Ordinal Social ranking: simulation for CP-majority rule

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Abstract. We study the problem of how to find a social ranking over individuals given a ranking over coalitions formed by them, or in other words, how to rank individuals based on their ability to influence the strength of a group containing them. We are interested in the use of *ceteris paribus* majority principle for social ranking and extend the results of two previous articles ([4, 3]). We analyse the behavior of the CP-majority rule with respect to Condorcet cycles and propose a linear programming model for the learning its approximation.

1 Introduction

In this article, we are interested in ranking individuals using their performances in different groups/coalitions of individuals. Such a problem can be common in some real world situations, for instance, ranking researchers in a scientific department by taking into account their impact across different working groups, finding the most influential political party regarding different coalitions, or finding the ingredient that makes best meals concerning different association of ingredients.

As it is sometimes hard to express preferences using numbers, we consider only ordinal informations : we suppose that we have an order over coalitions in form of a binary relation, that we call a *power relation*, and we look for a ranking over individuals who form such coalitions, that we call a *social ranking*.

Example 1. *Let's consider four individuals* $N = \{1, 2, 3, 4\}$. *We have the power relation over group's performances :* $1234 \succ 123 \succ 124 \succ 134 \succ 12 \succ 13 \succ 234 \succ 14 \succ 2 \succ 3 \succ 1 \succ 23 \succ 24 \succ 23 \succ 4$. $13 \succ 234$ *means that the group composed by* 1 *and* 3 *is better than the group formed by* 2*,* 3 *and* 4*. Our aim is to find pairwise comparisons between different individuals in order to answer questions such as who is the most influential individual between* 1 *and* 2*? In this article we are specially interested in finding a complete preorder over individuals.*

The problem that we address in this paper is recent. Previous works on it are all related to the axiomatic aspects of the social ranking rules: Bernardi and her colleagues ([1]) axiomatically characterized a social ranking solution based on the idea that the most influential individuals are those appearing more frequently in the highest positions in the ranking of coalitions; Moretti and Ozturk ([4]) presented some impossibility results on a set of three axioms inspired from social choice theory and finally Haret and his colleagues ([3]) showed an axiomatization of a social ranking rule based on the majority principle and analyzed it within some domain restriction conditions.

In this article we are interested in computational aspects of a particular social ranking rule which is the *CP-majority rule*. CP-majority social ranking is based on *ceteris paribus* comparisons between coalitions: looking for a pairwise comparison between i and j, the only information that we use is the comparisons between $S \cup \{i\}$ and $S \cup \{j\}$ where S is a coalition containing neither i nor j.

In [4] authors showed that when a social ranking uses only *ceteris paribus* comparisons, three intuitive axioms can not be verified simultaneously (Independence of irrelevant coalitions, dominance and symmetry) if a social ranking is asked to be transitive. A majority rule based on *ceteris paribus* (CP-majority rule) comparisons verifies these three axioms but does not guarantee the transitivity of the social ranking solution. In [3] authors characterized CP-majority rule (which may provide social ranking with cycles) using three axioms: equality of coalitions, positive responsiveness and neutrality and they presented a domain restriction which guarantees the transitivity of the social ranking. Briefly, CP-majority rule says that i is better than j because there are more coalitions S ($S \in 2^{N \setminus \{i,j\}}$) such that $S \cup \{i\} \succ S \cup \{j\}$. Coalitions S (($S \in 2^{N \setminus \{i,j\}}$)) can be seen as voters for i and j.

This article is based on simulation results on CP-majority rule. After analyzing the probability to have transitive social ranking with CP-majority, we propose an "approximation" of CP-majority rule where a minimum number of coalitions are removed in order to satisfy the transitivity, we call this rule *CP-majority with maximum coalitions*. The article concludes with a learning approach for this last rule based on a linear programming model.

2 Ceteris Paribus majority

2.1 Notations

We have a finite set of individuals $N = \{1, 2, ..., n\}$. We are given a power relation $\succ$ representing a binary relation on the power set 2^N. We suppose $\succ$ transitive and asymmetric (and complete for our simulations). We denote by $B(2^N)$ the set of all possible power relations. $S \succ T$ means that coalition S is preferred to coalition T.

We are looking for a social ranking, denoted by R, which must be a complete preorder (reflexive, transitive and complete). iRj means that i is at least as good as j, with i and j in N. We denote by $\mathcal{T}(N)$ the set of all total preorders on N.

A social ranking solution is a function $R\colon B(2^N) \longrightarrow \mathcal{T}(N)$ associating to each power relation $\succ \in B(2^N)$ a total preorder $R^{\succ}$ over the elements of N. By this definition, the notion $iR^{\succ}j$ means that applying the social ranking solution to the power relation $\succ$ gives the result that i is at least as good as j. We denote the asymmetric part of $R^{\succ}$ by $P^{\succ}$: $iP^{\succ}j \iff iR^{\succ}j$ and not $jR^{\succ}i$ (i is preferred to j). $I^{\succ}$ represents the symmetric part of $R^{\succ}$: $iI^{\succ}j \iff iR^{\succ}j$ and $jR^{\succ}i$ (i is indifferent to j).

2.2 Basic notions

As we mentioned, we are interested in a solution based on the principle of *Ceteris Paribus* (CP), which we can translate to "everything else being equal". Formally, given a power relation $\succ \in B(2^N)$ and two elements $i, j \in N$ we define : $D_{ij}(\succ) = \{S \in 2^{N\setminus\{i,j\}} : S \cup \{i\} \succ S \cup \{j\}\}$. We denote the cardinalities of $D_{ij}(\succ)$ as $d_{ij}^{\succ}$.

Then we apply CP-majority rule : if i is preferred to j over more coalitions than j against i, then $iP^{\succ}j$. If there is no majority, $iI^{\succ}j$

Definition 1 (CP-majority)**.** *Let $\succ \in B(2^N)$. The ceteris paribus majority relation (CP-majority) is the binary relation $R^{\succ} \subseteq N \times N$ such that for all $i, j \in N$:*

$$iR^{\succ}j \Leftrightarrow d_{ij}(\succ) \geq d_{ji}(\succ).$$

Example 2. *Let's consider four individuals $N = \{1, 2, 3, 4\}$. We have the power relation over group's performances : $1234 \succ 123 \succ 124 \succ 134 \succ 12 \succ 13 \succ 234 \succ 14 \succ 2 \succ 3 \succ 1 \succ 23 \succ 24 \succ 23 \succ 4$. We have $1P^{\succ}2$ because $S \cup \{1\}$ win 3 times for $S = \{34\}, \{4\}, \{3\}$ against 1 for $S \cup \{2\}$ (when $S = \emptyset$)*

2.3 Some remarks

1) CP-majority rule considers all coalitions equally important. For instance, the number of individuals within a coalition has no influence on the importance of the coalition.

2) Consider a set of n individuals, then there are $(2^n - 1)!$ possible complete power relations (permutation of all subsets of N except the empty set). If the power relation is a complete order, the number of coalitions being considered for a particular pairwise comparison (between i and j) is $2^{(n-2)}$ since all the subsets of N without i and j can vote. This number is equal to the number of *ceteris paribus* comparisons used for CP-majority. For instance, consider $N = \{1, 2, 3, 4\}$, coalitions which can "vote" (playing the role of voter) for the comparison between 1 and 2 are: $\emptyset$, 3, 4, 34. The number of coalitions being voter for at least one couple i and j is equal to $\sum_{i=0}^{n-2} \binom{n}{i}$. Indeed, all coalitions with a size inferior or equal to $n-2$ can be a voter for at least one couple. Coalitions of the size $n-1$ can not vote because they contain i or j. In example 2, the only coalition that can vote are $\emptyset$, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34. Table 1 illustrates some of these values.

n	Number of possible $\succ$	Number of coalitions S being voter at least for one couple	Number of coalitions S voting for a precise couple
2	6	4	1
3	5040	11	2
4	$\sim 10^{12}$	26	4
5	$\sim 10^{33}$	57	8
6	$\sim 10^{87}$	120	16

Table 1. Number of possible $\succ$ and voting coalitions in function of n

Note that the *ceteris paribus* principle can be seen as an interpretation of our problem in the context of social choice theory, with groups of individuals (coalitions) playing the role of voters: in Example 3, groups 45, 3 and 4 may be seen as voters for the comparison of candidates 1 and 2. Nevertheless, our framework differs from a classical voting scenario in that candidates can also be voters and voters are not identical for different pairwise comparisons : in the comparison $12 \succ 23$, the coalition containing only 2 acts as a voter, while in the comparison $245 \succ 345$, 2 is a candidate. Coalition 45 acts as a voter comparing 2 and 3 but can not be a voter if we want to compare 4 with another individual since it contains 4.

3) CP-majority makes use of limited quantity of information coming from the power relation. For instance, only comparisons over sets having the same size are used.

One of the consequences of such a remark is the fact that different power relations may provide exactly the same information about CP-majority rule if they have exactly the same *ceteris paribus* comparisons. As a result, they provide the same social ranking solution. For instance, power relation $\succ$ such that $1 \succ 2 \succ 3 \succ 12 \succ 13 \succ 23$ and power relation $\sqsupset$ such that $12 \sqsupset 13 \sqsupset 23 \sqsupset 1 \sqsupset 2 \sqsupset 3$ share the same *ceteris paribus* comparisons necessary for CP-majority rule. We call such power relations *CP-equivalent* and say that they share the same *CP-information table*. Table 2 presents the information table of $\succ$ and $\sqsupset$.

1 versus 2	1 versus 3	2 versus 3
$1 \succ 2$ $1 \sqsupset 2$	$1 \succ 3$ $1 \sqsupset 3$	$2 \succ 3$ $2 \sqsupset 3$
$13 \succ 23$ $13 \sqsupset 23$	$12 \succ 23$ $12 \sqsupset 23$	$12 \succ 13$ $12 \sqsupset 13$

Table 2. Information table of $\succ$ and $\sqsupset$

Definition 2 (CP-equivalence)**.** *Let N be a set of individuals and $\succ$ and $\sqsupset$ be two power relations on 2^N. $\succ$ and $\sqsupset$ are CP-equivalent if and only if*

$$\forall S, \forall i, j, S \cup \{i\} \succ S \cup \{j\} \Leftrightarrow S \cup \{i\} \sqsupset S \cup \{j\}$$

The number of different CP-information tables (sets of CP-equivalent power relations) is $\prod_{i=0}^{n} \binom{n}{i}!$ (product of total orders formed by coalitions of the same size). Table 3 shows number of possible CP-information tables. In the rest of our paper, we make use of CP-information tables when we want to simulate different power relations with different social rankings.

n	Number of possible CP-information table
2	2
3	36
4	414720
5	$\sim 10^{16}$
6	$\sim 10^{48}$

Table 3. Number of possible CP-information tables in function of n

Table 7 of annexe shows all the possible CP-information tables for $n = 3$.

4) The *ceteris paribus* majority solution is grounded in intuitive and appealing principles. However, it turns out that strict Condorcet-like cycles are possible for more than two candidates, similarly to classical voting theory as it is shown in the following example.

Example 3. *Consider the following power relation :* $2345 \succ 245 \succ 1234 \succ 13 \succ 12 \succ 23 \succ 145 \succ 35 \succ 24 \succ 14$. *CP-majority implies that* $3P^{\succ}2$ *(since* $13 \succ 12$*),* $2P^{\succ}1$ *(since* $245 \succ 145$, $24 \succ 14$ *and* $13 \succ 23$*), but* $1P^{\succ}3$ *(since* $12 \succ 23$*). So we have a Condorcet cycle.*

Condorcet-like cycles may be a source of difficulty for a choice (how to find the best(s) candidate(s)) or a ranking (how to order candidates) problem. In the following we analyze the probability of having Condorcet-like cycles and propose a modification of the CP-majority rule guaranteeing the transitivity of the social ranking.

3 CP-majority and transitive social ranking

3.1 Probability of Condorcet-like cycles

The aim of this section is to analyze the behavior of the CP-majority rule with respect to transitivity of the social ranking solution. Since we provide some statistics on this question, we consider only power relations in form of total preorders.

We want to know the probability to have one or several Condorcet's winners in function of the number of candidates. We call an individual a *CP-Condorcet winner* if she is at least as good as all the other candidates using the CP-Majorty rule. To find those probabilities we have done simulations for n= 3, 4, 5 and 6. Due to the small amount of possible power relations (or more precisely CP-information tables) for n = 3 and 4, we manage to simulate them all. For n = 5 and 6 we have randomly tested 100 000 different CP-information tables. Our simulations are inspired from a study done in the context of social choice theory where preferences of voters are supposed to be equally chosen (see Impartial cultures, [5]). This hypothesis means that each power relation has the same probability to occur. Note that some domain restriction forbidding some types of power relations may be meaningful in some contexts. For instance, in [3] authors relaxed the axiom of the universality of the domain and analysed the effect of single-peaked-like domain restriction for social ranking. They showed that their definition of single peakness guarantees the transitivity of the social ranking. In this section we do not suggest any restriction in the domain of power relations ($B(2^N)$).

Our results are presented in Figure 1 and Table 4.

The number of voters correspond to the number S of coalitions voting for a precise couple. Having 0 Condorcet winner means that the solution found is not transitive and there is a cycle in the social ranking. We can see that the higher the number of individuals, the lower the probability to have a transitive social order. We also observed that the probability to have several Condorcet winners decreases with the number of individuals.

Note that if we are interested in finding the best individual(s), having a transitive social ranking may not be mandatory since cycles may occur in the bottom of the ranking.

As we already mentioned, our framework has some similarities with voting procedures of social choice theory. Nevertheless, as it is underlined in point 2 of Section 2.3, there are some differences because of the fact that in our framework candidates play the role of voters for other candidates and the voting coalitions are different for each pairwise comparison. Condorcet paradox (having no Condorcet winner) is a classical paradox of social choice theory. We

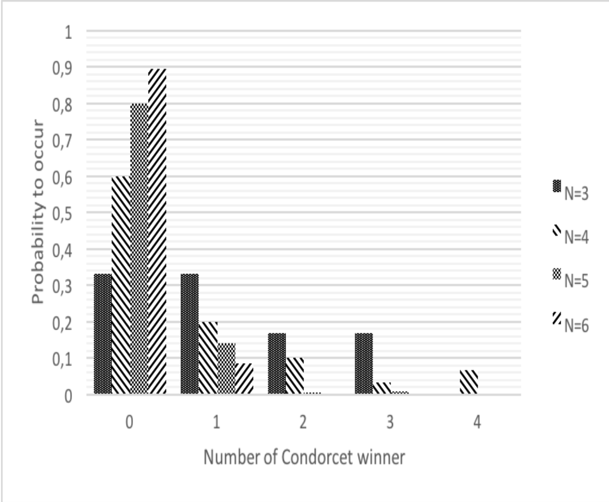

Figure 1. Probability to have one or more Condorcet winners

N	3	4	5	6
Nbr of voters	2	4	8	16
Nbr of different CP-information tables	36 (all)	414 720 (all)	100 000	100 000
% of Condorcet winner	33.33	26.66	13.93	19.5
% of transitive solution	66.66	40	19.5	10.5

Table 4. Probability to have Condorcet winner and a transitive social ranking

compare our results to the probability of the existence of Condorcet winner in Impartial Cultures, found by Gerhlein and Fishburn ([5]). Impartial culture means that each voters have a uniformly distributed probability to vote for each candidates. Figure 2 shows the theoretical results of Gerhlein and Fishburn. Note that in our framework, when there are n candidates, there are $2^{(n-2)}$ coalitions voting for each pairwise comparison.

It is easy to notice that there is a remarkable difference between our results and those found by Gehrlein and Fishburn. Such a gap is due to the amount of indifferences in our social ranking solution $R^{\succ}$. Indeed, the number of voters for $aR^{\succ}b$ being equal to 2^{N-2}, is even and provides indifferences. In fact, the probability to have indifferences in a social ranking is high as shown in the Figure3. The probabilities of this table are found thanks to a simulation : we have simulated 100 000 power relations (more precisely CP-information tables) for each $n = 4$, $n = 5$ and $n = 6$ and calculated the number of indifferences (binary relations in form $iI^{\succ}j$). The maximal number of strict preferences or indifferences is respectively 6, 10, and 15 (if there are n candidates, there are $\frac{n(n-1)}{2}$ possible comparisons). We can see that the numbers of strict preferences follow the same distribution law and leave place to many indifferences. Gehrlein and Fishburn avoided this problem by considering only odd number of voters.

To prevent indifferences, we have decided to do the same simulations than above but without taking into account coalition formed by

Probability of a Simple Majority Winner under IC
for n Voters and m Alternatives

n	m = 3	4	5[a]	6[a]
3	0.94444	0.88889	0.84000	0.79778
5	0.93056	0.86111	0.80048	0.74865
7	0.92498	0.84997	0.78467	0.72908
9	0.92202	0.84405	0.77628	0.71873
11	0.92019	0.84037	0.77108	0.71231
13	0.91893	0.83786	0.76753	0.70794
15	0.91802	0.83604	0.76496	0.70476
17	0.91733	0.83466	0.76300	0.70235
19	0.91678	0.83357	0.76146	0.70046
21	0.91635	0.83269	0.76023	0.69895
23	0.91599	0.83197	0.75921	0.69769
25	0.91568	0.83137	0.75835	0.69664
27	0.91543	0.83085	0.75763	0.69575
29	0.91521	0.83041	0.75700	0.69498
31	0.91501	0.83003	0.75646	0.69431
33	0.91484	0.82969	0.75598	0.69373
35	0.91470	0.82939	0.75556	0.69321
37	0.91456	0.82913	0.75519	0.69275
39	0.91444	0.82889	0.75485	0.69234
41	0.91434	0.82867	0.75455	0.69196
43	0.91424	0.82848	0.75427	0.69162
45	0.91415	0.82830	0.75402	0.69132
47	0.91407	0.82814	0.75379	0.69104
49	0.91399	0.82799	0.75358	0.69078
⋮	⋮	⋮	⋮	⋮
Limit	0.91226	0.82452	0.74869	0.68476

Figure 2. Probability to have a Condorcet Winner [5]

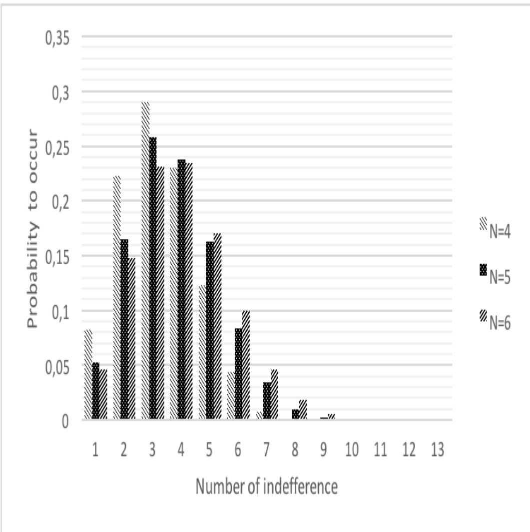

Figure 3. Probability to have indifference

the empty set $S = \emptyset$. In other words, we do not use the comparisons between singletons which are in the power relation[1]. This allows us to have $2^{N-2} - 1$ voters for each comparison making indifferences impossible. The new probabilities are presented in Table 5.

Having an odd number of coalitions voting for each comparison, we obtain very similar probabilities to those of Gehrlein and Fishburn. Even if there are similarities between our framework and social choice the correspondence is not immediate because of the second point of Section 2.3.

[1] Note that $\emptyset$ is the only "coalition" S which is able to compare all couple of candidates, hence its cancellation allows us to have an odd number of voters for every comparisons

N	3	4	5	6
Nbr of voters	1	3	7	15
Nbr of combination tested	6	17280	100 000	100 000
% of Condorcet winner	100	88.89	78.60	70.76
% of transitive solution	100	83.33	58.67	35.96

Table 5. Probability to have Condorcet winner and a transitive social ranking without coalition $S = \{\emptyset\}$

3.2 Removing some coalitions for a transitive social ranking using CP-majority

Simulations of the previous section show that the social ranking solution derived from CP-majority may not be transitive. Nevertheless, when one desires to use *ceteris paribus* comparisons, CP-majority appears as a very intuitive and natural rule. Moreover, as it is shown in [3], CP-majority is the only social ranking rule which satisfies the neutrality, the equality of coalitions and the positive responsiveness. Hence, we thought that it may be interesting to keep the basic principles of CP-majority and to propose an approximation of this rule which guarantees the transitivity.

Our idea is the following: we relax the equality of coalition axiom (all coalitions have the same importance, including $\emptyset$) and try to find the minimum number of coalitions to remove in order to guarantee the transitivity of the social ranking solution by CP-majority rule. We call this new rule *CP-majority with maximum coalitions*.

Example 4. *Consider the following power relation :* $13 \succ 23 \succ 12 \succ 1 \succ 2 \succ 3$. *Applying the CP-majority rule we obtain* $1P^{\succ}2$, $1I^{\succ}3$ *and* $2I^{\succ}3$*, which is not transitive. By removing the coalition* 2 *from the rule, we obtain* $1P^{\succ}3$*, which makes our social order transitive (we have* $1P^{\succ}2I^{\succ}3$*).*

In order to resolve our new problem we make use of linear programming where the objective function maximizes the number of coalitions playing the role of voters and the linear constraints guarantee the transitivity of the social ranking.

The linear programming :
$Max \sum_s S_s$

$$s.t. \begin{cases} \sum_s P_{Sij} \times S_s \geq -M(1 - R_{ij}) & \forall i, j \\ R_{a_1 a_2} + R_{a_2 a_3} + ... + R_{a_{n-1} a_n} - R_{a_k a_{k-1}} < n - 1 \end{cases}$$

* For all $a_1, a_2, ..., a_n$ forming a cycle, for all $k \in \{a_1, a_2, ..., a_n\}$ and for all n.

with:

$(i.)$ Power relation :

$$P_{Sij} = \begin{cases} 1 & \text{if } S \cup \{i\} \succ S \cup \{j\} \\ -1 & \text{if } S \cup \{j\} \succ S \cup \{i\} \end{cases}$$

$(ii.)$ Social ranking :

$$R_{ij} = \begin{cases} 1 & \text{if } iP^{\succ}j \\ 0 & \text{otherwise} \end{cases}$$

$(iii.)$ Decision variables for coalition :

$$S_s = \begin{cases} 1 & \text{if the coalition } S_s \text{ is kept} \\ 0 & \text{otherwise} \end{cases}$$

M is a constant large enough so that the first constraint is satisfied when $R_{ij} = 0$.

Remark that there may be more than one solution satisfying our constraints (for instance the minimum number of coalitions to remove may be 1 with many possibilities, removing coalition $\{1\}$ or coalition $\{23\}$, etc.). Our LP chooses just one solution.

We use our LP in order to do some simulations to have probabilities on the number of coalitions to remove. We have randomly simulated 10 000 power relations for $n = 4$ and $n = 5$ with or without empty set and found the minimum number of coalitions to remove. Results are shown in the following Figure 4 and Table 6.

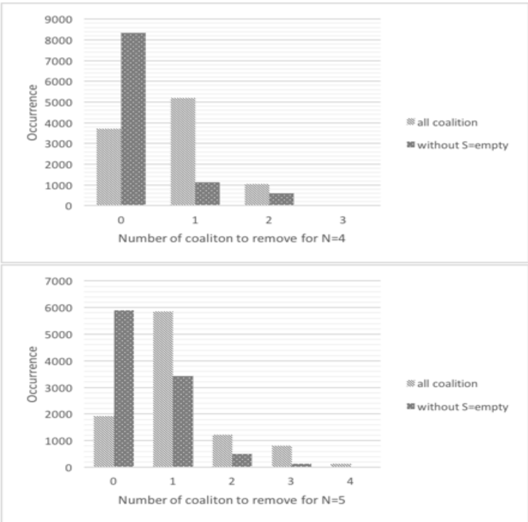

Figure 4. Minimum number of coalitions to remove to have a transitive social order with 10 000 power relations

S	∅		1	2	3	4
probability	40.35		11.13	10.99	11.09	15.18
S	12	13	14	23	24	34
probability	4.20	4.53	4.15	4.60	6.10	5.73

Table 6. Probability for a coalition to be removed (in %), for $N = 4$ (10 000 simulations)

As we can see, most of our solutions can be transitive by removing up to 2 coalitions for $n = 4$ and 4 for $n = 5$. We can also observe that removing the empty $S = \emptyset$ more than double the probability of being a transitive social ranking for $n = 4$ and nearly triple it for $n = 5$. Table 6 shows that we remove more frequently the coalitions that vote for the highest number of couples.

4 Learning a *CP-majority with maximum coalitions*

In this section we make the assumption that power relations and their social rankings are given. Our goal is to find a common sub-rule (which coalition(s) to keep or to remove) based on the CP-majority rule.

Our aim is to see to what extend our LP model is able to find a common CP-majority sub-rule : the same set of coalitions playing the role of voters for power relation+social ranking couples.

We realized three different tests:

- data (power relation + social ranking) resulting from a common CP-majority with maximum coalitions rule (the same coalitions are kept for each power relation) : Subsection 4.1.
- data (power relation + social ranking) resulting from CP-majority with maximum coalitions rules sharing a subset of coalitions (for instance half of the kept coalitions are common to the rules...) : Subsection 4.2.
- data (power relation + social ranking) without any particular CP-majority rule : Subsection 4.3

4.1 Data sharing the same rule

Firstly, we have generated random power relations and applied a unique random sub-rule (CP-majority using a subset of coalitions) to obtain associated social ranking[2]. The unique rule is generated through a function; every coalition has probability p to stay in the rule. If $p = 0.5$ every coalition has 50% of chance to stay in the rule. Then our goal is to find which coalitions have been eliminated, as shown in Figure 5.

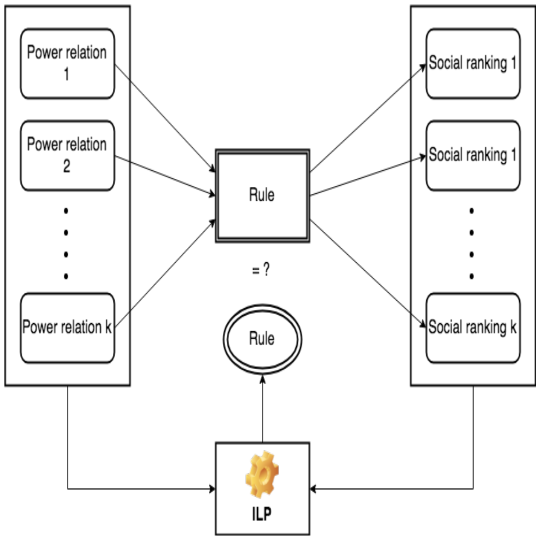

Figure 5. Learning rules

Then we have applied the following ILP and compared our results with the real rule.

$$Max \sum_s S_s$$

$$s.t. \begin{cases} \sum^S P_{Sijk} \times S_s \geq -M(1 - R_{ijk}) \, \forall i, j, k \\ \sum^S P_{Sijk} \times S_s \leq -1 + MR_{ijk} \, \forall i, j, k \end{cases}$$

With:

[2] As we don't modify the rule, we can have social rankings that are not transitive

i) Power relation :

$$P_{Sijk} = \begin{cases} 1 & \text{if } S \cup \{i\} \succ S \cup \{j\} \text{ for the power relation } k \\ -1 & \text{if } S \cup \{j\} \succ S \cup \{i\} \text{ for the power relation } k \end{cases}$$

ii) Social ranking :

$$R_{ijk} = \begin{cases} 1 & \text{if } iP^{\succ}j \text{ in the social ranking } k \\ 0 & \text{otherwise} \end{cases}$$

iii) Decision Variables (common to all power relations!) :

$$S_s = \begin{cases} 1 & \text{if } S_s \text{ is kept} \\ 0 & \text{otherwise} \end{cases}$$

M is a constant large enough so that the first (resp. second) constraint is verified when $R_{ijk} = 0$ (resp. $R_{ijk} = 1$).

We made our simulations on different sizes of training data k. We have made 500 tests for each different number of power relation k used to learn and different p, for $n = 4$. When there is at least one different coalition between the real sub-rule and what we find, we consider it as a defeat. Figure 6 illustrates our results and Figure 7 shows learning time for one rule.

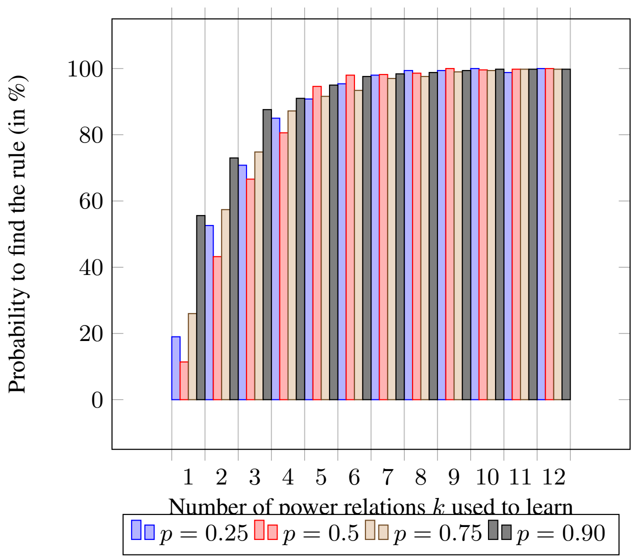

Figure 6. probability to find the CP-majority rule with exact subset of voting coalition for $n = 4$

When our data is very well structured, we can see that our model is more efficient with extreme values for p. It doesn't need a large number of power relations k to have satisfying results; for k greater than 8, it has more than 99% of chance to find the correct sub-rule. The time of processing seems to increase linearly.

4.2 Data sharing the same subset of coalitions

Now, we are interested in introducing noise in our data. The goal is to find a common subset of coalitions that is used by different sub-rules even if there may be other additional coalitions in the sub-rules applied to some of the power relations.

Firstly, we have randomly generated a subset of coalitions that we call *common CP-subset*. Then we have randomly generated

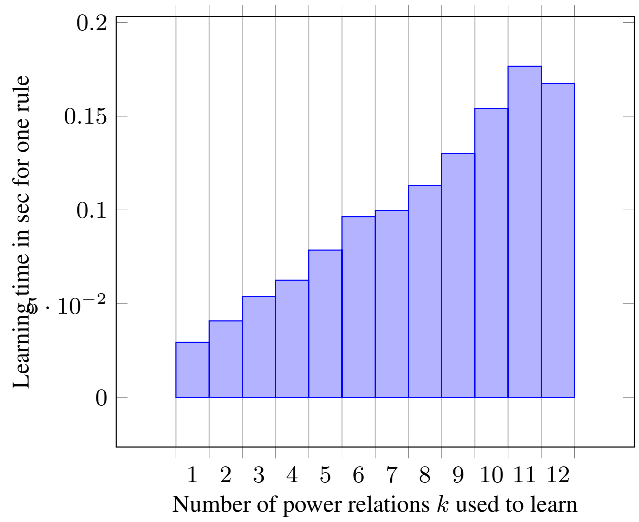

Figure 7. Time to learn a rule (for $n = 4$)

power relations, and we applied the CP-majority rule using *common CP-subset* plus a random coalition to obtain associated social ranking. The random coalition that we add represents "the noise". Our goal is to learn a sub-rule that minimizes the distance between social rankings of the input data and the ones generated by the learned sub-rule. We expect to find a common sub-rule that contains all the elements of the *common CP-subset*.

Example 5. *Let us give a small illustration for 4 individuals : We randomly generate a* common CP-subset *of the size* 4*, for instance,* $1, 3, 12, 34$*. Then we generate 3 random power relations. To create our social rankings (one for each power relation), we use three different sub-rules containing the* common CP-subset *plus one different coalition. The choice of the additional coalitions is random. For instance, we may have at the end three different sub-rules : the first sub-rule with coalitions* $1, 2, 3, 12, 34$*, the second with* $1, 3, 12, 23, 34$ *and the third with* $\emptyset, 1, 3, 12, 34$*. We expect to learn a sub-rule using at least* $1, 3, 12, 34$ *and minimizing the distance of generated social rankings to initial ones.*

To do so, we have used the distance to the Kemeny consensus [2], between the social ranking generated from the rule we want to find (learned sub-rule) and those in the input data. The Kemeny distance is a way to calculate the distance between two rankings. We take all binary relations $iR^{\succ}_{expected}j$ from our expected social rankings and compare them with the learned social rankings $iR^{\succ}_{learned}j$. If $iR^{\succ}_{expected}j \neq iR^{\succ}_{learned}j$ then we increase the distance by one.

Example 6. *Let's have the social ranking* $(1) : 0P^{\succ}_1 1P^{\succ}_1 2P^{\succ}_1 3$*.*
The distance with $(2) : 1P^{\succ}_2 0P^{\succ}_2 2P^{\succ}_2 3$ *is 1 because* $0P^{\succ}_1 1$ *and* $1P^{\succ}_2 0$*.*
The distance with $(3) : 2P^{\succ}_3 1P^{\succ}_3 0P^{\succ}_3 3$ *is 3 because* $0P^{\succ}_1 1$ *and* $1P^{\succ}_3 0$*,* $0P^{\succ}_2 2$ *and* $2P^{\succ}_3 0$ *and* $1P^{\succ}_1 2$ *and* $2P^{\succ}_3 1$*.*

Our goal is to minimize the general Kemeny distance between given social rankings and those we generated with our learned sub-rule. We have used the following ILP.

$Min \sum_{ijk} V_{ijk}$

$$s.t. \begin{cases} \sum_s P_{Sijk} \times S_s \geq -M(1-R_{ijk})\ \forall ijk \\ \sum_S P_{Sijk} \times S_s \leq -1 + MR_{ijk}\ \forall ijk \\ R_{ijk} + R_{jik} = 1\ \forall ijk \\ O_{ijk} - R_{ijk} - V_{ijk} \leq 0\ \forall ijk \\ O_{ijk} - R_{ijk} + V_{ijk} \geq 0\ \forall ijk \end{cases}$$

With:

$i)$ Power relation :

$$P_{Sijk} = \begin{cases} 1 & \text{if } S \cup \{i\} \succ S \cup \{j\} \text{ for the power relation } k \\ -1 & \text{if } S \cup \{j\} \succ S \cup \{i\} \text{ for the power relation } k \end{cases}$$

$ii)$ Objective social ranking :

$$O_{ijk} = \begin{cases} 1 & \text{if } iP^{\succ} j \text{in the social ranking } O_k \\ 0 & \text{otherwise} \end{cases}$$

$iii)$ Decision Variables social ranking :

$$R_{ijk} = \begin{cases} 1 & \text{if } iP \succ j \text{ in the social ranking } k \\ 0 & \text{otherwise} \end{cases}$$

$iv)$ Decision variables for coalition :

$$S_s = \begin{cases} 1 & \text{if } S_s \text{ is kept} \\ 0 & \text{otherwise} \end{cases}$$

$v)$ Decision variables Kemeny distance :

$$V_{ijk} = \begin{cases} 1 & \text{if } R_{ijk} \neq O_{ijk} \\ 0 & \text{otherwise} \end{cases}$$

M is a sufficiently large constant.

We made our simulations on different sizes of training data. We have done 1000 tests for $n = 4$, for different sizes of *common CP-subset*. If the sub-rule obtained contains all the elements of the *common CP-subset* we consider it as a success. Our results are shown in the Figure 8.

As we can see, we are able to learn the *common CP-subset* with few power relations, but it seems that there exists a limit to our learning : we can not succeed in 10% of the cases. The bigger is the *common CP-subset*, the more difficult it is to find it.

We calculated the average number of coalitions in the rule found by our LP on our 1000 test for $n = 4$ (see Figure 8 in the annexe). The average number of coalition found by our LP is close to the number of coalitions that compose the *common CP-subset*.

4.3 Data with different rules (CP-majority with different coalitions)

Now we want to study the case were someone is not consistent in the use of a sub-rule. We have random power relations and social rankings, but social rankings haven't been generated by a specific sub-rule. For simplification, social ranking are transitive and contains no indifferences. As social rankings haven't been obtained through a unique sub-rule, it is very unlikely to find a single sub-rule that satisfies all the transitions from the power relations to the social rankings. For this reason, when a sub-rule is impossible to find we will modify our expected social ranking. The goal is to find a rule that, given the power relations, gives an approximation of the given social rankings as shown in Figure 9.

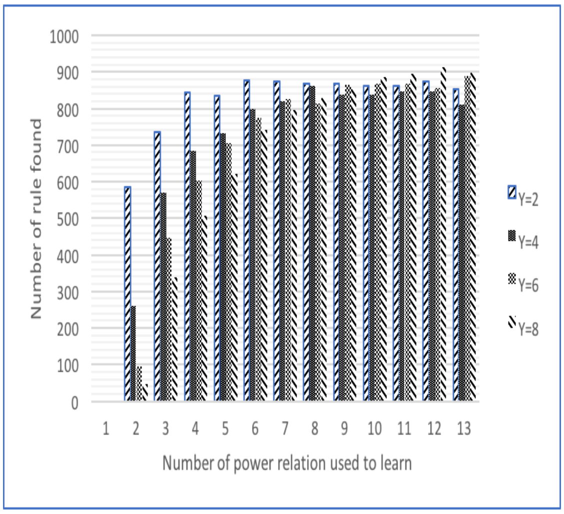

Figure 8. Learning rules with noise for $n = 4$, Y being the number of coalitions shared by all power relations, with 1000 simulations

Example 7. *We generate 3 random power relations for $N = 4$. To create our social rankings (one for each power relation), we use three different sub-rules of size y. For instance for $y = 4$, we can have the first sub-rule: $\emptyset, 4, 12, 34$, the second $1, 3, 23, 34$ and the third $2, 3, 13, 24$. We want to find a sub rule that minimizing the distance of generated social rankings to initial ones.*

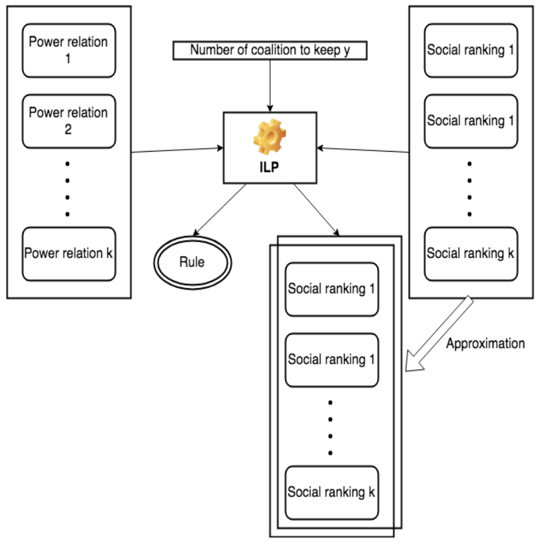

Figure 9. Learning rules with approximation

Our results for $n = 4$ are presented in the Figure 10. We use the same LP model as Subsection 4.2 in order to minimize the Kemeny distance.

As we can see the more coalitions we want to keep, the higher will be the Kemeny distance. For $Y = 8$, the Kemeny distance seems to tend to 3, knowing that for $n = 4$ the maximal distance is 6 (if the social order from our rule is the exact opposite of the one given), our learning model is not efficient. It is also harder to satisfy a lot of

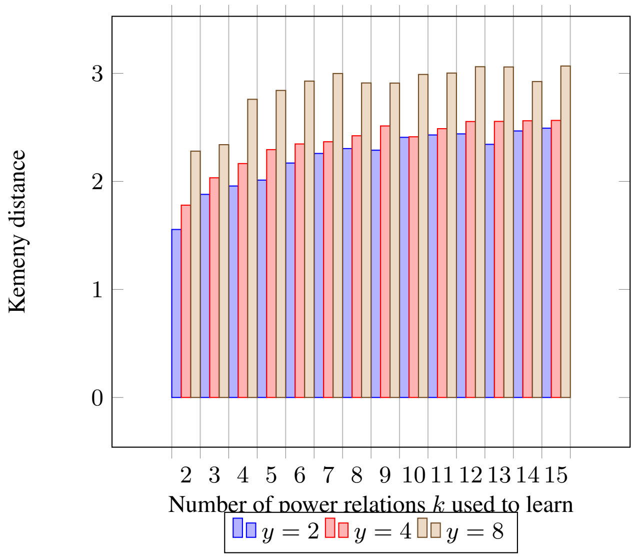

Figure 10. Kemeny distance between the real social ranking and the approximative one for $n = 4$

the social rankings. Such results are not surprising since our results are similar to a random selection when there is no structure in the data.

5 Conclusion and future work

In this paper, we presented some new results on the feasibility and the expected results of the implementation of the CP-Majority principle for social ranking. We analyzed the probability of having Condorcet cycles and presented an LP model in order to have a transitive social ranking as close as possible to a CP-majority rule. We addressed also the learning of a CP-majority like rule using a subset of coalitions as voters. We obtained interesting results for small n. Further simulations must be done with bigger n and different types of data.

Moreover, we only studied complete power relations, but it may be unlikely to happen in real life. An in-depth study of CP-Majority principle on incomplete power relations is another interesting problem to study. Another direction is the analysis of the consequences of a small change in the power relation to our social ranking, similar to a sensibility analysis.

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6 Annexe

Order on 1	Order on 1	$0R1$	$0R2$	$1R2$	Order
$0 \succ 1 \succ 2$	$01 \succ 02 \succ 12$	$0P^{\succ}1$	$0P^{\succ}2$	$1P^{\succ}2$	$0P^{\succ}1P^{\succ}2$
	$01 \succ 12 \succ 02$	$0I^{\succ}1$	$0P^{\succ}2$	$1P^{\succ}2$	$0I^{\succ}1P^{\succ}2$
	$02 \succ 01 \succ 12$	$0P^{\succ}1$	$0P^{\succ}2$	$1I^{\succ}2$	$0P^{\succ}1I^{\succ}2$
	$02 \succ 12 \succ 01$	$0P^{\succ}1$	$0I^{\succ}2$	$1P^{\succ}2$	
	$12 \succ 01 \succ 02$	$0I^{\succ}1$	$0I^{\succ}2$	$1P^{\succ}2$	
	$12 \succ 02 \succ 01$	$0I^{\succ}1$	$0I^{\succ}2$	$1I^{\succ}2$	$0I^{\succ}1I^{\succ}2$
$0 \succ 2 \succ 1$	$01 \succ 02 \succ 12$	$0P^{\succ}1$	$0P^{\succ}2$	$1I^{\succ}2$	$0P^{\succ}1I^{\succ}2$
	$01 \succ 12 \succ 02$	$0I^{\succ}1$	$0I^{\succ}2$	$1P^{\succ}2$	
	$02 \succ 01 \succ 12$	$0P^{\succ}1$	$0P^{\succ}2$	$2P^{\succ}1$	$0P^{\succ}2P^{\succ}1$
	$02 \succ 12 \succ 01$	$0P^{\succ}1$	$0I^{\succ}2$	$2P^{\succ}1$	$0I^{\succ}1P^{\succ}2$
	$12 \succ 01 \succ 02$	$0I^{\succ}1$	$0I^{\succ}2$	$1I^{\succ}2$	$0I^{\succ}1I^{\succ}2$
	$12 \succ 02 \succ 01$	$0I^{\succ}1$	$0I^{\succ}2$	$2P^{\succ}1$	
$1 \succ 0 \succ 2$	$01 \succ 02 \succ 12$	$0I^{\succ}1$	$0P^{\succ}2$	$1P^{\succ}2$	$0I^{\succ}1P^{\succ}2$
	$01 \succ 12 \succ 02$	$1P^{\succ}0$	$0P^{\succ}2$	$1P^{\succ}2$	$1P^{\succ}0P^{\succ}2$
	$02 \succ 01 \succ 12$	$0I^{\succ}1$	$0I^{\succ}2$	$1P^{\succ}2$	
	$02 \succ 12 \succ 01$	$0I^{\succ}1$	$0I^{\succ}2$	$1I^{\succ}2$	$0I^{\succ}1I^{\succ}2$
	$12 \succ 01 \succ 02$	$1P^{\succ}0$	$0I^{\succ}2$	$1P^{\succ}2$	$1P^{\succ}0I^{\succ}2$
	$12 \succ 02 \succ 01$	$0P^{\succ}1$	$0I^{\succ}2$	$1I^{\succ}2$	
$1 \succ 2 \succ 0$	$01 \succ 02 \succ 12$	$0I^{\succ}1$	$0I^{\succ}2$	$1P^{\succ}2$	
	$01 \succ 12 \succ 02$	$1P^{\succ}0$	$0I^{\succ}2$	$1P^{\succ}2$	$1P^{\succ}0I^{\succ}1$
	$02 \succ 01 \succ 12$	$0I^{\succ}1$	$0I^{\succ}2$	$1I^{\succ}2$	$0I^{\succ}I^{\succ}1I^{\succ}2$
	$02 \succ 12 \succ 01$	$0I^{\succ}1$	$2P^{\succ}0$	$1I^{\succ}2$	
	$12 \succ 01 \succ 02$	$1P^{\succ}0$	$2P^{\succ}0$	$1P^{\succ}2$	$1P^{\succ}2P^{\succ}0$
	$12 \succ 02 \succ 01$	$1P^{\succ}0$	$2P^{\succ}0$	$1I^{\succ}2$	$1I^{\succ}2P^{\succ}0$
$2 \succ 0 \succ 1$	$01 \succ 02 \succ 12$	$0P^{\succ}1$	$0I^{\succ}2$	$1I^{\succ}2$	
	$01 \succ 12 \succ 02$	$0I^{\succ}1$	$0I^{\succ}2$	$1I^{\succ}2$	$0I^{\succ}1I^{\succ}2$
	$02 \succ 01 \succ 12$	$0P^{\succ}1$	$0I^{\succ}2$	$2P^{\succ}1$	$0I^{\succ}2P^{\succ}1$
	$02 \succ 12 \succ 01$	$0P^{\succ}1$	$2P^{\succ}0$	$2P^{\succ}1$	$2P^{\succ}0P^{\succ}1$
	$12 \succ 01 \succ 02$	$0I^{\succ}1$	$2P^{\succ}0$	$1I^{\succ}2$	
	$12 \succ 02 \succ 01$	$0I^{\succ}1$	$2P^{\succ}0$	$2P^{\succ}1$	$2P^{\succ}0I^{\succ}1$
$2 \succ 0 \succ 1$	$01 \succ 02 \succ 12$	$0I^{\succ}1$	$0I^{\succ}2$	$1I^{\succ}2$	$0I^{\succ}1I^{\succ}2$
	$01 \succ 12 \succ 02$	$1P^{\succ}0$	$0I^{\succ}2$	$1I^{\succ}2$	
	$02 \succ 01 \succ 12$	$0I^{\succ}1$	$0I^{\succ}2$	$2P^{\succ}1$	
	$02 \succ 12 \succ 01$	$0I^{\succ}1$	$2P^{\succ}0$	$2P^{\succ}1$	$2P^{\succ}0I^{\succ}1$
	$12 \succ 01 \succ 02$	$1P^{\succ}0$	$2P^{\succ}0$	$1I^{\succ}2$	$1I^{\succ}2P^{\succ}0$
	$12 \succ 02 \succ 01$	$1P^{\succ}0$	$2P^{\succ}0$	$2P^{\succ}1$	$2P^{\succ}1P^{\succ}0$

Table 7. CP-information tables for $N = \{0, 1, 2\}$

Number of power relation used to learn	$Y = 2$	$Y = 4$	$Y = 6$	$Y = 8$
2	3.278	4.326	4.481	4.544
3	3.736	5.427	6.619	7.567
4	3.519	5.328	6.918	8.247
5	3.219	5.22	7.05	8.605
6	3.258	5.173	7.055	8.788
7	3.127	5.112	7.016	8.84
8	2.95	5.098	6.94	8.896
9	2.661	4.913	6.973	8.896
10	2.705	4.862	6.936	8.943
11	2.505	4.765	6.847	8.887
12	2.56	4.691	6.831	8.918
13	2.406	4.61	6.806	8.885

Table 8. Average number of coalition in learning rules with noise, Y: the nbr of coalitions shared by all power relations, tested on 1000 simulations