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Numerical Approximation of Optimal Strategies for Impulse Control of Piecewise Deterministic Markov Processes Application to Maintenance Optimisation

Benoîte de Saporta, François Dufour, Huilong Zhang

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Outline

Introduction
  Motivation
  Piecewise deterministic Markov processes

Impulse control for PDMPs

Numerical implementation

Conclusion
Introduction Motivation

Maintenance optimization

Equipments

- with several components
- subject to random degradation and failures

Maintenance optimization problem: find some optimal balance between

- repairing/changing components too often
- do nothing and wait for the total failure of the system

Optimize some criterion

- minimize a cost: repair, maintenance, unavailability penalty, failure penalty, ...
- maximize a reward: availability, production, ...
Maintenance optimization

Equipments
- with several components
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Maintenance optimization problem: find some optimal balance between
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Optimize some criterion
- minimize a cost: repair, maintenance, unavailability penalty, failure penalty, ... 
- maximize a reward: availability, production, ...
Impulse control problem

Impulse control

Select

- intervention dates
- new starting point for the process at interventions
to minimize a cost function

Piecewise deterministic Markov processes

General class of non-diffusion dynamic stochastic hybrid models: deterministic motion punctuated by random jumps.
[CD 89], [Davis 93], [dSDZ 14], ...
Starting point

\[ X_0 = (m, x) \]
Piecewise deterministic Markov processes

$X_t$ follows the deterministic flow until the first jump time $T_1 = S_1$

$$X_t = (m, \phi_m(x, t)), \quad \mathbb{P}_{(m,x)}(T_1 > t) = e^{-\int_0^t \lambda_m(\phi_m(x,s)) \, ds}$$
Piecewise deterministic Markov processes

Post-jump location \((m_1, x_{T_1})\) selected by the Markov kernel

\[
Q_m(\phi_m(x, T_1), \cdot)
\]
Piecewise deterministic Markov processes

$X_t$ follows the flow until the next jump time $T_2 = T_1 + S_2$

$X_{T_1 + t} = (m_1, \phi_{m_1}(x_{T_1}, t)), \quad t < S_2$
Post-jump location \((m_2, x_{T_2})\) selected by Markov kernel

\[ Q_{m_1}(\phi_{m_1}(x_{T_1}, S_2), \cdot) \ldots \]
Embedded Markov chain

\{X_t\} strong Markov process [Davis 93]

Natural embedded Markov chain

- \(Z_0\) starting point, \(S_0 = 0, S_1 = T_1\)
- \(Z_n\) new mode and location after \(n\)-th jump, \(S_n = T_n - T_{n-1}\), time between two jumps

Proposition

\((Z_n, S_n)\) is a discrete-time Markov chain

Only source of randomness of the PDMP
Mathematical definition of impulse control

Strategy $S = (\tau_n, R_n)_{n \geq 1}$

- $\tau_n$ intervention times
- $R_n$ new positions after intervention

Value function

$$J^S(x) = E_x^S \left[ \int_0^\infty e^{-\alpha s} f(Y_s) ds + \sum_{i=1}^{\infty} e^{-\alpha \tau_i} c(Y_{\tau_i}, Y_{\tau_i^+}) \right]$$

$$V(x) = \inf_{S \in \mathcal{S}} J^S(x)$$

- $f, c$ cost functions, $\alpha$ discount factor
- $Y_t$ controlled process, $\mathcal{S}$ set of admissible strategies
Dynamic programming

**Costa, Davis, 1988**

For any function $g \geq$ cost of the no-impulse strategy

- $v_0 = g$
- $v_n = L(v_{n-1})$

$$v_n(x) \xrightarrow{n \to \infty} V(x)$$

**dS, Dufour, Geeraert, 2017**

Construction of $\epsilon$-optimal strategies based on the dynamic programming operator
Dynamic programming

Jump-or-intervention operator

\[ v_n(Z_n) = L(Mv_{n+1}, v_{n+1})(Z_n) \]

\[ = \left( \inf_{t \leq t^*(Z_n)} \mathbb{E}\left[ F(Z_n, t) + e^{-\alpha S_{n+1}} v_{n+1}(Z_{n+1}) 1_{\{S_{n+1} < t \wedge t^*(Z_n)\}} \right] \right. \]

\[ + \left. e^{-\alpha t \wedge t^*(Z_n)} Mv_{n+1}(\phi(Z_n, t \wedge t^*(Z_n))) \mathbb{I}_{\{S_{n+1} \geq t \wedge t^*(Z_n)\} | Z_n} \right) \]

\[ \wedge \mathbb{E}\left[ F(Z_n, t^*(Z_n)) + e^{-\alpha S_{n+1}} v_{n+1}(Z_{n+1}) | Z_n \right] \]

with

\[ F(x, t) = \int_0^{t \wedge t^*(x)} e^{-\alpha s - \int_0^s \lambda(\phi(x, u)) du} f(\phi(x, s)) ds \]

\[ Mv_{n+1}(x) = \inf_{y \in U} \{ c(x, y) + v_{n+1}(y) \} \]
Approximation scheme - Value function

Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)
Approximation scheme - Value function

Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)

\[
\hat{v}_N(y^i) = g(\hat{Z}_0^i) \\
\hat{v}(\hat{Z}_1^i) = g(\hat{Z}_1^i) \\
\vdots \\
\hat{v}(\hat{Z}_{i-1}^i) = g(\hat{Z}_{i-1}^i) \\
\hat{v}(\hat{Z}_N^i) = g(\hat{Z}_N^i)
\]
Approximation scheme - Value function
Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)
Approximation scheme - Value function
Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)

\[ \tilde{v}_N(y^i) = g(\hat{Z}_0^i) \]

\[ \tilde{v}(\hat{Z}_1^i) = g(\hat{Z}_1^i) \rightarrow \tilde{v}(\hat{Z}_2^i) = g(\hat{Z}_2^i) \rightarrow \tilde{v}(\hat{Z}_1^i) \rightarrow \tilde{v}_N(y^i) \]

\[ \tilde{v}(\hat{Z}_1^i) = g(\hat{Z}_1^i) \rightarrow \tilde{v}(\hat{Z}_1^i) \rightarrow \tilde{v}_N(y^i) \]

\[ \vdots \]

\[ \vdots \]

\[ \vdots \]

\[ \vdots \]

\[ \tilde{v}_{N-1}(y^i) \]

\[ \tilde{v}_N(\hat{Z}_N) = g(\hat{Z}_N) \rightarrow \tilde{v}_{N-1}(\hat{Z}_{N-1}) \rightarrow \tilde{v}_{N-2}(y^i) \]

\[ \vdots \]

\[ \vdots \]

\[ \vdots \]

\[ \tilde{v}_0(x_0) \]
Approximation scheme - Value function

Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)
Approximation scheme - Value function

Based on time-dependent discretizations of the state space of $(Z_n, S_n)$
Approximation scheme - Value function
Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)
Approximation scheme - $\epsilon$-optimal strategy

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Approximation scheme - \( \epsilon \)-optimal strategy

Based on time-dependent discretizations of the state space of \((Z_n, S_n)\)

\[
\hat{v}_N(y^i) = g(\hat{Z}_0^i)
\]

\[
\hat{v}(\hat{Z}_1^i) = g(\hat{Z}_1^i) \rightarrow \hat{L}_1 \rightarrow \hat{v}_{N-1}(y^i)
\]

\[
\vdots \rightarrow \hat{L}_{N-3} \rightarrow \hat{v}_2(y^i)
\]

\[
\hat{v}(\hat{Z}_{N-3}^i) \rightarrow \hat{L}_{N-3} \rightarrow \hat{v}_2(y^i)
\]

\[
\hat{v}(\hat{Z}_1^i) \rightarrow \hat{L}_1 \rightarrow \hat{v}_1(y^i)
\]

\[
\hat{v}(\hat{Z}_1^i) \rightarrow \hat{L}_1 \rightarrow \hat{v}_1(y^i)
\]

\[
\vdots \rightarrow \hat{L}_1 \rightarrow \hat{v}_0(x_0)
\]
Equipment model

Typical model with 4 components

- Component 1: 2 states – stable $\xrightarrow{\text{Exponential}}$ failed
- Component 2: 2 states – stable $\xrightarrow{\text{Weibull}}$ failed
- Components 3 and 4: 3 states
  - Stable $\xrightarrow{\text{Weibull}}$ degraded $\xrightarrow{\text{Exponential}}$ failed
Maintenance operations

Possible maintenance operations

- All components, all states: do nothing
- Components 1 and 2, all states: change
- Components 3 and 4: change in all states, repair only in stable or degraded states
**Criterion to optimize**

Minimize the maintenance + unavailability costs

- **unavailability** cost proportional to time spend in **failed** state
- fixed cost for going to the workshop + repair < **change** costs
PDMP model of the equipment

- **Euclidean variables**: 5 time variables
  - functioning time of components 2, 3 and 4
  - calendar time
  - time spent in the workshop

- **Discrete variables**: 225 modes
  - state of the components / maintenance operations
Parameters to tune

- Number of points in the control grid (underlying continuous model)
- Number of point in the quantization grids for \((Z_n, S_n)\)
- Approximation horizon \(N\) such that \(v_N(x) - \mathcal{V}(x)\) small enough \(\approx\) allowed number of jumps + interventions
- bounding function \(g\)
- Time discretization step for \(\inf\)
Step 1: Exact simulation of the PDMP

Implementation of an exact simulator for reference strategies to serve as benchmark

- **Strategy 1**: do nothing
- **Strategy 2**: send equipment to workshop 1 day after failure, change all degraded components, change all failed ones
- **Strategy 3**: send equipment to workshop 1 day after degradation, change all degraded components, change all failed ones

<table>
<thead>
<tr>
<th>Strategy</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
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<tbody>
<tr>
<td>Mean cost</td>
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**Step 2 and 3: Discretisation of the control set \( U \) and the embedded Markov chain**

**Finite control set** \( U \)  
\[ \Rightarrow \text{discretize the functioning times at interventions} \]
\[ \Rightarrow \text{project the real times on the grid feasibly} \]

**Compromise between precision and computation time**

<table>
<thead>
<tr>
<th>Grid</th>
<th>Number of points</th>
<th>relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3 \times 3 \times 3 \times 5 )</td>
<td>246</td>
<td>0.10344</td>
</tr>
<tr>
<td>( 4 \times 4 \times 4 \times 5 )</td>
<td>331</td>
<td>0.0241</td>
</tr>
<tr>
<td>( 5 \times 5 \times 5 \times 5 )</td>
<td>592</td>
<td>0.0062</td>
</tr>
<tr>
<td>( 3 \times 3 \times 3 \times 11 )</td>
<td>615</td>
<td>0.0341</td>
</tr>
<tr>
<td>( 4 \times 4 \times 4 \times 11 )</td>
<td>923</td>
<td>0.0819</td>
</tr>
<tr>
<td>( 5 \times 5 \times 5 \times 11 )</td>
<td>1855</td>
<td>0.0186</td>
</tr>
<tr>
<td>( 6 \times 6 \times 6 \times 11 )</td>
<td>2110</td>
<td>0.0066</td>
</tr>
<tr>
<td>( 7 \times 7 \times 7 \times 11 )</td>
<td>2617</td>
<td>0.0071</td>
</tr>
<tr>
<td>( 8 \times 8 \times 8 \times 11 )</td>
<td>3359</td>
<td>0.0066</td>
</tr>
<tr>
<td>( 3 \times 3 \times 3 \times 21 )</td>
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<td>0.0034</td>
</tr>
<tr>
<td>( 4 \times 4 \times 4 \times 21 )</td>
<td>1899</td>
<td>0.0170</td>
</tr>
<tr>
<td>( 5 \times 5 \times 5 \times 21 )</td>
<td>2960</td>
<td>0.0095</td>
</tr>
<tr>
<td>( 6 \times 6 \times 6 \times 21 )</td>
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<td>0.0065</td>
</tr>
<tr>
<td>( 7 \times 7 \times 7 \times 21 )</td>
<td>5536</td>
<td>0.0059</td>
</tr>
<tr>
<td>( 8 \times 8 \times 8 \times 21 )</td>
<td>7111</td>
<td>0.0047</td>
</tr>
</tbody>
</table>
Step 4: Calibrating $N$ the number of allowed jumps + interventions

Horizon $N$ (number of iterations)
- 5 for Strategy 1
- up to 30 for Strategy 2 (mean 6)
- up to 25 for Strategy 3 (mean 6)
**Step 5: Approximation of the value function**

<table>
<thead>
<tr>
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<th>Strategy 2</th>
<th>Strategy 3</th>
<th>Approx. Value function</th>
</tr>
</thead>
<tbody>
<tr>
<td>19952</td>
<td>11389</td>
<td>8477</td>
<td>7076</td>
</tr>
</tbody>
</table>

- relative gain of 19.8% vs Strategy 5
Step 6: Optimally controlled trajectories

<table>
<thead>
<tr>
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<th>Strategy 2</th>
<th>Strategy 3</th>
<th>Approx. Value function</th>
<th>Optimally controlled traj.</th>
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<tbody>
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<td>19952</td>
<td>11389</td>
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<td>7076</td>
<td>6733</td>
</tr>
</tbody>
</table>

- numerical validation of the algorithm with various starting points: consistent results
Conclusion

Numerical method to derive a feasible $\epsilon$-optimal strategy

- rigorously validated [dSD 12, dSDG 17]
- with general error bounds for the approximation of the value function
- numerically demanding but viable in low dimensional examples
References

[CD 89] O. Costa, M. Davis *Impulse control of piecewise-deterministic processes*

[Davis 93] M. Davis, *Markov models and optimization*

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