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A NEW CRITERIUM FOR THE ERGODICITY OF HAMILTON-JACOBI-BELLMAN TYPE EQUATIONS

CARLO BIANCA AND CHRISTIAN DOGBE

Abstract. This paper deals with the link among the large-time behavior of a class of fully nonlinear partial differential equations, the concept of mean ergodicity of a dynamical system and the controllability problem. Specifically Abelian-Tauberian arguments are employed to develop a theory for the analysis of the ergodic mean behavior of systems of degenerate elliptic-parabolic equations and general systems of vector fields satisfying Hörmander's condition. A new criterium for ergodicity is established which is based on an asymptotic estimation of the rate of convergence. The new criterium is employed for the asymptotic analysis of Hamilton-Jacobi-Bellman type equations.

1. Introduction

The main aim of this paper is to analyze the large-time behavior of solutions of partial differential equations by employing arguments of stochastic differential equations and specifically the ergodicity framework. The motivation is twofold: on the one hand setting up the concept of ergodicity by revisiting the main definition in the field of dynamical systems and ergodic theory; on the other hand, developing a new ergodicity criterium for PDEs. The new criterium will be applied to the ergodic control problems. The term ergodic problem is motivated by the fact that for an uncontrolled system (i.e., an ordinary differential equation) the convergence property is equivalent to the ergodicity of that system. The ergodic problem has been widely studied in connection with homogenization or singular perturbation problems [5, 10]. The results of the present paper take advantage by the work of Lions [26].

Let Ω be a bounded domain of $\mathbb{R}^d$, $d \in \mathbb{N}^*$, and $\mathcal{S}_d(\mathbb{R})$ the space of the $d \times d$ real symmetric matrices. This paper focuses on the analysis of the following Cauchy problem:

$$\begin{cases} \partial_t u + F[u](x) = 0, & \text{in } (0,\infty) \times \Omega \\ u(0,x) = u_0 \in \mathscr{C}^0(\Omega), \end{cases} \tag{1.1}$$

where $u = u(t,x) : (0,\infty) \times \Omega \to \mathbb{R}$, $p = Du$ denotes the gradient vector of u, $X = D^2u$ denotes the Hessian matrix of second derivatives of the function u, and

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$F : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d(\mathbb{R}) \to \mathbb{R}$ represents the following fully nonlinear second-order partial differential operator:

$$F[u](x) \equiv F(x, r, p, X) := F(x, u, Du, D^2u). \tag{1.2}$$

It is worth stressing that the *fully nonlinear operator* means that the partial differential equation is nonlinear in the highest-order derivatives of a solution.
The main interest in the above defined model is related to the theory of stochastic processes, stochastic control theory and the kinetic theory (see for example [16]). In particular the most famous operator F is the following second-order differential operator:

$$F(x, u, Du, D^2u) = -\mathrm{Tr}(a(x)D^2u)(x) - b(x) \cdot Du, \quad x \in \Omega, \tag{1.3}$$

where the matrix (a_{ij}) is a nonnegative definite matrix (elliptic operator), the trace of aD^2u is defined by $\mathrm{Tr}(aD^2u)(x) = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}$, $b : \Omega \to \mathbb{R}^d$. A nonlinear second-order parabolic or elliptic problem is usually connected to an initial stochastic control problem. In particular the stochastic control problem related to Eq. (1.1) with F given by (1.3) can be derived. Accordingly, let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space and $X(t) \doteq X_t$ the random process solution of the following stochastic differential equation:

$$\begin{cases} dX_t = b(X_t)\, dt + \sigma(X_t) dW_t, \\ X_0 = u_0 \in \mathbb{R}^d, \end{cases} \tag{1.4}$$

where the drift coefficient $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are (at least) Lipschitz continuous functions, and W_t is the standard d-dimensional Brownian motion. Bearing all above in mind, the equation (1.4) admits a unique solution $(X_t)_{t\geqslant 0}$, $t \geqslant 0$, $x \in \mathbb{R}^d$, see, among others, Friedman [17]. Let $\mathscr{C}_b(\mathbb{R}^d)$ be the space of the bounded continuous functions, the stochastic representation reads:

$$S_t\varphi(x) := \mathbb{E}[\varphi(X(t, x)], \qquad x \in \mathbb{R}^d, \quad \varphi \in \mathscr{C}_b(\mathbb{R}^d) \tag{1.5}$$

where S_t, $t \geqslant 0$ is the corresponding transition semigroup and $\mathbb{E}$ denotes the conditional expectation.

The main interest of this paper is the asymptotic behavior (as $t \to \infty$) of the solution of the evolution equation (1.1). In general the asymptotic behavior is gained by employing regularity arguments of solutions and the strong maximum principle. The results are of great interest in the context of the ergodic/control problem. For such an evolution equation, it is expected that the large-time behavior of the solution of (1.1) has the following behavior:

$$u(t, x) = v(x) + \mathsf{c}t + o(1) \quad \text{locally uniformly for } x \in \mathbb{R}^d \text{ as } t \to \infty, \tag{1.6}$$

where $(\mathsf{c}, v) \in \mathbb{R} \times \mathscr{C}(\mathbb{R}^d)$ is solution, in the viscosity sense, of the following equation:

$$\mathsf{c} + F(x, v, Dv, D^2v) = 0. \tag{1.7}$$

In this context, the role of the following stationary problem is fundamental:

$$\varepsilon u_\varepsilon + F(x, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) = 0, \qquad x \in \mathbb{R}^d, \tag{1.8}$$

where $\varepsilon > 0$ and F is a fully nonlinear (possibly degenerate) elliptic operator. It is worth stressing that the stationary problem (1.8), also called the discounted approximation of the ergodic problem, arises in optimal control theory and differential game theory where ε is a discount factor.

The following proposition summarizes the important role of the stationary problem (1.8) (see [27] for the proof).

Proposition 1.1. *The following statements are equivalent.*

(i) *Let u_ε be solution of the stationary problem*

$$\varepsilon u_\varepsilon + F(x, u_\varepsilon, Du_\varepsilon, D^2 u_\varepsilon) = 0, \qquad x \in \mathbb{R}^d, \tag{1.9}$$

then $\varepsilon u_\varepsilon \to \mathtt{c}$ uniformly in x as $\varepsilon \to 0$, where $\mathtt{c}$ is a constant.

(ii) *Let u be solution of the Cauchy problem*

$$\begin{aligned} &\partial_t u + F(x, u, Du, D^2 u) = 0 \\ &u(0, x) = 0 \end{aligned} \tag{1.10}$$

then $u(t, x)/t$ converges uniformly in x to some constant as $t \to +\infty$.

In what follows we refer to [3] and the references cited therein for the ergodic stochastic control in $\mathbb{R}^d$. This problem is also studied in Meyn-Tweedie and [29] and in Down *et al.* [15] by using techniques coming from the theory of geometric ergodicity and probabilistic arguments. Instead of looking for steady states (1.1), the authors work directly with the paths of the diffusion process.

The main tool used in this paper is the relationship between ergodicity and controllability, which follows by means of the strong maximum principle. The strong maximum principle is usually employed, in the classical sense, for the second-order uniformly elliptic operator [18] but also for degenerate elliptic operators in the framework of viscosity solutions [35, 19, 30]. In general the compactification of the semigroup is used to obtain the result (compactness of strong maximum principle). An alternative approach is to use the traditional proof of Doob (see, e.g., [14, 13]) by looking at the estimated transition probability. Our approach will follows a purely PDE point of view. Moreover by employing the concept of fundamental solution and nonlinear analysis arguments, we derive a rate of convergence. It is worth stressing that the model investigated in the present paper shares several similarities with some processes occurring in the crowd/swarm behavior [12] and mean field game models [25].

The contents of the paper are outlined as follows. Section 2 contains preliminaries and fundamental results about ergodicity. In particular the relation between ergodicity and hypoellipticity is investigated by using a similar technique proposed in [26]. The above mentioned relation can be interpreted as a version of the Abelian theorem (see Theorem 2.4 below), which states that the averaged functional related to u converges to a constant if and only if the family $(\varepsilon u_\varepsilon)_{\varepsilon>0}$ converges uniformly to a constant. The most important result of this section is a Tauberian type result which transforms our problem to the study of the resolvent. Moreover a propagation of maxima of our problem in a self-contained way is proposed. Section 3 deals with the linking between ergodicity and controllability. Section 4 is devoted to the definition of a criterium for ergodicity, which is

based on the study of a convergence rate. Section 5 is devoted applications and specifically to Hamilton-Jacobi-Bellman equations. Finally, in section 6, we give a conclusion and some perspectives.

2. Background and preliminaries results

This section is devoted to the main definitions and results that will be used in the paper.

Definition 2.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of $\mathbb{R}^d$, $d \in \mathbb{N}^*$, $\mathcal{S}_d(\mathbb{R})$ the space of the $d \times d$ real symmetric matrices, and $F : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d(\mathbb{R}) \to \mathbb{R}$ a fully nonlinear operator.

(1) F is called elliptic if for any $(x, u, p, X) \in \times\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d$, we have
$$F(x, u, p, X + Y) > F(x, u, p, X). \tag{2.1}$$

(2) F is called degenerate elliptic if
$$F(x, r, p, Y) \leqslant F(x, s, p, X) \quad \text{whenever} \quad r \leqslant s \ \text{ and } \ X \leqslant Y, \tag{2.2}$$
where $X \leqslant Y$ means that $X - Y$ is a nonnegative definite symmetric matrix.

(3) F is said *uniformly elliptic* (with ellipticity constants $\lambda \leqslant \Lambda$) if
$$\lambda \mathrm{Tr}\,(Y) \leqslant F(x, u, p, X) - F(x, u, p, X + Y) \leqslant \Lambda\, \mathrm{Tr}\,(Y). \tag{2.3}$$

(4) $\partial_t + F$ is called degenerate parabolic (see e.g. [11]) if F is degenerate elliptic.

Some operators satisfying the conditions in definition 2.1 are now mentioned.

- The degenerate elliptic linear equation reads:
$$a_{ij}(x)\partial_{ij}u - b_i(x)\partial_i u - c(x)u(x) = f(x), \qquad x \in \mathbb{R}^d \tag{2.4}$$
where the matrix $a = (a_{ij}(x))$ is symmetric. Here and everywhere below we will use the implicit summation convention on repeated indices. The related operator F reads:
$$F(x, u, Du, D^2u) := \mathrm{Tr}(a(x)D^2u) - \sum b_i(x)p_i - c(x)r - f(x). \tag{2.5}$$
F is degenerate elliptic if and only if $a(x) \geqslant 0$. If a constant $C > 0$ exists such that $C\mathbf{I} \geqslant a(x) \geqslant C^{-1}\mathbf{I}$ for all $x \in \Omega$ where $\mathbf{I}$ is the identity matrix, F is *uniformly elliptic*. If $C(x)\mathbf{I} \geqslant a(x) \geqslant C^{-1}(x)\mathbf{I}$ for $C(x) > 0$ and any $x \in \Omega$, F is called *strictly elliptic*.
- The Hamilton-Jacobi-Bellman (in short HJB) and Isaacs equations are the fundamental partial differential equations employed for stochastic control and stochastic differential games. The natural setting involves a collection of second-order elliptic operators depending either on one parameter α (in the Hamilton-Jacobi-Bellman case) or two parameters α, β (in the Isaacs case), which lie in some index sets. Let $a_{ij}^{\alpha}(x)$, $a_{ij}^{\alpha,\beta} \in \mathcal{S}_d(\mathbb{R})$, the following operators can be defined:
$$\mathcal{L}^{\alpha}u := a_{ij}^{\alpha}(x)\partial_{ij}u - b_i^{\alpha}(x)\partial_i u - c^{\alpha}(x)u(x) + f^{\alpha}(x), \tag{2.6}$$

$$\mathcal{L}^{\alpha,\beta}u := a_{ij}^{\alpha,\beta}(x)\partial_{ij}u - b_i^{\alpha,\beta}(x)\partial_i u - c^{\alpha,\beta}(x)u(x) + f^{\alpha,\beta}(x), \tag{2.7}$$

where all coefficients are uniformly bounded functions. In (2.7), the operator F reads:

$$F(x,r,p,X) = \sup_{\alpha}\inf_{\beta}\{-\mathrm{Tr}(a^{\alpha,\beta}(x)X) + b^{\alpha,\beta}(x)\cdot p + c^{\alpha,\beta}(x)r - f^{\alpha,\beta}(x)\}$$

where $f^{\alpha}(\cdot) = f(\cdot,\alpha)$ and $f^{\alpha,\beta} = f(\cdot,\alpha,\beta)$ are families of given smooth functions. The Hamilton-Jacobi-Bellman equation (resp. Isaacs equations) which is *second-order, degenerate elliptic, fully nonlinear equations* reads:

$$\sup_{\alpha\in\mathcal{A}}\mathcal{L}^{\alpha}u = 0 \qquad (\text{resp.} \quad \sup_{\alpha}\inf_{\beta}\mathcal{L}^{\alpha,\beta}u = 0), \tag{2.8}$$

where $\mathcal{A}$ is a given set ($\alpha\in\mathcal{A}$, control), $a^{\alpha,\beta}(x) = \frac{1}{2}\sigma^{\alpha,\beta}(x)(\sigma^{\alpha,\beta}(x))^T$.
The Hamilton-Jacobi-Bellman equations are examples of *second-order, degenerate parabolic, fully nonlinear equations*. In more abstractly and more compactly, it is written:

$$\partial_t u + \sup_{\alpha\in\mathcal{A}}[\mathcal{L}^{\alpha}u - f^{\alpha}] = 0. \tag{2.9}$$

In what follows we set $\Gamma = \Omega\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}_d(\mathbb{R})$ and $(x,u,p,X)\in\Gamma$. When F in (2.6) is an affine function of the d variables, the equation (1.1) is called quasilinear; otherwise, it is called fully nonlinear.

2.1. Pucci's operator. In this subsection we introduce the so-called `Pucci`'s extremal operators (see [31]). Let X^+, X^- denote the positive and negative parts of $X\in\mathcal{S}_d(\mathbb{R})$, respectively, namely $X = X^+ - X^-$ and $\mathrm{Trace}(X^+)$ (respectively $\mathrm{Tr}(X^-)$) denotes the sum of the positive eigenvalues of X (respectively, $-X$). The maximal and minimal Pucci's extremal operators $\mathcal{P}^+_{\lambda,\Lambda}(X)$ and $\mathcal{P}^-_{\lambda,\Lambda}(X)$ are defined as follows:

$$\mathcal{P}^+_{\lambda,\Lambda}(X) := \sup_{M\in\mathcal{M}_{\lambda,\Lambda}}(-\mathrm{Tr}(MX)) \qquad \text{and} \qquad \mathcal{P}^-_{\lambda,\Lambda}(X) := \inf_{M\in\mathcal{M}_{\lambda,\Lambda}}(-\mathrm{Tr}(MX)), \tag{2.10}$$

where $0<\lambda\leqslant\Lambda$ and $\mathcal{M}_{\lambda,\Lambda} := \{M\in\mathcal{S}_d \,|\, \lambda I\leqslant M\leqslant\Lambda I\}$. The following equivalent definition of the Pucci extremal operators is often more convenient for calculations:

$$\mathcal{P}^+_{\lambda,\Lambda}(X) = -\lambda\sum_{\mu_j>0}\mu_j - \Lambda\sum_{\mu_j<0}\mu_j \qquad \text{and} \qquad \mathcal{P}^-_{\lambda,\Lambda}(X) = -\Lambda\sum_{\mu_j>0}\mu_j - \lambda\sum_{\mu_j<0}\mu_j, \tag{2.11}$$

where $\mu_1,\dots,\mu_n$ are the eigenvalues of X. We will consider this definition for future references. The elementary properties of the Pucci operators can be found in [9]. Here, we remark only that they are uniformly elliptic, $\mathcal{P}^+_{\lambda,\Lambda}$ is convex and $\mathcal{P}^+_{\lambda,\Lambda}$ concave.

This notion is one of the key tools we use to prove our results.

Bearing all above in mind, an equivalent way of writing (2.3) is : There exists $0<\lambda\leqslant\Lambda$ such that for every $X,Y\in\mathcal{S}_d(\mathbb{R})$,

$$\mathcal{P}^-_{\lambda,\Lambda}(X-Y)\leqslant F(X)-F(Y)\leqslant\mathcal{P}^+_{\lambda,\Lambda}(X).$$

Observe that (2.3) is satisfied for both $F = \mathcal{P}^-_{\lambda,\Lambda}$ and $F = \mathcal{P}^+_{\lambda,\Lambda}$ and these hypotheses imply

$$\mathcal{P}^-_{\lambda,\Lambda}(X) \leqslant F(X) \leqslant \mathcal{P}^+_{\lambda,\Lambda}(X), \qquad \text{for each } X \in \mathcal{S}_d.$$

Furthermore, an equivalent way of starting (2.3) is to assume that F is of the form

$$F(D^2u) = \sup_{\alpha\in\mathcal{A}} \left(-a^\alpha_{ij}\partial_{ij}u\right) \quad \text{or} \quad F(D^2u) = \inf_{\alpha\in\mathcal{A}} \left(-a^\alpha_{ij}\partial_{ij}u\right), \tag{2.12}$$

where ∂_{ij} denote the first and second partial derivatives with respect to x_i, x_j, α is the indice that belong to some sets X and Y, and the symmetric matrices satisfy the inequality $\lambda I \leqslant \{a^\alpha_{ij}\} \leqslant \Lambda I$. Hence the Pucci extremal operators are also given by formula:

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u) = \sup_{\lambda I\leqslant\{a^\alpha_{ij}\}\leqslant\Lambda I} a^\alpha_{ij}\partial_{ij}u \qquad \text{and} \qquad \mathcal{P}^-_{\lambda,\Lambda}(D^2u) = \inf_{\lambda I\leqslant\{a^\alpha_{ij}\}\leqslant\Lambda I} a^\alpha_{ij}\partial_{ij}u. \tag{2.13}$$

From (2.12) and (2.13), another examples of uniformly parabolic equations, especially in the theory of viscosity solution for fully nonlinear equations have the forms

$$\partial_t u + \mathcal{P}^+_{\lambda,\Lambda}(D^2u) = f(x), \qquad \partial_t u + \mathcal{P}^-_{\lambda,\Lambda}(D^2u) = f(x). \tag{2.14}$$

The reader is referred to papers [11, 22] for more information on the nonlinear degenerate elliptic equations.

2.2. Abelian-Tauberian arguments and characterization of ergodicity. We collect two of the basic ingredients used in the next sections to study the ergodic behavior of the model (1.1). In particular the mean ergodic theorem is mentioned.

The Abelian-Tauberian method is an important approach that can be used to characterize ergodicity. The method consists in transforming the ergodicity problem in the large-time behavior analysis of the Laplace transform of the integrated density of states. At this aim, the paper focuses on the methods that allow the derivation of the asymptotic equation.

Let u be the solution of the following equation:

$$\begin{cases} u_t + Au = 0 \qquad \text{in } \ \Omega \subseteq \mathbb{R}^d \\ u(0,x) = u_0 \in \mathscr{C}^0(\Omega), \end{cases} \tag{2.15}$$

where A denotes the following (possibly degenerate) elliptic operator with $\mathscr{C}^\infty(\Omega)$ coefficients:

$$A = -a_{ij}\partial_{ij} - b_i\partial_i. \tag{2.16}$$

It is worth stressing that the operator A is the generator of a semigroup associated to a parabolic PDE and it is said to be in divergence form.

It is worth pointing out that, under technical assumptions for the coefficients, for any function $u \in W^{2,1}(\mathbb{R}^d)$, the expressions Au and the adjoint A^*u are defined in the sense of generalized function[1]: For a function $\varphi \in \mathscr{C}^\infty_0(\mathbb{R}^d)$, we have (after

[1] $W^{p,1}(\mathbb{R}^d)$ denotes the Sobolev class of functions in $L^p(\mathbb{R}^d)$, equipped with its natural Sobolev norm

integration by parts):

$$\langle a_{ij}\partial_{ij}u, \varphi\rangle = -\int_{\mathbb{R}^d} \partial_j u\, \partial_i(a_{ij}\varphi)dx,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner-product in $\mathbb{R}^d$ with the Euclidean norm for a vector x: $|x| = \sqrt{\langle x, x\rangle} = \left(\sum_i x_i^2\right)^{1/2}$. In particular $\partial_i(a_{ij}\varphi) = \varphi\partial_i a_{ij} + a_{ij}\partial_i\varphi \in L^2(\mathbb{R}^d)$ and similarly $\partial_i(b_i u)$ and $\partial_i(a_{ij}\partial_j u)$ are meaningful. Hence the formal adjoint operator A^* reads:

$$A^*\psi := -\sum_{i,j}\partial_{ij}(a_{ij}\psi) + \sum_i \partial_i(b_i\psi), \tag{2.17}$$

where the summation convention is employed if confusion does not occur.

In order to analyze the asymptotic properties of (2.15), the following Cesàro time-averaged functional is considered:

$$C_u(t,x) := \frac{1}{t}\int_0^t u(s,x)\,ds, \qquad t > 0.$$

Let us briefly recall the definition of mean ergodicity (or weak ergodicity) and strong ergodicity on a compact set $\Omega \subset \mathbb{R}^d$.

Definition 2.2. (Cesàro mean ergodicity) Let Ω be a compact set of $\mathbb{R}^d$. The system described by Eq. (2.15) (or the associated semigroup of (1.4)) is said ergodic in the sense of Cesàro, if:

• A admits an unique invariant probability measure m, namely

$$\int e^{-tA}\,u_0 dm = \int u_0\,dm, \qquad \forall t > 0, \quad \text{and} \quad \int m(x)dx = 1.$$

• The solution $u(t,x)$ of Eq. (2.15) satisfies the following condition:

$$C_u(t,x) \xrightarrow[t\to\infty]{} \mathsf{c} := \int u_0\,dm, \quad \text{uniformly in } x. \tag{2.18}$$

Definition 2.3. (Strong ergodicity). The system described by Eq. (2.15) is said ergodic if A admits a unique invariant probability measure m such that

$$A^*m = 0 \quad \text{in} \quad \mathbb{R}^d, \qquad m \geqslant 0, \qquad \int dm = 1, \tag{2.19}$$

and the solution $u(t,x)$ of the Eq. (2.15) satisfies the following condition:

$$\lim_{t\to+\infty} u(t,x) = \mathfrak{c} := \int u_0\,dm, \tag{2.20}$$

uniformly in x.

It is worth stressing that in the context of the stochastic processes, strong ergodicity means that the process X_t converges, as $t \to +\infty$, to an unique stationary state.

The following result holds true. The result belongs to the class of the so-called Abelian-Tauberian theorems.

Theorem 2.4. *(Ergodic mean characterization). Let A be the (possibly degenerate) elliptic operator (2.16) and $u_\varepsilon \in \mathscr{C}^0(\Omega)$, for $\varepsilon > 0$, solution of the (resolvent) equation*

$$\varepsilon u_\varepsilon + A u_\varepsilon = f, \tag{2.21}$$

where f corresponds to u_0, or equivalently $u_\varepsilon = (\varepsilon I + A)^{-1} f$.
The following statements are equivalent.

(i) *The operator $C_u(t,x)$ converges uniformly to a constant as $t \to +\infty$.*
(ii) *$\varepsilon u_\varepsilon$ converges uniformly to a constant c as $\varepsilon \to 0$.*
(iii) *$\varepsilon u_\varepsilon$ is equicontinuous (compact in the space of continuous functions $\mathscr{C}^0(\Omega)$) and*

$$\begin{cases} Au = 0 \\ u \in \mathscr{C}^0(\Omega) \end{cases} \iff u = \mathsf{c}. \tag{2.22}$$

Before proving the Theorem 2.4, the following remark needs to be underlined.

Remark 2.5.

(i) The relationship between the evolution equation (2.15) and the stationary equation (2.22) can be explained as follows: u satisfies (2.15), but we are looking at a new function $u(T+s,x)$ with $T \leqslant s < \infty$, possibly at the price of extracting subsequence denoting T_n, converging uniformly to u (convergence uniformly in x and bounded in s belonging to any compact):

$$u(T_n + s, x) \xrightarrow[T_n \to \infty]{\text{converges unif.}} u(s,x),$$

solution of Eq. (2.22). Note that the uniform convergence serves to ensure that u is continuous.
The Abelian-Tauberian theorem (see, e.g., [32], Theorem 10.2) has an important rule. This theorem states that if f is a continuous function, and the following limit (Abel mean)

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^\infty e^{-\varepsilon t} f(x(t)) dt$$

exists, then the limit (Cesàro mean)

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x(t)) dt$$

also exists and the following equality holds:

$$\lim_{\varepsilon \to 0^+} \varepsilon \int_0^\infty e^{-\varepsilon t} f(x(t)) dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x(t)) dt,$$

provided that at least one side is meaningful. Thus it is clear the relation between $\lim_{\varepsilon \to 0^+} \varepsilon u_\varepsilon$ and $\lim_{T \to \infty} \dfrac{u(T,x)}{T}$. According to the theorem 2.4, the main problem is to understand the behavior of $\varepsilon u_\varepsilon$. Multiplying (2.21) by ε, one has

$$\varepsilon(\varepsilon u_\varepsilon) + A \varepsilon u_\varepsilon = \varepsilon f. \tag{2.23}$$

Observe that $\varepsilon f \to 0$ and by the equicontinuous of $\varepsilon u_\varepsilon$, possibly at the cost of extracting a subsequence, one has $\varepsilon u_\varepsilon \to u$ with u solution of $Au = 0$. From the informal point of view, the theorem 2.4 states the link between ε and $1/t$ in the Cesàro time-averaged functional and the operators

$$\frac{1}{T}\int_0^T e^{-As}ds, \quad (\varepsilon I + A)^{-1}.$$

(*ii*) The uniform convergence in Theorem 2.4 ensures that u is continuous which implies that u is constant. The continuity of u is important in this context. Indeed if we consider the following example (without control and boundary condition): $\Omega = [-1, 1]$, $a_{ij} = 0$ and $b(x) = -x$, the stationary equation reads $-xu'(x) = 0$. It is obvious that if u is continuous then u must be constant, i.e. $u \equiv u(0)$; but if u is not continuous there might be a jump at $t = 0$ (u is a Heaviside step function, constant for $x > 0$ and for $x < 0$).

(*iii*) The Definition 2.2 states the relationship between invariant measure and ergodicity, which is a key step in the study of the ergodic behavior of the underlying physical systems. The constant c can be represented in the form

$$\mathsf{c} = \int_{\mathbb{R}^d} u_0 dm, \qquad m \in \mathbb{P},$$

where $\mathbb{P}$ is a probability space and m is a probability measure. It is a positive linear form with respect to the initial condition u_0 that preserves the positivity. In addition, there is always at least one invariant measure m, which is not necessarily unique, but it still exists [6]. In the case of ergodicity this measure is unique because if the invariant measure m is unspecified, according to the definition of ergodicity, one has:

$$\int_{\mathbb{R}^d} u(t)dm = \int_{\mathbb{R}^d} u_0 dm \qquad \forall\, m \in \mathbb{P},$$

which, in the case of Cesàro, implies

$$\frac{1}{T}\int_0^T u(t)dm = \int u_0 dm.$$

By interchanging the integrals, one get

$$\frac{1}{T}\int_0^T\!\!\int u(t) \xrightarrow[t\to\infty]{\text{converges unif}} \text{constant}.$$

Therefore

$$\frac{1}{T}\int_0^T u(t)\int \mathsf{c}\, dm = \text{constant}.$$

The constant limit defines the action of any invariant measure on u_0 and therefore the invariant measure is automatically unique.

On the one hand, for the problem (2.15) if one consider the following probabilistic representation:

$$u(t, x) = \mathbb{E}_x[u_0(X_t)], \tag{2.24}$$

with a Markov process which is at least, unique in law, then

$$\|u(t,x)\| = \|E_x u_0(X_t)\| \leqslant \|u_0\|_{L^\infty}.$$

On the other hand, from Eq. (2.21), one has

$$\varepsilon u_\varepsilon = \varepsilon \int_0^\infty e^{-\varepsilon t} S(t) f dt, \qquad S(t) = e^{-tA},$$

and in light of the maximum principle, one get

$$\|\varepsilon u_\varepsilon\| \leqslant \|f\|_\infty \int_0^\infty \varepsilon e^{-\varepsilon t} dt = \|f\|_\infty.$$

Taking the supremum over all possible controlled trajectories, we end up with $\|\varepsilon u_\varepsilon\| \leqslant \|L\|_\infty$. Then the function $\varepsilon u_\varepsilon$ is uniformly bounded. Thus the arguments one uses on the solution at any time, turns on resolvents in a property where u_0 is replaced by f and the $u(t)$ is replaced by $\varepsilon u_\varepsilon$.

Proof of Theorem 2.4 ($\Rightarrow$) Assume that $C_u(t,x) := C_u(t)$ converges uniformly to 0 as $t \to +\infty$. The problem of the mean ergodicity reduces to the following question: Does $\varepsilon u_\varepsilon \xrightarrow[\varepsilon \to 0^+]{} 0$?

On the one hand, since $\varepsilon u_\varepsilon$ is equicontinuous, up to extraction of a subsequence $\varepsilon_n \xrightarrow[n\to\infty]{} 0$ such that

$$\varepsilon_n u_{\varepsilon_n} \xrightarrow[n\to\infty]{\text{converge unif.}} u$$

giving us our limit and the equation for u:

$$A(\varepsilon_n u_{\varepsilon_n}) + \varepsilon_n u_{\varepsilon_n} = \varepsilon_n f.$$

On the other hand

$$A(\varepsilon_n u_{\varepsilon_n}) \xrightarrow[n\to\infty]{\text{cv. unif.}} u, \qquad \varepsilon_n u_{\varepsilon_n} \xrightarrow[n\to\infty]{\text{cv. unif.}} u \qquad \varepsilon_n f \xrightarrow[n\to\infty]{\text{cv. unif.}} 0$$

therefore one has a continuous function such that $u = 0$. Hence, to prove ergodicity in the sense of Cesàro, we must prove (2.22).

Next, from the representation of u_ε, one has

$$u_\varepsilon = \int_0^\infty e^{-\varepsilon s} u(s) ds,$$

since the relation

$$\int_0^\infty e^{-\varepsilon s} e^{-As} ds = (\varepsilon I + A)^{-1},$$

holds if A is a constant or an operator. Integrating by part and using the estimate

$$\int_0^s u ds \simeq cs$$

yields

$$u_\varepsilon = \varepsilon \int_0^\infty e^{-\varepsilon s} \int_0^s u ds \simeq \varepsilon \mathsf{c} \int_0^\infty s e^{-\varepsilon s} ds.$$

Therefore

$$\varepsilon u_\varepsilon = -\varepsilon^2 \int_0^\infty e^{-\varepsilon s}\left(\int_0^s u\right)ds \simeq \varepsilon^2 c \int_0^\infty s e^{-\varepsilon s} ds \ \to \ \mathtt{c}.$$

($\Leftarrow$) *Let us show the reverse.* By using the variation of constants formula, we get that the solutions of equation

$$Au_\varepsilon = f - \varepsilon u_\varepsilon \tag{2.25}$$

are given by

$$u_\varepsilon = \int_0^T S(t)(f - \varepsilon u_\varepsilon) + S(T)u_\varepsilon \tag{2.26}$$

where $S(\cdot)$ is the solution of linear semigroup of contraction in L^∞ to the homogeneous problem (2.25), s is replaced by t and t by T. Since u_ε is independent of time, and since $\partial_t u_\varepsilon = 0$, one can write

$$\frac{\partial u_\varepsilon}{\partial t} + Au_\varepsilon = f - \varepsilon u_\varepsilon. \tag{2.27}$$

We then recovers $S(T)$ of the initial condition. Now, the solution at time T is given by

$$\frac{1}{T}u_\varepsilon = \frac{1}{T}\int_0^T S(t)(f - \varepsilon u_\varepsilon) + \frac{1}{T}S(T)u_\varepsilon, \tag{2.28}$$

from which we deduce that the formulas (2.26) and (2.27) are equivalents. The reverse is that we know that $\varepsilon u_\varepsilon \ \to \ c$ and we are interested in what takes place on $\frac{1}{T}u_\varepsilon$. We claim that

$$\frac{1}{T}\int_0^T S(t)f \ \xrightarrow[T\to\infty]{} \ \text{constant} = \mathtt{c}$$

where c is the constant we seek; that is the mean of f. Observe that, when T is large enough, and $\varepsilon u_\varepsilon$ independently of T, as ε becomes small, there is modulus of continuity $\omega(\varepsilon)$ such that

$$\frac{1}{T}\int_0^T S(t)f = c + \omega(\varepsilon) + \frac{1}{T}(u_\varepsilon - S(T)u_\varepsilon) \tag{2.29}$$

where $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$ (and independently of the time). This is just the convergence of $\varepsilon u_\varepsilon$ and since the semigroup preserves the bounds, we can estimate the expansion (2.29). Since

$$u_\varepsilon \leqslant C(\varepsilon),$$

one can check classically that

$$\frac{1}{T}(u_\varepsilon - S(T)u_\varepsilon) \leqslant \frac{C(\varepsilon)}{T},$$

equivalently, as $T \to \infty$, for any ε one has Cesàro-means, namely

$$\left|\frac{1}{T}\int_0^T S(t)f - c\right| \leqslant \omega(\varepsilon) + \frac{C(\varepsilon)}{T}, \tag{2.30}$$

which is the key estimate needed to complete the ergodic result. The estimate (2.30) shows that, when T tends to infinity, one has information on Cesàro averages and completes the proof of the Theorem 2.4. ■

It is worth stressing that the Abelian-Tauberian theorem states that in order to study the mean ergodicity of Cesàro, it is sufficient to look at the resolvent and understand why $\varepsilon u_\varepsilon$ converges towards a constant.

Remark 2.6. The existence of solutions of the resolvent equation (2.21) can be established by employing the maximum principle. However if A is degenerate the problem is still difficult, although existence of solutions to the resolvent equation is still ensured. In the non-degenerate case the situation is easier; if A is a uniformly elliptic operator, the resolvent set is well defined, indeed the first eigenvalue of A which 0 (the first eigenfunctions are constants). From a formal point of view, this problem is strongly related to Krein-Rutman's theorems (see [21], Theorem 6.1).

2.3. Strong maximum principle and propagation of maxima. This section contains some preliminaries that will allow to establish the connection with ergodicity/controllability through the propagation of maxima, and the hypoellipticity (in the sense of Hörmander [7]). As it is well known, the maximum principle provides a routine method for proving the uniqueness of classical solutions.

Our Strong Maximum Principle is the following.

Theorem 2.7. *Let $a = (a_{ij})$ be a symmetric and uniformly elliptic matrix. Let u be a viscosity subsolution of the following equation:*

$$Au = \sum_{ij} a_{ij}\partial_{ij}u + \sum_j b_j \partial_j u = 0. \tag{2.31}$$

Assume that u takes a maximum at a point $x_0 \in \Omega \subset \mathbb{R}^d$, i.e.

$$u(x) \leqslant u(x_0) \qquad \forall\, x \in \Omega. \tag{2.32}$$

Then u is constant in Ω almost everywhere.

Proof. Recall that, the probabilistic representation of the solution of Eq. (2.31) is given by

$$u(x) = \mathbb{E}_x[u(X_t)],$$

with the process $u(X_t)$. Assume that u takes a maximum at x_0, i.e.

$$u(x_0) = \max_x u.$$

Since we are working on the compact space and that u is continuous, for the maximum point x_0, one has

$$\max u(x) = \mathbb{E}_{x_0}[u(X_t)] \leqslant u \qquad \text{almost everywhere in } x \in \Omega.$$

This means that if there is a maximum point and all trajectories emanating from this point, then, on all these trajectories or almost everywhere, u is constant:

$$\forall\, t \qquad \max u(X_t^{x_0}) = u(x_0) = \max u.$$

We can now use the same arguments to show that for a minimum point y_0, one has

$$u(y_0) = \min u$$

and then

$$\min u(X_t^{y_0}) = u(y_0) = \min u, \qquad \text{almost everywhere,}$$

which guarantees that u is a constant and the claim is proved. ∎

Accordingly to the strong maximum principle, for a solution of an elliptic equation the extrema can be attained in the interior if and only if the function u is a constant. We will use the above strong maximum principle to solve the ergodic problem in the next sections.

In order to quantify the directions in which the maximum (or minimum) propagates, a controllability problem is set. The idea dates back to the paper by Bony [7], where Hörmander operators are considered. For Eq. (1.4) the following matrix σ is considered:

$$a = \frac{1}{2}\sigma\sigma^T, \qquad a_{ij}^{\alpha} = \frac{1}{2}\sigma_i^{\alpha}\sigma_j^{\alpha}, \qquad \sigma^{\alpha} = \begin{pmatrix}\sigma_1^{\alpha} & \dots & \sigma_d^{\alpha}\end{pmatrix}^T, \tag{2.33}$$

where σ^T denotes the transpose matrix of σ, σ^{α} a vector field and d is the dimension of the space. Suppose σ and b are bounded infinitely differentiable functions and let A denotes the generator of the corresponding Markov process on $\mathbb{R}^d$, possibly degenerate, written in non-divergence form (2.16). Before we get to the heart of the matter, it is interesting to write A in the form of a square matrix. Define the following vector fields:

$$X_{\ell}^{\alpha} = \sum_{i=1}^{d} \sigma_{i\ell}^{\alpha}\frac{\partial}{\partial x_i}, \qquad Y = \sum_{i=1}^{d} b_i\frac{\partial}{\partial x_i}, \qquad 1 \leqslant \ell \leqslant d, \tag{2.34}$$

where X_{ℓ}^{α}, Y are first-order differential operators with $\mathscr{C}^{\infty}$ coefficients. Then, it is helpful to rewrite the operator A in geometrical form by setting

$$A = -X_0 - \frac{1}{2}(X^{\alpha})^2 \quad \text{with} \quad X_0 \equiv \beta(x)\cdot\nabla, \tag{2.35}$$

where $X^2u := X(Xu)$. We recall that A hypoelliptic means that every distributional solution of $Au = 0$ is a $\mathscr{C}^{\infty}$ function.
The relationship between the vector field b in Eq. (2.16) and β in Eq. (2.35) follows. Writing the operator A in (2.16) in Hörmander form, one obtains:

$$\begin{aligned} Au &= -\frac{1}{2}(\sigma^{\alpha}\cdot\nabla)(\sigma^{\alpha}\nabla) - \beta_i\partial_i u \\ &= -a_{ij}\partial_{ij}u - \frac{1}{2}[(\sigma^{\alpha}\cdot\nabla)\sigma^{\alpha}]\cdot\nabla, \end{aligned} \tag{2.36}$$

with

$$\sigma^{\alpha}\nabla_x\cdot\sigma^{\alpha}\nabla_x = \sum_{i,j=1}^{d}\sigma_{i\ell}^{\alpha}\frac{\partial}{\partial x_i}\left(\sigma_{j\ell}^{\alpha}\frac{\partial}{\partial x_j}\right), \qquad 1 \leqslant \ell \leqslant d.$$

This means that we can rewritten (2.16) in the form

$$\frac{1}{2}\sigma^{\alpha}\nabla_x\cdot\sigma^{\alpha}\nabla_x + b\cdot\nabla_x = \frac{1}{2}\sum_{\ell=1}^{d} X_\ell^2 + X_0. \tag{2.37}$$

Therefore equation (2.36) can be written as

$$Au = -\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{i=1}^{d}\left(b - \frac{1}{2}\sigma'\sigma\right)_i\frac{\partial}{\partial x_i} \tag{2.38}$$

where $\sigma'\sigma$ is given by

$$(\sigma'\sigma)_i = \sigma_{i\ell}^{ij}\sigma^{\ell j}.$$

Finally, we deduce that

$$\beta = b - \frac{1}{2}(\sigma^{\alpha}\cdot\nabla)\sigma^{\alpha}. \tag{2.39}$$

Example 2.8. (The Fokker-Planck equation). We may associate to (1.4) two partial differential equations. The Fokker-Planck (or forward Kolmogorov) equation

$$\partial_t u + \partial_i(ub_i) - \frac{1}{2}\partial_{ij}(\sigma_{ik}\sigma_{jk}u) = 0 \tag{2.40}$$

and its adjoint equation, the backward Kolmogorov equation

$$\partial_t u - b_i\partial_i u - \frac{1}{2}\sigma_{ik}\sigma_{jk}\partial_{ij}u = 0, \tag{2.41}$$

where repeated indices denotes summation. Eq. (2.41) may be written as

$$-\frac{1}{2}\sigma_{ik}\sigma_{jk}\partial_{ij}u = -\frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j u) - \frac{1}{2}\partial_i(\partial_j(\sigma_{ik}\sigma_{jk})u), \tag{2.42}$$

then the Fokker-Planck equation (2.41) may be written as

$$\partial_i u + \partial_i\left(u\left(b_i - \frac{1}{2}\partial_j(\sigma_{ik}\sigma_{jk})\right)\right) - \frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j u) = 0 \tag{2.43}$$

which is equivalent to

$$\partial_t u + \partial_i(\beta\, u) - \frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_i u) = 0. \tag{2.44}$$

3. The Link between Ergodicity and Controllability

This section is devoted to show the link between the reachability/controllability properties of the system and the ergodic behavior of the solutions.

Definition 3.1. Let $\mathscr{C}(x)$ be the set of points $z\in\Omega$ which is controllable from x, i.e.

$$\mathscr{C}(x) = \begin{cases} z\in\Omega \;\mid\; \exists\, T>0,\ \exists\,\alpha(\cdot):(0,\infty)\to\mathcal{A}, \text{ and there exists } u\in L^2([0,T]) \\ \text{such that} \quad \dot{x} = \sigma_i^{\alpha(t)}u_i(t) + \beta_i^{\alpha(t)}; \quad x(0)=x, \quad \text{satisfies} \quad x(T)=z \end{cases} \tag{3.1}$$

and $\mathscr{C}^+(x)$ the closure of $\mathscr{C}(x)$; i.e.

$$\mathscr{C}^+(x) = \overline{\mathscr{C}(x)}, \quad \text{for all } x\in\Omega, \tag{3.2}$$

where the bar denotes the closure in the uniform topology.

Remark 3.2. The set $\mathscr{C}^+(x)$ is the closed trajectories, namely the set of points z that can be reached at any time T, with a control α_t and $u \in L^2$, where one solves the associated differential equation starting from x and by arriving in z. The quantities σ^α and β in (3.1) are defined in the Hörmander operator (2.39). The set $\mathscr{C}^+(x)$ characterizes the set of points where the maximum propagates from x.

The transfer mechanism between stochastic system (1.4) and deterministic control systems (3.1) is furnished by the so-called support theorems [33] which states that if P_x is the probability in $\mathscr{C}(\mathbb{R}^+, \mathbb{R}^d)$ for x_t with initial value $x \in \mathbb{R}^d$, and P_t, $t \in \mathbb{R}^+$, the semigroup corresponding to $P(t, \cdot, \cdot)$, then

$$\operatorname{supp}(P_x) = \mathscr{C}^+(x). \tag{3.3}$$

Actually, $\operatorname{supp}(P_x)$ is the law of the process P_x, starting in x_t; that is, the points which we will reach with a positive probability. Clearly (3.3) resembles an ergodicity property. The support Theorem of Stroock and Varadhan [34] is a bridge between diffusions and nonlinear (deterministic) control systems. Let $X_0, X_i,\ i = 1, \dots, r$, be $\mathscr{C}^\infty$ vector fields on a $\mathscr{C}^\infty$ manifold M of dimension d. Let $\mathscr{L}A(X_0, X_i)$ be the Lie algebra of vector fields generated by X_0, X_i, i.e. the smallest subalgebra of the vector fields on M that contains the X_0, X_i, and is closed under the Lie bracket operation $[X, Y]$, where in local components $x = (x_1, \dots, x_d)$ on M, reads:

$$[X, Y]_i = \sum_{j=1}^{d} \left(X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \right).$$

The second-order operator (2.35) is hypoelliptic (see [20]) on M, if

$$\dim \mathscr{L}(x) = d \qquad \text{for all } x \in \Omega, \tag{3.4}$$

i.e. the Lie algebra $\mathscr{L}A(Y, X_i)$ spans the whole tangent space T_xM for all $x \in M$. As a consequence of assumption (3.4) we have exact controllability in the interior of control sets. Under (3.4), it was shown in [7] that any solution of $Au = 0$ vanishing in neighborhood of a point of Ω must vanish in the hole of Ω. Using Bony's technique, one can show that $\mathscr{C}^+(x)$ contains all $\phi \in \mathscr{C}([0, \infty), \mathbb{R}^d)$ such that

$$\phi(t) = x_0 + \int_{t_0}^{t} Z(u, \phi(u))du + \int_{t_0}^{t} Y(u, \phi(u))du, \qquad t \geqslant t_0, \tag{3.5}$$

where Z is an element of the Lie algebra $\mathscr{L}A(X_1, \dots, X_d)$ generated by $X_1, \dots, X_d$.

Observe that condition (3.4) implies that the control system (3.1): (3.1) is accessible, i.e. $\mathscr{C}^+(x) \neq \emptyset$ for all $x \in \Omega$. This information gives another possibility for observing an ergodic behavior.
The following theorem show the connection between ergodicity and controllability.

Theorem 3.3. *The following statements are equivalent.*

(i) *The system (2.15) is mean ergodic in the sense of Cesàro implies that*

$$\begin{cases} Au = 0 \\ u \in \mathscr{C}^0(\Omega) \end{cases} \iff u = c.$$

(ii) *For every pair of points* $(x_0, y_0) \in \Omega$, $\forall\, x_0 \neq y_0$, *one has* $\mathscr{C}_{x_0} \cap \mathscr{C}_{y_0} \neq \emptyset$, *where* x_0 *is a maximum point and* y_0 *a minimum point and* $\mathscr{C}_x$ *is the closure of the set of points* $z \in \Omega$ *which is controllable from* x, *i.e.*

$$\{z \in \Omega \ s.t.\ \exists\, T \geqslant 0,\ \exists\, u \in L^2(0,T;\Omega)\ \ s.t. \quad \dot{x} = \sigma^\alpha(x)u_\alpha(t) + \beta(t), \quad x(0) = x,$$
$$satisfies \quad x(T) = z\},$$

where, u_α *are the coordinates of* u, $\mathscr{C}_{x_0}$ *is the set of accessibility and* $\mathscr{C}_{y_0}$ *is the set of controllability.*

Before going into the proof, some remarks are needed. Specifically we take the maximum at some point x_0 and this maximum point is propagated on certain set which is exactly the set $\mathscr{C}(x)$. Roughly speaking, when one takes a curve along the trajectory in $\mathscr{C}(x)$, according to the differential equation in (3.1), the first derivative of this curve, along this curve is equal to 0: It is a maximum point and then the gradient is equals to zero. This is a weak maximum principle; the second derivative with respect to time is zero. The Theorem 3.3 actually states that, when one takes a curve along the path of $\mathscr{C}(x)$, from the strong maximum principle one can integrate along this path and that remains constant. We make these ideas precise below.

Proof of Theorem 3.3. The proof is based on the analytical approach of Bony [7] which uses the relationship between the strong maximum principle and the propagation of maxima. Observe that if u takes a maximum at $x_0 \in \Omega$, then

$$\begin{cases} Au = 0, \\ u(x_0) = \max\limits_{\Omega}(u) \end{cases}$$

and one obtains:

$$\nabla u(x_0) = 0$$

but also all the operators of the second order vanish. For simplicity, we will say that if $a = \dfrac{1}{2}\sigma_i^\alpha \sigma_j^\alpha$, from the weak maximum principle, we will know

$$a_{ij}\partial_{ij}u(x_0) = 0. \tag{3.6}$$

Roughly speaking, the gradient of u vanishes at x_0, and the Hessian of u is non-positive definite at x_0, i.e.

$$\nabla u(x_0) = 0 \qquad \text{and} \quad \nabla^2 u(x_0) \leqslant 0.$$

In light of the Hopf's demonstration of strong maximum principle, one deduces that the fields the vector fields σ^α are always tangent to the closed set, which is the set of all points containing the connected component of the closed set containing x_0. This ensures a certain propagation along the integral curves of the vector field. Now, by virtue of Bony's interior principles [7], by choosing a maximum point x_0, for a regular function u, it follows that $Au = 0$, and $u(x_0) = \max(u)$ which implies $u(x) \equiv u(x_0)$ on $\mathscr{C}(x_0)$. ∎

4. Ergodicity and rate of convergence

This section is concerned with the problem (1.1) and specifically with the prove that the rate of convergence is related to the uniform ellipticity or hypoellipticity. Consider the controlled diffusion processes of the form (1.4). In the general setting, assume that X_t is a solution of (1.4) such that the distribution of x_0 is absolutely continuous and has the density $v(x)$. Then X_t has also the density $u(t,x)$ and satisfies the so called, Fokker-Planck (or forward Kolmogorov) equation

$$\begin{cases} \partial_t u + Au = 0 \\ u(0,x) = u_0 \in \mathscr{C}_b(\Omega) \end{cases} \tag{4.1}$$

where A is defined by (2.16), with coefficients $a_{ij}(x) > 0 \ \forall x$, $b_i(x)$ and $\sigma_{ij}(x)$ connected with the coefficients of SDE (1.4). The solution $u(t,x)$ of the Cauchy problem (4.1) represents the conditional transition probabilities of the process X_t, which solves SDE (1.4). Namely, the Feynman-Kac representation formula of (1.4) is

$$u(t,x) = \mathbb{E}[u_0(X_t)|X_0 = x_0]. \tag{4.2}$$

The easiest way to formulate precisely what we mean is to use the mathematical formulation of dynamic programming principle (see, e.g., A. Bensoussan-J. L. Lions [4], Krylov [23]). Accordingly, we define

$$M(R) \equiv \sup_{z \in B(R)} v(z), \qquad m(R) \equiv \inf_{z \in B(R)} v(z),$$

where $B(R)$ denotes the closed ball in $\mathbb{R}^d$ with center O, radius R, and we set

$$\operatorname*{osc}_{B(R)} v \equiv M(R) - m(R), \qquad 0 < R < R_0, \tag{4.3}$$

which denotes the oscillation function of v on $B(R)$.
The following theorem holds true.

Theorem 4.1. *Assumes that the equation (4.1) is hypoelliptic or uniformly elliptic. Then, $\forall u_0$ periodic and bounded function on $\mathbb{R}^d$, there is a constant $\overline{u} := \int u_0 dm$, such that the following estimate holds*

$$\sup_{\Omega} |u(t,x) - \overline{u}| \leqslant Ce^{-rt}\|u_0\|_{L^\infty}, \tag{4.4}$$

where r is a positive constant independent of x and t.

Note that the convergence in (4.4) implies mean ergodicity of the corresponding semigroup $(S_t)_{t \geqslant 0}$. Hence, Theorem 4.1 rests on the following lemma which we now establish.

Lemma 4.2. *Let $u(t) = S(t)u_0$ be the semigroup associated to (4.1). Then, if $u_0 \in L^\infty$*

$$\exists\, \kappa \in]0,1] \quad \text{such that} \quad osc(S(1)u_0) \leqslant \kappa \ osc(u_0), \tag{4.5}$$

where $osc(u)$ is defined in (4.3).

Before turning to a description of the proof, let us make some remarks. Modulo compactness, the strictly decreasing of the oscillation means that one looks at the maximum and minimum at the time 1, which are propagated in a certain set and one wants to say essentially that the only case where the oscillation does not decrease is the case where it is already constant; therefore everything is null. This corresponds essentially to say that

$$\exists\, T_0, \quad \forall\, x \neq y, \quad \exists\, T \leqslant T_0, \quad \mathscr{C}(x,T) \cap \mathscr{C}(y,T) \neq \emptyset$$

where

$$\mathscr{C}(x,T) = \texttt{cls}\{z \mid \text{ s.t. } \exists\, u \in L^2(0,T;\mathbb{R}^d) \text{ s.t. } \dot{x} = \sigma^\alpha(x)u_\alpha(t) + \beta(t) \\ x(0) = x, \quad x(T) = z\}.$$

This means $\mathscr{C}(x,T)$ crosses $\mathscr{C}(y,T)$ and will ensure that

$$\operatorname{osc}\left(S(t)u_0\right) \leqslant \kappa \operatorname{osc}(u_0),$$

if u_0 belongs to a family of continuous functions on a compact. Exploiting the above idea, one can use a method to ensure that there is an exponential convergence for ergodicity. We make these ideas precise below.

Proof of Lemma 4.2. Our proof is an adaptation of the methods developed by Lions [26] and the proof is obtained by contradiction. Observe that, adding or subtracting a constant if necessary to u_0, the estimate (4.5) remains invariant since the oscillation does not vary. Thus

- one can always renormalize the essential minimum of u_0 to be equal to 0 (since $u_0 \in L^\infty$).
- Also multiplying u_0 by any positive factor, the estimate (4.5) remains invariant; therefore we can renormalize u_0 so that the maximum of u_0 is 1.

Hence we must prove that the oscillation at time 1 cannot be equal to 1 uniformly. Indeed, since u_0 is renormalized, choosing $u_0 \in [0,1]$ with oscillation 1, then $S(1)u_0$ is compact within the space of continuous functions, as a regulated real-valued function. These observations allows to prove the lemma by contradiction. In order to do so, we truncate the space $\mathbb{R}^d$ by a ball of radius R and center 0 and then pass to the limit. Up to extraction of a subsequence denoted (u_0^n) we get

$$\operatorname*{osc}_{B(R)}(u_0^n) = 1 \quad \text{such that} \quad \operatorname*{osc}_{B(R)}(S(1)u_0^n) \xrightarrow[n\to+\infty]{} 1,$$

with $u_0^n \in [0,1]$ and $S(1)u_0^n$ compact in the space of continuous functions. Possibly at the cost of extracting a subsequence, we can deduce that

$$u_0^n \xrightarrow[n\to\infty]{w-L^\infty_*} u_0, \quad \text{such that} \quad 0 \leqslant u_0 \leqslant 1$$

and local maximum/minimum values decrease/increase monotonically in time. Possibly u_0^n is constant. But more generally, one knows that

$$u^n(t,x) \xrightarrow[n\to\infty]{\text{uniformly}} u(t,x) = S(t)u_0, \quad t > 0, \text{ in } x,$$

with $0 \leqslant u(t,x) \leqslant 1$, provided that

$$\underset{B(R)}{\operatorname{osc}}\,(S(1)u_0) = 1,$$

in view of the uniform convergence (in the $\max(u)$ at time 1, as well as for $\min(u)$ also). We now apply, the strong maximum principle to obtain

$$\max_{B(R)}(S(1)u_0) \neq 1 \qquad \text{and} \qquad \min_{B(R)}(S(t)u_0) \neq 0.$$

If this is not true, u is constant, this is because $S(1)u_0 \in [0,1]$ and also $u(t,x) \in [0,1]$ and oscillation of $u(t,x)$ is 1. For any continuous function u belonging to $[0,1]$ such that $\operatorname{osc}(u) = 1$, one has $\max(u) = 1$ and $\min(u) = 0$. This infers, thanks to the strong maximum principle, that if

$$\max_{x\in B(R)} u(x) = 1 \quad \Rightarrow u \equiv 1 \qquad \text{and} \qquad \min_{x\in B(R)} u(x) = 0 \quad \Rightarrow u \equiv 0, \tag{4.6}$$

then a contradiction. Thus (4.4) the proof of Lemma 4.2 is reached. ■

5. Applications to Hamilton-Jacobi-Bellman type equations

This section is concerned with some diffusion processes where ergodicity can be stated by the results of the present paper.

5.1. Mean-field games theory.

From the modeling viewpoint, the equation (4.1) is able to describe the penalization of congestion problem.
Let Ω be a bounded, smooth domain of $\mathbb{R}^d$, the interest is to minimize the following cost functional:

$$J(t,x) := \mathbb{E}\int_t^T \left\{\psi(X_T) + \frac{1}{q}|\alpha_s|^q\right\} ds, \tag{5.1}$$

where $\mathbb{E}$ denotes the average, $|\alpha_s|^q$ measures the cost of speed, $\frac{1}{q}|\alpha_s|^q$ is a Lagrangian depending on the position of the agent and the control; q is a parameter, ψ is the terminal cost. According to Eq (5.1) the cost of movement of an agent is affected by the density of neighborhood agents. The process X_t - the state process or the controlled process - is the solution of the following generic stochastic differential equation:

$$dX_t = \sigma dW_t - \alpha_t dt. \tag{5.2}$$

Let u be the Bellman (value) function, namely:

$$u(x) = \inf_{\alpha\in\mathcal{A}} J(x,\alpha), \qquad x \in \Omega, \tag{5.3}$$

where the inferior is taken over all admissible systems. According to the dynamical programming principle of R. Bellman, the value function $u \in \mathscr{C}^2(\Omega)$ is solution of the following second-order quasilinear elliptic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + H(\nabla u) = 0 \\ u(0,x) = u_0, \end{cases} \tag{5.4}$$

where H is the convex conjugate function of the Lagrangian, which reads:

$$\sup_{\alpha}\left\{\alpha\cdot\nabla_x u-\frac{|\alpha|^q}{q}\right\}=H(\nabla u). \tag{5.5}$$

The hamiltonian (5.5) is the convex conjugate function of Lagrangian, thus

$$(\alpha)^{q-1}=\nabla u, \tag{5.6}$$

where ∇u is the optimal control, with the following notation: For any p,

$$(p)^{q-1}:=|p|^{q-2}p.$$

With this notation and from (5.6) one has

$$\alpha=(\nabla u)^{1/q-1}.$$

Inserting the value of α in (5.5) leads to

$$\sup_{\alpha}\left\{\alpha\cdot\nabla_x u-\frac{|\alpha|^q}{q}\right\}=\sup_{\alpha}\left\{\left(1-\frac{1}{q}\right)(\nabla u)^{1/q-1}\cdot\nabla u=\frac{|\nabla u|^p}{p},\right. \tag{5.7}$$

and setting $\nu=\sigma^2/2$ (from d'Itô formula), we have

$$\begin{cases}\dfrac{\partial u}{\partial t}-\nu\Delta u+\dfrac{1}{p}|\nabla u|^p=0\\ u(0,x)=u_0\end{cases} \tag{5.8}$$

with the notation $(\xi)^r=|\xi|^{r-1}\xi$, $p>1$ is the conjugate exponent of q i.e. $p=\dfrac{q}{q-1}$, $1<q<\infty$, where, we change the sense of the time: $t\to T-t$, such that the initial condition becomes $u(0,x)=u_0$. The equation (5.8) is a very particular case of the so-called Hamilton-Jacobi-Bellman equation. If $p=2$ it is possible to prove that there exists a unique $\lambda\in\mathbb{R}$ such that:

(i) $u(t,x)/t$ converges uniformly to a constant $\lambda\in\mathbb{R}$;
(ii) $u(0,x)-\lambda t\to v$ solution of the following ergodic problem:

$$v-\nu\Delta v+\frac{1}{p}|\nabla v|^p=0.$$

5.2. The linear case.

In this subsection, we assume that Ω is a bounded domain of $\mathbb{R}^d$ and u_0 a periodic function in the variable x. The corresponding probability density $u(t,x)$ is solution of the forward Kolmogorov (Fokker-Planck) equation (2.40). Note that Eq. (2.40) is a linear partial differential equation, while (1.4) is a (generally non-linear) system of stochastic ordinary differential equations.

In what follows we assume that for all $i=1,\dots,d$, and $j=0,\dots m$, the functions σ_{ij} are sufficiently smooth and have bounded derivatives of all orders, and the coefficients of the matrix σ are also bounded. Moreover we assume that there exists:

$$\text{a bounded measurable } \sigma:[0,+\infty[\times\mathbb{R}^d\to\mathcal{S}_d(\mathbb{R})\qquad\text{such that}\qquad a=\sigma\sigma^T. \tag{5.9}$$

It is worth noting that in this case the existence and uniqueness of strong solutions is not always ensured. A related result is the following.

Lemma 5.1. *Assume that*

$$a = \sigma\sigma^T, \qquad \exists\ \nu \ \text{ such that } \ a \geqslant \nu I, \tag{5.10}$$

and that the initial condition $u_0 \in L^\infty(\mathbb{R}^n)$. Let X_t be solution of the stochastic differential equation (1.4). Then the function

$$u(x) = \inf_{\substack{\text{all choices}\\ of\ \alpha}} \mathbb{E}[u_0(X_t)] \tag{5.11}$$

is solution of the following Cauchy problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \inf\limits_{A\in\mathcal{A}_{c_0,\nu}} \operatorname{tr}(AD^2u), \\ u|_{t=0} = u_0 \not\equiv 0 \end{cases} \tag{5.12}$$

where α is the control, $A = (a_{ij}^\alpha)$ are constant matrices satisfying, for all t and x, the following uniform ellipticity condition:

$$\nu I \leqslant \{a_{ij}^\alpha\} \leqslant c_0 I. \tag{5.13}$$

Moreover the function u converges to some constant $\mathtt{c}$:

$$u(t) \xrightarrow[t\to\infty]{} \mathtt{c} = \int u_0 m, \quad m \geqslant 0, \quad \int m = 1. \tag{5.14}$$

Proof. The parabolic case follows by the elliptic setting. Let us note that in the absence of regularity, the resolution of the problem (1.4) poses problem as well as the resolution of the associated partial differential equation. The two keys points for proving the ergodicity underlying Lemma 5.1 are the maximum principle and the Pucci's extremal operators which will provide a partial differential equation. We prove the claims by the following three steps.

Step 1. We begin by normalizing u_0. Adding a constant to u_0, we may assume without loss of generality that $\texttt{ess-inf}\, u_0 = 0$. Since the equation is invariant, subtracting the infimum reduce to the case where the infimum is 0 and multiplying u_0 by what is needed ensure that $\texttt{ess-sup}\, u_0 = 1$. One can check classically that $\texttt{ess-sup} \geqslant 0$. If $\texttt{ess-sup} u_0 = 0$, everything remains constant and the inequality (4.5) is true with everything we want and thus as soon as the $\texttt{ess-sup}$ is not zero, one divides by a constant, this is because the equation (1.4) is a linear problem with the solution given

$$u(t,x) = \mathbb{E}_x[u_0(x_t)], \tag{5.15}$$

thus giving a trajectorial interpretation of the system.

Step 2. As we have no partial differential equations, we will introduce a non-decrease and non-increase function u by writing a stochastic control problem. Let us consider the upper and lower envelopes of u defined by

$$\overline{u} =: \sup_{\substack{\text{all controls s.t.}\\ c_0 I \geqslant a_{ij} \geqslant \nu I}} \mathbb{E}_x[u_0(x_t)] \quad \text{and} \quad \underline{u} := \inf_{\substack{\text{all controls s.t.}\\ c_0 I \geqslant a_{ij} \geqslant \nu I}} \mathbb{E}_x[u_0(x_t)]$$

where x_t is trajectory of the following stochastic differential equation

$$dx_t = \sqrt{2\sigma(t,\omega)}\, dW_t + b_i(x_t)dt, \tag{5.16}$$

namely $\sigma = a^{1/2}$ is a control. Here, $a^{1/2}$ is the positive square root of the matrix $(a_{ij}(x))$. By virtue of the definition of $\underline{u}$ and $\overline{u}$, one has $\underline{u} \leqslant \overline{u}$ in $\mathbb{R}^d \times [0,T]$ and since $\underline{u}$ and $\overline{u}$ are viscosity solutions, by strong comparison property, one has

$$\underline{u}(x,t) \leqslant u(t,x) \leqslant \overline{u}(x,t).$$

Rephrasing the stochastic differential equation (5.16), we said that: one chooses all adapted processes such as their square lies between $c_0 I$ and νI and one takes any process with which the square satisfies this terminal, and defines

$$dx_t = \sigma(t,\omega)dW_t + b(x_t,\omega)dt. \tag{5.17}$$

The problem is formulated as follows: one chooses a process and one looks at all these processes, one maximizes and thus $\overline{u}$ is certainly larger and of course if there is a solution (that it is strong or weak in a certain space) which is one of the elements of the collection, once one has a solution, one says that $\sigma(x_t)$ are certainly a process of x_t and of ω. The advantage is that now look for a stochastic control problem which should satisfy a certain partial differential equation called the Hamilton-Jacobi-Bellman equation, this is because one has a Markov property. This equation brings up the operator defined in (2.10). The function $\overline{u}$ solves the evolution equation

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial t} + \inf\limits_{\nu I \leqslant a_{ij}^{\alpha} \leqslant c_0 I} \left(-a_{ij}^{\alpha} \partial_{ij} \overline{u} \right) = 0 \\ \overline{u}|_{t=0} = u_0, \end{cases} \tag{5.18}$$

where the infimum is taken over all controls define above, a_{ij}^{α} (α is our control) are constant matrices. Eq. (5.18) can be rewritten in more abstractly and more compactly form

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial t} + \mathcal{P}^{+}(\overline{u}) = 0 \\ \overline{u}|_{t=0} = u_0, \end{cases} \tag{5.19}$$

where $\mathcal{P}^{+}$ denotes the extremal Pucci operators. One obtains now the envelope of all the generators through the infimum. Finally, we derive general parabolic Bellman problem of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = F(D^2 u) = \inf\limits_{A \in \mathcal{A}_{\nu,c_0}} \{ -\mathrm{Tr}(AD^2 u) \}, \qquad (x,t) \in \Omega, \\ u|_{t=0} = u_0 \not\equiv 0, \end{cases} \tag{5.20}$$

with $\Omega = \{(x,t) : x \in \mathbb{R}^n,\, t > 0\}$ and $\mathcal{A}_{\nu,c_0}$ is the set of all such matrices $A \in \mathcal{S}_n$, i.e., positive definite matrices, the eigenvalues of which belong to $[\![\nu, c_0]\!]$.

Step 3. As a consequence of our analysis, it follows from the L^p-theory developed in Krylov and Safonov [24], and Caffarelli [8, 9] that

- when $u_0 \in L^{\infty}$ the compactification is obtained (this is the parabolic version of Caffareli's results), for all strictly positive times, one has Hölder's estimates and Eq. (5.18) admits instantaneously a $\mathscr{C}^{1,1}$-solutions, thus one can apply the strong maximum principle.

- Therefore the maximum of $\overline{u}$ is obliged to lower proportionally to the maximum of u_0 otherwise the solution is constant.
- Therefore we can apply the lemma and we will actually have that the maximum of $\overline{u}(x,t)$ decreases strictly proportionally to the maximum of u_0 otherwise it is constant and everything is zero and the minimum of $\overline{u}$ grows strictly proportionally to the minimum of u_0 and one has compactness. Indeed, by using the representation of the solutions to write

$$\begin{aligned} \max_x \mathbb{E}[u_0(X_t)] &\leqslant \mathbb{E}[\max_x u_0(X_t^x)] \\ &\leqslant \max_z u_0(z). \end{aligned}$$

  Therefore, between the time 0 and time t, the maximum decreases; one has semigroup property. It means that the function is decreasing; if one changes u_0, we have the same estimate on the minimum.

Conclusion. Thanks to the Hamilton-Jacobi-Bellman equation (5.19), and in light of the Theorem 4.1, we obtain the following estimation

$$\left\| u(t) - \int u_0 m \right\|_\infty \leqslant Ce^{-rt}. \tag{5.21}$$

This means that the convergence of u to some constant:

$$u(t) \xrightarrow[t\to\infty]{} \mathsf{c} = \int u_0 m, \tag{5.22}$$

where m is a probability measure. The constant c is a linear, positive with respect to the initial condition u_0, preserving positivity. The required conclusion follows immediately from inequalities (5.21) and (5.22) and complete the proof of the Lemma 5.1. ■

5.3. The nonlinear case.

This section is devoted to the nonlinear case and specifically it consists of two subsections for taking into account the uniformly elliptic case (the matrix $a = \frac{1}{2}\sigma\sigma^T$ is definite positive) and the degenerate case. The type of result expected here is the limit (1.6)-(1.7).

5.3.1. *Uniformly elliptic framework.*

In this subsection we consider the case of nonlinear elliptic equations on a bounded domain. Specifically the Theorem 4.1 is generalized to the case of nonlinear operators of the form (1.2) in the uniformly elliptic framework. Accordingly we consider the following second-order partial differential equation:

$$\begin{cases} \partial_t u + F(x, u, Du, D^2u) = 0 \\ u(0,x) = u_0, \end{cases} \tag{5.23}$$

where the functional F defined by (2.6) is fully nonlinear in the sense that it is a second order equation in which the nonlinearity involves the second derivatives,

where $\mathcal{A}$ denotes a set of controls. Eq. (5.23) is usually called Hamilton-Jacobi-Bellman (briefly, HJB) equation. We assume that there exist matrices σ^α such that $a = \frac{1}{2}\sigma^\alpha(\sigma^\alpha)^T(x)$ and

$$\nu I \leqslant a_{ij}^\alpha(x) \leqslant c_0 I \tag{5.24}$$

where $0 < \nu \leqslant c_0$ positive constants, I the $d \times d$ identity matrix and a_{ij}^α, $b_i^\alpha \in L^\infty(\Omega)$.
Let CP the following Cauchy problem:

$$\texttt{CP}: \quad \begin{cases} \dfrac{\partial u}{\partial t} + \mathscr{A}_\alpha u = f \\ u(0,x) = 0, \end{cases} \tag{5.25}$$

where $\mathscr{A}_\alpha := -a_{ij}^\alpha(x)\partial_{ij} - b_i^\alpha(x)\partial_i$. Setting $u_t = v$ and differentiating Eq. (5.25) with respect to t, Eq. (5.25) rewrites as

$$\begin{cases} \dfrac{\partial v}{\partial t} + \mathscr{A}_\alpha v = 0 \\ v(0,x) = f. \end{cases} \tag{5.26}$$

It is natural to think that the solution $u(t,x)$ of the problem (5.23) when it exists, will behave as follows:

$$u(t,x) = v(x) + o_t(1), \tag{5.27}$$

where v is the unique solution of the associated stationary problem

$$F(x, v, Dv, D^2v) + \mathtt{c} = 0, \qquad v(x) = v_0. \tag{5.28}$$

However the problem (5.28) does not always admit solutions. In what follows we show that the unique solution u of (5.23) has the behavior (5.27) when the limiting problem (5.28) has a solution. If the equation (5.28) does not admit solution, we show that the asymptotic behavior of u is of the following type:

$$u(t,x) = \mathtt{c}t + v(x) + o_t(1), \tag{5.29}$$

where the constant $\mathtt{c}$ and the function v defined on Ω come from the associated "ergodic problem" consisting in finding a couple $(\mathtt{c}, v)$ such as u satisfied the equation (5.28).
The estimate (5.29) raises obvious questions concerning the existence of solutions of (5.26). Thus the question we address in this section is: What happens to the limit $u(t,x)/t$ when $t \to \infty$ (since obtaining u requires integration v; the passage from (5.26) to (5.25))? Is

$$\frac{u(t)}{t} \xrightarrow[t\to\infty]{\text{converges uniformly}} \mathtt{c}? \tag{5.30}$$

This question is equivalent to the Abelian-Tauberian Theorem question: Does

$$(5.30) \quad \Longleftrightarrow \quad \varepsilon u_\varepsilon \xrightarrow[\varepsilon\to 0]{\text{converge uniformly}} \mathtt{c},$$

where u_ε is a solution of the associate approximate stationary equation

$$\varepsilon u_\varepsilon + F(x, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) = 0?$$

By standard theory, c fulfills

$$\mathsf{c} = \lim_{t\to+\infty} \frac{u(t,x)}{t} = \lim_{\varepsilon\to 0^+} \varepsilon u_\varepsilon(x) \quad \text{uniformly in } x, \tag{5.31}$$

where u and $v_\varepsilon := \varepsilon u_\varepsilon$ are respectively the solution to problem (5.23) and to equation

$$v_\varepsilon + F(x, v_\varepsilon, Dv_\varepsilon, D^2 v_\varepsilon) = 0, \qquad x \in \mathbb{R}^d. \tag{5.32}$$

Hence, the behavior of the solution $u(t,x)$ of (5.23) helps to study the limit (5.31). In other words, the study of (5.31) leads to the introduction of the notion of ergodicity used in the theory of dynamical systems. It is the objective of what follows to verify these facts.

In what follows the following condition is assumed.

(**H1**) We assume that the coefficients $a_{ij} \equiv a_{ij}(t,x)$.

As consequence the main result of this section is the follows.

Theorem 5.2. *Let (H1) holds. Then, there exists an unique* c*, and* $r > 0$ *such that*

$$\|\partial_t u - \mathsf{c}\|_\infty \leqslant C_0 e^{-rt}. \tag{5.33}$$

We deduce that

$$\|u(t) - \mathsf{c}t\|_\infty \leqslant C_0 e^{-rt}, \tag{5.34}$$

where C_0 *is a constant. In addition, there exists a unique, up to additive constant, solution* v *of the problem*

$$F(x, v, Dv, D^2 v) + \mathsf{c} = 0. \tag{5.35}$$

Let us try to give an outline of the main ideas behind the proof of Theorem 5.2. The existence of v is classical since v represents (up to a multiplication by a constant) the solution of the stationary ergodic problem. When the equation is nondegenerate, which will be an assumption when looking for (1.6), the structure of the solutions of (5.28) is simpler: the solution of (5.28) is unique up to translations by constants. Then Cauchy problem (5.25) is nondegenerate when it is uniformly parabolic, i.e., the diffusion is elliptic, namely (5.24) holds. The new point here is the new method we use to obtain the rate of convergence through the method developed in Lemma 4.2. Applying the comparison principle to the Cauchy problem

$$\begin{cases} u_t + F(x, u, Du, D^2 u) = 0 \\ u(0,x) = 0 \end{cases} \tag{5.36}$$

then one has

$$\|u(t,\cdot) - \mathsf{c}t - v\|_{L^\infty} \leqslant \|v\|_{L^\infty}.$$

Letting $t \to +\infty$, one get

$$\frac{u(t,x)}{t} \to \mathsf{c} \quad \text{as } t \to +\infty, \text{ uniformly on } x.$$

Therefore, F is ergodic and $\mathsf{c} = -F$. The result of Theorem 5.2 follows the idea of Lions [1] to study the asymptotic behavior of these kind of equations.

Proof of Theorem 5.2. Differentiating the partial differential equation (5.23) with respect to the variable t, we obtain

$$\frac{\partial v}{\partial t} + F'_A \cdot D^2 v + F'_p \cdot Dv = 0, \tag{5.37}$$

where $F'_A := a_{ij}(x,t)$ and $F'_p := b(x,t)$ denotes the first partial derivative of F with respect to its ij-th entry (resp. in p). On the one hand, by (5.24) we know that F'_A is uniformly elliptic, with ellipticity positive constants ν and c_0 independent of the regularity of u. This property ensures that, for all $X, Y \in \mathcal{S}_d$ with $Y \geqslant 0$ (nonnegative definite)

$$\inf_{\nu I \leqslant M \leqslant c_0 I} (-\mathrm{Tr}(MX)) \leqslant F(X+Y) - F(X) \leqslant \sup_{\nu I \leqslant M \leqslant c_0 I} (-\mathrm{Tr}(MX)).$$

This means that the oscillations of the function F are trapped by such extremal operators introduced by Pucci. On the other hand, observe that Equation (5.37) is a parabolic equation which is time dependent. So we must slightly change the assumptions of Lemma 4.2 replacing $a_{ij}(x,t)$ with $a_{ij}(x,t+n)$. To be more precise, we assume that:

$$\text{there exists } \kappa \in [0,1] \quad (\text{independent of } n) \quad \text{such that} \quad \operatorname{osc}(n+1) \leqslant \kappa \operatorname{osc}(n). \tag{5.38}$$

This means to write Lemma 4.2 on $[0,1]$, but by shifting the coefficients for times which vary between n and $n+1$ or times that vary between 0 and 1. Thus Theorem 4.1 tells us that v converges exponentially fast to a constant and consequently

$$\text{there exists } r > 0, \text{ and } \mathsf{c} > 0 \text{ such that} \quad \|u_t - \mathsf{c}\| \leqslant C_0 e^{-rt}.$$

The time derivative satisfies an equation of the type we had already looked for and we did not require the regularity on the coefficients. Formally, we have at least one non-degenerate parabolic equation solved by $v(t)$ and right away we know that entails exponential convergence. From (5.34), the solution u asymptotically behaves like $u \sim \mathsf{c}t + v$, namely

$$u(t,x) = v(x) + \mathsf{c}t + o_t(1), \quad \text{as} \quad t \to \infty,$$

where (c, v) is a solution of the stationary ergodic problem (5.35). Hence

$$\|u(t) - \mathsf{c}t - v\| \leqslant Ce^{-rt},$$

which ends the proof. ∎

5.3.2. *Degenerate framework.*

An other question we address in this paper, is the connections with the large time behavior of the solutions of the controlled Itô stochastic differential equation in $\mathbb{R}^d$

$$\begin{cases} dX_t = \sigma^{\alpha_t}(X_t) dW_t + f^{\alpha_t}(X_t) dt, & t > 0, \\ X_0 = x. \end{cases} \tag{5.39}$$

This problem is interesting when the diffusion process described by (5.39) is degenerate, namely, the matrix $a^\alpha := \frac{1}{2}\sigma^\alpha (\sigma^\alpha)^T$ is merely positive semidefinite for

some α. This will be a situation where the associated Legendre transform F is

$$F(x,p,A)=\sup_{\alpha\in\mathcal{A}}\left(a_{ij}^{\alpha}(x)A_{ij}-b_i^{\alpha}p_i-f_i^{\alpha}(x)\right), \tag{5.40}$$

where $A_{ij}:=\partial_{ij}$. We impose the following conditions on the drift term b and the first spatial derivative of σ are uniformly Lipschitz continuous in x. For σ and b, we form the operator

$$\frac{1}{2}\sigma^{\alpha}\nabla_x\cdot\sigma^{\alpha}\nabla_x+b\cdot\nabla_x=X_0+\frac{1}{2}\sum_{\ell=1}^{d}X_{\ell}^2, \tag{5.41}$$

by the prescription given in subsection 2.3. Roughly speaking, there is a collection parameterized by α which describes a set $\mathcal{A}$ of operators and the envelope of these operators is taken; this is Pucci type operators. The set $\mathcal{A}$ is the set of all matrix between νI and $C_0 I$. In (5.40), the coefficients depend on x and are all uniformly in α and regular in x.

Thus we had simply taken a collection of operators as within the framework linear and taken their envelope by sup. That corresponds to minimize the expectations of all the processes associated with these operators with the possibility constantly of choosing like dynamics one or the other of these diffusions. Thus one can start and choose that at any moment in a random way; one starts with a dynamics then one switch over the second ... etc. one finds at the end a process; one calculates its expectation and thus one minimizes compared to any possible choice. This gives an infinitesimal equation which is no longer parabolic equation for a semigroup process to a diffusion process, but what it gives us is the envelope of all these operators and all these equations through the operation of sup.

The methods and arguments discussed in the preceding sections link the theory of viscosity solutions and the fact that u_ε (resp. $u(T,x)$) is the unique continuous viscosity solution of the equation of Hamilton-Jacobi-Bellman associated (HJB), that means,

$$\varepsilon u_\varepsilon+\sup_{\alpha\in\mathcal{A}}(A^{\alpha}u-f^{\alpha})=0,\qquad x\in\Omega \tag{5.42}$$

respectively

$$\begin{cases}\partial_t u+\sup_{\alpha\in\mathcal{A}}(A^{\alpha}u-f^{\alpha})=0, & (x,t)\in\Omega\times(0,\infty)\\ u|_{t=0}\equiv 0, & x\in\Omega\end{cases} \tag{5.43}$$

and A^{α} is operator A with parameter α defined in (2.16). Using the Hamiltonian, we can reformulate the associated F as

$$F(x,p,A)=\sup_{\alpha\in\mathcal{A}}[-\mathrm{Tr}(a^{\alpha}\cdot A)-b^{\alpha}\cdot p-f_{\alpha}]. \tag{5.44}$$

We will answer the question raised in previous sections regarding the limit of the functions, as $t\to+\infty$, of the Eq. (5.43). It is well known that in L^{∞} framework, it holds

$$\max_x(\varepsilon u_\varepsilon)\ \xrightarrow[\varepsilon\to 0]{\text{converges}}\ M\qquad\text{et}\qquad\min_x(\varepsilon u_\varepsilon)\ \xrightarrow[\varepsilon\to 0]{\text{converges}}\ \mathfrak{m}. \tag{5.45}$$

Actually

$$\max_x(\varepsilon u_\varepsilon) \text{ is decreasing}, \quad \text{and} \quad \min_x(\varepsilon u_\varepsilon) \text{ is increasing}.$$

This ensures that if one looked at the evolution problem (5.43), the ergodic theorem states that

$$\max_x t^{-1}u(t) \xrightarrow[t\to\infty]{\text{converges}} M \qquad \min_x t^{-1}u(t) \xrightarrow[t\to\infty]{\text{converges}} \mathfrak{m}.$$

This attempt to replicate what we have already done, that is the convergence of the maximum.

Convergence of $\varepsilon u_\varepsilon$ and characterization of the limit number. Here we investigate the regions where u_ε uniformly converges to some constant function. This can be guaranteed by suitable controllability assumptions on our system. We will adapt the arguments of the preceding section. We assume that

(**H2**) $\mathscr{C}^+(x) = \Omega$, for all $x \in \Omega \Rightarrow v = \text{Constant}$.

This assumption corresponds to the requirement that one has controllability starting from any point. Note also that (H2) implies that the system is ergodic (convergence to a constant). Next, in order to apply the tools involved in these theories some compactness properties of the process (or semigroup) defined by the equation are requested. Roughly speaking, to study the ergodic problem, we introduce additional hypothesis ensuring the characterization of ergodicity.

Following the subsection 2.2, we assume that

(**H3**) $(u(t,x)/t)_{t>0}$ (resp. $\varepsilon u_\varepsilon$) is equicontinuous on $\mathbb{R}^d$ (resp. uniformly continuous, uniformly in $\varepsilon > 0$).

The above assumption means that $(u(t,x)/t)_{t>0}$ is compact. Our next theorem is a resolution of the limit problem. It claims that the family $(\varepsilon u_\varepsilon)$ converges to a constant.

Theorem 5.3. *Let Assumptions (H2)-(H3) hold and $a_{ij}^\alpha \in \mathscr{C}^\infty(\mathbb{R}^d)$ for all $\alpha \in \mathcal{A}$. Then $v_\varepsilon := \varepsilon u_\varepsilon$ (resp. $\dfrac{1}{t}u(t,\cdot)$) converges uniformly on $\mathbb{R}^d$ to a constant $\mathtt{c}$ as ε goes to 0^+ (resp. $t \to \infty$). Furthermore, $\mathtt{c}$ is the unique constant such that there exists a solution v to equation*

$$F(x, v, Dv, D^2v) + \mathtt{c} = 0, \qquad x \in \Omega \subseteq \mathbb{R}^d. \tag{5.46}$$

Let us try to give an outline of the main ideas behind the proof. The proof is delicate when the matrix A is degenerate since the structure of the solutions of (5.46) can be very complicated. One can obtain easily in general that the ergodic constant c is unique but there can be a lot of solutions of (5.46). It is obvious that, given a solution v of (5.46), all the translations by constants are still solutions but there can exist even solutions which are not translation of a given one by constants, see Lions *et al.* [27]. Actually, in a linear framework, this issue was solved by using a point of maximum and a point of minimum and said that if one can check from a point of a maximum and check a point from a point of minimum and get as close as one wants, the assertion is proved: that is to say, the maximum is equal to the

minimum because the maximum is propagated. But, here there is a non-linear equation and we cannot work with a point of minimum, because they are not at all the same operators of our case. Actually, the operator who counts, it is that which maximizes here and in another point, it is another operator who counts. Thus to compare things does not makes no sense. However, this argument breaks down in the nonlinear case. Therefore, we cannot work at the same time with the properties of the maximum and the minimum. In other words, which we did in the linear case breaks down in the non-linear case. On the other hand, we can try to work only with the maximum, that is to use the hypothesis (H2). The strong maximum principle is essential to prove the existence of the ergodic number $\mathtt{c}$ in theorem 5.3.

Proof of Theorem 5.3. From equicontinuity assumption (H3), and setting

$$v_\varepsilon = \varepsilon u_\varepsilon,$$

the problem is now to determine the behavior of v_ε as ε goes to 0. We can consequently reinterpret Eq. (5.43) as follows. Replacing u_t with v in the equation (5.43), then the problem reduces to the study the following equation:

$$u + \sup_\alpha (A^\alpha u - f^\alpha) = 0. \tag{5.47}$$

Multiplying the equation (5.47) by ε, is recovered

$$\sup_\alpha (A^\alpha v_\varepsilon - \varepsilon f^\alpha) + \varepsilon v_\varepsilon = 0. \tag{5.48}$$

From the Arzelà-Ascoli Theorem, upon letting $\varepsilon \to 0$, possibly at the cost of extracting a subsequence, we have that

$$v_\varepsilon = \varepsilon u_\varepsilon \xrightarrow[\varepsilon \to 0]{\text{converges uniformly}} v,$$

which, in view of the structural stability of viscosity solutions is a solution of the homogeneous equation

$$\sup_\alpha (A^\alpha v) = 0. \tag{5.49}$$

This infers that

$$A^\alpha v = 0, \quad \text{on } \mathbb{R}^d, \quad \forall\, \alpha \in \mathcal{A}. \tag{5.50}$$

From hypothesis (H3), for any sequence $\varepsilon_n > 0$ going to 0, there exists a subsequence, still denoted by ε_n, such that $\varepsilon_n u_{\varepsilon_n}$ converges uniformly on $\mathbb{R}^d$ to some v which satisfies (5.50).
As an immediate consequence of (5.49), we deduce that

$$A^\alpha v \leqslant 0, \quad \text{in } \mathbb{R}^d, \quad \forall\, \alpha \in \mathcal{A},$$

and as well-known this is equivalent to

$$A^\alpha v \leqslant 0, \quad \text{in } \mathscr{D}'(\mathbb{R}^d), \quad \forall\, \alpha \in \mathcal{A}.$$

Then regularizing v by conclusion if necessary, one obtains $A^\alpha v = 0$ in $\mathscr{D}'(\mathbb{R}^d)$, $\forall\, \alpha \in \mathcal{A}$. Next, we conclude the proof by showing that v is a constant. The strong maximum principle states that v attains a maximum at $x_0 \in \Omega$; then v is a constant function. From (5.45) the maximum of v is identified:

$$\max(v) = M,$$

and thus, we are going to be able to conclude that v is constant. In order to do so, we have to examine how the maximum is propagated. By virtue of assumption (H2), the propagation set $\mathscr{C}^+(x)$ of the maximum at x, is the entire space $\mathbb{R}^d$, on which we will have the possibility of taking any control, and Eq. (5.49) corresponds to

$$v = \inf_{\alpha_t} \mathbb{E}[v(x_t)],$$

where x_t is a process corresponding to a control $\alpha(t,\omega)$. This implies a strong maximum principle for a class of operators that are not strictly elliptic and there is propagation, not only for one operator, but for all operators. We may then apply the strong maximum principle to deduce that v is constant and our claim is proven. ■

6. Conclusions and perspectives

This paper has been devoted to the link between the asymptotic analysis of PDE and the ergodicity problem in dynamical systems theory. According to Abel-Tauberian type theorems, the proof of mean ergodicity relies on the limit of $(\varepsilon u_\varepsilon)_{\varepsilon>0}$, when ε tends to zero, and on the analysis of the resolvent equation. The paper proposes the link among the asymptotic behavior, ergodicity, and controllability as criterium for the analysis of nonlinear PDEs. To the best of our knowledge, these features (especially the last one) have never been tackled up in literature. The proof of the main results of the present paper is based on the method introduced in [26]. In particular the method proposed in this paper works clearly for the second order fully parabolic integro-differential equation such as those appearing in stochastic control of jump diffusion process. One of the novelties of our work is that the proposed methods allow to approach also non-linear problems, especially the degenerate case where controllability sets still play an important role. These questions were tackled by many authors but from the stochastic point of view [33, 2, 36, 28].

Concerning our results, some open problems can be addressed. For instance, ergodicity has been investigated under the equicontinuity assumption. A question thus arises: Can equicontinuity be relaxed? It is worth stressing that the assumption that the family $(u_\varepsilon)_{\varepsilon>0}$ is locally equibounded and equicontinuous has allowed to employ the Ascoli-Arzela theorem and standard diagonal arguments in order to conclude the existence of a solution of (1.8).
The following problem set the motivation of this question. Specifically, we consider the following equation

$$\begin{cases} -a_{ij}(x)\partial_{ij}u_\varepsilon + H(\nabla u_\varepsilon) + \varepsilon u_\varepsilon = f \\ a_{ij} = \frac{1}{2}\sigma\sigma^T \end{cases} \tag{6.1}$$

where $\epsilon > 0$, $H(\nabla u_\varepsilon) = |\nabla u_\epsilon|$ where $|\cdot|$ is Euclidean norm with periodic boundaries conditions, σ and f are regular, the matrix (a_{ij}) is assumed symmetric, positive or null matrices and not necessarily non-degenerate. What about the ergodic problem? Is that $(u_\varepsilon)_{\varepsilon>0}$ equicontinuous? $(u_\varepsilon)_{\varepsilon>0}$ converges to a constant? Clearly,

for $a_{ij} = 0$, one has $|\varepsilon u_\varepsilon| \leqslant C$, that is $v_\varepsilon := \varepsilon u_\varepsilon$ is bounded (see e.g.[27]). For $a_{ij} > 0$, Eq. (6.1) is uniformly elliptic and $|v_\varepsilon| \leqslant C$ is bounded from

$$-a_{ij}(x)\partial_{ij}v_\epsilon + |\nabla v_\epsilon| = \varepsilon f.$$

Hence, the answer of our question is positive. But, what happens if the matrix (a_{ij}) is not uniformly elliptic or $a_{ij} = 0$? In one-dimensional case, it is not hard to show that Eq. (6.1) writes as follows:

$$-au_\varepsilon'' + |u_\varepsilon'| + \varepsilon u_\varepsilon = f, \tag{6.2}$$

which is bounded and thereby, it is not very complicated to convince ourselves that this involve that u_ε' is bounded, independently of ε, thus making possible to obtain a priori estimates.

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