Nonlocal damage formulation with evolving internal length: the Eikonal nonlocal approach

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ABSTRACT: The proposed contribution presents and investigates the numerical properties of a Eikonal Nonlocal (ENL) continuum damage model. According to this approach, nonlocal interactions between material points are controlled by geodesic distances obtained as solutions of an isotropic time-independent Eikonal equation with a damage dependent metric function. Nonlocal interactions in two-dimensional damaged domains are illustrated first. A numerical formulation for modeling damage dependent non-local interactions within mechanical computations is presented then. It is obtained by using a Fast-Marching Method for updating damage dependent nonlocal interactions throughout the strain localization process. Numerical results of quasi-static simulations involving the failure of quasi-brittle materials in isotropic media are presented. Regularization properties of the proposed model are demonstrated. Furthermore, it is shown that the proposed formulation allows for reducing several parasite effects classically associated with Integral Nonlocal (INL) formulations (damage spreading over large damaged bands, damage diffusion near notches and free-edges, etc).

1 INTRODUCTION

During the development of nonlinearities, the softening behavior of some materials (e.g. ductile failure in metals, quasi-brittle failure in concrete, etc.) leads to the appearance of a localization process zone finite in size. Several theories were proposed to provide a description of these phenomena (G. & Bazant 1987, Pijaudier-Cabot & Benallal 1993, Bažant & Jirásek 2002, Frémond & Nedjar 1996, Peerlings, Geers, de Borst, & Brekelmans 2001, Miehe, Welschinger, & Hofacker 2010, Moës, Stolz, Bernard, & Chevaugeon 2011). Their common feature consists in the introduction of an internal length expressing nonlocal interactions in the localization process zone. Furthermore, these methods allow to avoid problems of non-objective results (mesh dependency) that can appear when using a finite element method for the solution of the quasi-static boundary value problem.

Integral non-local (INL) formulations on the internal variables of the constitutive model (G. & Bazant 1987), in particular, are widely used due to their simplicity of implementation, strong theoretical background and numerical robustness. According to this approach the thermodynamic variable driving the damage evolution process on a material point is computed by weighted averaging of the corresponding local field over the entire domain. Averaging is performed through a nonlocal weighting function (e.g. a Gaussian distribution function), such that the higher is the Cartesian distance between two material points lower is their interaction. A main drawback of this assumption consists however, in nonphysical interactions of material points across damaged bands, cracks and holes. Indeed, any couple of material points such that the Cartesian distance between them is the same interact in the same way. From a numerical viewpoint, this induces some parasite effects, such as damage spreading over a large damaged band, damage diffusion near notches and free-edges, etc. Enhancements of the initial methods were proposed in order to face these limitations and to describe more and more precisely strain localization processes in softening media. Among a lot of papers in the literature, several propose an evolution of the internal length based on phenomenological considerations (Geers, De Borst, Brekelmans, & Peerlings 1998, Pijaudier-Cabot, Haidar, & Dubé 2004, Simone, Wells, & Sluys 2003, Nguyen 2011, Giry, Dufour, & Mazars 2011, Saroukhani, Vafadari, & Simone 2013).

Theoretically derived in (Desmorat, Gatuingt, & Jirásek 2015) and numerically implemented/studied in (Rastiello, Giry, Gatuingt, & Desmorat 2017), the Eikonal Non-Local (ENL) formulation provides a novel interpretation of damage dependent evolving non-local interactions. From a mathematical point of view, interaction distances between material points are computed by solving an isotropic time-independent Eikonal equation (a stationary Hamilton-Jacobi equation) with a damage dependent Riemannian metric function. From a differential geometry viewpoint, this approach leads to consider that damage induces a curvature of the Riemannian space in which interaction distances are computed. This space is thus no more Euclidean, and distances increase eventually tending to infinity. From a numerical viewpoint, ENL damage models can be implemented by coupling: 1) a nonlinear Finite Element Method (FEM) for solving the continuum damage mechanics problem; 2) a Fast-Marching Method (FMM) (Sethian 1996) for evaluating damage dependent interaction distances over the computed structure. This mathematical/physical framework allows for directly modeling evolving interactions throughout the localization process.

In this paper, a simple ENL Damage model and its numerical implementation are presented first. Then, after discussing on nonlocal interaction in damaged media some simple quasi-static strain localization problems are simulated in order to show the main features of the proposed formulation.

2 ENL DAMAGE FORMULATION

Consider a n-dimensional domain $\mathcal{B}$ and suppose that the mechanical behavior of its constituting material can be described through an isotropic Continuum Damage Model (CDM) with a single scalar variable $d \in [0, 1]$. Under small-strain conditions, the second order stress tensor ($\boldsymbol{\sigma}$) is written as:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\epsilon}, d) = (1 - d)\left[2\mu\boldsymbol{\epsilon} + \lambda(\text{tr}\boldsymbol{\epsilon})\mathbf{I}\right] \tag{1}$$

where tr($\bullet$) is the trace operator, (λ, μ) are the homogeneous Lamé parameters, $\boldsymbol{\epsilon}$ is the second order small strain tensor and $\mathbf{I}$ is the second order identity tensor.

The nonlocal damage criterion function is defined as[1]:

$$f^{\text{NL}} = f(\epsilon_{\text{eq}}^{\text{NL}}, \kappa) = \epsilon_{\text{eq}}^{\text{NL}} - \kappa \tag{2}$$

where $\epsilon_{\text{eq}}^{\text{NL}} = \epsilon_{\text{eq}}^{\text{NL}}(\boldsymbol{x})$ is a nonlocal equivalent strain measure (Mazars 1984, de Vree, Brekelmans, & van Gils 1995) and κ is an internal variable. This latter starts at a (damage) threshold level κ_0 and is updated by requiring that $f^{\text{NL}} = 0$ during damage growth, while $\dot{\kappa} = 0$ at unloading and when $f^{\text{NL}} < 0$. It is therefore updated as:

$$\kappa = \max_t \epsilon_{\text{eq}}^{\text{NL}} \tag{3}$$

where $t \in [0, T]$ is a pseudo-time variable. Damage growth is finally supposed to follow the exponential evolution law:

$$d = g(\kappa) = 1 - \frac{\kappa_0}{\kappa}\exp\left(-\frac{\langle \kappa - \kappa_0 \rangle}{\kappa_c - \kappa_0}\right) \tag{4}$$

where κ_c is the equivalent strain level controlling the shape of the damage evolution function.

2.1 *Nonlocal (NL) strain field*

The nonlocal field $\epsilon_{\text{eq}}^{\text{NL}}$ is computed by weighted averaging of its local couterpart ($\epsilon_{\text{eq}} = \epsilon_{\text{eq}}(\boldsymbol{x})$) over $\mathcal{B}$. Provided a material point occupying the position $\boldsymbol{x}_x$, the averaging formula reads:

$$\epsilon_{\text{eq}}^{\text{NL}} = \epsilon_{\text{eq}}^{\text{NL}}(\boldsymbol{x}_x) = \frac{\int_{\mathcal{B}} \phi\left(\xi_{xs}\right)\, \epsilon_{\text{eq}}(\boldsymbol{x}_s)\, \mathrm{d}v}{\int_{\mathcal{B}} \phi\left(\xi_{xs}\right)\, \mathrm{d}v} \tag{5}$$

where $\phi = \phi(\xi_{xs}) \geq 0$ is a nonlocal weighting function and ξ_{xs} is the ratio of the interaction distance between material points $\boldsymbol{x}_s$ and $\boldsymbol{x}_x$ to the characteristic/internal length ℓ_c. A typical choice of function ϕ is the Gaussian distribution function:

$$\phi = \phi(\xi_{xs}) = \exp(-\xi_{xs}^2/2) \tag{6}$$

2.1.1 *Integral NL model*

In INL formulations, interactions are controlled by Euclidean distances between material points. The length ratio ξ_{xs} thus reads:

$$\xi_{xs} = \xi_{xs}^{\text{INL}} = \frac{\ell_{xs}}{\ell_c} = \frac{\|\boldsymbol{x}_s - \boldsymbol{x}_x\|}{\ell_c} \tag{7}$$

where $\|\bullet\|$ denotes the Euclidean norm. As a consequence of this assumption, any pair of material points $(\boldsymbol{x}_x, \boldsymbol{x}_s)$ and $(\boldsymbol{x}_x, \boldsymbol{x}_s')$ such that $\ell_{xs} = \ell_{xs'}$ interact in the same way. A physical request is, however, that material points across cracks, holes and highly damaged zones do not interact (or at least reduce their interactions).

2.1.2 *Eikonal NL model*

The ENL formulation models this effect by considering that damage induces a curvature of the Riemannian space in which interaction distances are computed. In this framework, interactions between $\boldsymbol{x}_x$ and other material points $\boldsymbol{x} \in \mathcal{B}$ are controlled by an effective/geodesic distances field $\tilde{\ell}(\boldsymbol{x})$ approximating the

[1] This represents a simple modeling assumption, introduced for sake of illustration. The proposed ENL formulation can be applied without modifications to different non-local damage models.

viscosity solution of the time-independent isotropic Eikonal equation:

$$\begin{cases} m(\boldsymbol{x})\,||\nabla\tilde{\ell}(\boldsymbol{x})|| = 1 & \boldsymbol{x} \in \mathcal{B}, \\ \tilde{\ell}(\boldsymbol{x}_x) = 0 \end{cases} \tag{8}$$

where $\nabla(\bullet)$ is the gradient operator and $m(\boldsymbol{x})$ denotes a damage dependent isotropic Riemannian metric function:

$$m(\boldsymbol{x}) = \sqrt{1 - d(\boldsymbol{x})} > 0 \qquad \boldsymbol{x} \in \mathcal{B} \tag{9}$$

This leads to rewrite the length ratio ξ_{xs} as:

$$\xi_{xs} = \xi_{xs}^{\mathrm{ENL}} = \frac{\tilde{\ell}_{xs}}{\ell_c} \geq \xi_{xs}^{\mathrm{INL}} = \frac{\ell_{xs}}{\ell_c} \tag{10}$$

Uni-dimensional example The influence of damage on nonlocal interactions can be illustrated easily by studying simple one-dimensional (1D) case. Consider an inhomogeneous field $d = d(x)$ defined over a bar $\mathcal{B} = \{x \in \mathbb{R} : x \in (0, L)\}$ and compute the geodesic interaction distance between two material points (x_x and $x_s > x_x$) pertaining to $\mathcal{B}$. Under these conditions, the Eikonal problem (8) reads:

$$\begin{cases} \sqrt{1 - d(x)}\,|\mathrm{d}_x\tilde{\ell}(x)| = 1 & x \in \mathcal{B} \\ \tilde{\ell}(x_x) = 0 \end{cases} \tag{11}$$

where d_x denotes the total derivation with respect to variable x and $|\bullet|$ is the absolute value operator.

Integrating (11) between x_x and x_s leads to write:

$$\tilde{\ell}_{xs} = \int_{x_x}^{x_s} \frac{\mathrm{d}x}{\sqrt{1 - d(x)}} \geq \ell_{xs} = x_s - x_x > 0 \tag{12}$$

According to (12), $\tilde{\ell}_{xs}$ coincides with ℓ_{xs} in undamaged conditions (i.e., the INL setting is recovered) and increases progressively when damaging occurs. As a consequence, material points separated by highly damaged zones could no longer interact (or reduce their interactions).

3 NONLOCAL INTERACTIONS IN TWO-DIMENSIONS

Similar considerations apply to non-local interactions in two-dimensional (2D) domains. When the damage field $d(\boldsymbol{x})$ is not uniform, however, no general closed form viscosity solutions to the Eikonal equation exist. For that reason, numerical solution procedures (Bertsekas 1993, Zhao 2005, Sethian 1996) are needed. In this work, we use a FMM (Sethian 1996) based on a second order approximation of the term $\nabla\tilde{\ell}(\boldsymbol{x})$ over a regular grid of points (Rouy & Tourin 1992).

As a representative example, we consider a square plate $\mathcal{B}$ and compute interaction distances from point $\boldsymbol{x}_x$ under three different conditions: 1) $\mathcal{B}$ is undamaged; 2) $\mathcal{B}$ is crossed by a sharp crack, i.e., $d \to 1^-$ along a line of points and is null elsewhere; 3) $\mathcal{B}$ is a holed plate, i.e., $d \to 1^-$ on points located inside the hole and is null otherwise.

The computed geodesic distances fields and resulting nonlocal weighting functions ϕ are depicted in Figure 1. As expected, one observes that:

1. Geodesic and Cartesian distance fields coincide when damage is null (as in the 1D case). The resulting function ϕ is a Gaussian distribution function with center in $\boldsymbol{x}_x$ (as it is classically assumed in INL formulations);

2. When the domain is damaged (or an hole is present), the Riemannian space in which distances are computed is deformed (i.e., $m(\boldsymbol{x}) \neq 1$). Shortest paths between $\boldsymbol{x}_x$ and points $\boldsymbol{x}_s \in \mathcal{B}$ are no more straight lines and interaction distances increase. As in the 1D case, under some conditions, material points separated by highly damaged zones no-more interact (the resulting weighting functions ϕ are truncated).

4 FEM-FMM NUMERICAL FORMULATION

The implementation of the ENL method into a nonlinear finite element code can be achieved in a non intrusive (as less intrusive as possible) way. The main ingredients of the numerical formulation proposed in (Rastiello, Giry, Gatuingt, & Desmorat 2017) can be summarized as follows:

1. Quasi-static equilibrium equations are solved thanks to a standard FE formulation. Provided the displacement and damage fields at time step t_n, the solution at time $t_{n+1} = t_n + \Delta t_{n+1} > t_n$ is searched iteratively by using a secant algorithm. An explicit integration scheme is adopted for updating the damage field at the Gauss point level throughout global iterations. For the Gauss point occupying the position $\boldsymbol{x}_x$, at the global iteration $k+1$, one computes:

$$d_{n+1}^{k+1}(\boldsymbol{x}_x) = \max\left(g(\epsilon_{\mathrm{eq},n+1}^{\mathrm{NL},k}), d_{n+1}^{k}\right) \tag{13}$$

where $d_{n+1}^{k} = d_{n+1}^{k}(\boldsymbol{x}_x)$;

2. Interaction distances between integration points are evaluated through a second-order accurate FMM. For this purpose, independent FM grids are defined gauss point by gauss point. They are centered on the considered Gauss point ($\boldsymbol{x}_x$) and are $2\ell_c \times 2\ell_c$ in size. Provided the finite difference approximation of the gradient term, grid spacing (h) is adapted grid-by-grid in order to ensure minimizing errors in distances computation;

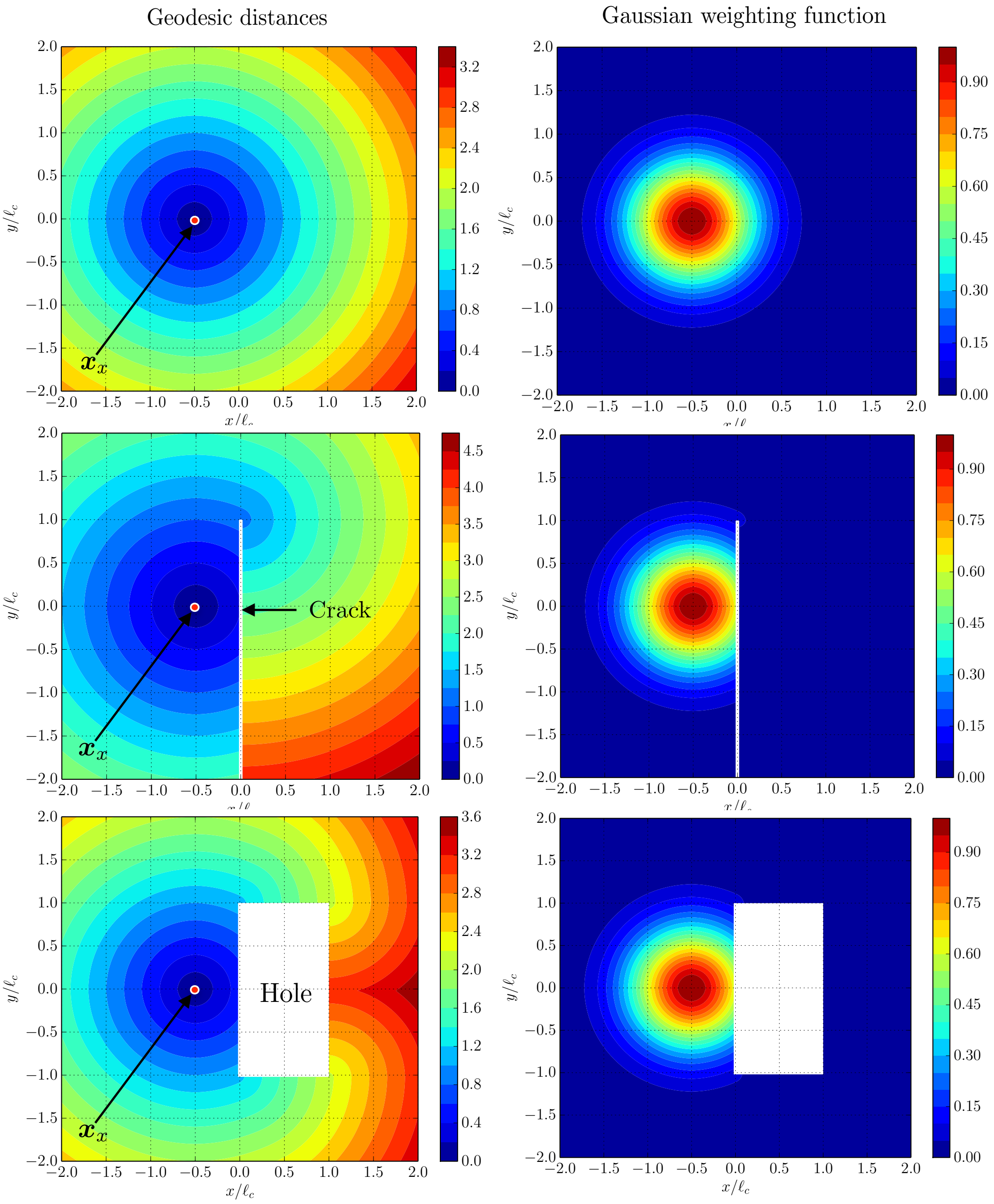

Figure 1: Geodesic distances field $\tilde{\ell}(\boldsymbol{x})$ for a square plate and its influence on the Gaussian weighting function $\phi(\xi)$ centered on the point $\boldsymbol{x}_x$: a) undamaged medium; b) cracked medium; c) holed plate. Geodesic distances are computed over a regular grid comprising $40'401 = 201 \times 201$ vertex (grid spacing $h = 4\ell_c/200$)

3. The discretized metric function to be used for computing interaction distances is obtained after projection of the damage field from the FE mesh to FM grids;

4. Geodesic distances between gauss points are then computed at the beginning of each time step by solving:

$$\begin{cases} \sqrt{1 - d_n(\boldsymbol{x})}\,||\nabla\tilde{\ell}_{n+1}(\boldsymbol{x})|| = 1, & \boldsymbol{x} \in \mathcal{B} \\ \tilde{\ell}_{n+1}(\boldsymbol{x}_x) = 0 \end{cases} \tag{14}$$

   where $d_n(\boldsymbol{x})$ is the damage field a time step t_n;

5. Interaction distances are then kept constant until convergence at time t_{n+1}. They are then used to update the equivalent nonlocal strain field $\epsilon_{\text{eq}}^{\text{NL}}$ driving damage evolution.

5 STRAIN LOCALIZATION EXAMPLES

In this section, 2D quasi-static strain localization problems in quasi-brittle continua are simulated to illustrate the main features (regularization, damage evolution, ...) of the ENL damage formulation. In computations, a simple yield criterion function written in the equivalent Mazars strain space is adopted for sake of simplicity.

5.1 *Tie-specimen under tensile loading*

A tie-specimen submitted to a tensile loading is considered first (Figure 2). The domain is discretized through three FE meshes comprising 26, 51 or 101 linear quadrangular FEs to study mesh sensitivity. Strain localization is forced on the center of the specimen by introducing a weak finite element. Provided the chosen material parameters (Figure 2), the resulting structural responses are unstable in the post-peak phase of load for every considered mesh. For that reason, the external load is controlled indirectly using path-following method based upon controlling the mean relative horizontal displacement of two vertical lines of nodes symmetrically placed ($\pm L/20$) with respect to the vertical symmetry axis of the specimen.

5.1.1 *Representative responses*

Computations are performed assuming plane strains conditions and considering the local, standard INL and ENL formulations for sake of comparison. Representative structural responses obtained for a mesh comprising $51 = 51 \times 1$ elements are compared in Figure 2. Damage and equivalent strain distributions along a horizontal line (parallel to the loading direction) for different time stations are depicted in Figure. 3. Numerical results evidence that:

- When damage is small, the global and local response provided by the ENL formulation is very close to that obtained using the INL formulation. In this phase, the metric field is approximatively equal to unity and effective geodesic distances do not strongly differ from Euclidean ones.

- When damage increases, geodesic distances increase and become larger than Euclidean distances. Non-local interactions progressively reduce and the ENL global response tends progressively to that obtained through the local damage model.

- When $d \rightarrow 1^-$ on the weak finite element, the response provided by the ENL formulation becomes equivalent to that obtained in a local setting. Interaction distances between gauss integration points across the damaged zone tend to infinity and non-local interactions vanish. By this way, no damage evolution occurs even through the sample elongation continues to increase. As it is well known, this effect cannot be modeled through a INL formulation.

5.1.2 *Regularization features*

As shown in Figure 4, the ENL formulation ensures the objectivity of the obtained solution with respect to the spatial discretization of the computed structure. The global force-displacement responses obtained through three different FE meshes are in good agreement in the whole range of displacements.

Small differences in global responses can be observed only on final simulation phases, when the weaker element is almost fully damaged and no more interacts with its neighbors. A tendency toward a mesh convergence can be however evidenced, thus demonstrating the regularization properties of the proposed ENL formulation. Damage profiles obtained for the three FE meshes are also similar for any damage level (Figure 5) .

5.2 *Wedge-splitting test*

A wedge-splitting test (Brühwiler & Wittman 1990) is simulated to study the damage propagation process in a 2D context. A vertically notched sample 100 mm width and 100 mm in height is solicited by imposing increasing horizontal displacements of two vertical bearing surfaces. A sub-vertical damage propagation (form the notch to the bottom of the specimen) is thus induced. The computational domain is discretized by using a finite element mesh comprising 2510 linear quadrilateral finite elements. Computations are performed under plane strain conditions, considering both ENL and INL formulations.

Damage fields obtained corresponding to an advanced phase of the test are compared in Figure 6. This allows showing that the damage field predicted by the INL formulation is spread over a large damaged band, whereas the ENL formulation allows reducing this diffusion. In that case, the damaged band in less wide and d attains unity values on the symmetry axis only. Once this condition is attained, the damage field no more evolves because no interactions occur between integration points located across the symmetry axis. This also ensure that damage diffusion in the backward of the notch is strongly reduced.

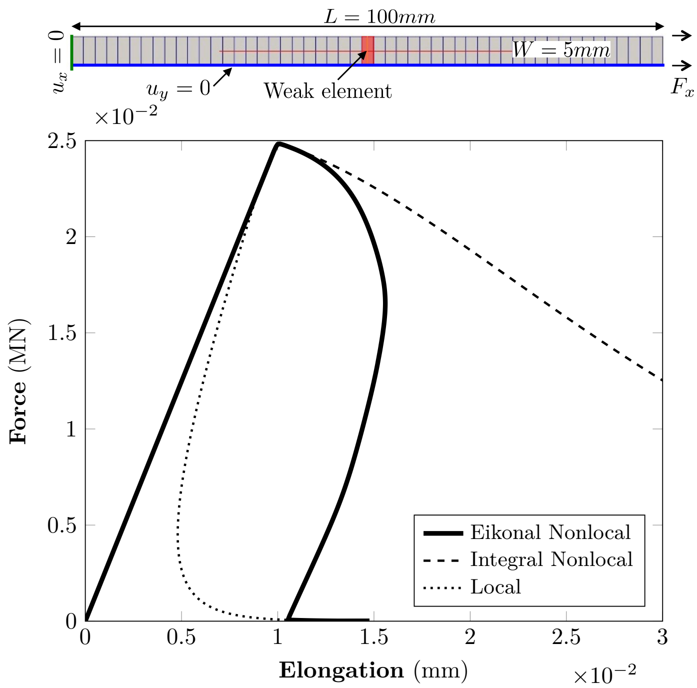

Figure 2: Tensile test. Comparison among representative global force - displacement responses obtained through the local, INL and ENL damage evolution models (FE mesh comprising 51 elements). The specimen in 100 mm in length and 5 mm in width. Constitutive model parameters are as follows: $E = 100$ MPa (Young's modulus), $\nu = 0$, $\kappa_0 = 0.0001$, $\kappa_c = 0.001$ and $\ell_c = 20$ mm.

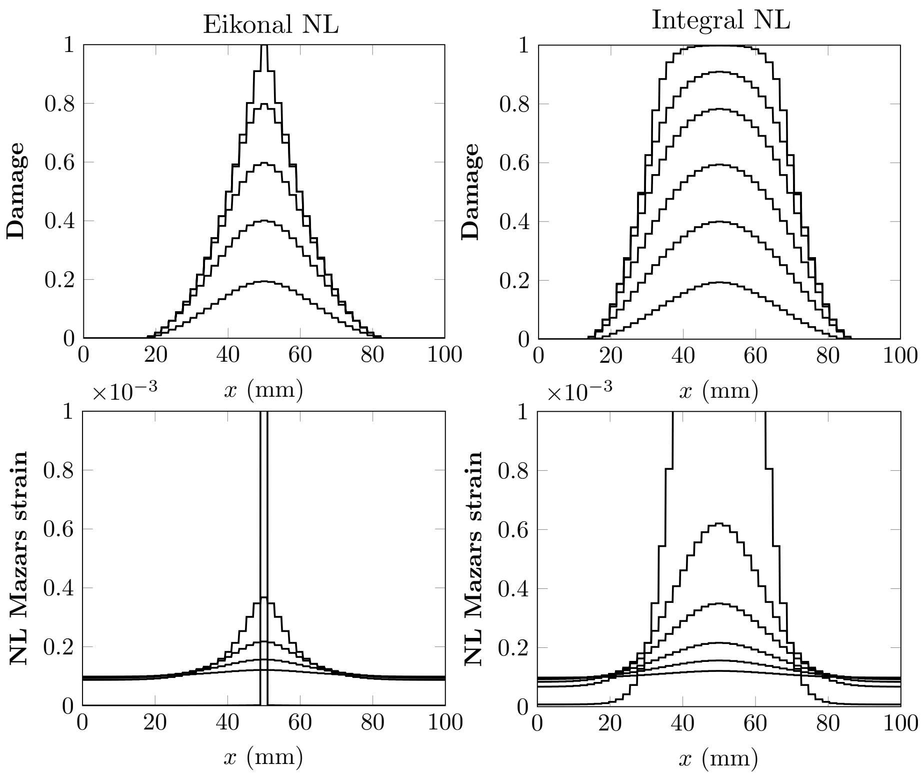

Figure 3: Tensile test, evolution of damage and non-local equivalent strain fields throughout two representative simulations carried out considering ENL (left) and INL (right) damage formulations

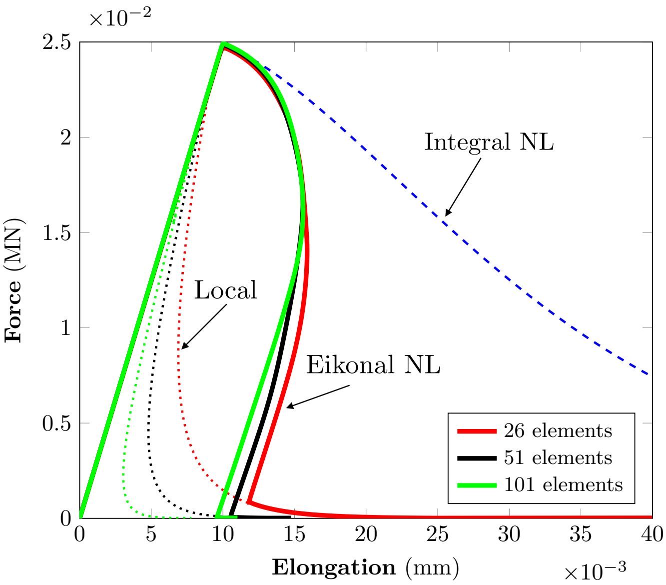

Figure 4: Tensile test, FE mesh sensitivity study. Global responses obtainted through three different FE meshes.

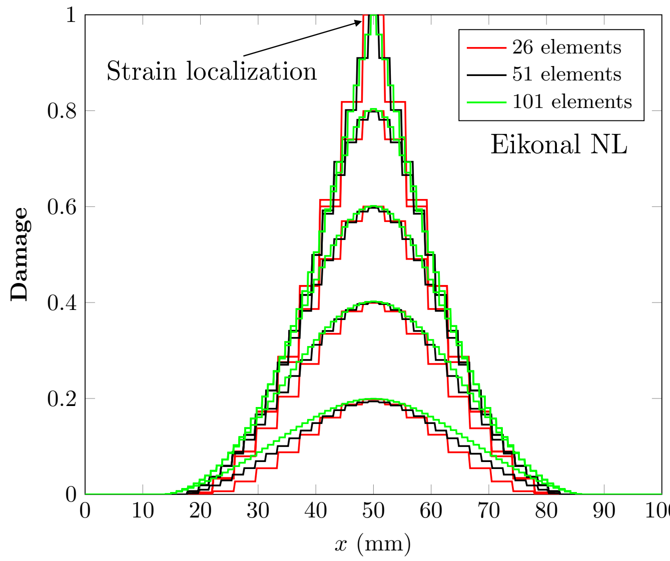

Figure 5: Tensile test, FE mesh sensitivity study. Damage distributions obtainted through three different FE meshes.

6 CONCLUSIVE REMARKS

A simple Eikonal Nonlocal (ENL) continuum damage model was presented in this paper. According to this approach (Desmorat, Gatuingt, & Jirásek 2015, Rastiello, Giry, Gatuingt, & Desmorat 2017), nonlocal interactions between material points are controlled by geodesic distances obtained as solutions of an isotropic time-independent Eikonal equation with a damage dependent metric function. In a differential geometry context, the ENL framework considers that the Riemannian space in which interaction distances are computed is curved due to damage. In other words, interaction distances are no more Euclidean, as in the INL formulation, but evolves depending on the damage field evolution. The numerical implementation was obtained by using a Fast-Marching Method (Sethian 1996) for updating damage dependent nonlocal interactions throughout a quasi-static Finite Elements computation (Rastiello, Giry, Gatuingt, & Desmorat 2017). Two simple test-cases were performed in order to show the main features of the ENL formulation. Regularization properties of the proposed model were demonstrated. Furthermore, it was shown that the proposed formulation allows for reducing several parasite effects classically associated with INL formulations (damage spreading over large damaged bands, damage diffusion near notches and

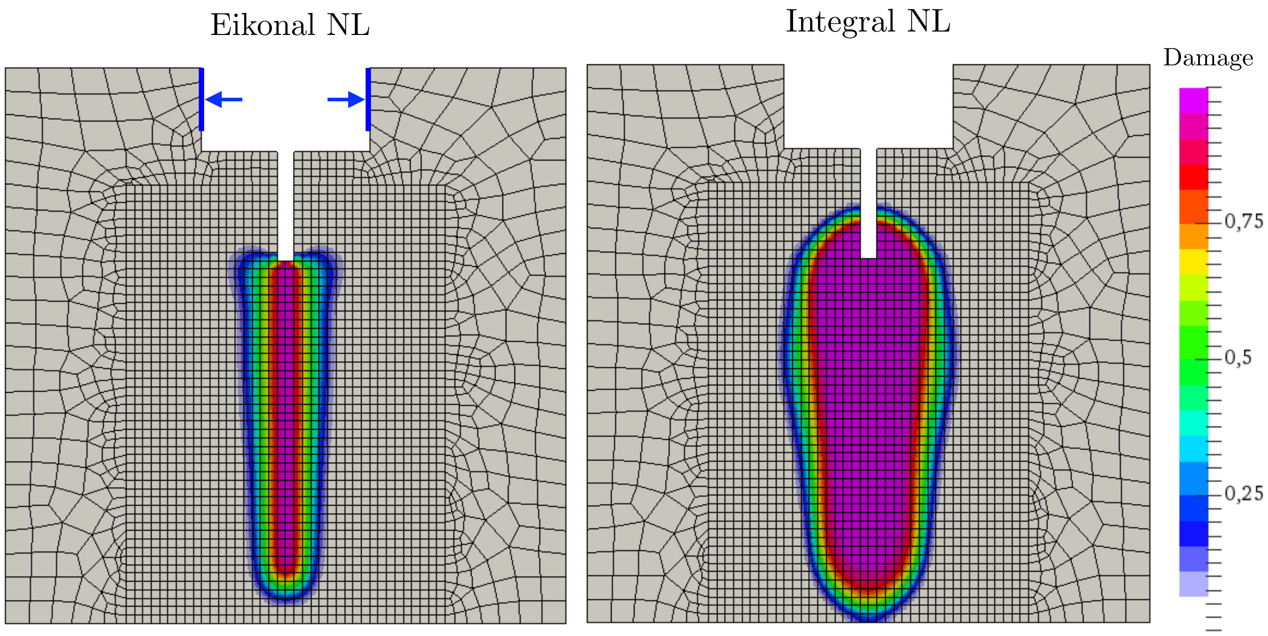

Figure 6: Wedge splitting test: comparison among damage fields computed by considering INL and ENL damage evolution models for the same imposed displacement level. Material parameters are assigned as follows: $E = 100$ MPa, $\nu = 0$, $\epsilon_0 = 0.0001$, $\epsilon_c = 0.0005$, $\ell_c = 20$ mm.

free-edges, etc).

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