Beam equation with saturating piezoelectric controls

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Abstract: This paper deals with a controlled beam equation for which the input is subject to magnitude saturation. The partial differential equation describes the dynamics of the deflection of the beam with respect to the rest position. The input is the voltage applied on an actuator located in a given interval of the space domain. Two kinds of control are considered: a static output feedback law and a dynamical output feedback control law. In both cases, the saturated control is indeed applied to the beam equation. By closing the loop with such a nonlinear control, it is thus obtained a nonlinear partial differential equation, which is the generalization of the classical beam equation. The well-posedness is proven by using nonlinear semigroups techniques. Considering a generalized sector condition to tackle the control nonlinearity, the semi-global asymptotic stabilization system is proven by Lyapunov-based arguments.

Keywords: infinite-dimensional systems, saturation, well-posedness, semi-global asymptotic stability, output feedback control, Lyapunov function

1. INTRODUCTION

Many mechanical systems should be modeled by partial differential equation providing the possibility to measure and to control all modes of the dynamic, e.g., all modes for flexible structure (see e.g., Géradin and Rixen (2014)). For many application, smart sensors and actuators equip the flexible structures as for the control of vibration of ski materials (Lind and Sanders (2013)) or of tennis rackets (Moheimani and Fleming (2006)). Many of these devices are equipped with piezoelectric sensors and actuators that allow to measure and to control the vibration.

The goal of this paper is to consider the beam equation. This is a partial differential differential equation describing the evolution of the deflection of a 1D beam that is attached on one extremity and free at the other one. This beam is assumed to be equipped with collocated piezoelectric sensor and actuator, as considered in Halim and Moheimani (2002); Meirovitch (1975). This kind of smart inputs may be subject to amplitude saturations, so that, when closing the loop with an output feedback law, it asks to study a nonlinear infinite-dimensional system. Omitting the saturation map in the study of the closed-loop system properties may be very restrictive since it is known, even for finite-dimensional systems, that the saturation maps are nonlinearities that may introduce limit cycles, new equilibrium points and performance degradations (see e.g., Tarbouriech et al. (2011); Zaccarian and Teel (2011) for recent textbooks on this subject and how to improve the performance of saturating finite-dimensional systems).

This paper studies the stabilization of the beam equation, in closed loop with two classes of output feedback laws: static and dynamic ones. The tuning gains are designed and the obtained controllers yield an asymptotic stability property that differs for each class of controllers: first a global asymptotic stability is obtained for the system in closed loop with the saturating static controllers, and then a semi-global asymptotic stabilization can be reached when designing saturating dynamic controllers. For both cases, the well-posedness of the infinite-dimensional Cauchy problem is proven using abstract theory of systems.

To the best of our knowledge this paper is the first work dealing with saturating controllers for the beam equation. Many works are related to this study, as Prieur et al. (2016) where similar results are obtained for the wave equation in presence of saturation nonlinearities for bounded and unbounded control operators. See also Alabau-Boussouira (2002); Haraux (1986); Marx et al. (2018); Lasiecka and Seidman (2003) for close results exploiting Lyapunov techniques and dissipative properties mainly for the wave equation. Moreover, consider Marx et al. (2017) for a study on the Korteweg-de Vries equation, that is a nonlinear partial differential equation controlled by a saturating controller.

In this paper, two approaches are developed for the well-posedness and the stability proofs. First the abstract theory of nonlinear semigroups is used for the closed-loop systems (see in particular Miyadera (1992)), and then Lyapunov techniques are used (as in Karafyllis and Krstic (2019); Bastin and Coron (2016); Bribiesca Argomedo et al. (2012) to cite just a few references). When designing controllers, only weak Lyapunov functions are designed so that it is necessary to apply a LaSalle invariance principe in the considered infinite-dimensional setting (see Slemrod (1989); Luo et al. (1999) for an introduction of this LaSalle invariance principle for infinite dimensional dynamical systems).

Due to space limitation, some proofs are omitted. They will be in the journal version of this conference paper and can be reached here:

`http://www.gipsa-lab.grenoble-inp.fr/~christophe.prieur/NOLCOS19_complete.pdf`

This paper is organized as follows. First the partial differential equation of the beam is introduced and some preliminary computations are done in Section 2. Then saturating output feedback laws are introduced, and static controllers are designed in Section 3. Dynamic controllers are then considered in Section 4. In these two sections, the well-posedness and the asymptotic stability are established (in a different sense). Some concluding remarks are collected in Section 5.

2. PRELIMINARIES

In this section, we recall some ingredients on the asymptotic stability of the beam equation through a linear static output feedback law.

We are first interested in the following partial differential equation, for all $0 < x < \pi$ and for all $t \geq 0$

$$w_{tt}(x,t) + w_{xxxx}(x,t) = u(t)\frac{d}{dx}[\delta_\eta(x) - \delta_\xi(x)] \ , \quad (1)$$

$$w(0,t) = w_x(0,t) = w_{xx}(\pi,t) = w_{xxx}(\pi,t) = 0 \ , \quad (2)$$

$$w(x,0) = w^0(x), \ w_t(x,0) = w^1(x) \ , \quad (3)$$

where $w(x,t)$ is the deflection of the beam with respect to the rest position, at point x in $[0,\pi]$ and at time t, $u(t)$ is the voltage applied on a actuator located on the interval $[\eta,\xi]$. See Halim and Moheimani (2002); Meirovitch (1975) for this model involving the two Dirac functions δ_η and δ_ξ respectively at η and δ. In (2) the boundary equations come from the assumption that the beam is clamped at the end $x = 0$ and free at the other extremity. In (3), the initial conditions depend on the initial deflection and initial speed deflection. See Figure 1 for an illustration of the considered beam.

Assume that a collocated piezoelectric sensor is attached to the beam, so that not all the deflection is known but only the following output is measured, for all $t \geq 0$,

$$y = w_{xt}(\eta,t) - w_{xt}(\xi,t) \ . \quad (4)$$

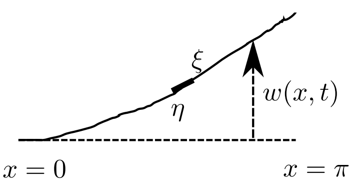

Fig. 1. A clamped-free beam subject to a piezoelectric actuator

The energy is defined as follows:

$$E(w) = \frac{1}{2}\int_0^\pi (w_{xx}^2 + w_t^2)\,dx \ . \quad (5)$$

An informal computation along the solutions to (1)-(3) gives [1]

$$\begin{aligned}\dot{E}(w) &= \int_0^\pi (w_{xx}w_{xxt} + w_t(-w_{xxxx} + u(t)\frac{d}{dx}[\delta_\eta(x) \\ &\quad -\delta_\xi(x)]))\,dx \\ &= \int_0^\pi (w_{xx}w_{xxt} - w_{xxt}w_{xx})\,dx \\ &\quad -[w_t w_{xxx}]_0^\pi + [w_{xt}w_{xx}]_0^\pi \\ &\quad -u(t)\int_0^\pi w_{xt}[\delta_\eta(x) - \delta_\xi(x)]\,dx \\ &\quad +u(t)[w_t(\delta_\eta(x) - \delta_\xi(x))]_0^\pi \\ &= -u(t)(w_{xt}(\eta) - w_{xt}(\xi))\end{aligned}$$

where the PDE (1) has been used to get the first equation and two integrations by parts have been used for the second equation with the boundary conditions (2) and the definitions of the Dirac function δ_η and δ_ξ have been used for the third equation.

This computation leads to let

$$u(t) := k(w_{xt}(\eta) - w_{xt}(\xi))$$

where $k > 0$ is a tuning parameter. Recalling (4), the previous controller is a linear output feedback law, so that along the solutions to the closed-loop system

$$w_{tt}(x,t) + w_{xxxx}(x,t) =$$
$$k(w_{xt}(\eta) - w_{xt}(\xi))\frac{d}{dx}[\delta_\eta(x) - \delta_\xi(x)], \quad (6)$$

$$w(0,t) = w_x(0,t) = w_{xx}(\pi,t) = w_{xxx}(\pi,t) = 0, \quad (7)$$

$$w(x,0) = w^0(x), \ w_t(x,0) = w^1(x), \quad (8)$$

we have

$$\dot{E} = -k(w_{xt}(\eta) - w_{xt}(\xi))^2 \ .$$

Summing up, we have the stability property of the system (6)-(8) for any non-critical points (η,ξ). To understand what are critical points, it is necessary to recall some useful facts about the *free* evolution of (1)-(3) (i.e., when $k = 0$), as described in Crépeau and Prieur (2006). To do that we consider the homogeneous Cauchy problem

$$\phi_{tt} + \phi_{xxxx} = 0, \quad (9)$$

$$\phi(0,t) = \phi_x(0,t) = \phi_{xx}(\pi,t) = \phi_{xxx}(\pi,t) = 0, \quad (10)$$

$$\phi(.,0) = \phi^0, \ \phi_t(.,0) = \phi^1. \quad (11)$$

Let $A_0 : \mathcal{D}(A) \to L^2(0,\pi)$ be the (open-loop, linear) operator with domain

$$\mathcal{D}(A_0) := \{\phi \in H^4(0,\pi);\ \phi(0) = \phi_x(0) = \phi_{xx}(\pi) = \phi_{xxx}(\pi) = 0\}$$

and defined by $A_0\phi = \phi_{xxxx}$. Define the set $\mathcal{V} = \{w \in H^2(0,\pi),\ w(0) = w'(0) = 0\}$ with the hermitian product $\langle w_1, w_2\rangle_\mathcal{V} = \int_0^\pi w_1'' \overline{w_2''} dx$ and the set $\mathcal{H} = \mathcal{V} \times L^2(0,\pi)$. The operator A_0^{-1} is compact and symmetric on $\mathcal{H}$, hence there exists a countable orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A_0^{-1}. The following result (Crépeau and Prieur, 2006, Lemma 2.1) provides useful results about the eigenvectors of A_0.

Proposition 1. *The $L^2(0,\pi)$-normalized eigenfunctions of A_0 are the functions $(\psi_n)_{n\geq 1}$, defined by*

[1] See Le Gall et al. (2007) for the proof that this computation makes sense along appropriate solutions.

$$\psi_n(x) = \gamma_n\big(\cos(\alpha_n x) - \cosh(\alpha_n x) + \mu_n(\sinh(\alpha_n x) - \sin(\alpha_n x))\big), \quad (12)$$

where α_n is the n-th positive root of

$$1 + \cos(\alpha_n \pi)\cosh(\alpha_n \pi) = 0, \quad (13)$$

and

$$\mu_n = \frac{\cos(\alpha_n\pi) + \cosh(\alpha_n\pi)}{\sin(\alpha_n\pi) + \sinh(\alpha_n\pi)},$$

$$\gamma_n = \frac{1}{\sqrt{\pi}},$$

We are now in position to define the critical set. To do that let $L = \xi - \pi$ be the length of the actuator/sensor. For any $n \geq 1$ and any $L \in (0, \pi]$, let

$$\mathcal{S}_n(L) := \{\eta \in [0, \pi - L],\ \psi_n'(\eta) - \psi_n'(\eta + L) = 0\}, \quad (14)$$

and set

$$\mathcal{S}(L) := \cup_{n\geq 1}\mathcal{S}_n(L).$$

The asymptotic stability for the PDE (1) with the boundary conditions (2) is proven in Le Gall et al. (2007) and is recalled here:

Proposition 2. *(Le Gall et al. (2007)) The system* (1)-(2) *is asymptotically stable in $\mathcal{H}$ if and only if $k > 0$ and $\eta \notin \mathcal{S}(L)$.*

3. A SATURATING STATIC OUTPUT FEEDBACK LAW

3.1 Problem statement and preliminary computations

In this section, we consider the case where the input in (1) may be subject to saturations. To be specific, consider the following output (4) and the controlled PDE

$$w_{tt}(x,t) + w_{xxxx}(x,t) = \mathtt{sat}(u(t))\frac{d}{dx}[\delta_\eta(x) - \delta_\xi(x)], \quad (15)$$

with the boundary conditions (2) and the initial conditions (3).

A formal computation along the solutions to (15), (2) and (3) yields

$$\dot{E}(w) = -\mathtt{sat}(u(t))(w_{xt}(\eta) - w_{xt}(\xi)) .$$

Letting, as for the linear case, $u(t) = k(w_{xt}(\eta) - w_{xt}(\xi))$ for any positive tuning parameter k, we get

$$\dot{E}(w) = -\mathtt{sat}(k(w_{xt}(\eta) - w_{xt}(\xi)))(w_{xt}(\eta) - w_{xt}(\xi)) \quad (16)$$

and since $\mathtt{sat}(s)s \geq 0$ for any $s \in \mathbb{R}$, we have the stability property.

The problem under consideration in this section is to prove the asymptotic stability by being more formal (by stating first a well-posedness result and then the asymptotic stability of the closed-loop system). To do that we first introduce the nonlinear operator defining the PDE (15) with the boundary conditions (2).

To specify this operator, let us adapt some computations of Le Gall et al. (2007) and we have to see for which function w the right-hand side of (15) belongs to $L^2(0,\pi)$. Let us recall some notations of Le Gall et al. (2007). If w is any function in $H^1(0,\eta)\cap H^1(\eta,\xi)\cap H^1(\xi,\pi)$, we define $\{w_x\} \in L^2(0,\pi)$ by

$$\{w_x\}(x) := \begin{cases} w_x^{\mathcal{D}'(0,\eta)}(x) & \text{if } x \in (0,\eta), \\ w_x^{\mathcal{D}'(\eta,\xi)}(x) & \text{if } x \in (\eta,\xi), \\ w_x^{\mathcal{D}'(\xi,\pi)}(x) & \text{if } x \in (\xi,\pi). \end{cases}$$

We set also $[w]_\eta := w(\eta^+) - w(\eta^-)$, and $[w]_\xi := w(\xi^+) - w(\xi^-)$. Then it follows that

$$w_x = \{w_x\} + [w]_\eta\delta_\eta + [w]_\xi\delta_\xi \quad \text{in } \mathcal{D}'(0,\pi).$$

Assume now that $w \in H^2(0,\pi)$ and that $v \in H^2(0,\pi)$, and define $\varpi \in \mathcal{D}'(0,\pi)$ by $\varpi := -w_{xxxx} + \mathtt{sat}(k(v_x(\eta) - v_x(\xi)))\frac{d}{dx}(\delta_\eta - \delta_\xi)$. If $\varpi \in L^2(0,\pi)$, then the restriction of u to each of the intervals $(0,\eta)$, (η,ξ) and (ξ,π) has also to be a square integrable function. The same conclusion holds for w_{xxxx}, hence $w \in H^4(0,\eta)\cap H^4(\eta,\xi)\cap H^4(\xi,\pi)$. We may then compute the first space-derivatives of w and then ϖ. We obtain

$$\begin{aligned} w_x &= \{w_x\} + [w]_\eta\delta_\eta + [w]_\xi\delta_\xi = \{w_x\} \\ w_{xx} &= \{w_{xx}\} + [w_x]_\eta\delta_\eta + [w_x]_\xi\delta_\xi = \{w_{xx}\}, \\ w_{xxx} &= \{w_{xxx}\} + [w_{xx}]_\eta\delta_\eta + [w_{xx}]_\xi\delta_\xi, \\ w_{xxxx} &= \{w_{xxxx}\} + [w_{xxx}]_\eta\delta_\eta + [w_{xxx}]_\xi\delta_\xi + [w_{xx}]_\eta\frac{d}{dx}\delta_\eta \\ &\quad + [w_{xx}]_\xi\frac{d}{dx}\delta_\xi, \end{aligned} \quad (17)$$

and

$$\begin{aligned} \varpi = &-\{w_{xxxx}\} - [w_{xxx}]_\eta\delta_\eta - [w_{xxx}]_\xi\delta_x - [w_{xx}]_\eta\frac{d}{dx}\delta_\eta \\ &-[w_{xx}]_\xi\frac{d}{dx}\delta_\xi + \mathtt{sat}(k(v_x(\eta) - v_x(\xi)))\frac{d}{dx}(\delta_\eta - \delta_\xi). \end{aligned}$$

Then ϖ is in $L^2(0,\pi)$ provided that all the coefficients in front of the Dirac masses vanish, i.e.

$$[w_{xx}]_\eta = \mathtt{sat}(k(v_x(\eta) - v_x(\xi))) = -[w_{xx}]_\xi$$

and

$$[w_{xxx}]_\eta = [w_{xxx}]_\xi = 0.$$

We are now in a position to define the operator associated to the PDE (15) with the boundary conditions (2). If we introduce $v := w_t$ and define the operator A with domain

$$\begin{aligned} \mathcal{D}(A) = \{\ & z = (w,v)\ ,\ (w,v) \in H^2(0,\pi)^2, \\ & w \in H^4(0,\eta)\cap H^4(\eta,\xi)\cap H^4(\xi,\pi), \\ & w(0) = w_x(0) = w_{xx}(\pi) = w_{xxx}(\pi) = 0, \\ & v(0) = v_x(0) = 0, \\ & [w_{xx}]_\eta = \mathtt{sat}(k(v_x(\eta) - v_x(\xi))) = -[w_{xx}]_\xi\,, \\ & [w_{xxx}]_\eta = [w_{xxx}]_\xi = 0 \Big\} \end{aligned}$$

and defined by

$$\begin{aligned} Az &= \left(v, -w_{xxxx} + \mathtt{sat}(k(v_x(\eta) - v_x(\xi)))\frac{d}{dx}(\delta_\eta - \delta_\xi)\right) \\ &= (v, -\{w_{xxxx}\})\,, \end{aligned}$$

then the PDE (15) with the boundary conditions (2) may be seen as the initial value problem for the following abstract first-order evolution equation in $\mathcal{H}$

$$\begin{cases} \dfrac{dz}{dt} = Az,\ t > 0 \\ z(0) = z^0. \end{cases} \quad (18)$$

First let us prove the well-posedness of this Cauchy problem, and then prove the asymptotic stability. This is done successively in the next two subsections.

3.2 Well-posedness of (18)

To prove the well-posedness of the Cauchy Problem (18), we follow the steps of Prieur et al. (2016) and we prove successively that 1) A is closed; 2) A is dissipative; 3) A satisfies a range condition.

Let us prove these properties successively.

Lemma 1. *The operator A is closed.*

Proof. To do that, let us pick a sequence $(w_n, v_n)_{n\in\mathbb{N}}$ that converges to (w,v) in $\mathcal{D}(A)$, and such that the sequence $(A(w_n,v_n)^\top)_{n\in\mathbb{N}}$ converges to (y_w, y_v) in $\mathcal{H}$. Then we need to prove that $A(w,v)^\top = (y_w,y_v)$ in $\mathcal{H}$. By definition of A, the first line of $A(w_n,v_n)$ is v_n and it converges to v. Thus $v = y_w$ in $\mathcal{V}$. To prove that $-\{w_{xxxx}\} = y_v$ in $L^2(0,\pi)$, we prove that, for all f in $C_0^\infty(0,\pi)$, $\langle -\{w_{xxxx}\}, f\rangle_{L^2} = \langle y_v, f\rangle_{L^2}$, in other words, we need to prove that

$$\int_0^\pi \big(-w_{xx}f_{xx} + \mathtt{sat}(k(v_x(\eta)-v_x(\xi)))(f_x(\eta)-f_x(\xi))\big)dx = \int_0^\pi y_v f dx \ . \tag{19}$$

Since $(w_n,v_n)_{n\in\mathbb{N}}$ tends to (w,v) in $H^2(0,\pi)^2$, then $(w_{kxx})_{k\in\mathbb{N}}$ tends to w in $L^2(0,\pi)$ and $(\mathtt{sat}(k(v_{nx}(\eta) - v_{nx}(\xi))_{n\in\mathbb{N}}$ tends to $\mathtt{sat}(k(v_x(\eta)-v_x(\xi))$ in $\mathbb{R}$. Therefore (19) holds. This completes the proof of Lemma 1. □

Lemma 2. *For any $z = (w,v) \in \mathcal{D}(A)$ we have that*

$$\begin{aligned}\langle Az, z\rangle_{\mathcal{H}} &= 2i\ \mathrm{Im}(\int_0^\pi v_{xx}\overline{w_{xx}}\,dx) \\ &\quad -\mathtt{sat}(k(v_x(\eta)-v_x(\xi))\overline{(k(v_x(\eta)-v_x(\xi))}.\end{aligned}$$

In particular, if $k \geq 0$ $\mathrm{Re}\,\langle Az,z\rangle_{\mathcal{H}} \leq 0$, i.e. A is dissipative.

Proof. Pick any pair of functions $(w,v)\in\mathcal{H}$. Then

$$\langle A(w,v),(w,v)\rangle_{\mathcal{H}} = \int_0^\pi v_{xx}\overline{w_{xx}}\,dx - \int_0^\pi \{w_{xxxx}\}\overline{v}\,dx.$$

After some integrations by parts on the intervals $(0,\eta)$, (η,ξ), (ξ,π), we obtain that

$$\begin{aligned}&-\int_0^\pi \{w_{xxxx}\}\overline{v} \\ &= -\int_0^\eta \{w_{xxxx}\}\overline{v} - \int_\eta^\xi \{w_{xxxx}\}\overline{v} - \int_\xi^\pi \{w_{xxxx}\}\overline{v} \\ &= -\int_0^\pi w_{xx}\overline{v_{xx}}\,dx + [w_{xx}\overline{v_x}]_{x=0}^\eta + [w_{xx}\overline{v_x}]_{x=\eta}^\xi \\ &\quad +[w_{xx}\overline{v_x}]_{x=\xi}^\pi \\ &= -\int_0^\pi w_{xx}\overline{v_{xx}}\,dx - [w_{xx}]_\eta\overline{v_x(\eta)} - [w_{xx}]_\xi\overline{v_x(\xi)}.\end{aligned}$$

Hence

$$\begin{aligned}&\langle A(w,v),(w,v)\rangle_{\mathcal{H}} \\ &= \int_0^\pi (v_{xx}\overline{w_{xx}} - w_{xx}\overline{v_{xx}})\,dx - [w_{xx}]_\eta\overline{v_x(\eta)} - [w_{xx}]_\xi\overline{v_x(\xi)}\ .\end{aligned}$$

Using the domain of A, this completes the proof of Lemma 2. □

Lemma 3. *The following range condition holds*

$$\mathcal{D}(A) \subset \mathtt{Ran}(I-\lambda A)$$

for all $\lambda > 0$ sufficiently small, where $\mathtt{Ran}(I-\lambda A)$ is the range of the operator $I-\lambda A$.

Proof. To prove this range condition, let us pick (w,v) in $\mathcal{D}(A)$ and prove that there exists $(\tilde{w},\tilde{v})$ in $\mathcal{D}(A)$ such that

$$(I-\lambda A)(\tilde{w},\tilde{v}) = (w,v)\ .$$

This latter equation is equivalent to $\tilde{v} = w$ and $\tilde{v} - \lambda\{\tilde{w}_{xxxx}\} = v$, which is equivalent to find $\tilde{w}$ in $H^2(0,\pi)$ such that $\tilde{w}_{xxxx} = \frac{1}{\lambda}(w-v) + \mathtt{sat}(k(w_x(\eta)-w_x(\xi)))\frac{d}{dx}(\delta_\eta - \delta_\xi)$ in the distribution sense. It is possible to find such a function in $H^2(0,\pi)$ satisfying also the boundary conditions so that $(\tilde{w},\tilde{v})$ is in $\mathcal{D}(A)$. This concludes the proof of Lemma 3. □

Since A is dissipative (due to Lemma 2), it follows, from (Miyadera, 1992, Thm 4.2), that A generates a semigroup of contractions $T(t)$. Moreover, by (Miyadera, 1992, Thm 4.5), for all $(w^0,v^0)^\top$ in $\mathcal{D}(A)$, $T(t)(w^0,v^0)^\top$ is differentiable for $t>0$ and is a solution to the Cauchy Problem (18). Moreover due to (Miyadera, 1992, Thm 4.10), it is the unique solution to this Cauchy problem. This constitues the proof of the following theorem.

Theorem 1. *If $k \geq 0$, then, for all z^0 in $\mathcal{D}(A)$, the Cauchy Problem (18) is well-posed.*

3.3 Asymptotic stability of (18)

In this section, we prove the global asymptotic stability. First we consider a Lyapunov function candidate and then we apply the LaSalle invariance principle.

Let us consider E defined by (5). Due to Theorem 1, for all z^0 in $\mathcal{D}(A)$, the formal computation (16) makes sense, along the solutions to (18) and we have the stability property. The global asymptotic stability is the aim of the second main result of this paper:

Theorem 2. *The system (18) is globally asymptotically stable, that is, for all initial conditions z^0 in $\mathcal{D}(A)$, the solution to (18) satisfies, the following stability property $\forall t \geq 0$,*

$$\begin{aligned}&\|w(.,t)\|_{H^2(0,\pi)} + \|v(.,t)\|_{L^2(0,\pi)} \\ &\quad\leq \|w^0\|_{H^2(0,\pi)} + \|w^1\|_{L^2(0,\pi)}\ ,\end{aligned}$$

together with the attractivity property

$$\|w(.,t)\|_{H^2(0,\pi)} + \|v(.,t)\|_{L^2(0,\pi)} \to 0, \quad as\ t\to\infty\ ,$$

if and only if $k>0$ and $\eta \notin \mathcal{S}(L)$.

Proof. The proof of the *only if part* follows from the proof of (Le Gall et al., 2007, Theorem 2), and is skipped from this conference paper.

Let us focus on the sufficient part of the proof, that is assume that $k>0$ and $\eta\notin\mathcal{S}(L)$. To prove the global asymptotic stability, we use the LaSalle invariance principle. To do that we first note that the canonical embedding from $\mathcal{D}(A)$ to $\mathcal{H}$ is compact.

The graph norm is defined as, for all (w,v) in $\mathcal{D}(A)$,

$$\|(w,v)\|^2_{\mathcal{D}(A)} = \|(w,v)\|^2_{\mathcal{D}(A_l)}\int_0^1 v_{xx}v_{xx}dx + \int_0^1 \{w_{xxxx}\}^2 dx$$

Due to Le Gall et al. (2007), the inclusion of $\mathcal{D}(A_l)$ is also compact (where A_l is the linear operator corresponding to the dynamics without saturation), we get that the canonical embedding from $\mathcal{D}(A)$ to $\mathcal{H}$ is also compact. We obtain that the set $\mathrm{orb}(w^0, w^1) := \{(w(t), v(t)) | \; t \geq 0\}\}$ is precompact in $\mathcal{H}$ for any $(w^0, w^1) \in \mathcal{D}(A)$. Therefore, the $\omega-$limit set of (w^0, w^1), defined as

$$\omega(w^0, w^1) = \{z \in \mathcal{H}, \exists (t_n) \to \infty, \lim_{n\to\infty} S(t_n)(w^0, w^1) = z\},$$

is nonempty. On the other hand, according to LaSalle's invariance principle (see Slemrod (1989)), for any $(\phi^0, \phi^1) \in \omega(w^0, w^1)$, we have that $S(t)(\phi^0, \phi^1) = (\phi(t), \phi_t(t)) \in \omega(w^0, w^1)$ and $E(\phi(t), \phi_t(t)) = E(\phi^0, \phi^1)$. The above relation and (16) imply that ϕ is a solution of the homogeneous Cauchy Problem (9)-(11) and fulfills

$$\phi_{xt}(\eta, t) - \phi_{xt}(\xi, t) = 0 \qquad \forall t \geq 0.$$

The rest of the proof follows the proof of Theorem 2 in Le Gall et al. (2007). Let us recall it, just for a seek of completeness. It is easily seen that if $\phi^0 = \sum_{k\geq 1} \phi_k^0 \psi_k$, and $\phi^1 = \sum_{k\geq 1} \phi_k^1 \psi_k$, then the solution $\phi = \phi(x, t)$ to (9)-(11) reads

$$\phi(x,t) = \sum_{k=1}^{+\infty} \left(\phi_k^0 \cos(\alpha_k^2 t) + \frac{\phi_k^1}{\alpha_k^2} \sin(\alpha_k^2 t) \right) \psi_k(x). \quad (20)$$

Derivating w.r.t. x and t in (20), we obtain

$$0 \equiv \phi_{xt}(\eta, t) - \phi_{xt}(\xi, t) = -\sum_{k=1}^{+\infty} \left(\alpha_k^2 \phi_k^0 \sin(\alpha_k^2 t) - \phi_k^1 \cos(\alpha_k^2 t) \right) \left(\psi_k'(\eta) - \psi_k'(\xi) \right).$$

Since $\alpha_{k+1}^2 - \alpha_k^2 \to \infty$, we infer from a generalization of Ingham's inequality that for any $T > 0$

$$0 = \int_0^T |\phi_{xt}(\eta, t) - \phi_{xt}(\xi, t)|^2 dt \geq C_T \sum_{k=1}^{+\infty} \left(|\alpha_k^2 \phi_k^0|^2 + |\phi_k^1|^2 \right) |\psi_k'(\eta) - \psi_k'(\xi)|^2.$$

Therefore, if $\eta \notin \mathcal{S}(L)$, then $\phi_k^0 = \phi_k^1 = 0$ for all $k \geq 1$ and $S(t)(w^0, w^1) \to (\phi^0, \phi^1) = (0, 0)$ in $\mathcal{H}$. Conversely, if $\eta \in \mathcal{S}_k(L)$ for some $k \geq 1$, then any state of the form $(\phi^0, \phi^1) = (\phi_k^0 \psi_k, \phi_k^1 \psi_k)$ gives rise to a solution to (1)-(3) (or (9)-(11)) whose energy does not tend to 0. Therefore we have the proof of the asymptotic stability for positive k and for non-critical locations of the actuator.

This concludes the proof of Theorem 2. □

4. A SATURATING DYNAMIC OUTPUT FEEDBACK LAW

4.1 Problem statement and preliminary computations

In this section, we consider the case where the input may be subject to saturation and that it is a dynamical controller. To be specific, consider again the PDE (15) with the boundary conditions (2) and the initial conditions (3), and assume that $u(t)$ is a solution to the following scalar linear ordinary differential equation:

$$\tau \dot{u} + u = u_e \quad (21)$$

where u_e is a to-be-computed control that should depend on the output (4) only.

An informal computation along the solutions to (15), (2) and (3) yields

$$\dot{E}(w) = -\mathtt{sat}(u)(w_{xt}(\eta) - w_{xt}(\xi)) \quad (22)$$

Consider the Lyapunov function candidate

$$V = E + \frac{1}{2}(u - k\sigma)^2, \quad (23)$$

where $\sigma = w_{xt}(\eta) - w_{xt}(\xi)$. Along the solutions to (15), (2) and (21), it holds (at least formally)

$$\begin{aligned} \dot{V} &= -\mathtt{sat}(u)\sigma + (u - k\sigma)(\dot{u} - k\dot{\sigma}) \\ &= \frac{1}{k}\mathtt{sat}(u)(u - k\sigma) - \frac{1}{k}\mathtt{sat}(u)u + (u - k\sigma)(\dot{u} - k\dot{\sigma}) \\ &= -\frac{1}{k}\mathtt{sat}(u)u + (u - k\sigma)(\frac{1}{k}\mathtt{sat}(u) - \frac{1}{\tau}u + \frac{1}{\tau}u_e - k\dot{\sigma}) \; . \end{aligned}$$

This motivates the following definition for the control u_e, for any $K > 0$,

$$u_e = -K\tau(u - k\sigma) + u + k\tau\dot{\sigma} - \frac{\tau}{k}\mathtt{sat}(u) \quad (24)$$

which makes

$$\dot{V} = -\frac{1}{k}\mathtt{sat}(u)u - K(u - k\sigma)^2 \; . \quad (25)$$

To give a sense to the quantity $\dot{\sigma}$, we note that, using (15), at least formally:

$$\begin{aligned} \dot{\sigma} &= w_{ttx}(\eta) - w_{ttx}(\xi) \\ &= \Big(w_{x5} - \mathtt{sat}(u)\frac{d^2}{dx^2}[\delta_\eta - \delta_\xi]\Big)(\eta) \\ &\quad -\Big(w_{x5} - \mathtt{sat}(u)\frac{d^2}{dx^2}[\delta_\eta - \delta_\xi]\Big)(\xi) \end{aligned} \quad (26)$$

where w_{x5} denotes the 5-th derivative of w with respect to x. To check that we prove that its space primitive, that is

$$w_{x4} - \mathtt{sat}(u)\frac{d}{dx}[\delta_\eta - \delta_\xi]$$

is in $H^1(0, \pi)$. To check this latter property, we come back to (17), and using in particular $\frac{d}{dx}\mathtt{sat}(u) = 0$, we compute

$$w_{x4} - \mathtt{sat}(u)\frac{d}{dx}[\delta_\eta - \delta_\xi] = \{w_{x4}\} + [w_{x3}]_\eta \delta_\eta + [w_{x3}]_\xi \delta_\xi + [w_{xx}]_\eta \frac{d}{dx}\delta_\eta + [w_{xx}]_\xi \frac{d}{dx}\delta_\xi$$

and

$$\begin{aligned} &-\mathtt{sat}(u)\frac{d}{dx}[\delta_\eta - \delta_\xi] w_{x5} - \mathtt{sat}(u)\frac{d^2}{dx^2}[\delta_\eta - \delta_\xi] \\ &= \{w_{x5}\} + [w_{x4}]_\eta \delta_\eta + [w_{x4}]_\xi \delta_\xi \\ &+ 2[w_{x3}]_\eta \frac{d}{dx}\delta_\eta + 2[w_{x3}]_\xi \frac{d}{dx}\delta_\xi \\ &+ [w_{xx}]_\eta \frac{d^2}{dx^2}\delta_\eta + [w_{xx}]_\xi \frac{d^2}{dx^2}\delta_\xi \\ &- \mathtt{sat}(u)[\frac{d^2}{dx^2}\delta_\eta - \frac{d^2}{dx^2}\delta_\xi] \end{aligned}$$

Then $w_{x5} - \mathtt{sat}(u)\frac{d^2}{dx^2}[\delta_\eta - \delta_\xi]$ is in $L^2(0, \pi)$ provided that all the coefficients in front of the Dirac masses vanish, i.e.

$$[w_{x4}]_\eta = [w_{x4}]_\xi = 0 \; ,$$
$$[w_{xx}]_\eta = \mathtt{sat}(u) = -[w_{xx}]_\xi \; ,$$

and

$$[w_{x3}]_\eta = [w_{x3}]_\xi = 0 \ .$$

Therefore we may rewrite the PDE (15) with the boundary conditions (2) coupled with the ODE (21) in closed loop with (24) by introducing an extension of the operator A defined in Section 3.1, and by incorporating the dynamics (21). More precisely the operator A_e is defined on the following subset of the Hilbert space $\mathcal{H}_e$=$\mathcal{H} \times \mathcal{V} \times \mathbb{R}$:

$$\begin{aligned}\mathcal{D}(A_e) = \{\ & z_e = (w, v, u) \ \ (w, v, u) \in H^2(0,\pi)^2 \times \mathbb{R}, \\ & w \in H^4(0,\eta) \cap H^4(\eta,\xi) \cap H^4(\xi,\pi), \\ & w(0) = w_x(0) = w_{xx}(\pi) = w_{xxx}(\pi) = 0, \\ & v(0) = v_x(0) = 0, \quad [w_{xx}]_\eta = \mathtt{sat}(u) = -[w_{xx}]_\eta , \\ & [w_{x3}]_\eta = [w_{x3}]_\xi = 0, \quad [w_{x4}]_\eta = [w_{x4}]_\xi = 0 \Big\}\end{aligned}$$

and defined by

$$\begin{aligned}A_e z_e = \Bigg(& v, -w_{xxxx} + \mathtt{sat}(u)\frac{d}{dx}[\delta_\eta - \delta_\xi] \ , \\ & -\frac{1}{\tau}u - K\tau(u - k\sigma) + u + k\tau\dot{\sigma} - \frac{\tau}{k}\mathtt{sat}(u)\Bigg) .\end{aligned}$$

where $\dot{\sigma}$ is defined in (26). We rewrite the PDE (15) with the boundary conditions (2) coupled with the ODE (21) in closed loop with (24) as the abstract first-order evolution equation in $\mathcal{H}$

$$\begin{cases} \dfrac{dz}{dt} = A_e z_e, \ t > 0 \\ z_e(0) = (w^0, w^1, u^0). \end{cases} \tag{27}$$

We equip the Hilbert space $\mathcal{H}_e$ with not the usual scalar product, but with the one adapted to the Lyapunov function candidate V defined in (23), that is for all (w_1, v_1, u_1) and (w_2, v_2, u_2) in $\mathcal{H}_e$:

$$\begin{aligned}&\langle (w_1, v_1, u_1), (w_2, v_2, u_2)\rangle_{\mathcal{H}\times\mathbb{R}} \\ &= \int_0^\pi w_{1xx}\overline{w_{2xx}}dx + \int_0^\pi v_1\overline{v_2}dx + (u_1 - k\sigma_1)\overline{(u_2 - k\sigma_2)}\end{aligned}$$

where $\sigma_i = v_{ix}(\eta) - v_{ix}(\xi)$ for i in $\{1, 2\}$. Note that, by the trace theorem on $H^2(0,\pi)$, and since v_i is in V, σ_i is well defined.

4.2 Well-posedness of (27)

The well-posedness of (27) could be proven using similar computations as for A. In particular the proof of the dissipativity of A_e for $k \geq 0$ and $K \geq 0$ follows from the proof of Lemma 2 and the informal computation done to get (25). We get

Theorem 3. *If $k \geq 0$ and $K \geq 0$, then, for all z^0 in $\mathcal{D}(A_e)$, the Cauchy Problem (27) is well posed.*

4.3 Asymptotic stability of (27)

Let us come back to (25). To deduce an asymptotic stability from this property, we introduce the notation $\varphi = \mathtt{sat}(u) - u$ and we use the following local sector condition (see Gomes da Silva Jr and Tarbouriech (2005) for a proof of this sector condition, generalizing the global sector condition of (Khalil, 2002, Chap. 10))

$$\forall u, \ |(1 - c)u| \leq 1, \ \Rightarrow \varphi(\varphi + cu) \leq 0$$

where c is a new degree of freedom. This inequality implies with (25) the following inequality

$$\begin{aligned}\dot{V} &\leq -\frac{1}{k}\mathtt{sat}(u)u - K(u - k\sigma)^2 - 2\varphi(\varphi + cu) \\ &\leq -\frac{1}{k}\varphi u - \frac{1}{k}u^2 - K(u - k\sigma)^2 - 2\varphi(\varphi + cu) \\ &\leq \begin{pmatrix} u \\ u - k\sigma \\ \varphi \end{pmatrix}^\top \begin{pmatrix} -\frac{1}{k} & 0 & -\frac{1}{2k} - c \\ \star & -K & 0 \\ \star & \star & -2 \end{pmatrix} \begin{pmatrix} u \\ u - k\sigma \\ \varphi \end{pmatrix} .\end{aligned} \tag{28}$$

Given $k > 0$, and $K > 0$ one can show that there exists c such that

$$M := \begin{pmatrix} -\frac{1}{k} & 0 & -\frac{1}{2k} - c \\ \star & -K & 0 \\ \star & \star & -2 \end{pmatrix} \leq 0 \ .$$

For example, pick $c = -\frac{1}{2k}$ so that $M \leq 0$.

This later inequality is instrumental for the proof of the following result.

Theorem 4. *For all $k > 0$, and for all $r > 0$, there exists $K > 0$, such that the system (27) is asymptotically stable with a basin of attraction containing all initial conditions in $D(A_e)$ satisfying*

$$\|A_e z_e(t = 0)\|_{\mathcal{H}_e} \leq r \ .$$

More precisely, for all such initial conditions, we have the following stability property

$$\|A_e z_e(t)\|_{\mathcal{H}_e} \leq \|A_e z_e(t = 0)\|_{\mathcal{H}_e} \ ,$$

together with the attractivity property

$$\|z_e(t)\|_{\mathcal{H}_e} \to_{t\to\infty} 0 \ .$$

Before proving this result, let us note that it can be seen as a *semi-global asymptotic stabilization* by a saturating dynamic output feedback law for (1)-(3). Indeed, the control gain $K > 0$ depends on the size of the set of initial conditions.

Proof. We need to check the condition $|(1 - c)u| \leq 1$ for sufficiently small initial condition and for all positive time. To do that let us first note that, using the dissipativity of A_e, it holds

$$t \mapsto \|z_e(t)\|_{\mathcal{H}_e} \ , \quad t \mapsto \|A_e z_e(t)\|_{\mathcal{H}_e} \tag{29}$$

are non-increasing along the (strong) solutions to (27).

Therefore

$$\begin{aligned}\|A_e z_e(t = 0\|_{\mathcal{H}_e}^2 &= \int_0^\pi \{w_{xxxx}(t = 0)\}^2 dx \\ &\quad + \int_0^\pi w_{txx}(t = 0)^2 dx + (u - \sigma)(t = 0)^2 \\ &\geq (u - \sigma)(t = 0)^2\end{aligned} \tag{30}$$

and we have, for all $t \geq 0$,

$$|(u - \sigma)(t)|^2 \leq \|A_e z_e(t)\|_{\mathcal{H}_e}^2 \leq \|A_e z_e(t = 0)\|_{\mathcal{H}_e}^2 \ . \tag{31}$$

Pick $r > 0$ and consider an initial condition such that $\|A_e z_e(t = 0)\|_{\mathcal{H}_e}^2 \leq r^2$.

Lemma 4. *There exists $C_1 > 0$ such that, for all $t \geq 0$,*

$$|\sigma(t)| \leq C_1 \|A_e z_e(t)\|_{\mathcal{H}_e} \ . \tag{32}$$

Proof. Recall that $\sigma(t) = w_{tx}(\eta, t) - w_{tx}(\xi, t)$. Due to (30), it holds

$$\|A_e z_e(t=0)\|_{\mathcal{H}_e} \geq \int_0^\pi w_{txx}^2 dx \ .$$

Moreover, using $w_x(0) = 0$ and $w_x(\pi) = 0$, we have

$$\begin{aligned}
|w_{tx}(\eta) - w_{tx}(\xi)|^2 &= |\int_0^\eta w_{txx}dx + \int_0^\pi w_{txx}dx|^2 \\
&\leq 2|\int_0^\eta w_{txx}dx|^2 + 2|\int_0^\pi w_{txx}dx|^2 \\
&\leq 2\eta^2 \int_0^\eta w_{txx}^2 dx \\
&\quad +2(\pi-\xi)^2 \int_\xi^\pi w_{txx}^2 dx \\
&\leq 2\max\{\eta^2, (\pi-\xi)^2\} \int_0^\pi w_{txx}^2 dx
\end{aligned}$$

Therefore with (30) we get (32), with

$$C_1 = \sqrt{2\max\{\eta^2, (\pi-\xi)^2\}}.$$

This concludes the proof of Lemma 4. □

Since $u^2 \leq 2(u-\sigma)^2 + 2\sigma^2$, with (31) and (32) in Lemma 4, we get the existence of $C_2 > 0$ such that, if

$$\|A_e z_e(t=0)\|_{\mathcal{H}_e} \leq r$$

then

$$|u|^2 \leq 2r^2 + 2C_1^2 r^2 \ .$$

Therefore we have the following implication

$$\begin{aligned} &\|A_e z_e(t=0)\|_{\mathcal{H}_e} \leq r \\ &\Rightarrow |(1-c)u| \leq 1 \end{aligned} \tag{33}$$

with $c = 1 - (r\sqrt{2(C_1^2+1)})^{-1}$ where C_1 is given in Lemma 4. Note that

$$c < 1 \ . \tag{34}$$

Moreover let us prove the following simple result

Lemma 5. *Given $K > 0$, and $c > 0$, there exists $k > 0$ such that $M \leq 0$.*

Proof. To prove that, it is equivalent to check that

$$\begin{pmatrix} -K & 0 & 0 \\ \star & -\frac{1}{k} & -\frac{1}{2k} - c \\ \star & \star & -2 \end{pmatrix} \leq 0$$

and, since $K > 0$ and $k > 0$, it is sufficient to check that there exists $k > 0$ such that

$$\frac{2}{k} - (\frac{1}{2k} + c)^2 > 0 \ .$$

This is true as soon as there exists $X > 0$ such that

$$\frac{1}{4}X^2 - (-c+2)X + c^2 < 0 \ . \tag{35}$$

The discriminant is $\Delta = (-c+2)^2 - c^2 = 4 - 4c > 0$, due to (34). There are two distincts zeros of the previous polynomial, whose product and sum are positive (due to (34)). Thus there exists $X > 0$ such that (35) holds. This concludes the proof of Lemma 5. □

With Lemma 5, the implication (33) and (28), we get that, along the solutions to (27), it holds $\dot{V} \leq 0$ for all initial conditions in $D(A_e)$ satisfying

$$\|A_e z_e\|_{\mathcal{H}_e} \leq r \ .$$

We conclude with the asymptotic stability thanks to the LaSalle invariance principle as in the proof of Theorem 2. Since the asymptotic stability holds for all $r > 0$, we deduce the semi-global asymptotic stability.

This concludes the proof of Theorem 4. □

5. CONCLUSION

A beam equation has been considered in this paper. The boundary conditions of the partial differential equation modeling the dynamics of the deflection corresponded to a beam that was attached to one extremity and free on the other. The beam was assumed to be equipped with collocated piezoelectric sensor and actuator. The actuator may be saturated so that the input function may be nonlinear, and the closed-loop system became a nonlinear infinite-dimensional system. Both static and dynamic output feedback laws have been designed yielding to different closed-loop systems. Abstract theory was applied to state the well-posedness of the nonlinear system, and to prove both stability results (one global asymptotic stability and one semi-global asymptotic stability).

This work lets some questions open. In particular the design of a strict Lyapunov function may be fruitful to state exponential convergence conditions.

REFERENCES

Alabau-Boussouira, F. (2002). Indirect boundary stabilization of weakly coupled hyperbolic systems. *SIAM Journal on Control and Optimization*, 41(2), 511–541.

Bastin, G. and Coron, J.M. (2016). *Stability and Boundary Stabilization of 1-D Hyperbolic Systems*, volume 88 of *Progress in Nonlinear Differential Equations and Their Applications*. Springer.

Bribiesca Argomedo, F., Witrant, E., and Prieur, C. (2012). D 1-input-to-state stability of a time-varying nonhomogeneous diffusive equation subject to boundary disturbances. In *American Control Conference*, 2978–2983. Montréal, Canada.

Crépeau, E. and Prieur, C. (2006). Control of a clamped-free beam by a piezoelectric actuator. *ESAIM: Control, Optimisation and Calculus of Variations*, 12, 545–563.

Géradin, M. and Rixen, D. (2014). *Mechanical vibrations: theory and application to structural dynamics*. John Wiley & Sons.

Gomes da Silva Jr, J. and Tarbouriech, S. (2005). Anti-windup design with guaranteed regions of stability: an LMI-based approach. *IEEE Transactions on Automatic Control*, 50(1), 106–111.

Halim, D. and Moheimani, S. (2002). Spatial H_2 control of a piezoelectric laminate beam: experimental implementation. *IEEE Transactions on Control Systems Technology*, 10(4), 533–546.

Haraux, A. (1986). *Nonlinear vibrations and the wave equation*, volume 20. Springer.

Karafyllis, I. and Krstic, M. (2019). *Input-to-State Stability for PDEs*. Communications and Control Engineering. Springer.

Khalil, H. (2002). *Nonlinear Systems*. Prentice-Hall, 3rd edition.

Lasiecka, I. and Seidman, T. (2003). Strong stability of elastic control systems with dissipative saturating feedback. *Systems & Control Letters*, 48(3-4), 243–252.

Le Gall, P., Prieur, C., and Rosier, L. (2007). Output feedback stabilization of a clamped-free beam. *Internat. J. Control*, 80(8), 1201–1216.

Le Gall, P., Prieur, C., and Rosier, L. (2007). Stabilization of a clamped-free beam with collocated piezoelectric sensor/actuator. *Int. J. Tomogr. Stat*, 6(S07), 104–109.

Lind, D. and Sanders, S. (2013). *The physics of skiing: skiing at the triple point*. Springer Science & Business Media.

Luo, Z.H., Guo, B.Z., and Morgul, O. (1999). *Stability and stabilization of infinite dimensional systems and applications*. Communications and Control Engineering. Springer-Verlag, New York.

Marx, S., Cerpa, E., Prieur, C., and Andrieu, V. (2017). Global stabilization of a korteweg-de vries equation with saturating distributed control. *SIAM Journal on Control and Optimization*, 55(3), 1452–1480.

Marx, S., Chitour, Y., and Prieur, C. (2018). Stability analysis of dissipative systems subject to nonlinear damping via lyapunov techniques. *arXiv preprint arXiv:1808.05370*.

Meirovitch, L. (1975). *Elements of vibration analysis*. McGraw-Hill Companies.

Miyadera, I. (1992). *Nonlinear Semigroups*. Translations of mathematical monographs. American Mathematical Society.

Moheimani, S. and Fleming, A.J. (2006). *Piezoelectric transducers for vibration control and damping*. Springer.

Prieur, C., Tarbouriech, S., and Gomes da Silva Jr, J.M. (2016). Wave equation with cone-bounded control laws. *IEEE Transactions on Automatic Control*, 61(11), 3452–3463.

Slemrod, M. (1989). Feedback stabilization of a linear control system in Hilbert space with an a priori bounded control. *Mathematics of Control, Signals, and Systems*, 2(3), 265–285.

Tarbouriech, S., Garcia, G., Gomes da Silva Jr, J.M., and Queinnec, I. (2011). *Stability and Stabilization of Linear Systems with Saturating Actuators*. Springer, London.

Zaccarian, L. and Teel, A. (2011). *Modern Anti-windup Synthesis: Control Augmentation for Actuator Saturation*. Princeton Series in Applied Mathematics. Princeton University Press.