Gevrey regularity for a system coupling the Navier-Stokes system with a beam equation

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Abstract

We analyse a bi-dimensional fluid-structure interaction system composed by a viscous incompressible fluid and a beam located at the boundary of the fluid domain. Our main result is the existence and uniqueness of strong solutions for the corresponding coupled system. The proof is based on a the study of the linearized system and a fixed point procedure. In particular, we show that the linearized system can be written with a Gevrey class semigroup. The main novelty with respect to previous results is that we do not consider any approximation in the beam equation.

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Contents

1 Introduction

In this article, we are interested in the interaction of a Navier-Stokes fluid and of a beam. More precisely, we consider a planar domain $\mathcal{F}(t)$ where evolves a viscous incompressible fluid modeled by the classical Navier-Stokes system. The fluid domain is a transformation of an infinite horizontal strip through the displacement of a beam located at its upper side. More precisely, we write for any $\eta : \mathbb{R} \to (-1, \infty)$

$$\mathcal{F}_{\#}(\eta) \stackrel{\text{def}}{=} \left\{(x_1, x_2) \in \mathbb{R}^2,\ x_1 \in \mathbb{R},\ x_2 \in (0, 1 + \eta(x_1))\right\}, \tag{1.1}$$

$$\Gamma_{\#}(\eta) \stackrel{\text{def}}{=} \{(s, 1 + \eta(s)),\ s \in \mathbb{R}\}, \quad \Gamma_{\text{fix},\#} \stackrel{\text{def}}{=} \mathbb{R} \times \{0\}. \tag{1.2}$$

Then the fluid domain writes $\mathcal{F}(t) \stackrel{\text{def}}{=} \mathcal{F}_{\#}(\eta(t))$ where $\eta(t, s)$ $(t > 0,\ s \in \mathbb{R})$ is the vertical displacement of the beam. If we denote by v and p the velocity and the pressure of the fluid, our system is the following

$$\begin{cases} \partial_t v + (v \cdot \nabla)v - \operatorname{div} \mathbb{T}(v, p) = 0, \quad t > 0,\ x \in \mathcal{F}_{\#}(\eta(t)), \\ \operatorname{div} v = 0 \quad t > 0,\ x \in \mathcal{F}_{\#}(\eta(t)), \\ v(t, s, 1 + \eta(t, s)) = (\partial_t \eta)(t, s) e_2 \quad t > 0,\ s \in \mathbb{R}, \\ v = 0 \quad t > 0,\ x \in \Gamma_{\text{fix},\#}, \\ v\ Le_1 - \text{periodic} \quad t > 0, \\ \partial_{tt}\eta + \alpha_1 \partial_{ssss}\eta - \alpha_2 \partial_{ss}\eta = -\widetilde{\mathbb{H}}_\eta(v, p), \quad t > 0,\ s \in \mathbb{R}, \\ \eta\ L - \text{periodic} \quad t > 0, \end{cases} \tag{1.3}$$

with the initial conditions

$$\eta(0) = \eta_1^0, \quad \partial_t \eta(0) = \eta_2^0 \quad \text{and} \quad v(0, \cdot) = v^0. \tag{1.4}$$

We focus in this article in the case of a periodic solution in the direction e_1, where (e_1, e_2) is the canonical basis of $\mathbb{R}^2$. We have also used the following notation:

$$\mathbb{T}(v, p) \stackrel{\text{def}}{=} 2\nu D(v) - pI_2, \quad D(v) = \frac{1}{2}\left(\nabla v + (\nabla v)^*\right), \tag{1.5}$$

$$\widetilde{\mathbb{H}}_\eta(v, p)(t, s) \stackrel{\text{def}}{=} \left\{(1 + |\partial_s \eta(t, s)|^2)^{1/2} \left[\mathbb{T}(v, p)n\right](t, s, 1 + \eta(t, s)) \cdot e_2\right\}. \tag{1.6}$$

Above, the constant $\nu > 0$ is the viscosity and the vector fields n is the unit exterior normal to $\mathcal{F}_{\#}(\eta(t))$ and in particular, on $\Gamma_{\#}(\eta(t))$,

$$n(t, x_1, x_2) = \frac{1}{\sqrt{1 + |\partial_s \eta(t, x_1)|^2}} \begin{bmatrix} -\partial_s \eta(t, x_1) \\ 1 \end{bmatrix}. \tag{1.7}$$

The constants α_1 and α_2 are assumed to satisfy

$$\alpha_1 > 0, \quad \alpha_2 \geqslant 0. \tag{1.8}$$

Due the spatial periodicity, we also use the following notation (see Figure 1): for any $\eta : [0, L] \to (-1, \infty)$,

$$\mathcal{F}(\eta) \stackrel{\text{def}}{=} \left\{(x_1, x_2) \in \mathbb{R}^2,\ x_1 \in (0, L),\ x_2 \in (0, 1 + \eta(x_1))\right\}, \tag{1.9}$$

$$\Gamma(\eta) \stackrel{\text{def}}{=} \{(s, 1 + \eta(s)),\ s \in (0, L)\}, \quad \Gamma_{\text{fix}} \stackrel{\text{def}}{=} (0, L) \times \{0\}. \tag{1.10}$$

A formal calculation on system (1.3) shows that

$$\frac{d}{dt} \int_0^L \eta(t, s)\ ds = 0.$$

For simplicity, and without loss of generality, we assume that the mean value of η_1^0 is zero and thus

$$\int_0^L \eta(t, s)\ ds = 0 \quad (t \geqslant 0). \tag{1.11}$$

This leads us to consider the following spaces

$$L^p_{\#}(0, L) \stackrel{\text{def}}{=} \{f \in L^p_{\text{loc}}(\mathbb{R})\ ;\ f(\cdot + L) = f\} \quad (p \in [1, \infty]), \tag{1.12}$$

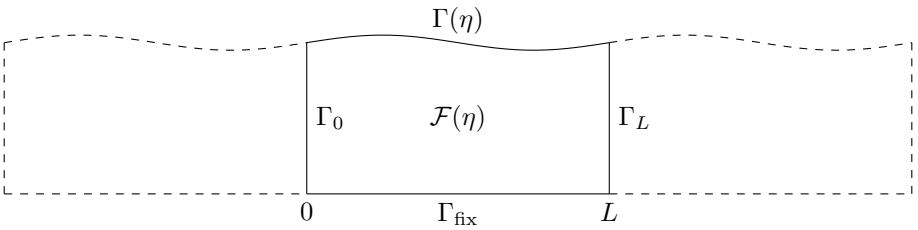

Figure 1: Our geometry

$$L^p_{\#,0}(0,L) \stackrel{\text{def}}{=} \left\{ f \in L^p_{\#}(0,L) \ ; \ \int_0^L f(s)\ ds = 0 \right\} \quad (p \in [1,\infty]), \tag{1.13}$$

$$H^\alpha_{\#,0}(0,L) \stackrel{\text{def}}{=} H^\alpha_{\text{loc}}(\mathbb{R}) \cap L^2_{\#,0}(0,L) \quad (\alpha \geqslant 0), \tag{1.14}$$

and the orthogonal projection

$$M : L^2_{\#}(0,L) \to L^2_{\#,0}(0,L).$$

We also define the operator for the structure:

$$\mathcal{H}_{\mathcal{S}} \stackrel{\text{def}}{=} L^2_{\#,0}(0,L), \quad \mathcal{D}(A_1) \stackrel{\text{def}}{=} H^4_{\#,0}(0,L), \tag{1.15}$$

$$A_1 : \mathcal{D}(A_1) \to \mathcal{H}_{\mathcal{S}}, \quad \eta \mapsto \alpha_1 \partial_{ssss}\eta - \alpha_2 \partial_{ss}\eta. \tag{1.16}$$

One can check that for any $\alpha \in [0,1]$,

$$\mathcal{D}(A_1^\alpha) = H^{4\alpha}_{\#,0}(0,L). \tag{1.17}$$

Note that from relation (1.11), the pressure is not determined up to a constant as in the usual Navier-Stokes system. Indeed, from the beam equation in (1.3) and relation (1.11) we deduce

$$\int_0^L \left(p(t,s,\eta(t,s)) - 2\nu \left\{ (1+|\partial_s \eta|^2)^{1/2} \left[D(v)n \right](t,s,1+\eta(t,s)) \cdot e_2 \right\} \right) ds = 0$$

that can determine the constant for the pressure. In what follows, we only write the projection of the beam equation in $\mathcal{H}_{\mathcal{S}}$:

$$\partial_{tt}\eta + A_1 \eta = -\mathbb{H}_\eta(v,p), \quad t > 0, \tag{1.18}$$

where

$$\mathbb{H}_\eta(v,p) \stackrel{\text{def}}{=} M \widetilde{\mathbb{H}}_\eta(v,p). \tag{1.19}$$

In what follows, we need spaces of the form $H^1(0,T;L^\alpha(\mathcal{F}(\eta)))$ and $L^2(0,T;H^k(\mathcal{F}(\eta)))$, etc. with $T \leqslant \infty$. If $\eta(t,\cdot) > -1$ ($t \in (0,T)$), then $v \in H^1(0,T;L^\alpha(\mathcal{F}(\eta)))$ if $y \mapsto v(t,y_1,y_2(1+\eta(t,y_1))$ is in $H^1(0,T;L^\alpha(\mathcal{F}(0)))$ and similarly, $v \in L^2(0,T;H^k(\mathcal{F}(\eta)))$ if $y \mapsto v(t,y_1,y_2(1+\eta(t,y_1)))$ is in $L^2(0,T;H^k(\mathcal{F}(0)))$. We have used the following notation: L^α for the Lebesgue spaces, H^k for the Sobolev spaces and C_b for the set of the continuous and bounded maps. We use the bold notation for the spaces of vector fields: $\mathbf{L}^\alpha = (L^\alpha)^2$, $\mathbf{H}^k = (H^k)^2$ etc.

As explained above, our aim is to show the existence and uniqueness of strong solutions for system (1.3). We mean by a strong solution of system (1.3) a strong solution for the fluid equations: that is

$$v \in L^2(0,T;\mathbf{H}^2(\mathcal{F}(\eta)) \cap C_b([0,T];\mathbf{H}^1(\mathcal{F}(\eta))) \cap H^1(0,T;\mathbf{L}^2(\mathcal{F}(\eta))), \quad p \in L^2(0,T;H^1(\mathcal{F}(\eta))) \tag{1.20}$$

and the first four equations of (1.3) are satisfied almost everywhere or in the trace sense and a solution for the structure equation with the regularity

$$\begin{aligned} \eta &\in L^2(0,T;H^{7/2}_{\#,0}(0,L)) \cap C_b([0,T];H^{5/2}_{\#,0}(0,L)) \cap H^1(0,T;H^{3/2}_{\#,0}(0,L)), \\ \partial_t \eta &\in L^2(0,T;H^{3/2}_{\#,0}(0,L)) \cap C_b([0,T];H^{1/2}_{\#,0}(0,L)) \cap H^1(0,T;(H^{1/2}_{\#,0}(0,L))'), \end{aligned} \tag{1.21}$$

and (1.18) holds in $L^2(0,T;H^{1/2}_{\#,0}(0,L)')$.

Remark 1.1. *From the third equation of* (1.3)*, we see that the regularity of v and of $\partial_t\eta$ are related by trace theorems. The regularity* (1.21) *of $\partial_t\eta$ that we consider is thus quite natural when looking at the regularity* (1.20) *of v.*

We are now in position to state our main results. We assume there exists $\varepsilon > 0$ such that

$$\eta_1^0 \in H^{3+\varepsilon}_{\#,0}(0,L), \quad \eta_2^0 \in H^{1+\varepsilon}_{\#,0}(0,L) \tag{1.22}$$

with

$$\eta_1^0 > -1 \quad \text{in } \mathbb{R}. \tag{1.23}$$

and

$$v^0 \in \mathbf{H}^1(\mathcal{F}(\eta_1^0)), \tag{1.24}$$

with

$$\operatorname{div} v^0 = 0 \text{ in } \mathcal{F}_\#(\eta_1^0), \tag{1.25}$$

$$v^0 \ Le_1 - \text{periodic}, \tag{1.26}$$

$$v^0(s, 1+\eta_1^0(s)) = \eta_2^0(s)e_2 \quad s \in \mathbb{R}, \tag{1.27}$$

$$v^0 = 0 \quad \text{on } \Gamma_{\text{fix},\#}. \tag{1.28}$$

We first give the existence and uniqueness of strong solutions for small times.

Theorem 1.2. *Assume $[v^0,\ \eta_1^0,\ \eta_2^0]$ satisfies* (1.22)–(1.28)*. Then there exist $T > 0$ and $C_0 > 0$ such that if*

$$\|\eta_1^0\|_{H^2(0,L)} \leqslant C_0 \tag{1.29}$$

there exists a strong solution (η, v, p) of (1.3) *with*

$$\eta(t,\cdot) > -1 \quad t \in [0,T]. \tag{1.30}$$

This solution is unique locally: if $(\eta^{(*)}, v^{(*)}, p^{(*)})$ is another solution, there exists $T^* > 0$ such that

$$(\eta^{(*)}, v^{(*)}, p^{(*)}) = (\eta, v, p) \quad \text{on } [0, T^*].$$

We can also obtain the following result

Theorem 1.3. *There exists $C_0 > 0$ such that for any $[v^0,\ \eta_1^0,\ \eta_2^0]$ satisfying* (1.22)–(1.28) *and*

$$\left\|[v^0,\ \eta_1^0,\ \eta_2^0]\right\|_{\mathbf{H}^1(\mathcal{F}(\eta_1^0))\times H^{3+\varepsilon}_{\#,0}(0,L)\times H^{1+\varepsilon}_{\#,0}(0,L)} \leqslant C_0,$$

there exists a solution (η, v, p) of (1.3) *with*

$$\eta(t,\cdot) > -1 \quad t \in [0,\infty). \tag{1.31}$$

Let us give some comments about our main results. Theorem 1.2 states a local in time existence of strong solutions whereas Theorem 1.3 gives the global existence of strong solutions for small initial conditions. We do not recover the global in time existence of strong solutions as for the standard Navier-Stokes equations. Moreover, we see that there is a loss of regularity for $(\eta, \partial_t\eta)$: we have the continuity of $(\eta, \partial_t\eta)$ in $H^{5/2}_{\#,0}(0,L)\times H^{1/2}_{\#,0}(0,L)$ but we need to impose that at initial time, it belongs to $H^{3+\varepsilon}_{\#,0}(0,L)\times H^{1+\varepsilon}_{\#,0}(0,L)$ for some $\varepsilon > 0$. Finally, the uniqueness holds only locally in time. All the above points are due to the coupling of the Navier-Stokes equations with the beam equation which modify the nature of the Navier-Stokes system. More precisely, the linearized system (3.23) that we consider in Section 3 is composed by a Stokes system and a beam equation and the corresponding semigroup is not analytic but only of Gevrey class (see Section 5). This is stated in Theorem 5.1 and is a part of our main results.

Another important remark is that in this work we have focused on a particular geometry: our linear result, Theorem 5.1, is proved in the case where the fluid domain is a rectangle. This explains why we need the smallness conditions (1.29) in Theorem 1.2. This hypothesis on the geometry implies the commutativity of some operators (see Proposition 4.6) and this simplifies the resolvent estimates in Section 4. The result should hold true in a general geometry, but the corresponding proof should be more involved. This will be the subject of a forthcoming paper.

The model presented above, mainly system (1.3) was proposed in [16] for a model a blood flow in a vessel. It is important to remark that in their model, the beam equation is damped by a term of the form $-\delta\partial_{tss}\eta$. More precisely in (1.3), the beam equation is replaced by

$$\partial_{tt}\eta + \alpha_1\partial_{ssss}\eta - \alpha_2\partial_{ss}\eta - \delta\partial_{tss}\eta = -\widetilde{\mathbb{H}}_\eta(v,p). \tag{1.32}$$

Several authors have studied this model: [5] (existence of weak solutions), [2], [14] and [10] (existence of strong solutions), [17] (stabilization of strong solutions), [1] (stabilization of weak solutions around a stationary state). In all these works, the damping term plays an important role. In particular, with this term, the linearized system is parabolic that is the underlying semigroup is analytic.

Up to our knowledge, there exists only one result in the case without damping, that is (1.32) for $\delta = 0$: the existence of weak solutions is obtained in [9] (by passing to the limit $\delta \to 0$). Note that recently in [11], the authors show the existence of local strong solutions for a structure described by either a wave equation ($\alpha_1 = \delta = 0$ and $\alpha_2 > 0$ in (1.32)) or a beam equation with inertia of rotation ($\alpha_1 > 0$, $\alpha_2 = \delta = 0$ and with an additional term $-\partial_{ttss}\eta$ in (1.32)).

In particular, they do not treat our case ($\alpha_1 > 0$, $\alpha_2 \geqslant 0$ and $\delta = 0$ in (1.32)) and therefore, our work gives the first results of existence and uniqueness of strong solutions in the case of an undamped beam equation. Moreover, we develop a new general approach for the analysis of fluid-structure interaction systems based on Gevrey class semigroups. More precisely, our idea consists in linearizing the problem and in showing that the linearized system (that is (3.23) in our case) is of Gevrey class (see [19] for this notion). We then derive some regularity properties for our linear system and perform a fixed point argument to deduce the well-posedness of system (1.3). Several works considered Gevrey class semigroups, but this is usually done for elastic structures: [6], [7], [18], [8], [22], etc.

The outline of the article is as follows: in Section 2, we perform a change of variables to write system (1.3) in a cylindrical domain. The rest of the paper concerns the resulting system. Section 3 presents the linear system associated with this nonlinear system, and we show in Section 4 the Gevrey class of this system by estimating the resolvent of the corresponding operator. In Section 5, we prove some regularity properties of the linear system. Part of this section is general for any Gevrey class systems. Finally, Section 6 is devoted to the proof of the main results, Theorem 1.2 and Theorem 1.3, by using a fixed point argument.

Notation

We complete here some notation that we use all along the paper. We denote by $\mathcal{L}(X_1, X_2)$ the space of the bounded linear operators from X_1 to X_2. We also set for $T \in (0,\infty]$

$$W(0,T;X_1,X_2) \stackrel{\text{def}}{=} \left\{ w \in L^2(0,T;X_1);\, \frac{dw}{dt} \in L^2(0,T;X_2) \right\}.$$

We recall (see [3, Rem. 4.1 p. 156 and Prop. 4.3 p. 159]) the following embediing

$$W(0,\infty;X_1,X_2) \hookrightarrow C_b([0,\infty),[X_1,X_2]_{1/2}), \tag{1.33}$$

where $[X_1,X_2]_\theta$ denotes the complex interpolation method.

Finally, we use C as a generic positive constant that does not depend on the other terms of the inequality. The value of the constant C may change from one appearance to another.

2 The system written in a fixed domain

We transform the system (1.3) written in the non cylindrical domain

$$\bigcup_{t>0} \{t\} \times \mathcal{F}(\eta(t))$$

into a system written in the domain

$$(0,\infty) \times \mathcal{F},$$

where

$$\mathcal{F} \stackrel{\text{def}}{=} \mathcal{F}(0) = (0,L) \times (0,1). \tag{2.1}$$

We also define

$$\mathcal{F}_\# \stackrel{\text{def}}{=} \mathbb{R}\times(0,1), \quad \Gamma \stackrel{\text{def}}{=} \Gamma(0) = (0,L)\times\{1\}, \quad \Gamma_\# \stackrel{\text{def}}{=} \Gamma_\#(0) = \mathbb{R}\times\{1\}, \tag{2.2}$$

and

$$\Gamma_{\rm b} \stackrel{\text{def}}{=} \Gamma\cup\Gamma_{\rm fix}, \quad \Gamma_{{\rm b},\#} \stackrel{\text{def}}{=} \Gamma_\#\cup\Gamma_{{\rm fix},\#} = \partial\mathcal{F}_\#, \tag{2.3}$$

where $\Gamma_{\rm fix}$, $\Gamma_{{\rm fix},\#}$ are defined in (1.2) and (1.10).

Using the particular geometry of the problem, one can consider the general changes of variables

$$X_{\eta^1,\eta^2} : \mathcal{F}(\eta^1)\to\mathcal{F}(\eta^2), \quad (y_1,y_2)\mapsto\left(y_1, y_2\frac{1+\eta^2(y_1)}{1+\eta^1(y_1)}\right), \tag{2.4}$$

whose inverse is X_{η^2,η^1}. Our change of variables is thus defined by

$$X(t,\cdot) \stackrel{\text{def}}{=} X_{0,\eta(t)} : (y_1,y_2)\mapsto(y_1, y_2(1+\eta(t,y_1)))\,, \tag{2.5}$$

$$Y(t,\cdot) \stackrel{\text{def}}{=} X(t,\cdot)^{-1} = X_{\eta(t),0} : (x_1,x_2)\mapsto\left(x_1, x_2\frac{1}{1+\eta(t,x_1)}\right). \tag{2.6}$$

We write

$$a \stackrel{\text{def}}{=} {\rm Cof}(\nabla Y)^*, \quad b \stackrel{\text{def}}{=} {\rm Cof}(\nabla X)^*. \tag{2.7}$$

We set

$$w(t,y) \stackrel{\text{def}}{=} b(t,y)v(t,X(t,y)) \quad \text{and} \quad q(t,y) \stackrel{\text{def}}{=} p(t,X(t,y)), \tag{2.8}$$

so that

$$v(t,x) = a(t,x)w(t,Y(t,x)) \quad \text{and} \quad p(t,x) = q(t,Y(t,x)). \tag{2.9}$$

Remark 2.1. *Note that we use the cofactor matrix* ${\rm Cof}(\nabla X)$ *of* ∇X *in* (2.8)*. Such a change of variables allows us to keep the divergence free condition and the structure of the boundary conditions (see [13], [4], [1, Lemma 2.3]).*

Then some calculation yields

$$\begin{aligned}[b\Delta v(X)]_\alpha = \sum_{i,j,k} b_{\alpha i}\frac{\partial^2 a_{ik}}{\partial x_j^2}(X)w_k + 2\sum_{i,j,k,\ell} b_{\alpha i}\frac{\partial a_{ik}}{\partial x_j}(X)\frac{\partial w_k}{\partial y_\ell}\frac{\partial Y_\ell}{\partial x_j}(X)\\ + \sum_{j,\ell,m}\frac{\partial^2 w_\alpha}{\partial y_\ell\partial y_m}\frac{\partial Y_\ell}{\partial x_j}(X)\frac{\partial Y_m}{\partial x_j}(X) + \sum_{j,\ell}\frac{\partial w_\alpha}{\partial y_\ell}\frac{\partial^2 Y_\ell}{\partial x_j^2}(X),\end{aligned} \tag{2.10}$$

$$[b\,[(v\cdot\nabla)v]\,(X)]_\alpha = \sum_{i,j,k,m} b_{\alpha i}\frac{\partial a_{ik}}{\partial x_j}(X)a_{j,m}(X)w_k w_m + \frac{1}{\det(\nabla X)}\,[(w\cdot\nabla)w]_\alpha\,, \tag{2.11}$$

$$[b\nabla p(X)]_\alpha = \det(\nabla X)\sum_{k,i}\frac{\partial q}{\partial y_k}\frac{\partial Y_\alpha}{\partial x_i}(X)\frac{\partial Y_k}{\partial x_i}(X), \tag{2.12}$$

$$b\partial_t v(X) = b(\partial_t a)(X)w + \partial_t w + (\nabla w)(\partial_t Y)(X). \tag{2.13}$$

For the other equations of (1.3), we need to use in a more precise way (2.5) and (2.6). We have the following formulas

$$\nabla X(t,y_1,y_2) = \begin{bmatrix} 1 & 0\\ y_2\partial_s\eta & 1+\eta\end{bmatrix}, \quad b(t,y_1,y_2) = \begin{bmatrix} 1+\eta & 0\\ -y_2\partial_s\eta & 1\end{bmatrix}, \tag{2.14}$$

$$\nabla Y(t,x_1,x_2) = \begin{bmatrix} 1 & 0\\ -x_2\dfrac{\partial_s\eta}{(1+\eta)^2} & \dfrac{1}{1+\eta}\end{bmatrix}, \quad a(t,x_1,x_2) = \begin{bmatrix} \dfrac{1}{1+\eta} & 0\\ x_2\dfrac{\partial_s\eta}{(1+\eta)^2} & 1\end{bmatrix}. \tag{2.15}$$

Thus the boundary condition of (1.3) on $\Gamma_\#(\eta)$ rewrites

$$w(s,1) = \partial_t\eta(t,s)e_2 \quad t>0,\ s\in\mathbb{R}.$$

We also deduce from (1.5), (1.6), (1.7), (1.19) and (2.5) that

$$\begin{aligned}\mathbb{H}_\eta(v,p)(t,s) &= M\left\{[\nu(\nabla v+(\nabla v)^*)-pI_2]\circ X(t,s,1)\begin{bmatrix}-\partial_s\eta(t,s)\\1\end{bmatrix}\cdot e_2\right\}\\ &= -Mq(t,s,1)+M\Bigg\{\Bigg[\nu\sum_k\frac{\partial a_{ik}}{\partial x_j}(X)w_k+\nu\sum_{k,\ell}a_{ik}(X)\frac{\partial w_k}{\partial y_\ell}\frac{\partial Y_\ell}{\partial x_j}(X)+\nu\sum_k\frac{\partial a_{jk}}{\partial x_i}(X)w_k\\ &\qquad+\nu\sum_{k,\ell}a_{jk}(X)\frac{\partial w_k}{\partial y_\ell}\frac{\partial Y_\ell}{\partial x_i}(X)\Bigg]_{i,j}(t,s,1)\begin{bmatrix}-\partial_s\eta(t,s)\\1\end{bmatrix}\cdot e_2\Bigg\}. \qquad (2.16)\end{aligned}$$

Moreover, we recall that

$$\mathbb{H}_0(w,q)(t,s)=M\left\{[\mathbb{T}(w,q)e_2](t,s,1)\cdot e_2\right\}. \qquad (2.17)$$

In particular,

$$\begin{aligned}\widehat{G}(\eta,w)(t,s) &\stackrel{\text{def}}{=} \mathbb{H}_0(w,q)(t,s)-\mathbb{H}_\eta(v,p)(t,s)\\ &= \nu M\Bigg\{2\sum_{k,\ell}\left[\delta_{2,k}\delta_{2,\ell}-a_{2k}(X)\frac{\partial Y_\ell}{\partial x_2}(X)\right]\frac{\partial w_k}{\partial y_\ell}\\ &\quad+\partial_s\eta\left[\sum_{k,\ell}a_{2k}(X)\frac{\partial w_k}{\partial y_\ell}\frac{\partial Y_\ell}{\partial x_1}(X)+\sum_{k,\ell}a_{1k}(X)\frac{\partial w_k}{\partial y_\ell}\frac{\partial Y_\ell}{\partial x_2}(X)\right]\\ &\quad+\sum_k\left(\partial_s\eta\left[\frac{\partial a_{2k}}{\partial x_1}(X)+\frac{\partial a_{1k}}{\partial x_2}(X)\right]-2\frac{\partial a_{2k}}{\partial x_2}(X)\right)w_k\Bigg\}. \qquad (2.18)\end{aligned}$$

We also define

$$\begin{aligned}\widehat{F}_\alpha(\eta,w,q) &\stackrel{\text{def}}{=} \nu\sum_{i,j,k}b_{\alpha i}\frac{\partial^2 a_{ik}}{\partial x_j^2}(X)w_k+2\nu\sum_{i,j,k,\ell}b_{\alpha i}\frac{\partial a_{ik}}{\partial x_j}(X)\frac{\partial w_k}{\partial y_\ell}\frac{\partial Y_\ell}{\partial x_j}(X)\\ &\quad+\nu\sum_{j,\ell,m}\frac{\partial^2 w_\alpha}{\partial y_\ell\partial y_m}\left(\frac{\partial Y_\ell}{\partial x_j}(X)\frac{\partial Y_m}{\partial x_j}(X)-\delta_{\ell,j}\delta_{m,j}\right)+\nu\sum_{j,\ell}\frac{\partial w_\alpha}{\partial y_\ell}\frac{\partial^2 Y_\ell}{\partial x_j^2}(X)\\ &\quad-\sum_{k,i}\frac{\partial q}{\partial y_k}\left(\det(\nabla X)\frac{\partial Y_\alpha}{\partial x_i}(X)\frac{\partial Y_k}{\partial x_i}(X)-\delta_{\alpha,i}\delta_{k,i}\right)\\ &\quad-\sum_{i,j,k,m}b_{\alpha i}\frac{\partial a_{ik}}{\partial x_j}(X)a_{jm}(X)w_kw_m-\frac{1}{\det(\nabla X)}\left[(w\cdot\nabla)w\right]_\alpha\\ &\quad-\left[b(\partial_t a)(X)w\right]_\alpha-\left[(\nabla w)(\partial_t Y)(X)\right]_\alpha, \qquad (2.19)\end{aligned}$$

Then system (1.3), (1.4) rewrites,

$$\begin{cases}\partial_t w-\operatorname{div}\mathbb{T}(w,q)=\widehat{F}(\eta,w,q) \quad \text{in } (0,\infty)\times\mathcal{F}_\#,\\ \operatorname{div} w=0 \quad \text{in } (0,\infty)\times\mathcal{F}_\#,\\ w(t,s,1)=(\partial_t\eta)(t,s)e_2 \quad t>0,\ s\in\mathbb{R},\\ w=0 \quad t>0,\ y\in\Gamma_{\text{fix},\#},\\ w\ Le_1-\text{periodic} \quad t>0,\\ \partial_{tt}\eta+A_1\eta=-\mathbb{H}_0(w,q)+\widehat{G}(\eta,w), \quad t>0,\end{cases} \qquad (2.20)$$

with the initial conditions

$$\eta(0)=\eta_1^0,\quad \partial_t\eta(0)=\eta_2^0 \quad \text{and} \quad w(0,y)=w_0(y)\stackrel{\text{def}}{=}b(0,y)v_0(X(0,y)),\quad y\in\mathcal{F}. \qquad (2.21)$$

Remark 2.2. *We can notice that in the reference geometry we use here (where $n=e_2$ on Γ),*

$$\mathbb{H}_0(w,q)(t,s)=-Mq(t,s,1).$$

This can be done by standard calculation. Nevertheless, this particular form of $\mathbb{H}_0$ is never used in what follows.

Using (2.14) and (2.15), we can precise some terms of the above nonlinearities:

$$a(X) = \begin{bmatrix} \dfrac{1}{1+\eta} & 0 \\ y_2 \dfrac{\partial_s \eta}{1+\eta} & 1 \end{bmatrix}, \quad \nabla Y(X) = \begin{bmatrix} 1 & 0 \\ -y_2 \dfrac{\partial_s \eta}{1+\eta} & \dfrac{1}{1+\eta} \end{bmatrix}, \tag{2.22}$$

$$\nabla Y(X) - I_2 = \begin{bmatrix} 0 & 0 \\ -y_2 \dfrac{\partial_s \eta}{1+\eta} & \dfrac{-\eta}{1+\eta} \end{bmatrix}, \quad \det(\nabla X) = 1+\eta, \tag{2.23}$$

$$\frac{\partial a}{\partial x_1}(X) = \begin{bmatrix} \dfrac{-\partial_s \eta}{(1+\eta)^2} & 0 \\ y_2 \dfrac{\partial_{ss}\eta(1+\eta) - 2(\partial_s \eta)^2}{(1+\eta)^2} & 0 \end{bmatrix}, \quad \frac{\partial a}{\partial x_2}(X) = \begin{bmatrix} 0 & 0 \\ \dfrac{\partial_s \eta}{(1+\eta)^2} & 0 \end{bmatrix}, \tag{2.24}$$

$$\frac{\partial^2 a}{\partial x_1 \partial x_2}(X) = \begin{bmatrix} 0 & 0 \\ \dfrac{\partial_{ss}\eta(1+\eta) - 2(\partial_s \eta)^2}{(1+\eta)^3} & 0 \end{bmatrix}, \tag{2.25}$$

$$\frac{\partial^2 a}{\partial x_1^2}(X) = \begin{bmatrix} \dfrac{-\partial_s \eta}{(1+\eta)^2} & 0 \\ y_2 \dfrac{\partial_{sss}\eta(1+\eta)^2 - 6(1+\eta)\partial_s\eta\partial_{ss}\eta + 6(\partial_s\eta)^3}{(1+\eta)^3} & 0 \end{bmatrix}, \quad \frac{\partial^2 a}{\partial x_2^2}(X) = 0, \tag{2.26}$$

$$\frac{\partial}{\partial x_1}\nabla Y(X) = \begin{bmatrix} 0 & 0 \\ y_2 \dfrac{-\partial_{ss}\eta(1+\eta) + 2(\partial_s \eta)^2}{(1+\eta)^2} & \dfrac{-\partial_s \eta}{(1+\eta)^2} \end{bmatrix}, \quad \frac{\partial}{\partial x_2}\nabla Y(X) = \begin{bmatrix} 0 & 0 \\ -\dfrac{\partial_s \eta}{(1+\eta)^2} & 0 \end{bmatrix}, \tag{2.27}$$

$$\partial_t a(X) = \begin{bmatrix} \dfrac{-\partial_t \eta}{(1+\eta)^2} & 0 \\ y_2 \dfrac{\partial_{ts}\eta(1+\eta) - 2\partial_s\eta\partial_t\eta}{(1+\eta)^2} & 0 \end{bmatrix}, \quad \partial_t Y(X) = \begin{bmatrix} 0 \\ -y_2 \dfrac{\partial_t \eta}{1+\eta} \end{bmatrix}. \tag{2.28}$$

3 The linear system

In this section, we consider a linear system associated with (2.20). This linear system is similar to the one introduced in [1] in the case of a damped beam equation and we use several results of this previous work.

We recall that $\mathcal{F}$, $\mathcal{F}_\#$, Γ, $\Gamma_\#$, Γ_{b}, $\Gamma_{\mathrm{b},\#}$ are defined by (2.1), (2.2) and (2.3), that L^α and H^k stand for Lebesgue spaces and Sobolev spaces respectively, and that we use the bold notation for the spaces of vector fields: $\mathbf{L}^\alpha = (L^\alpha)^2$, $\mathbf{H}^k = (H^k)^2$ etc. We also recall that $L^p_\#(0,L)$, $L^p_{\#,0}(0,L)$ and $H^\alpha_{\#,0}(0,L)$ are defined by (1.12)–(1.14).

We consider the following functional spaces:

$$L^p_\#(\Gamma_{\mathrm{b}}) \stackrel{\text{def}}{=} \{f \in L^p_{\mathrm{loc}}(\Gamma_{\mathrm{b},\#}) \ ; \ f(\cdot + Le_1) = f\} \quad (p \in [1,\infty]), \tag{3.1}$$

$$\mathbf{L}^p_{\#,0}(\Gamma_{\mathrm{b}}) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{L}^p_\#(\Gamma_{\mathrm{b}}) \ ; \ \int_{\Gamma_{\mathrm{b}}} f \cdot n \ d\gamma = 0 \right\} \quad (p \in [1,\infty]), \tag{3.2}$$

$$H^\alpha_\#(\Gamma_{\mathrm{b}}) \stackrel{\text{def}}{=} H^\alpha_{\mathrm{loc}}(\Gamma_{\mathrm{b},\#}) \cap L^2_\#(\Gamma_{\mathrm{b}}) \quad (\alpha \geqslant 0), \tag{3.3}$$

$$\mathbf{V}^\alpha_\#(\Gamma_{\mathrm{b}}) \stackrel{\text{def}}{=} \mathbf{H}^\alpha_\#(\Gamma_{\mathrm{b}}) \cap \mathbf{L}^2_{\#,0}(\Gamma_{\mathrm{b}}) \quad (\alpha \geqslant 0), \tag{3.4}$$

$$L^p_\#(\mathcal{F}) \stackrel{\text{def}}{=} \{f \in L^p_{\mathrm{loc}}(\mathcal{F}_\#) \ ; \ f(\cdot + Le_1) = f\} \quad (p \in [1,\infty]), \tag{3.5}$$

$$H^\alpha_\#(\mathcal{F}) \stackrel{\text{def}}{=} H^\alpha_{\mathrm{loc}}(\mathcal{F}_\#) \cap L^2_\#(\mathcal{F}) \quad (\alpha \geqslant 0), \tag{3.6}$$

$$\mathbf{V}^\alpha_\#(\mathcal{F}) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{H}^\alpha_\#(\mathcal{F}) \ ; \ \operatorname{div} f = 0 \right\}, \tag{3.7}$$

$$\mathbf{V}^\alpha_{\#,n}(\mathcal{F}) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{H}^\alpha_\#(\mathcal{F}) \ ; \ \operatorname{div} f = 0, \quad f \cdot n = 0 \quad \text{on } \Gamma_b \right\} \quad (\alpha \in [0, 1/2)), \tag{3.8}$$

$$\mathbf{V}^\alpha_{\#,n}(\mathcal{F}) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{H}^\alpha_\#(\mathcal{F}) \ ; \ \operatorname{div} f = 0, \quad f = 0 \quad \text{on } \Gamma_b \right\} \quad (\alpha \in (1/2, 1]). \tag{3.9}$$

We introduce the operator $\Lambda : L^2_\#(0,L) \to \mathbf{L}^2_\#(\Gamma_{\rm b})$ defined by

$$\begin{aligned} &(\Lambda\eta)(y) = (M\eta(y_1))\, e_2 \quad \text{if} \quad y \in \Gamma, \\ &(\Lambda\eta)(y) = 0 \qquad\quad \text{if} \quad y \in \Gamma_{\rm fix}. \end{aligned} \tag{3.10}$$

The adjoint $\Lambda^* : \mathbf{L}^2_\#(\Gamma_{\rm b}) \to L^2_\#(0,L)$ of Λ is given by

$$(\Lambda^* v)(s) = M\,(v(s,1)\cdot e_2) \quad (s \in (0,L)). \tag{3.11}$$

Note that for any $\alpha \in [0,4]$,

$$\Lambda \in \mathcal{L}(H^\alpha_\#(0,L), \mathbf{V}^\alpha_\#(\Gamma_{\rm b})) \tag{3.12}$$

and

$$\Lambda^* \in \mathcal{L}(\mathbf{H}^\alpha_\#(\Gamma_{\rm b})), \mathcal{D}(A_1^{\alpha/4})). \tag{3.13}$$

We recall that A_1 is defined in (1.15), (1.16) and satisfies (1.17). Moreover,

$$\|\Lambda\eta\|_{\mathbf{H}^\alpha_\#(\Gamma_{\rm b})} \geqslant c(\alpha)\|A_1^{\alpha/4}\eta\|_{\mathcal{H}_\mathcal{S}} \quad (\eta \in \mathcal{D}(A_1^{\alpha/4})). \tag{3.14}$$

Note that $\mathbb{H}_0$ (see (2.17)) can be written in a simpler way as

$$\mathbb{H}_0(w,q) = \Lambda^* \left\{ \mathbb{T}(w,q) n_{|\Gamma_{\rm b,\#}} \right\}. \tag{3.15}$$

We consider the space $\mathbf{L}^2_\#(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_\mathcal{S}$ equipped with the scalar product:

$$\left\langle \left[w^{(1)}, \eta_1^{(1)}, \eta_2^{(1)}\right], \left[w^{(2)}, \eta_1^{(2)}, \eta_2^{(2)}\right] \right\rangle = \int_\mathcal{F} w^{(1)} \cdot w^{(2)}\, dy + \left(A_1^{1/2}\eta_1^{(1)}, A_1^{1/2}\eta_1^{(2)}\right)_{\mathcal{H}_\mathcal{S}} + \left(\eta_2^{(1)}, \eta_2^{(2)}\right)_{\mathcal{H}_\mathcal{S}},$$

and we introduce the following spaces:

$$\mathcal{H} \stackrel{\rm def}{=} \left\{ [w,\eta_1,\eta_2] \in \mathbf{L}^2_\#(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_\mathcal{S} \ ;\ w\cdot n = (\Lambda\eta_2)\cdot n \text{ on } \Gamma_{\rm b,\#},\ \text{div } w = 0 \text{ in } \mathcal{F} \right\}, \tag{3.16}$$

$$\mathcal{V} \stackrel{\rm def}{=} \left\{ [w,\eta_1,\eta_2] \in \mathbf{H}^1_\#(\mathcal{F}) \times \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4}) \ ;\ w = \Lambda\eta_2 \text{ on } \Gamma_{\rm b,\#},\ \text{div } w = 0 \text{ in } \mathcal{F} \right\}.$$

We denote by P_0 the orthogonal projection from $\mathbf{L}^2_\#(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_\mathcal{S}$ onto $\mathcal{H}$. We have the following regularity result on P_0:

Lemma 3.1. *For any $s \in [0,1]$,*

$$P_0 \in \mathcal{L}(\mathbf{H}^s_\#(\mathcal{F}) \times \mathcal{D}(A_1^{1/2+s/4}) \times \mathcal{D}(A_1^{s/4})), \tag{3.17}$$

and

$$P_0 \in \mathcal{L}(\mathbf{L}^2_\#(\mathcal{F}) \times \mathcal{D}(A_1^{3/8}) \times \mathcal{D}(A_1^{1/8})'). \tag{3.18}$$

Proof. We have proven in [1, Proposition 3.1 and Proposition 3.2] relation (3.17) and that for any $[w,\eta_1,\eta_2] \in \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_\mathcal{S}$,

$$P_0 \begin{bmatrix} w \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} w - \nabla p \\ \eta_1 \\ \eta_2 + \Lambda^*(pn) \end{bmatrix}$$

where the pressure function $p \in H^1(\mathcal{F})$ obeys $\displaystyle\int_\mathcal{F} p\, dy = 0$ and is solution to the Neumann problem:

$$\begin{cases} \forall q \in H^1(\mathcal{F}) \ \text{ such that } \displaystyle\int_\mathcal{F} q\, dy = 0, \\ \displaystyle\int_\mathcal{F} \nabla p \cdot \nabla q\, dy + \left(\Lambda^*(pn), \Lambda^*(qn)\right)_{\mathcal{H}_\mathcal{S}} = \int_\mathcal{F} w \cdot \nabla q\, dy - \left(\eta_2, \Lambda^*(qn)\right)_{\mathcal{H}_\mathcal{S}}. \end{cases} \tag{3.19}$$

From (3.19) and by using the trace inequality $\|p\|_{H^{1/2}(\Gamma)} \leqslant C\|\nabla p\|_{\mathbf{L}^2(\mathcal{F})}$ and (3.13) for $\alpha = 1/2$, we deduce that

$$\begin{aligned} \|\nabla p\|^2_{\mathbf{L}^2(\mathcal{F})} + \|\Lambda^*(pn)\|^2_{\mathcal{H}_\mathcal{S}} &\leqslant C(\|w\|_{\mathbf{L}^2(\mathcal{F})}\|\nabla p\|_{\mathbf{L}^2(\mathcal{F})} + \|A_1^{-1/8}\eta_2\|_{\mathcal{H}_\mathcal{S}}\|A_1^{1/8}\Lambda^*(pn)\|_{\mathcal{H}_\mathcal{S}}), \\ &\leqslant C(\|w\|_{\mathbf{L}^2(\mathcal{F})} + \|A_1^{-1/8}\eta_2\|_{\mathcal{H}_\mathcal{S}})\|\nabla p\|_{\mathbf{L}^2(\mathcal{F})}, \end{aligned}$$

from which, we obtain (3.18). □

We now define the linear operator $A_0 : \mathcal{D}(A_0) \subset \mathcal{H} \to \mathcal{H}$:

$$\mathcal{D}(A_0) \stackrel{\text{def}}{=} \mathcal{V} \cap \left[\mathbf{H}^2_{\#}(\mathcal{F}) \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2})\right], \tag{3.20}$$

and for $\left[w, \eta_1, \eta_2\right] \in \mathcal{D}(A_0)$, we set

$$\widetilde{A}_0 \begin{bmatrix} w \\ \eta_1 \\ \eta_2 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \nu \Delta w \\ \eta_2 \\ -A_1 \eta_1 - \Lambda^*(2\nu D(w)n) \end{bmatrix} \tag{3.21}$$

and

$$A_0 \stackrel{\text{def}}{=} P_0 \widetilde{A}_0. \tag{3.22}$$

Remark 3.2. *As already pointed out in Remark 2.2, by using the particular geometry of $\mathcal{F}$, we can simplify the above expression since $\Lambda^*(2\nu D(w)n) = 0$. This simplification is not used in this paper and several results of the next sections remain true for a general geometry. The important consequence of our geometry in our work corresponds to the commutativity of some operators (see Proposition 4.6).*

By using the above operators, we can rewrite the following linear system

$$\begin{cases} \partial_t w - \operatorname{div} \mathbb{T}(w, q) = F \quad \text{in } (0, \infty) \times \mathcal{F}_{\#}, \\ \operatorname{div} w = 0 \quad \text{in } (0, \infty) \times \mathcal{F}_{\#}, \\ w(t, s, 1) = (\partial_t \eta)(t, s) e_2 \quad (t, s) \in (0, \infty) \times \mathbb{R}, \\ w(t, \cdot) \ Le_1 - \text{periodic} \quad t \in (0, \infty), \\ \partial_{tt} \eta + A_1 \eta = -\mathbb{H}_0(w, q) + G \quad \text{in } (0, \infty), \end{cases} \tag{3.23}$$

with the initial conditions

$$w(0, \cdot) = w^0, \quad \eta(0, \cdot) = \eta_1^0, \quad \partial_t \eta(0, \cdot) = \eta_2^0, \tag{3.24}$$

as follows

$$\frac{d}{dt} \begin{bmatrix} w \\ \eta \\ \partial_t \eta \end{bmatrix} = A_0 \begin{bmatrix} w \\ \eta \\ \partial_t \eta \end{bmatrix} + P_0 \begin{bmatrix} F \\ 0 \\ G \end{bmatrix}, \quad \begin{bmatrix} w \\ \eta \\ \partial_t \eta \end{bmatrix} (0) = \begin{bmatrix} w^0 \\ \eta_1^0 \\ \eta_2^0 \end{bmatrix}. \tag{3.25}$$

We have the following result (see [1, Proposition 3.4, Proposition 3.5 and Remark 3.6]).

Proposition 3.3. *The operator A_0 defined by (3.20)–(3.22) has compact resolvents, it is the infinitesimal generator of a strongly continuous semigroup of contractions on $\mathcal{H}$ and it is exponentially stable on $\mathcal{H}$.*

We have also the following result (see [1, Proposition 3.8]).

Proposition 3.4. *For $\theta \in [0, 1]$, the following equalities hold*

$$\mathcal{D}((-A_0)^\theta) = \left[\mathbf{H}^{2\theta}_{\#}(\mathcal{F}) \times \mathcal{D}(A_1^{1/2+\theta/2}) \times \mathcal{D}(A_1^{\theta/2})\right] \cap \mathcal{H} \quad \textit{if } \theta \in (0, 1/4), \tag{3.26}$$

$$\mathcal{D}((-A_0)^\theta) = \left\{[w, \eta_1, \eta_2] \in \left[\mathbf{H}^{2\theta}_{\#}(\mathcal{F}) \times \mathcal{D}(A_1^{1/2+\theta/2}) \times \mathcal{D}(A_1^{\theta/2})\right] \cap \mathcal{H}\,;\, w = \Lambda \eta_2 \textit{ on } \Gamma_{\mathrm{b},\#}\right\} \quad \textit{if } \theta \in (1/4, 1]. \tag{3.27}$$

4 Gevrey type resolvent estimates for A_0

We introduce the notation

$$\mathbb{C}^+ \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C}\ ;\ \operatorname{Re}(\lambda) \geqslant 0\}, \tag{4.1}$$

and

$$\mathbb{C}^+_\alpha \stackrel{\text{def}}{=} \left\{\lambda \in \mathbb{C}^+\ ;\ |\lambda| > \alpha\right\}. \tag{4.2}$$

The goal of this section it to prove the following result on the operator A_0 defined by (3.20)–(3.22).

Theorem 4.1. *There exists $C > 0$ such that for all $\lambda \in \mathbb{C}^+$*

$$|\lambda|^{1/2} \left\|(\lambda - A_0)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant C. \tag{4.3}$$

Moreover, there exists $C > 0$ such that for all $\lambda \in \mathbb{C}^+$ and for all

$$\left[f, g, h\right] \in \mathcal{H} \cap \left(\mathbf{L}^2_\#(\mathcal{F}) \times \mathcal{D}(A_1^{5/8}) \times \mathcal{D}(A_1^{1/8})\right),$$

the following estimates hold

$$\left\|(\lambda - A_0)^{-1} \begin{bmatrix} f \\ g \\ h \end{bmatrix}\right\|_{\mathbf{H}^2_\#(\mathcal{F})\times\mathcal{D}(A_1^{7/8})\times\mathcal{D}(A_1^{3/8})} \leqslant C \left\|\begin{bmatrix} f \\ g \\ h \end{bmatrix}\right\|_{\mathbf{L}^2_\#(\mathcal{F})\times\mathcal{D}(A_1^{5/8})\times\mathcal{D}(A_1^{1/8})} \tag{4.4}$$

and

$$|\lambda| \left\|(\lambda - A_0)^{-1} \begin{bmatrix} f \\ g \\ h \end{bmatrix}\right\|_{\mathbf{L}^2_\#(\mathcal{F})\times\mathcal{D}(A_1^{3/8})\times\mathcal{D}(A_1^{1/8})'} \leqslant C \left\|\begin{bmatrix} f \\ g \\ h \end{bmatrix}\right\|_{\mathbf{L}^2_\#(\mathcal{F})\times\mathcal{D}(A_1^{5/8})\times\mathcal{D}(A_1^{1/8})}. \tag{4.5}$$

Remark 4.2. *A consequence of* (4.3) *is the following resolvent estimate,*

$$\sup_{\tau\in\mathbb{R}} |\tau|^{1/2} \|(\imath\tau - A_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} < +\infty,$$

which implies in particular that $(e^{tA_0})_{t\geqslant 0}$ is of Gevrey class δ for all $\delta > 2$ (see [19]), namely, for all compact $K \subset (0, +\infty)$ and $\theta > 0$ there exists $C = C(\theta, K)$ such that for all $t \in K$ and all $n \in \mathbb{N}$,

$$\left\|\frac{d^n e^{tA_0}}{dt^n}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant C\theta^n (n!)^\delta.$$

In order to prove Theorem 4.1, we rewrite the resolvent equation in a more convenient way. Assume $\lambda \in \mathbb{C}^+$ and $[f, g, h] \in \mathcal{H}$. We set $[v, \eta_1, \eta_2] := (\lambda - A_0)^{-1}[f, g, h]$ so that

$$\begin{cases} \lambda v - \operatorname{div} \mathbb{T}(v, p) = f \quad \text{in } \mathcal{F}_\#, \\ \operatorname{div} v = 0 \quad \text{in } \mathcal{F}_\#, \\ v = \Lambda\eta_2 \quad \text{on } \Gamma_{\mathrm{b},\#}, \\ v \ Le_1 - \text{periodic}, \\ \lambda\eta_1 - \eta_2 = g, \\ \lambda\eta_2 + A_1\eta_1 = -\Lambda^* \left\{\mathbb{T}(v, p) n_{|\Gamma_{\mathrm{b},\#}}\right\} + h. \end{cases} \tag{4.6}$$

For all $\lambda \in \mathbb{C}^+$, we define the solution (w_η, q_η) (that depends on λ) of

$$\begin{cases} \lambda w_\eta - \operatorname{div} \mathbb{T}(w_\eta, q_\eta) = 0 \quad \text{in } \mathcal{F}_\#, \\ \operatorname{div} w_\eta = 0 \quad \text{in } \mathcal{F}_\#, \\ w_\eta = \Lambda\eta \quad \text{on } \Gamma_{\mathrm{b},\#}, \\ w_\eta \ Le_1 - \text{periodic}. \end{cases} \tag{4.7}$$

We also define the operator

$$D_0(\lambda)\eta \stackrel{\text{def}}{=} w_\eta. \tag{4.8}$$

We denote by $\mathbb{A}$ the Stokes operator:

$$\mathcal{D}(\mathbb{A}) \stackrel{\text{def}}{=} \mathbf{V}^1_{\#,n}(\mathcal{F}) \cap \mathbf{H}^2_\#(\mathcal{F}), \quad \mathbb{A} \stackrel{\text{def}}{=} \nu\mathbb{P}\Delta : \mathcal{D}(\mathbb{A}) \to \mathbf{V}^0_{\#,n}(\mathcal{F}), \tag{4.9}$$

where $\mathbb{P} : \mathbf{L}^2_\#(\mathcal{F}) \to \mathbf{V}^0_{\#,n}(\mathcal{F})$ is the Leray projection operator.

Using the above notation, we can decompose the solution of (4.6) as

$$v = D_0(\lambda)\eta_2 + (\lambda - \mathbb{A})^{-1}\mathbb{P}f,$$

that is $(v,p) = (w_{\eta_2}, q_{\eta_2}) + (\widehat{v},\widehat{p})$ where (w_{η_2}, q_{η_2}) satisfies (4.7) with $\eta = \eta_2$ and where $(\widehat{v},\widehat{p})$ satisfies

$$\left\{\begin{array}{c} \lambda\widehat{v} - \operatorname{div}\mathbb{T}(\widehat{v},\widehat{p}) = f \quad \text{in } \mathcal{F}_{\#}, \\ \operatorname{div}\widehat{v} = 0 \quad \text{in } \mathcal{F}_{\#}, \\ \widehat{v} = 0 \quad \text{on } \Gamma_{\mathrm{b},\#}, \\ \widehat{v}\ Le_1 - \text{periodic}. \end{array}\right. \tag{4.10}$$

In that case, system (4.6) becomes

$$\left\{\begin{array}{c} \lambda\eta_1 - \eta_2 = g \\ \lambda\eta_2 + A_1\eta_1 = -\Lambda^*\left\{\mathbb{T}(\widehat{v},\widehat{p})n_{|\Gamma_{\mathrm{b},\#}}\right\} - \Lambda^*\left\{\mathbb{T}(w_{\eta_2},q_{\eta_2})n_{|\Gamma_{\mathrm{b},\#}}\right\} + h. \end{array}\right. \tag{4.11}$$

This leads us to define the operators $\mathcal{T}(\lambda) \in \mathcal{L}(\mathbf{L}^2_{\#}(\mathcal{F}), \mathcal{D}(A_1^{1/8}))$ and $L(\lambda) \in \mathcal{L}(\mathcal{D}(A_1^{3/8}), \mathcal{D}(A_1^{1/8}))$ (see Proposition 4.3 and (4.31) below) by

$$\mathcal{T}(\lambda)f \stackrel{\text{def}}{=} -\Lambda^*\left\{\mathbb{T}(\widehat{v},\widehat{p})n_{|\Gamma_{\mathrm{b},\#}}\right\}, \tag{4.12}$$

$$L(\lambda)\eta \stackrel{\text{def}}{=} \Lambda^*\left\{\mathbb{T}(w_\eta,q_\eta)n_{|\Gamma_{\mathrm{b},\#}}\right\}, \tag{4.13}$$

so that system (4.11) can be written

$$\left\{\begin{array}{c} \lambda\eta_1 - \eta_2 = g \\ \lambda\eta_2 + L(\lambda)\eta_2 + A_1\eta_1 = \mathcal{T}(\lambda)f + h. \end{array}\right. \tag{4.14}$$

We thus introduce the operator

$$\mathcal{A}(\lambda) \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -I \\ A_1 & L(\lambda) \end{bmatrix} \tag{4.15}$$

and study the equation

$$(\lambda + \mathcal{A}(\lambda)) \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} g \\ \widehat{h} \end{bmatrix}, \tag{4.16}$$

where

$$\widehat{h} \stackrel{\text{def}}{=} \mathcal{T}(\lambda)f + h.$$

In what follows, we need the following operator:

$$V(\lambda) = \lambda^2 I + \lambda L(\lambda) + A_1. \tag{4.17}$$

We will prove that $V(\lambda)$ is invertible (see Proposition 4.8 below), so we can compute the inverse of $\lambda + \mathcal{A}(\lambda)$

$$(\lambda + \mathcal{A}(\lambda))^{-1} = \begin{bmatrix} \dfrac{I - V^{-1}(\lambda)A_1}{\lambda} & V^{-1}(\lambda) \\ -V^{-1}(\lambda)A_1 & \lambda V^{-1}(\lambda) \end{bmatrix}, \tag{4.18}$$

and the inverse of $\lambda - A_0$:

$$(\lambda - A_0)^{-1} = \begin{bmatrix} (\lambda - \mathbb{A})^{-1}\mathbb{P} + \lambda D_0(\lambda)V^{-1}(\lambda)\mathcal{T}(\lambda) & -D_0(\lambda)V^{-1}(\lambda)A_1 & \lambda D_0(\lambda)V^{-1}(\lambda) \\ V^{-1}(\lambda)\mathcal{T}(\lambda) & \dfrac{I - V^{-1}(\lambda)A_1}{\lambda} & V^{-1}(\lambda) \\ \lambda V^{-1}(\lambda)\mathcal{T}(\lambda) & -V^{-1}(\lambda)A_1 & \lambda V^{-1}(\lambda) \end{bmatrix}. \tag{4.19}$$

4.1 Preliminaries

We first recall a standard result on the operator $\mathbb{A}$ (see (4.9)) and on the operator $\mathcal{T}$ (see (4.12)).

Proposition 4.3. *Let $\theta \in [0,1]$ and $f \in \mathbf{L}^2_{\#}(\mathcal{F})$. There exists $C > 0$ such that for any solution $(\widehat{v},\widehat{p})$ of (4.10) and for any $\lambda \in \mathbb{C}^+$,*

$$\|\widehat{v}\|_{\mathbf{H}^{2\theta}_{\#}(\mathcal{F})} \leqslant C|\lambda|^{\theta-1}\|f\|_{\mathbf{L}^2_{\#}(\mathcal{F})}. \tag{4.20}$$

In particular, there exists $C > 0$ such that for any $\lambda \in \mathbb{C}^+$,

$$\|(\lambda - \mathbb{A})^{-1}\mathbb{P}\|_{\mathcal{L}(\mathbf{L}^2_{\#}(\mathcal{F}),\mathbf{H}^2_{\#}(\mathcal{F}))} + |\lambda|\|(\lambda - \mathbb{A})^{-1}\mathbb{P}\|_{\mathcal{L}(\mathbf{L}^2_{\#}(\mathcal{F}))} \leqslant C, \tag{4.21}$$

and

$$\|\mathcal{T}(\lambda)\|_{\mathcal{L}(\mathbf{L}^2_{\#}(\mathcal{F}),\mathcal{D}(A_1^{1/8}))} \leqslant C. \tag{4.22}$$

Proof. Using that the Stokes operator $\mathbb{A}$ (defined by (4.9)) is the infinitesimal generator of an analytic semigroup and that $\mathbb{C}^+ \subset \rho(\mathbb{A})$, we have the following properties

$$\|(-\mathbb{A})^\theta(\lambda-\mathbb{A})^{-1}g\|_{\mathbf{L}^2_\#(\mathcal{F})} \leqslant C|\lambda|^{\theta-1}\|g\|_{\mathbf{L}^2_\#(\mathcal{F})} \quad (g\in \mathbf{V}^0_{\#,n}(\mathcal{F}),\ \lambda\in\mathbb{C}^+,\ \theta\in[0,1]).$$

This can be proven by interpolation by showing the case $\theta=0$ and $\theta=1$. We deduce (4.20) (and then (4.21)) from the above estimate and from the embedding $\mathcal{D}((-\mathbb{A})^\theta)\hookrightarrow \mathbf{H}^{2\theta}_\#(\mathcal{F})$.

We deduce in particular

$$\|\nabla\widehat{p}\|_{\mathbf{L}^2_\#(\mathcal{F})} \leqslant C\|\widehat{v}\|_{\mathbf{H}^2_\#(\mathcal{F})} + |\lambda|\|\widehat{v}\|_{\mathbf{L}^2_\#(\mathcal{F})} + \|f\|_{\mathbf{L}^2_\#(\mathcal{F})} \leqslant C\|f\|_{\mathbf{L}^2_\#(\mathcal{F})}.$$

Using (3.13) and (4.12), we obtain

$$\|\mathcal{T}(\lambda)f\|_{\mathcal{D}(A_1^{1/8})} \leqslant C\|\mathbb{T}(\widehat{v},\widehat{p})n\|_{\mathbf{H}^{1/2}_\#(\Gamma_{\mathrm{b},\#})} \leqslant C\|f\|_{\mathbf{L}^2_\#(\mathcal{F})}.$$

□

Next we show the following result on the operator $D_0(\lambda)$ defined by (4.8).

Proposition 4.4. *For any $\lambda\in\mathbb{C}^+$, the operator $D_0(\lambda)$ satisfies*

$$D_0(\lambda)\in\mathcal{L}(\mathcal{D}(A_1^{3/8}),\mathbf{H}^2_\#(\mathcal{F}))\cap\mathcal{L}(\mathcal{D}(A_1^{1/8})',\mathbf{L}^2_\#(\mathcal{F})).$$

More precisely, for $\theta\in[0,1]$, there exists a constant $C>0$ such that the operator $D_0(\lambda)$ defined by (4.8) *satisfies*

$$\|D_0(\lambda)\eta\|_{\mathbf{H}^{2\theta}_\#(\mathcal{F})} \leqslant C\left(\|A_1^{\theta/2-1/8}\eta\|_{\mathcal{H}_\mathcal{S}} + |\lambda|^\theta\|A_1^{-1/8}\eta\|_{\mathcal{H}_\mathcal{S}}\right) \quad (\eta\in\mathcal{D}(A_1^{\theta/2-1/8}),\ \lambda\in\mathbb{C}^+). \tag{4.23}$$

Proof. Using (3.12) and standard Stokes regularity results, we obtain the first part of the proof.

In order to prove (4.23), we first decompose the solution (w_η,q_η) of (4.7) as

$$(w_\eta,q_\eta)=(v_\eta,p_\eta)+(z_\eta,\zeta_\eta)$$

with

$$\begin{cases} -\operatorname{div}\mathbb{T}(v_\eta,p_\eta)=0 \quad \text{in } \mathcal{F}_\#,\\ \operatorname{div} v_\eta=0 \quad \text{in } \mathcal{F}_\#,\\ v_\eta=\Lambda\eta \quad \text{on } \Gamma_{\mathrm{b},\#},\\ v_\eta\ Le_1-\text{periodic},\end{cases} \quad \text{and} \quad \begin{cases} \lambda z_\eta-\operatorname{div}\mathbb{T}(z_\eta,\zeta_\eta)=-\lambda v_\eta \quad \text{in } \mathcal{F}_\#,\\ \operatorname{div} z_\eta=0 \quad \text{in } \mathcal{F}_\#,\\ z_\eta=0 \quad \text{on } \Gamma_{\mathrm{b},\#},\\ z_\eta\ Le_1-\text{periodic}.\end{cases} \tag{4.24}$$

Using again standard Stokes regularity results, we deduce that for any $\theta\in[0,1]$, there exists a positive constant C independent of λ such that

$$\|v_\eta\|_{\mathbf{H}^{2\theta}_\#(\mathcal{F})} \leqslant C\|A_1^{\theta/2-1/8}\eta\|_{\mathcal{H}_\mathcal{S}}. \tag{4.25}$$

On the other hand, from (4.20) in Proposition 4.3 we deduce

$$\|z_\eta\|_{\mathbf{H}^{2\theta}_\#(\mathcal{F})} \leqslant C|\lambda|^\theta\|v_\eta\|_{\mathbf{L}^2_\#(\mathcal{F})}.$$

Combining the above estimates with (4.25), we obtain

$$\|z_\eta\|_{\mathbf{H}^{2\theta}_\#(\mathcal{F})} \leqslant C|\lambda|^\theta\|A_1^{-1/8}\eta\|_{\mathcal{H}_\mathcal{S}}.$$

Then (4.23) follows by combining the above inequalities with (4.25). □

Using Proposition 4.4, we can define for $\lambda\in\mathbb{C}^+$,

$$K(\lambda)\in\mathcal{L}(\mathcal{D}(A_1^{1/8})',\mathcal{D}(A_1^{1/8})),\quad G(\lambda)\in\mathcal{L}(\mathcal{D}(A_1^{1/8}),\mathcal{D}(A_1^{1/8})')$$

by

$$\langle K(\lambda)\eta,\zeta\rangle_{\mathcal{D}(A_1^{1/8}),\mathcal{D}(A_1^{1/8})'} \stackrel{\text{def}}{=} \int_\mathcal{F} w_\eta\cdot\overline{w_\zeta}\,dy \tag{4.26}$$

and

$$\langle G(\lambda)\eta,\zeta\rangle_{\mathcal{D}(A_1^{1/8})',\mathcal{D}(A_1^{1/8})} \stackrel{\text{def}}{=} 2\int_{\mathcal{F}} Dw_\eta : D\overline{w_\zeta}\ dy. \tag{4.27}$$

Note that we have

$$K(\lambda)\eta = -\Lambda^*\{\mathbb{T}(\varphi_\eta,\pi_\eta)n|_{\Gamma_{\mathrm{b},\#}}\} \tag{4.28}$$

where

$$\begin{cases} \lambda\varphi_\eta - \operatorname{div}\mathbb{T}(\varphi_\eta,\pi_\eta) = w_\eta \quad \text{in } \mathcal{F}_\#, \\ \operatorname{div}\varphi_\eta = 0 \quad \text{in } \mathcal{F}_\#, \\ \varphi_\eta = 0 \quad \text{on } \Gamma_{\mathrm{b},\#}, \\ \varphi_\eta\ Le_1 - \text{periodic}, \end{cases} \tag{4.29}$$

and where w_η is the solution of (4.7).

If $\eta \in \mathcal{D}(A_1^{3/8})$, then we can write

$$2\int_{\mathcal{F}} Dw_\eta : D\overline{w_\zeta}dy = 2\int_0^L \Lambda^*((Dw_\eta)n)\ \overline{\zeta}\ ds - \int_{\mathcal{F}} \Delta w_\eta \cdot \overline{w_\zeta}dy,$$

and, with Proposition 4.4, we deduce that

$$G(\lambda) \in \mathcal{L}(\mathcal{D}(A_1^{3/8}),\mathcal{D}(A_1^{1/8})). \tag{4.30}$$

The operators $K(\lambda)$ and $G(\lambda)$ are related to the operator $L(\lambda)$ defined by (4.13): multiplying (4.7) by $\overline{w_\zeta}$ and integrating by part, we deduce that

$$L(\lambda) = \lambda K(\lambda) + G(\lambda). \tag{4.31}$$

Proposition 4.5. *The operators $K(\lambda) \in \mathcal{L}(\mathcal{D}(A_1^{1/8})',\mathcal{D}(A_1^{1/8}))$ and $G(\lambda) \in \mathcal{L}(\mathcal{D}(A_1^{1/8}),\mathcal{D}(A_1^{1/8})')$ defined above are non negative and self-adjoint. There exist $0<\rho_1<\rho_2$ such that for any $\lambda \in \mathbb{C}^+$, we have*

$$\rho_1\|A_1^{1/8}\eta\|^2_{\mathcal{H}_\mathcal{S}} \leqslant \langle G(\lambda)\eta,\eta\rangle_{\mathcal{D}(A_1^{1/8})',\mathcal{D}(A_1^{1/8})} \leqslant \rho_2\left(\|A_1^{1/8}\eta\|^2_{\mathcal{H}_\mathcal{S}} + |\lambda|\|A_1^{-1/8}\eta\|^2_{\mathcal{H}_\mathcal{S}}\right) \quad (\eta\in\mathcal{D}(A_1^{1/8})), \tag{4.32}$$

$$0 \leqslant \langle K(\lambda)\eta,\eta\rangle_{\mathcal{D}(A_1^{1/8}),\mathcal{D}(A_1^{1/8})'} \leqslant \rho_2\|A_1^{-1/8}\eta\|^2_{\mathcal{H}_\mathcal{S}} \quad (\eta\in\mathcal{D}(A_1^{1/8})'). \tag{4.33}$$

Proof. First, by definition, $K(\lambda)$, $G(\lambda)$ are symmetric and

$$\begin{cases} \langle G(\lambda)\eta,\eta\rangle_{\mathcal{D}(A_1^{1/8})',\mathcal{D}(A_1^{1/8})} = 2\displaystyle\int_{\mathcal{F}} |Dw_\eta|^2\ dy \quad (\eta\in\mathcal{D}(A_1^{1/8})), \\ \langle K(\lambda)\eta,\eta\rangle_{\mathcal{D}(A_1^{1/8}),\mathcal{D}(A_1^{1/8})'} = \displaystyle\int_{\mathcal{F}} |w_\eta|^2\ dy \quad (\eta\in\mathcal{D}(A_1^{1/8})). \end{cases} \tag{4.34}$$

In particular, they are non negative and since $K(\lambda)$ is bounded, it is self-adjoint.

From (4.34), (3.14) and from the trace theorem, there exists a constant $C>0$ such that

$$\langle G(\lambda)\eta,\eta\rangle_{\mathcal{H}_\mathcal{S}} \geqslant C\|w_\eta\|^2_{\mathbf{H}^{1/2}_\#(\Gamma)} = C\|\Lambda\eta\|^2_{\mathbf{H}^{1/2}_\#(\Gamma_b)} \geqslant C\|A_1^{1/8}\eta\|^2_{\mathcal{H}_\mathcal{S}}.$$

This yields the left inequality in (4.32).

The other estimates are a consequence of (4.34) and (4.23).

Finally, from the Lax-Milgram lemma we deduce that $G(\lambda) : \mathcal{D}(A_1^{1/8}) \to \mathcal{D}(A_1^{1/8})'$ is onto. Since it is also symmetric we conclude that $G(\lambda)$ is self-adjoint (see e.g. [21, Proposition 3.2.4]). □

The next result is crucial in our analysis and is due to the particular shape of the domain $\mathcal{F}$.

Proposition 4.6. *Assume $\lambda \in \mathbb{C}^+$. Then for $\alpha \in \mathbb{R}$*

$$L(\lambda)A_1^\alpha = A_1^\alpha L(\lambda), \quad K(\lambda)A_1^\alpha = A_1^\alpha K(\lambda), \quad G(\lambda)A_1^\alpha = A_1^\alpha G(\lambda). \tag{4.35}$$

Proof. Using the definition (3.10) of the operator Λ and (3.11), we first notice that

$$\partial_{y_1}\Lambda\eta = \Lambda(\partial_s\eta) \quad (\eta \in H^1_\#(0,L)), \quad \partial_s\Lambda^* v = \Lambda^*(\partial_{y_1} v) \quad (v \in \mathbf{H}^1_\#(\Gamma_\mathrm{b})).$$

Thus, we can differentiate system (4.7) with respect to y_1 and we obtain that

$$\partial_{y_1} w_\eta = w_{\partial_s\eta}.$$

We thus deduce $L(\lambda)A_1 = A_1 L(\lambda)$. Using (4.28), we show similarly that $K(\lambda)A_1 = A_1 K(\lambda)$ and from (4.31), we finally deduce $G(\lambda)A_1 = A_1 G(\lambda)$. □

Corollary 4.7. *Let ρ_2 the constant introduced in Proposition 4.5. For any $\lambda \in \mathbb{C}^+$, we have*

$$\|G(\lambda)\eta\|^2_{\mathcal{H}_S} \leqslant 2\rho_2^2 \left(\|A_1^{1/4}\eta\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{-1/4}\eta\|^2_{\mathcal{H}_S}\right) \quad (\eta \in \mathcal{D}(A_1^{1/4})). \tag{4.36}$$

Proof. First, since $G(\lambda) : \mathcal{D}(A_1^{1/8}) \to \mathcal{D}(A_1^{1/8})'$ is positive and self-adjoint, we can define its square root $G(\lambda)^{1/2}$. Using that $[\mathcal{D}(A_1^{1/8}), \mathcal{D}(A_1^{1/8})']_{1/2} = \mathcal{H}_S$, we have $G(\lambda)^{1/2} \in \mathcal{L}(\mathcal{H}_S, \mathcal{D}(A_1^{1/8})')$ and $G(\lambda)^{1/2} \in \mathcal{L}(\mathcal{D}(A_1^{1/8}), \mathcal{H}_S)$.

The right inequality in (4.32) yields

$$\|G(\lambda)^{1/2}\eta\|^2_{\mathcal{H}_S} \leqslant \rho_2 \left(\|A_1^{1/8}\eta\|^2_{\mathcal{H}_S} + |\lambda|\|A_1^{-1/8}\eta\|^2_{\mathcal{H}_S}\right) \quad (\eta \in \mathcal{D}(A_1^{1/8})),$$

and using the identity $G(\lambda)^{1/2}G(\lambda)^{1/2} = G(\lambda)$ and the fact that $G(\lambda)$ and A_1 commute, we find

$$\|G(\lambda)\eta\|^2_{\mathcal{H}_S} \leqslant \rho_2^2 \left(\|A_1^{1/4}\eta\|^2_{\mathcal{H}_S} + 2|\lambda|\|\eta\|^2_{\mathcal{H}_S} + |\lambda|^2\|A_1^{-1/4}\eta\|^2_{\mathcal{H}_S}\right) \quad (\eta \in \mathcal{D}(A_1^{1/4})).$$

The conclusion follows from $2\|\eta\|^2_{\mathcal{H}_S} = 2(A_1^{1/4}\eta, A_1^{-1/4}\eta)_{\mathcal{H}_S} \leqslant (|\lambda|^{-1}\|A_1^{1/4}\eta\|_{\mathcal{H}_S} + |\lambda|\|A_1^{-1/4}\eta\|_{\mathcal{H}_S})$. □

According to (4.31), for $\lambda \in \mathbb{C}^+$ the operator $V(\lambda)$ that is (formally) introduced in (4.17) can be defined as the following unbounded operator on $\mathcal{H}_S$:

$$\mathcal{D}(V(\lambda)) = \mathcal{D}(A_1) \quad \text{and} \quad V(\lambda)\eta = \lambda^2(\eta + K(\lambda)\eta) + \lambda G(\lambda)\zeta + A_1\zeta. \tag{4.37}$$

Proposition 4.8. *For all $\lambda \in \mathbb{C}^+$ the operator $V(\lambda)$ is an isomorphism from $\mathcal{D}(A_1)$ onto $\mathcal{H}_S$.*

Proof. The case $\lambda = 0$ is straightforward since $V(0) = A_1$. In what follows we suppose $\lambda \in \mathbb{C}^+$ and $\lambda \neq 0$.

We can write

$$V(\lambda) = \left[\lambda^2(A_1^{-1} + K(\lambda)A_1^{-1}) + \lambda G(\lambda)A_1^{-1} + I\right] A_1,$$

with $K(\lambda)A_1^{-1}$, $G(\lambda)A_1^{-1} \in \mathcal{L}(\mathcal{H}_S, \mathcal{D}(A_1^{1/8}))$ (see (4.30)).

It is sufficient to show that $V(\lambda)A_1^{-1} \in \mathcal{L}(\mathcal{H}_S)$ is invertible. Since $\lambda^2(A_1^{-1} + K(\lambda)A_1^{-1}) + \lambda G(\lambda)A_1^{-1}$ is a compact operator, we can use the Fredholm alternative: assume $\xi \in \ker(V(\lambda)A_1^{-1})$ and let us write $\eta \stackrel{\text{def}}{=} A_1^{-1}\xi$. Then

$$\begin{aligned} 0 = \operatorname{Re}\left(V(\lambda)A_1^{-1}\xi, \lambda A_1^{-1}\xi\right)_{\mathcal{H}_S} &= \operatorname{Re}(V(\lambda)\eta, \lambda\eta)_{\mathcal{H}_S} \\ &= \operatorname{Re}\lambda|\lambda|^2(\eta + K(\lambda)\eta, \eta)_{\mathcal{H}_S} + |\lambda|(G(\lambda)\eta, \eta)_{\mathcal{H}_S} + \operatorname{Re}\lambda\|A_1^{1/2}\eta\|^2_{\mathcal{H}_S} \geqslant \rho_1|\lambda|\|A_1^{1/8}\eta\|^2_{\mathcal{H}_S}, \end{aligned} \tag{4.38}$$

and thus $\xi = 0$. We thus deduce that $V(\lambda)$ is an isomorphism from $\mathcal{D}(A_1)$ onto $\mathcal{H}_S$. □

4.2 Estimation of $V^{-1}(\lambda)$

The following section is devoted to the estimation (in terms of λ) of the inverse of the operator $V(\lambda)$ for $\lambda \in \mathbb{C}_1^+$. We recall that the notation $\mathbb{C}_1^+$ is introduced in (4.2).

We also recall that K and G are defined by (4.26) and (4.27) and from Proposition 4.5, we have

$$\sup_{\lambda \in \mathbb{C}^+} \|K\|_{\mathcal{L}(\mathcal{H}_\mathcal{S})} + \|A_1^{1/8} K(\lambda) A_1^{1/8}\|_{\mathcal{L}(\mathcal{H}_\mathcal{S})} < +\infty. \tag{4.39}$$

To obtain estimates for $V(\lambda)$ we first consider the following "approximation"

$$V_K(\lambda) \stackrel{\text{def}}{=} \lambda^2 (I + K(\lambda)) + 2\rho\lambda A_1^{1/4} + A_1, \tag{4.40}$$

where $\rho > 0$ is a given parameter. The estimates on $V(\lambda)$ will then be deduced by a perturbation argument.

Theorem 4.9. *For all $\lambda \in \mathbb{C}_1^+$ the operator $V_K(\lambda) : \mathcal{D}(A_1) \to \mathcal{H}_\mathcal{S}$ is an isomorphism and for $\theta \in [0,1]$ the following estimates hold*

$$\sup_{\lambda \in \mathbb{C}_1^+} |\lambda|^{3/2-2\theta} \|A_1^\theta V_K^{-1}(\lambda)\|_{\mathcal{L}(\mathcal{H}_\mathcal{S})} < +\infty. \tag{4.41}$$

Proof. The proof of the invertibility of $V_K(\lambda)$ can be done in the same way as for $V(\lambda)$ (see Proposition 4.8).

We only prove (4.41) for $\theta = 0$ and $\theta = 1$, the other cases are obtained by interpolation. Let us consider $\lambda \in \mathbb{C}_1^+$ and $\eta \in \mathcal{D}(A_1)$. We first develop the expression of $V_K(\lambda)$ in (4.40):

$$\begin{aligned}\left\| \frac{V_K(\lambda)\eta}{\lambda^2} \right\|_{\mathcal{H}_\mathcal{S}}^2 &= \|\eta + K(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}}^2 + 4\rho^2 \left\| \frac{A_1^{1/4}\eta}{\lambda} \right\|_{\mathcal{H}_\mathcal{S}}^2 + \left\| \frac{A_1\eta}{\lambda^2} \right\|_{\mathcal{H}_\mathcal{S}}^2 + 2\,\mathrm{Re}\left(\eta + K(\lambda)\eta, \frac{A_1\eta}{\lambda^2} \right)_{\mathcal{H}_\mathcal{S}} \\ &\quad + 4\rho\,\mathrm{Re}\left(K(\lambda)\eta, \frac{A_1^{1/4}\eta}{\lambda} \right)_{\mathcal{H}_\mathcal{S}} + 4\rho\,\mathrm{Re}\left(\eta, \frac{A_1^{1/4}\eta}{\lambda} \right)_{\mathcal{H}_\mathcal{S}} + 4\rho\,\mathrm{Re}\left(\frac{A_1^{1/4}\eta}{\lambda}, \frac{A_1\eta}{\lambda^2} \right)_{\mathcal{H}_\mathcal{S}}. \end{aligned} \tag{4.42}$$

Since $\mathrm{Re}\,\lambda \geqslant 0$ we have $\mathrm{Re}(1/\lambda) \geqslant 0$ and we deduce,

$$\mathrm{Re}\left(\frac{A_1^{1/4}\eta}{\lambda}, \frac{A_1\eta}{\lambda^2} \right)_{\mathcal{H}_\mathcal{S}} + \mathrm{Re}\left(\eta, \frac{A_1^{1/4}\eta}{\lambda} \right)_{\mathcal{H}_\mathcal{S}} = \mathrm{Re}\left(\frac{1}{\lambda} \right) \left\| \frac{A_1^{5/8}\eta}{\lambda} \right\|_{\mathcal{H}_S}^2 + \mathrm{Re}\left(\frac{1}{\lambda} \right) \left\| \frac{A_1^{1/8}\eta}{\lambda} \right\|_{\mathcal{H}_S}^2 \geqslant 0. \tag{4.43}$$

Using the fact that $K(\lambda)$ and $A_1^{1/8}$ commute (see Proposition 4.6), $\mathrm{Re}(1/\lambda) \geqslant 0$ and (4.33), we deduce

$$\mathrm{Re}\left(K(\lambda)\eta, \frac{A_1^{1/4}\eta}{\lambda} \right)_{\mathcal{H}_\mathcal{S}} = \mathrm{Re}\left(\frac{1}{\lambda} \right) \left(K(\lambda) A_1^{1/8}\eta, A_1^{1/8}\eta \right)_{\mathcal{H}_\mathcal{S}} \geqslant 0. \tag{4.44}$$

Using again that $K(\lambda)$ and $A_1^{1/8}$ commute, Proposition 4.5 and (4.39), we deduce

$$\left| (\eta + K(\lambda)\eta, A_1\eta)_{\mathcal{H}_\mathcal{S}} \right| = \left| \left(A^{1/4}\eta + A_1^{1/8} K(\lambda) A_1^{1/8}\eta, A_1^{3/4}\eta \right)_{\mathcal{H}_\mathcal{S}} \right| \leqslant C_1 \|A_1^{1/4}\eta\|_{\mathcal{H}_\mathcal{S}} \|A_1^{3/4}\eta\|_{\mathcal{H}_\mathcal{S}}. \tag{4.45}$$

Combining (4.42), (4.43), (4.44), (4.45) yields

$$\left\| \frac{V_K(\lambda)\eta}{\lambda^2} \right\|_{\mathcal{H}_\mathcal{S}}^2 \geqslant \|\eta + K(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}}^2 + \left\| \frac{A_1\eta}{\lambda^2} \right\|_{\mathcal{H}_\mathcal{S}}^2 - 2 \left| \left(\eta + K(\lambda)\eta, \frac{A_1\eta}{\lambda^2} \right)_{\mathcal{H}_\mathcal{S}} \right| \left(1 - \frac{2\rho^2}{C_1} \frac{\|A_1^{1/4}\eta\|_{\mathcal{H}_\mathcal{S}}}{\|A_1^{3/4}\eta\|_{\mathcal{H}_\mathcal{S}}} \right). \tag{4.46}$$

Note that if $\|A_1^{3/4}\eta\|_{\mathcal{H}_\mathcal{S}} \leqslant \dfrac{2\rho^2}{C_1} \|A_1^{1/4}\eta\|_{\mathcal{H}_\mathcal{S}}$, then we deduce from (4.46), (4.33) and $|\lambda| > 1$ that

$$\|V_K(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}}^2 \geqslant |\lambda|^3 \|\eta\|_{\mathcal{H}_\mathcal{S}}^2 + |\lambda|^{-1} \|A_1\eta\|_{\mathcal{H}_\mathcal{S}}^2 \quad (\lambda \in \mathbb{C}_1^+),$$

which yields (4.41) for $\theta = 0$ and $\theta = 1$.

We can thus focus on the case $\|A_1^{3/4}\eta\|_{\mathcal{H}_\mathcal{S}} > \dfrac{2\rho^2}{C_1}\|A_1^{1/4}\eta\|_{\mathcal{H}_\mathcal{S}}$. We deduce from (4.46), by using the Cauchy-Schwarz inequality, that

$$\left\|\frac{V_K(\lambda)\eta}{\lambda^2}\right\|_{\mathcal{H}_\mathcal{S}}^2 \geqslant \frac{2\rho^2}{C_1}\left(\|\eta+K(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}}^2+\left\|\frac{A_1\eta}{\lambda^2}\right\|_{\mathcal{H}_\mathcal{S}}^2\right)\frac{\|A_1^{1/4}\eta\|_{\mathcal{H}_\mathcal{S}}}{\|A_1^{3/4}\eta\|_{\mathcal{H}_\mathcal{S}}}. \tag{4.47}$$

Now, we use a standard inequality (see e.g. [15, Theorem 6.10, p. 73]):

$$\left\|\frac{A_1^{3/4}\eta}{|\lambda|^{3/2}}\right\|_{\mathcal{H}_\mathcal{S}}^2 \leqslant C\left(\|\eta\|_{\mathcal{H}_\mathcal{S}}^2+\left\|\frac{A_1\eta}{\lambda^2}\right\|_{\mathcal{H}_\mathcal{S}}^2\right)$$

and (4.33) to obtain

$$\left\|\frac{V_K(\lambda)\eta}{\lambda^2}\right\|_{\mathcal{H}_\mathcal{S}}^2 \geqslant C|\lambda|^{-3}\|A_1^{3/4}\eta\|_{\mathcal{H}_\mathcal{S}}\|A_1^{1/4}\eta\|_{\mathcal{H}_\mathcal{S}}. \tag{4.48}$$

On the other hand, we obtain directly from (4.42), (4.43), (4.44), (4.45) the relation

$$\left\|\frac{V_K(\lambda)\eta}{\lambda^2}\right\|_{\mathcal{H}_\mathcal{S}}^2 \geqslant \|\eta+K(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}}^2+4\rho^2\left\|\frac{A_1^{1/4}\eta}{\lambda}\right\|_{\mathcal{H}_\mathcal{S}}^2+\left\|\frac{A_1\eta}{\lambda^2}\right\|_{\mathcal{H}_\mathcal{S}}^2-\frac{2C_1}{|\lambda|^2}\|A_1^{1/4}\eta\|_{\mathcal{H}_\mathcal{S}}\|A_1^{3/4}\eta\|_{\mathcal{H}_\mathcal{S}}. \tag{4.49}$$

Combining (4.48) and (4.49) and taking $\lambda\in\mathbb{C}_1^+$, we conclude that,

$$\left\|\frac{V_K(\lambda)\eta}{\lambda^2}\right\|_{\mathcal{H}_\mathcal{S}}^2 \geqslant \frac{C}{|\lambda|}\left(\|\eta+K(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}}^2+\left\|\frac{A_1\eta}{\lambda^2}\right\|_{\mathcal{H}_\mathcal{S}}^2+4\rho^2\left\|\frac{A_1^{1/4}\eta}{\lambda}\right\|_{\mathcal{H}_\mathcal{S}}^2\right) \tag{4.50}$$

and thus, with (4.33),

$$\|V_K(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}}^2 \geqslant C\left(|\lambda|^3\|\eta\|_{\mathcal{H}_\mathcal{S}}^2+|\lambda|^{-1}\|A_1\eta\|_{\mathcal{H}_\mathcal{S}}^2\right) \quad (\lambda\in\mathbb{C}_1^+). \tag{4.51}$$

Consequently (4.41) is proved for $\theta=0$ and $\theta=1$. □

Corollary 4.10. *For $\theta\in[-1/4,0]$ the following estimate holds*

$$\sup_{\lambda\in\mathbb{C}_1^+}|\lambda|^{3/2-2\theta}\|A_1^\theta V_K^{-1}(\lambda)\|_{\mathcal{L}(\mathcal{H}_\mathcal{S})}<+\infty. \tag{4.52}$$

Proof. From Theorem 4.9 we have the estimate if $\theta=0$. Hence, if we show the case $\theta=-1/4$ we will then obtain the cases $\theta\in(-1/4,0)$ by interpolation.

Using (4.40) and the fact that $K(\lambda)$ and $A_1^{-1/4}$ commute (see Proposition 4.6) we deduce

$$\lambda^2(I+K(\lambda))A_1^{-1/4}V_K^{-1}(\lambda)=A_1^{-1/4}-2\rho\lambda V_K^{-1}(\lambda)-A_1^{3/4}V_K^{-1}(\lambda)$$

and thus with (4.41) for $\theta=3/4$, $\theta=0$ and with (4.33), we obtain

$$\begin{aligned}|\lambda|^2\|A_1^{-1/4}V_K^{-1}(\lambda)\|_{\mathcal{L}(\mathcal{H}_\mathcal{S})} &\leqslant |\lambda|^2\|(I+K(\lambda))A_1^{-1/4}V_K^{-1}(\lambda)\|_{\mathcal{L}(\mathcal{H}_\mathcal{S})}\\ &\leqslant C\left(\|A_1^{-1/4}\|_{\mathcal{L}(\mathcal{H}_\mathcal{S})}+|\lambda|\|V_K^{-1}(\lambda)\|_{\mathcal{L}(\mathcal{H}_\mathcal{S})}+\|A_1^{3/4}V_K^{-1}(\lambda)\|_{\mathcal{L}(\mathcal{H}_\mathcal{S})}\right)\leqslant C(1+|\lambda|^{-1/2})\leqslant C.\end{aligned}$$

□

Comparing $V(\lambda)$ and $V_K(\lambda)$, we prove the following

Theorem 4.11. *For $\theta\in[-1/4,1]$ the following estimate holds*

$$\sup_{\lambda\in\mathbb{C}_1^+}|\lambda|^{3/2-2\theta}\|A_1^\theta V^{-1}(\lambda)\|_{\mathcal{L}(\mathcal{H}_\mathcal{S})}<+\infty. \tag{4.53}$$

Proof. First, we take in the definition (4.40) of $V_K(\lambda)$ a constant $\rho \in (0, \rho_1/4)$, where ρ_1 is defined in Proposition 4.5, and we set

$$S(\lambda) \stackrel{\text{def}}{=} G(\lambda) - 2\rho A_1^{1/4}.$$

From Proposition 4.5, we deduce that $S(\lambda) \in \mathcal{L}(\mathcal{D}(A_1^{1/8}), \mathcal{D}(A_1^{1/8})')$ is a positive self-adjoint operator satisfying

$$(\rho_1 - 2\rho)^{1/2} \|A_1^{1/8}\eta\|_{\mathcal{H}_\mathcal{S}} \leqslant \|S^{1/2}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}} \leqslant C\left(\|A_1^{1/8}\eta\|_{\mathcal{H}_\mathcal{S}} + |\lambda|^{1/2}\|A_1^{-1/8}\eta\|_{\mathcal{H}_\mathcal{S}}\right) \quad (\eta \in \mathcal{D}(A_1^{1/8})). \tag{4.54}$$

Moreover, from (4.40) and (4.37), $V(\lambda) - V_K(\lambda) = \lambda S(\lambda)$ and in particular,

$$V_K^{-1}(\lambda) - V^{-1}(\lambda) = V_K^{-1}(\lambda)(V(\lambda) - V_K(\lambda))V^{-1}(\lambda) = \lambda V_K^{-1}(\lambda)S(\lambda)V^{-1}(\lambda)$$

and then

$$[I + \lambda V_K^{-1}(\lambda)S(\lambda)]V^{-1}(\lambda) = V_K^{-1}(\lambda). \tag{4.55}$$

Let us prove

$$\forall \eta \in \mathcal{H}_\mathcal{S}, \quad \|S^{1/2}(\lambda)V^{-1}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}} \leqslant \|S^{1/2}(\lambda)V_K^{-1}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}}. \tag{4.56}$$

For that, suppose that $\zeta \in \mathcal{D}(A_1)$ and $f \in \mathcal{H}_S$ satisfy the equation

$$\zeta + \lambda V_K^{-1}(\lambda)S(\lambda)\zeta = f. \tag{4.57}$$

We multiply (4.57) by $S(\lambda)\zeta$:

$$\|S(\lambda)^{1/2}\zeta\|_{\mathcal{H}_\mathcal{S}}^2 + \left(\lambda V_K^{-1}(\lambda)S(\lambda)\zeta, S(\lambda)\zeta\right)_{\mathcal{H}_\mathcal{S}} = \left(S(\lambda)^{1/2}f, S(\lambda)^{1/2}\zeta\right)_{\mathcal{H}_\mathcal{S}}. \tag{4.58}$$

Writting $\xi = V_K^{-1}(\lambda)S(\lambda)\zeta$ and using (4.40) we obtain that for any $\lambda \in \mathbb{C}^+$,

$$\begin{aligned}\mathrm{Re}\langle \lambda V_K^{-1}(\lambda)S(\lambda)\zeta, S(\lambda)\zeta\rangle_{\mathcal{H}_\mathcal{S}} &= \mathrm{Re}\,(\lambda\xi, V_K(\lambda)\xi)_{\mathcal{H}_\mathcal{S}} \\ &= \mathrm{Re}(\lambda)\|\lambda(I + K(\lambda))^{1/2}\xi\|_{\mathcal{H}_\mathcal{S}}^2 + 2\rho\|\lambda A_1^{1/8}\xi\|_{\mathcal{H}_\mathcal{S}}^2 + \mathrm{Re}(\lambda)\|A_1^{1/2}\xi\|_{\mathcal{H}_\mathcal{S}}^2 \geqslant 0,\end{aligned}$$

and with (4.58) it gives the estimate

$$\|S(\lambda)^{1/2}\zeta\|_{\mathcal{H}_\mathcal{S}} \leqslant \|S(\lambda)^{1/2}f\|_{\mathcal{H}_\mathcal{S}}.$$

Applying the above estimate to (4.55) implies (4.56).

Combining (4.56) with (4.54) and using that A_1 commutes with $S(\lambda)$ and $K(\lambda)$ yield for all $\eta \in \mathcal{H}_\mathcal{S}$,

$$\begin{aligned}\|S(\lambda)V^{-1}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}} &= \|S(\lambda)^{1/2}S(\lambda)^{1/2}V^{-1}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}} \\ &\leqslant C\left(\|A_1^{1/8}S(\lambda)^{1/2}V^{-1}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}} + |\lambda|^{1/2}\|A_1^{-1/8}S(\lambda)^{1/2}V^{-1}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}}\right) \\ &= C\left(\|S(\lambda)^{1/2}V^{-1}(\lambda)A_1^{1/8}\eta\|_{\mathcal{H}_\mathcal{S}} + |\lambda|^{1/2}\|S(\lambda)^{1/2}V^{-1}(\lambda)A_1^{-1/8}\eta\|_{\mathcal{H}_\mathcal{S}}\right) \\ &\leqslant C\left(\|S(\lambda)^{1/2}V_K^{-1}(\lambda)A_1^{1/8}\eta\|_{\mathcal{H}_\mathcal{S}} + |\lambda|^{1/2}\|S(\lambda)^{1/2}V_K^{-1}(\lambda)A_1^{-1/8}\eta\|_{\mathcal{H}_\mathcal{S}}\right) \\ &\leqslant C\left(\|A_1^{1/4}V_K^{-1}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}} + |\lambda|^{1/2}\|V_K^{-1}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}} + |\lambda|\|A_1^{-1/4}V_K^{-1}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}}\right).\end{aligned}$$

Applying estimates (4.41) and (4.52) on the above estimate gives

$$\forall \eta \in \mathcal{H}_\mathcal{S}, \quad |\lambda|\|S(\lambda)V^{-1}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}} \leqslant C\|\eta\|_{\mathcal{H}_\mathcal{S}}. \tag{4.59}$$

Coming back to equality (4.55) we deduce that for $\eta \in \mathcal{H}_\mathcal{S}$, $\theta \in [-1/4, 1]$,

$$\|A_1^\theta V^{-1}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}} \leqslant \|A_1^\theta V_K^{-1}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}} + |\lambda|\|A_1^\theta V_K^{-1}(\lambda)S(\lambda)V^{-1}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}}.$$

Then using estimates (4.41) and (4.52) we obtain

$$|\lambda|^{3/2-2\theta}\|A_1^\theta V^{-1}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}} \leqslant C\|\eta\|_{\mathcal{H}_\mathcal{S}} + C|\lambda|\|S(\lambda)V^{-1}(\lambda)\eta\|_{\mathcal{H}_\mathcal{S}}.$$

Combining the above estimate with (4.59) yields (4.53). □

4.3 Proof of Theorem 4.1

Proof. First, the exponential stability of $(e^{A_0 t})_{t\geqslant 0}$ (see Proposition 3.3) and standard results (see [3, p.101, Theorem 2.5]) yield that $\|(\lambda - A_0)^{-1}\|_{\mathcal{L}(\mathcal{H})}$ is uniformly bounded for $\lambda \in \mathbb{C}^+$. This implies that

$$\sup_{\lambda\in\mathbb{C}^+, |\lambda|\leqslant 1} \|A_0(\lambda - A_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} + |\lambda|\|(\lambda - A_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Using (3.16) and (3.20), we deduce (4.3), (4.4) and (4.5) for $\lambda \in \mathbb{C}^+$ with $|\lambda| \leqslant 1$. In the remaining part of the proof, we can thus assume $\lambda \in \mathbb{C}_1^+$ (see (4.2)).

In order to prove (4.3), we recall formula (4.19) for $(\lambda - A_0)^{-1}$ and we see that we need to estimate

$$\begin{aligned}
&|\lambda|^{1/2}\|(\lambda - \mathbb{A})^{-1}\mathbb{P}f\|_{\mathbf{L}^2_\#(\mathcal{F})},\ |\lambda|^{3/2}\|D_0(\lambda)V^{-1}(\lambda)\mathcal{T}(\lambda)f\|_{\mathbf{L}^2_\#(\mathcal{F})},\\
&\qquad |\lambda|^{1/2}\|D_0(\lambda)V^{-1}(\lambda)A_1 g\|_{\mathbf{L}^2_\#(\mathcal{F})},\ |\lambda|^{3/2}\|D_0(\lambda)V^{-1}(\lambda)h\|_{\mathbf{L}^2_\#(\mathcal{F})}\\
&\quad |\lambda|^{1/2}\|A_1^{1/2}V^{-1}(\lambda)\mathcal{T}(\lambda)f\|_{\mathcal{H}_\mathcal{S}},\ |\lambda|^{-1/2}\|A_1^{1/2}(I - V^{-1}(\lambda)A_1)g\|_{\mathcal{H}_\mathcal{S}},\ |\lambda|^{1/2}\|A_1^{1/2}V^{-1}(\lambda)h\|_{\mathcal{H}_\mathcal{S}}\\
&\qquad |\lambda|^{3/2}\|V^{-1}(\lambda)\mathcal{T}(\lambda)f\|_{\mathcal{H}_\mathcal{S}},\ |\lambda|^{1/2}\|V^{-1}(\lambda)A_1 g\|_{\mathcal{H}_\mathcal{S}},\ |\lambda|^{3/2}\|V^{-1}(\lambda)h\|_{\mathcal{H}_\mathcal{S}}
\end{aligned} \tag{4.60}$$

by

$$C(\|f\|_{\mathbf{L}^2_\#(\mathcal{F})} + \|A_1^{1/2}g\|_{\mathcal{H}_\mathcal{S}} + \|h\|_{\mathcal{H}_\mathcal{S}})$$

for some constant C independent of λ. Combining (4.21), (4.22), (4.23) (for $\theta = 0$), (4.53) (for $\theta = 0, 1/2, 1$), we deduce (4.3). Note that here and in the following we use several times that A_1 and $V(\lambda)$ commute.

In a similar way, to prove (4.4), we need to estimate

$$\begin{aligned}
&\|(\lambda - \mathbb{A})^{-1}\mathbb{P}f\|_{\mathbf{H}^2_\#(\mathcal{F})},\ |\lambda|\|D_0(\lambda)V^{-1}(\lambda)\mathcal{T}(\lambda)f\|_{\mathbf{H}^2_\#(\mathcal{F})},\\
&\qquad \|D_0(\lambda)V^{-1}(\lambda)A_1 g\|_{\mathbf{H}^2_\#(\mathcal{F})},\ |\lambda|\|D_0(\lambda)V^{-1}(\lambda)h\|_{\mathbf{H}^2_\#(\mathcal{F})}\\
&\quad \|A_1^{7/8}V^{-1}(\lambda)\mathcal{T}(\lambda)f\|_{\mathcal{H}_\mathcal{S}},\ |\lambda|^{-1}\|A_1^{7/8}(I - V^{-1}(\lambda)A_1)g\|_{\mathcal{H}_\mathcal{S}},\ \|A_1^{7/8}V^{-1}(\lambda)h\|_{\mathcal{H}_\mathcal{S}}\\
&\qquad |\lambda|\|A_1^{3/8}V^{-1}(\lambda)\mathcal{T}(\lambda)f\|_{\mathcal{H}_\mathcal{S}},\ \|A_1^{3/8}V^{-1}(\lambda)A_1 g\|_{\mathcal{H}_\mathcal{S}},\ |\lambda|\|A_1^{3/8}V^{-1}(\lambda)h\|_{\mathcal{H}_\mathcal{S}}
\end{aligned} \tag{4.61}$$

by

$$C(\|f\|_{\mathbf{L}^2_\#(\mathcal{F})} + \|A_1^{5/8}g\|_{\mathcal{H}_\mathcal{S}} + \|A_1^{1/8}h\|_{\mathcal{H}_\mathcal{S}})$$

We use (4.21), (4.23) (for $\theta = 1$) and (4.53) for $\theta = 1/4, -1/4, 3/4$ to estimate all the terms except

$$|\lambda|^{-1}\|A_1^{7/8}(I - V^{-1}(\lambda)A_1)g\|_{\mathcal{H}_\mathcal{S}} = |\lambda|^{-1}\|A_1^{1/4}V^{-1}(\lambda)(\lambda^2(I + K(\lambda)) + \lambda G(\lambda))A_1^{5/8}g\|_{\mathcal{H}_\mathcal{S}}.$$

Here we have used the expression (4.37) of $V(\lambda)$. Using (4.33), (4.36) and (4.53) for $\theta = 0, 1/4, 1/2$, we deduce the result.

Let us prove (4.5). Let $[w, \eta_1, \eta_2] \in \mathcal{H} \cap \mathbf{H}^2_\#(\mathcal{F}) \times \mathcal{D}(A_1^{7/8}) \times \mathcal{D}(A_1^{3/8})$. First, from the continuity of $\Lambda^* : \mathbf{H}^{1/2}_\#(\Gamma_\mathrm{b}) \to \mathcal{D}(A_1^{1/8})$ and a trace inequality we have,

$$\|\Lambda^*(2D(w)n)\|_{\mathcal{D}(A_1^{1/8})'} \leqslant C\|\Lambda^*(2D(w)n)\|_{\mathcal{D}(A_1^{1/8})} \leqslant C\|D(w)n\|_{\mathbf{H}^{1/2}_\#(\Gamma_\mathrm{b})} \leqslant C\|w\|_{\mathbf{H}^2_\#(\mathcal{F})}.$$

From (3.21), (3.22), (3.18) and the above estimate we deduce,

$$\begin{aligned}
\left\| A_0 \begin{bmatrix} w \\ \eta_1 \\ \eta_2 \end{bmatrix} \right\|_{\mathbf{L}^2(\mathcal{F})\times\mathcal{D}(A_1^{3/8})\times\mathcal{D}(A_1^{1/8})'} &= \left\| P_0 \begin{bmatrix} \nu\Delta w \\ \eta_2 \\ -A_1\eta_1 - \Lambda^*(2\nu D(w)n) \end{bmatrix} \right\|_{\mathbf{L}^2_\#(\mathcal{F})\times\mathcal{D}(A_1^{3/8})\times\mathcal{D}(A_1^{1/8})'}\\
&\leqslant C \left\| \begin{bmatrix} \nu\Delta w \\ \eta_2 \\ -A_1\eta_1 - \Lambda^*(2\nu D(w)n) \end{bmatrix} \right\|_{\mathbf{L}^2_\#(\mathcal{F})\times\mathcal{D}(A_1^{3/8})\times\mathcal{D}(A_1^{1/8})'} \leqslant C \left\| \begin{bmatrix} w \\ \eta_1 \\ \eta_2 \end{bmatrix} \right\|_{\mathbf{H}^2_\#(\mathcal{F})\times\mathcal{D}(A_1^{7/8})\times\mathcal{D}(A_1^{3/8})}.
\end{aligned}$$

Then with formula $\lambda(\lambda - A_0)^{-1} = A_0(\lambda - A_0)^{-1} + I$, the above inequality and (4.4) yield (4.5). □

5 Regularity result for the linear system

We use the results of the previous section to deduce a regularity result on the linear system (3.23)–(3.24), or equivalently (3.25). We recall that A_0 is defined by (3.20)-(3.22). The main result of this section is the following theorem

Theorem 5.1. *Assume $F \in L^2(0,\infty;\mathbf{L}^2_\#(\mathcal{F}))$ and $G \in L^2(0,\infty;\mathcal{D}(A_1^{1/8}))$. Assume $[w^0,\ \eta_1^0,\ \eta_2^0] \in \mathcal{V}$ and $\eta_1^0 \in \mathcal{D}(A_1^{3/4+\varepsilon})$, $\eta_2^0 \in \mathcal{D}(A_1^{1/4+\varepsilon})$ for some $\varepsilon > 0$. Then the solution $[w,\ q,\ \eta,\ \partial_t\eta]$ of (3.25) satisfies*

$$w \in L^2(0,\infty;\mathbf{H}^2_\#(\mathcal{F})) \cap C_b([0,\infty);\mathbf{H}^1_\#(\mathcal{F})) \cap H^1(0,\infty;\mathbf{L}^2_\#(\mathcal{F})), \quad q \in L^2(0,\infty;H^1_\#(\mathcal{F})) \tag{5.1}$$

$$\eta \in L^2(0,\infty;\mathcal{D}(A_1^{7/8})) \cap C_b([0,\infty);\mathcal{D}(A_1^{5/8})) \cap H^1(0,\infty;\mathcal{D}(A_1^{3/8})), \tag{5.2}$$

and

$$\partial_t\eta \in L^2(0,\infty;\mathcal{D}(A_1^{3/8})) \cap C_b([0,\infty);\mathcal{D}(A_1^{1/8})) \cap H^1(0,\infty;\mathcal{D}(A_1^{1/8})'). \tag{5.3}$$

In order to prove this result, we first consider the general linear system

$$\mathbf{Y}' = A_0\mathbf{Y} + \mathbf{F}, \quad \mathbf{Y}(0) = \mathbf{Y}^0, \tag{5.4}$$

and we use the resolvent estimates obtained in Theorem 4.1.

5.1 Regularity results for Gevrey linear systems

In this subsection, we only assume that $(A_0, \mathcal{D}(A_0))$ is the infinitesimal generator of a strongly continuous semigroup on a Hilbert space $\mathcal{H}$ satisfying the resolvent estimate

$$\sup_{\tau\in\mathbb{R}} |\tau|^{1/2}\|(\imath\tau - A_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} < +\infty. \tag{5.5}$$

This implies that the semigroup is of Gevrey class δ for all $\delta > 2$ (see Remark 4.2). For sake of simplicity, we also assume that $(e^{tA_0})_{t\geqslant 0}$ is exponentially stable and thus (see, for instance, [3, p.101]),

$$\imath\mathbb{R} \subset \rho(A), \quad \sup_{\tau\in\mathbb{R}} \|(\imath\tau - A_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} < +\infty. \tag{5.6}$$

Lemma 5.2. *Assume $\mathbf{F} \in L^2(0,+\infty;\mathcal{H})$ and $\mathbf{Y}^0 = 0$. The solution of (5.4) satisfies*

$$\mathbf{Y} \in W(0,+\infty;\mathcal{D}((-A_0)^{1/2}), \mathcal{D}((-A_0)^{*1/2})')$$

and there exists a constant $C > 0$, independent on $\mathbf{F}$ and $\mathbf{Y}$, such that

$$\|\mathbf{Y}\|_{W(0,+\infty;\mathcal{D}((-A_0)^{1/2}),\mathcal{D}((-A_0)^{*1/2})')} \leqslant C\|\mathbf{F}\|_{L^2(0,+\infty;\mathcal{H})}.$$

Proof. Combining (5.6) and (5.5) we first deduce that

$$\|(\imath\tau - A_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant C(1 + |\tau|^{1/2})^{-1}.$$

On the other hand, from the relation $A_0(\imath\tau - A_0)^{-1} = -I + \imath\tau(\imath\tau - A_0)^{-1}$ and estimate (5.5) we also have

$$\|A_0(\imath\tau - A_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant C(1 + |\tau|^{1/2}).$$

From the two above estimates we deduce,

$$\|(-A_0)^{1/2}(\imath\tau - A_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant C. \tag{5.7}$$

Let us now consider the solution of (5.4) with $\mathbf{F} \in L^2(0,+\infty;\mathcal{H})$ and $\mathbf{Y}^0 = 0$. We extend $\mathbf{Y}$ and $\mathbf{F}$ by zero in $(-\infty, 0)$ and we denote by $\widehat{\mathbf{Y}}$ and $\widehat{\mathbf{F}}$ their Fourier transforms. We deduce from (5.4) that

$$(\imath\tau - A_0)\widehat{\mathbf{Y}}(\tau) = \widehat{\mathbf{F}}(\tau) \quad (\tau \in \mathbb{R}).$$

Combining this relation with (5.7) yields

$$\mathbf{Y} \in L^2(0,+\infty;\mathcal{D}((-A_0)^{1/2})), \quad \text{with} \quad \|\mathbf{Y}\|_{L^2(0,+\infty;\mathcal{D}((-A_0)^{1/2}))} \leqslant C\|\mathbf{F}\|_{L^2(0,+\infty;\mathcal{H})}.$$

From (5.4), the above relations imply

$$\mathbf{Y}' \in L^2(0,+\infty;\mathcal{D}((-A_0^*)^{1/2})') \quad \text{with} \quad \|\mathbf{Y}'\|_{L^2(0,+\infty;\mathcal{D}((-A_0^*)^{1/2})')} \leqslant C\|\mathbf{F}\|_{L^2(0,+\infty;\mathcal{H})}.$$

□

Lemma 5.3. *Assume* $\mathbf{F} = 0$. *For all* $k \in \mathbb{N}^*$, *there exists* $C = C(k) > 0$ *such that for any* $\mathbf{Y}^0 \in \mathcal{H}$ *the solution of* (5.4) *satisfies*

$$t^{k+1}\|(-A_0)^{k/2}\mathbf{Y}\|_{\mathcal{H}} \leqslant C\|\mathbf{Y}^0\|_{\mathcal{H}} \quad \textit{for all } t \in \mathbb{R}^+. \tag{5.8}$$

Proof. Since $(e^{tA_0})_{t\geqslant 0}$ is exponentially stable, $\mathbf{Y} \in L^2(0,+\infty;\mathcal{H})$ and there exists $C > 0$ such that

$$\|\mathbf{Y}\|_{L^2(0,\infty;\mathcal{H})} \leqslant C\|\mathbf{Y}^0\|_{\mathcal{H}}.$$

In what follows, we denote by Υ^k the map in $[0,\infty)$ defined by $\Upsilon^k(t) = t^k$ for $t \geqslant 0$. Since $\Upsilon^1\mathbf{Y}$ satisfies

$$(\Upsilon^1\mathbf{Y})' = A_0(\Upsilon^1\mathbf{Y}) + \mathbf{Y}, \quad (\Upsilon^1\mathbf{Y})(0) = 0$$

we deduce from Lemma 5.2 that $\Upsilon^1\mathbf{Y} \in W(0,+\infty;\mathcal{D}((-A_0)^{1/2}),\mathcal{D}((-A_0)^{*1/2})')$ with

$$\|\Upsilon^1\mathbf{Y}\|_{W(0,+\infty;\mathcal{D}((-A_0)^{1/2}),\mathcal{D}((-A_0)^{*1/2})')} \leqslant C\|\mathbf{Y}\|_{L^2(0,\infty;\mathcal{H})} \leqslant C\|\mathbf{Y}^0\|_{\mathcal{H}}.$$

Next, we observe that $\Upsilon^2(-A_0)^{1/2}\mathbf{Y}$ satisfies

$$(\Upsilon^2(-A_0)^{1/2}\mathbf{Y})' = A_0(\Upsilon^2(-A_0)^{1/2}\mathbf{Y}) + 2(-A_0)^{1/2}\Upsilon^1\mathbf{Y}, \quad (\Upsilon^2\mathbf{Y})(0) = 0,$$

and we deduce from Lemma 5.2 with the above estimate

$$\|\Upsilon^2\mathbf{Y}\|_{W(0,+\infty;\mathcal{D}(A_0),\mathcal{H})} \leqslant C\|\mathbf{Y}^0\|_{\mathcal{H}}.$$

By induction, we deduce

$$\Upsilon^{k+1}\mathbf{Y} \in W(0,+\infty;\mathcal{D}((-A_0)^{(k+1)/2},\mathcal{D}((-A_0)^{(k-1)/2})$$

for $k \geqslant 1$, with

$$\|\Upsilon^{k+1}\mathbf{Y}\|_{W(0,+\infty;\mathcal{D}((-A_0)^{(k+1)/2},\mathcal{D}((-A_0)^{(k-1)/2})} \leqslant C\|\mathbf{Y}^0\|_{\mathcal{H}}. \tag{5.9}$$

From [3, Prop. 6.1 p. 171]), we have

$$[\mathcal{D}((-A_0)^{(k+1)/2},\mathcal{D}((-A_0)^{(k-1)/2})]_{1/2} = \mathcal{D}((-A_0)^{k/2}). \tag{5.10}$$

Finally, (5.8) follows from (1.33), (5.9) and (5.10). □

We now improve (5.8) by using interpolation results.

Lemma 5.4. *Assume* $\mathbf{F} = 0$ *and* $\mathbf{Y}^0 \in \mathcal{D}((-A_0)^{1/4+\varepsilon})$ *for* $\varepsilon > 0$. *The solution of system* (5.4) *belongs to*

$$W(0,+\infty;\mathcal{D}((-A_0)^{1/2}),\mathcal{D}((-A_0)^{*1/2})')$$

and there exists a constant $C > 0$, *independent on* $\mathbf{Y}_0$, *such that*

$$\|\mathbf{Y}\|_{W(0,+\infty;\mathcal{D}((-A_0)^{1/2}),\mathcal{D}((-A_0)^{*1/2})')} \leqslant C\|\mathbf{Y}^0\|_{\mathcal{D}((-A_0)^{1/4+\varepsilon})}.$$

Proof. Let us consider $k \in \mathbb{N}$ such that $k > 4$ and $k > 1/\varepsilon$. We have in particular

$$\mathbf{Y}^0 \in \mathcal{D}((-A_0)^{1/4+\varepsilon}) \subset \mathcal{D}((-A_0)^{1/4+1/k}).$$

Since $(e^{tA_0})_{t\geqslant 0}$ is exponentially stable, there exists $C > 0$ such that

$$\|(-A_0)^{1/4+1/k}e^{tA_0}\mathbf{Y}^0\|_{\mathcal{H}} \leqslant C\|(-A_0)^{1/4+1/k}\mathbf{Y}^0\|_{\mathcal{H}} \quad (t \geqslant 0).$$

On the other hand, from (5.8) we deduce

$$t^{k+1}\|(-A_0)^{k/2+1/4+1/k}e^{A_0t}\mathbf{Y}^0\|_{\mathcal{H}} \leqslant C\|(-A_0)^{1/4+1/k}\mathbf{Y}^0\|_{\mathcal{H}} \quad (t \geqslant 0).$$

Using that $k > 4$, we can interpolate the above estimates and we obtain

$$t^{\frac{1}{2}-\left(\frac{3}{2k}+\frac{2}{k^2}\right)}\|(-A_0)^{1/2}e^{A_0t}\mathbf{Y}^0\|_{\mathcal{H}} \leqslant C\|(-A_0)^{1/4+1/k}\mathbf{Y}^0\|_{\mathcal{H}} \quad (t \geqslant 0).$$

This concludes the proof. □

5.2 Proof of Theorem 5.1

We now come back to our particular case where A_0 is defined by (3.20)-(3.22). The hypotheses of the previous subsection are still valid but we use the particular structure of A_0 and the resolvent estimates (4.4) and (4.5) to obtain some better regularity results.

Lemma 5.5. *Assume* $\mathbf{F} \in L^2\left(0,+\infty;\mathcal{H}\cap\left(\mathbf{L}^2_\#(\mathcal{F})\times\mathcal{D}(A_1^{5/8})\times\mathcal{D}(A_1^{1/8})\right)\right)$ *and* $\mathbf{Y}^0=0$. *Then, the solution of system* (5.4) *satisfies*

$$\mathbf{Y}\in W(0,+\infty;\mathbf{H}^2_\#(\mathcal{F})\times\mathcal{D}(A_1^{7/8})\times\mathcal{D}(A_1^{3/8}),\mathbf{L}^2_\#(\mathcal{F})\times\mathcal{D}(A_1^{3/8})\times\mathcal{D}(A_1^{1/8})')$$

and there exists a constant $C>0$, *independent on* $\mathbf{F}$ *and* $\mathbf{Y}$, *such that*

$$\|\mathbf{Y}\|_{W(0,+\infty;\mathbf{H}^2_\#(\mathcal{F})\times\mathcal{D}(A_1^{7/8})\times\mathcal{D}(A_1^{3/8}),\mathbf{L}^2_\#(\mathcal{F})\times\mathcal{D}(A_1^{3/8})\times\mathcal{D}(A_1^{1/8})')}\leqslant C\|\mathbf{F}\|_{L^2(0,+\infty;\mathbf{L}^2_\#(\mathcal{F})\times\mathcal{D}(A_1^{5/8})\times\mathcal{D}(A_1^{1/8}))}.$$

Proof. We extend $\mathbf{Y}$ and $\mathbf{F}$ by zero for $t<0$ and we denote by $\widehat{\mathbf{Y}}$ and $\widehat{\mathbf{F}}$ their Fourier transforms. Since $\mathbf{Y}^0=0$, we obtain from (5.4) that

$$(\imath\tau-A_0)\widehat{\mathbf{Y}}(\tau)=\widehat{\mathbf{F}}(\tau)\quad(\tau\in\mathbb{R}),$$

and we deduce the result from (4.4) and (4.5). □

Assume $\mathbf{F}=0$. Writing $\mathbf{Y}=[w,\ \eta,\ \partial_t\eta]$ and $\mathbf{Y}^0=[w^0,\ \eta_1^0,\ \eta_2^0]$, (5.4) is equivalent to

$$\left\{\begin{array}{c}\partial_t w-\operatorname{div}\mathbb{T}(w,q)=0\quad\text{in }(0,\infty)\times\mathcal{F}_\#,\\ \operatorname{div}w=0\quad\text{in }(0,\infty)\times\mathcal{F}_\#,\\ w=\Lambda(\partial_t\eta)\quad\text{on }(0,\infty)\times\Gamma_{\mathrm{b},\#},\\ w\ Le_1-\text{periodic}\quad\text{in }(0,\infty),\\ \partial_{tt}\eta+A_1\eta=-\Lambda^*\{\mathbb{T}(w,q)n_{|\Gamma_{\mathrm{b},\#}}\}\quad\text{in }(0,\infty),\\ w(0,\cdot)=w^0,\quad\eta(0,\cdot)=\eta_1^0,\quad\partial_t\eta(0,\cdot)=\eta_2^0.\end{array}\right.\tag{5.11}$$

Lemma 5.6. *Assume* $[w^0,\ \eta_1^0,\ \eta_2^0]\in\mathcal{D}((-A_0)^{1/2+\varepsilon})$ *for* $\varepsilon>0$. *The solution of system* (5.11) *satisfies*

$$w\in W(0,\infty;\mathbf{H}^2_\#(\mathcal{F}),\mathbf{L}^2_\#(\mathcal{F}))\quad q\in L^2(0,\infty;H^1_\#(\mathcal{F})),\tag{5.12}$$

and

$$\eta\in L^2(0,\infty;\mathcal{D}(A_1^{7/8}))\quad\partial_t\eta\in L^2(0,\infty;\mathcal{D}(A_1^{3/8}))\quad\partial_{tt}\eta\in L^2(0,\infty;\mathcal{D}(A_1^{1/8})').\tag{5.13}$$

Proof. Since $(-A_0)^{1/4}\mathbf{Y}$ satisfies (5.4) with $\mathbf{F}=0$ and the initial datum $(-A_0)^{1/4}\mathbf{Y}_0\in\mathcal{D}((-A_0)^{1/4+\varepsilon})$ then from Lemma 5.4 we deduce

$$[w,\ \eta,\ \partial_t\eta]\in W(0,+\infty;\mathcal{D}((-A_0)^{3/4}),\mathcal{D}((-A_0)^{*1/4})')\tag{5.14}$$

and thus

$$[w,\ \eta,\ \partial_t\eta]\in H^{3/4}(0,+\infty;\mathcal{H}).\tag{5.15}$$

From (3.27), we have

$$\mathcal{D}((-A_0)^{3/4})=\mathcal{V}\cap\left[\mathbf{H}^{3/2}_\#(\mathcal{F})\times\mathcal{D}(A_1^{7/8})\times\mathcal{D}(A_1^{3/8})\right],\tag{5.16}$$

and therefore we obtain from (5.14) that $\eta\in L^2(0,\infty;\mathcal{D}(A_1^{7/8}))$, $\partial_t\eta\in L^2(0,\infty;\mathcal{D}(A_1^{3/8}))$

We have $w^0\in\mathbf{H}^1_\#(\mathcal{F})$, $\operatorname{div}w^0=0$ and $w^0=\Lambda(\eta_2^0)$ on $\Gamma_{\mathrm{b},\#}$. Moreover, from (5.14), (5.15) and (3.12),

$$\Lambda(\partial_t\eta)\in L^2(0,\infty;\mathbf{V}^{3/2}_\#(\Gamma_\mathrm{b}))\cap H^{3/4}(0,\infty;\mathbf{V}^0_\#(\Gamma_\mathrm{b}))$$

and thus, from standard result on the Stokes system, we obtain (5.12). In particular,

$$\mathbb{T}(w,q)n_{|\Gamma_{\mathrm{b},\#}}\in L^2(0,\infty;\mathbf{H}^{1/2}_\#(\Gamma_b)),$$

and from (3.13), $\Lambda^*\{\mathbb{T}(w,q)n_{\Gamma_{\mathrm{b},\#}}\}\in L^2(0,\infty;\mathcal{D}(A_1^{1/8}))$, and thus $\partial_{tt}\eta\in L^2(0,\infty;\mathcal{D}(A_1^{1/8})')$. □

We are now in position to prove Theorem 5.1.

Proof of Theorem 5.1. We first consider a lifting operator $\mathcal{R} \in \mathcal{L}(\mathcal{D}(A_1^{1/4+\varepsilon}), \mathbf{V}_{\#}^{3/2+4\varepsilon}(\mathcal{F}))$ such that,

$$\mathcal{R}\eta = \begin{cases} \Lambda\eta & \text{on } \Gamma_{\#}, \\ 0 & \text{on } \Gamma_{\mathrm{fix},\#}. \end{cases}$$

In particular, $w^0 - \mathcal{R}\eta_2^0 \in \mathbf{V}_{\#,0}^1(\mathcal{F})$. Since $F \in L^2(0,\infty;\mathbf{L}_{\#}^2(\mathcal{F}))$, we deduce the existence and uniqueness of

$$w^{(1)} \in W(0,\infty;\mathbf{H}_{\#}^2(\mathcal{F}),\mathbf{L}_{\#}^2(\mathcal{F})), \quad q^{(1)} \in L^2(0,\infty;H_{\#}^1(\mathcal{F})) \tag{5.17}$$

satisfying

$$\begin{cases} \partial_t w^{(1)} - \operatorname{div}\mathbb{T}(w^{(1)},q^{(1)}) = F \quad \text{in } (0,\infty)\times\mathcal{F}_{\#}, \\ \operatorname{div} w^{(1)} = 0 \quad \text{in } (0,\infty)\times\mathcal{F}_{\#}, \\ w^{(1)} = 0 \quad \text{in } (0,\infty)\times\Gamma_{\#}, \\ w^{(1)} \ Le_1 - \text{periodic}, \\ w^{(1)}(0,\cdot) = w^0 - \mathcal{R}\eta_2^0 \quad \text{in } \mathcal{F}_{\#}. \end{cases} \tag{5.18}$$

From (3.13), we also deduce that $\Lambda^*\{\mathbb{T}(w^{(1)},q^{(1)})n_{|\Gamma_{\mathrm{b},\#}}\} \in L^2(0,\infty;\mathcal{D}(A_1^{1/8}))$.

Now we set

$$w^{(2)} \stackrel{\text{def}}{=} w - w^{(1)}, \quad q^{(2)} \stackrel{\text{def}}{=} q - q^{(1)}$$

and (3.23)–(3.24) is transformed into

$$\begin{cases} \partial_t w^{(2)} - \operatorname{div}\mathbb{T}(w^{(2)},q^{(2)}) = 0 \quad \text{in } (0,\infty)\times\mathcal{F}_{\#}, \\ \operatorname{div} w^{(2)} = 0 \quad \text{in } (0,\infty)\times\mathcal{F}_{\#}, \\ w^{(2)} = \Lambda(\partial_t\eta) \quad \text{in } (0,\infty)\times\Gamma_{\#}, \\ w^{(2)} \ Le_1 - \text{periodic}, \\ \partial_{tt}\eta + A_1\eta = -\Lambda^*\left\{\mathbb{T}(w^{(2)},q^{(2)})n_{|\Gamma_{\mathrm{b},\#}}\right\} - \Lambda^*\left\{\mathbb{T}(w^{(1)},q^{(1)})n_{|\Gamma_{\mathrm{b},\#}}\right\} + G, \\ w^{(2)}(0,\cdot) = \mathcal{R}\eta_2^0 \quad \text{in } \mathcal{F}_{\#}, \quad \eta(0,\cdot) = \eta_1^0, \quad \partial_t\eta(0,\cdot) = \eta_2^0. \end{cases} \tag{5.19}$$

We can write (5.19) as

$$\frac{d}{dt}\begin{bmatrix} w^{(2)} \\ \eta \\ \partial_t\eta \end{bmatrix} = A_0\begin{bmatrix} w^{(2)} \\ \eta \\ \partial_t\eta \end{bmatrix} + P_0\begin{bmatrix} 0 \\ 0 \\ -\Lambda^*\left\{\mathbb{T}(w^{(1)},q^{(1)})n_{|\Gamma_{\mathrm{b},\#}}\right\} + G \end{bmatrix}, \quad \begin{bmatrix} w^{(2)} \\ \eta \\ \partial_t\eta \end{bmatrix}(0) = \begin{bmatrix} \mathcal{R}\eta_2^0 \\ \eta_1^0 \\ \eta_2^0 \end{bmatrix}. \tag{5.20}$$

From (3.17), we have

$$P_0\begin{bmatrix} 0 \\ 0 \\ -\Lambda^*\left\{\mathbb{T}(w^{(1)},q^{(1)})n_{|\Gamma_{\mathrm{b},\#}}\right\} + G \end{bmatrix} \in \mathcal{H} \cap \left(\mathbf{H}_{\#}^{1/2}(\mathcal{F}) \times \mathcal{D}(A_1^{5/8}) \times \mathcal{D}(A_1^{1/8})\right).$$

On the other hand, from the definition of $\mathcal{R}$ and from (3.27), we obtain

$$\begin{bmatrix} \mathcal{R}\eta_2^0 \\ \eta_1^0 \\ \eta_2^0 \end{bmatrix} \in \mathcal{D}((-A_0)^{1/2+2\varepsilon}).$$

Combining Lemma 5.5 and Lemma 5.6, we deduce (5.2), (5.3) and

$$w^{(2)} \in W(0,\infty;\mathbf{H}_{\#}^2(\mathcal{F}),\mathbf{L}_{\#}^2(\mathcal{F})), \quad q^{(2)} \in L^2(0,\infty;H_{\#}^1(\mathcal{F})).$$

Combining this with (5.17), we conclude the proof of the theorem. □

6 Fixed Points

We prove here Theorem 1.2 and Theorem 1.3.

6.1 Local in time existence

In order to solve (2.20)–(2.21), we use a fixed point argument. We define for $R, T > 0$

$$\mathfrak{B}_{R,T} \stackrel{\text{def}}{=} \left\{ (F,G) \in L^2(0,T;\mathbf{L}^2_\#(\mathcal{F})) \times L^2(0,T;H^{1/2}_{\#,0}(0,L)) \ ; \ \|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))} \leqslant R \right\}.$$

For $T = \infty$, we simply write $\mathfrak{B}_R \stackrel{\text{def}}{=} \mathfrak{B}_{R,\infty}$.

Assume $(F,G) \in \mathfrak{B}_{R,T}$. Then we consider the solution (w,η) of system (3.23)-(3.24) or equivalently (3.25). In particular, from Theorem 5.1, we have

$$w \in L^2(0,T;\mathbf{H}^2_\#(\mathcal{F})) \cap C([0,T];\mathbf{H}^1_\#(\mathcal{F})) \cap H^1(0,T;\mathbf{L}^2_\#(\mathcal{F})), \quad q \in L^2(0,T;H^1_\#(\mathcal{F})) \tag{6.1}$$

$$\eta \in L^2(0,T;H^{7/2}_{\#,0}(0,L)) \cap C([0,T];H^{5/2}_{\#,0}(0,L)) \cap H^1(0,T;H^{3/2}_{\#,0}(0,L)), \tag{6.2}$$

$$\partial_t \eta \in L^2(0,T;H^{3/2}_{\#,0}(0,L)) \cap C([0,T];H^{1/2}_{\#,0}(0,L)) \cap H^1(0,T;H^{1/2}_{\#,0}(0,L)'), \tag{6.3}$$

with

$$\begin{aligned} &\|w\|_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap C([0,T];\mathbf{H}^1(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))} + \|q\|_{L^2(0,T;H^1(\mathcal{F}))} \\ &\quad + \|\eta\|_{L^2(0,T;H^{7/2}(0,L))\cap C([0,T];H^{5/2}(0,L))} + \|\partial_t\eta\|_{L^2(0,T;H^{3/2}(0,L))\cap C([0,T];H^{1/2}(0,L))} \\ &\qquad \leqslant C\left(R + \|[w^0,\eta_1^0,\eta_2^0]\|_{\mathbf{H}^1(\mathcal{F})\times H^{3+\varepsilon}(0,L)\times H^{1+\varepsilon}(0,L)}\right), \end{aligned} \tag{6.4}$$

for a constant C independent of R and T.

In what follows, we take R such that

$$R \geqslant \|[w^0,\eta_1^0,\eta_2^0]\|_{\mathbf{H}^1(\mathcal{F})\times H^{3+\varepsilon}(0,L)\times H^{1+\varepsilon}(0,L)}. \tag{6.5}$$

We show below that for T small enough, we can construct the change of variables defined in Section 2 and thus consider the mapping

$$\mathcal{Z} : (F,G) \mapsto (\widehat{F}(\eta,w,q), \widehat{G}(\eta,w,q)) \tag{6.6}$$

where the maps $\widehat{F}$ and $\widehat{G}$ are defined by (2.19) and (2.18), and (w,η,p) is solution of system (3.23)-(3.24).

First we notice that by interpolation, (6.4) yields

$$\|\eta\|_{H^{3/4}(0,T;H^2(0,L))} + \|\eta\|_{L^4(0,T;H^3(0,L))} + \|\partial_t\eta\|_{L^4(0,T;H^1(0,L))} + \|w\|_{L^8(0,T;\mathbf{H}^{5/4}(\mathcal{F}))} \leqslant CR. \tag{6.7}$$

We recall that η_1^0 satisfies (1.23). Using the Sobolev embeddings, there exists a positive constant ε_0 such that

$$\eta_1^0 > -1 + \varepsilon_0. \tag{6.8}$$

We first start with a series of useful results:

Lemma 6.1. *There exists $T_0 \stackrel{\text{def}}{=} T_0(\varepsilon_0/R) > 0$ and $C = C(\varepsilon_0) > 0$ such that for all $T \in (0,T_0)$,*

$$\left\| \frac{1}{1+\eta} \right\|_{L^\infty(0,T;L^\infty(0,L))} \leqslant C. \tag{6.9}$$

In particular, for any $n_1, n_2, n_3 \geqslant 0$ and for all $T \in (0,T_0)$,

$$\left\| \frac{\eta^{n_1}(\partial_s\eta)^{n_2}}{(1+\eta)^{n_3}} \right\|_{L^\infty(0,T;L^\infty(0,L))} \leqslant CR^{n_1+n_2}, \tag{6.10}$$

for a constant C independent of T and R.

Proof. First, combining the continuous embedding $H^{3/2}(0,L) \hookrightarrow L^\infty(0,L)$ and (6.4),(6.5) we deduce

$$\|\eta - \eta_1^0\|_{L^\infty(0,T;L^\infty(0,L))} \leqslant CT^{1/2}\|\partial_t\eta\|_{L^2(0,T;H^{3/2}(0,L))} \leqslant CT^{1/2}R.$$

Then, from (6.8) and the above relation, we can choose $T_0 > 0$ proportional to $(\varepsilon_0/R)^2$ such that $\eta(t) > -1 + \varepsilon_0/2$ for all $t \in (0,T_0)$, and the first estimate is proved.

For the second estimate, we first observe that the continuous embedding $H^{5/2}(0,L) \hookrightarrow W^{1,\infty}(0,L)$ with (6.4), (6.5) implies

$$\|\eta\|_{L^\infty(0,T;L^\infty(0,L))} + \|\partial_s\eta\|_{L^\infty(0,T;L^\infty(0,L))} \leqslant CR.$$

Then the conclusion follows with (6.9). □

Lemma 6.2. *There exists a constant C depending on T_0 such that for all $T \in (0, T_0)$,*

$$\|\eta\|_{L^\infty(0,T;H^2(0,L))} + \|\partial_s \eta\|_{L^\infty(0,T;L^\infty(0,L))} \leqslant C(T^{1/6}R + \|\eta_1^0\|_{H^2(0,L)}). \tag{6.11}$$

Proof. Using Proposition A.1, we have

$$\|\eta\|_{L^\infty(0,T;H^2(0,L))} \leqslant C\|\eta_1^0\|_{H^2(0,L)} + CT^{1/6}\|\eta\|_{H^{3/4}(0,T;H^2(0,L)}.$$

We then combine this with (6.7) to obtain the first estimate. The second estimate is then deduced from the continuous embedding $H^1(0,L) \hookrightarrow L^\infty(0,L)$. □

Using Lemma 6.1 and the expressions (2.14), (2.22), (2.24), (2.26) and (2.27), we deduce that

$$\|\nabla X\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4} + \|b\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4} + \|a(X)\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4} + \|\nabla Y(X)\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4} \leqslant C(1+R), \tag{6.12}$$

$$\left\|\frac{\partial a}{\partial x_2}(X)\right\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4} + \left\|\frac{\partial^2 a}{\partial x_2^2}(X)\right\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4} + \left\|\frac{\partial}{\partial x_2}\nabla Y(X)\right\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4} \leqslant CR. \tag{6.13}$$

Using the expressions (2.24)–(2.27) and (6.7), (6.10), we have

$$\left\|\frac{\partial a}{\partial x_1}(X)\right\|_{L^4(0,T;L^\infty(\mathcal{F}))^4} + \left\|\frac{\partial^2 a}{\partial x_1 \partial x_2}(X)\right\|_{L^4(0,T;L^\infty(\mathcal{F}))^4} + \left\|\frac{\partial}{\partial x_1}\nabla Y(X)\right\|_{L^4(0,T;L^\infty(\mathcal{F}))^4} \leqslant C(R+R^2), \tag{6.14}$$

$$\left\|\frac{\partial^2 a}{\partial x_1^2}(X)\right\|_{L^4(0,T;L^2(\mathcal{F}))^4} \leqslant C(R+R^3). \tag{6.15}$$

Using the expression (2.23) and (6.11) with $H^2(0,L) \hookrightarrow L^\infty(0,L)$ we deduce,

$$\|\nabla Y(X) - I_2\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4} + \|\det(\nabla X) - 1\|_{L^\infty(0,T;L^\infty(\mathcal{F}))} \leqslant C(T^{1/6}R + \|\eta_1^0\|_{H^2(0,L)}). \tag{6.16}$$

Using the expressions (2.28) and (6.7), (6.10) with $H^1(0,L) \hookrightarrow L^\infty(0,L)$ we deduce,

$$\|\partial_t a(X)\|_{L^4(0,T;L^2(\mathcal{F}))^4} \leqslant C(R+R^2), \tag{6.17}$$

$$\|(\partial_t Y)(X)\|_{L^4(0,T;\mathbf{L}^\infty(\mathcal{F}))} \leqslant CR. \tag{6.18}$$

Using (6.4), (6.7) with $\mathbf{H}^{5/4}(\mathcal{F}) \hookrightarrow \mathbf{L}^\infty(\mathcal{F})$ we deduce,

$$\|w\|_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))} + \|p\|_{L^2(0,T;H^1(\mathcal{F}))} \leqslant CR, \tag{6.19}$$

$$\|w\|_{L^4(0,T;\mathbf{H}^1(\mathcal{F}))} \leqslant CT^{1/4}\|w\|_{L^\infty(0,T;\mathbf{H}^1(\mathcal{F}))} \leqslant CT^{1/4}R, \tag{6.20}$$

$$\|w\|_{L^4(0,T;\mathbf{L}^\infty(\mathcal{F}))} \leqslant CT^{1/8}\|w\|_{L^8(0,T;\mathbf{H}^{5/4}(\mathcal{F}))} \leqslant CT^{1/8}R, \tag{6.21}$$

$$\|w\otimes w\|_{L^4(0,T;L^2(\mathcal{F}))^4} \leqslant T^{1/4}\|w\otimes w\|_{L^\infty(0,T;L^2(\mathcal{F}))^4} \leqslant CT^{1/4}\|w\|^2_{L^\infty(0,T;\mathbf{H}^1(\mathcal{F}))} \leqslant CT^{1/4}R^2, \tag{6.22}$$

$$\|(w\cdot\nabla)w\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} \leqslant T^{1/4}\|(w\cdot\nabla)w\|_{L^4(0,T;\mathbf{L}^2(\mathcal{F}))} \leqslant CT^{1/4}\|w\|^2_{L^8(0,T;\mathbf{H}^{5/4}(\mathcal{F}))} \leqslant CT^{1/4}R^2. \tag{6.23}$$

From (2.19), (6.12)–(6.22), we obtain for some $N_1 \geqslant 2$,

$$\left\|\widehat{F}_\alpha(\eta,w,q)\right\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))} \leqslant C(T^{1/8}R + \|\eta_1^0\|_{H^2(0,L)})(R+R^{N_1}). \tag{6.24}$$

On the other hand, in order to estimate $\widehat{G}(\eta,w)$ in $L^2(0,T;H^{1/2}_{\#,0}(0,L))$, we need to estimate the following expressions in $L^2(0,T;L^2(\mathcal{F}))$ (and use the trace theorem)

$$\frac{\partial}{\partial y_m}\left(\partial_s\eta\frac{\partial a_{ik}}{\partial x_j}(X)w_k\right) = \delta_{1,m}\partial_{ss}\eta\frac{\partial a_{ik}}{\partial x_j}(X)w_k + \partial_s\eta\sum_n\frac{\partial^2 a_{ik}}{\partial x_j\partial x_n}(X)\frac{\partial X_n}{\partial y_m}w_k + \partial_s\eta\frac{\partial a_{ik}}{\partial x_j}(X)\frac{\partial w_k}{\partial y_m} \tag{6.25}$$

$$\frac{\partial}{\partial y_m}\left(\partial_s\eta a_{ik}(X)\frac{\partial w_k}{\partial y_\ell}\frac{\partial Y_\ell}{\partial x_j}(X)\right)=\delta_{1,m}\partial_{ss}\eta a_{ik}(X)\frac{\partial w_k}{\partial y_\ell}\frac{\partial Y_\ell}{\partial x_j}(X)+\partial_s\eta\sum_n\frac{\partial a_{ik}}{\partial x_n}(X)\frac{\partial X_n}{\partial y_m}\frac{\partial w_k}{\partial y_\ell}\frac{\partial Y_\ell}{\partial x_j}(X)$$
$$+\partial_s\eta a_{ik}(X)\frac{\partial^2 w_k}{\partial y_\ell\partial y_m}\frac{\partial Y_\ell}{\partial x_j}(X)+\partial_s\eta\sum_n a_{ik}(X)\frac{\partial w_k}{\partial y_\ell}\frac{\partial^2 Y_\ell}{\partial x_j\partial x_n}(X)\frac{\partial X_n}{\partial y_m}\quad(6.26)$$

$$\frac{\partial}{\partial y_m}\left(\left(\delta_{2,k}\delta_{2,l}-a_{2k}(X)\frac{\partial Y_\ell}{\partial x_2}(X)\right)\frac{\partial w_k}{\partial y_\ell}\right)=-\sum_n\frac{\partial a_{2k}}{\partial x_n}(X)\frac{\partial X_n}{\partial y_m}\frac{\partial w_k}{\partial y_\ell}\frac{\partial Y_\ell}{\partial x_2}(X)$$
$$+\left(\delta_{2,k}\delta_{2,l}-a_{2k}(X)\frac{\partial Y_\ell}{\partial x_2}(X)\right)\frac{\partial^2 w_k}{\partial y_\ell\partial y_m}-a_{2k}(X)\sum_n\frac{\partial w_k}{\partial y_\ell}\frac{\partial^2 Y_\ell}{\partial x_2\partial x_n}(X)\frac{\partial X_n}{\partial y_m}.\quad(6.27)$$

From (6.4), (6.10)-(6.16), (6.20), (6.21) we deduce that for some $N_2\geqslant 2$,

$$\left\|\widehat{G}(\eta,w)\right\|_{L^2(0,T;H^{1/2}_{\#,0}(0,L))}\leqslant C(T^{1/8}R+\|\eta_1^0\|_{H^2(0,L)})(R+R^{N_2}).\quad(6.28)$$

This shows that if $(F,G)\in\mathfrak{B}_{R,T}$, then

$$\|\mathcal{Z}(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))}\leqslant C(T^{1/8}R+\|\eta_1^0\|_{H^2(0,L)})(R+R^N),$$

for some $N\geqslant 2$. In particular, for T and $\|\eta_1^0\|_{H^2(0,L)}$ small enough,

$$\mathcal{Z}(F,G)\in\mathfrak{B}_{R,T}.$$

Assume now $(F^{(1)},G^{(1)}),(F^{(2)},G^{(2)})\in\mathfrak{B}_{R,T}$, and let us denote by $(\eta^{(1)},w^{(1)},q^{(1)})$, $(\eta^{(2)},w^{(2)},q^{(2)})$ the corresponding solutions of (3.23)-(3.24) given by Theorem 5.1. They satisfy in particular (6.4) and (6.7). By setting

$$F\stackrel{\text{def}}{=}F^{(1)}-F^{(2)},\quad G\stackrel{\text{def}}{=}G^{(1)}-G^{(2)},\quad \eta\stackrel{\text{def}}{=}\eta^{(1)}-\eta^{(2)},\quad w\stackrel{\text{def}}{=}w^{(1)}-w^{(2)},\quad q\stackrel{\text{def}}{=}q^{(1)}-q^{(2)}$$

we have also from Theorem 5.1,

$$\|w\|_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap C([0,T];\mathbf{H}^1(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))}+\|q\|_{L^2(0,T;H^1(\mathcal{F}))}$$
$$+\|\eta\|_{L^2(0,T;H^{7/2}(0,L))\cap C([0,T];H^{5/2}(0,L))}$$
$$+\|\partial_t\eta\|_{L^2(0,T;H^{3/2}(0,L))\cap C([0,T];H^{1/2}(0,L))}\leqslant C\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))},\quad(6.29)$$

and

$$\|\eta\|_{H^{3/4}(0,T;H^2(0,L))}+\|\eta\|_{L^4(0,T;H^3(0,L))}+\|\partial_t\eta\|_{L^4(0,T;H^1(0,L))}$$
$$+\|w\|_{L^8(0,T;\mathbf{H}^{5/4}(\mathcal{F}))}\leqslant C\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))},\quad(6.30)$$

for a constant C independent of R and T.

Using Proposition A.1 and (6.29), we have

$$\|\eta\|_{L^\infty(0,T;H^2(0,L))}\leqslant CT^{1/6}\|\eta\|_{H^{3/4}(0,T;H^2(0,L)}\leqslant CT^{1/6}\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))}.$$

Then, combining this estimate with Lemma 6.1 for $\eta^{(i)}$, $i=1,2$, we obtain the following

Lemma 6.3. *For any nonnegative integers n_1, n_1, n_2 there exists $C>0$ such that for all $T\in(0,T_0)$,*

$$\left\|\frac{(\eta^{(1)})^{n_1}(\partial_s\eta^{(1)})^{n_2}}{(1+\eta^{(1)})^{n_3}}-\frac{(\eta^{(2)})^{n_1}(\partial_s\eta^{(2)})^{n_2}}{(1+\eta^{(2)})^{n_3}}\right\|_{L^\infty(0,T;L^\infty(0,L))}$$
$$\leqslant C(1+R^{n_1+n_2})T^{1/6}\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))}.\quad(6.31)$$

Using (6.4), (6.5), (6.7), (6.10) for $\eta^{(i)}$, $i=1,2$, and (6.29)-(6.31) and the expressions (2.14), (2.22)–(2.28), we deduce that,

$$\left\|\nabla Y^{(1)}(X^{(1)})-\nabla Y^{(2)}(X^{(2)})\right\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4}+\left\|\det(\nabla X^{(1)})-\det(\nabla X^{(2)})\right\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4}$$
$$\leqslant C(1+R)T^{1/6}\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))},\quad (6.32)$$

$$\|\nabla X^{(1)}-\nabla X^{(2)}\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4}+\|b^{(1)}-b^{(2)}\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4}+\|a^{(1)}(X^{(1)})-a^{(2)}(X^{(2)})\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4}$$
$$\leqslant C(1+R)\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))},\quad (6.33)$$

$$\left\|\frac{\partial}{\partial x_2}\nabla Y^{(1)}(X^{(1)})-\frac{\partial}{\partial x_2}\nabla Y^{(2)}(X^{(2)})\right\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4}+\left\|\frac{\partial a^{(1)}}{\partial x_2}(X^{(1)})-\frac{\partial a^{(2)}}{\partial x_2}(X^{(2)})\right\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4}$$
$$+\left\|\frac{\partial^2 a^{(1)}}{\partial x_2^2}(X^{(1)})-\frac{\partial^2 a^{(2)}}{\partial x_2^2}(X^{(2)})\right\|_{L^\infty(0,T;L^\infty(\mathcal{F}))^4}\leqslant C(1+R)\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))},\quad (6.34)$$

$$\left\|\frac{\partial a^{(1)}}{\partial x_1}(X^{(1)})-\frac{\partial a^{(2)}}{\partial x_1}(X^{(2)})\right\|_{L^4(0,T;L^\infty(\mathcal{F}))^4}+\left\|\frac{\partial^2 a^{(1)}}{\partial x_1\partial x_2}(X^{(1)})-\frac{\partial^2 a^{(2)}}{\partial x_1\partial x_2}(X^{(2)})\right\|_{L^4(0,T;L^\infty(\mathcal{F}))^4}$$
$$+\left\|\frac{\partial}{\partial x_1}\nabla Y^{(1)}(X^{(1)})-\frac{\partial}{\partial x_1}\nabla Y^{(2)}(X^{(2)})\right\|_{L^4(0,T;L^\infty(\mathcal{F}))^4}$$
$$\leqslant C(1+R^2)\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))},\quad (6.35)$$

$$\left\|\frac{\partial^2 a^{(1)}}{\partial x_1^2}(X^{(1)})-\frac{\partial^2 a^{(2)}}{\partial x_1^2}(X^{(2)})\right\|_{L^4(0,T;L^2(\mathcal{F}))^4}\leqslant C(1+R^3)\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))},\quad (6.36)$$

$$\left\|\partial_t a^{(1)}(X^{(1)})-\partial_t a^{(2)}(X^{(2)})\right\|_{L^4(0,T;L^2(\mathcal{F}))^4}\leqslant C(1+R^2)\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))},\quad (6.37)$$

$$\left\|(\partial_t Y^{(1)})(X^{(1)})-(\partial_t Y^{(2)})(X^{(2)})\right\|_{L^4(0,T;\mathbf{L}^\infty(\mathcal{F}))}\leqslant C(1+R)\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))},\quad (6.38)$$

and with

$$w^{(1)}\otimes w^{(1)}-w^{(2)}\otimes w^{(2)}=w^{(1)}\otimes w+w\otimes w^{(2)},$$
$$(w^{(1)}\cdot\nabla)w^{(1)}-(w^{(2)}\cdot\nabla)w^{(2)}=(w^{(1)}\cdot\nabla)w+(w\cdot\nabla)w^{(2)},$$

and (6.19)-(6.21) for $(w^{(i)},p^{(i)})$, $i=1,2$, (6.29) and (6.30) we deduce,

$$\left\|w^{(1)}\otimes w^{(1)}-w^{(2)}\otimes w^{(2)}\right\|_{L^4(0,T;L^2(\mathcal{F}))^4}\leqslant T^{1/4}\left\|w^{(1)}\otimes w^{(1)}-w^{(2)}\otimes w^{(2)}\right\|_{L^\infty(0,T;L^2(\mathcal{F}))^4}$$
$$\leqslant CT^{1/4}\left(\left\|w^{(1)}\right\|_{L^\infty(0,T;\mathbf{H}^1(\mathcal{F}))}+\left\|w^{(2)}\right\|_{L^\infty(0,T;\mathbf{H}^1(\mathcal{F}))}\right)\|w\|_{L^\infty(0,T;\mathbf{H}^1(\mathcal{F}))}$$
$$\leqslant CRT^{1/4}\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))}\quad (6.39)$$

$$\left\|(w^{(1)}\cdot\nabla)w^{(1)}-(w^{(2)}\cdot\nabla)w^{(2)}\right\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))}\leqslant T^{1/4}\left\|(w^{(1)}\cdot\nabla)w^{(1)}-(w^{(2)}\cdot\nabla)w^{(2)}\right\|_{L^4(0,T;\mathbf{L}^2(\mathcal{F}))}$$
$$\leqslant CT^{1/4}\left(\left\|w^{(1)}\right\|_{L^8(0,T;\mathbf{H}^{5/4}(\mathcal{F}))}+\left\|w^{(2)}\right\|_{L^8(0,T;\mathbf{H}^{5/4}(\mathcal{F}))}\right)\|w\|_{L^8(0,T;\mathbf{H}^{5/4}(\mathcal{F}))}$$
$$\leqslant CRT^{1/4}\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))}.\quad (6.40)$$

Using the above estimates, (2.19), (6.12)–(6.21) for $(w^{(i)}, p^{(i)}, \eta^{(i)})$, $i = 1, 2$, and (2.18)–(6.27) and (6.11), we obtain

$$\begin{aligned}\left\|\widehat{F}_\alpha(\eta^{(1)}, w^{(1)}, q^{(1)}) - \widehat{F}_\alpha(\eta^{(2)}, w^{(2)}, q^{(2)})\right\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))} \\ \leqslant C(1+R^N)(T^{1/8}R + \|\eta_1^0\|_{H^2(0,L)})\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))},\end{aligned} \tag{6.41}$$

$$\begin{aligned}\left\|\widehat{G}(\eta^{(1)}, w^{(1)}) - \widehat{G}(\eta^{(2)}, w^{(2)})\right\|_{L^2(0,T;H^{1/2}_{\#,0}(0,L))} \\ \leqslant C(1+R^N)(T^{1/8}R + \|\eta_1^0\|_{H^2(0,L)})\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))},\end{aligned} \tag{6.42}$$

for some $N \geqslant 1$. Thus, if $(F^{(i)}, G^{(i)}) \in \mathfrak{B}_{R,T}$, $i = 1, 2$, then

$$\begin{aligned}\|\mathcal{Z}(F^{(1)}, G^{(1)}) - \mathcal{Z}(F^{(2)}, G^{(2)})\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))} \\ \leqslant C(1+R^N)(T^{1/8}R + \|\eta_1^0\|_{H^2(0,L)})\|(F,G)\|_{L^2(0,T;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T;H^{1/2}_{\#,0}(0,L))}.\end{aligned}$$

In particular, for T and $\|\eta_1^0\|_{H^2(0,L)}$ small enough, $\mathcal{Z}$ is a contraction on $\mathfrak{B}_{R,T}$. Using the Banach fixed point theorem, we deduce the existence and uniqueness of $(F, G) \in \mathfrak{B}_{R,T}$ such that

$$\mathcal{Z}((F,G)) = (F,G).$$

The corresponding solution (η, w, q) of system (3.23)-(3.24) is a solution of (2.20)–(2.21).

6.2 Uniqueness

Let us consider another solution $(\eta^{(*)}, w^{(*)}, q^{(*)})$ of (2.20)–(2.21) on $(0, T)$ with $T > 0$. If we write

$$F^{(*)} \stackrel{\text{def}}{=} \widehat{F}(\eta^{(*)}, w^{(*)}, q^{(*)}), \quad G^{(*)} \stackrel{\text{def}}{=} \widehat{G}(\eta^{(*)}, w^{(*)}),$$

then we have

$$(F^{(*)}, G^{(*)}) \in L^2(0,T;\mathbf{L}^2_\#(\mathcal{F})) \times L^2(0,T;H^{1/2}_{\#,0}(0,L))$$

and

$$\mathcal{Z}((F^{(*)}, G^{(*)})) = (F^{(*)}, G^{(*)}).$$

Moreover, according to the Lebesgue theorem we have

$$\lim_{T^*\to 0}\left\|(F^{(*)}, G^{(*)})\right\|_{L^2(0,T^*;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,T^*;H^{1/2}_{\#,0}(0,L))} = 0$$

and thus for $T^* \leqslant T$ small enough,

$$(F^{(*)}, G^{(*)}) \in \mathfrak{B}_{R,T^*}.$$

Since $\mathcal{Z}$ is a contraction on $\mathfrak{B}_{R,T^*}$, we deduce that $(\eta^{(*)}, w^{(*)}, q^{(*)}) = (\eta, w, q)$. This ends the proof of Theorem 1.2.

6.3 Small data

We can now consider the case of small initial conditions and $T = \infty$. The proof is similar to the proof of Section 6.1. We assume $(F, G) \in \mathfrak{B}_R$. Then, from Theorem 5.1, the system (3.23)-(3.24) admits a unique solution (η, w, q) with the estimates

$$\begin{aligned}\|w\|_{L^2(0,\infty;\mathbf{H}^2_\#(\mathcal{F}))\cap C_b([0,\infty);\mathbf{H}^1_\#(\mathcal{F}))\cap H^1(0,\infty;\mathbf{L}^2_\#(\mathcal{F}))} + \|q\|_{L^2(0,\infty;H^1_\#(\mathcal{F}))} \\ + \|\eta\|_{L^2(0,\infty;H^{7/2}(0,L))\cap C_b([0,\infty);H^{5/2}(0,L))} + \|\partial_t\eta\|_{L^2(0,\infty;H^{3/2}(0,L))\cap C_b([0,\infty);H^{1/2}(0,L))} \\ \leqslant C\left(R + \|[w^0, \eta_1^0, \eta_2^0]\|_{\mathbf{H}^1(\mathcal{F})\times H^{3+\varepsilon}(0,L)\times H^{1+\varepsilon}(0,L)}\right),\end{aligned} \tag{6.43}$$

for a constant C independent of R.

In what follows, we take R such that (6.5) is satisfied and we choose R small enough so that we can construct the change of variables defined in Section 2. Thus we can consider the mapping

$$\mathcal{Z} : (F, G) \in \mathfrak{B}_R \mapsto (\widehat{F}(\eta, w, q), \widehat{G}(\eta, w, q)) \in \mathfrak{B}_R \tag{6.44}$$

where the maps $\widehat{F}$ and $\widehat{G}$ are defined by (2.19) and (2.18).

First we notice that by interpolation, (6.43) and (6.5) yield

$$\|\eta\|_{H^{3/4}(0,\infty;H^2(0,L))} + \|\eta\|_{L^4(0,\infty;H^3(0,L))} + \|\partial_t\eta\|_{L^4(0,\infty;H^1(0,L))} + \|w\|_{L^4(0,\infty;\mathbf{H}^{3/2}(\mathcal{F}))} \leqslant CR. \tag{6.45}$$

Then it implies,

$$\|w\|_{L^4(0,\infty;\mathbf{H}^1(\mathcal{F}))} \leqslant C\,\|w\|_{L^4(0,\infty;\mathbf{H}^{3/2}(\mathcal{F}))} \leqslant CR, \tag{6.46}$$

$$\|w\|_{L^4(0,\infty;\mathbf{L}^\infty(\mathcal{F}))} \leqslant C\,\|w\|_{L^4(0,\infty;\mathbf{H}^{3/2}(\mathcal{F}))} \leqslant CR, \tag{6.47}$$

$$\|w \otimes w\|_{L^4(0,\infty;L^2(\mathcal{F}))^4} \leqslant C\,\|w\|^2_{L^4(0,\infty;\mathbf{H}^{3/2}(\mathcal{F}))} \leqslant CR^2, \tag{6.48}$$

$$\|(w \cdot \nabla)w\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} \leqslant C\,\|w\|^2_{L^4(0,\infty;\mathbf{H}^{3/2}(\mathcal{F}))} \leqslant CR^2. \tag{6.49}$$

Then with similar calculations than in Section 6.1, by using (6.46)-(6.49) instead of (6.20)-(6.23), we can show that if $(F, G) \in \mathfrak{B}_R$, then

$$\|\mathcal{Z}(F, G)\|_{L^2(0,\infty;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,\infty;H^{1/2}_{\#,0}(0,L))} \leqslant C(R^2 + R^{N_1}),$$

for some $N_1 \geqslant 2$. In particular, for R small enough,

$$\mathcal{Z}(F, G) \in \mathfrak{B}_R.$$

We can also show that, if $(F^{(i)}, G^{(i)}) \in \mathfrak{B}_R$, $i = 1, 2$, then

$$\begin{aligned}\|\mathcal{Z}(F^{(1)}, G^{(1)}) - \mathcal{Z}(F^{(2)}, G^{(2)})\|_{L^2(0,\infty;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,\infty;H^{1/2}_{\#,0}(0,L))} \\ \leqslant C(R + R^{N_2})\|(F, G)\|_{L^2(0,\infty;\mathbf{L}^2_\#(\mathcal{F}))\times L^2(0,\infty;H^{1/2}_{\#,0}(0,L))},\end{aligned}$$

for some $N_2 \geqslant 2$. In particular, for R small enough, $\mathcal{Z}$ is a contraction on $\mathfrak{B}_R$. Using the Banach fixed point theorem, we deduce the existence and uniqueness of $(F, G) \in \mathfrak{B}_R$ such that

$$\mathcal{Z}((F, G)) = (F, G).$$

The corresponding solution (η, w, q) of system (3.23)-(3.24) is a solution of (2.20)–(2.21).

A A technical result

In this section, X denotes a Hilbert space and $C > 0$ denotes a generic constant independant on $T > 0$.

Proposition A.1. *Let $\varepsilon \in (0, 1/2)$. There exists $C > 0$ such that for all $v \in H^{1/2+\varepsilon}(0, T; X)$,*

$$\|v\|_{L^\infty(0,T;X)} \leqslant (1 + T^{1/(2(1-\varepsilon))})\|v(0)\|_X + CT^{\varepsilon/(2(1-\varepsilon))}\|v\|_{H^{1/2+\varepsilon}(0,T;X)}.$$

Proof. Let $u \in H^{1/2+\varepsilon}(0, T; X)$ such that $u(0) = 0$. We define as follows

$$\tilde{u}(t) = \begin{cases} u(t) & \text{if } t \in [0, T), \\ u(2T - t) & \text{if } t \in [T, 2T), \\ 0 & \text{if } t \geqslant T. \end{cases}$$

In particular, we have

$$\forall t \in [0, 2T], \quad \tilde{u}(2T - t) = \tilde{u}(t) \quad \text{and} \quad \forall t > 2T \quad \tilde{u}(t) = 0 \tag{A.1}$$

and since $\tilde{u}(0) = \tilde{u}(2T) = 0$ we have $\tilde{u} \in H^{1/2+\varepsilon}(0, +\infty; X)$.

First, let us prove,

$$\|\tilde{u}\|_{H^{1/2+\varepsilon/2}(0,+\infty;X)} \leqslant C\|u\|_{H^{1/2+\varepsilon/2}(0,T;X)}. \tag{A.2}$$

For that, we start with the following calculations (where we use (A.1)),

$$\begin{aligned}
\|\tilde{u}\|^2_{H^{1/2+\varepsilon/2}(0,+\infty;X)} &= \int_0^{+\infty} \|\tilde{u}(s)\|_X^2 ds + \int_0^{+\infty}\int_0^{+\infty} \frac{\|\tilde{u}(s)-\tilde{u}(\tau)\|_X^2}{|s-\tau|^{2+\varepsilon}} ds d\tau \\
&= \int_0^{2T} \|u(s)\|_X^2 ds + \int_0^{2T}\int_0^{2T} \frac{\|\tilde{u}(s)-\tilde{u}(\tau)\|_X^2}{|s-\tau|^{2+\varepsilon}} ds d\tau + 2\int_0^{2T}\int_{2T}^{+\infty} \frac{\|\tilde{u}(s)\|_X^2}{|s-\tau|^{2+\varepsilon}} d\tau ds \\
&= \|\tilde{u}\|^2_{H^{1/2+\varepsilon/2}(0,2T;X)} + \frac{2}{1+\varepsilon}\int_0^{2T} \frac{\|\tilde{u}(2T-s)\|_X^2}{(2T-s)^{1+\varepsilon}} ds \\
&= \|\tilde{u}\|^2_{H^{1/2+\varepsilon/2}(0,2T;X)} + \frac{2}{1+\varepsilon}\int_0^{2T} \frac{\|\tilde{u}(s)\|_X^2}{s^{1+\varepsilon}} ds. \qquad \text{(A.3)}
\end{aligned}$$

Thus, we recall the following generalized Hardy's inequality, that can be obtained from [20, 3.2.6, (6) p.261] or from the proof of [12, Thm 1.4.4.4],

$$\int_0^1 \frac{|v(\xi)|^2}{\xi^{1+\varepsilon}} d\xi \leqslant C\int_0^1\int_0^1 \frac{|v(\xi)-v(\zeta)|^2}{|\xi-\zeta|^{2+\varepsilon}} d\xi d\zeta \quad \forall v \in H_0^{1/2+\varepsilon/2}(0,1).$$

By applying the above inequality to $v(\xi) = \|\tilde{u}(\xi 2T)\|$ we deduce,

$$\begin{aligned}
\int_0^{2T} \frac{\|\tilde{u}(s)\|_X^2}{s^{1+\varepsilon}} ds &= (2T)^{-\varepsilon}\int_0^1 \frac{\|\tilde{u}(\xi 2T)\|_X^2}{\xi^{1+\varepsilon}} d\xi \leqslant C(2T)^{-\varepsilon}\int_0^1\int_0^1 \frac{|\|\tilde{u}(\xi 2T)\|_X - \|\tilde{u}(\zeta 2T)\|_X|^2}{|\xi-\zeta|^{2+\varepsilon}} d\xi d\zeta \\
&\leqslant C(2T)^{-\varepsilon}\int_0^1\int_0^1 \frac{\|\tilde{u}(\xi 2T)-\tilde{u}(\zeta 2T)\|_X^2}{|\xi-\zeta|^{2+\varepsilon}} d\xi d\zeta = C\int_0^{2T}\int_0^{2T} \frac{\|\tilde{u}(s)-\tilde{u}(\tau)\|_X^2}{|s-\tau|^{2+\varepsilon}} ds d\tau.
\end{aligned}$$

By combining the above inequality with (A.3) we obtain

$$\|\tilde{u}\|^2_{H^{1/2+\varepsilon/2}(0,+\infty;X)} \leqslant C\|\tilde{u}\|^2_{H^{1/2+\varepsilon/2}(0,2T;X)}.$$

Moreover, by using (A.1) we deduce

$$\begin{aligned}
\|\tilde{u}\|^2_{H^{1/2+\varepsilon/2}(0,2T;X)} &= \int_0^{2T} \|\tilde{u}(s)\|_X^2 ds + \int_0^{2T}\int_0^{2T} \frac{\|\tilde{u}(s)-\tilde{u}(\tau)\|_X^2}{|s-\tau|^{2+\varepsilon}} ds d\tau \\
&= 2\int_0^{T} \|u(s)\|_X^2 ds + 2\int_0^{T}\int_0^{T} \frac{\|u(s)-u(\tau)\|_X^2}{|s-\tau|^{2+\varepsilon}} ds d\tau + 2\int_0^{T}\int_T^{2T} \frac{\|\tilde{u}(s)-\tilde{u}(\tau)\|_X^2}{|s-\tau|^{2+\varepsilon}} ds d\tau \\
&= 2\|u\|^2_{H^{1/2+\varepsilon/2}(0,T;X)} + 2\int_0^{T}\int_0^{T} \frac{\|u(s)-u(\tau)\|_X^2}{|s+\tau-2T|^{2+\varepsilon}} ds d\tau.
\end{aligned}$$

Since $|s+\tau-2T| = T-s+T-\tau \geqslant |s-\tau|$ the last above integral is bounded by $\|u\|_{H^{1/2+\varepsilon/2}(0,T;X)}$ and (A.2) follows.

Next, the continuous embedding $H^{1/2+\varepsilon/2}(0,+\infty;X) \hookrightarrow L^\infty(0,+\infty;X)$ guarantees the existence of $C>0$ independent on T such that

$$\|u\|_{L^\infty(0,T;X)} = \|\tilde{u}\|_{L^\infty(0,+\infty;X)} \leqslant C\|\tilde{u}\|_{H^{1/2+\varepsilon/2}(0,+\infty;X)}.$$

Then with (A.2),

$$\|u\|_{L^\infty(0,T;X)} \leqslant C\|u\|_{H^{1/2+\varepsilon/2}(0,T;X)}.$$

Next, if we now suppose that $u \in H^1(0,T;X)$ and $u(0)=0$, then $u(t) = \displaystyle\int_0^t u'(s)ds$ with the Cauchy-Schwarz inequality yields

$$\|u\|_{L^\infty(0,T;X)} \leqslant CT^{1/2}\|u\|_{H^1(0,T;X)}.$$

Then by combining the two last inequalities with an interpolation argument we obtain

$$\|u\|_{L^\infty(0,T;X)} \leqslant CT^{\varepsilon/(2(1-\varepsilon))}\|u\|_{H^{1/2+\varepsilon}(0,T;X)}.$$

Finally, suppose that $v \in H^{1/2+\varepsilon}(0,T;X)$ and apply the above inequality to $u(t) = v(t)-v(0)$. With $\|v(0)\|_{L^\infty(0,T;X)} = \|v(0)\|_X$ and $\|v(0)\|_{H^{1/2+\varepsilon}(0,T;X)} = \|v(0)\|_{L^2(0,T;X)} = T^{1/2}\|v(0)\|_X$ we obtain the result. □

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