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Budget Rules, Distortionnary Taxes, and Aggregate Instability: A reappraisal

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Abstract

In a seminal contribution, Schmitt-Grohé and Uribe (JPE, 1997), showed that the balanced-budget rule (BBR) produces aggregate instability in an exogenous growth model with labor tax-based adjustment. The present paper challenges this result in an endogenous growth framework with a more general budget rule, involving deficit and debt in the long-run and making the BBR a special case. We show that the emergence of aggregate instability dramatically depends on the level of public spending. In particular, low public spending ensures determinacy. However, in the case of high public spending, multiplicity arises, with four potential equilibria: two high-growth BGPs, a low-growth trap, and a "catastrophic" equilibrium where the economy asymptotically collapses. In addition, when the ratio of public spending is sufficiently large, a subcritical Hopf bifurcation appears around the low-growth trap, giving rise to a homoclinic orbit going around the neighborhood of the catastrophic equilibrium. A calibration exercise confirms that these results are obtained for realistic values of parameters.

Keywords: Budget rules; Indeterminacy; Distortionary taxation; Public Debt; Endogenous growth; Bifurcation.

1. Introduction

Understanding aggregate fluctuations is at the core of macroeconomics. In the context of the emergence of externality-based endogenous growth models in the late 1980s, Benhabib and Farmer (1994) showed how labor demand externalities generating increasing returns may produce indeterminacy[2]. Backed-up by the evaluation performed by Farmer and Guo (1994), showing the capacity of such models to replicate the features of the US business cycle, these models imposed the study of endogenous fluctuations as a major research topic in macroeconomics.

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[2]The literature refers to local indeterminacy (an infinity of possible paths towards a given equilibrium) and global indeterminacy (several possible paths towards different equilibria, starting from given initial conditions). Many terms are used to characterize this property, including aggregate instability, sunspots, sink, animal spirits, or self-fulfilling prophecies (Benhabib and Farmer, 1999). This paper is concerned with both local and global indeterminacy.

Among the different potential sources of indeterminacy, few received so much attention as the fiscal policy, and important debates surround the effects of public spending and taxes in terms of aggregate fluctuations. In a seminal contribution, Schmitt-Grohé and Uribe (1997) (hereafter SGU) reveal that fiscal policy can be a source of aggregate fluctuations. In the neoclassical growth model, SGU notably show that under a balanced-budget rule (hereafter BBR) the use of endogenous labor-income distortionary taxes to finance fixed wasteful government spending may lead to *aggregate instability*, defined as the local indeterminacy of the perfect-foresight equilibrium. On the contrary, exploring SGUs proposition to remove indeterminacy, Guo and Harrison (2004) show that financing endogenous public spending with fixed tax rates on labor (or capital income) under a BBR turns the equilibrium into determinate (precisely, saddle-path stable). Starting from these two influential results, a rich and expanding literature aims at identifying the different channels of fiscal policy-driven (in)determinacy.

However, so far little is known regarding the implications of fiscal deficits for aggregate fluctuations. The existing literature rests on a BBR, and does not account for public debt. Yet, the presence of public deficits and debt characterizes most developed countries since the mid-1970.[3] In addition, starting the 1980, many economies adopted fiscal rules constraining deficit and/or debt.[4] As such, the study of aggregate instability in indebted economies is a major challenge facing economic theory.

This paper addresses this challenge. To introduce public debt and deficit, we relax the BBR hypothesis. A number of recent works have shown that endogenous growth setups are a useful framework for reporting on continuous grow of public debt in the long run.[5] In these lines, we pursue the research program opened by SGU, and explore the issue of (in)determinacy as related to public deficits and debt, in a Romer-type endogenous growth model.

Our results are as follows.

First, we show that, under endogenous growth, SGUs findings must be amended on two grounds. On the one hand, aggregate instability only occurs if public spending is high. In the opposite case with low public spending, the perfect-foresight balanced-growth path

[3]The deficit-to-GDP ratio was around 2.5% on average in OECD countries in the period 1970-2005, and this ratio increased since the Great Recession (according to the 2017 IMF's World Economic Outlook, average general government gross debt in ratio of GDP in developed countries rose from around 72% in 2007 to roughly 105% in 2007; and the imbalances triggered by the public debt were at the core of the 2012 Eurozone debt crisis).

[4]For example, in eurozone countries, the Stability and Growth Pact (SGP) imposes deficit and debt ceilings. More generally, among all types of fiscal rules, debt and deficit rules were enacted in more than 60 countries by 2012, namely roughly three times more than expenditure rules, and more than six times more than revenue rules (see Schaechter et al., 2012; Combes et al., 2017).

[5]In exogenous growth setups, public debt is only transitory (see section 3 in SGU). With endogenous growth, in contrast, public debt can grow in the long run; see, e.g., Minea and Villieu (2012), Boucekkine et al. (2015), Nishimura et al. (2015a), Nishimura et al. (2015b), Menuet et al. (2017).

(hereafter BGP) is unique and well-determined, such that there is no aggregate instability. On the other hand, in the case with high public spending, our aggregate-instability result covers a broader class of mechanisms than in SGU, because it may rely on local or *global* indeterminacy.

Second, we subsequently extend and generalize these findings to the presence of deficit and debt. We confirm that aggregate instability emerge only when public spending (in ratio of GDP) are sufficiently large; however, accounting for deficit and debt yields two additional equilibria, and the model now displays four potential equilibria: two high-growth BGPs, a low-growth trap (with growth close to zero), and a "catastrophic" equilibrium where the economy asymptotically collapses.

Third, when the ratio of public spending is sufficiently large, a subcritical Hopf bifurcation appears around the low-growth trap, giving rise to a homoclinic orbit going around the neighborhood of the catastrophic equilibrium.

Interestingly, although indeterminacy can be avoided with sufficiently small public spending, the economy cannot reach the highest BGP in such a case. This BGP can be achieved only if public spending is large enough, but at the price of aggregate instability (local and global indeterminacy).

Regarding the related literature, Benhabib and Farmer (1999) provide a comprehensive discussion of the different sources of indeterminacy, notably regarding the role of externalities and increasing social returns. Our model belongs to this class, since a human capital externality generates increasing returns. Besides, our paper is close to a rich and expanding literature aiming at identifying the different channels of fiscal policy-driven (in)determinacy. Taking SGU and Guo and Harrison (2004) setups as benchmarks, these channels can be roughly divided into three categories. The first one relates to the way taxes are modeled; examples include taxes on consumption, instead of labor (Giannitsarou, 2007), or progressive taxation[6] (Guo and Lansing, 1998; Christiano and Harrison, 1999). Second, the way public spending are modeled is also of importance. Growth- or utility-enhancing, instead of wasteful public spending, can either support determinacy (Chen, 2006) or indeterminacy (Guo and Harrison, 2008, with exogenous growth, and Cazzavillan, 1996; Palivos et al., 2003; Park and Philippopoulos, 2004 with endogenous growth). Third, the (de)stabilizing effects of fiscal policy may significantly differ when departing from SGU and GH setups aside from alternative assumptions on taxes and public spending.[7]

[6]For example, Guo and Lansing (1998) find that a progressive income tax can stabilize the economy. This result does no longer hold when progressive taxes are combined with growth-enhancing (Chen and Guo, 2013b,a) or utility-enhancing public spending (Chen and Guo, 2014), or in the presence of heterogeneous agents Bosi and Seegmuller (2010).

[7]Such departures include non-separable utility function (Linnemann, 2008; Nourry et al., 2013; Abad et al., 2017), CES production function (Guo and Lansing, 2009; Ghilardi and Rossi, 2014), two-sector models (Nishimura et al., 2013; Chang et al., 2015), or an open economy (Huang et al., 2017).

With respect to this literature, we remain faithful to SGUs setup, and conserve wasteful public spending, endogenous flat-rate taxes on endogenous labor, and an additive utility function. Therefore, our (in)determinacy results are not triggered by the channels previously emphasized. On the methodological side, moving from exogenous to endogenous growth dramatically changes SGUs conclusions regarding the effects of labor taxes. In addition, our analysis provides a rich environment for studying complicated dynamics in a simple two-dimensional system.

Moreover, our results do not depend on the famous Benhabib-Farmer-Guo condition for indeterminacy (Benhabib and Farmer, 1994; Farmer and Guo, 1994), namely that the increasing labor demand must be positively sloped and steeper than the labor supply.[8] In our model with constant returns-to-scale and decreasing returns in all private factors, the labor demand is a decreasing function of the wage, bur nonetheless consistent with indeterminacy.

Finally, our findings highlight a new channel in the emergence of indeterminacy, through public spending. Indeed, large public spending (in percent of GDP) is a necessary, but not sufficient, condition for the birth of aggregate instability, resulting from either local or global indeterminacy.

Our results have important policy implications. First, contrary to SGUs main finding, (labor) taxes alone are not found to be a source of indeterminacy; indeed the amount of public spending has crucial implications in de(stabilizing) a growing economy. Second, in the presence of public debt, increasing public spending with the aim of reaching a higher BGP may bring the economy close to a low-growth trap in the long run. Finally, with large public spending, the presence of a subcritical orbit triggers possible large oscillations around the low-growth trap. Such finding can shed some light on the concept of debt "super-cycles" in the post-crisis low-growth context (see Rogoff, 2015).

The paper is organized as follows. Section 2 presents the model, section 3 analyzes the no-debt special case, and section 4 solves the model in the general case. Sections 5 and 6 look at local and global dynamics, respectively. Section 7 provides a calibration exercise. Finally, section 8 concludes the paper.

2. The model

We consider a simple continuous-time endogenous-growth model with N representative individuals and a government. Each representative agent consists of a household and a competitive firm. All agents are infinitely-lived and have perfect foresight. Population remains fixed over time, and we denote individual quantities by lower case letters, and aggregate quantities by corresponding upper case letters, namely $X = Nx$ for all variable X.

[8] The survey of Benhabib and Farmer (1999) provides a thorough discussion of this condition.

2.1. Households

The representative household starts at the initial period with a positive stock of capital (k_0), and chooses the path of consumption $\{c_t\}_{t\geq 0}$, hours worked $\{l_t\}_{t\geq 0}$, and capital $\{k_t\}_{t>0}$, so as to maximize the present discount value of its lifetime utility. We follow SGU's specification, namely

$$U = \int_0^\infty e^{-\rho t} \left\{ \log(c_t) - \frac{B}{1+\varepsilon} l_t^{1+\varepsilon} \right\} dt, \tag{1}$$

where $\rho \in (0,1)$ is the subjective discount rate, $\varepsilon \geq 0$ the constant elasticity of intertemporal substitution in labour, and $B > 0$ a scale parameter.

Households use labor income ($w_t l_t$, where w_t is the hourly wage rate) and capital revenues ($q_t k_t$, where q_t is the rental rate of capital), to consume (c_t), invest ($\dot{k}_t$), and buy government bonds (d_t), which return the real interest rate r_t. They pay taxes on wage income ($\tau_t w_t l_t$, where τ_t is the wage tax rate) and lump-sum taxes π_t (in equilibrium, π_t is the share Π_t/N of total lump-sump taxes Π_t); hence the following budget constraint

$$\dot{k}_t + \dot{d}_t = r_t d_t + q_t k_t + (1-\tau_t) w_t l_t - c_t - \pi_t. \tag{2}$$

The first order conditions for the maximization of the household's programme give rise to the familiar Keynes-Ramsey rule (with $q_t = r_t$ in competitive equilibrium)

$$\frac{\dot{c}_t}{c_t} = r_t - \rho, \tag{3}$$

and to the static relation

$$(1-\tau_t) w_t / c_t = B l_t^{\varepsilon}. \tag{4}$$

Eq. (4) means that, at each period t, the marginal gain of hours worked (the net real wage $(1-\tau_t)w_t$, expressed in terms of marginal utility of consumption $1/c_t$) just equals the marginal cost ($B l_t^{\varepsilon}$).

Finally, the optimal path of consumption has to verify the set of transversality conditions

$$\lim_{t\to+\infty} \{\exp(-\rho t)\ u'(c_t)\, k_t\} = 0 \text{ and } \lim_{t\to+\infty} \{\exp(-\rho t)\ u'(c_t)\, d_t\} = 0,$$

ensuring that lifetime utility U is bounded.[9]

[9] On the BGP associated to constant growth and interest rates (γ^* and r^*, respectively), transversality conditions correspond to the no-Ponzi game constraint $\gamma^* < r^*$. Such condition ensures that public debt will be repaid in the long run, and does not preclude the possibility that $\gamma > r$ in the short run.

2.2. Firms

Output of the individual firm (y_t) is produced using a constant returns-to-scale technology with a human capital externality, namely $y_t = \tilde{A} k_t^{\alpha} h_t^{1-\alpha}$, where k_t and h_t respectively stand for physical and human capital, $\tilde{A} > 0$ is a scale parameter, and $\alpha \in (0,1)$ is the elasticity of output to private capital.

According to Romer (1986), human capital is produced both by raw labor (or training activity) l_t, and by the economy-wide stock of knowledge X_t that generates positive technological spillovers onto firms' productivity, namely $h_t = X_t l_t$. We assume that knowledge is produced by a simple Cobb-Douglas technology depending on aggregate levels of physical and human capital: $X_t = H_t^{\beta} K_t^{1-\beta}$, where $\beta \in (0,1)$ is a measure of human capital efficiency in the accumulation of knowledge. At aggregate level, we then obtain $H_t = K_t L_t^{1/(1-\beta)} = K_t L_t^{1+\phi}$, with $1 + \phi = 1/(1-\beta) \geq 1$.[10]

As usual, the production function exhibits constant returns-to-scale at the individual level, and decreasing returns in all private factors. Thus, the first order conditions for profit maximization (relative to private factors) are

$$r_t = \alpha \frac{y_t}{k_t}, \tag{5}$$

$$w_t = (1-\alpha) \frac{y_t}{l_t}. \tag{6}$$

At the aggregate level, the knowledge externality will allow reaching an endogenous growth path, because the social return of capital is not decreasing. Effectively, the aggregate production function is

$$Y_t = \tilde{A} K_t L_t^{(1+\phi)(1-\alpha)}. \tag{7}$$

2.3. The government

The government provides public expenditures G_t, levies taxes, and borrows from households. Fiscal deficit is financed by issuing debt ($\dot{D}_t$); hence, the following budget constraint

$$\dot{D}_t = r_t D_t + G_t - \tau_t w_t L_t - \Pi_t. \tag{8}$$

We shall assume that the government claims a fraction g of aggregate output for public spending $G_t = g Y_t$. As in SGU, public expenditure has no effect on utility or production (i.e. wastefull public spending).[11] In addition revenues retrieved from lump-sum taxation

[10] Human capital externalities, i.e. the fact that your coworkers' human capital makes you more productive, are well documented in empirical literature (see, e.g. Rauch, 1993; Moretti, 2004, who find very significant estimates of human capital externalities). Alternative models of endogenous growth, based on the Lucas (1988)'s archetype, consider the formation of human capital through individual training decisions that compete with productive activities.

[11] For a model with productive expenditure, see, e.g. Menuet et al. (2017).

are assumed to be a constant fraction of aggregate output $\Pi_t = vY_t$. At this stage, there are two exogenous parameters (g and v) and two endogenous policy instruments in Eq.(7): public debt (D_t), and the tax rate (τ_t). To close the model, one instrument has to be exogenously specified. To this end, we suppose that the government follows a fiscal rule

$$\dot{D}_t = \theta Y_t, \tag{9}$$

where θ is the deficit-to-GDP ratio.[12]

2.4. Equilibrium

To find endogenous growth solutions, we deflate all growing variables by the capital stock to obtain long-run stationary ratios, namely (we henceforth omit time indexes): $y_k := Y/K$, $c_k := C/K$ and $d_k = D/K$.

From (4), (5), and (7), we obtain the equilibrium level of output

$$y_k = A\left(\frac{(1-\alpha)(1-\tau)}{c_k}\right)^{\psi}, \tag{10}$$

where $\psi := \frac{(1+\phi)(1-\alpha)}{1+\varepsilon-(1+\phi)(1-\alpha)} > 0$,[13] and $A := \tilde{A}\left(\frac{\tilde{A}N^{\varepsilon}}{B}\right)^{\psi}$.

The inverse relationship between the consumption ratio and the output ratio in Eq (10) comes from the labor market equilibrium (4). As the consumption ratio increases, the marginal utility of consumption decreases, thus inducing the representative household to substitute leisure for working hours (since $\varepsilon \geq 0$, leisure and consumption are complement in equilibrium). As a result, the equilibrium labor supply and output are reduced. The same arises following an increase in the tax rate, which reduces net real wage.

As the production function exhibits constant returns-to-scale at the individual level, and decreasing returns in all private factors, labor demand is normal, i.e. decreasing with real wage. Thus our indeterminacy results do not rest on a positively-sloped labor-demand curve, contrasting with Benhabib and Farmer (1994); Farmer and Guo (1994).[14]

[12] Such a deficit rule is discussed in Minea and Villieu (2012); Menuet et al. (2017).

[13] The denominator is strictly positive under the sufficient unnecessary condition $\beta < \alpha$, that we assume throughout the paper.

[14] In Benhabib and Farmer, 1994, p. 30, a necessary condition for indeterminacy is that (using our notations): $(1+\phi)(1-\alpha) > 1+\varepsilon$. This implies that the aggregate labor demand has to be increasing with real wages (see Eqs. (6) and (7) with $(1+\phi)(1-\alpha)-1 > \varepsilon \geq 0$). For the labor demand to slope up with real wages, increasing returns must be important, as discussed by Benhabib and Farmer (1994) and Schmitt-Grohé (1997). In our model, as we have seen, we assume $(1+\phi)(1-\alpha) < 1+\varepsilon$, such that labor demand is normal, i.e. decreasing with real wages. We nevertheless obtain indeterminacy, thanks to the constant social return of capital at equilibrium. In addition, in our model, indeterminacy is consistent with lowly-increasing social returns, as illustrated by our quantitative analysis in section 7 (see Benhabib and Farmer, 1999, for a synthesis of several ways to obtain indeterminacy with small increasing returns).

The optimal aggregate consumption behaviour is, from (3) and (4),

$$\frac{\dot{C}}{C} = \alpha y_k - \rho. \tag{11}$$

Using Eqs. (6) and (9), the tax rate on wages is endogenously determined by

$$\tau = \frac{\alpha d_k + g - v - \theta}{1 - \alpha}, \tag{12}$$

and the path of the capital stock is given by the goods market equilibrium

$$\frac{\dot{K}}{K} = (1 - g)y_k - c_k. \tag{13}$$

From (6) and (8), the path of public debt is $\dot{D} = \alpha y_k D + (g - v)Y - \tau(1 - \alpha)Y$. Hence, the reduced-form of the model is obtained by Eqs. (5)-(6)-(9)-(11)-(12)-(13), namely

$$\begin{cases} \dfrac{\dot{c}_k}{c_k} = (\alpha + g - 1)y_k - \rho + c_k, \\ \\ \dfrac{\dot{d}_k}{d_k} = \dfrac{\theta y_k}{d_k} - (1 - g)y_k + c_k, \end{cases} \tag{14}$$

where, from Eqs. (10) and (12)

$$y_k = A\left(\frac{\bar{d} - \alpha d_k}{c_k}\right)^{\psi} =: y_k(c_k, d_k), \tag{15}$$

with $\bar{d} := 1 + v - \alpha - g + \theta \geq \alpha d_k \geq 0$.[15]

At equilibrium, any increase in the deficit ratio reduces the output ratio. Indeed, by increasing wage taxation, d_k discourages labor supply. The same mechanism applies in case of increases in public spending, through coefficient $\bar{d}$.

We define a BGP as a path on which consumption, capital, output and public debt grow at the same (endogenous) rate, namely: $\dot{c}_k = \dot{d}_k = 0$ in (14). Thus, for any steady-state i, we have: $\gamma^i := \dot{C}/C = \dot{K}/K = \dot{Y}/Y = \dot{D}/D$, while the real interest rate (r^i) is constant.

We determine the steady-state solutions of the model in section 4, and analyze local and global dynamics in sections 5 and 6, respectively. Beforehand, for the stake of clarity, let us turn our attention to a simple special case without public debt.

[15] Notice that $\bar{d} \geq \alpha d_k$ is a necessary condition for the tax rate on wage to be less than one.

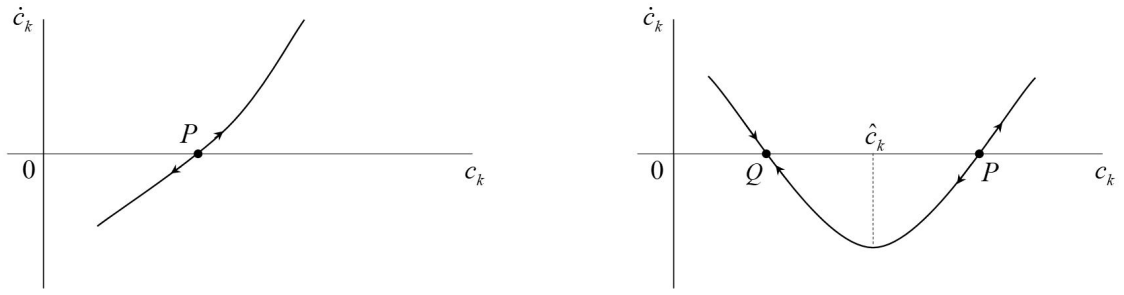
3. A preliminary analysis: the case without public debt

Without public debt, our model is similar to Schmitt-Grohé and Uribe (1997), but in an endogenous growth context. In this special case, $\theta = d_k = 0$ at each instant, and the reduced form of the model (14) boils down to

$$\frac{\dot{c}_k}{c_k} = \frac{\dot{C}}{C} - \frac{\dot{K}}{K} = (\alpha + g - 1)y_k(c_k) - \rho + c_k, \tag{16}$$

where $y_k(c_k) = A\left(\frac{\bar{d}}{c_k}\right)^{\psi}$.

To fix ideas, suppose that $\psi = 1$. Ignoring (for the moment) the degenerate solution $c_k = 0$, relation $\dot{c}_k = 0$ then has two real solutions: $c_k^P = \frac{1}{2}\left[\rho + \sqrt{\rho^2 - 4A\bar{d}\,(\alpha + g - 1)}\right]$ and $c_k^Q = \frac{1}{2}\left[\rho - \sqrt{\rho^2 - 4A\bar{d}\,(\alpha + g - 1)}\right]$, provided that $\rho^2 > 4A\bar{d}\,(\alpha + g - 1)$. If $\alpha + g < 1$, c_k^Q is negative, and there is one unique positive long-run solution c_k^P. As $d\dot{c}_k/dc_k > 0$ this solution is unstable (see Figure 1a). If $\alpha + g > 1$, both solutions are positive, and $\dot{c}_k$ has a minimum at $\hat{c}_k = \sqrt{A\bar{d}\,(\alpha + g - 1)}$. Therefore, c_k^P is unstable and c_k^Q is stable (see Figure 1b). Since the consumption ratio c_k is a jump variable, the steady state P associated to c_k^P is locally determined, while the steady state Q associated to c_k^Q is locally undetermined.



1a: $\alpha < 1 - g$ 1b: $\alpha > 1 - g$

Figure 1: Phase portrait in the no-debt case

The intuition of these results is as follows. A necessary and sufficient condition for a BGP to be stable is that $d\dot{c}_k/dc_k < 0$. Yet, in Eq. (16), an increase in the consumption ratio c_k exerts two effects.

First, capital accumulation ($\dot{K}/K$) is reduced, because consumption is higher (*direct effect*). This rises the law of motion of the consumption ratio $\dot{c}_k/c_k$.

Second, the output ratio y_k decreases, as we have seen (*indirect effect*), with two consequences: (i) the return of capital ($r = \alpha y_k$) is reduced, which affects the consumption path ($\dot{C}/C$) in the Keynes-Ramsey relationship; (ii) simultaneously, the capital path

$(\dot{K}/K)$ is also affected in the goods market equilibrium, through the public spending puncture $((1-g)y_k)$.

If $1-g>\alpha$, the second consequence (ii) outweighs the first (i), so that the indirect effect also rises the law of motion of the consumption ratio $\dot{c}_k/c_k$. In this case, both the direct and the indirect effects play in the same direction, and we have: $d\dot{c}_k/dc_k>0$; hence the local instability of steady state P. To circumvent this unstable dynamics, the consumption ratio must jump initially to its long-run value c_k^P. This makes P locally determined, as in Figure 1a (there is no aggregate instability in the form of SGU, 1997).

If $1-g<\alpha$, the first consequence (i) outweighs the second (ii), so that the indirect effect leads to a decrease in the law of motion of the consumption ratio $\dot{c}_k/c_k$. In this case, the direct and the indirect effects play in opposite directions, and we cannot, *a priori*, assert unambiguously the sign of $d\dot{c}_k/dc_k$. However the condition $d\dot{c}_k/dc_k=0$ is precisely the minimum $\hat{c}_k$ of $\dot{c}_k$, such that the direct and the indirect effects offset each other. Clearly, since $c_k^Q<\hat{c}_k<c_k^P$ (see Figure 1b), the direct effect dominates at point P, while the indirect effect dominates at point Q.[16] Thus $d\dot{c}_k/dc_k>0$ in the vicinity of P, and $d\dot{c}_k/dc_k<0$ in the vicinity of Q. This makes steady-state P locally determined, and steady-state Q locally undetermined.

Indeed, in the neighborhood of Q, any initial value of the consumption ratio can be chosen, which makes the adjustment path to Q subject to sunspots.[17] If, during the transition path, households expect a high return of capital, they reduce initial consumption to increase saving. This reduces the marginal utility of leisure and rises labor supply, which will, in turn, produce a high return of capital. If they expect a low return of capital and increase initial consumption, this expectation will also be validated at equilibrium, because the marginal utility of leisure will rise, thus discouraging labor supply. Such self-fulling prophecies are the mechanism that drives the indeterminacy of point Q.

Our findings challenge Schmitt-Grohé and Uribe (1997)'s result. In a neoclassical exogenous growth model, SGU show that *aggregate instability*, defined as the local indeterminacy of the perfect-foresight equilibrium, occurs when taxes are levied on labor income. In our endogenous growth setup, their analysis needs to be amended on two levels. First, in the case with low public spending ($g<1-\alpha$), the perfect-foresight BGP is unique and well-determined, such that there is no aggregate instability. Second, in the case with high public spending ($g>1-\alpha$), our aggregate-instability result covers a broader class of mechanisms than in SGU, because it relies both on local *and on global* indeterminacy. Indeed, as there are two possible reachable BGPs in the long run (P and Q), and c_k is a jumpable variable, not only the transition path to solution Q is undeter-

[16] The consumption ratio is very small at this steady state, so that the direct effect does not have much strength.

[17] In deterministic perfect-foresight models, local indeterminacy can be associated to the existence of sunspot equilibria (see, e.g., Woodford, 1986a,b; Matsuyama, 1991; Benhabib and Farmer, 1999).

mined, but also the long-run solution towards which the economy converges in the long run (due to the multipliticy of BGPs).

The following sections extend and generalize these findings in the general version of the model with public debt.

4. Steady states in the general case with public debt

With public debt, long-run endogenous growth solutions can be described by two relations between c_k and d_k.

The first one is the $\dot{c}_k = 0$ locus, which comes from the Keynes-Ramsey relation (11) and the IS equilibrium (13)

$$d_k = d_1(c_k) = \frac{1}{\alpha}\left\{\bar{d} - c_k\left(\frac{\rho - c_k}{(\alpha + g - 1)A}\right)^{1/\psi}\right\}. \tag{17}$$

The second relation is the $\dot{d}_k = 0$ locus, related to the government's budget constraint (8), and the deficit rule (9)

$$\theta y_k(c_k, d_k) = [(1-g)y_k(c_k, d_k) - c_k]d_k. \tag{18}$$

In this section, we establish analytical results for the case $\theta = 0$, which characterizes the balanced-budget rule, associated with no deficit (but possibly to a positive inherited public debt, i.e. $d_{k0} \geq 0$). Numerical simulations in section 7 show that our results continue to hold for reasonable values of the deficit ratio $\theta > 0$.

Steady-state solutions are characterized by the crossing-point of Eqs. (17) and (18). A trivial solution, denoted by point D, is associated to $c_k = 0 =: c_k^D$ and $d_k = \bar{d}/\alpha := d_k^D$ (in this case, we have $y_k^D = 0$). The couple (c_k^D, d_k^D) is such that the economy asymptotically vanishes. Although this "catastrophic" solution might be seen as not economically attractive, it cannot be rejected without assessing local and global dynamics of the model, as we will see.[18]

Let us now study the solutions associated to (strictly) positive consumption and output. First of all, if $\theta = 0$, Eq. (17) leads to

$$\gamma(c_k, d_k)d_k = 0,$$

where $\gamma := (1-g)y_k(c_k, d_k) - c_k$ is the economic growth rate. Disregarding negative long-run growth rate, this condition implies that either the long-run public debt is positive and the associated growth rate is zero (*zero-growth solution*, such as $d_k > 0 \Rightarrow \gamma(c_k, d_k) = 0$), or the long-run growth rate is positive, with zero public debt ($\gamma(c_k, d_k) > 0 \Rightarrow d_k = 0$).

[18] Clearly, households' preferences are defined only for $c_t > 0$, but the steady state D can be asymptotically reached with $\lim_{t\to+\infty} c_t = 0^+$.

The following theorem describes the long-run solutions of the model.

Theorem 1. *The long-run equilibria are characterized by the following regimes.*

- *Regime $\mathcal{L}$ (low public spending): $g < 1-\alpha$. There is one positive growth solution (point P), and one no-growth solution (point M).*
- *Regime $\mathcal{H}$ (high public spending): $g > 1-\alpha$. There are two critical levels A_1 and A_2 (with $0 < A_1 < A_2$), such that regime $\mathcal{H}$ is subdivided between three cases.*
  - *Regime $\mathcal{H}_1$: $A < A_1$. There is one positive growth solution (point Q), and the degenerate solution (D).*
  - *Regime $\mathcal{H}_2$: $A_1 < A < A_2$. There are two positive growth solutions (points P and Q), the no-growth solution (point M), and the degenerate solution (D).*
  - *Regime $\mathcal{H}_3$: $A > A_2$. There is one no-growth solution (point M), and the degenerate solution (D).*

Proof. We adopt a three-step proof. The first two steps analyze the no-growth (M) and the positive growth solutions (P and Q), respectively, and the last step completes the description of the equilibria.

i. No-growth solution. This solution is characterized by $(1-g)y_k(c_k, d_k) = c_k > 0$. In addition, from the Keynes-Ramsey relation (11), we have: $\gamma = \dot{C}/C = 0 \Leftrightarrow \alpha y_k(c_k, d_k) = \rho$. Consequently, the no-growth solution (denoted by the point M in Figure 1) is characterized by $c_k^M = (1-g)\rho/\alpha$, and, from (17),

$$d_k^M = \frac{1}{\alpha}\left\{\bar{d} - c_k^M \left(\frac{\rho}{\alpha A}\right)^{1/\psi}\right\}. \tag{19}$$

Outstandingly, by Eq. (15), the equilibrium condition $(1-g)y_k(c_k, d_k) = c_k$ leads to the decreasing relation: $c_k = [(1-g)A]^{1/(\psi+1)}[\bar{d} - \alpha d_k]^{\psi/(1+\psi)}$, whose maximum is reached at $d_k = 0$ and $c_k = [(1-g)A]^{1/(\psi+1)}\bar{d}^{\psi/(1+\psi)} =: \bar{c}_k$. Those steady states associated with $c_k > \bar{c}_k$ are such that long-run economic growth is negative, which we exclude.

ii. Positive growth solutions. As $d_k = 0$, we have, using Eq. (17),

$$\Phi(c_k) := \bar{d} - c_k \left(\frac{c_k - \rho}{(1-\alpha-g)A}\right)^{1/\psi} = 0. \tag{20}$$

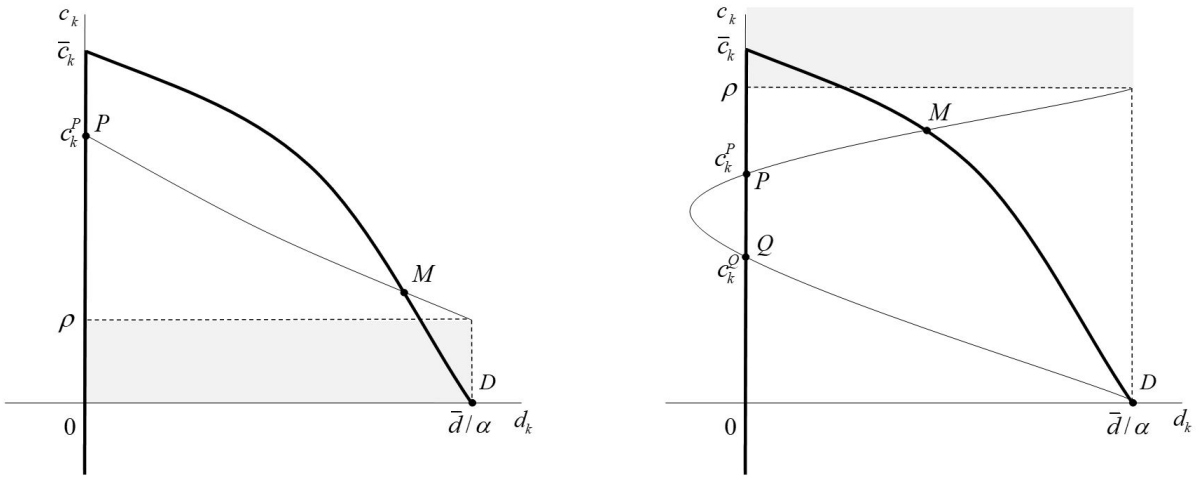
As demonstrated in Appendix 1, if $g < 1-\alpha$, there is a unique root (denoted by point P in Figure 2a), for all $A > 0$. In contrast, if $g > 1-\alpha$, there is a threshold A_2, such that there are two roots (denoted by points P and Q in Figure 2b) iff $A < A_2$. These results generalize the findings obtained in the simple model of section 3.

iii. Existence. The point M characterizes a solution if: (i) $d_k^M > 0$, namely if $\bar{d} > \frac{\rho}{\alpha}\left(\frac{\rho}{\alpha A}\right)^{1/\psi}$, and (ii) $c_k^P < \bar{c}_k$, where $\bar{c}_k := \left[(1-g)A\bar{d}^{\psi}\right]^{1/(1+\psi)}$. To sump up, the set of steady-states are fully characterized by four configurations.

- $g < 1-\alpha$. There are two steady-states: the no growth (M) and the positive growth (P) solutions[19] (see Figure 2a) – defining the regime $\mathcal{L}$.

- $g > 1-\alpha$. As shown in Appendix 1, there is a threshold A_1, where $0 < A_1 < A_2$, such that: $c_k^P < \bar{c}_k$ iff $A > A_1$. Therefore, we can distinguish three subcases according to the value of A.

  - If $A < A_1$, as $c_k^P > \bar{c}_k$, there are two steady-states: the degenerate (D), and the positive growth solution Q – defining the regime $\mathcal{H}_1$ (see Figure 4.2 in section 6).
  - If $A_1 < A < A_2$, as $c_k^P < \bar{c}_k$, there are four steady-states: the degenerate (D), the two positive-growth (Q and P), and the no-growth (M) solutions (see Figure 2b) – defining the regime $\mathcal{H}_2$.
  - If $A > A_2$, as $c_k^P < \bar{c}_k$, there are two steady-states: the degenerate (D), and the no-growth (M) solutions – defining the regime $\mathcal{H}_3$ (see Figures 6 in section 6). □

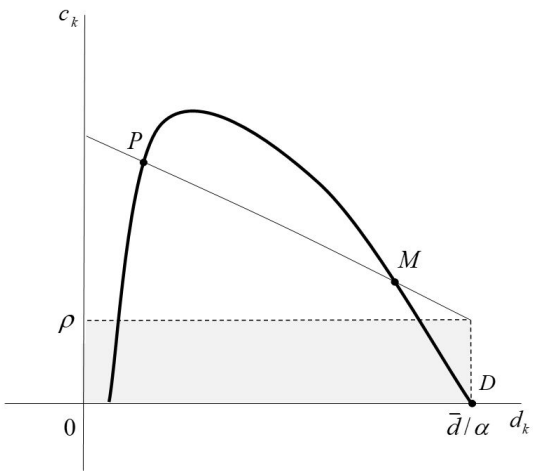


2a: Regime $\mathcal{L}$ ($g < 1-\alpha$)

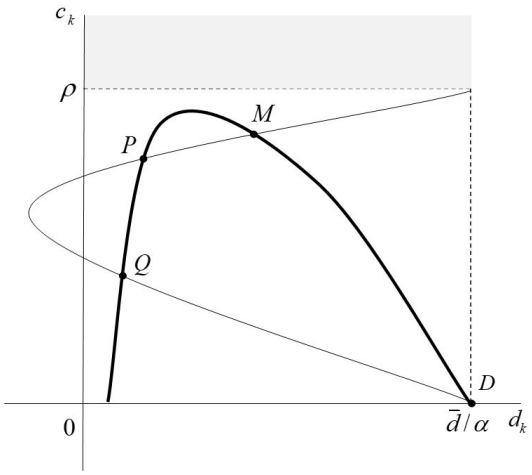
2b: Regime $\mathcal{H}_2$ ($g > 1-\alpha$)

Figure 2.1: The Steady States ($\theta = 0$)

[19]The point M is the unique crossing-point of Eqs. (17) and (18) in the open interval $d_k \in (0, \bar{d}/\alpha)$, provided that ρ is small enough. Effectively, we have (i) $d_k^M > 0$, since $\bar{d} > 0 \approx \frac{\rho}{\alpha}\left(\frac{\rho}{\alpha A}\right)^{1/\psi}$, and, from (20), (ii) $c_k^P = [(1-\alpha-g)A\bar{d}^{\psi}]^{1/(1+\psi)} < [(1-g)A\bar{d}^{\psi}]^{1/(1+\psi)} =: \bar{c}_k$.



2c: Regime $\mathcal{L}$ ($g < 1 - \alpha$)



2d: Regime $\mathcal{H}_2$ ($g > 1 - \alpha$)

Figure 2.2: The Steady States ($\theta > 0$)

Notice that, since $c_k^Q < c_k^P$ and $d_k^Q = d_k^P = 0$, the output ratio and the return of capital are higher at point Q than at point P. The BGP Q is then associated with a higher growth rate than P.

For positive values of the deficit ratio (namely $\theta > 0$), results are qualitatively unchanged, as show Figures 2c and 2d. The major change is that the $\dot{d}_k = 0$ locus is now a hump-shaped curve, which intersect the $\dot{c}_k = 0$ locus, two, three, of four times, depending on parameters. The existence conditions of the different long-run solutions are only slightly amended, as shows our numeric results in section 7.

Fundamentally, the multiplicity that arises in our model comes from the interaction between two non-linear relationships linking the consumption and the deficit ratios.

The first one is related to the government's budget constraint ($\theta y_k = \gamma d_k$) that can be rewritten as $\theta = \dot{D}/Y = d_k[(1 - g) - c_k/y_k]$. This relation describes a bell-curve bewteen c_k and d_k. Indeed, there are two conflicting forces as d_k increases. First, the deficit-to-output ratio rises $\dot{D}/Y$ (direct effect). Second, the tax rate increases, leading to a decrease of y_k, namely an increase of c_k/y_k in the bracketed term. This in turn reduces equilibrium economic growth, hence the deficit-to-output ratio $\dot{D}/Y$ (indirect effect).

As the deficit-to-output ratio is constant at θ on the BGP, the increase of d_k exerts a nonlinear effect on the consumption ratio c_k. Consequently, due to the budget rule, each value of c_k is consistent with two values of d_k: a small value associated to high economic growth, and a high value associated to low growth.

The second relation comes from the assumption of the balanced-growth in the long run (namely $\dot{K}/K = \dot{C}/C = \gamma$). According to the Keynes-Ramsey relationship (12), and

the IS equilibrium (13), this condition amounts to $(1-g)y_k - c_k = \alpha y_k - \rho$, or

$$c_k - \rho = (1 - g - \alpha)y_k(c_k, d_k). \tag{21}$$

As we have seen, y_k negatively depends on d_k and c_k through the labour market equilibrium (see Eq. (15)). The relation between c_k and d_k then crucially depends on the sign of $1 - g - \alpha$, as emphasized in section 3. If $g < 1 - \alpha$ (regime $\mathcal{L}$), an increase of c_k decreases the RHS of (21), and rises the LHS, thus generating an unambiguously monotonic decreasing relation between c_k and d_k, as depicted in Figures 1a-2a. If $g > 1-\alpha$ (regime $\mathcal{H}$), both sides of Eq. (21) positively depend on c_k, which produces a non-monotonic relationship between c_k and d_k. Consequently, any deficit ratio is associated with two consumption ratios.

The role of the condition $g > (<)1 - \alpha$ is intuitive. As d_k increases, the growth rates of consumption $(\dot{C}/C)$ and private capital $(\dot{K}/K)$ decreases, through an adverse effect on output. However, the impact of the output ratio on the growth rate of consumption depends on the return of capital (α), while its impact on the growth rate of capital depends on public spending $(1 - g)$. Therefore, if $1 - g > (<)\alpha$, the investment-goods sector is more (less) sensitive than the consumption-goods sector to a change of d_k, and the consumption ratio c_k must adjust in order to restore the equality $\dot{K}/K = \dot{C}/C$ along the BGP.

Let us now study local dynamics.

5. Local dynamics

By linearization, in the neighborhood of steady-state i, $i \in \mathscr{S} = \{D, M, P, Q\}$, the system (14) behaves according to $(\dot{c}_k, \dot{d}_k) = \mathbf{J}^i(c_k - c_k^i, d_k - d_k^i)$, where $\mathbf{J}^i$ is the Jacobian matrix. The reduced-form includes one jump variable (the consumption ratio c_{k0}) and one pre-determined variable (the public-debt ratio d_{k0}, since initial stocks of public debt D_0 and private capital K_0 are predetermined). Thus, for BGP i to be well determined, $\mathbf{J}^i$ must contain two opposite-sign eigenvalues. Using (14), when $\theta = 0$, we compute

$$\mathbf{J}^i = \begin{pmatrix} CC^i & CD^i \\ DC^i & DD^i \end{pmatrix},$$

where

$$CC^i = c_k^i[1 + (\alpha + g - 1)yc^i], \tag{22}$$
$$CD^i = c_k^i(\alpha + g - 1)yd^i, \tag{23}$$
$$DD^i = -\gamma^i - (1 - g)yd^i d_k^i, \tag{24}$$
$$DC^i = -(1 - g)yc^i d_k^i + d_k^i, \tag{25}$$

with, using (15),

$$yc^i := \frac{\partial y_k^i}{\partial c_k^i} = -\frac{\psi y_k^i}{c_k^i} < 0, \text{ and } yd^i := \frac{\partial y_k^i}{\partial d_k^i} = -\frac{\alpha\psi y_k^i}{\bar{d} - \alpha d_k^i} < 0. \tag{26}$$

Hence, the trace and the determinant of the Jacobian matrix are, respectively

$$\mathrm{Tr}(\mathbf{J}^i) = c_k^i[1 + (\alpha + g - 1)yc^i] - \gamma^i - (1 - g)yd^i d_k^i, \tag{27}$$

$$\det(\mathbf{J}^i) = -c_k^i[\gamma^i + (1 - g)d_k^i yd^i + (\alpha + g - 1)(\gamma^i yc^i + d_k^i yd^i)]. \tag{28}$$

The following theorem establishes the topological behaviour of each steady-state.

Theorem 2. *(Local Stability)*

- *In regime $\mathcal{L}$, M is locally over-determined (unstable), and P is locally determined (saddle-point stable).*
- *In regime $\mathcal{H}$, D is locally determined (saddle-point stable). The other steady-states are characterized as follows.*
  - *In regime $\mathcal{H}_1$, Q is locally under-determined (stable).*
  - *In regime $\mathcal{H}_2$, P is locally determined (saddle-point stable), Q is locally under-determined (stable), and M is locally over-determined (unstable).*
  - *In regime $\mathcal{H}_3$, there is a critical public spending level $g^h > 1 - \alpha$, such that*
    - *If $1 - \alpha < g < g^h$, M is locally over-determined (unstable),*
    - *If $g = g^h$, a Hopf bifurcation occurs,*
    - *If $g^h < g$, M is locally under-determined (stable).*

Proof. We study the local stability of steady-states in each regime.

(i) Regime $\mathcal{L}$.

At steady-state P, we have $d_k^P = 0$, thus: $\mathrm{Tr}(\mathbf{J}^P) = c_k^P[1 + (\alpha + g - 1)yc^P] - \gamma^P$, and $\det(\mathbf{J}^P) = -c_k^P\gamma^P[1 + (\alpha + g - 1)yc^P]$. As $g < 1 - \alpha$ and $yc^P < 0$, $\det(\mathbf{J}^P) < 0$, namely there are two opposite-sign eigenvalues. Consequently, P is saddle-point stable.

At steady-state M, $c_k^M > 0$, $d^M > 0$ and $\gamma^M = \alpha y_k^M - \rho = 0$, namely $\mathrm{Tr}(\mathbf{J}^M) = c_k^M[1 + (\alpha + g - 1)yc^M] - (1 - g)yd^M d_k^M$, and $\det(\mathbf{J}^M) = -\alpha c_k^M d_k^M yd^M > 0$. As $yc^M < 0$ and $yd^M < 0$, we have $\det(\mathbf{J}^M) > 0$, and $\mathrm{Tr}(\mathbf{J}^M) > 0$, and there are two positive eigenvalues. Consequently, M is locally unstable.

(ii) Regime $\mathcal{H}$.

First, Appendix 2 shows that the degenerate point D is a saddle-point.

Second, let us consider the two solutions with positive economic growth. At steady-states P and Q, we have $d_k^i = 0$, thus: $\mathrm{Tr}(\mathbf{J}^i) = c_k^i[1 + (\alpha + g - 1)yc^i] - \gamma^i$, and $\det(\mathbf{J}^P) = -c_k^i\gamma^i[1 + (\alpha + g - 1)yc^i]$ for $i = P, Q$. Since $DC^i = 0$, there is one negative

eigenvalue ($\lambda_1^i = -\gamma^i$) and one eigenvalue that changes sign, depending on the considered equilibrium ($\lambda_2^i = c_k^i[1+(\alpha+g-1)yc^i]$). With $yc^i = -\psi y_k^i/c_k^i$ and $c_k^i = \rho-(\alpha+g-1)y-k^i$ at steady states $i = P, Q$, we obtain $\lambda_2^i := \lambda_2(c_k^i) = c_k^i+\psi(c_k^i-\rho)$. Thus $\lambda_2^i(\hat{c}_k) = 0$, where $\hat{c}_k := \psi\rho/(1+\psi)$ is the minimum of Φ on $[0, \rho)$, see Eq. (20). Since $c_k^Q < \hat{c}_k$ and $c_k^P > \hat{c}_k$, it follows that $\lambda_2^Q < 0$ and $\lambda_2^P > 0$. Consequently, P is characterized by two opposite-sign eigenvalues and is locally determined (saddle-point stable), while Q is characterized by two negative eigenvalues and is locally undetermined (stable). This analysis generalizes the simple case of section 3.

Third, in the neighbourhood of the no-growth trap (point M), the dynamics are more complicated. As $yc^M = -\psi/(1-g) < 0$ and $yd^M < 0$, we have $\det(\mathbf{J}^M) > 0$, as in the case $g < 1-\alpha$. However, the Trace of the Jacobian matrix now changes sign at

$$1-(1-g)yd^M d_k^M/c_k^M = (1-\alpha-g)yc^M. \tag{29}$$

Therefore, if $\mathrm{Tr}(\mathbf{J}^M) > 0$, $\mathbf{J}^M$ has two positive eigenvalues and M is over-determined, while if $\mathrm{Tr}(\mathbf{J}^M) < 0$, $\mathbf{J}^M$ has two negative eigenvalues and M is under-determined. The Hopf bifurcation arises for $\mathrm{Tr}(\mathbf{J}^M) = 0$. At this point, a periodic orbit through a local change in the stability properties of M appears. Since $DD^M > 0$, a necessary condition for $\mathrm{Tr}(\mathbf{J}^M)$ to change sign is $CC^M < 0$. Hence the public spending ratio must be higher than g^m, where: $g^m := \frac{1+\psi(1-\alpha)}{1+\psi} > 1-\alpha$. Then, by inspection of relations (26)-(29), we can establish the following lemma.

Lemma 1. *The Hopf bifurcation occurs at the unique value $g^h > g^m > 1-\alpha$, such that*

$$g^h = \frac{A_0(1-\alpha\psi)+\psi(1-\alpha+v)}{A_0+\psi}.$$

where $A_0 := A(\rho/\alpha A)^{(1+\psi)/\psi}$.[20]

□

Figure 3 illustrates the topological behavior of point M according to the public spending ratio.

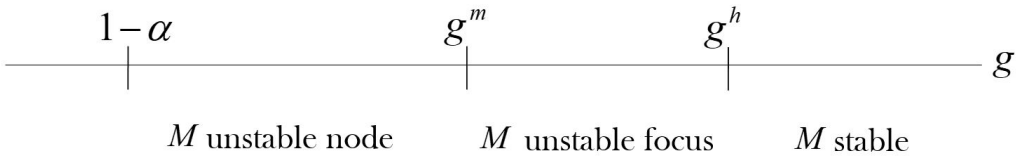


Figure 3: Topological behavior of the no-growth solution

[20]To ensure that $g^h > g^m$, we assume that $\alpha(1+\psi A_0) < v(1+\psi)$.

An important feature of the model is that the Hopf bifurcation can arise only in the case where the steady states associated with positive BGP (namely, P and Q) do not exist, namely in regime $\mathcal{H}_3$. Effectively, the Hopf bifurcation exists only in the case $g^h > g^m$, such that the $\dot{c}_k = 0$ locus must be positively sloped at M, as in Figure 6.2 below. This excludes the existence of points P and Q.

The major difference between regimes $\mathcal{H}$ and $\mathcal{L}$ is the presence or not of the catastrophic equilibrium D. In regime $\mathcal{L}$ this point is not relevant, because it cannot be reached unless the economy is initially at zero. Effectively, for all positive value of c_k, $\mathcal{L}$ is defined only if $c_k > \rho$. Therefore, as negative consumption ratios are excluded, steady state D cannot be reached. This is no longer the case in regime $\mathcal{H}$, since this regime is characterized by $c_k < \rho$.

As regards the local stability of the positive growth solutions P and Q, the difference between the two equilibria only comes from the sign of the term $CC^i = \partial \dot{c}_k / \partial c_k$ in the Jacobian matrix. As in section 3, steady state Q is locally undetermined because $CC^Q < 0$; hence the law of motion of consumption is stable (so as the law of motion of public debt). On the contrary, as $CC^P > 0$, the law of motion of consumption is unstable in the neighborhood of point P, which makes P a determined (saddle-path stable) solution.

Finally, the dynamics around the no-growth solution can be explained by the law of motion of public debt, which becomes unstable in the neighborhood of point M (in the sense of $DD^M = \partial \dot{d}_k / \partial d_k > 0$). Effectively, with zero growth, the snowball effect of the debt burden cannot be avoided. Thus, if $CC^M > 0$, the no-growth solution is locally unstable, while a cyclical dynamic appears if $CC^M < 0$. In the latter case, if public spending is high enough ($g > g^h$), the no-growth solution becomes locally stable and can be reached, but at the price of (possibly large) oscillations during the transition path.

Thanks to local analysis, we can now turn to global dynamics.

6. Global dynamics

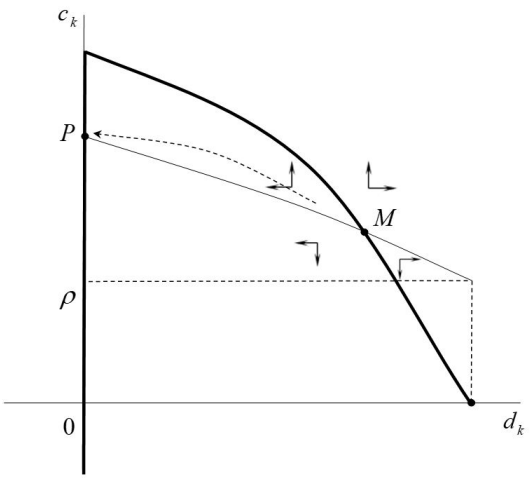
According to local analysis, we can distinguish four cases, depending on parameters, and especially the public spending ratio.

Regime $\mathcal{L}$ – In this case, there are two steady-states, but only the high BGP (P) can be reached in the long-run, as the no-growth trap (M) is unstable. Thus, there is no local or global indeterminacy.[21] As in case without public debt studied in section 3, if public spending is low (regime $\mathcal{L}$), the unique equilibrium (namely, the positive growth BGP

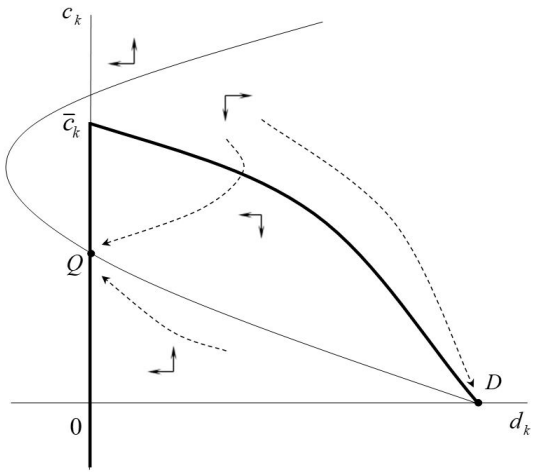
[21] The initial public debt ratio exerts a threshold effect: if $d_{k0} < d_k^M$, for any predetermined d_{k0}, the consumption ratio c_{k0} jumps to place the economy on the saddle-path that converges towards P, which defines the unique long-run equilibrium. In contrast, if $d_{k0} > d_k^M$, there is no long-run solution.

P), is locally and globally well determined. Therefore, in contrast with SGU, aggregate instability disappears (see Figure 4.1).

Regime $\mathcal{H}_1$ – In this regime, there are two steady states (since we exclude steady states with negative growth, i.e. above $\bar{c}_k$). One is associated to positive economic growth (Q, with low consumption and deficit ratios) and is stable, while the "catastrophic" equilibrium D is saddle-path stable. Consequently, there is both local (in the vicinity of Q), and global indeterminacy (see Figure 4.2).



4.1: Regime $\mathcal{L}$



4.2: Regime $\mathcal{H}_1$

Figure 4: Global dynamics

Regime $\mathcal{H}_2$ – This regime can be seen as the synthesis of the two preceding. As we have seen, there are four steady states, whose properties are unchanged: P and D are saddle-path stable, Q is stable, and M is unstable. Thus, this regime is characterized by a local indeterminacy (in the vicinity of Q), and global indeterminacy that comes from two facts. First, given a (predetermined) public debt ratio $d_{k0} < d_k^M$, the initial consumption ratio can jump on the unique transition path that converges towards P or on one of the multiple paths that converges towards Q. Second, if $d_{k0} > d_k^M$, one of the latter paths can still be reached, as well as the unique path that converges towards the catastrophic equilibrium D (see Figure 5). Yet, in both cases, the long-run equilibrium of the economy is subject to "animal spirits", in the form of optimistic or pessimistic views of households at the initial time.

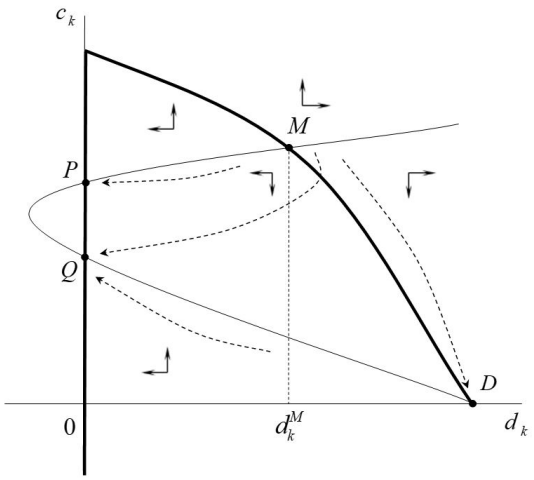


Figure 5: Global dynamics – Regime $\mathcal{H}_2$

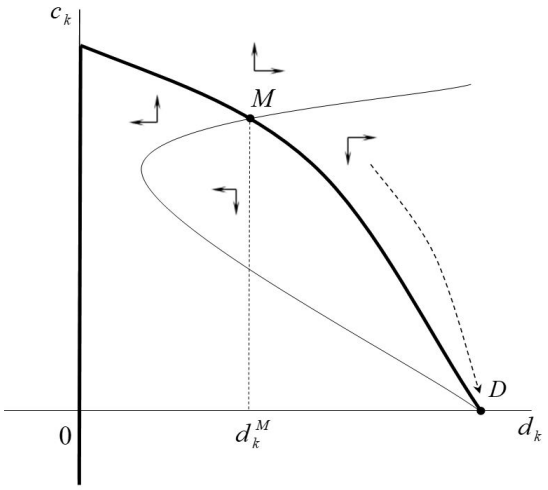
The mechanisms that make Q locally undetermined are the same as those outlined in the no-debt case (section 3). Focusing here on global dynamics, the multiplicity that arises in regimes $\mathcal{H}_1$ and $\mathcal{H}_2$ results, as usual in non-linear models with forward-looking households, from self-fulfilling prophecies that generate multiple equilibrium growth paths in the future.

Suppose, e.g., that, at the initial time, households expect low public debt in steady-state. This implies that the expected tax rate is low, and the expected net return of capital is high. At the initial time, households then increase their savings, such that the initial consumption ratio (c_{k0}) is low, and the initial hours worked will be high. This means that, in equilibrium, labor supply will also be high, generating large fiscal resources and low public debt in the future (along P and Q BGPs). Conversely, following the same mechanism, high expected public debt is self-fulfilling, and may lead to the no-growth solutions M or D. In other words, by their consumption-leisure tradeoff at the initial time, households can, in equilibrium, validate any expectation on the BGP that can be reached in the future.

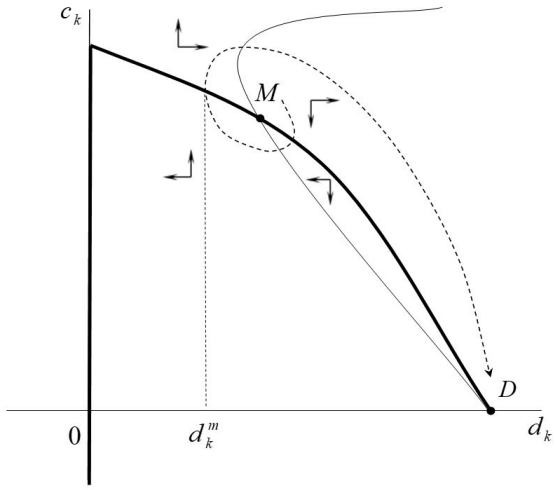
Two results deserve particular attention. First, despite that the low regime is well-determined, the higher growth solution (Q) cannot be reached. This explain why governments can be induced to increase public spending until reaching regime $\mathcal{H}$. Second, in regime $\mathcal{H}_2$, indeterminacy cannot be avoided, unless the positive long-run solution disappears. Effectively, one cannot obtain positive BGP P without the undetermined solution Q. Thus, local and global indeterminacy can be viewed as the price that must be paid to generate a positive long-run growth solution.

Regime $\mathcal{H}_3$ – In this case, $A > A_2$ and long-run solutions with positive growth are eliminated.[22] Thus, there are only two steady states, M and D. The latter is still locally determined, but the topological behavior of the no-growth trap M depends on the level of the public spending ratio. We can distinguish two situations.

(i) If $1 - \alpha < g < g^h$, there is no local and global indeterminacy. The no-growth trap M is an unstable node (if $1 - \alpha < g < g^m$, see Figure 6.1) or focus (if $g^m < g < g^h$, see Figure 6.2), and, provided that $d_{k0} > d_k^M$ or $d_{k0} > d_k^m$ (where d_k^m is the leftmost point of the unique trajectory that converges to D), the economy converges towards the catastrophic equilibrium D.[23]



6.1: $1 - \alpha < g < g^m$



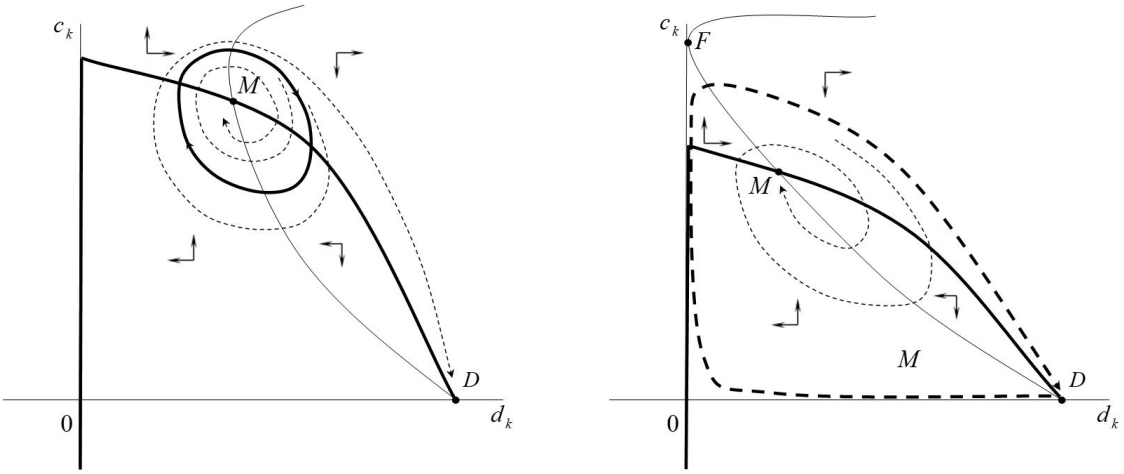
6.2: $g^m < g < g^h$

Figure 6: Global dynamics – Regime $\mathcal{H}_3$

(ii) If $g^h < g$, on the contrary, the no-growth solution M is stable, thus there is both local and global indeterminacy. At $g = g^h$, the Hopf bifurcation arises. Simulations on ©matcont show that the first Lyapunov coefficient is positive, defining a sub-critical bifurcation (see the numeric section below). Therefore, when g takes values slightly higher than g^h, an unstable closed orbit births; as g increases, this orbit becomes larger. This unstable cycle defines a separatrix orbit between the paths that converge towards the no-growth trap M (inside the closed orbit), and those that diverge from it. Among the latter, there is a unique path that goes towards the catastrophic equilibrium D. Thus, given a predetermined debt ratio, any path that converges toward M or D, or remains on the cycle, can be reached (Figure 7.1).

[22] Since A_2 negatively depends on g for reasonable values of parameters (see Appendix 1), such a case is more likely to arise for relatively high public spending ratios.

[23] If $d_{k0} < d_k^m$, the model has no solution in the long-run.



7.1: $g^h < g$ 7.2: $g = g^{max}$

Figure 7: Global dynamics – Regime $\mathcal{H}_3$

Moreover, as g rises, the closed orbit becomes larger, until it includes point D, as in Figure 7.2. This defines a subcritical Hopf-homoclinic bifurcation. In such a case, the economy can reach either the no-growth trap with large oscillations during the transition period or the homoclinic orbit that passes "through" D, with a cycle of (asymptotically) infinite period.[24]

Fundamentally, the interaction between Households' saving behavior and fiscal rules can give rise to very complicated dynamics, especially when public spending is "high". The next section shows that these results hold for reasonable values of parameters.

7. A quantitative exploration

In this numerical section, we generalize our analytical results by considering positive deficit rules $\theta \geq 0$. Our numerical results are based on reasonable values for parameters. We interpret the time period as 5-year average. We choose $\rho = 0.05$, corresponding to a 1% annual risk free (real) interest rate and the labor elasticity of substitution is fixed at $\varepsilon = 0$, thus characterizing an infinite Frisch elasticity. This choice corresponds to Schmitt-Grohé and Uribe (1997). Regarding the technology, we set $A \in (0.01, 0.1)$ to obtain realistic rates of economic growth, and the capital share in the production function is $\alpha = 0.75$, close to the value (0.715) used by Gomme et al. (2011). As the model is driven by an AK technology, capital should be interpreted broadly as a composite of physical and human capital. If the share of human capital is about, e.g., 60%, the share

[24]The birth of the homoclinic orbit is linked to the point F, where the $\dot{c} = 0$ locus intersects with the $c_k = 0$ axis. In such a situation, all orbits are forced to turn around on $\dot{c} = 0$, and can not escape up.

of output going to physical capital will be 30%. The measure of human capital intensity in the accumulation of knowledge is set to $\beta \in (0, 0.75)$ (with a benchmark at 0.5), to produce the different cases that can arise in Regime $\mathcal{H}_3$.[25]

Regarding the government's behavior, the deficit ratio is $\theta \in (0, 0.03)$, consistent with long-run average values in the US or OECD from 1950 to 2015, and the value of the public spending ratio will be scanned over a large range of values to verify the presence (or not) of a Hopf bifurcation in Regime $\mathcal{H}_3$. In our benchmark calibration, lump-sum taxation is assumed to be 20% of GDP (namely, $v = 0.2$). For these parameters' values, the corresponding rate of wage taxation (in percent of GDP) is between 10% and 16.5%, depending on the equilibrium considered.

Table 1 shows that our analytical results are qualitatively unchanged with a positive deficit rule. Our numeric results are rather realistic. The annual real interest rate is between 1% and 15%, depending on the considered equilibrium, and the long-run rate of economic growth in the different steady-states are hard-nosed: between 1% and 3% at point P, between 8% and 10% at point Q, and close to zero at point M.

	M			**P**			**Q**		
	$\theta = 0$	$\theta = 1\%$	$\theta = 2.5\%$	$\theta = 0$	$\theta = 1\%$	$\theta = 2.5\%$	$\theta = 0$	$\theta = 1\%$	$\theta = 2.5\%$
c_k	0.0467	0.0463	0.0454	0.0408	0.0413	0.0422	0.0092	0.00912	0.009
d_k	0.117	0.12	0.12	0	0.0216	0.0584	0	0.0145	0.0363
$\tau wl/Y$	18%	17.77%	16.65%	10%	10.62%	11.88%	10%	10.09%	10.22%
$Debt/GDP$	175%	158%	128%	0	12.41%	37.57%	0	1.77%	4.42%
γ *(per year)*	0	0.13%	0.4%	1.70%	1.56%	1.30%	9.32%	9.35%	9.38%

Table 1: Steady-state values in the benchmark calibration (Regime $\mathcal{H}_3$).
($g = 0.3$, $\alpha = 0.75$, $A = 0.05$, $\rho = 0.05$, $v = 0.2$, $\varepsilon = 0$, and $\beta = 0.5$)

The tax rate on wage is around 40%, and as a percent of GDP, total tax revenues (lump-sum plus wage taxes) are around 30% at points P and Q, and 37% at point M, consistent with estimates for developed countries.[26]

In the benchmark calibration, regime $\mathcal{L}$ appears for $g < 0.25$, while, in the opposite case, regime $\mathcal{H}_1$ prevails for $A < A_1 \approx 0.01$, regime $\mathcal{H}_3$ for $A > A_2 \approx 0.083$, and regime $\mathcal{H}_2$ in the intermediate interval. Figure 8 depicts the values of the consumption ratio in the different steady states as a function of A.

Interestingly, regime $\mathcal{H}_2$ can occur without the need of high social returns-to-scale in the aggregate production function. With $g = 0.15$, $A = 0.051$, $\rho = 0.05$, $v = 0.2$, $\varepsilon = 0$,

[25] The corresponding values for ϕ are $\phi \in (0, 3)$.

[26] In 2017, the tax wedge on wage – the sum of personal income tax, employee and employer social security contributions plus any payroll tax less cash transfers expressed as a percentage of labour costs – was 36% on average in OECD countries.

for example, regime $\mathcal{H}_2$ is consistent with $\beta = 0$ and $\alpha = 0.99$, namely for almost constant returns-to-scale $(1 + (1 + \rho)(1 - \alpha) = 1.01)$. Therefore, in our model, multiplicity can arise even if returns-to-scale are close to constant, as empirical evidence suggests (see, e.g. Basu and Fernald, 1997). In contrast Benhabib and Farmer (1994) need increasing returns in excess of 1.43.

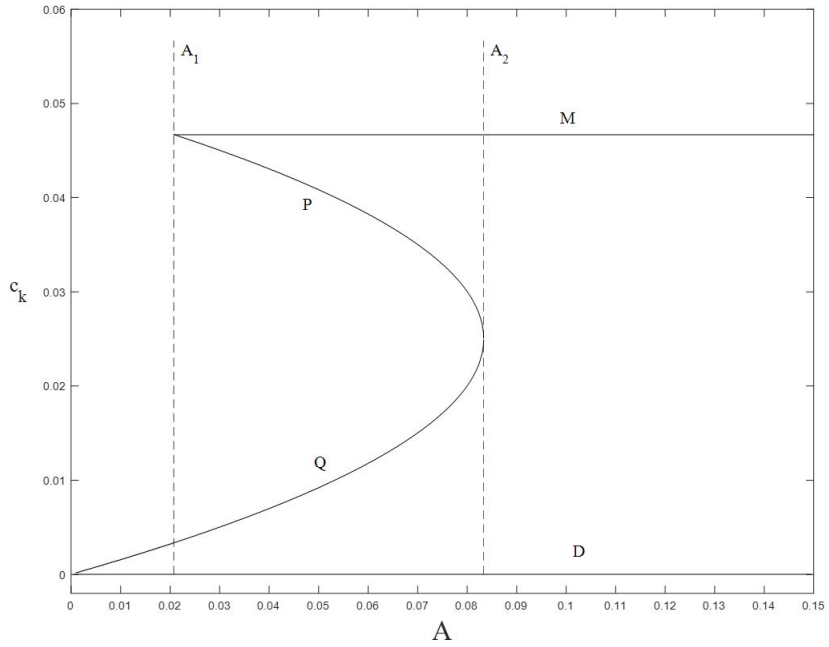


Figure 8: Bifurcation diagram as a function of A (regime $\mathcal{H}$)

From a cyclical analysis perspective, the most interesting configuration is the regime $\mathcal{H}_3$. This regime occurs when human capital externalities are high enough (in our simulations below, we take $\beta = 0.7$). The value of the public spending ratio that corresponds to the Hopf bifurcation around the low-growth trap is $g^h \simeq 40.547\%$, such that, for slightly higher values of g, a closed orbit appears (the corresponding Lyapunov coefficient is equal to 105, confirming the presence of a sub-critical Hopf bifurcation). As g increases, the (unstable) cycle becomes larger, as in Figure 9. Increasing g further, the periods of these periodic orbits tend to infinity, and, at the limit, we find the homoclinic orbit, which connects the saddle equilibrium to itself. This homoclinic orbit is composed of the unstable and stable manifolds of the saddle equilibrium D. This homoclinic orbit is obtained for $g = g^H \simeq 40.783\%$. If g rises further, point D can no longer be reached.

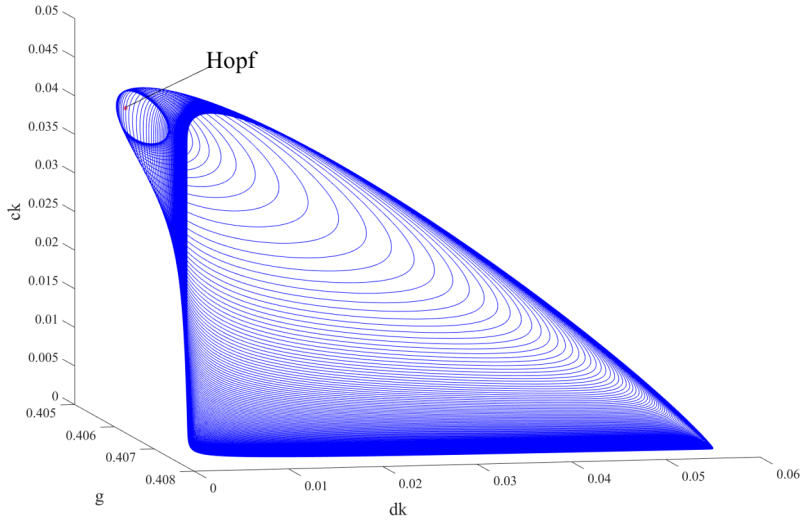


Figure 9: The family of cycles as g becomes larger

Figure 10a and 10b depict two typical configurations for $g = 40.65\% \in (g^h, g^H)$. There is a periodic orbit such that the public debt ratio oscillates between 0.018 and a value close to (but strictly superior than) zero. As the cycle is unstable, inside the orbit the economy converges to the no-growth trap M (Figure 10a), while outside the orbit it goes to the catastrophic equilibrium D (Figure 10b). In the first case, the public debt ratio remains relatively weak in the long-run, while it takes a high value (and the debt-to-GDP ratio becomes virtually infinite) in the second case.

Clearly, since the initial consumption ratio is a free jumpable variable, both situations are possible, departing from a predetermined public debt ratio $d_{k0} < \bar{d}/\alpha$. The economy then is subject to households' optimistic or pessimistic views on the future.

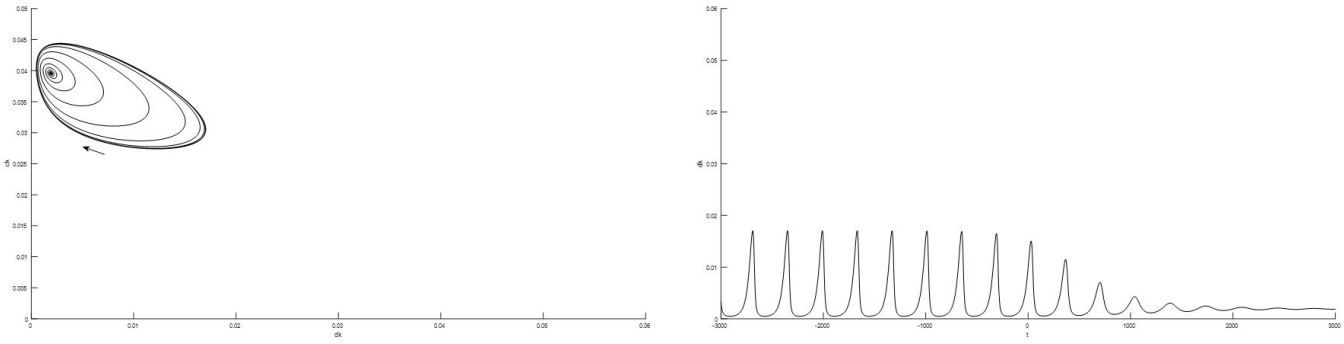


Figure 10a: Inside the periodic orbit: exit from the cycle to M ($g = 0.4065$)

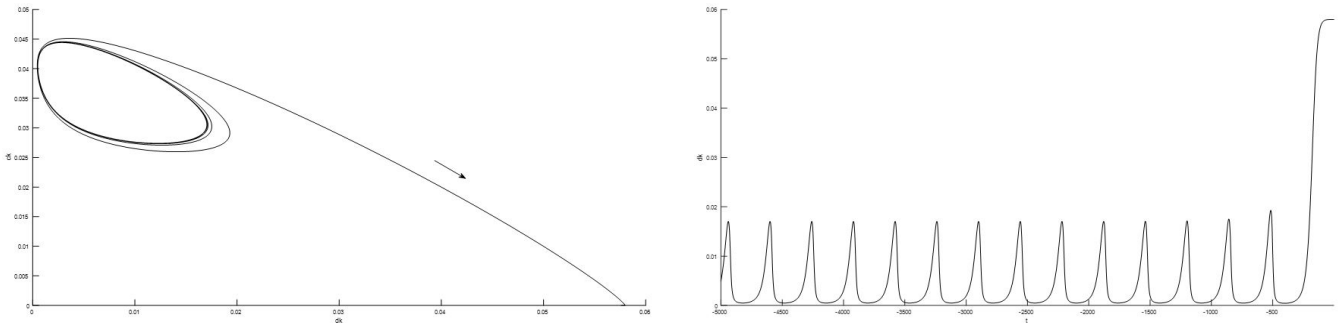


Figure 10b: Outside the periodic orbit: exit from the cycle to D ($g = 0.4065$)

As g rises, the amplitude of the cycle increases, but its frequency decreases. At the limit, when g approaches g^H, the period of the cycle becomes asymptotically infinite, and the economy converges more and more slowly towards the catastrophic equilibrium D. For $g = 0.407827$, e.g., the cycle already lasts more than 1000 periods (see Figure 11). From an empirical point of view, it would be difficult, then, to distinguish the cyclical nature of the economy, and not to consider it as a monotonic convergence towards D.

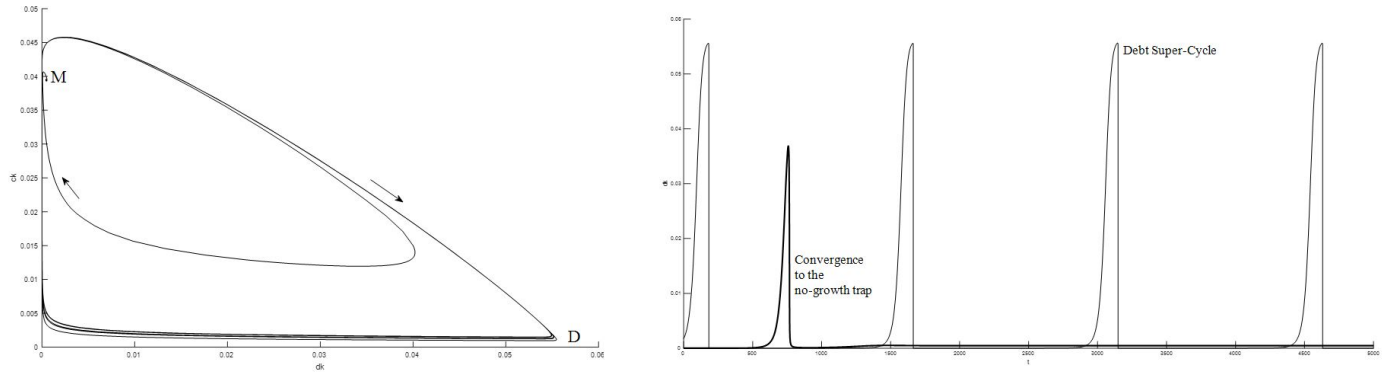


Figure 11: A debt "Supercycle" close to the homoclinic orbit

This configuration recalls the notion of "debt super-cycle" developed by Rogoff (2015), suggesting that economies experiment large oscillating debt cycles during crisis periods, rather than secular stagnation. Our model also develops two alternative views of the future: a convergence to a low-growth trap, or a large debt cycle characterized by long periods of economic atony, as in Figure 11.[27] As shows Figure 11, debt super-cycles generate sudden periodic debt crises and thus capture the "heart attack" (in Rogoff's words) experienced by the global economy during the Great Recession.

[27] If the first view can be associated to a stagnationist perspective (Hansen, 1939), the second is closer to the harrodian's vision (Harrod, 1939; Domar, 1944).

8. Conclusion

The seminal contribution of SGU motivated a large literature analyzing the consequences of fiscal policy in terms of aggregate fluctuations. Compared to this rich and influencing literature that rests on a balanced-budget rule (BBR), we adopt an endogenous growth framework that allows dealing with more general budget rules, in which the possible existence of deficit and debt in the long-run turn the BBR into a special case. Under this more general framework, our results are threefold.

First, SGUs indeterminacy result is modified in the presence of endogenous growth, even with a BBR. Indeed, the equilibrium is determined provided that exogenous public spending is below a certain threshold. Conversely, indeterminacy arises above this threshold, but under more complex forms than in SGU; in this case, two BGPs exist, and our model exhibits both local and global indeterminacy. Second, although the equilibrium is still determined with low public spending when we relax the BBR, the presence of deficit and debt yields up to four equilibria in the long run when public spending is above a threshold. Third, when public spending is sufficiently large, complex dynamics emerge around the low-BGP trap (i.e. a subcritical Hopf bifurcation), which triggers a homoclinic orbit going around the neighborhood of the catastrophic equilibrium.

Our model has several policy implications. Compared to SGU, public spending presents a crucial role for aggregate fluctuations, despite being held exogenous. Indeed, indeterminacy can be avoided provided that public spending is maintained to a sufficiently low level. However, in this case, the economy cannot reach the highest BGP. On the contrary, this BGP may be attained provided that public spending is large. However, in this case, the model displays local and global indeterminacy: attempting to put the economy on the highest BGP comes with the cost of aggregate fluctuations, and even a possible growth trap. Finally, when public spending are sufficiently large, our model can comfort recent propositions asserting the presence of a debt super-cycle in our current indebted economies (Rogoff, 2015), since large oscillations arise around the low-growth trap.

From a general perspective, our analysis opens the door for reassessing the consequences of fiscal policy in terms of aggregate fluctuations, in the presence of a major component of nowadays Governments fiscal policies, namely deficits and debt, on at least two grounds. On the one hand, some of the conclusions of the previous literature may have to be revisited in the presence of deficit and debt. Evaluating the (in)determinacy effects of endogenous public spending, progressive taxes, or alternative specifications of preferences are some handful examples. On the other hand, the complex effects triggered by our simple budget rule make the case for exploring alternative fiscal rules, all the more given their increase popularity since the recent crisis (see, e.g., Combes et al., 2017; Menuet et al., 2017). These two possible directions are left for future research.

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Annexe 1: Characterization of positive growth solutions

By (20), positive growth solutions are given by

$$\Phi(c_k) := \bar{d} - c_k \left(\frac{c_k - \rho}{(1-\alpha-g)A} \right)^{1/\psi} = 0. \tag{.1}$$

(a) *If* $g < 1-\alpha$. Clearly, $\Phi \in C^1((\rho, +\infty))$ and Φ is a decreasing function. As $\Phi(\rho) = \bar{d} > 0$, and $\lim_{c_k \to +\infty} \Phi(c_k) = -\infty$, according to the Intermediate Value Theorem, there is a unique point $\check{c}_k \in (\rho, +\infty)$, such that $\Phi(\check{c}_k) = 0$. The point $P = (0, \check{c}_k)$ characterizes a steady-state if and only if $c_k^P < \bar{c}_k = [(1-g)\bar{d}^{\psi}]^{1/(1+\psi)}$, which is true for small discount rates (see footnote 6).

(b) *If* $g > 1-\alpha$. In this case, $\Phi \in C^1([0, \rho))$, and

$$\Phi'(c_k) = \left[\frac{\rho - c_k}{(\alpha+g-1)A} \right]^{-1+1/\psi} \left[\frac{1+\psi}{\psi(\alpha+g-1)A} \right] (c_k - \hat{c}_k),$$

where $\hat{c}_k = \psi\rho/(1+\psi) < \rho$ is the minimum of Φ on $[0, \rho)$.

Consequently, $\Phi'(c_k) < 0$ if $c_k \in [0, \hat{c}_k)$ and $\Phi'(c_k) > 0$ if $c_k \in (\hat{c}_k, \rho)$. As $\Phi(0) = \Phi(\rho) = \bar{d} > 0$, according to the Intermediate Value Theorem, there are two roots: $c_k^Q \in (0, \hat{c}_k)$ and $c_k^P \in (\hat{c}_k, \rho)$ if and only if

$$\Phi(\hat{c}_k) = \bar{d} - \hat{c}_k \left(\frac{\rho - \hat{c}_k}{(\alpha+g-1)A} \right)^{1/\psi} < 0 \tag{.2}$$

If (.2) is true, $Q = (c_k^Q, 0)$ is a steady-state, and, if $c_k^P < \bar{c}_k$, $P = (c_k^P, 0)$ is also a steady-state. As shown by the following lemma, these two existence conditions can be expressed according to the value of A.

Lemma 2. *Let* $g > 1-\alpha$. *There are two critical levels* A_1, A_2 $(0 < A_1 < A_2)$ *such that:*

- *If* $A < A_1$, *there are two roots, and* $c_k^P > \bar{c}_k$.
- *If* $A_1 < A < A_2$, *there are two roots, and* $c_k^P < \bar{c}_k$.
- *If* $A > A_2$, *there is no root.*

Proof. First, from (11), as $\hat{c}_k$ does not depend on A, $A \mapsto \Phi(\hat{c}_k)$ is an increasing continuous function on $(0, +\infty)$, where $\lim_{A \to 0^+} \Phi(\hat{c}_k) = -\infty$, and $\lim_{A \to +\infty} \Phi(\hat{c}_k) = \bar{d} > 0$. Consequently, according to the Intermediate Value Theorem, there is a unique value $A_2 > 0$, such that $\Phi(\hat{c}_k) < 0$ (namely, there are two roots: regimes $\mathcal{H}_1$ or $\mathcal{H}_2$) if $A < A_2$; and $\Phi(\hat{c}_k) > 0$ (there are no roots: regime $\mathcal{H}_3$) if $A > A_2$. As Figure A1 depicts, the value A_2 is such that the two positive growth solutions (P and Q) coincide

$(A = A_2 \Rightarrow\ c_k^P = c_k^Q = \hat{c}_k)$; hence

$$A_2 = \left(\frac{\hat{c}_k}{\bar{d}}\right)^{\psi} \left(\frac{\rho - \hat{c}_k}{\alpha + g - 1}\right)$$

Second, we compute

$$c_k^P < \overline{c}_k := [(1-g)A\bar{d}^{\psi}]^{1/(1+\psi)} \Leftrightarrow \Phi(\overline{c}_k, A) = \bar{d} - \overline{c}_k \left(\frac{\rho - \overline{c}_k}{(\alpha + g - 1)A}\right)^{1/\psi} > 0$$

On the one hand, $\overline{c}_k$ is an increasing continuous function with respect to A ($\bar{d}$ does not depend on A). On the other hand, we have $\partial_A \Phi(\overline{c}_k, A) > 0$, and $\partial_{c_k} \Phi(\overline{c}_k, A)$ as $\overline{c}_k > \hat{c}_k$;[28] hence Φ is an increasing function with respect to A. Besides, using $\kappa := [(1-g)\bar{d}^{\psi}]^{1/(1+\psi)}$, we compute

$$\Phi(\overline{c}_k, A) = \bar{d} - \frac{\kappa}{A^{1/(\psi(1+\psi))}} \left(\frac{\rho - \kappa A^{1/(1+\psi)}}{\alpha + g - 1}\right) \to -\infty \text{ when } A \to 0,$$

and, $\lim_{A \to +\infty} \Phi(\overline{c}_k, A) = \bar{d} + \left(\frac{\kappa^{1+\psi}}{\alpha+g-1}\right)^{1/\psi} > 0$. Consequently, according to the Intermediate Value Theorem, there is a unique value $A_1 > 0$, such that: $c_k^P < \overline{c}_k$ if $A > A_1$ and $c_k^P > \overline{c}_k$ if $A < A_1$. As Figure A1 shows, A_1 is such that the higher positive growth solution (P) and the no growth solution coincide ($A = A_1 \Rightarrow\ c_k^P = c_k^M = \overline{c}_k$).

Third, we have $A_1 < A_2$, because $\overline{c}_k > \hat{c}_k$. □

Consequently, if $A < A_1 < A_2$, solutions P and Q are present, but P is not a crossing-point, since $c_k^P > \overline{c}_k$. In this case, there is only one positive-growth steady-state: Q (regime $\mathcal{H}_1$). If $A_1 < A < A_2$, P and Q are crossing-points, and characterize positive-growth solutions (regime $\mathcal{H}_2$). Finally, if $A > A_2$, P and Q do not exist, and there is no positive-growth solution (regime $\mathcal{H}_3$).

In this way, there is a bifurcation at $A = A_1$ and $A = A_2$, as depicted in Figure 1A. Indeed, at $A = A_1$, the system changes from regime $\mathcal{H}_1$ to regime $\mathcal{H}_2$, and at $A = A_2$, the system changes from $\mathcal{H}_2$ to $\mathcal{H}_3$. The bifurcation-diagram will be numerically characterized in Section 6.

[28] Indeed, $\hat{c}_k$ positively and linearly depends on ρ, while $\overline{c}_k$ is independent of ρ. Thus, if ρ is small enough, if follows that $\overline{c}_k > \hat{c}_k$.

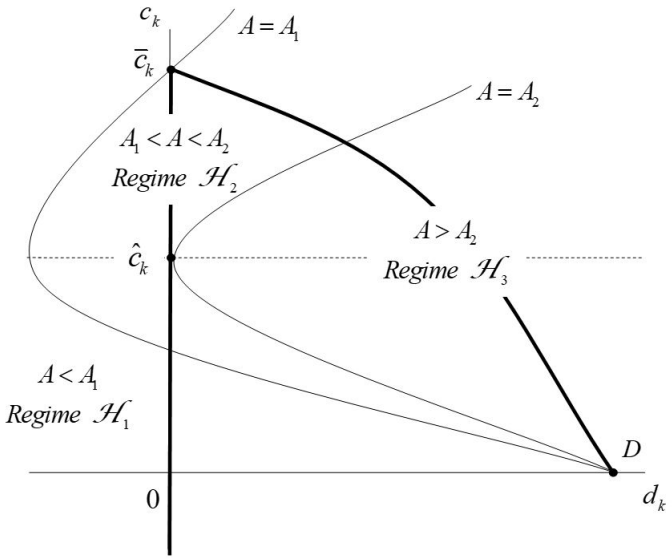


Figure 2: Existence ($g > 1 - \alpha$, $\theta = 0$)

Annexe 2: Topological behaviour of point D

To show that $D = (0, d_k^D)$ is a saddle-point, we must ensure that: (i) for $c_k = 0$, $\dot{d}_k < 0$ if $d_k < d_k^D$; (ii) for $d_k = d_k^D := \bar{d}/\alpha$, $\dot{c}_k < 0$ if $c_k > 0$ (we exclude the case $c_k < 0$ and $d_k > d_k^D$, since y_k is not defined in the latter case).

(i) Let $m > 0$. By considering the point $(0, d_k^D - m)$, according to the global stability analysis of section 5 (see Figure 4), the deficit ratio decreases, because $(0, d_k^D - m)$ lies below the $\dot{d}_k = 0$ locus.

(ii) Let $h > 0$. By considering the point (h, d_k^D), according to the global stability analysis, the consumption ratio decreases, because (h, d_k^D) lies above the $\dot{d}_k = 0$ locus.

Finally, we conclude that the point $(0, d_k^D)$ is a saddle-point.