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The steady-state Navier-Stokes equations in thin structures lead to some elliptic second order equation for the macroscopic pressure on a graph. At the nodes of the graph the pressure satisfies Kirchoff-type junction conditions. In the non-steady case the problem for the macroscopic pressure on the graph becomes nonlocal in time. In the paper we study the existence and uniqueness of a solution to such one-dimensional model on the graph for a pipe-wise network. We also prove the exponential decay of the solution with respect to the time variable in the case when the data decay exponentially with respect to time. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4891249]

I. INTRODUCTION

The Newtonian fluid flows in tube structures were considered in Refs. 3 and 5. Such flow domains are connected finite unions of thin finite cylinders (in the 2*D* case, respectively, thin rectangles).

Each tube structure may be schematically represented by its graph. Letting the thickness of tubes to zero we find out that tubes degenerate to segments (see Figs. 1 and 2).

It is known that the steady-state Navier-Stokes equations in thin structures lead to some elliptic second order equation for the macroscopic pressure on a graph (see Refs. 3–5 and 1) (for general theory of differential equations on graphs see Refs. 2 and 8). At the nodes of the graph the pressure satisfies some Kirchhoff-type junction conditions. In the non-steady case the problem for the macroscopic pressure on the graph becomes nonlocal in time. In the present paper we introduce the 1D-model on the graph for a non-stationary pipe-wise network. We study the existence and uniqueness of a solution to this problem and prove its exponential decay with respect to the time variable in the case when the data decay exponentially with respect to time.

II. GRAPHS

Let $O_1, O_2, \dots, O_N$ be N different points in $\mathbb{R}^n$, $n = 2, 3,$ and $e_1, e_2, \dots, e_M$ be M closed segments each connecting two of these points (i.e., each $e_j = \overline{O_{i_j} O_{k_j}}$, where $i_j, k_j \in \{1, \dots, N\}$, $i_j \neq k_j$). All points O_i are supposed to be the ends of some segments e_j. The segments e_j are called edges of the graph. A point O_i is called node if it is the common end of at least two edges and O_i is called vertex if it is the end of the only one edge. Any two edges e_j and e_i can intersect only at the common node. The set of vertices is supposed to be non-empty.

Denote $\mathcal{B} = \bigcup_{j=1}^{N} e_j$ the union of edges and assume that $\mathcal{B}$ is a connected set. The graph $\mathcal{G}$ is defined as the collection of nodes, vertices, and edges.

The union of all edges having the same end point in O_l is called the bundle $\mathcal{B}^{(l)}$.

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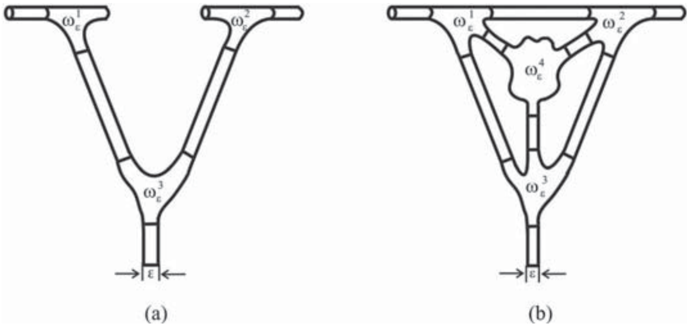

FIG. 1. Fluid flow domains: Tube structures.

Let e be some edge, $e = \overline{O_i O_j}$. Consider two Cartesian coordinate systems in $\mathbb{R}^n$. The first one has the origin in O_i and the axis $O_i x_n^{(e)}$ has the direction of the ray $[O_i O_j)$; the second one has the origin in O_j and the opposite direction, i.e., $O_i \tilde{x}_n^{(e)}$ is directed over the ray $[O_j O_i)$.

Further in various situations we will chose one or another coordinate system denoting the local variable in both cases as x^e and pointing out which end is taken as the origin of the coordinate system.

III. FORMULATION OF THE STEADY-STATE PROBLEM ON THE GRAPH

Let $H^1(\mathcal{B})$ be the set of all continuous on $\mathcal{B}$ functions such that for any edge e they belong to $H^1((0, |e|))$. Introducing the inner product

$$(p, q)_{H^1(\mathcal{B})} = \sum_{i=1}^{M} \int_0^{|e_i|} \Big(p^{(e_i)} q^{(e_i)} + \frac{\partial p^{(e_i)}}{\partial x_n^{(e_i)}} \frac{\partial q^{(e_i)}}{\partial x_n^{(e)}} \Big) d x_n^{(e_i)}$$

it can be easily checked that $H^1(\mathcal{B})$ is a Hilbert space.

Consider the following steady-state problem set on the graph $\mathcal{B}$. Given real constants Ψ_l, $l = 1, \ldots, N$, positive constants κ_{e_i} and functions $F^{(e_i)} \in L_2(\mathcal{B})$, $i = 1, \ldots, M$, find a function $p \in H^1(\mathcal{B})$ such that equations

$$\begin{aligned} -\frac{\partial}{\partial x_n^{(e)}} \Big(\kappa_e \frac{\partial p}{\partial x_n^{(e)}} (x_n^{(e)}) \Big) &= f^{(e)}(x_n^{(e)}),\ x_n^{(e)} \in (0, |e|), \\ -\sum_{e: O_l \in e} \Big(\kappa_e \frac{\partial p}{\partial x_n^{(e)}} \Big)(0) &= \Psi_l,\ l = 1, \ldots, N_1, \\ -\Big(\kappa_e \frac{\partial p^{(e)}}{\partial x_n^{(e)}} \Big)(0) &= \Psi_l,\ l = N_1 + 1, \ldots, N, \end{aligned} \tag{3.1}$$

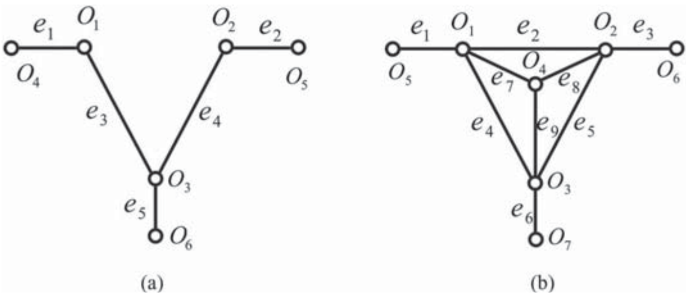

FIG. 2. Graphs of tube structures.

hold for all edges $e = e_i$, $i = 1, \ldots, M$, i.e., the first equation holds on every edge e of the graph, the second equation holds at every node and the sum is taken over all edges of the bundle $\mathcal{B}^{(l)}$, the third equation holds at every vertex O_l. In conditions $(3.1)_2$ and $(3.1)_3$ the local coordinate system has the origin O_l.

This problem describes the one-dimensional steady state flow in a pipe-wise network. Here p stands for the macroscopic pressure and the right-hand sides describe given sources distributed in the edges or concentrated at the nodes and vertices of the graph. The pressure is supposed to be continuous on the graph $\mathcal{B}$. Alternatively, one can consider the condition of prescribed known jumps of the pressure at the nodes instead of the continuity condition. Evidently, this new problem can be reduced to the previous one by a change of the unknown function (and respectively, of the right-hand sides).

Problem (3.1) admits a variational formulation. By a weak solution of problem (3.1) we call a function $p \in H^1(\mathcal{B})$ satisfying for any test function $q \in H^1(\mathcal{B})$ the integral identity

$$\sum_{i=1}^{M} \int_0^{|e_i|} \kappa_{e_i} \frac{\partial p^{(e_i)}}{\partial x_n^{(e_i)}} \frac{\partial q^{(e_i)}}{\partial x_n^{(e_i)}} dx_n^{(e_i)} = \sum_{i=1}^{M} \int_0^{|e_i|} f^{(e_i)} q^{(e_i)} dx_n^{(e_i)} + \sum_{l=1}^{N} \Psi_l q(O_l), \tag{3.2}$$

where $q^{(e_i)} = q|_{e_i}$.

Consider now a subspace $H^1_{0,N}(\mathcal{B})$ of $H^1(\mathcal{B})$ consisting of functions vanishing in one vertex, say in O_N.[9] By standard arguments one can prove the Poincaré-Friedrichs inequality for functions $u \in H^1_{0,N}(\mathcal{B})$.

Lemma 3.1. Let $u \in H^1_{0,N}(\mathcal{B})$. The following estimate

$$\sum_{i=1}^{M} \int_0^{|e_i|} \left|u^{(e_i)}\right|^2 dx_n^{(e_i)} \le C_{PF} \sum_{i=1}^{M} \int_0^{|e_i|} \left|\frac{\partial u^{(e_i)}}{\partial x_n^{(e_i)}}\right|^2 dx_n^{(e_i)} \tag{3.3}$$

holds. Here $C_{PF} = \dfrac{L^2}{2}$, $L = \sum_{i=1}^{M} |e_i|$.

Moreover,

$$\sup_{x \in \mathcal{B}} |u(x)| \le \sum_{i=1}^{M} \int_0^{|e_i|} \left|\frac{\partial u^{(e_i)}}{\partial x_n^{(e_i)}}\right| dx_n^{(e_i)} \le \Big(L \sum_{i=1}^{M} \int_0^{|e_i|} \left|\frac{\partial u^{(e_i)}}{\partial x_n^{(e_i)}}\right|^2 dx_n^{(e_i)}\Big)^{1/2}. \tag{3.4}$$

One can look for the solution of problem (3.2) which additionally satisfies the condition $p(O_N) = 0$, i.e., $p \in H^1_{0,N}(\mathcal{B})$. In this case we have to take the test functions $q \in H^1_{0,N}(\mathcal{B})$ and the last sum in integral identity (3.2) is taken from 1 to $N - 1$.

Theorem 3.1. *Let $f \in L_2(\mathcal{B})$. Problem (3.2) has a unique solution $p \in H^1_{0,N}(\mathcal{B})$ if and only if the compatibility condition*

$$\sum_{i=1}^{M} \int_0^{|e_i|} f^{(e_i)} dx_n^{(e_i)} + \sum_{l=1}^{N} \Psi_l = 0 \tag{3.5}$$

holds. For the solution p the following inequality

$$\|p\|_{H^1(\mathcal{B})} \le \frac{1 + C_{PF}}{\kappa_{min}} \Big(\|f\|_{L_2(\mathcal{B})} + \sqrt{LN}\Big(\sum_{l=1}^{N-1} |\Psi_l|^2\Big)^{1/2}\Big) \tag{3.6}$$

holds. Here $\kappa_{min} = \min_{i \in \{1,\ldots,M\}} \kappa_{e_i}$, $\|f\|^2_{L_2(\mathcal{B})} = \sum_{i=1}^{M} \int_0^{|e_i|} |f^{(e_i)}|^2 dx_n^{(e_i)}$.

The proof of this theorem is a standard application of Riesz theorem on the representation of linear bounded functionals in Hilbert spaces.

Corollary 3.1. Let the condition (3.5) be valid. Then problem (3.2) admits a unique (up to an additive constant) solution $p \in H^1_2(\mathcal{B})$. If p is normalized so that $p \in H^1_{0N}(\mathcal{B})$, then the following *estimate*

$$\|p\|^2_{H^1(\mathcal{B})} \leq C\Big(\|f\|^2_{L_2(\mathcal{B})} + \sum_{l=1}^{N} |\Psi_l|^2\Big) \tag{3.7}$$

holds. Here

$$C = \frac{L^2/2+1}{\kappa_{min}} \max\{1, \sqrt{LN}\}, \quad \kappa_{min} = \min_{i\in\{1,\dots,M\}} \kappa_{e_i}. \tag{3.8}$$

Proof. By the Poincaré-Friedrichs inequality (3.3) we have

$$\kappa_{min}\|p\|^2_{H^1(\mathcal{B})} \leq \Big(\frac{L^2}{2}+1\Big)\sum_{i=1}^{M} \kappa_{e_1} \int_0^{|e_i|} \Big|\frac{\partial p^{(e_i)}}{\partial x_n^{(e_i)}}\Big|^2 dx_n^{(e_i)}$$

$$= \Big(\frac{L^2}{2}+1\Big)\Big(\sum_{i=1}^{M}\int_0^{|e_i|} f^{(e_i)}(x_n^{(e_i)})p^{(e_i)}(x_n^{(e_i)})dx_n^{(e_i)} + \sum_{l=1}^{N}\Psi_l p(O_l)\Big)$$

$$\leq \Big(\frac{L^2}{2}+1\Big)\Big(\|f\|_{L_2(\mathcal{B})}\|p\|_{H^1(\mathcal{B})} + \sqrt{N}\Big(\sum_{l=1}^{N}\Psi_l^2\Big)^{1/2}\max_{x\in\mathcal{B}}|\widehat{p}(x)|\Big)$$

$$\leq \Big(\frac{L^2}{2}+1\Big)\Big(\|f\|_{L_2(\mathcal{B})} + \sqrt{LN}\Big(\sum_{l=1}^{N}\Psi_l^2\Big)^{1/2}\Big)\|p\|_{H^1(\mathcal{B})}.$$

So, we have proved inequality (3.7) with the constant C defined by (3.8). □

IV. OPERATOR RELATING THE PRESSURE DROP AND THE FLUX IN AN INFINITE TUBE; THE NON-STEADY CASE

With every edge e_j we associate a bounded domain $\sigma^j \subset \mathbb{R}^{n-1}$ having Lipschitz boundary $\partial\sigma^j$, $j = 1, \dots, M$, and the operator $L^{(e)}$ relating the pressure drop $\mathcal{S}(\tau)$ and the flux (flow rate) $\mathcal{H}(\tau)$ in an infinite cylindric pipe with the cross-section $\sigma^{(e)}$. Namely, consider the following initial boundary value problem for the heat equation: for given $\mathcal{S} \in L_2(0, +\infty)$ find $\mathcal{V} \in L_2(0, +\infty; H_0^1(\sigma^{(e)}))$ with $\frac{\partial\mathcal{V}}{\partial\tau} \in L_2(0, +\infty; L_2(\sigma^{(e)}))$ such that

$$\begin{gathered}\frac{\partial\mathcal{V}}{\partial\tau}(y^{(e)\prime}, \tau) - \nu\Delta'_{y^{(e)\prime}}\mathcal{V}(y^{(e)\prime}, \tau) = \mathcal{S}(\tau),\ y^{(e)\prime} \in \sigma^{(e)}, \tau > 0,\\ \mathcal{V}(y^{(e)\prime}, \tau)|_{\partial\sigma^{(e)}} = 0, \quad \tau > 0,\\ \mathcal{V}(y^{(e)\prime}, 0) = 0,\ y^{(e)\prime} \in \sigma^{(e)},\end{gathered} \tag{4.1}$$

and denote

$$L^{(e)}\mathcal{S}(\tau) = \int_{\sigma^{(e)}} \mathcal{V}(y^{(e)\prime}, \tau)dy^{(e)\prime}.$$

Evidently, $L^{(e)}$ is bounded linear operator acting from $L_2(0, +\infty)$ to $H_0^1(0, +\infty)$. It was proved in Refs. 6 and 7 that the operator $L^{(e)}$ admits the bounded inverse $(L^{(e)})^{-1} : H_0^1(0, +\infty) \mapsto L_2(0, +\infty)$, i.e., there holds the following

Theorem 4.1. *Let $\mathcal{H} \in H_0^1(0, +\infty)$ be given. There exists a unique pair $(\mathcal{V}, \mathcal{S})$ satisfying (in the sense of distributions) Eq. (4.1) and the flux condition*

$$\int_{\sigma^{(e)}} \mathcal{V}(y^{(e)\prime}, \tau)dy^{(e)\prime} = \mathcal{H}(\tau).$$

Moreover, $\mathcal{V} \in L_2(0, +\infty; H_0^1(\sigma^{(e)}))$, $\dfrac{\partial \mathcal{V}}{\partial \tau} \in L_2(0, +\infty; L_2(\sigma^{(e)}))$, $\mathcal{S} \in L_2(0, +\infty)$, *and the following estimate*

$$\|\mathcal{V}\|_{L_2(0,\hat{T};H_0^1(\sigma^{(e)}))} + \left\|\frac{\partial \mathcal{V}}{\partial \tau}\right\|_{L_2(0,\hat{T};L_2(\sigma^{(e)}))} + \|\mathcal{S}\|_{L_2(0,\hat{T})} \le c\|\mathcal{H}\|_{H^1(0,\hat{T})} \tag{4.2}$$

holds for any $\hat{T} > 0$ *with the constant c independent of* $\hat{T}$.

Thus, the following estimate

$$C_L^{-1}\|Q\|_{L_2(0,\hat{T})} \le \|L^{(e)}Q\|_{H^1(0,\hat{T})} \le C_L\|Q\|_{L_2(0,\hat{T})} \quad \forall Q \in L_2(0, \hat{T}), \tag{4.3}$$

holds. In (4.3) $C_L > 0$ is a constant independent of $\hat{T}$.

V. FORMULATION OF THE NON-STEADY PROBLEM ON THE GRAPH

Consider the following non-steady problem set on the graph $\mathcal{B}$. Let $\hat{T} > 0$. Given functions $\Psi_l \in H_0^1(0, +\infty)$, $l = 1, \ldots, N$, and functions $F^{(e_i)} \in H_0^1(0, +\infty; L^2(\mathcal{B}))$, $i = 1, \ldots, M$, find a function $p \in L_2(0, \hat{T}; H^1(\mathcal{B}))$ such that equations

$$\begin{aligned} -\frac{\partial}{\partial x_n^{(e)}}\Big(L^{(e)}\frac{\partial p}{\partial x_n^{(e)}}(x_n^{(e)}, \tau)\Big) &= f^{(e)}(x_n^{(e)}, \tau),\ x_n^{(e)} \in (0, |e|),\\ -\sum_{e:O_l\in e}\Big(L^{(e)}\frac{\partial p}{\partial x_n^{(e)}}\Big)(0, \tau) &= \Psi_l(\tau),\ l = 1, \ldots, N_1,\\ -\Big(L^{(e)}\frac{\partial \hat{p}^{(e)}}{\partial x_n^{(e)}}\Big)(0, \tau) &= \Psi_l(\tau),\ l = N_1 + 1, \ldots, N, \end{aligned} \tag{5.1}$$

hold for all $t \in (0, \hat{T})$ and for all edges $e = e_i$, $i = 1, \ldots, M$. Here the right-hand sides $f^{(e)}(x_n^{(e)}, \tau)$, $\Psi_l(\tau)$, $l = 1, \ldots, N$, depend on the time variable τ. Note that, applying the operator $L^{(e)}$ to $\dfrac{\partial p}{\partial x_n^{(e)}}(x_n^{(e)}, \tau)$ we treat the variable $x_n^{(e)}$ as a parameter.

This problem describes the one-dimensional flow in a pipe-wise network. Here p stands for the macroscopic pressure and the right-hand sides describe given non-steady sources distributed in the edges or concentrated at the nodes and vertices of the graph. The pressure is supposed to be continuous on the graph $\mathcal{B}$. Alternatively, as for the steady state problem, one can consider the condition of prescribed known jumps of the pressure at the nodes instead of the continuity condition. Evidently, this new problem can be reduced to the previous one by a change of the unknown function (and respectively, of the right-hand sides).

VI. EXISTENCE AND UNIQUENESS OF A SOLUTION TO THE PROBLEM ON THE GRAPH

Differentiating relations (5.1) with respect to time τ, we get an equivalent problem having the following variational formulation

$$a_{\hat{T}}(p, \psi) = b_{\hat{T}}(\psi)\forall\psi \in L_2(0, \hat{T}; H^1(\mathcal{B})), \tag{6.1}$$

where

$$a_{\hat{T}}(p, \psi) = \sum_{i=1}^{M}\int_0^{\hat{T}}\int_0^{|e_i|}\frac{\partial(L^{(e_i)}p)_\tau}{\partial x_n^{(e_i)}}\frac{\partial \psi}{\partial x_n^{(e_i)}}dx_n^{(e_i)}d\tau$$

and

$$b_{\hat{T}}(\psi) = \sum_{i=1}^{M}\int_0^{\hat{T}}\int_0^{|e_i|}(f^{(e_i)})_\tau\psi dx_n^{(e_i)}d\tau + \sum_{l=1}^{N}\int_0^{\hat{T}}(\Psi_l)_\tau(\tau)\psi(O_l, \tau)d\tau.$$

Here $g_\tau = \dfrac{\partial g}{\partial \tau}$.

Theorem 6.1. *Let $\Psi_l \in H_0^1(0, +\infty)$, $l = 1, \ldots, N$, and $f^{(e_i)} \in H_0^1(0, +\infty; H^1(\mathcal{B}))$, $i = 1, \ldots, M$. Problem (5.1) admits a unique solution $p \in L_2(0, \hat{T}; H^1(\mathcal{B}))$ vanishing at the vertex O_N if and only if (3.5) holds for almost all $\tau \in (0, \hat{T})$.*[10]

Proof. The main idea of the proof is the Lax-Milgram lemma argument and the application of inequality (4.3) on any edge e. Indeed, consider the subspace $H^1_{0,N}(\mathcal{B})$ of functions of $H^1(\mathcal{B})$ vanishing at the vertex O_N. We check directly using (4.3) that $a_{\hat{T}}$ is a bilinear form continuous with respect to the norm of $L_2(0, \hat{T}; H^1_{0,N}(\mathcal{B}))$, and that $b_{\hat{T}}$ is bounded linear functional. The coerciveness of $a_{\hat{T}}$ is a consequence of the following estimates, which hold for any edge e and the solution $(\mathcal{V}, \mathcal{S})$ to problem (4.1):

$$\int_0^{\hat{T}} (L^{(e)}\mathcal{S})_\tau \mathcal{S} d\tau = \int_0^{\hat{T}} \int_{\sigma^{(e)}} (\mathcal{V}(y^{(e)\prime}, \tau))_\tau \mathcal{S}(\tau)\, dy^{(e)\prime} d\tau$$

$$= \int_0^{\hat{T}} \int_{\sigma^{(e)}} \left\{ |(\mathcal{V}(y^{(e)\prime}, \tau))_\tau|^2 + \frac{\nu}{2} \frac{\partial}{\partial \tau} \left|\nabla_{y^{(e)\prime}} \mathcal{V}(y^{(e)\prime}, \tau)\right|^2 \right\} dy^{(e)\prime} d\tau$$

$$\geq \int_0^{\hat{T}} \int_{\sigma^{(e)}} |(\mathcal{V}(y^{(e)\prime}, \tau))_\tau|^2 dy^{(e)\prime} d\tau$$

$$\geq \frac{1}{|\sigma^{(e)}|} \int_0^{\hat{T}} \Big(\int_{\sigma^{(e)}} (\mathcal{V}(y^{(e)\prime}, \tau))_\tau dy^{(e)\prime} \Big)^2 d\tau = \frac{1}{|\sigma^{(e)}|} \int_0^{\hat{T}} |(L^{(e)}\mathcal{S})_\tau|^2 d\tau$$

$$\geq \frac{1}{2|\sigma^{(e)}|} \int_0^{\hat{T}} |(L^{(e)}\mathcal{S})_\tau|^2 d\tau + \frac{1}{\hat{T}^2 |\sigma^{(e)}|} \int_0^{\hat{T}} |L^{(e)}\mathcal{S}|^2 d\tau$$

$$\geq \min\Big(\frac{1}{2}, \frac{1}{\hat{T}^2}\Big) \frac{C_L^{-2}}{|\sigma^{(e)}|} \int_0^{\hat{T}} |\mathcal{S}(\tau)|^2 d\tau.$$

Thus,

$$a_{\hat{T}}(p, p) \geq \min\Big(\frac{1}{2}, \frac{1}{\hat{T}^2}\Big) \frac{C_L^{-2}}{\min_{i=1,\ldots,M} |\sigma^{(e_i)}|} (p, p)_{\hat{T}}, \tag{6.2}$$

where

$$(p, \psi)_{\hat{T}} = \sum_{i=1}^{M} \int_0^{\hat{T}} \int_0^{|e_i|} \frac{\partial p}{\partial x_n^{(e_i)}} \frac{\partial \psi}{\partial x_n^{(e_i)}} dx_n^{(e_i)} d\tau.$$

Thus, all conditions of the Lax-Milgram lemma are verified in the Hilbert space $L_2(0, \hat{T}; H^1_{0N}(\mathcal{B}))$. So, we have proved the existence and uniqueness of the solution to problem (6.1) projected on the subspace $L_2(0, \hat{T}; H^1_{0,N}(\mathcal{B}))$. Finally, as usually, one can check directly that if condition (3.5) is valid, then (6.1) still holds for any test function from the larger space $L_2(0, \hat{T}; H^1(\mathcal{B}))$. Indeed, let ψ belongs to $L_2(0, \hat{T}; H^1(\mathcal{B}))$; decompose it in a sum $\psi(x_n^{(e_i)}, \tau) = \tilde{\psi}(x_n^{(e_i)}, \tau) + \psi(O_N, \tau)$, where $\tilde{\psi} \in L_2(0, \hat{T}; H^1_{0,N}(\mathcal{B}))$. Taking into account that $\tilde{\psi}$ satisfies (6.1), that the $x_n^{(e_i)}$-derivative of $\psi(O_N, \tau)$ is equal to zero (so that $a_{\hat{T}}(p, \psi(O_N, \tau)) = 0$), and that, due to (3.5), $b_{\hat{T}}(\psi(O_N, \tau)) = 0$, we get (6.1) for all $\psi \in L_2(0, \hat{T}; H^1_{0,N}(\mathcal{B}))$. The necessity of condition (3.5) follows from the identity (6.1) written for $\psi = \psi(\tau)$, an arbitrary function of $L_2(0, \hat{T})$. The theorem is proved. □

Remark 6.1. Estimate (6.2) yields the inequality

$$\sqrt{(p, p)_{\hat{T}}} \leq C_* \hat{T}^2 \quad \forall \hat{T} > 1, \tag{6.3}$$

where the constant $C_* \geq 0$ is independent of $\hat{T}$.

VII. EXPONENTIAL DECAY IN TIME OF THE SOLUTION

Assume that the data of problem (5.1) decay exponentially as $\tau \to \infty$. Denote $\mathcal{L}_{2,\beta}(0, +\infty)$ the space of functions $f \in L_2(0, +\infty)$ such that

$$\int_0^{+\infty} |f(\tau)|^2 \exp\{2\beta\tau\} d\tau < \infty,$$

$\mathcal{H}^1_\beta(0, +\infty)$ the space of functions $f \in H^1(0, +\infty)$ such that $f, f' \in \mathcal{L}_{2,\beta}(0, +\infty)$. Let $\mathcal{H}^1_{0,\beta}(0, +\infty)$ be the subspace of $\mathcal{H}^1_\beta(0, +\infty)$ consisting of functions vanishing at $\tau = 0$.

Theorem 7.1. *Let p be solution to problem (6.1) for all $\hat{T} > 0$ and let $\Psi_l \in \mathcal{H}^1_{0,\beta}(0, +\infty)$, $l = 1, \ldots, N$, $f^{(e_i)} \in \mathcal{H}^1_{0,\beta}(0, +\infty; L_2(e_i))$, $i = 1, \ldots, M$. Then $p \in \mathcal{L}_{2,\beta_1}(0, +\infty; H^1(\mathcal{B}))$ with some positive β_1.*

The proof of this theorem is based on the uniform with respect to $\hat{T}$ bounds for the bilinear and linear forms

$$a^{(\gamma)}_{\hat{T},\beta}(p, \psi) = \int_0^{\hat{T}} \exp\{2\beta\tau\} a_{\gamma,\tau}(p, \psi) d\tau, \; b^{(\gamma)}_{\hat{T},\beta}(\psi) = \int_0^{\hat{T}} \exp\{2\beta\tau\} b_{\gamma,\tau}(\psi) d\tau,$$

with $\gamma = \beta$, where

$$a_{\gamma,\tau}(p, \psi) = \sum_{i=1}^{M} \int_0^{|e_i|} \frac{\partial\{(L^{(e_i)} p)_\tau + \gamma L^{(e_i)} p\}}{\partial x_n^{(e_i)}} \frac{\partial \psi}{\partial x_n^{(e_i)}} dx_n^{(e_i)},$$

$$b_{\gamma,\tau}(\psi) = \sum_{i=1}^{M} \int_0^{|e_i|} \{(f^{(e_i)})_\tau + \gamma f^{(e_i)}\} \psi dx_n^{(e_i)} + \sum_{l=1}^{N} \{(\Psi)_\tau(\tau) + \gamma \Psi(\tau)\} \psi(O_l, \tau).$$

In order to get these bounds, we shall use the following lemma:

Lemma 7.1. Let $H \in H^1(0, \hat{T})$. Then for any $\gamma > 0$ the following inequality

$$\int_0^{\hat{T}} H^2(\tau) \exp\{2\gamma\tau\} d\tau \leq \frac{1}{\gamma^2} \int_0^{\hat{T}} (H'(\tau))^2 \exp\{2\gamma\tau\} d\tau + \frac{1}{\gamma} H^2(\hat{T}) \exp\{2\gamma\hat{T}\}$$

holds.

Proof. Integrating by parts and using the Young inequality we get

$$\int_0^{\hat{T}} H^2(\tau) \exp\{2\gamma\tau\} d\tau =$$

$$= \frac{1}{2\gamma}(H^2(\hat{T})\exp\{2\gamma\hat{T}\} - H^2(0)) - 2\int_0^{\hat{T}} H(\tau)H'(\tau)\frac{\exp\{2\gamma\tau\}}{2\gamma}d\tau$$

$$\leq 2\int_0^{\hat{T}}\{\frac{\gamma}{2}H^2(\tau) + \frac{1}{2\gamma}(H'(\tau))^2\}\frac{\exp\{2\gamma\tau\}}{2\gamma}d\tau + \frac{1}{2\gamma}H^2(\hat{T})\exp\{2\gamma\hat{T}\}.$$

Hence,

$$\int_0^{\hat{T}} H^2(\tau)\exp\{2\gamma\tau\}d\tau \leq \frac{1}{2}\int_0^{\hat{T}} H^2(\tau)\exp\{2\gamma\tau\}d\tau +$$

$$+\frac{1}{2\gamma^2}\int_0^{\hat{T}}(H'(\tau))^2\exp\{2\gamma\tau\}d\tau + \frac{1}{2\gamma}H^2(\hat{T})\exp\{2\gamma\hat{T}\}$$

and the lemma is proved. □

From Lemma 7.1 immediately follows the Poincaré–Friedrichs inequality for weighted spaces.

Corollary 7.1. (Poincaré–Friedrichs inequality for weighted spaces). Let $H \in H^1(0, +\infty)$. Then for any $\gamma > 0$ the following inequality

$$\int_0^{+\infty} H^2(\tau)\exp\{2\gamma\tau\}d\tau \leq \frac{1}{\gamma^2}\int_0^{+\infty}(H'(\tau))^2\exp\{2\gamma\tau\}d\tau$$

holds (this inequality makes sense if the right-hand side is finite).

Lemma 7.2. Let $f \in L_{2,loc}(0, +\infty)$ and let there exist a constant $C > 0$ and a real number α such that for any $\hat{T} > 1$, $\int_0^{\hat{T}} f^2(\tau)d\tau \leq C\hat{T}^\alpha$. Then for any $\gamma > 0$,

$$\int_0^{+\infty} f^2(\tau)\exp\{-\gamma\tau\}d\tau < +\infty.$$

Proof. For any integer $N > 0$,

$$\int_0^N f^2(\tau)\exp\{-\gamma\tau\}d\tau \leq \sum_{j=0}^{N-1}\int_j^{j+1} f^2(\tau)d\tau\exp\{-\gamma j\} \leq C\sum_{j=0}^{N-1}(j+1)^\alpha\exp\{-\gamma j\}.$$

Evidently, this sum is uniformly bounded with respect to N. □

Proof of Theorem 7.1. For any $\mathcal{S} \in L_2(0, \hat{T})$ and for every $e = e_i$ we have

$$I^{(e)}(\mathcal{S}) = \int_0^{\hat{T}}\Big((L^{(e)}\mathcal{S})_\tau(\tau)\mathcal{S}(\tau) + \beta(L^{(e)}\mathcal{S})(\tau)\mathcal{S}(\tau)\Big)\exp\{2\beta\tau\}d\tau$$

$$= \int_0^{\hat{T}}\int_{\sigma^{(e)}}\Big\{(\mathcal{V}(y^{(e)\prime}, \tau))_\tau + \beta\mathcal{V}(y^{(e)\prime}, \tau)\Big\}dy^{(e)\prime}\mathcal{S}(\tau)\exp\{2\beta\tau\}d\tau$$

$$= \int_0^{\hat{T}} \int_{\sigma^{(e)}} \Big\{ |(\mathcal{V}(y^{(e)\prime}, \tau))_\tau|^2 + \frac{\nu}{2} \frac{\partial}{\partial \tau} (\nabla_{y^{(e)\prime}} \mathcal{V}(y^{(e)\prime}, \tau))^2$$

$$+\beta\nu |\nabla_{y^{(e)\prime}} \mathcal{V}(y^{(e)\prime}, \tau)|^2 + \frac{\beta}{2} \frac{\partial}{\partial \tau} (\mathcal{V}(y^{(e)\prime}, \tau))^2 \Big\} dy^{(e)\prime} \exp\{2\beta\tau\} d\tau.$$

Integrating by parts with respect to the time variable in the second and the fourth terms, we get

$$I^{(e)}(\mathcal{S}) = \int_0^{\hat{T}} \int_{\sigma^{(e)}} \Big\{ |(\mathcal{V}(y^{(e)\prime}, \tau))_\tau|^2 - \nu\beta |\nabla_{y^{(e)\prime}} \mathcal{V}(y^{(e)\prime}, \tau)|^2$$

$$+\beta\nu |\nabla_{y^{(e)\prime}} \mathcal{V}(y^{(e)\prime}, \tau)|^2 - \beta^2 |\mathcal{V}(y^{(e)\prime}, \tau)|^2 \Big\} dy^{(e)\prime} \exp\{2\beta\tau\} d\tau$$

$$+ \int_{\sigma^{(e)}} \frac{\nu}{2} |\nabla_{y^{(e)\prime}} \mathcal{V}(y^{(e)\prime}, \hat{T})|^2 dy^{(e)\prime} \exp\{2\beta\hat{T}\} + \int_{\sigma^{(e)}} \frac{\beta}{2} |\mathcal{V}(y^{(e)\prime}, \hat{T})|^2 dy^{(e)\prime} \exp\{2\beta\hat{T}\}.$$

Simplifying the second and the third terms and applying Lemma 7.1 to the first term yield

$$I^{(e)}(\mathcal{S}) \geq \beta^2 \int_0^{\hat{T}} \int_{\sigma^{(e)}} |\mathcal{V}(y^{(e)\prime}, \tau)|^2 dy^{(e)\prime} \exp\{2\beta\tau\} d\tau$$

$$-\beta \int_{\sigma^{(e)}} |\mathcal{V}(y^{(e)\prime}, \hat{T})|^2 dy^{(e)\prime} \exp\{2\beta\hat{T}\} + \frac{\nu}{2} \int_{\sigma^{(e)}} |\nabla_{y^{(e)\prime}} \mathcal{V}(y^{(e)\prime}, \hat{T})|^2 dy^{(e)\prime} \exp\{2\beta\hat{T}\}$$

$$+\frac{\beta}{2} \int_{\sigma^{(e)}} |\mathcal{V}(y^{(e)\prime}, \hat{T})|^2 dy^{(e)\prime} \exp\{2\beta\hat{T}\} - \beta^2 \int_0^{\hat{T}} \int_{\sigma^{(e)}} |\mathcal{V}(y^{(e)\prime}, \tau)|^2 dy^{(e)\prime} \exp\{2\beta\tau\} d\tau$$

$$= \frac{\nu}{2} \int_{\sigma^{(e)}} |\nabla_{y^{(e)\prime}} \mathcal{V}(y^{(e)\prime}, \hat{T})|^2 dy^{(e)\prime} \exp\{2\beta\hat{T}\} - \frac{\beta}{2} \int_{\sigma^{(e)}} |\mathcal{V}(y^{(e)\prime}, \hat{T})|^2 dy^{(e)\prime} exp\{2\beta\hat{T}\}.$$

Applying now the Poincaré-Friedrichs inequality in the domain $\sigma^{(e)}$ with constant C_{PF} we get for $\beta < \dfrac{\nu}{2C_{PF}}$:

$$I^{(e)}(\mathcal{S}) \geq \frac{\beta}{2} \int_{\sigma^{(e)}} |\mathcal{V}(y^{(e)\prime}, \hat{T})|^2 dy^{(e)\prime} \exp\{2\beta\hat{T}\}$$

$$\geq \frac{\beta}{2} \frac{1}{|\sigma^{(e)}|} \Big(\int_{\sigma^{(e)}} \mathcal{V}(y^{(e)\prime}, \hat{T}) dy^{(e)\prime} \Big)^2 \exp\{2\beta\hat{T}\}$$

$$\geq \frac{\beta}{2} \frac{1}{|\sigma^{(e)}|} |L^{(e_i)} p^{(e_i)}(\hat{T})|^2 \exp\{2\beta\hat{T}\}.$$

So, for all $\hat{T} > 1$,

$$a^{(\beta)}_{\hat{T},\beta}(p, p) \geq \frac{\beta}{2} \frac{1}{\max\limits_e |\sigma^{(e)}|} \exp\{2\beta\hat{T}\} \sum_{i=1}^{M} \int_0^{|e_i|} \Big| \frac{\partial (L^{(e_i)} p)}{\partial x_n^{(e_i)}} \Big|^2 dx_n^{(e_i)}.$$

On the other hand,

$$a^{(\beta)}_{\hat{T},\beta}(p,p) = b^{(\beta)}_{\hat{T},\beta}(p)$$

and for any positive $\delta \leq \beta$,

$$b^{(\beta)}_{\hat{T},\beta}(\hat{p}) \leq \sqrt{\int_0^{\hat{T}} \exp\{2\delta\tau\} \sum_{i=1}^{M} \int_0^{|e_i|} \left((f^{(e_i)})_\tau(\tau) + \beta f^{(e_i)}(\tau)\right)^2 dx_n^{(e_i)} d\tau} \times$$

$$\times \sqrt{\int_0^{\hat{T}} \exp\{-2\delta\tau\} \sum_{i=1}^{M} \int_0^{|e_i|} \left|\frac{\partial p}{\partial x_n^{(e_i)}}\right|^2 dx_n^{(e_i)} d\tau} +$$

$$+ \sqrt{\int_0^{\hat{T}} \exp\{2\delta\tau\} \sum_{l=1}^{N} \left((\Psi_l(O_l,\tau))_\tau + \beta\Psi_l(O_l,\tau)\right)^2 d\tau} \times$$

$$\times \sqrt{\int_0^{\hat{T}} \exp\{-2\delta\tau\} \sum_{l=1}^{N} |p(O_l,\tau)|^2 d\tau}$$

is bounded uniformly with respect to $\hat{T}$ due to (6.3) and Lemma 7.2.

So, for any edge e_i, $i = 1, \ldots, M$,

$$\exp\{2\beta\hat{T}\} \sum_{i=1}^{M} \int_0^{|e_i|} \left|\frac{\partial(L^{(e_i)} p^{(e_i)})}{\partial x_n^{(e_i)}}\right|^2 dx_n^{(e_i)}$$

is bounded uniformly. This implies that for any edge e_i, $i = 1, \ldots, M$, the inclusion

$$\frac{\partial(L^{(e_i)} p^{(e_i)})}{\partial x_n^{(e_i)}} \in \mathcal{L}_{2,\beta_2}(0,+\infty; L_2(e_i)),$$

holds for every $\beta_2 < \beta$. Consequently, there exists a positive β_1 such that

$$\frac{\partial p^{(e_i)}}{\partial x_n^{(e_i)}} \in \mathcal{L}_{2,\beta_1}(0,+\infty; L_2(e_i)).$$

So, finally, by Lemma 3.1,

$$p \in \mathcal{L}_{2,\beta_1}(0,+\infty; H^1(\mathcal{B})).$$

Theorem 7.1 is proved. □

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[9] Notice that functions from $H^1_{0,N}(\mathcal{B})$ are not assumed to be zero in the remaining vertices O_l, $l = N_1 + 1, \ldots, N - 1$.
[10] Note that due to the hypothesis of the theorem, this condition is equivalent to its time derivative, and so, to the relation $b_{\hat{T}}(\phi) = 0 \ \ \forall \phi \in L_2(0, \hat{T})$ (or for all ϕ from any subspace which is dense in $L_2(0, \hat{T})$), i.e.,

$$\sum_{i=1}^{M} \int_0^{\hat{T}} \int_0^{|e_i|} (f^{(e_i)})_\tau (x_n^{(e_i)}, \tau) dx_n^{(e_i)} \phi(\tau) d\tau + \sum_{l=1}^{N} \int_0^{\hat{T}} (\Psi_l)_\tau (\tau) \phi(\tau) d\tau = 0.$$