k:N} > \varepsilon x)}{\mathbb{P}(X_{N-k:N} > x)} \xrightarrow{x \rightarrow \infty} 0$$

since $\mathbb{P}\left(\sum_{i=1}^{N-k-1} X_{i:n} > x\right) = o(\mathbb{P}(X_{N-k:N} > x))$ from Step 2 and since $X_{N-k:N}$ is regularly varying with index $(k+1)\alpha$ from Step 1. On the other hand, regular variation of $X_{N-k:N}$ also yields

$$(6.11) \quad \frac{\mathbb{P}(X_{N-k:N} > (1+\varepsilon)x)}{\mathbb{P}(X_{N-k:N} > \varepsilon x)} \rightarrow (1+\varepsilon)^{-(k+1)\alpha}.$$

Combining Equations (6.10) and (6.11), we obtain

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}\left(\sum_{i=1}^{N-k} X_{i:N} > x\right)}{\mathbb{P}(X_{N-k:N} > x)} \leq (1+\varepsilon)^{-(k+1)\alpha} \xrightarrow{\varepsilon \rightarrow 0} 1.$$

This proves Equation (6.8) and concludes the proof of Proposition 4.7. \square

PROOF OF COROLLARY 4.8. Proposition 4.7 together with Equation 4.5 imply that, for all $\varepsilon > 0$,

$$\begin{aligned} n^{k+1} \mathbb{P}(d(a_n^{-1}R, \mathbb{J}_k) > \varepsilon) &= n^{k+1} \mathbb{P}(a_n^{-1} \sum_{i=1}^{N-k+1} X_{i:N} > \varepsilon) \\ &\sim n^{k+1} \mathbb{P}(a_n^{-1} X_{N+1-k:N} > \varepsilon) = n^{k+1} \mathbb{P}(d(a_n^{-1}R, \mathbb{D}_k) > \varepsilon/2) \\ &\rightarrow \frac{1}{k!} \mathbb{E}[N^{[k]}] \varepsilon^{-\alpha k}. \end{aligned}$$

Since $\{d(a_n^{-1}R, \mathbb{D}_k) > \varepsilon/2\} \subset \{d(a_n^{-1}R, \mathbb{J}_k) > \varepsilon\}$, we deduce

$$(6.12) \quad \begin{aligned} & n^{k+1} \mathbb{P} \left(d(a_n^{-1}R, \mathbb{J}_k) > \varepsilon, d(a_n^{-1}R, \mathbb{D}_k) \leq \varepsilon/2 \right) \\ & = n^{k+1} \left(\mathbb{P} \left(d(a_n^{-1}R, \mathbb{J}_k) > \varepsilon \right) - \mathbb{P} \left(d(a_n^{-1}R, \mathbb{D}_k) > \varepsilon/2 \right) \right) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

We need to prove that, as $n \rightarrow \infty$,

$$(6.13) \quad n^{k+1} \mathbb{E} [f(a_n^{-1}R)] \longrightarrow \int_{\mathbb{D}} f(r) \mu_{k+1}^{\#}(dr)$$

for all continuous bounded functions $f : \mathbb{D} \rightarrow \mathbb{R}$ with support bounded away from \mathbb{J}_k . Let $\varepsilon > 0$ such that f vanishes on an ε -neighborhood of \mathbb{J}_k . We have

$$\begin{aligned} & n^{k+1} \mathbb{E} [f(a_n^{-1}R)] \\ & = n^{k+1} \mathbb{E} \left[f(a_n^{-1}R) \mathbb{1}_{\{d(a_n^{-1}R, \mathbb{D}_k) > \varepsilon/2\}} \right] + n^{k+1} \mathbb{E} \left[f(a_n^{-1}R) \mathbb{1}_{\{d(a_n^{-1}R, \mathbb{J}_k) > \varepsilon, d(a_n^{-1}R, \mathbb{D}_k) \leq \varepsilon/2\}} \right] \\ & = n^{k+1} \mathbb{E} \left[f(a_n^{-1}R) \mathbb{1}_{\{d(a_n^{-1}R, \mathbb{D}_k) > \varepsilon/2\}} \right] + o(1). \end{aligned}$$

The $o(1)$ term is justified by Equation (6.12) and the boundedness of f . Theorem 4.1 Equation (4.2) provides the asymptotic

$$n^{k+1} \mathbb{E} \left[f(a_n^{-1}R) \mathbb{1}_{\{d(a_n^{-1}R, \mathbb{D}_k) > \varepsilon/2\}} \right] \longrightarrow \int_{\mathbb{D}} f(r) \mathbb{1}_{\{d(r, \mathbb{D}_k) > \varepsilon/2\}} \mu_{k+1}^{\#}(dr)$$

because the function $r \in \mathbb{D} \mapsto f(r) \mathbb{1}_{\{d(r, \mathbb{D}_k) > \varepsilon/2\}}$ is bounded and continuous $\mu_{k+1}^{\#}(dr)$ -a.e. with support bounded away from \mathbb{D}_k . Note that, by Lemma 6.1, the discontinuity set is included in $\{r \in \mathbb{D} : \Delta_{k+1}(r) = \varepsilon/2\}$ and that the Lebesgue measure involved in the definition of $\mu_{k+1}^{\#}$ has no atom. Finally, we have

$$\int_{\mathbb{D}} f(r) \mathbb{1}_{\{d(r, \mathbb{D}_k) > \varepsilon/2\}} \mu_{k+1}^{\#}(dr) = \int_{\mathbb{D}} f(r) \mu_{k+1}^{\#}(dr)$$

because $f(r) \mathbb{1}_{\{d(r, \mathbb{D}_k) \leq \varepsilon/2\}} = 0$ $\mu_{k+1}(dr)$ -a.e. This concludes the proof of Equation (6.13) and Corollary 4.8. \square

PROOF OF PROPOSITION 4.9. The regular variation result stated in point *i*) for $R_k^- = \sum_{i=1}^{N-k} X_{i:N}$ follows from Proposition 4.7.

The conditional limit theorem stated in point *ii*) is a consequence of Corollary 4.8 together with Proposition 2.3. Indeed, since $R_k^- = d(R, \mathbb{J}_k)$, we have

$$\mathbb{P}(x^{-1}R \in \cdot \mid R_k^- > x) = \mathbb{P}(x^{-1}R \in \cdot \mid x^{-1}R \in A)$$

with $A = \{r \in \mathbb{D} : d(r, \mathbb{J}_k) > 1\}$. Then, Corollary 4.8 together with Proposition 2.3 imply

$$\mathbb{P}(x^{-1}R \in \cdot \mid x^{-1}R \in A) \xrightarrow{d} \frac{\mu_{k+1}^{\#}(A \cap \cdot)}{\mu_{k+1}^{\#}(A)}$$

because A is bounded away from \mathbb{J}_k and such that $\mu_{k+1}^{\#}(A) = M_{k+1}([0, \infty)^{k+1}) > 0$ and $\mu_{k+1}^{\#}(\partial A) = \mu_{k+1}^{\#}(\{r \in \mathbb{D} : d(r, \mathbb{J}_k) = 1\}) = 0$. Using the definition (4.3) of $\mu_{k+1}^{\#}$, the

expression of the limiting distribution $\mu_{k+1}^\#(A \cap \cdot) / \mu_{k+1}^\#(A)$ is easily deduced as in statement *ii*).

Proof of statement *iii*). The heuristic is that if the residual risk is large at t_0 , then there are at least $k+1$ large claims between 0 and t_0 . If the residual risk is even larger at t_1 , then some large claim must have occurred between t_0 and t_1 , leading to a total of at least $k+2$ large claims between 0 and t_1 . This is formalized as follows. Noting that $R_k^-(t_0) = d(R_k(\cdot \wedge t_0), \mathbb{J}_k)$ and similarly $R_k^-(t_1) = d(R_k(\cdot \wedge t_1), \mathbb{J}_k)$, we consider $A_\varepsilon = \{r \in \mathbb{D} : d(r(\cdot \wedge t_0), \mathbb{J}_k) \in (1, 1 + \varepsilon)\}$ and $B_u = \{r \in \mathbb{D} : d(r(\cdot \wedge t_1), \mathbb{J}_k) > u\}$. With these notations, for $\varepsilon > 0$ and $u > 0$, we have

$$\mathbb{P}(R_k^-(t_1) > ux \mid x < R_k^-(t_0) < (1 + \varepsilon)x) = \frac{\mathbb{P}(x^{-1}R \in A_\varepsilon \cap B_u)}{\mathbb{P}(x^{-1}R \in A_\varepsilon)}.$$

Because A_ε and $A_\varepsilon \cap B_u$ are bounded away from \mathbb{J}_k and \mathbb{J}_{k+1} respectively and because the continuity of M_{k+1} and M_{k+2} ensures that A and B are continuity sets for $\mu_{k+1}^\#$ and $\mu_{k+2}^\#$ respectively, Corollary 4.8 implies

$$\mathbb{P}(x^{-1}R \in A_\varepsilon) \sim \mu_{k+1}^\#(A_\varepsilon)(1 - F(x))^{k+1}$$

and

$$\mathbb{P}(x^{-1}R \in A_\varepsilon \cap B_u) \sim \mu_{k+2}^\#(A_\varepsilon \cap B_u)(1 - F(x))^{k+2}.$$

As a consequence, the quotient satisfies

$$(6.14) \quad \lim_{x \rightarrow \infty} \frac{\mathbb{P}(R_k^-(t_1) > ux \mid x < R_k^-(t_0) < (1 + \varepsilon)x)}{1 - F(x)} = \frac{\mu_{k+2}^\#(A_\varepsilon \cap B_u)}{\mu_{k+1}^\#(A_\varepsilon)}.$$

The quantity $\mu_{k+1}^\#(A_\varepsilon)$ is computed considering $k+1$ claims $((s_i, z_i))_{1 \leq i \leq k+1}$ with occurrence times $s_i \leq t_0$ and magnitudes z_i satisfying $1 < \min_{1 \leq i \leq k+1} z_i < 1 + \varepsilon$, yielding the integral form

$$\begin{aligned} \mu_{k+1}^\#(A_\varepsilon) &= \frac{1}{(k+1)!} \int_{E^{k+1}} \mathbb{1}_{\{1 < \min_{1 \leq i \leq k+1} z_i < 1 + \varepsilon, s_i \leq t_0, 1 \leq i \leq k+1\}} M_{k+1}(ds) \mu^{\otimes(k+1)}(dz) \\ &= \frac{1}{(k+1)!} M_{k+1}([0, t_0]^{k+1}) \int_{(1, \infty)^{k+1}} \mathbb{1}_{\{1 < \min_{1 \leq i \leq k+1} z_i < 1 + \varepsilon\}} \mu^{\otimes(k+1)}(dz). \end{aligned}$$

In the integral, the domain can be restricted to $(1, \infty)^{k+1}$ because the minimum is larger than 1 in the indicator function. Since $1_{z > 1} \mu(dz)$ is the α -Pareto distribution, the integral is equal to

$$\mathbb{P}\left(\min_{1 \leq i \leq k+1} Z_i < 1 + \varepsilon\right) = 1 - (1 + \varepsilon)^{-(k+1)\alpha} \sim (k+1)\alpha\varepsilon \quad \text{as } \varepsilon \rightarrow 0,$$

where $(Z_i)_{1 \leq i \leq k+1}$ are independent random variables with standard α -Pareto distribution. We deduce

$$(6.15) \quad \mu_{k+1}^\#(A_\varepsilon) \sim \frac{1}{k!} M_{k+1}([0, t_0]^{k+1}) \alpha\varepsilon.$$

The quantity $\mu_{k+2}^\#(A_\varepsilon \cap B_u)$ is computed considering $k+2$ claims $((s_i, z_i))_{1 \leq i \leq k+2}$ with $k+1$ of those occurring before t_0 and one between t_0 and t_1 . The s_i are not ordered and there are $k+2$ cases according to which event occurs last. By symmetry, we consider only the case when the last event corresponds to the index $i = k+2$. The magnitudes must then satisfy

$1 < \min_{i \leq i \leq k+1} z_i < 1 + \varepsilon$ so that A_ε is satisfied and $z_{1:k+2} + z_{2:k+2} > u$ so that B_u is satisfied. This yields,

$$(6.16) \quad \mu_{k+2}^\#(A_\varepsilon \cap B_u) = \frac{(k+2)}{(k+2)!} M_{k+2} \left([0, t_0]^{k+1} \times (t_0, t_1] \right) I(\varepsilon)$$

with

$$I(\varepsilon) = \int_{(0, \infty)^{k+2}} \mathbb{1}_{\{1 < \min_{i \leq i \leq k+1} z_i < 1 + \varepsilon, z_{1:k+2} + z_{2:k+2} > u\}} \mu^{\otimes(k+2)}(dz).$$

We claim that, as $\varepsilon \rightarrow 0$,

$$(6.17) \quad I(\varepsilon) \sim \left((u-1)^{-(k+1)\alpha} + ((u-1)^{-\alpha} - 1)_+ \right) (k+1)\alpha\varepsilon.$$

Then, Equation (4.6) follows from Equations (6.14-6.17).

It only remains to prove Equation (6.17). The integral $I(\varepsilon)$ can be decomposed into two integrals $I_1(\varepsilon) + I_2(\varepsilon)$, corresponding to $z_{k+2} > 1$ and $z_{k+2} < 1$ respectively. In the case $z_{k+2} > 1$, the z_i 's are larger than 1 and

$$I_1(\varepsilon) = \mathbb{P} \left(1 < \min_{i \leq i \leq k+1} Z_i < 1 + \varepsilon, Z_{1:k+2} + Z_{2:k+2} > u \right).$$

We decompose $I_1(\varepsilon)$ as a sum of two terms according to $Z_{k+2} < \min_{i \leq i \leq k+1} Z_i$ or $Z_{k+2} > \min_{i \leq i \leq k+1} Z_i$. The first contribution is upper bounded by

$$\mathbb{P} \left(1 < Z_{k+2} < \min_{i \leq i \leq k+1} Z_i < 1 + \varepsilon \right) = o(\varepsilon^2)$$

and is negligible. The second contribution corresponds to

$$\begin{aligned} & \mathbb{P} \left(1 < \min_{i \leq i \leq k+1} Z_i < 1 + \varepsilon, Z_{1:k+2} + Z_{2:k+2} > u, Z_{k+2} > \min_{i \leq i \leq k+1} Z_i \right) \\ &= \mathbb{P} (1 < Z_{1:k+2} < 1 + \varepsilon, Z_{1:k+2} + Z_{2:k+2} > u, R_{k+2} \neq 1) \end{aligned}$$

where R_{k+2} denotes the rank of Z_{k+2} among Z_1, \dots, Z_{k+2} and $(Z_{i:k+2})_{1 \leq i \leq k+2}$ the order statistics. Since R_{k+2} is uniform on $\{1, \dots, k+2\}$ and independent of the order statistics, we obtain

$$\begin{aligned} I_1(\varepsilon) &= \mathbb{P}(1 < Z_{1:k+2} < 1 + \varepsilon, Z_{1:k+2} + Z_{2:k+2} > u, R_{n+2} \neq 1) + o(\varepsilon^2) \\ &= \frac{k+1}{k+2} \mathbb{P}(1 < Z_{1:k+2} < 1 + \varepsilon, Z_{1:k+2} + Z_{2:k+2} > u) + o(\varepsilon^2). \end{aligned}$$

The joint density of $(Z_{1:k+2}, Z_{2:k+2})$ is given by (see for instance [Ahsanullah et al. \(2013\)](#), Chapter 2)

$$(k+1)(k+2)\alpha^2 z_1^{-\alpha-1} z_2^{-(k+1)\alpha-1} \mathbb{1}_{\{1 < z_1 < z_2\}},$$

so that, for $u > 1$,

$$\begin{aligned} & \mathbb{P}(1 < Z_{1:k+2} < 1 + \varepsilon, Z_{1:k+2} + Z_{2:k+2} > u) \\ &= \int_{z_1=1}^{1+\varepsilon} \int_{z_2=(u-z_1) \vee z_1}^{\infty} (k+1)(k+2)\alpha^2 z_1^{-\alpha-1} z_2^{-(k+1)\alpha-1} dz_1 dz_2 \\ &= \int_{z_1=1}^{1+\varepsilon} (k+2)\alpha z_1^{-\alpha-1} ((u-z_1) \vee z_1)^{-(k+1)\alpha} dz_1 \\ &= (k+2)\alpha(u-1)^{-(k+1)\alpha} \varepsilon + o(\varepsilon). \end{aligned}$$

Gathering the previous estimates, we obtain

$$I_1(\varepsilon) \sim (u-1)^{-(k+1)\alpha} (k+1)\alpha\varepsilon.$$

We now consider the contribution $I_2(\varepsilon)$ that corresponds to the case $z_{k+2} < 1$. Then, $z_{1:k+2} = z_{k+2}$ and $z_{2:k+2} = \min_{1 \leq i \leq k+1} z_i$ so that

$$\begin{aligned} I_2(\varepsilon) &= \int_{(1,\infty)^{k+1}} \int_{z_{k+2}=0}^1 \mathbb{1}_{\{1 < \min_{1 \leq i \leq k+1} z_i < 1+\varepsilon, z_{k+2} > u - \min_{1 \leq i \leq k+1} z_i\}} \mu^{\otimes(k+2)}(dz) \\ &= \mathbb{E} \left[\left((u - \min_{1 \leq i \leq k+1} Z_i)^{-\alpha} - 1 \right)_+ \mathbb{1}_{\{1 < \min_{1 \leq i \leq k+1} Z_i < 1+\varepsilon\}} \right] \\ &= ((u-1)^{-\alpha} - 1)_+ (k+1)\alpha\varepsilon + o(\varepsilon). \end{aligned}$$

Note that the main term vanishes when $u \geq 2$ which reflects the fact the residual loss cannot double if $z_{k+2} < \min_{1 \leq i \leq k+1} z_i$. The asymptotic results for $I_1(\varepsilon)$ and $I_2(\varepsilon)$ imply Equation (6.17) and the proof is complete. \square

7. Proofs related to Section 2.

7.1. Proof of Proposition 2.4.

PROOF OF PROPOSITION 2.4. We proceed as in the proof of Theorem 4.2 in Billingsley (1968). Consider $r > 0$ and a μ -continuity set $B \in \mathcal{B}(E \setminus F^r)$. Possibly replacing r by a smaller value, we can assume that $E \setminus F^r$ is a μ -continuity set (Theorem 2.2 (i) in Hult and Lindskog (2006) ensures that all but countably many $r > 0$ have this property). We have

$$n^k \mathbb{P}(a_n^{-1}X \in B) \leq n^k \mathbb{P}(a_n^{-1}X_{n,m} \in B^\varepsilon) + n^k \mathbb{P}(d(a_n^{-1}X_{n,m}, a_n^{-1}X) > \varepsilon, a_n^{-1}X \in B).$$

For $\varepsilon < r/2$, $B^\varepsilon \in \mathcal{B}(E \setminus F^{r/2})$ and Assumption *i*) implies

$$\limsup_{n \rightarrow \infty} n^k \mathbb{P}(a_n^{-1}X_{n,m} \in B^\varepsilon) \leq \mu_m(\text{cl}B^\varepsilon),$$

so that

$$\limsup_{n \rightarrow \infty} \mathbb{P}(a_n^{-1}X \in B) \leq \mu_m(\text{cl}B^\varepsilon) + \limsup_{n \rightarrow \infty} n^k \mathbb{P}(d(a_n^{-1}X_{n,m}, a_n^{-1}X) > \varepsilon, a_n^{-1}X \in B).$$

Letting $m \rightarrow \infty$ in the right-hand side and using assumptions ii) and iii), we deduce,

$$\limsup_{n \rightarrow \infty} n^k \mathbb{P}(a_n^{-1}X \in B) \leq \mu(\text{cl}B^\varepsilon).$$

Since B is a μ -continuity set, letting $\varepsilon \downarrow 0$, we obtain by monotone convergence

$$\limsup_{n \rightarrow \infty} n^k \mathbb{P}(a_n^{-1}X \in B) \leq \mu(\text{cl}B) = \mu(B).$$

In order to obtain a lower bound, we use completion and write

$$n^k \mathbb{P}(a_n^{-1}X \in B) = n^k \mathbb{P}(a_n^{-1}X \in E \setminus F^r) - n^k \mathbb{P}(a_n^{-1}X \in E \setminus F^r \cap B^c).$$

Thanks to the previous bound on the limsup,

$$\begin{aligned} \liminf_{n \rightarrow \infty} n^k \mathbb{P}(a_n^{-1} X \in B) &\geq \liminf_{n \rightarrow \infty} n^k \mathbb{P}(a_n^{-1} X \in E \setminus F^r) - \limsup_{n \rightarrow \infty} n^k \mathbb{P}(a_n^{-1} X \in E \setminus F^r \cap B^c) \\ &\geq \liminf_{n \rightarrow \infty} n^k \mathbb{P}(a_n^{-1} X \in E \setminus F^r) - \mu(E \setminus F^r \cap B^c). \end{aligned}$$

Moreover,

$$\begin{aligned} &\mathbb{P}(a_n^{-1} X_{n,m} \in E \setminus F^{r+\varepsilon}) \\ &= \mathbb{P}(a_n^{-1} X \in F^r, a_n^{-1} X_{n,m} \in E \setminus F^{r+\varepsilon}) + \mathbb{P}(a_n^{-1} X \in E \setminus F^r, a_n^{-1} X_{n,m} \in E \setminus F^{r+\varepsilon}) \\ &\leq \mathbb{P}(d(a_n^{-1} X_{n,m}, a_n^{-1} X) > \varepsilon, a_n^{-1} X_{n,m} \in E \setminus F^{r+\varepsilon}) + \mathbb{P}(a_n^{-1} X \in E \setminus F^r), \end{aligned}$$

whence

$$\mathbb{P}(a_n^{-1} X \in E \setminus F^r) \geq \mathbb{P}(a_n^{-1} X_{n,m} \in E \setminus F^{r+\varepsilon}) - \mathbb{P}(d(a_n^{-1} X_{n,m}, a_n^{-1} X) > \varepsilon, a_n^{-1} X_{n,m} \in E \setminus F^{r+\varepsilon}).$$

Using assumptions i), ii) and iii) with the same reasoning as above, we get

$$\liminf_{n \rightarrow \infty} n^k P(a_n^{-1} X \in E \setminus F^r) \geq \mu(\text{int} E \setminus F^{r+\varepsilon}).$$

Letting $\varepsilon \downarrow 0$, monotone convergence entails

$$\liminf_{n \rightarrow \infty} n^k P(a_n^{-1} X \in E \setminus F^r) \geq \mu(\text{int} E \setminus F^r) = \mu(E \setminus F^r)$$

as $E \setminus F^r$ is a μ -continuity set. Finally, we have obtained

$$\mu(B) \leq \liminf_{n \rightarrow \infty} n^k P(a_n^{-1} X \in B) \leq \limsup_{n \rightarrow \infty} n^k P(a_n^{-1} X \in B) = \mu(B),$$

proving the desired convergence $n^k P(a_n^{-1} X \in \cdot) \rightarrow \mu(\cdot)$ in $\mathbb{M}(E \setminus F)$. \square

7.2. Proof of Theorem 2.5. The proof of Theorem 2.5 requires a good understanding of the distance to the cone \mathcal{N}_k in \mathcal{N} . The following lemma generalizes inequalities (3.3) and (3.4) in [Dombry et al. \(2018\)](#). It characterizes the distance of a point measure $\pi \in \mathcal{N}$ to the cone \mathcal{N}_k . For $\pi = \sum_{i \geq 1} \varepsilon_{(t_i, x_i)}$, we define $\|\pi\|_{k+1}$ the $(k+1)$ -th largest distance within $\{d_{\mathcal{X}}(0, x_i), i \geq 1\}$, with the convention $\|\pi\|_{k+1} = 0$ if π has less than k points.

LEMMA 7.1. *Let ρ be the distance defined in (2.1) and let $\pi \in \mathcal{N}$. Then, for all $k \geq 0$,*

$$\frac{1}{2} (\|\pi\|_{k+1} \wedge 1) \leq \rho(\pi, \mathcal{N}_k) \leq \|\pi\|_{k+1}.$$

PROOF OF LEMMA 7.1. For $\pi \in \mathcal{N}$, let $r_0 = \|\pi\|_{k+1}$ and denote π_0 the restriction of π to $\{(t, x) \in E : d_{\mathcal{X}}(0, x) > r_0\}$. By definition of r_0 , π_0 has at most k points, that is $\pi_0 \in \mathcal{N}_k$. For all $r > r_0$, the restrictions π^r and π_0^r coincide so that $\rho_r(\pi^r, \pi_0^r) = 0$ and

$$\rho(\pi, \pi_0) = \int_0^\infty \{\rho_r(\pi^r, \pi_0^r) \wedge 1\} e^{-r} dr \leq \int_0^{r_0} e^{-r} dr \leq r_0 = \|\pi\|_{k+1}.$$

Since $\pi_0 \in \mathcal{N}_k$, $\rho(\pi, \mathcal{N}_k) \leq \|\pi\|_{k+1}$, proving the right-hand side of the inequality.

On the other hand, for $r < r_0$, the restriction π^r has at least $k + 1$ points. If $\psi \in \mathcal{N}_k$, that is ψ is a point measure on $E \setminus F$ with at most k points, then the restrictions π^r and $\psi^{(r)}$ do not have the same number of points. Then, a straightforward application of Lemma B.1 in [Dombry et al., 2018](#) leads to

$$\rho_r(\pi^r, \psi^r) \geq |\pi(E \setminus F^r) - \psi(E \setminus F^r)| \geq 1.$$

We deduce

$$\rho(\pi, \psi) \geq \int_0^{r_0} e^{-r} dr = 1 - e^{-\|\pi\|_{k+1}} \geq \frac{1}{2} (\|\pi\|_{k+1} \wedge 1)$$

and, $\psi \in \mathcal{N}_k$ being arbitrary,

$$\rho(\pi, \mathcal{N}_k) \geq \frac{1}{2} (\|\pi\|_{k+1} \wedge 1).$$

This proves the left-hand side of the inequality and concludes the proof. \square

PROOF OF THEOREM 2.5. The proof of Theorem 2.5 is very similar to the proof of Theorem A.1 in [Dombry et al. \(2018\)](#), except that Equations (3.3) and (3.4) are replaced here by Lemma 7.1. This implies, in the same way as Equation (A1) in [Dombry et al. \(2018, Theorem A.1\)](#),

$$(7.1) \quad (\mathcal{N} \setminus \mathcal{N}_k^r) \subset \{\pi \in \mathcal{N} : \|\pi\|_{k+1} > r\} \subset (\mathcal{N} \setminus \mathcal{N}_k^{r/2}), \quad r \leq 1.$$

We note also that

$$\{\pi \in \mathcal{N} : \|\pi\|_{k+1} > r\} = \{\pi \in \mathcal{N} : \pi(E \setminus F^r) \geq k + 1\}.$$

According to the Portmanteau Theorem for \mathbb{M} -convergence ([Lindskog et al., 2014, Theorem 2.1](#)), the convergence $\mu_n^* \rightarrow \mu^*$ in $\mathbb{M}(\mathcal{N} \setminus \mathcal{N}_k)$ is equivalent to the convergence of the restrictions $\mu_n^{*r_i} \rightarrow \mu^{*r_i}$ in $\mathbb{M}_b(\mathcal{N} \setminus \mathcal{N}_k^{r_i})$ for each i , for some sequence $r_i \downarrow 0$. Using Equation (7.1), it is more convenient to consider restrictions $\tilde{\mu}_n^{*r_i}, \tilde{\mu}^{*r_i}$ to the subsets $\{\pi(E \setminus F^{r_i}) \geq k + 1\}$ rather than restrictions $\mu_n^{*r_i}, \mu^{*r_i}$ to the subsets $\mathcal{N} \setminus \mathcal{N}_k^{r_i}$. The convergence $\mu_n^* \rightarrow \mu^*$ in $\mathbb{M}(\mathcal{N} \setminus \mathcal{N}_k)$ is hence equivalent to the weak convergence of $\tilde{\mu}_n^{*r_i} \rightarrow \tilde{\mu}^{*r_i}$ for each i , for some sequence $r_i \downarrow 0$. We then appeal to the characterization of weak convergence in terms of finite-dimensional distributions or Laplace functional ([Zhao, 2016, Theorem 3.10 and Corollary 3.11](#)).

In the remainder of the proof, we simply point out some differences with respect to the proof of Theorem A.1 in [Dombry et al. \(2018\)](#).

Proof of (i) \Rightarrow (ii): For $p \geq 1$, let $A_1, \dots, A_p \in \mathcal{B}_\mu^*$ and $m_1, \dots, m_p \in \mathbb{N}$ such that $\sum_{i=1}^p m_i \geq k + 1$. Define the event $\mathcal{A} := \{\pi(A_i) = m_i, 1 \leq i \leq p\}$. For r small enough, $A_1, \dots, A_p \in E \setminus F^r$ and we have

$$\mathcal{A} \subset \{\pi : \pi(E \setminus F^r) \geq k + 1\} \subset \mathcal{N} \setminus \mathcal{N}_k^r,$$

whence $\mu_n^{*r}(\mathcal{A}) = \mu_n^*(\mathcal{A})$ and $\mu^{*r}(\mathcal{A}) = \mu^*(\mathcal{A})$. The convergence $\mu_n^* \rightarrow \mu^*$ in $\mathbb{M}(\mathcal{N} \setminus \mathcal{N}_k)$ implies the weak convergence $\mu_n^{*r} \rightarrow \mu^{*r}$ for small r , which implies according to [Zhao \(2016\)](#) the finite-dimensional convergence

$$\mu_n^*(\mathcal{A}) = \mu_n^{*r}(\mathcal{A}) \longrightarrow \mu^{*r}(\mathcal{A}) = \mu^*(\mathcal{A}).$$

This proves (ii).

Proof of (ii) \Rightarrow (iii): By Lemma 7.1, (ii) implies the finite-dimensional convergence $\tilde{\mu}_n^{*r} \xrightarrow{fidi} \tilde{\mu}^{*r}$ for all $r > 0$ such that $\{\pi \in \mathcal{N} : \pi(E \setminus F^r) \geq k + 1\} \in \mathcal{B}_{\mu^*}$. Let $r_i \rightarrow 0$ be a sequence of such continuity points. From Zhao (2016), convergence of the finite-dimensional distribution is equivalent to convergence of the Laplace functional

$$\int_{\mathcal{N}} e^{-\pi(f)} \tilde{\mu}_n^{*r_i}(\mathrm{d}\pi) \longrightarrow \int_{\mathcal{N}} e^{-\pi(f)} \tilde{\mu}^{*r_i}(\mathrm{d}\pi).$$

This proves (iii).

Proof of (iii) \Rightarrow (i): Similar to Dombry et al. (2018) where the approximating functions can be taken bounded Lipschitz. \square

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