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MATCHING CELLS

Gaël Meigniez

ABSTRACT. A (complete) matching of the cells of a triangulated manifold can be thought as a combinatorial or discrete version of a nonsingular vector field. This note gives several methods for constructing such matchings.

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On a triangulated manifold (all triangulations being understood smooth), a "complete matching" (for short we just say "matching") is a partition of the set of the cells into pairs such that in each pair, one of the two cells is a hyperface of the other. Such objects are regarded as a combinatorial equivalent to nonsingular vector fields — a viewpoint inspired by Forman's works [4], see also [5]. The present note intends to provide some methods for the construction of matchings, either allowing oneself to subdivide the triangulation, or not. We feel that the methods are more important than the existence results themselves. A first approach is algebraic, playing with Hall's "marriage theorem" and cellular homology; a second one is geometric: a matching is deduced from an ambiant nonsingular vector field transverse to the cells, or from a round handle decomposition of the manifold.

In a first time, manifolds are not mandatory, nor simplices. Consider generally a polyhedral cellular complex X (the cells are convex polyhedra, finiteness is understood everywhere) and a subcomplex $Y \subset X$. Write $\Sigma(X,Y)$ for the set of the cells of X not lying in Y. Call two cells *incident* to each other if one is a hyperface of the other.

DEFINITION 1. A *matching* on X *relative to* Y*,* or a matching *on the pair* (X,Y)*,* is a partition of $\Sigma(X,Y)$ into incident pairs.

As usual, for $Y=\emptyset$ we write $\Sigma(X)$ instead of $\Sigma(X,\emptyset)$ and we speak of "a matching on X".

The cases of the complexes of dimension 1 and of the triangulations of surfaces will easily follow from a few general remarks.

REMARK 2 (Euler characteristic). *If* (X,Y) *is matchable, then the relative Euler characteristic* $\chi(X,Y)=\chi(X)-\chi(Y)$ *vanishes.*

REMARK 3 (Collapse). *Every collapse of a polyhedral complex X onto a subcomplex Y gives a matching on X rel. Y.*

Indeed, a collapse is nothing but filtration of X by subcomplexes X_n, $0 \leq n \leq N$, such that $X_0 = Y$ and $X_N = X$ and $\Sigma(X_n, X_{n-1})$ consists, for each $1 \leq n \leq N$, of exactly two incident cells. (More precisely, an *orbit* in a matching is defined as a finite sequence σ_0, σ_1, $\dots \in \Sigma(X, Y)$ such that for every odd k, the cell σ_{k-1} is a hyperface of σ_k and its mate; and the cell σ_{k+1} is a hyperface of σ_k and not its mate. A collapse of X onto Y amounts to a matching of X rel. Y without *cyclic* orbit.)

REMARK 4 (Top-dimensional cycle). *Every cellulation of the circle admits two matchings.*

More generally, let X be a polyhedral cellulation of a manifold; let ℓ be a simple loop in the 1-skeleton of the dual cellulation; let $Y \subset X$ be the union of the cells of X disjoint from ℓ. Then, X admits two matchings rel. Y.

EXAMPLE 5 (Graphs). *Every connected graph whose Euler characteristic vanishes is matchable.*

Indeed, such a graph collapses onto a circle.

EXAMPLE 6 (Surfaces). *Let M be a compact, connected 2-manifold such that $\chi(M) = 0$. Then, every polyhedral cellulation X of M is matchable absolutely, and relatively to ∂M.*

Proof. First case: M is the annulus or the Möbius strip. Then, the 1-skeleton of the cellulation dual to X contains an essential simple loop ℓ such that M cut along ℓ is an annulus or two annuli. So, the union $Y \subset X$ of the cells of X disjoint from ℓ collapses onto ∂M. The pair (X, Y) is matchable (Remark 4), the pair $(Y, \partial M)$ is matchable (Remark 3), and $X|\partial M$ is matchable (Remark 4).

Second case: M is the 2-torus or the Klein bottle. Then, the 1-skeleton of the cellulation dual to X contains an essential simple loop ℓ such that M cut along ℓ is an annulus. Consider the union $Y \subset X$ of the cells of X disjoint from ℓ. The pair (X, Y) is matchable (Remark 4) and the annulus Y is matchable (first case). $\square$

Next, recall Hall's so-called "marriage theorem". Let $\Sigma := \Sigma_0 \sqcup \Sigma_1$ be a finite, $\mathbf{Z}/2\mathbf{Z}$-graded set and let I be a symmetric relation in Σ, of degree 1. For every subset $A \subset \Sigma$, denote by $|A|$ its cardinality, and denote by $I(A) \subset \Sigma$ the subset of the elements I-related to at least one element of A. A *matching* on Σ w.r.t. I is a partition of Σ into I-related pairs.

THEOREM 7 (Hall [6]). *The following properties are equivalent:*

(1) The relation I is matchable;
(2) One has $|A| \leq |I(A)|$ for every $A \subset \Sigma$;
(3) $|\Sigma_1| = |\Sigma_0|$ and one has $|A| \leq |I(A)|$ for every $A \subset \Sigma_0$.

Also recall that the Ford-Fulkerson algorithm [3][2] computes a matching, if any, in time $O(|\Sigma|^2|I|)$, thus giving some (moderate) effectiveness to the existence results below.

Coming back to a pair of polyhedral complexes (X, Y), we write $\Sigma_0(X, Y)$ (resp. $\Sigma_1(X, Y)$) for the set of the cells of X of even (resp. odd) dimension not lying in Y. Some counterexamples of unmatchable complexes will follow from the *trivial* sense of Hall's criterium.

EXAMPLE 8. *A connected simplicial 2-complex whose Euler characteristic vanishes, unmatchable as well as its subdivisions.*

Let S (resp. T) be a triangulated 2-sphere (resp. circle); let $X := S * T * T$ be the bouquet, at some common vertex v, of S with two T's. Then, $\chi(X) = 0$, but X does not admit any matching. Indeed, for $A := \Sigma_0(S, v)$, one has $I(A) = \Sigma_1(S)$, hence $|I(A)| = |A| - 1$. The same holds for any subdivision of X.

EXAMPLE 9. *An unmatchable triangulated closed 4-manifold whose Euler characteristic vanishes.*

Let n be even and at least 4. Let M_0 be a closed $(n-1)$-manifold which is the boundary of a compact n-manifold. Let X_0 be a triangulation of M_0. It is easy to make two compact n-manifolds M_i, $i = 1, 2$, bounded by M_0, and whose Euler characteristics verify :

$$\chi(M_2) = -\chi(M_1) > |\Sigma_0(X_0)|$$

Extend X_0 to some triangulation X_1 of M_1 and to some triangulation X_2 of M_2. Let M (resp. X) be the union of M_1 (resp. X_1) with M_2 (resp. X_2) over M_0 (resp. X_0). Then $\chi(M) = 0$, but its triangulation X does not admit any matching. Indeed, the set $A := \Sigma_0(X_2, X_0)$ has $I(A) = \Sigma_1(X_2)$; hence:

$$|I(A)| = |\Sigma_0(X_2)| - \chi(X_2) = |A| + |\Sigma_0(X_0)| - \chi(X_2) < |A|$$

On the other hand, in dimension 3, E. Gallais has proved that every closed 3-manifold admits a matchable triangulation [5].

QUESTION 10. *Is* every *triangulation of every closed 3-manifold matchable? Is every triangulation of every closed odd-dimensional manifold matchable?*

LEMMA 11 (Acyclic pair). *If $H_*(X, Y) = 0$, then the pair (X, Y) is matchable.*

Rational coefficients are understood everywhere; one could as well use $\mathbf{Z}/2\mathbf{Z}$.

Proof. This is an application of Hall's criterium. For $n \geq 0$, consider as usual the set Σ^n of the n-dimensional cells of X not lying in Y; the chain vector space C_n of basis Σ^n; the differential $\partial_n : C_n \to C_{n-1}$; and its kernel Z_n. Consider the union X_n of Y with the $(n-1)$-skeleton of X and with some n-cells which span a linear subspace complementary to Z_n in C_n. The sequence (X_n) is a filtration of the pair (X, Y) by subcomplexes, and $H_*(X_n, X_{n-1}) = 0$.

One is thus reduced to the case where moreover, $\Sigma(X, Y) = \Sigma^n \cup \Sigma^{n-1}$ for some $n \geq 1$. Note that necessarily, $|\Sigma^n| = |\Sigma^{n-1}|$. For every $A \subset \Sigma(X, Y)$, let $\langle A \rangle \subset C_*$ denote the spanned linear subspace; recall that $I(A) \subset \Sigma(X, Y)$ is the set of the cells incident to at least one cell belonging to A; hence $\partial\langle A \rangle \subset \langle I(A) \rangle$. If $A \subset \Sigma^n$, since ∂_n is linear and one-to-one:

$$|A| = \dim(\langle A \rangle) = \dim(\partial\langle A \rangle) \leq |I(A)|$$

By the equivalence of (1) with (3) in the marriage theorem, the pair (X, Y) is matchable. $\square$

COROLLARY 12 (Subdivision). *Let (X, Y) be a pair of polyhedral complexes. Assume that (X, Y) is matchable.*

Then, every polyhedral subdivision (X', Y') of (X, Y) is also matchable.

Proof. Consider a matching on (X, Y). For each matched pair $\sigma, \tau \in \Sigma(X, Y)$ with $\tau \subset \sigma$, consider the union $\hat{\partial}\sigma := \partial\sigma \setminus Int(\tau)$ of the other hyperfaces of σ. The restriction $(X'|\sigma, X'|\hat{\partial}\sigma)$ is a pair of polyhedral complexes, which admits a matching by Lemma 11. Clearly, the collection of all these partial matchings constitutes a global matching for the pair of complexes (X', Y'). $\square$

COROLLARY 13 (Rational homology sphere). *Let M be a rational homology sphere of odd dimension n.*

Then, every polyhedral cellulation X of M is matchable.

Proof. One can assume that $n \geq 3$. Fix a $(n-1)$-cell σ of X and a hyperface $\tau \subset \sigma$. Consider in X the union Y of τ with the cells of X not containing τ. First, the pair (X, Y) is matchable (Remark 4). Second, $H_*(Y, \partial\sigma) = 0$, hence the pair $(Y, \partial\sigma)$ is matchable (Lemma 11). Third, the polyhedral complex $\partial\sigma$, being homeomorphic to the $(n-2)$-sphere, is matchable by induction on n. $\square$

COROLLARY 14 (Betti number 1). *Let M be a closed 3-manifold whose first Betti number is 1.*

Then, every polyhedral cellulation X of M is matchable.

Proof. The 1-skeleton of X and the 1-skeleton of the dual cellulation contain respectively two homologous essential simple loops ℓ, ℓ^*. Consider the union $Y \subset X$ of the cells of X disjoint from ℓ^*. First, the pair (X, Y) is matchable (Remark 4). Second, $H_*(Y, \ell) = 0$, hence the pair (Y, ℓ) is matchable (Lemma 11). Third, the circle ℓ is matchable (Remark 4). □

Now, consider a triangulation X of a compact manifold M of dimension $n \geq 1$ with smooth boundary ∂M (maybe empty). If a nonsingular vector field ∇ on M is transverse to every $(n-1)$-simplex of X, we say for short that ∇ is *transverse to X*. Note that in particular, ∇ is then transverse to ∂M; thus ∂M splits as the disjoint union of $\partial_s(M, \nabla)$, where ∇ enters M, with $\partial_u(M, \nabla)$, where ∇ exits M.

THEOREM 15 (Transverse nonsingular vector field). *If the nonsingular vector field ∇ is transverse to the triangulation X, then X is matchable rel. $\partial_u(M, \nabla)$.*

Proof. Because of the transversality, for every simplex $\sigma \in \Sigma(X)$ of dimension less than n and not contained in $\partial_u(M, \nabla)$ (resp. $\partial_s(M, \nabla)$), there is a unique *downstream* (resp. *upstream*) n-simplex $d(\sigma)$ (resp. $u(\sigma)) \in \Sigma^n(X)$ containing σ and such that the vector field ∇ enters $d(\sigma)$ (resp. exits $u(\sigma)$) at every point of $Int(\sigma) := \sigma \setminus \partial\sigma$. For $\sigma \in \Sigma^n(X)$, we put $d(\sigma) := u(\sigma) := \sigma$.

Consider any n-simplex $\delta \in \Sigma^n(X)$ and any face $\sigma \subset \delta$ (the case $\sigma = \delta$ is included.) We call σ *stable* (resp. *unstable*) with respect to δ if $\sigma \not\subset \partial_u(M, \nabla)$ (resp. $\sigma \not\subset \partial_s(M, \nabla)$) and if $d(\sigma) = \delta$ (resp. $u(\sigma) = \delta$). Note that

- Every *hyperface* of δ is either stable or unstable;
- δ has at least one stable hyperface (for degree reasons);
- σ is stable if and only if every hyperface of δ containing σ is stable.

Next, for each $\delta \in \Sigma^n(X)$ we pick arbitrarily a base vertex $v(\delta)$ *in the intersection $\partial_-\delta$ of the unstable hyperfaces of* δ (here of course, it is mandatory that δ is a simplex rather than a general convex polytope.) To this choice there corresponds canonically a matching, as follows. For every simplex $\sigma \in \Sigma(X, \partial_u(M, \nabla))$ we define its mate $\bar{\sigma}$ by:

(1) If $v(d(\sigma)) \in \sigma$ then $\bar{\sigma}$ is the hyperface of σ opposed to $v(d(\sigma))$;
(2) If $v(d(\sigma)) \notin \sigma$ then $\bar{\sigma}$ is the join of σ with $v(d(\sigma))$.

These rules do define a matching on the pair $(X, \partial_u(M, \nabla))$: the point here is that $\bar{\sigma}$ is also a stable face of $d(\sigma)$. Indeed, if not, then $\bar{\sigma}$ would be contained in some unstable hyperface η of $d(\sigma)$; but in both cases (1) and (2) above, this would imply that σ itself would be contained in η, a contradiction. In other words, $d(\bar{\sigma}) = d(\sigma)$: the map $\sigma \mapsto \bar{\sigma}$ induces locally, for each n-simplex δ, an involution in the set of the stable faces of δ; and thus globally a matching on $\Sigma(X, \partial_u(M, \nabla))$.

Note — It can be suggestive, for $n = 2$ and $n = 3$ and for each $0 \leq i \leq n-1$, to figure out in $\mathbf{R}^n$, endowed with the parallel vector field $\nabla := -\partial/\partial x_n$, a linear n-simplex δ in general position with respect to ∇ and such that $\dim(\partial_- \delta) = i$; to list the stable faces and the unstable faces; to choose a base vertex $v \in \partial_- \delta$; and to compute the corresponding matching between the stable faces. □

In particular, the Hall cardinality conditions also constitute some *combinatorial* necessary conditions for a triangulation to admit a transverse nonsingular vector field. For example, in Example 9, not only X does not admit any transverse nonsingular vector field (which is obvious since such a field would be transverse to M_0, in contradiction with $\chi(M_1, M_0) \neq 0$), but this holds also for every triangulation of M combinatorially isomorphic with X (e.g. every jiggling of X).

COROLLARY 16 (Matchable subdivision). *Let M be a compact connected manifold with smooth boundary, and let $\partial_0 M$ be a union of connected components of ∂M such that $\chi(M, \partial_0 M) = 0$.*

Then, every triangulation X of M admits a subdivision matchable rel. $\partial_0 M$.

Either of the two sets $\partial_0 M$ and $\partial_1 M := \partial M \setminus \partial_0 M$ may be empty, or both.

Proof. Since $\chi(M, \partial_0 M) = 0$, there is on M a nonsingular vector field ∇ transverse to ∂M, which exits M through $\partial_0 M$, and which enters M through $\partial_1 M$. Then, by W. Thurston's famous Jiggling lemma [7], one has on M a triangulation X' which is combinatorially isomorphic to some iterated crystalline subdivision of X, and which is transverse to ∇. By Theorem 15, X' is matchable. □

Finally, we give an alternative construction for Corollary 16 which works in every dimension, but 3. Note that, by Corollary 12 and the Hauptvermutung for smooth triangulations, it is enough to construct *one* triangulation of M matchable relatively to $\partial_0 M$.

Since $n \geq 4$ and $\chi(M, \partial_0 M) = 0$, the pair $(M, \partial_0 M)$ admits a *round handle decomposition* [1]. One chooses a triangulation of M for which each handle is a subcomplex. Hence, one is reduced to the case of a round handle

$$M = \mathbf{S}^1 \times \mathbf{D}^i \times \mathbf{D}^{n-i-1}$$

$$\partial_0 M = \mathbf{S}^1 \times \mathbf{S}^{i-1} \times \mathbf{D}^{n-i-1}$$

for some $0 \leq i \leq n-1$ (we agree that $\mathbf{S}^{-1} = \emptyset$). Since $\mathbf{D}^i \times \mathbf{D}^{n-i-1}$ retracts by deformation onto the union of $\mathbf{D}^i \times 0$ with $\mathbf{S}^{i-1} \times \mathbf{D}^{n-i-1}$, by Lemma 11 the proof is reduced to the case where $M = \mathbf{S}^1 \times \mathbf{D}^i$ and $\partial_0 M = \mathbf{S}^1 \times \mathbf{S}^{i-1}$. In that case, let X be any triangulation of M. The 1-skeleton of the dual subdivision contains a simple loop ℓ homologous to the core $\mathbf{S}^1 \times 0$. Consider the union $Y \subset X$ of the cells of X disjoint from ℓ. On the one hand, the pair (X, Y) is matchable (Remark 4). On the other hand, since $H_*(Y, \partial_0 M) = 0$, the pair $(Y, \partial_0 M)$ is matchable (Lemma 11).

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