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Numerical Approximation of Optimal Strategies for Impulse Control of Piecewise Deterministic Markov Processes
Application to Maintenance Optimisation

Benoîte de Saporta, François Dufour, Huilong Zhang
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Outline

Introduction
Motivation
Piecewise deterministic Markov processes

Impulse control for PDMPs

Numerical implementation

Conclusion
Maintenance optimization

Equipments

- with several components
- subject to random degradation and failures

Maintenance optimization problem: find some optimal balance between

- repairing/changing components too often
- do nothing and wait for the total failure of the system

Optimize some criterion

- minimize a cost: repair, maintenance, unavailability penalty, failure penalty, ...
- maximize a reward: availability, production, ...
Maintenance optimization

Equipments

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Maintenance optimization problem: find some optimal balance between

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Impulse control problem

**Impulse control**

Select

- intervention dates
- new *starting point* for the process at interventions

*to minimize* a cost function

**Piecewise deterministic Markov processes**

General class of *non-diffusion* dynamic stochastic *hybrid* models: *deterministic* motion punctuated by *random* jumps.

[CD 89], [Davis 93], [dSDZ 14], ...
Introducing Piecewise deterministic Markov processes

Starting point

\[ X_0 = (m, x) \]
Piecewise deterministic Markov processes

$X_t$ follows the deterministic flow until the first jump time $T_1 = S_1$

$$X_t = (m, \phi_m(x, t)),$$  \[ \mathbb{P}_{(m,x)}(T_1 > t) = e^{- \int_0^t \lambda_m(\phi_m(x,s)) \, ds} \]
Introduction Piecewise deterministic Markov processes

Piecewise deterministic Markov processes

Post-jump location \((m_1, x_{T_1})\) selected by the Markov kernel

\[
Q_m(\phi_m(x, T_1), \cdot)
\]
**Piecewise deterministic Markov processes**

\[ X_t \text{ follows the flow until the next jump time } T_2 = T_1 + S_2 \]

\[ X_{T_1+t} = (m_1, \phi_{m_1}(x_{T_1}, t)), \quad t < S_2 \]
Piecewise deterministic Markov processes

Post-jump location \((m_2, x_{T_2})\) selected by Markov kernel

\[ Q_{m_1}(\phi_{m_1}(x_{T_1}, S_2), \cdot) \ldots \]
Embedded Markov chain

{X_t} strong Markov process [Davis 93]

Natural embedded Markov chain

- Z_0 starting point, S_0 = 0, S_1 = T_1
- Z_n new mode and location after n-th jump, S_n = T_n - T_{n-1}, time between two jumps

Proposition

(Z_n, S_n) is a discrete-time Markov chain
Only source of randomness of the PDMP
Mathematical definition of impulse control

Strategy $S = (\tau_n, R_n)_{n \geq 1}$
- $\tau_n$ intervention times
- $R_n$ new positions after intervention

Value function

$$J^S(x) = E^S_x \left[ \int_0^\infty e^{-\alpha s} f(Y_s) ds + \sum_{i=1}^{\infty} e^{-\alpha \tau_i} c(Y_{\tau_i}, Y_{\tau_i^+}) \right]$$

$$\mathcal{V}(x) = \inf_{S \in \mathcal{S}} J^S(x)$$

- $f$, $c$ cost functions, $\alpha$ discount factor
- $Y_t$ controlled process, $\mathcal{S}$ set of admissible strategies
Dynamic programming

Costa, Davis, 1988
For any function $g \geq$ cost of the no-impulse strategy

- $v_0 = g$
- $v_n = L(v_{n-1})$

$v_n(x) \xrightarrow[n \to \infty]{} V(x)$

dS, Dufour, Geeraert, 2017
Construction of $\epsilon$-optimal strategies based on the dynamic programming operator
Dynamic programming

Jump-or-intervention operator

\[
\begin{align*}
v_n(Z_n) &= L(Mv_{n+1}, v_{n+1})(Z_n) \\
&= \left( \inf_{t \leq t^*(Z_n)} \mathbb{E}\left[ F(Z_n, t) + e^{-\alpha S_{n+1}} v_{n+1}(Z_{n+1}) 1\{S_{n+1} < t \land t^*(Z_n)\} \right] \\
&\quad + e^{-\alpha t \land t^*(Z_n)} Mv_{n+1}(\phi(Z_n, t \land t^*(Z_n))) 1\{S_{n+1} \geq t \land t^*(Z_n)\} | Z_n \right) \\
&\quad \land \mathbb{E}\left[ F(Z_n, t^*(Z_n)) + e^{-\alpha S_{n+1}} v_{n+1}(Z_{n+1}) | Z_n \right]
\end{align*}
\]

with

\[
F(x, t) = \int_0^{t \land t^*(x)} e^{-\alpha s - \int_0^s \lambda(\phi(x, u))du} f(\phi(x, s)) ds
\]

\[
Mv_{n+1}(x) = \inf_{y \in U} \left\{ c(x, y) + v_{n+1}(y) \right\}
\]
Approximation scheme - Value function

Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)
Approximation scheme - Value function

Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)

\[
\hat{v}_N(y^i) = g(\hat{Z}^i_0) \\
\hat{v}(\hat{Z}^i_1) = g(\hat{Z}^i_1) \\
\hat{v}(\hat{Z}^i_2) = g(\hat{Z}^i_2) \\
\vdots \\
\hat{v}(\hat{Z}^i_{N-2}) = g(\hat{Z}^i_{N-2}) \\
\hat{v}(\hat{Z}^i_{N-1}) = g(\hat{Z}^i_{N-1}) \\
\hat{v}_N(\hat{Z}_N) = g(\hat{Z}_N)
\]
Approximation scheme - Value function

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Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)

\[
\begin{align*}
\tilde{v}_N(y^i) &= g(\hat{Z}_0^i) \\
\hat{v}_N &= g(\hat{Z}_N) \\
\end{align*}
\]
Approximation scheme - Value function

Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)

\[
\tilde{v}_N(y^i) = g(\hat{Z}_0^i)
\]

\[
\tilde{v}(\hat{Z}_1^i) = g(\hat{Z}_1^i)
\]

\[
\tilde{v}(\hat{Z}_2^i) = g(\hat{Z}_2^i)
\]

\[
\vdots
\]

\[
\tilde{v}_2(y^i)
\]

\[
\tilde{v}_1(y^i)
\]

\[
\tilde{v}_0(x_0)
\]
Approximation scheme - $\epsilon$-optimal strategy

Based on time-dependent discretizations of the state space of $(Z_n, S_n)$
Approximation scheme - $\epsilon$-optimal strategy
Based on time-dependent discretizations of the state space of $(Z_n, S_n)$

\[
\hat{v}_N(y^i) = g(\hat{Z}_i)
\]

\[
\hat{v}(\hat{Z}_1^i) = g(\hat{Z}_1^i) \rightarrow \hat{L}_1 \rightarrow \hat{v}_{N-1}(y^i)
\]

\[
\vdots \quad \vdots \quad \vdots
\]

\[
\hat{v}(\hat{Z}_{N-3}^i) \rightarrow \hat{L}_{N-3} \quad \ldots \quad \hat{L}_1 \rightarrow \hat{v}_{2}(y^i)
\]

\[
\hat{v}(\hat{Z}_1^i) \rightarrow \hat{L}_1 \rightarrow \hat{v}_{1}(y^i)
\]

\[
\hat{v}_1(\hat{Z}_1) \rightarrow \hat{L}_1 \rightarrow \hat{v}_0(x_0)
\]
Approximation scheme - $\epsilon$-optimal strategy

Based on time-dependent discretizations of the state space of $(Z_n, S_n)$

\[ \tilde{v}_N(y^i) = g(\hat{Z}_i^0) \]

\[ \vdots \]

\[ \tilde{v}(\hat{Z}_1^i) = g(\hat{Z}_1^i) \]

\[ L_1^i \rightarrow \tilde{v}_{N-1}(y^i) \]

\[ \vdots \]

\[ \tilde{v}(\hat{Z}_{N-3}^i) \rightarrow L_{N-3}^i \]

\[ \vdots \]

\[ \tilde{v}(\hat{Z}_{N-3}^i) \rightarrow \hat{L}_{N-3} \]

\[ \vdots \]

\[ \tilde{v}_1(\hat{Z}_1^i) \rightarrow \hat{L}_1 \]

\[ \tilde{v}_1(y^i) \]

\[ \vdots \]

\[ \tilde{v}_0(x_0) \]
Approximation scheme - $\epsilon$-optimal strategy

Based on time-dependent discretizations of the state space of $(Z_n, S_n)$
Approximation scheme - $\epsilon$-optimal strategy

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Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)
Approximation scheme - $\epsilon$-optimal strategy

Based on time-dependent discretizations of the state space of $(Z_n, S_n)$

\[
\tilde{v}_N(y^i) = g(\hat{Z}^i_0)
\]
\[
\hat{v}(\hat{Z}^i_1) = g(\hat{Z}^i_1) \xrightarrow{\hat{L}^i_1} \tilde{v}_{N-1}(y^i)
\]
\[
\vdots \quad \vdots \quad \vdots
\]
\[
\hat{v}(\hat{Z}^i_{N-3}) \xrightarrow{\hat{L}^i_{N-3}} \cdots \hat{v}_2(y^i) \xrightarrow{\hat{L}^i_1} \hat{v}_1(y^i)
\]
\[
\hat{v}_1(\hat{Z}_1) \xrightarrow{\hat{L}_1} \hat{v}_0(x_0)
\]
Approximation scheme - $\epsilon$-optimal strategy

Based on time-dependent discretizations of the state space of $(Z_n, S_n)$

\[
\tilde{v}_N(y^i) = g(\hat{Z}^i_0)
\]

\[
\hat{v}(\hat{Z}^i_1) = g(\hat{Z}^i_1) \rightarrow \hat{l}_1 \rightarrow \tilde{v}_{N-1}(y^i)
\]

\[
\vdots \quad \vdots \quad \vdots
\]

\[
\hat{v}(\hat{Z}^i_{N-3}) \rightarrow \hat{l}_{N-3} \rightarrow \tilde{v}_{N-2}(y^i)
\]

\[\vdots\]

\[
\hat{v}(\hat{Z}^i_{N-2}) \rightarrow \hat{l}_{N-2} \rightarrow \tilde{v}_{N-3}(y^i)
\]

\[
\hat{v}(\hat{Z}^i_{N-1}) \rightarrow \hat{l}_{N-1} \rightarrow \tilde{v}_{N-4}(y^i)
\]

\[\vdots\]

\[
\hat{v}(\hat{Z}^i_N) \rightarrow \hat{l}_N \rightarrow \tilde{v}_N(y^i)
\]

\[
\hat{v}(\hat{Z}^i_1) \rightarrow \hat{l}_1 \rightarrow \tilde{v}_1(y^i)
\]

\[
\hat{v}_1(\hat{Z}_1) \rightarrow \hat{l}_1 \rightarrow \tilde{v}_0(x_0)
\]
Equipment model

Typical model with 4 components

- Component 1: 2 states – stable $\xrightarrow{\text{Exponential}}$ failed
- Component 2: 2 states – stable $\xrightarrow{\text{Weibull}}$ failed
- Components 3 and 4: 3 states
  - stable $\xrightarrow{\text{Weibull}}$ degraded $\xrightarrow{\text{Exponential}}$ failed
Maintenance operations

Possible maintenance operations

- All components, all states: do nothing
- Components 1 and 2, all states: change
- Components 3 and 4: change in all states, repair only in stable or degraded states
Criterion to optimize

Minimize the maintenance + unavailability costs

- **unavailability** cost proportional to time spend in **failed** state
- fixed cost for going to the workshop + repair < **change** costs
PDMP model of the equipment

- **Euclidean variables**: 5 time variables
  - functioning time of components 2, 3 and 4
  - calendar time
  - time spent in the workshop

- **Discrete variables**: 225 modes
  - state of the components / maintenance operations
Parameters to tune

- Number of points in the control grid (underlying continuous model)
- Number of points in the quantization grids for \((Z_n, S_n)\)
- Approximation horizon \(N\) such that \(v_N(x) - \mathcal{V}(x)\) small enough \(\simeq\) allowed number of jumps + interventions
- Bounding function \(g\)
- Time discretization step for inf
Step 1: Exact simulation of the PDMP

Implementation of an exact simulator for reference strategies to serve as benchmark

- **Strategy 1**: do nothing
- **Strategy 2**: send equipment to workshop 1 day after failure, change all degraded components, change all failed ones
- **Strategy 3**: send equipment to workshop 1 day after degradation, change all degraded components, change all failed ones

<table>
<thead>
<tr>
<th>Strategy</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<td>Mean cost</td>
<td>19952</td>
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Step 1: Exact simulation of the PDMP

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Step 2 and 3: Discretisation of the control set $U$ and the embedded Markov chain

Finite control set $U$
$\implies$ discretize the functioning times at interventions
$\implies$ project the real times on the grid feasibly

Compromise between precision and computation time

Tests on strategy 3

<table>
<thead>
<tr>
<th>Grid</th>
<th>Number of points</th>
<th>relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3 \times 3 \times 3 \times 5$</td>
<td>246</td>
<td>0.10344</td>
</tr>
<tr>
<td>$4 \times 4 \times 4 \times 5$</td>
<td>331</td>
<td>0.0241</td>
</tr>
<tr>
<td>$5 \times 5 \times 5 \times 5$</td>
<td>592</td>
<td>0.0062</td>
</tr>
<tr>
<td>$3 \times 3 \times 3 \times 11$</td>
<td>615</td>
<td>0.0341</td>
</tr>
<tr>
<td>$4 \times 4 \times 4 \times 11$</td>
<td>923</td>
<td>0.0819</td>
</tr>
<tr>
<td>$5 \times 5 \times 5 \times 11$</td>
<td>1855</td>
<td>0.0186</td>
</tr>
<tr>
<td>$6 \times 6 \times 6 \times 11$</td>
<td>2110</td>
<td>0.0066</td>
</tr>
<tr>
<td>$7 \times 7 \times 7 \times 11$</td>
<td>2617</td>
<td>0.0071</td>
</tr>
<tr>
<td>$8 \times 8 \times 8 \times 11$</td>
<td>3359</td>
<td>0.0066</td>
</tr>
<tr>
<td>$3 \times 3 \times 3 \times 21$</td>
<td>1230</td>
<td>0.0034</td>
</tr>
<tr>
<td>$4 \times 4 \times 4 \times 21$</td>
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<td>0.0170</td>
</tr>
<tr>
<td>$5 \times 5 \times 5 \times 21$</td>
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<td>$6 \times 6 \times 6 \times 21$</td>
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<td>0.0065</td>
</tr>
<tr>
<td>$7 \times 7 \times 7 \times 21$</td>
<td>5536</td>
<td>0.0059</td>
</tr>
<tr>
<td>$8 \times 8 \times 8 \times 21$</td>
<td>7111</td>
<td>0.0047</td>
</tr>
</tbody>
</table>
Step 4: Calibrating $N$ the number of allowed jumps + interventions

Horizon $N$ (number of iterations)

- 5 for Strategy 1
- up to 30 for Strategy 2 (mean 6)
- up to 25 for Strategy 3 (mean 6)
Step 5: Approximation of the value function

<table>
<thead>
<tr>
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<th>Strategy</th>
<th>Approx. Value function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>7076</td>
</tr>
<tr>
<td>19952</td>
<td>11389</td>
<td>8477</td>
<td></td>
</tr>
</tbody>
</table>

- Relative gain of 19.8% vs Strategy 5
Step 6: Optimally controlled trajectories

<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>7076</td>
<td>6733</td>
</tr>
<tr>
<td>19952</td>
<td>11389</td>
<td>8477</td>
<td>7076</td>
<td>6733</td>
</tr>
</tbody>
</table>

- numerical **validation** of the algorithm with various starting points: consistent results
Conclusion

Numerical method to derive a feasible $\epsilon$-optimal strategy

- rigorously validated [dSD 12, dSDG 17]
- with general error bounds for the approximation of the value function
- numerically demanding but viable in low dimensional examples
References

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