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► **To cite this version:**

Gabriel Lepetit-Aimon, Dominique Fober, Yann Orlarey, Stéphane Letz. INScore expressions to compose symbolic scores. International Conference on Technologies for Music Notation and Representation, 2016, Cambridge, United Kingdom. pp.137-143. hal-02158908

**HAL Id: hal-02158908**

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Submitted on 18 Jun 2019

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# INSCORE EXPRESSIONS TO COMPOSE SYMBOLIC SCORES

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## ABSTRACT

INScore is an environment for the design of augmented interactive music scores turned to non-conventional use of music notation. The environment allows arbitrary graphic resources to be used and composed for the music representation. It supports symbolic music notation, described using Guido Music Notation or MusicXML formats. The environment has been extended to provided score level composition using a set of operators that consistently take scores as arguments to compute new scores as output. INScore API supports now *score expressions* both at OSC and at scripting levels. The work is based on a previous research that solved the issues of the notation consistency across scores composition. This paper focuses on the language level and explains the different strategies to evaluate score expressions.

## 1. INTRODUCTION

Contemporary music creation poses numerous challenges to the music notation. Spatialized music, new instruments, gesture based interactions, real-time and interactive scores, are among the new domains that are now commonly explored by artists. Classical music notation doesn't cover the needs of these new musical forms and numerous research and approaches have recently emerged, testifying to the maturity of the music notation domain, in the light of computer tools for music notation and representation. Issues like writing spatialized music [1], addressing new instruments [2] or new interfaces [3] (to cite just a few), are now subject of active research and proposals.

Interactive music and real-time scores are also representative of an expanding domain in the music creation field. The advent of the digital score and the maturation of the computer tools for music notation and representation constitute the basement for the development of this musical form, which is often grounded on non-traditional music representation [4] [5] but may also use the common music notation [6, 7].

In order to address the notation challenges mentioned above, INScore [8, 9] has been designed as an environment opened

to non-conventional music representation (although it supports symbolic notation), and turned to real-time and interactive use [10, 11]. It is clearly focused on music representation only and in this way, differs from tools integrated into programming environments like Bach [12] or MaxScore [13].

INScore has been extended with *score expressions* that provide symbolic scores composition features (e.g., putting scores in sequence or in parallel). Building new scores from existing scores at symbolic level is not new. Haskell is providing such features [14]. Freeman and Lee proposed score composition operations in a real-time and interactive notation context [15]. Regarding the score operations used by INScore, they are imported from a previous work [16] that was focusing on the music notation consistency through arbitrary scores composition.

The novelty of the proposed approach relies on the dynamic aspects of the scores composition operations, as well as on the persistence of the score expressions. A score may be composed as an arbitrary graph of score expressions and equipped with a fine control over the changes propagation.

The paper introduces first the score composition expressions. Next, the different evaluation strategies are explained and illustrated with examples. The articulation with the INScore environment is presented in detail and followed by concrete use cases. An extension of the primary scores composition design to *score expressions* composition is next introduced. A generalisation of this approach to the whole set of INScore graphic objects is finally considered in the concluding section.

## 2. LANGUAGE SPECIFICATION

The main idea behind the project is to design a relevant language that provides easy to use tools to compose and to manipulate symbolic scores. Indeed, as all the operators have already been defined in a previous work [16], the point is to imagine a handy way to use them from INScore but above all, to benefit of the dynamic aspects of the INScore environment.

### 2.1 The operators

All the operators have a common interface: regardless their actual definition, they always take two scores as input to produce a score as output. The scores are expressed using the Guido Music Notation format [GMN][17]. A few low-level score manipulation operations are defined (which ap-

operation	arguments	description
seq	$s1\ s2$	puts the scores $s1$ and $s2$ in sequence
par	$s1\ s2$	puts the scores $s1$ and $s2$ in parallel
rpar	$s1\ s2$	puts the scores $s1$ and $s2$ in parallel, right aligned
top	$s1\ s2$	takes the $n$ first voices of $s1$ , where $n$ is the number of voices of $s2$
bottom	$s1\ s2$	cuts the $n$ first voices of $s1$ , where $n$ is the number of voices of $s2$
head	$s1\ s2$	takes the head of $s1$ on $s2$ duration
evhead	$s1\ s2$	takes the $n$ first events of $s1$ , where $n$ is $s2$ events count
tail	$s1\ s2$	cuts the head of $s1$ on $s2$ duration
evtail	$s1\ s2$	cuts the $n$ first events of $s1$ , where $n$ is $s2$ events count
transpose	$s1\ s2$	transposes $s1$ so its first note of its first voice match $s2$ one
duration	$s1\ s2$	stretches $s1$ to the duration of $s2$ (note that this operation may produce durations that are not displayable)
pitch	$s1\ s2$	applies the pitches of $s1$ to $s2$ in a loop
rhythm	$s1\ s2$	applies the rhythm of $s1$ to $s2$ in a loop

**Table 1.** INScore operators

ply perfectly to INScore language’s philosophy) with a deterministic behaviour (none of the operators implement random operations). Basically, these operations apply to the time domain (putting scores in sequence, in parallel, cutting parts of a score, time stretching), or to the score structure (extracting voices). A few additional operations are provided: transposition and application of a score’s rhythm or pitch to another score. The small set of operators is not a real limitation, as the uniformity between their inputs and output make them easy to combine into pipeline designs, creating more high-level operations. The selected basic operators are not intended to cover the composition process (a real programming language like Open Music [18] would be required) but to provide tools for dynamic symbolic score computation, especially in the context of music performance.

See Table 1 for the definition of all operators. Note that there is no constraint on the input scores. For the `par` and `rpar` operations, the shortest score (if any) is suffixed or prefixed with the necessary duration to obtain the same length. These extensions appear as empty staves, which is easily expressed using the GMN language.

## 2.2 Designing a creative language

In the context of software used for artistic creation like INScore, designing a language is not trivial. Like any other creative tools, the *score expressions* language shall inevitably frame the creation process through which the artist must go. To that extent, conceiving a language is actually designing a creative ”work-flow” that the users shall then adopt.

The continuity between inputs and outputs through Guido operators allows to compose a music by successively transform and aggregate scores fragments. This process (applying transformations on various materials and combining them into a whole creation) is similar to electro-acoustic creative processes where, after choosing sound material, the composer applies effects... and mixes them until this raw musical materials become unrecognisable.

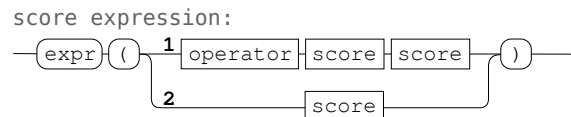
Adapting this approach to the symbolic music notation

would not only make the language easy to learn for composer but could offer great tools for composition: carving and assembling score samples using structural operators, placing the musical structure in the center of the creative process. In some ways, the art wouldn’t emerge from the quality of the raw score fragments but from the process that transforms, shapes, and links them together.

It’s with this perspective and emphasis of the structure that the *score expressions* syntax has been defined. In particular, these expressions should make use of various heterogeneous materials including *score expressions* or existing score objects.

## 2.3 Score expressions syntax

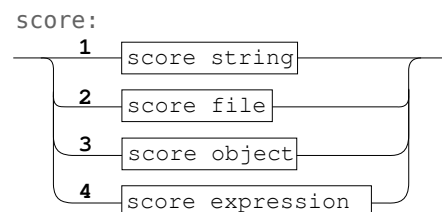
*Score expressions* can be defined using two syntaxes:



1. The classic syntax reflects the way Guido operators actually work: two scores are combined into one, according to the operator.
2. The alternate syntax defines an expression using a single score, which can be useful to duplicates objects e.g. to provide different views (see section 6.2).

Note that the leading `expr` token is present to disambiguate parenthesis that are already used in INScore scripts with messages lists.

Both of the syntaxes make use of `score` arguments. *Score expressions* are quite permissive regarding to their type:



1. `score string`: are GMN or MusicXML strings.
2. `score file`: refers to a score file that should contain GMN or MusicXML data. File path complies to INScore file handling and could indicate an absolute, a relative path or a URL.
3. `score object`: refers to an existing INScore object using a relative or absolute OSC address. The object must be a guido, musicxml or piano-roll object, as well as guido and piano-roll streams.
4. `score expression`: `score expressions` can be used as arguments of `score expressions` (in this case the `expr` token is optional).

Here is an example of a `score expression` that puts a score in parallel with 2 scores in sequence:

```
expr( par score.gmn (seq "[c]" score) )
```

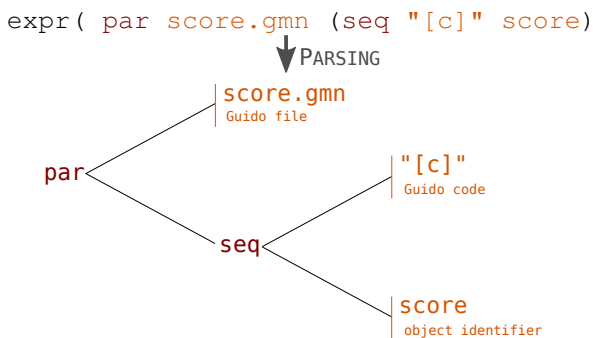
Note that some operations could take more than 2 scores as arguments. For example, the sequence (`seq`) or parallel (`par`) operations could apply to arbitrary lists of arguments and higher-order operations could be defined, similarly to the functional programming *fold* (or *reduce*) high-order function [19]. The current syntax doesn't support *folding* but this may be considered in the future. For example, that would allow to write `(seq a b c)` instead of `(seq a (seq b c))`.

### 3. EVALUATION SPECIFICATION

The `score expressions` language is first transformed into an internal memory representation. In a second step, this representation is evaluated to produce Guido Music Notation [GMN] [17] strings as output, that are finally passed to the INScore object as specific data.

#### 3.1 Internal representation of `score expressions`

When encountering an `score expressions`, the INScore parser creates a tree representation of it: arguments are stored as leaves and operators as nodes (Figure 1). This tree form allows to easily store, manipulate, assemble and evaluate `score expressions`.



**Figure 1.** Parsing `score expressions` into tree form.

The tree representation is strictly matching the expression string. Type specification of arguments is the only difference, whereas types are implicit in `score expressions`, arguments are explicitly stored as GMN code or file or identifier... in the tree form.

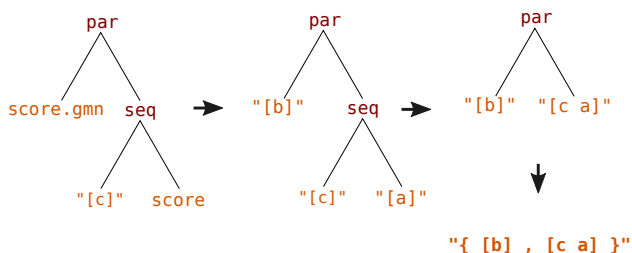
Once the internal representation has been constructed by the parser, it is stored with the newly defined object, ready for evaluation.

#### 3.2 `Score expressions` evaluation process

The evaluation process goes through every nodes of the expression tree using a depth first post-order traversal, reducing all of them into GMN code. A node evaluation is type dependent (Figure 2).

Evaluation of:

- a GMN file gives its content,
- a GMN string returns the string,
- a MusicXML file returns its content converted to GMN code,
- a MusicXML string returns the string converted to GMN code,
- an object identifier gives its GMN code,
- an operator node returns the application of the operator to the GMN code given as parameters.



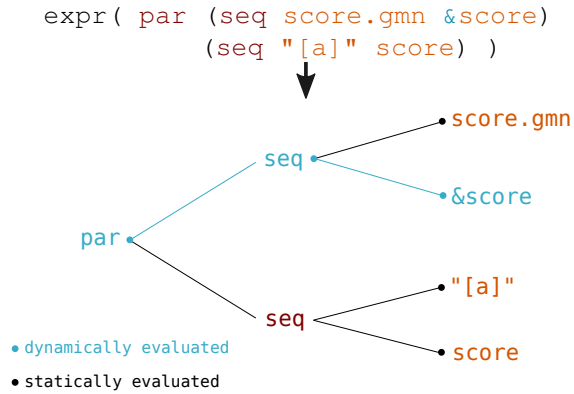
**Figure 2.** Simple evaluation of an expression tree, where `score` is defined as `[a]` and `score.gmn` contains `[b]`.

This evaluation scheme avoid recursion issues (e.g., a score that modifies itself using an expression based on its own content) since the caller object is modified only at the end of the evaluation process. All arguments are referentially transparent by default: each argument is evaluated once and its value is then considered constant.

#### 3.3 Dynamic evaluation of `score expressions`

Referential transparency (i.e., static evaluation) can be a huge limitation. For example, working with guido stream, one could want to maintain the result of a `score expression` up to date to the stream's actual state. Thus *variable arguments* have been introduced using a `&` prefix: a variable argument is always evaluated regardless of previous values (Figure 3). Only arguments subject to changes (`score` object or `score file`) can be declared *variable*.

A tree that contains a variable argument is a *dynamic tree*. When a variable argument is encountered on a tree branch,



**Figure 3.** Propagation of dynamic evaluation. `&score` is updated to the actual value of `score` when re-evaluating, while `score` keeps the value computed on the first evaluation. Thus, on re-evaluation the lower `seq` operation will not be computed again.

the dynamic tree attribute is propagated up to the tree root. During the evaluation process, only the dynamic parts of a tree is recomputed. For optimisation, INScore checks if a variable argument value has changed and recomputes the corresponding operator only when needed.

Using variable arguments, an expression tree with arbitrary variable parts can be described: that may be viewed as building a symbolic score with arbitrary aggregation of static and variable parts.

#### 4. SCORE EXPRESSIONS API IN INSCORE

In order to fully integrate score operators, the implementation relies on INScore existing features. As a result, *score expressions* support URLs as file arguments, interaction events and benefit of web features. Interaction events have been extended notably for the purpose of dynamic evaluation (see section 4.3).

##### 4.1 Declaring score expressions

Both `gmn` and `pianoroll` objects can be defined with *score expressions* using an extension of the `set` message. The evaluation of the expression is actually triggered by the target object when the `set` message is processed.

```

/ITL/scene/score set gmn expr(score.gmn);
/ITL/scene/pr set pianoroll expr(&score);

```

The previous example creates two objects: `score` is a symbolic representation of the GMN file `score.gmn`, and `pr` is a piano roll representation of `score` (here dynamically evaluated due to the `&` prefix).

##### 4.2 Score expressions specific messages

Objects that are based on *score expressions* support additional messages:

- `reeval`: triggers the re-evaluation of the expression tree taking account of the static and dynamic parts.

- `renew`: triggers the re-evaluation of the expression tree regardless of existing constant values.

All these messages are available through the `expr` message:

```

/ITL/scene/score expr reeval;
/ITL/scene/score expr renew;

```

Finally, the *score expression* of an object can be retrieved with the `get expr` message:

```

/ITL/scene/score get expr;

```

#### 4.3 Events typology extension

INScore interaction features are based on the association between an event and arbitrary set of OSC messages [10]. These messages are sent when the event occurs (e.g., a mouse down). INScore events typology has been extended with a `newData` event, that is triggered when the value of the target object changes, either due to a `set` or `reeval` message, or because data has been written in a stream object.

Using the `expr reeval` message in conjunction with the `newData` event, may trigger the automatic reevaluation of an expression when an object changes. With the example below, changing the content of `score` will fire the `newData` event and the associated `expr reeval` message is automatically sent to `copy` that updates its content accordingly.

```

/ITL/scene/score set gmn "[a]";
/ITL/scene/copy set gmn expr(&score);
/ITL/scene/score watch newData
  (/ITL/scene/copy expr reeval);

```

In order to catch infinite loop issues, `newData` event handling is postponed to the next INScore time slot. As a result, updating the whole scene after changing the value of an object can take several event loop (if one object is referring to another object, itself referring to another one...) and during this process the INScore's graphic scene could go through transitory states. However, if objects are defined with recursive references and are auto-updated using this mechanism, INScore will still be able to update the score (without freezing).

#### 5. COMPOSING SCORE EXPRESSIONS

While the expressions already presented allow to compose symbolic scores, it is also possible to compose *score expressions* which are stored in the referred objects using the prefix `~`. Indeed, whereas `score` and `&score` refer to the object's value, `~score` refers to the *score expression* used to define `score`. In practical, before the first evaluation, all arguments prefixed by `~` are replaced by a copy of the expression tree from the corresponding objects. It allows to easily make use of previously defined *score expressions* to create more complex ones.

Figure 4 illustrates how the expression tree is expanded with the example below.

```

/ITL/scene/score set gmn
  expr(seq "[a]" &sample);
/ITL/scene/score set gmn
  expr(seq (seq ~score "[b]") ~score);

```

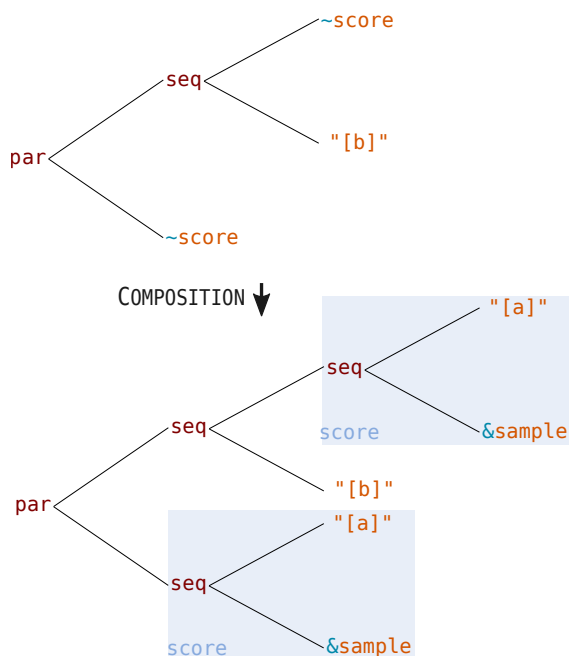


Figure 4. Composing *score expressions*

## 6. EXAMPLES

### 6.1 Canon structure

A simple but still well-known music structure is of course the canon. Creating such structure from a `score` is quite easy using *score expressions*.

With the example below, the first line creates a `score` object based on a GMN file. It is then transposed to a fifth and a second voice is added, delayed of a half note. Because transposing according to a specific interval is not a basic guido operator (the transposition interval is computed from the 2 scores arguments), one should combine `transpose` with `seq` and `evtail` to prepend the score with a note, transpose the whole score using this note and finally remove it.

```

/ITL/scene/score set gmnf score.gmn;

# Transposing score
/ITL/scene/canon set gmn
  expr(evtail
    (transpose (seq "[c]" score) "[g]"
      "[a]"
    )
  );

# Putting score in sequence with it
/ITL/scene/canon set gmn
  expr(seq score canon);

# Adding a second voice delayed
/ITL/scene/canon set gmn
  expr(par canon (seq "[-/2]" canon));

```

The result is a simple canon:

Original score :



Canon :



Figure 5. Canon result

### 6.2 Multiform synced scores

*Score expressions* is a great tools to duplicate and dynamically transform scores, keeping every copies synced to the original.

```

/ITL/scene/stream set gmnstream
  '[\meter<"4/4">]';

/ITL/saxo/score set gmn
  expr( evtail
    (transpose
      (seq
        "[e&1]"
        &/ITL/scene/stream )
      "[c2]" )
    "[a]"
  );

/ITL/audience/score set pianoroll
  expr( &/ITL/scene/stream);

/ITL/scene/stream watch newData
  (/ITL/*/score expr reeval);

```

The previous example creates 2 copies of the GMN stream object `stream`, one transposed for the saxophone and one displayed as a piano roll, intended as a visual support for the audience. Both are displayed in different scenes. The

last line ensure the update of the copies when `stream` is modified. The `/ITL/scene/stream` argument is re-evaluated due to the `&` prefix. The result is illustrated in figure 6.

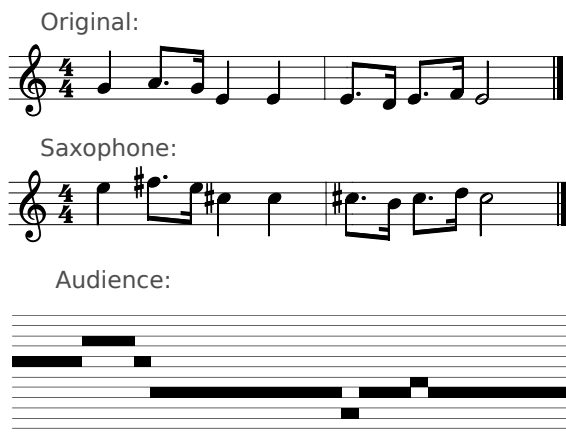


Figure 6. Multiform scores result

### 6.3 Mixing dynamic and static scores

This example illustrates how dynamic and static symbolic scores can be mixed and transformed in real-time. In a first step, we create a stream (named `stream`) intended to be written in real-time and a static score (named `static`).

```
/ITL/scene/stream set gmnstream
  '\meter<"4/4">';
/ITL/scene/static set gmn
  '\meter<"4/4"> g e f a f d c/2';
```

In a second step, the last two bars of the stream are extracted and store in a new object named `tail`. Since the 'tail' operation cuts the head of the score using the second argument, the duration of this argument is expressed as the tail of the stream using the desired duration (2 whole notes). Note that `tail` expression is using a reference to the stream in order to be updated each time data is written to the stream.

```
/ITL/scene/tail set gmn
  expr(tail &stream
    (tail &stream '[a*2]'));;
```

The final result is simply obtained using the 'par' and 'transpose' operations. It makes use of references to `tail` but the `static` object is embedded statically. Note that `tail` is used as an intermediate object intended to optimise the computation and to facilitate reading of the expression. It can be hidden from the overall score without affecting the result.

```
/ITL/scene/score set gmn
  expr(par &tail
    (transpose static &tail));;
```

Activation of the score dynamic computation makes use of the `newData` event watched by the `stream` object, that inform `tail` and `score` that their expressions need to be re-evaluated.

```
/ITL/scene/stream watch newData
  (/ITL/scene/part expr reeval,
  /ITL/scene/score expr reeval
  );
```

## 7. CONCLUSIONS

Combining a simple set of operators with the powerful features of INScore (like URL support, full OSC compatibility, interaction support...), *score expressions* fully integrate symbolic score composition into an interactive and augmented music score environment. They suggest a creative process based upon musical structures and scores aggregation by giving the possibility to compose various score materials including score objects. Above all, *score expressions* provide a handy way to manipulate scores regardless to their origin (files, URL, streams...) or their representation (traditional music notation or piano roll) and to design dynamic scores based on arbitrary score composition.

In future work, we're considering extending the *score expressions* to all the INScore objects. Such an approach - composing arbitrary graphic resources using a musical semantic - raises issues that are non-trivial to solve. Indeed, if the operations on the time domain could be applied to any object due to their common time dimension, transformations in the pitch domain or based on structured time (like rhythm) implies to extend the musical semantic of the graphics space.

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