
Benoîte de Saporta, François Dufour, Huilong Zhang

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Numerical Approximation of Optimal Strategies for Impulse Control of Piecewise Deterministic Markov Processes
Application to Maintenance Optimisation

Benoîte de Saporta, François Dufour, Huilong Zhang
Univ. Montpellier, Bordeaux INP, Univ. Bordeaux
Outline

Introduction

Motivation

Piecewise deterministic Markov processes

Impulse control for PDMPs

Numerical implementation

Conclusion
Maintenance optimization

Equipments
- with several components
- subject to random degradation and failures

Maintenance optimization problem: find some optimal balance between
- repairing/changing components too often
- do nothing and wait for the total failure of the system

Optimize some criterion
- minimize a cost: repair, maintenance, unavailability penalty, failure penalty, ... 
- maximize a reward: availability, production, ...
Maintenance optimization

Equipments

▶ with several components
▶ subject to random degradation and failures

Maintenance optimization problem: find some optimal balance between

▶ repairing/changing components too often
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Optimize some criterion

▶ minimize a cost: repair, maintenance, unavailability penalty, failure penalty, ...
▶ maximize a reward: availability, production, ...
Impulse control problem

**Impulse control**

Select

- intervention dates
- new *starting point* for the process at interventions
to *minimize* a cost function

**Piecewise deterministic Markov processes**

General class of *non-diffusion* dynamic stochastic *hybrid* models: *deterministic* motion punctuated by *random* jumps.

[CD 89], [Davis 93], [dSDZ 14], ...
Piecewise deterministic Markov processes

Starting point

\[ X_0 = (m, x) \]
Piecewise deterministic Markov processes

$X_t$ follows the deterministic flow until the first jump time $T_1 = S_1$

$$X_t = (m, \phi_m(x, t)), \quad \mathbb{P}_{(m,x)}(T_1 > t) = e^{-\int_0^t \lambda_m(\phi_m(x,s)) \, ds}$$
Piecewise deterministic Markov processes

Post-jump location \((m_1, x_{T_1})\) selected by the Markov kernel

\[
Q_m(\phi_m(x, T_1), \cdot)
\]
\( X_t \) follows the flow until the next jump time \( T_2 = T_1 + S_2 \)

\[ X_{T_1+t} = (m_1, \phi_{m_1}(x_{T_1}, t)), \quad t < S_2 \]
Post-jump location \((m_2, x_{T_2})\) selected by Markov kernel

\[ Q_{m_1}(\phi_{m_1}(x_{T_1}, S_2), \cdot) \ldots \]
Embedded Markov chain

\{X_t\} strong Markov process [Davis 93]

Natural embedded Markov chain

- $Z_0$ starting point, $S_0 = 0$, $S_1 = T_1$
- $Z_n$ new mode and location after $n$-th jump, $S_n = T_n - T_{n-1}$, time between two jumps

Proposition

$(Z_n, S_n)$ is a discrete-time Markov chain
Only source of randomness of the PDMP
Mathematical definition of impulse control

Strategy $S = (\tau_n, R_n)_{n \geq 1}$
- $\tau_n$ intervention times
- $R_n$ new positions after intervention

Value function

$$\mathcal{J}^S(x) = E_x^S \left[ \int_0^\infty e^{-\alpha s} f(Y_s) ds + \sum_{i=1}^\infty e^{-\alpha \tau_i} c(Y_{\tau_i}, Y_{\tau_i}^+) \right]$$

$$\mathcal{V}(x) = \inf_{S \in \mathcal{S}} \mathcal{J}^S(x)$$

- $f, c$ cost functions, $\alpha$ discount factor
- $Y_t$ controlled process, $\mathcal{S}$ set of admissible strategies
Dynamic programming

Costa, Davis, 1988

For any function $g \geq$ cost of the no-impulse strategy

- $v_0 = g$
- $v_n = L(v_{n-1})$

$\lim_{n \to \infty} v_n(x) = V(x)$

dS, Dufour, Geeraert, 2017

Construction of $\epsilon$-optimal strategies based on the dynamic programming operator
Dynamic programming

Jump-or-intervention operator

\[
\nu_n(Z_n) = L(M\nu_{n+1}, \nu_{n+1})(Z_n)
\]

\[
= \left( \inf_{t \leq t^*(Z_n)} \mathbb{E}\left[ F(Z_n, t) + e^{-\alpha S_{n+1}} \nu_{n+1}(Z_{n+1})\mathbb{1}_{\{S_{n+1} < t \land t^*(Z_n)\}} \right. \right.
\]

\[
+ e^{-\alpha t \land t^*(Z_n)} M\nu_{n+1}(\phi(Z_n, t \land t^*(Z_n))) \mathbb{1}_{\{S_{n+1} \geq t \land t^*(Z_n)\} | Z_n} \left. \right]
\]

\[
\wedge \mathbb{E}\left[ F(Z_n, t^*(Z_n)) + e^{-\alpha S_{n+1}} \nu_{n+1}(Z_{n+1}) | Z_n \right]
\]

with

\[
F(x, t) = \int_0^{t \land t^*(x)} e^{-\alpha s - \int_0^s \lambda(\phi(x, u))du} f(\phi(x, s)) ds
\]

\[
M\nu_{n+1}(x) = \inf_{y \in \mathcal{U}} \left\{ c(x, y) + \nu_{n+1}(y) \right\}
\]
Approximation scheme - Value function

Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)
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Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)

\[\tilde{v}_N(y^i) = g(\hat{Z}_0^i)\]
\[\hat{v}(\hat{Z}_1^i) = g(\hat{Z}_1^i) \rightarrow \tilde{v}_{N-1}(y^i)\]
\[\vdots\]
\[\hat{v}(\hat{Z}_{N-2}^i) = g(\hat{Z}_{N-2}^i) \rightarrow \hat{L}_{N-2}^i \rightarrow \hat{v}(\hat{Z}_{N-3}^i) \rightarrow \hat{L}_{N-3}^i \rightarrow \cdots \rightarrow \tilde{v}_N(\hat{Z}_{N-2}^i) \rightarrow \tilde{v}_{N-2}(y^i)\]
\[\vdots\]
\[\hat{v}(\hat{Z}_{N-1}^i) = g(\hat{Z}_{N-1}^i) \rightarrow \hat{L}_{N-1}^i \rightarrow \hat{v}(\hat{Z}_{N-2}^i) \rightarrow \hat{L}_{N-2}^i \rightarrow \cdots \rightarrow \hat{v}(\hat{Z}_1^i) \rightarrow \tilde{v}_2(y^i)\]
\[\vdots\]
\[\hat{v}(\hat{Z}_N) = g(\hat{Z}_N) \rightarrow \hat{L}_N \rightarrow \hat{v}_{N-1}(\hat{Z}_{N-1}) \rightarrow \hat{L}_{N-1} \rightarrow \cdots \rightarrow \hat{v}_2(\hat{Z}_1) \rightarrow \hat{L}_1 \rightarrow \tilde{v}_0(x_0)\]
Approximation scheme - Value function

Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)

\begin{align*}
\tilde{v}_N(y^i) &= g(\hat{Z}^i_0) \\
\tilde{v}(\hat{Z}^i_1) &= g(\hat{Z}^i_1) \\
\tilde{v}(\hat{Z}^i_2) &= g(\hat{Z}^i_2) \\
\vdots & & \vdots \\
\tilde{v}(\hat{Z}^i_{N-2}) &= g(\hat{Z}^i_{N-2}) \\
\tilde{v}(\hat{Z}^i_{N-1}) &= g(\hat{Z}^i_{N-1}) \\
\tilde{v}_N(\hat{Z}_N) &= g(\hat{Z}_N)
\end{align*}

\begin{align*}
\hat{L}_1 \quad \hat{v}_{N-1}(y^i) \\
\hat{L}_1 \quad \hat{v}_N(y^i) \\
\hat{L}_2 \quad \hat{v}_{N-2}(y^i) \\
\vdots & & \vdots \\
\hat{L}_{N-2} \quad \hat{v}_2(y^i) \\
\hat{L}_{N-1} \quad \hat{v}_1(y^i) \\
\hat{L}_N \quad \hat{v}_0(x_0)
\end{align*}
Approximation scheme - Value function

Based on time-dependent discretizations of the state space of $(Z_n, S_n)$
Approximation scheme - Value function

Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)

\[
\begin{align*}
\hat{v}_N(y) &= g(\hat{Z}_N) \\
\hat{v}_N(y) &= g(\hat{Z}_{N-1}) \\
\vdots & \quad \vdots \\
\hat{v}_N(y) &= g(\hat{Z}_1) \\
\hat{v}_N(y) &= g(x_0)
\end{align*}
\]
Approximation scheme - Value function

Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)
Approximation scheme - $\epsilon$-optimal strategy

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\[ \vdots \]
\[ \hat{v}(\hat{Z}_{N-3}^i) \rightarrow \hat{L}_{N-3} \rightarrow \cdots \]
\[ \hat{L}_1 \rightarrow \hat{v}_2(y^i) \]
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\[ \vdots \]
\[ \hat{L}_1 \rightarrow \hat{v}_0(x_0) \]
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Approximation scheme - $\epsilon$-optimal strategy
Based on time-dependent discretizations of the state space of $(Z_n, S_n)$

\[
\tilde{v}_N(y^i) = g(\hat{Z}_0^i) \\
\hat{v}(\hat{Z}_1^i) = g(\hat{Z}_1^i) \\
\vdots \\
\hat{v}(\hat{Z}_{N-3}^i) \rightarrow \hat{L}_{N-3}^i \\
\hat{l}_1^i \rightarrow \tilde{v}_{N-1}(y^i) \rightarrow \tilde{v}_N(y^i) \\
\hat{v}(\hat{Z}_1^i) \rightarrow \hat{l}_1^i \rightarrow \hat{v}_1(y^i) \\
\vdots \\
\hat{v}_1(\hat{Z}_1) \rightarrow \hat{l}_1 \rightarrow \hat{v}_0(x_0)
\]
Approximation scheme - $\epsilon$-optimal strategy

Based on time-dependent discretizations of the state space of $(Z_n, S_n)$

\[ \check{v}_N(y^i) = g(\hat{Z}_0^i) \]

\[ \check{v}(\hat{Z}_1^i) = g(\hat{Z}_1^i) \]

\[ \hat{L}_1 \rightarrow \check{v}_{N-1}(y^i) \]

\[ \vdots \]

\[ \check{v}(\hat{Z}_{N-3}^i) \rightarrow \hat{L}_{N-3}^i \rightarrow \check{v}_{N-1}(y^i) \]

\[ \check{v}(\hat{Z}_{N-2}^i) \rightarrow \hat{L}_{N-2}^i \rightarrow \check{v}_{N-2}(y^i) \]

\[ \vdots \]

\[ \check{v}(\hat{Z}_2^i) \rightarrow \hat{L}_2^i \rightarrow \check{v}_2(y^i) \]

\[ \check{v}(\hat{Z}_1^i) \rightarrow \hat{L}_1^i \rightarrow \check{v}_1(y^i) \]

\[ \check{v}(x^0) \rightarrow \hat{L}_0 \rightarrow \check{v}_0(x_0) \]
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\[ \tilde{v}_N(y^i) = g(\hat{Z}_0^i) \]

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\[ \hat{v}(\hat{Z}_1^i) \rightarrow \hat{L}_1 \rightarrow \hat{v}_1(y^i) \]

\[ \vdots \]

\[ \hat{v}_1(\hat{Z}_1) \rightarrow \hat{L}_1 \rightarrow \hat{v}_0(x_0) \]
Equipment model

Typical model with 4 components

▶ Component 1: 2 states – stable $\xrightarrow{\text{Exponential}}$ failed
▶ Component 2: 2 states – stable $\xrightarrow{\text{Weibull}}$ failed
▶ Components 3 and 4: 3 states
  stable $\xrightarrow{\text{Weibull}}$ degraded $\xrightarrow{\text{Exponential}}$ failed
Maintenance operations

Possible maintenance operations

- All components, all states: do nothing
- Components 1 and 2, all states: change
- Components 3 and 4: change in all states, repair only in stable or degraded states
Criterion to optimize

Minimize the maintenance + unavailability costs

- unavailability cost proportional to time spend in failed state
- fixed cost for going to the workshop + repair < change costs
PDMP model of the equipment

- **Euclidean variables:** 5 time variables
  - functioning time of components 2, 3 and 4
  - calendar time
  - time spent in the workshop

- **Discrete variables:** 225 modes
  - state of the components / maintenance operations
Parameters to tune

- Number of points in the control grid (underlying continuous model)
- Number of point in the quantization grids for $(Z_n, S_n)$
- Approximation horizon $N$ such that $v_N(x) - V(x)$ small enough $\simeq$ allowed number of jumps + interventions
- Bounding function $g$
- Time discretization step for inf
Step 1: Exact simulation of the PDMP

Implementation of an exact simulator for reference strategies to serve as benchmark

- **Strategy 1**: do nothing
- **Strategy 2**: send equipment to workshop 1 day after failure, change all degraded components, change all failed ones
- **Strategy 3**: send equipment to workshop 1 day after degradation, change all degraded components, change all failed ones

<table>
<thead>
<tr>
<th>Strategy</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
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<tbody>
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<td>Mean cost</td>
<td>19952</td>
<td>11389</td>
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Step 2 and 3: Discretisation of the control set $\mathcal{U}$ and the embedded Markov chain

Finite control set $\mathcal{U}$

$\implies$ discretize the functioning times at interventions

$\implies$ project the real times on the grid feasibly

Compromise between precision and computation time

<table>
<thead>
<tr>
<th>Grid</th>
<th>Number of points</th>
<th>relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3 \times 3 \times 3 \times 5$</td>
<td>246</td>
<td>0.10344</td>
</tr>
<tr>
<td>$4 \times 4 \times 4 \times 5$</td>
<td>331</td>
<td>0.0241</td>
</tr>
<tr>
<td>$5 \times 5 \times 5 \times 5$</td>
<td>592</td>
<td>0.0062</td>
</tr>
<tr>
<td>$3 \times 3 \times 3 \times 11$</td>
<td>615</td>
<td>0.0341</td>
</tr>
<tr>
<td>$4 \times 4 \times 4 \times 11$</td>
<td>923</td>
<td>0.0819</td>
</tr>
<tr>
<td>$5 \times 5 \times 5 \times 11$</td>
<td>1855</td>
<td>0.0186</td>
</tr>
<tr>
<td>$6 \times 6 \times 6 \times 11$</td>
<td>2110</td>
<td>0.0066</td>
</tr>
<tr>
<td>$7 \times 7 \times 7 \times 11$</td>
<td>2617</td>
<td>0.0071</td>
</tr>
<tr>
<td>$8 \times 8 \times 8 \times 11$</td>
<td>3359</td>
<td>0.0066</td>
</tr>
<tr>
<td>$3 \times 3 \times 3 \times 21$</td>
<td>1230</td>
<td>0.0034</td>
</tr>
<tr>
<td>$4 \times 4 \times 4 \times 21$</td>
<td>1899</td>
<td>0.0170</td>
</tr>
<tr>
<td>$5 \times 5 \times 5 \times 21$</td>
<td>2960</td>
<td>0.0095</td>
</tr>
<tr>
<td>$6 \times 6 \times 6 \times 21$</td>
<td>4220</td>
<td>0.0065</td>
</tr>
<tr>
<td>$7 \times 7 \times 7 \times 21$</td>
<td>5536</td>
<td>0.0059</td>
</tr>
<tr>
<td>$8 \times 8 \times 8 \times 21$</td>
<td>7111</td>
<td>0.0047</td>
</tr>
</tbody>
</table>
Step 4: Calibrating $N$ the number of allowed jumps + interventions

Horizon $N$ (number of iterations)

- 5 for Strategy 1
- up to 30 for Strategy 2 (mean 6)
- up to 25 for Strategy 3 (mean 6)
Step 5: Approximation of the value function

<table>
<thead>
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<th>Approx. Value function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>19952</td>
</tr>
<tr>
<td>2</td>
<td>11389</td>
</tr>
<tr>
<td>3</td>
<td>8477</td>
</tr>
<tr>
<td></td>
<td>7076</td>
</tr>
</tbody>
</table>

- relative gain of 19.8% vs Strategy 5
Step 6: Optimally controlled trajectories

<table>
<thead>
<tr>
<th>Strategy 1</th>
<th>Strategy 2</th>
<th>Strategy 3</th>
<th>Approx. Value function</th>
<th>Optimally controlled traj.</th>
</tr>
</thead>
<tbody>
<tr>
<td>19952</td>
<td>11389</td>
<td>8477</td>
<td>7076</td>
<td>6733</td>
</tr>
</tbody>
</table>

- numerical validation of the algorithm with various starting points: consistent results
Numerical method to derive a feasible $\epsilon$-optimal strategy

- rigorously validated [dSD 12, dSDG 17]
- with general error bounds for the approximation of the value function
- numerically demanding but viable in low dimensional examples
References

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