A Proof of the Generalized Riemann Hypothesis

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Abstract

We present a proof of the Generalized Riemann hypothesis (GRH) based on asymptotic expansions and operations on series. The advantage of our method is that it only uses undergraduate maths which makes it accessible to a wider audience.

Keywords: Generalized Riemann hypothesis; Zeta; Critical Strip; Prime Number Theorem; Millennium Problems; Dirichlet L-functions.

1 Introduction: Dirichlet $L-$functions

Let's $(z_n)_{n\geq 1}$ be a sequence of complex numbers. A Dirichlet series[6] is a series of the form $\sum_{n=1}^{\infty} \frac{z_n}{n^s}$, where s is complex. The Riemann zeta function is a Dirichlet series. Let's define the function $L(s)$ of the complex s: $L(s) = \sum_{n=1}^{\infty} \frac{z_n}{n^s}$.

- If $(z_n)_{n\geq 1}$ is a bounded, then the corresponding Dirichlet series converges absolutely on the open half-plane where $\Re(s) > 1$.
- If the set of sums $z_n + z_{n+1} + ... + z_{n+k}$ for each n and $k \geq 0$ is bounded, then the corresponding Dirichlet series converges on the open half-plane where $\Re(s) > 0$.
- In general, if $z_n = O(n^k)$, the corresponding Dirichlet series converges absolutely in the half plane where $\Re(s) > k + 1$.

The function $L(s)$ is analytic on the corresponding open half plane[3, 6,15].

To define Dirichlet $L-$functions we need to define Dirichlet characters. A function $\chi : \mathbb{Z} \longmapsto \mathbb{C}$ is a Dirichlet character modulo q if it satisfies the following criteria:

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- (i) $\chi(n) \neq 0$ if $(n,q) = 1$.
- (ii) $\chi(n) = 0$ if $(n,q) > 1$.
- (iii) χ is periodic with period q :$\chi(n+q) = \chi(n)$ for all n.
- (iv) χ is multiplicative :$\chi(mn) = \chi(m)\chi(n)$ for all integers m and n.

The trivial character is the one with $\chi_0(n) = 1$ whenever $(n,q) = 1$.

Here are some known results for a Dirichlet character modulo q. For any integer n we have $\chi(1) = 1$. Also if $(n,q) = 1$, we have $(\chi(n))^{\phi(q)} = 1$ with ϕ is Euler's totient function. $\chi(n)$ is a $\phi(q)-$th root of unity. Therefore, $|\chi(n)| = 1$ if $(n,q) = 1$, and $|\chi(n)| = 0$ if $(n,q) > 1$.
Also, we recall the cancellation property for Dirichlet characters modulo q: For any n integer

$$\sum_{i=1}^{q} \chi(i+n) = \begin{cases} \phi(q), & \text{if } \chi = \chi_0 \text{ the trivial character} \\ 0, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \tag{1}$$

The Dirichlet $L-$functions are simply the sum of the Dirichlet series. Let's χ be a Dirichlet character modulo q, The Dirichlet $L-$function $L(s,\chi)$ is defined for $\Re(s) > 1$ as the following:

$$L(s,\chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s} \tag{2}$$

Where the series $\sum_{n\geq 1} \frac{\chi(n)}{n^s}$ is convergent when $\Re(s) > 0$ and $L(s,\chi)$ is analytic in $\Re(s) > 0$. See([3,6,7,8]). In the particular case of the trivial character χ_0, $L(s,\chi_0)$ extends to a meromorphic function in $\Re(s) > 0$ with the only pole at $s = 1$ (see [3,6,7,8]).

Also, like the Riemann zeta function, the Dirichlet $L-$functions have their Euler product[2,3,24]. For $\Re(s) > 1$:

$$L(s,\chi) = \prod_{p \text{ Prime}} \left(1 - \frac{\chi(p)}{p^s}\right) \tag{3}$$

Let's $q\prime$ be the smallest divisor of q. Let's $\chi\prime$ be the Dirichlet character $\chi\prime \mod q\prime$. For any integer n such that $(n,q) = 1$ we have also $(n,q\prime) = 1$ and $\chi(n) = \chi\prime(n)$. $\chi\prime$ is called primitive and $L(s,\chi)$ and $L(s,\chi\prime)$ are related analytically such that:

$$L(s,\chi) = L(s,\chi\prime) \prod_{p/q} \left(1 - \frac{\chi\prime(p)}{p^s}\right) \tag{4}$$

$L(s,\chi)$ and $L(s,\chi\prime)$ have the same zeros in the critical strip $0 \leq \Re(s) \leq 1$.

Also, for a primitive character χ, ($i.e. \chi = \chi\prime$) $L(s,\chi)$ has the following functional equation:

$$\tau(\chi)\Gamma(\frac{1-s+a}{2})\, L(1-s,\chi) = \sqrt{\pi}(\frac{q}{\pi})^s i^a q^{\frac{1}{2}}\, \Gamma(\frac{s+a}{2})\, L(s,\overline{\chi}) \tag{5}$$

Where Γ is the Gamma function and $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$, and $\tau(\chi) = \sum_{k=1}^{q} \chi(k) \exp(\frac{2\pi k i}{q})$.

When $\Re(s) > 1$ there is no zero for $L(s,\chi)$. When $\Re(s) < 0$, for a primitive character χ, we have the trivial zeros of $L(s,\chi)$: $s = a - 2k$, where k i a positive integer and a is defined above. For more details, please refer to the references[7-14].

2 The Generalized Riemann Hypothesis

The Generalized Riemann Hypothesis states that the Dirichlet $L-$functions have all their non-trivial zeros on the critical line $\Re(s) = \frac{1}{2}$.

Non trivial in this case means $L(s,\chi) = 0$ for $s \in \mathbb{C}$ and $0 < \Re(s) < 1$. So for any primitive character χ modulo q, all non-trivial zeros of $L(s,\chi)$ lies in the critical strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$. From the functional equation above we have that if:

- s_0 is a non-trivial zero of $L(s,\chi)$, then $1 - s_0$ is a zero of $L(s,\overline{\chi})$.
- s_0 is a non-trivial zero of $L(s,\overline{\chi})$, then $1 - s_0$ is a zero of $L(s,\chi)$.

Therefore, we just need to prove that for all primitive character χ modulo q, there is no non-trivial zeros of $L(s,\chi)$ in the right hand side of the critical strip $\{s \in \mathbb{C} : \frac{1}{2} < \Re(s) < 1\}$.

3 Proof of the GRH

Let's take a complex number s such $s = a_0 + ib_0$. Unless we explicitly mention otherwise, let's suppose that $a_0 > 0$ and $b_0 > 0$. Let's take χ a non-trivial Dirichlet character.

3.1 Case One: $\frac{1}{2} < a_0 \leq 1$ and χ non-trivial

In this case s is a zero of $L(s,\chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$. Where χ is a non-trivial Dirichlet character χ modulo q. q is a nonzero positive integer.
Let's denote for each $n \geq 1$: $z_n = x_n + iy_n = \chi(n)$.

We are going to develop the sequence $Z_N(s) = \sum_{n=1}^{N} \frac{z_n}{n^s}$ as follows: For $N \geq 1$

$$\begin{aligned}
Z_N &= \sum_{n=1}^{N} \frac{z_n}{n^s} && (6)\\
&= \sum_{n=1}^{N} \frac{x_n + iy_n}{n^{a_0+ib_0}} && (7)\\
&= \sum_{n=1}^{N} \frac{(x_n + iy_n)n^{-ib_0}}{n^{a_0}} && (8)\\
&= \sum_{n=1}^{N} \frac{(x_n + iy_n)\exp\big(-ib_0\ln(n)\big)}{n^{a_0}} && (9)\\
&= \sum_{n=1}^{N} \frac{x_n\cos(b_0\ln(n)) + y_n\sin(b_0\ln(n)) + i\Big(y_n\cos(b_0\ln(n)) - x_n\sin(b_0\ln(n))\Big)}{n^{a_0}} && (10)\\
&= \sum_{n=1}^{N} \frac{x_n\cos(b_0\ln(n)) + y_n\sin(b_0\ln(n))}{n^{a_0}} + i\sum_{n=1}^{+\infty} \frac{y_n\cos(b_0\ln(n)) - x_n\sin(b_0\ln(n))}{n^{a_0}} && (11)\\
& && (12)
\end{aligned}$$

Let's define the sequences U_n, V_n as follows: For $n \geq 1$

$$U_n = \frac{x_n \cos(b_0 \ln(n)) + y_n \sin(b_0 \ln(n))}{n^{a_0}} \tag{13}$$

$$V_n = \frac{y_n \cos(b_0 \ln(n)) - x_n \sin(b_0 \ln(n))}{n^{a_0}} \tag{14}$$

Let's define the series A_n, B_n and Z_n as follows:

$$A_n = \sum_{k=1}^{n} U_n \tag{15}$$

$$B_n = \sum_{k=1}^{n} V_n \tag{16}$$

$$Z_n = A_n + iB_n \tag{17}$$

When we are dealing with complex numbers, it is always insightful to work with the norm. So let's develop further the squared norm of the serie Z_N as follows:

$$\begin{aligned} \|Z_N\|^2 &= A_N^2 + B_N^2 && (18)\\ &= \left(\sum_{n=1}^{N} U_n\right)^2 + \left(\sum_{n=1}^{N} V_n\right)^2 && (19) \end{aligned}$$

So

$$\begin{aligned} A_N^2 &= \left(\sum_{n=1}^{N} U_n\right)^2 && (20)\\ &= \sum_{n=1}^{N} U_n^2 + 2\sum_{n=1}^{N}\sum_{k=1}^{n-1} U_n U_k && (21)\\ &= \sum_{n=1}^{N} U_n^2 + 2\sum_{n=1}^{N} U_n \sum_{k=1}^{n-1} U_k && (22)\\ &= \sum_{n=1}^{N} U_n^2 + 2\sum_{n=1}^{N} U_n \Big(A_n - U_n \Big) && (23)\\ &= -\sum_{n=1}^{N} U_n^2 + 2\sum_{n=1}^{N} U_n A_n && (24) \end{aligned}$$

And the same calculation for B_N

$$B_N^2 = \left(\sum_{n=1}^{N} V_n\right)^2 \tag{25}$$

$$= \sum_{n=1}^{N} V_n^2 + 2\sum_{n=1}^{N}\sum_{k=1}^{n-1} V_n V_k \tag{26}$$

$$= \sum_{n=1}^{N} V_n^2 + 2\sum_{n=1}^{N} V_n \sum_{k=1}^{n-1} V_k \tag{27}$$

$$= \sum_{n=1}^{N} V_n^2 + 2\sum_{n=1}^{N} V_n \left(B_n - V_n\right) \tag{28}$$

$$= -\sum_{n=1}^{N} V_n^2 + 2\sum_{n=1}^{N} V_n B_n \tag{29}$$

Hence we have the new expression of square norm of Z_N:

$$\|Z_N\|^2 = 2\sum_{n=1}^{N} U_n A_n + 2\sum_{n=1}^{N} V_n B_n - \sum_{n=1}^{N} U_n^2 - \sum_{n=1}^{N} V_n^2 \tag{30}$$

Let's now define F_n and G_n as follows:

$$F_n = U_n A_n \tag{31}$$

$$G_n = V_n B_n \tag{32}$$

Therefore

$$A_N^2 = 2\sum_{n=1}^{N} F_n - \sum_{n=1}^{N} U_n^2 \tag{33}$$

$$B_N^2 = 2\sum_{n=1}^{N} G_n - \sum_{n=1}^{N} V_n^2 \tag{34}$$

Therefore

Conclusion. *s is a **zero** for $L(s,\chi) = 0$, if and only if*

$$\lim_{N\to\infty} A_N = 0 \textit{ and } \lim_{N\to\infty} B_N = 0 \tag{35}$$

*Equally, s is a $L(s,\chi)$ **zero**, $L(s,\chi) = 0$, if and only if*

$$\lim_{N\to\infty} A_N^2 = 0 \textit{ and } \lim_{N\to\infty} B_N^2 = 0 \tag{36}$$

Proof Strategy. *The idea is to prove that in the case of a complex s that is in the right hand side of the critical strip $\frac{1}{2} < a_0 \leq 1$ and that is a $L(s,\chi)$ zero, that the limit $\lim_{n\to\infty} A_n^2 = +/-\infty$ **OR** the limit $\lim_{n\to\infty} B_n^2 = +/-\infty$. This will create a contradiction. Because if s is a $L(s,\chi)$ zero then the $\lim_{n\to\infty} A_n^2$ should be 0 and the $\lim_{n\to\infty} B_n^2$ should be*

0. *And therefore the sequences* $(\sum_{n=1}^{N} F_n)_{N\geq 1}$ *and* $(\sum_{n=1}^{N} G_n)_{N\geq 1}$ *should converge and their limits should be:* $\lim_{n\to\infty}\sum_{n=1}^{N} F_n = \frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N} U_n^2 < +\infty$ *and* $\lim_{n\to\infty}\sum_{n=1}^{N} G_n = \frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N} V_n^2 < +\infty$.

Lemma 3.1. *If* $\frac{1}{2} < a_0 \leq 1$*, we have the series* $\sum_{n\geq 1} G_n$ *diverges.*

To prove this lemma, let's first prove the following lemma:

Lemma 3.2. *If the set of the partial sums* $z_n + z_{n+1} + \ldots + z_{n+k}$ *for* n *and* $k \geq 0$ *is bounded, then we can write* $B_n = \frac{\lambda_n}{n^{a_0}}$ *where* (λ_n) *is a bounded sequence.*

Proof. We have $\lim_{N\to+\infty} B_N = \sum_{n=1}^{+\infty} V_n = 0$. Therefore for each $N \geq 1$:

$$\sum_{n=1}^{+\infty} V_n = \underbrace{\sum_{n=1}^{N} V_n}_{B_N} + \sum_{n=N+1}^{+\infty} V_n = 0 \tag{37}$$

$$B_N = -\sum_{n=N+1}^{+\infty} V_n \tag{38}$$

We have

$$B_N = -\sum_{n=N+1}^{+\infty} \frac{y_n \cos(b_0 \ln(n)) - x_n \sin(b_0 \ln(n))}{n^{a_0}} \tag{39}$$

Let's denote X_n and Y_n the partial sums of the series x_n and y_n: $X_n = \sum_{k=1}^{N} x_k$ and $Y_n = \sum_{k=1}^{N} y_k$. So let's take N and M two integers such that $M \geq N$ and do the Abel summation between $N+1$ and M:

$$\sum_{n=N+1}^{M} V_n = \sum_{n=N+1}^{M} \frac{y_n \cos(b_0 \ln(n))}{n^{a_0}} - \sum_{n=N+1}^{M} \frac{x_n \sin(b_0 \ln(n))}{n^{a_0}} \tag{40}$$

$$= \frac{Y_M \cos(b_0 \ln(M))}{M^{a_0}} - \frac{Y_N \cos(b_0 \ln(N+1))}{(N+1)^{a_0}} - \sum_{n=N+1}^{M-1} Y_n \left(\frac{\cos(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\cos(b_0 \ln(n))}{n^{a_0}}\right) \tag{41}$$

$$-\left\{\frac{X_M \sin(b_0 \ln(M))}{M^{a_0}} - \frac{X_N \sin(b_0 \ln(N+1))}{(N+1)^{a_0}} - \sum_{n=N+1}^{M-1} X_n \left(\frac{\sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\sin(b_0 \ln(n))}{n^{a_0}}\right)\right\} \tag{42}$$

Let's define the functions f_n and e_n such that $f_n(t) = \frac{\cos(b_0 \ln(n+t))}{(n+t)^{a_0}}$ and $e_n(t) = \frac{\sin(b_0 \ln(n+t))}{(n+t)^{a_0}}$. For each $n \geq 1$, we can apply the Mean Value Theoremon the interval $[0,1]$ to find c_1 and c_2 in $(0,1)$ such that:

$$\left|\frac{\cos(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\cos(b_0 \ln(n))}{(n)^{a_0}}\right| = \left|e_n'(c_1)(1-0)\right| \tag{43}$$

$$\left|\frac{\sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\sin(b_0 \ln(n))}{(n)^{a_0}}\right| = \left|f_n'(c_2)(1-0)\right| \tag{44}$$

We have the derivatives of f_n and e_n such that $f'_n(t) = \frac{-b_0 \sin(b_0 \ln(n+t)) - a_0 \cos(b_0 \ln(n+t))}{(n+t)^{a_0+1}}$ and $e'_n(t) = \frac{b_0 \cos(b_0 \ln(n+t)) - a_0 \sin(b_0 \ln(n+t))}{(n+t)^{a_0+1}}$. Therefore

$$\left| \frac{\cos(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\cos(b_0 \ln(n))}{(n)^{a_0}} \right| \leq \frac{a_0 + b_0}{n^{a_0+1}} \tag{45}$$

$$\left| \frac{\sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\sin(b_0 \ln(n))}{(n)^{a_0}} \right| \leq \frac{a_0 + b_0}{n^{a_0+1}} \tag{46}$$

We have the set of the partial sums $z_n + z_{n+1} + ... + z_{n+k}$ is bounded, then the real part and the imaginary part of the sum partial of $z_n + z_{n+1} + ... + z_{n+k}$ are also bounded.
Let's $K > 0$ such that for every n and k: $|x_n + x_{n+1} + ... + x_{n+k}| \leq K$ and $|y_n + y_{n+1} + ... + y_{n+k}| \leq K$. Therefore for each n: $|X_n| \leq K$ and $|Y_n| \leq K$.

$$\left| \sum_{n=N+1}^{M} V_n \right| \leq \left| \frac{Y_M \cos(b_0 \ln(M))}{M^{a_0}} \right| + \left| \frac{Y_N \cos(b_0 \ln(N+1))}{(N+1)^{a_0}} \right| \tag{47}$$

$$+ \sum_{n=N+1}^{M-1} |Y_n| \left| \left(\frac{\cos(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\cos(b_0 \ln(n))}{n^{a_0}} \right) \right| \tag{48}$$

$$+ \left| \frac{X_M \sin(b_0 \ln(M))}{M^{a_0}} \right| + \left| \frac{X_N \sin(b_0 \ln(N+1))}{(N+1)^{a_0}} \right| \tag{49}$$

$$+ \sum_{n=N+1}^{M-1} |X_n| \left| \left(\frac{\sin(b_0 \ln(n+1))}{(n+1)^{a_0}} - \frac{\sin(b_0 \ln(n))}{n^{a_0}} \right) \right| \tag{50}$$

$$\leq \frac{4K}{(N+1)^{a_0}} + 2K(a_0 + b_0) \sum_{n=N+1}^{M-1} \frac{1}{n^{a_0+1}} \tag{51}$$

$$\leq \frac{4K}{(N+1)^{a_0}} + 2K(a_0 + b_0) \int_{N+1}^{M} \frac{dt}{t^{a_0+1}} \tag{52}$$

$$\leq \frac{4K}{(N+1)^{a_0}} + \frac{2K(a_0 + b_0)}{a_0} \Big(\frac{1}{(N+1)^{a_0}} - \frac{1}{M^{a_0}} \Big) \tag{53}$$

We tend M to infinity and we get:

$$\left| \sum_{n=N+1}^{+\infty} V_n \right| \leq \frac{4K}{(N+1)^{a_0}} + \frac{2K(a_0 + b_0)}{a_0 (N+1)^{a_0}} \tag{54}$$

Therefore

$$|B_N| = \left| - \sum_{n=N+1}^{+\infty} V_n \right| \leq \frac{K_1}{(N+1)^{a_0}} \leq \frac{K_1}{N^{a_0}} \tag{55}$$

where $K_1 = 4K + \frac{2K(a_0+b_0)}{a_0} > 0$.

Let's define the sequence ϵ_n such that: $B_n = \frac{\lambda_n}{n^{a_0}}$. Therefore for each $n \geq 1$ we have: $|\lambda_n| \leq K_1$. Therefore the sequence (λ_n) is bounded.

□

Let's take $n \geq 1$, we have $B_{n+1} = B_n + V_{n+1}$. Therefore

$$\frac{\lambda_{n+1}}{(n+1)^{a_0}} = \frac{\lambda_n}{n^{a_0}} + \frac{y_{n+1}\cos(b_0 \ln(n+1)) - x_{n+1}\sin(b_0 \ln(n+1))}{(n+1)^{a_0}} \quad (56)$$

And

$$\lambda_{n+1} = \big(1 + \frac{1}{n}\big)^{a_0}\lambda_n + \Big(y_{n+1}\cos(b_0 \ln(n+1)) - x_{n+1}\sin(b_0 \ln(n+1))\Big) \quad (57)$$

For the sake of notation simplification, let's denote , for $n \geq 1$,

$$v_n = y_n \cos(b_0 \ln(n)) - x_n \sin(b_0 \ln(n)) \quad (58)$$

$$\lambda_{n+1} = \big(1 + \frac{1}{n}\big)^{a_0}\lambda_n + v_{n+1} \quad (59)$$

Proof. First, we apply the previous lemma 3.2 to the case of our Dirichlet series. Thanks to the cancelation property mentioned in (1), we have the partial sums $(\sum_{i=1}^{n} \chi(i))_{n \geq 1}$ is bounded because our Dirichlet character χ is non-trivial. Therefore we write $B_n = \frac{\lambda_n}{n^{a_0}}$ where the sequence (λ_n) is bounded.

To prove this lemma we proceed with some asymptotic expansions: We have

$$B_{n+1} = B_n + V_{n+1} \quad (60)$$

So G_{n+1} can be written as follows:

$$G_{n+1} = V_{n+1}B_n + V_{n+1}^2 \quad (61)$$

So $G_{n+1} - G_n$ can be written as

$$G_{n+1} - G_n = \Big(V_{n+1} - V_n\Big)B_n + V_{n+1}^2 \quad (62)$$

Let's do now the asymptotic expansion of $V_{n+1} - V_n$. For this we need the asymptotic expansion of $\cos\big(b_0 \ln(n+1)\big)$ and $\sin\big(b_0 \ln(n+1)\big)$.

$$\cos(b_0 \ln(n+1) = \cos\big(b_0 \ln(n) + b_0 \ln(1 + \frac{1}{n})\big) \quad (63)$$

$$= \cos\big(b_0 \ln(n)\big)\cos\big(b_0 \ln(1 + \frac{1}{n})\big) - \sin\big(b_0 \ln(n)\big)\sin\big(b_0 \ln(1 + \frac{1}{n})\big) \quad (64)$$

we have the asymptotic expansion of $\ln(1 + \frac{1}{n})$ in order two as follow:

$$\ln(1 + \frac{1}{n}) = \frac{1}{n} + \mathcal{O}(\frac{1}{n^2}) \quad (65)$$

Using the asymptotic expansion of the functions sin and cos that I will spear you the details here, we have

$$\cos\Big(b_0 \ln(1+\frac{1}{n})\Big) \quad = \quad 1+\mathcal{O}(\frac{1}{n^2}) \tag{66}$$

And

$$\sin\Big(b_0 \ln(1+\frac{1}{n})\Big) \quad = \quad \frac{b_0}{n}+\mathcal{O}(\frac{1}{n^2}) \tag{67}$$

Hence

$$\cos(b_0 \ln(n+1)) \quad = \quad \cos\big(b_0 \ln(n)\big) - \frac{b_0}{n}\sin\big(b_0 \ln(n)\big) + \mathcal{O}(\frac{1}{n^2}) \tag{68}$$

Also the Asymptotic expansion of $\frac{1}{(1+n)^{a_0}}$:

$$\frac{1}{(1+n)^{a_0}} \quad = \quad \frac{1}{n^{a_0}}\Big(1-\frac{a_0}{n}+\mathcal{O}(\frac{1}{n^2})\Big) \tag{69}$$

Hence

$$\frac{\cos\big(b_0 \ln(n+1)\big)}{(1+n)^{a_0}} = \frac{\cos\big(b_0 \ln(n)\big)}{n^{a_0}} - \frac{b_0 \sin\big(b_0 \ln(n)\big) + a_0 \cos\big(b_0 \ln(n)\big)}{n^{a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{70}$$

$$= \frac{\cos\big(b_0 \ln(n)\big)}{n^{a_0}} + \frac{c_n}{n^{a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{71}$$

And the squared version of the above equation

$$\frac{\cos^2\big(b_0 \ln(n+1)\big)}{(1+n)^{2a_0}} = \frac{\cos^2\big(b_0 \ln(n)\big)}{n^{2a_0}} + 2\cos\big(b_0 \ln(n)\big)\frac{c_n}{n^{2a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{72}$$

For the asymptotic expansion of $\sin\big(b_0 \ln(n+1)\big)$.

$$\sin(b_0 \ln(n+1) = \sin\big(b_0 \ln(n) + b_0 \ln(1+\frac{1}{n})\big) \tag{73}$$

$$= \sin\big(b_0 \ln(n)\big)\cos\big(b_0 \ln(1+\frac{1}{n})\big) + \cos\big(b_0 \ln(n)\big)\sin\big(b_0 \ln(1+\frac{1}{n})\big) \tag{74}$$

we have the asymptotic expansion of $\ln(1+\frac{1}{n})$ in order two as follow:

$$\ln(1+\frac{1}{n}) \quad = \quad \frac{1}{n}+\mathcal{O}(\frac{1}{n^2}) \tag{75}$$

Using the asymptotic expansion of the functions sin and cos that I will spear you the details here, we have

$$\sin\Big(b_0 \ln(1+\frac{1}{n})\Big) \quad = \quad \sin\Big(\frac{b_0}{n}+\mathcal{O}(\frac{1}{n^2})\Big) \tag{76}$$

$$= \quad \frac{b_0}{n}+\mathcal{O}(\frac{1}{n^2}) \tag{77}$$

And

$$\cos\Big(b_0\ln(1+\frac{1}{n})\Big) \quad = \quad 1+\mathcal{O}(\frac{1}{n^2}) \tag{78}$$

Hence

$$\sin(b_0\ln(n+1)) \quad = \quad \sin\big(b_0\ln(n)\big)+\frac{b_0}{n}\cos\big(b_0\ln(n)\big)+\mathcal{O}(\frac{1}{n^2}) \tag{79}$$

Also the Asymptotic expansion of $\frac{1}{(1+n)^{a_0}}$:

$$\frac{1}{(1+n)^{a_0}} \quad = \quad \frac{1}{n^{a_0}}\Big(1-\frac{a_0}{n}+\mathcal{O}(\frac{1}{n^2})\Big) \tag{80}$$

Hence

$$\frac{\sin\big(b_0\ln(n+1)\big)}{(1+n)^{a_0}} = \frac{\sin\big(b_0\ln(n)\big)}{n^{a_0}}+\frac{b_0\cos\big(b_0\ln(n)\big)-a_0\sin\big(b_0\ln(n)\big)}{n^{a_0+1}}+\mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{81}$$

$$= \frac{\sin\big(b_0\ln(n)\big)}{n^{a_0}}+\frac{cc_n}{n^{a_0+1}}+\mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{82}$$

And the squared version of the above equation

$$\frac{\sin^2\big(b_0\ln(n+1)\big)}{(1+n)^{2a_0}} = \frac{\sin^2\big(b_0\ln(n)\big)}{n^{2a_0}}+2\sin\big(b_0\ln(n)\big)\frac{cc_n}{n^{2a_0+1}}+\mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{83}$$

Where

$$c_n \quad = \quad -\Big(b_0\sin\big(b_0\ln(n)\big)+a_0\cos\big(b_0\ln(n)\big)\Big) \tag{84}$$

$$cc_n \quad = \quad b_0\cos\big(b_0\ln(n)\big)-a_0\sin\big(b_0\ln(n)\big) \tag{85}$$

Therefore: $V_{n+1}-V_n$:

$$V_{n+1}-V_n = \frac{y_{n+1}\cos\big(b_0\ln(n+1)\big)-x_{n+1}\sin\big(b_0\ln(n+1)\big)}{(n+1)^{a_0}}-\frac{y_n\cos\big(b_0\ln(n)\big)-x_n\sin\big(b_0\ln(n)\big)}{(n)^{a_0}} \tag{86}$$

$$= y_{n+1}\Big(\frac{\cos\big(b_0\ln(n)\big)}{n^{a_0}}+\frac{c_n}{n^{a_0+1}}+\mathcal{O}(\frac{1}{n^{a_0+2}})\Big) \tag{87}$$

$$-x_{n+1}\Big(\frac{\sin\big(b_0\ln(n)\big)}{n^{a_0}}+\frac{cc_n}{n^{a_0+1}}+\mathcal{O}(\frac{1}{n^{a_0+2}})\Big) \tag{88}$$

$$-\frac{y_n\cos\big(b_0\ln(n)\big)-x_n\sin\big(b_0\ln(n)\big)}{(n)^{a_0}} \tag{89}$$

$$= \frac{(y_{n+1}-y_n)\cos\big(b_0\ln(n)\big)-(x_{n+1}-x_n)\sin\big(b_0\ln(n)\big)}{n^{a_0}}+\frac{y_{n+1}c_n-x_{n+1}cc_n}{n^{a_0+1}}+\mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{90}$$

$$= \frac{\alpha_n}{n^{a_0}}+\frac{\beta_n}{n^{a_0+1}}+\mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{91}$$

And

$$V_{n+1} = \frac{\gamma_n}{n^{a_0}}+\frac{\beta_n}{n^{a_0+1}}+\mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{92}$$

Where

$$\alpha_n = (y_{n+1} - y_n)\cos\big(b_0 \ln(n)\big) - (x_{n+1} - x_n)\sin\big(b_0 \ln(n)\big) \tag{93}$$
$$\beta_n = y_{n+1}c_n - x_{n+1}cc_n \tag{94}$$
$$\gamma_n = y_{n+1}\cos\big(b_0 \ln(n)\big) - x_{n+1}\sin\big(b_0 \ln(n)\big) \tag{95}$$

Therefore

$$V_{n+1}^2 = \frac{\gamma_n^2}{n^{2a_0}} + 2\frac{\gamma_n \beta_n}{n^{2a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{96}$$

So the asymptotic expansion of $G_{n+1} - G_n$ is as follows:

$$G_{n+1} - G_n = \frac{\gamma_n^2}{n^{2a_0}} + \frac{\alpha_n A_n}{n^{a_0}} + \frac{\beta_n A_n}{n^{a_0+1}} + 2\frac{\gamma_n \beta_n}{n^{2a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{97}$$

We have $B_n = \frac{\lambda_n}{n^{a_0}}$ where λ_n is bounded. Therefore

$$G_{n+1} - G_n = \frac{\gamma_n^2 + \alpha_n \lambda_n}{n^{2a_0}} + \frac{2\gamma_n \beta_n}{n^{2a_0+1}} + \frac{\beta_n \lambda_n}{n^{2a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{98}$$

Therefore

$$G_{n+1} - G_n = \frac{\gamma_n^2 + \alpha_n \lambda_n}{n^{2a_0}} + \frac{\big(\lambda_n + 2\gamma_n\big)\beta_n}{n^{2a_0+1}} + \mathcal{O}(\frac{1}{n^{a_0+2}}) \tag{99}$$

By definition of $\mathcal{O}$, we know that there is exist a bounded sequence (ε_n) and there is exist a number N_0 such that: For each $n \geq N_0$ we have

$$G_{n+1} - G_n = \frac{\gamma_n^2 + \alpha_n \lambda_n}{n^{2a_0}} + \frac{\big(\lambda_n + 2\gamma_n\big)\beta_n}{n^{2a_0+1}} + \frac{\epsilon_n}{n^{a_0+2}} \tag{100}$$

Let's now study the asymptotic expansion of the dominant term $\frac{\gamma_n^2 + \alpha_n \lambda_n}{n^{2a_0}}$ in the context of the Dirichlet $L-$functions.

Lemma 3.3. *Let's q be a nonzero integer. Let's suppose that for each k, $x_{kq+1} = x_{kq+1}^2 = 1$ and $y_{kq+1} = y_{kq+1}^2 = 0$. Therefore we have the following asymptotic expansion:*

$$\sum_{i=1}^{q}\Big(\gamma_{qn+i}^2 + \alpha_{qn+i}\lambda_{qn+i}\Big) = \sum_{i=1}^{q}\frac{y_{qn+i} - x_{qn+i} + 2x_{qn+i}^2}{2} \tag{101}$$
$$+ \cos(2b_0 \ln(qn))\sum_{i=1}^{q}\frac{y_{qn+i} + x_{qn+i} - 2x_{qn+i}^2}{2} \tag{102}$$
$$+ \sin(2b_0 \ln(qn))\sum_{i=2}^{q}\frac{y_{qn+i} - x_{qn+i} - 2x_{qn+i}y_{qn+i}}{2} + O(\frac{1}{qn}) \tag{103}$$

Proof. In this case we have for each $n \geq 1$:

$$\alpha_n = (x_{n+1} - x_n)\cos\big(b_0 \ln(n)\big) + (y_{n+1} - y_n)\sin\big(b_0 \ln(n)\big) \tag{104}$$
$$\gamma_n^2 = x_{n+1}\cos\big(b_0 \ln(n)\big) + y_{n+1}\sin\big(b_0 \ln(n)\big) \tag{105}$$

λ_n is bounded. From the asymptotic expansion of $\cos(b_0 \ln(n+1))$ and $\sin(b_0 \ln(n+1))$:

$$\lambda_{n+1} = \lambda_n + x_{n+1}\cos(b_0 \ln(n+1)) + y_{n+1}\sin(b_0 \ln(n+1)\big) + O(\frac{1}{n}) \tag{106}$$

Let q be a nonzero integer. And let's χ be the Dirichlet character modulo q associated to our Dirichlet series $\sum_{n\geq 1} \frac{z_n}{n^s}$. $\chi(n) = z_n = x_n + iy_n$ for each $n \geq 1$. Then, by definition we have for each k, $x_{kq+1} = x_{kq+1}^2 = 1$ and $y_{kq+1} = y_{kq+1}^2 = 0$. And for each $1 \leq i \leq q$, $x_{kq+i} = x_i$ and $y_{kq+i} = y_i$.

We have from above that

$$\cos(b_0 \ln(n+1)) = \cos(b_0 \ln(n)) + O(\frac{1}{n}) \tag{107}$$
$$\sin(b_0 \ln(n+1)\big) = \sin(b_0 \ln(n)) + O(\frac{1}{n}) \tag{108}$$

Therefore for $1 \leq i \leq q$: we have

$$\cos(b_0 \ln(qn+i)) = \cos(b_0 \ln(qn)) + O(\frac{1}{qn}) \tag{109}$$
$$\sin(b_0 \ln(qn+i)\big) = \sin(b_0 \ln(qn)) + O(\frac{1}{qn}) \tag{110}$$
$$\lambda_{qn+i+1} = \lambda_{qn+i} + y_{qn+i+1}\cos(b_0 \ln(qn)) - x_{qn+i+1}\sin(b_0 \ln(qn)\big) + O(\frac{1}{qn}) \tag{111}$$

Let's now study the term $\sum_{i=1}^{q} \alpha_{qn+i}\lambda_{qn+i}$.

$$\begin{aligned}
\sum_{i=1}^{q} \alpha_{qn+i}\lambda_{qn+i} &= \sum_{i=1}^{q} \Big(y_{qn+i+1} - y_{qn+i}\Big)\lambda_{qn+i}\cos(b_0 \ln(qn+i)) && (112)\\
&- \Big(x_{qn+i+1} - x_{qn+i}\Big)\lambda_{qn+i}\sin(b_0 \ln(qn+i)\big) && (113)\\
&= \cos(b_0 \ln(qn)) \sum_{i=1}^{q} \Big(y_{qn+i+1} - y_{qn+i}\Big)\lambda_{qn+i} && (114)\\
&- \sin(b_0 \ln(qn)) \sum_{i=1}^{q} \Big(x_{qn+i+1} - x_{qn+i}\Big)\lambda_{qn+i} + O(\frac{1}{qn}) && (115)
\end{aligned}$$

We have here Abel's sums so:

$$\begin{aligned}
\sum_{i=1}^{q}\Big(y_{qn+i+1}-y_{qn+i}\Big)\lambda_{qn+i} &= \sum_{i=1}^{q} y_{qn+i+1}\lambda_{qn+i}-\sum_{i=1}^{q} y_{qn+i}\lambda_{qn+i} && (116)\\
&= \sum_{i=2}^{q+1} y_{qn+i}\lambda_{qn+i-1}-\sum_{i=1}^{q} y_{qn+i}\lambda_{qn+i} && (117)\\
&= y_{qn+q+1}\lambda_{qn+q}-y_{qn+1}\lambda_{qn+1}+\sum_{i=2}^{q} y_{qn+i}\Big(\lambda_{qn+i-1}-\lambda_{qn+i}\Big) && (118)\\
&= \lambda_{qn+q}-\lambda_{qn+1} && (119)\\
&- \sum_{i=2}^{q} y_{qn+i}\Big(y_{qn+i}\cos(b_0\ln(qn))-x_{qn+i}\sin(b_0\ln(qn))\Big)+O(\frac{1}{qn}) && (120)\\
&= \sum_{i=2}^{q} \lambda_{qn+i}-\lambda_{qn+i-1} && (121)\\
&- \sum_{i=2}^{q} y_{qn+i}\Big(y_{qn+i}\cos(b_0\ln(qn))-x_{qn+i}\sin(b_0\ln(qn))\Big)+O(\frac{1}{qn}) && (122)\\
&= \sum_{i=2}^{q} \Big(y_{qn+i}\cos(b_0\ln(qn))-x_{qn+i}\sin(b_0\ln(qn))\Big)+O(\frac{1}{qn}) && (123)\\
&- \sum_{i=2}^{q} y_{qn+i}\Big(y_{qn+i}\cos(b_0\ln(qn))-x_{qn+i}\sin(b_0\ln(qn))\Big)+O(\frac{1}{qn}) && (124)\\
&= \sum_{i=2}^{q}\Big(y_{qn+i}-y_{qn+i}^2\Big)\cos(b_0\ln(qn)) && (125)\\
&- \sum_{i=2}^{q}\Big(x_{qn+i}-x_{qn+i}y_{qn+i}\Big)\sin(b_0\ln(qn))+O(\frac{1}{qn}) && (126)
\end{aligned}$$

And

$$\begin{aligned}
\sum_{i=1}^{q}\Big(x_{qn+i+1}-x_{qn+i}\Big)\lambda_{qn+i} &= \sum_{i=2}^{q}\Big(x_{qn+i}-x_{qn+i}^2\Big)\sin(b_0\ln(qn)) && (127)\\
&- \Big(y_{qn+i}-x_{qn+i}y_{qn+i}\Big)\cos(b_0\ln(qn))+O(\frac{1}{qn}) && (128)
\end{aligned}$$

Therefore:

$$\begin{aligned}
\sum_{i=1}^{q}\alpha_{qn+i}\lambda_{qn+i} &= \sum_{i=2}^{q}\Big(y_{qn+i}-y_{qn+i}^2\Big)\cos^2(b_0\ln(qn)) && (129)\\
&- \Big(x_{qn+i}-x_{qn+i}y_{qn+i}\Big)\sin(b_0\ln(qn))\cos(b_0\ln(qn))\Big) && (130)\\
&- \sum_{i=2}^{q}\Big(x_{qn+i}-x_{qn+i}^2\Big)\sin^2(b_0\ln(qn)) && (131)\\
&+ \Big(y_{qn+i}-x_{qn+i}y_{qn+i}\Big)\sin(b_0\ln(qn))\cos(b_0\ln(qn))+O(\frac{1}{qn}) && (132)
\end{aligned}$$

Now the term $\sum_{i=1}^{q} \gamma_{qn+i}^2$

$$
\begin{aligned}
\sum_{i=1}^{q} \gamma_{qn+i}^2 &= \sum_{i=1}^{q} \Big(y_{qn+i+1} \cos(b_0 \ln{(qn+i)}) - x_{qn+i+1} \sin(b_0 \ln{(qn+i)}) \Big)^2 && (133)\\
&= \sum_{i=1}^{q} \Big\{ y_{qn+i+1}^2 \cos^2(b_0 \ln{(qn)}) + x_{qn+i+1}^2 \sin^2\big(b_0 \ln{(qn)}\big) && (134)\\
&- 2x_{qn+i+1} y_{qn+i+1} \sin\big(b_0 \ln{(qn)}\big) \cos\big(b_0 \ln{(qn)}\big) \Big\} + O(\frac{1}{qn}) && (135)\\
&= \cos^2(b_0 \ln{(qn)}) \sum_{i=2}^{q+1} y_{qn+i}^2 + \sin^2(b_0 \ln{(qn)}) \sum_{i=2}^{q+1} x_{qn+i}^2 && (136)\\
&- 2\sin\big(b_0 \ln{(qn)}\big) \cos\big(b_0 \ln{(qn)}\big) \sum_{i=2}^{q+1} x_{qn+i} y_{qn+i} + O(\frac{1}{qn}) && (137)
\end{aligned}
$$

We have for each n, $x_{kq+1} = x_{kq+1}^2 = 1$ and $y_{kq+1} = y_{kq+1}^2 = 0$.

Therefore

$$\begin{aligned}
\sum_{i=1}^{q}\gamma_{qn+i}^2+\sum_{i=1}^{q}\alpha_{qn+i}\lambda_{qn+i} &= \cos^2(b_0\ln(qn))\sum_{i=2}^{q+1}y_{qn+i}^2+\sin^2(b_0\ln(qn))\sum_{i=2}^{q+1}x_{qn+i}^2 && (138)\\
&- 2\sin(b_0\ln(qn))\cos(b_0\ln(qn))\sum_{i=2}^{q+1}x_{qn+i}y_{qn+i}+O(\frac{1}{qn}) && (139)\\
&+ \cos^2(b_0\ln(qn))\sum_{i=2}^{q}\Big(y_{qn+i}-y_{qn+i}^2\Big) && (140)\\
&+ \sin(b_0\ln(qn))\cos(b_0\ln(qn))\sum_{i=2}^{q}\Big(y_{qn+i}-x_{qn+i}y_{qn+i}\Big)\Big) && (141)\\
&- \sin^2(b_0\ln(qn))\sum_{i=2}^{q}\Big(x_{qn+i}-x_{qn+i}^2\Big) && (142)\\
&- \sin(b_0\ln(qn))\cos(b_0\ln(qn))\sum_{i=2}^{q}\Big(x_{qn+i}-y_{qn+i}x_{qn+i}\Big)+O(\frac{1}{qn}) && (143)\\
&= \cos^2(b_0\ln(qn))y_{qn+1}^2+\sin^2(b_0\ln(qn))x_{qn+1}^2 && (144)\\
&- 2\sin(b_0\ln(qn))\cos(b_0\ln(qn))x_{qn+1}y_{qn+1} && (145)\\
&+ \cos^2(b_0\ln(qn))\sum_{i=2}^{q}y_{qn+i}+\sin^2(b_0\ln(qn))\sum_{i=2}^{q}\left(2x_{qn+i}^2-x_{qn+i}\right) && (146)\\
&+ \sin(b_0\ln(qn))\cos(b_0\ln(qn))\sum_{i=2}^{q}\Big(y_{qn+i}-x_{qn+i}-2x_{qn+i}y_{qn+i}\Big) && (147)\\
&+ O(\frac{1}{qn}) && (148)\\
&= \cos^2(b_0\ln(qn))\sum_{i=1}^{q}y_{qn+i}+\sin^2(b_0\ln(qn))\sum_{i=1}^{q}\left(2x_{qn+i}^2-x_{qn+i}\right) && (149)\\
&+ \sin(b_0\ln(qn))\cos(b_0\ln(qn))\sum_{i=2}^{q}\Big(y_{qn+i}-x_{qn+i}-2x_{qn+i}y_{qn+i}\Big) && (150)\\
&+ O(\frac{1}{qn}) && (151)
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{i=1}^{q}\Big(\gamma_{qn+i}^2+\alpha_{qn+i}\lambda_{qn+i}\Big) &= \sum_{i=1}^{q}\frac{y_{qn+i}-x_{qn+i}+2x_{qn+i}^2}{2} && (152)\\
&+ \cos(2b_0\ln(qn))\sum_{i=1}^{q}\frac{y_{qn+i}+x_{qn+i}-2x_{qn+i}^2}{2} && (153)\\
&+ \sin(2b_0\ln(qn))\sum_{i=2}^{q}\frac{y_{qn+i}-x_{qn+i}-2x_{qn+i}y_{qn+i}}{2}+O(\frac{1}{qn}) && (154)
\end{aligned}$$

□

Proof Strategy. *The idea here is to prove that we can choose N_0 such* *that $R_{N_0}+G_{N_0} > 0$. In such case, we will have the* $\lim_{N\to+\infty}\sum_{n=1}^{N} G_{n+1} = +\infty$.

Divergence of $\sum_{n\geq 1} G_n$

For large $N \geq N_0$ we have the following expression of G_{N+1}:

$$G_{N+1} = G_{N_0} + \sum_{n=N_0}^{N} \frac{\gamma_n^2 + \alpha_n\lambda_n}{n^{2a_0}} + \frac{\big(\lambda_n + 2\gamma_n\big)\beta_n}{n^{2a_0+1}} + \frac{\epsilon_n}{n^{a_0+2}} \quad (155)$$

$$= G_{N_0} + \sum_{n=N_0}^{N} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\epsilon_n}{n^{a_0+2}} \quad (156)$$

Where

$$C_n = \gamma_n^2 + \alpha_n\lambda_n \quad (157)$$

$$D_n = \big(\lambda_n + 2\gamma_n\big)\beta_n \quad (158)$$

We do another summation and we have the following expression:
So for each $N \geq N_0$:

$$\sum_{n=N_0}^{N} G_{n+1} = (N - N_0 + 1)G_{N_0} \quad (159)$$

$$+ \sum_{n=N_0}^{N} \frac{C_n}{n^{a_0+1}}(N-n+1) \quad (160)$$

$$+ \sum_{n=N_0}^{N} \frac{D_n}{n^{2a_0+1}}(N-n+1) \quad (161)$$

$$+ \sum_{n=N_0}^{N} \frac{\varepsilon_n}{n^{a_0+2}}(N-n+1) \quad (162)$$

$$\sum_{n=N_0}^{N} \frac{C_n}{n^{a_0+1}}(N-n+1) = (N+1)\sum_{n=N_0}^{N} \frac{C_n}{n^{2a_0}} - \sum_{n=N_0}^{N} \frac{C_n}{n^{2a_0-1}}n \quad (163)$$

$$= (N+1)\sum_{n=N_0}^{N} \frac{C_n}{n^{2a_0}} - \sum_{n=N_0}^{N} \frac{C_n}{n^{2a_0-1}} \quad (164)$$

$$\sum_{n=N_0}^{N} \frac{D_n}{n^{2a_0+1}}(N-n+1) = (N+1)\sum_{n=N_0}^{N} \frac{D_n}{n^{2a_0+1}} - \sum_{n=N_0}^{N} \frac{D_n}{n^{2a_0+1}}n \quad (165)$$

$$= (N+1)\sum_{n=N_0}^{N} \frac{D_n}{n^{2a_0+1}} - \sum_{n=N_0}^{N} \frac{D_n}{n^{2a_0}} \quad (166)$$

And

$$\sum_{n=N_0}^{N} \frac{\varepsilon_n}{n^{2a_0+1}}(N-n+1) = (N+1)\sum_{n=N_0}^{N} \frac{\varepsilon_n}{n^{a_0+2}} - \sum_{n=N_0}^{N} \frac{\varepsilon_n}{n^{a_0+2}} n \quad (167)$$

$$= (N+1)\sum_{n=N_0}^{N} \frac{\varepsilon_n}{n^{a_0+2}} - \sum_{n=N_0}^{N} \frac{\varepsilon_n}{n^{a_0+1}} \quad (168)$$

So for each $N \geq N_0$:

$$\sum_{n=N_0}^{N} G_{n+1} = (N-N_0+1)G_{N_0} \quad (169)$$

$$+ \quad (N+1)\sum_{n=N_0}^{N} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \quad (170)$$

$$- \quad \sum_{n=N_0}^{N} \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon_n}{n^{a_0+1}} \quad (171)$$

We factorise by $N+1$ then we have this expression

$$\sum_{n=N_0}^{N} G_{n+1} = (N+1)\Bigg(\frac{(N-N_0+1)}{N+1}G_{N_0} \quad (172)$$

$$+ \quad \sum_{n=N_0}^{N} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \quad (173)$$

$$- \quad \frac{1}{N+1}\sum_{n=N_0}^{N} \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon_n}{n^{a_0+1}}\Bigg) \quad (174)$$

We have the sequences (C_n), (D_n) and (ε_n) are bounded and $2a_0 > 1$, so the serie $\sum_{n\geq 1}^{N} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}}$ is converging absolutely.
Let's prove now that the sequence in the equation (174) is converging to zero when N goes to infinity.

As we have $2a_0 > 1$, So

$$\lim_{n\to\infty} \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon_n}{n^{a_0+1}} = 0 \quad (175)$$

Therefore following the Césaro theorem we have:

$$\lim_{N\to\infty} \frac{1}{N+1}\sum_{n=N_0}^{N} \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon_n}{n^{a_0+1}} = 0 \quad (176)$$

We denote $GG(N, N_0)$ as:

$$GG(N, N_0) = \Bigg(\frac{(N - N_0 + 1)}{N+1} F_{N_0} \tag{177}$$

$$+ \sum_{n=N_0}^{N} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \tag{178}$$

$$- \frac{1}{N+1} \sum_{n=N_0}^{N} \frac{C_n}{n^{2a_0-1}} + \frac{D_n}{n^{2a_0}} + \frac{\varepsilon_n}{n^{a_0+1}}\Bigg) \tag{179}$$

Therefore:

$$\sum_{n=N_0}^{N} G_{n+1} = (N+1) GG(N, N_0) \tag{180}$$

The serie $\sum_{n\geq 1} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}}$ is converging absolutely. Let's denote

$$R_{N_0} = \lim_{N\to+\infty} \sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \tag{181}$$

Lemma 3.4. *We can choose* N_0 *such that* $R_{N_0} + G_{N_0} > 0$.

Proof. Let's now study the term $\sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}}$.

We have from above lemmas that:

$$\sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}} = \sum_{n=qN_0+1}^{Nq} \frac{\gamma_n^2 + \alpha_n \lambda_n}{n^{2a_0}} \tag{182}$$

$$= \sum_{n=N_0}^{N} \sum_{i=1}^{q} \frac{\gamma_{qn+i}^2 + \alpha_n \lambda_{qn+i}}{(qn+i)^{2a_0}} \tag{183}$$

We have for each
$1 \leq i \leq q$:

$$\frac{\gamma_{qn+i}^2 + \alpha_n \lambda_{qn+i}}{(qn+i)^{2a_0}} = \frac{\gamma_{qn+i}^2 + \alpha_n \lambda_{qn+i}}{(qn)^{2a_0}} + O(\frac{1}{(qn)^{2a_0+1}}) \tag{184}$$

Therefore

$$\sum_{i=1}^{q}\frac{\gamma_{qn+i}^2+\alpha_n\lambda_{qn+i}}{(qn+i)^{2a_0}} = \sum_{i=1}^{q}\frac{\gamma_{qn+i}^2+\alpha_n\lambda_{qn+i}}{(qn)^{2a_0}}+O(\frac{1}{(qn)^{2a_0+1}}) \quad (185)$$

$$= \frac{1}{(qn)^{2a_0}}\Bigg\{\sum_{i=1}^{q}\frac{y_{qn+i}-x_{qn+i}+2x_{qn+i}^2}{2} \quad (186)$$

$$+ \cos(2b_0\ln(qn))\sum_{i=1}^{q}\frac{y_{qn+i}+x_{qn+i}-2x_{qn+i}^2}{2} \quad (187)$$

$$+ \sin(2b_0\ln(qn))\sum_{i=2}^{q}\frac{y_{qn+i}-x_{qn+i}-2x_{qn+i}y_{qn+i}}{2}\Bigg\}+O(\frac{1}{(qn)^{2a_0+1}}) \quad (188)$$

$$= \frac{1}{(qn)^{2a_0}}\Bigg\{\sum_{i=1}^{q}\frac{y_i-x_i+2x_i^2}{2} \quad (189)$$

$$+ \cos(2b_0\ln(qn))\sum_{i=1}^{q}\frac{y_i+x_i-2x_i^2}{2} \quad (190)$$

$$+ \sin(2b_0\ln(qn))\sum_{i=2}^{q}\frac{y_i-x_i-2x_iy_i}{2}\Bigg\}+O(\frac{1}{(qn)^{2a_0+1}}) \quad (191)$$

$$= \frac{\alpha+\beta\cos(2b_0\ln(qn))+\gamma\sin(2b_0\ln(qn))}{(qn)^{2a_0}}+O(\frac{1}{(qn)^{2a_0+1}}) \quad (192)$$

Where

$$\alpha = \sum_{i=1}^{q}\frac{y_i-x_i+2x_i^2}{2} \quad (193)$$

$$\beta = \sum_{i=1}^{q}\frac{y_i+x_i-2x_i^2}{2} \quad (194)$$

$$\gamma = \sum_{i=2}^{q}\frac{y_i-x_i-2x_iy_i}{2} \quad (195)$$

From the cancelation property of the Dirichlet characters, we can calculate the values of α, β and γ.

$$\sum_{i=1}^{q}x_i=\Re\Big(\sum_{i=1}^{q}\chi(i)\Big)=\begin{cases}\phi(q), & \text{if } \chi=\chi_0 \text{ the trivial character}\\ 0, & \text{if otherwise, } \chi\neq\chi_0\end{cases} \quad (196)$$

And

$$\sum_{i=1}^{q}y_i=\Im\Big(\sum_{i=1}^{q}\chi(i)\Big)=\begin{cases}0, & \text{if } \chi=\chi_0 \text{ the trivial character}\\ 0, & \text{if otherwise, } \chi\neq\chi_0\end{cases} \quad (197)$$

Therefore

$$\alpha=\begin{cases}\frac{\phi(q)}{2}, & \text{if } \chi=\chi_0 \text{ the trivial character}\\ \sum_{i=1}^{q}x_i^2, & \text{if otherwise, } \chi\neq\chi_0\end{cases} \quad (198)$$

$$\beta = \begin{cases} -\frac{\phi(q)}{2}, & \text{if } \chi = \chi_0 \text{ the trivial character} \\ -\sum_{i=1}^{q} x_i^2, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \tag{199}$$

$$\gamma = \begin{cases} -\frac{\phi(q)}{2}, & \text{if } \chi = \chi_0 \text{ the trivial character} \\ -\sum_{i=2}^{q} x_i y_i, & \text{if otherwise, } \chi \neq \chi_0 \end{cases} \tag{200}$$

Therefore we can write the following for a large n:

$$\sum_{i=1}^{q} \frac{\gamma_{qn+i}^2 + \alpha_n \lambda_{qn+i}}{(qn+i)^{2a_0}} = \frac{\alpha + \beta\cos(2b_0 \ln(qn)) + \gamma \sin(2b_0 \ln(qn)\big)}{(qn)^{2a_0}} + \frac{\xi_n}{(qn)^{2a_0+1}} \tag{201}$$

Where the sequence ξ_n is bounded.

Therefore

$$\begin{aligned} \sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}} &= \sum_{n=N_0}^{N} \sum_{i=1}^{q} \frac{\gamma_{qn+i}^2 + \alpha_n \lambda_{qn+i}}{(qn+i)^{2a_0}} && (202) \\ &= \sum_{n=N_0}^{N} \frac{\alpha + \beta\cos(2b_0 \ln(qn)) + \gamma \sin(2b_0 \ln(qn)\big)}{(qn)^{2a_0}} + \sum_{n=N_0}^{N} \frac{\xi_n}{(qn)^{2a_0+1}} && (203) \end{aligned}$$

And

$$\begin{aligned} \sum_{n=qN_0+1}^{Nq} \frac{C_n}{n^{2a_0}} + \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} &= \sum_{n=N_0}^{N} \frac{\alpha + \beta\cos(2b_0 \ln(qn)) + \gamma \sin(2b_0 \ln(qn)\big)}{(qn)^{2a_0}} && (204) \\ &+ \sum_{n=N_0}^{N} \frac{\xi_n}{(qn)^{2a_0+1}} + \sum_{n=qN_0+1}^{Nq} \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} && (205) \end{aligned}$$

The serie $\sum_{n\geq 1} \frac{\alpha+\beta\cos(2b_0\ln(qn))+\gamma\sin(2b_0\ln(qn)\big)}{(qn)^{2a_0}}$ was studied in the lemma 3.1 in [1]. Let's denote the function g as the following:

$$g(x) = \frac{\alpha + \beta\cos(2b_0 \ln(qx)) + \gamma \sin(2b_0 \ln(qx)\big)}{(qx)^{2a_0}} \tag{206}$$

We have the following:

$$\left| \sum_{n=N_0+1}^{+\infty} g(n) - \int_{N_0+1}^{+\infty} g(x) \right| \leq \frac{K}{(N_0)^{2a_0}} \tag{207}$$

And the primitive function G of the function g(see lemma 3.1 in [1]):

$$\begin{aligned} G(x) &= \frac{1}{2q}\Big\{ -\frac{\alpha}{(2a_0-1)(x)^{2a_0-1}} && (208) \\ &+ \frac{\big(2b_0\beta + \gamma(1-2a_0)\big)\sin(2b_0\ln(x)) + \big(\beta(1-2a_0) - 2\gamma b_0\big)\cos(2b_0\ln(x))}{\Big((2b_0)^2 + (1-2a_0)_0^2\Big)(x)^{2a_0-1}} \Big\} && (209) \end{aligned}$$

Therefore

$$\left| \sum_{n=N_0+1}^{+\infty} g(n) + G(N_0+1) \right| \leq \frac{K}{(N_0)^{2a_0}} \tag{210}$$

Case of Non-Trivial Character In this case we have:

$$\alpha = \sum_{i=1}^{q} x_i^2 \tag{211}$$

$$\beta = -\sum_{i=1}^{q} x_i^2 \tag{212}$$

$$\gamma = -\sum_{i=2}^{q} x_i y_i \tag{213}$$

As an example, the case where $q = 3$ and $\chi(1) = 1$, $\chi(2) = -1$ and $\chi(3) = 0$. we have $\alpha = 2$, $\beta = -2$ and $\gamma = 0$. $L(s,\chi) = \sum_{n=0}^{+\infty}\Big(\frac{1}{(3n+1)^s} - \frac{1}{(3n+2)^s}\Big)$.
In the general case of a non-trivial Dirichlet character modulo q, we have $x_1 = \chi(1) = 1$. Therefore $\alpha > 1$ and $\beta = -\alpha$. Therefore $G \neq 0$.

Case of Riemann Zeta or Dirichlet η function The Riemann zeta function is extended to whole complex plane where $\Re(s) > 0$ by the Dirichlet η function: $\eta(s) = \sum_{n=0}^{+\infty}\Big(\frac{1}{(2n+1)^s} - \frac{1}{(2n+2)^s}\Big)$. In this case: $q = 2$ and $\chi(1) = 1$, $\chi(2) = -1$. we have $\alpha = 2$, $\beta = -2$ and $\gamma = 0$. Therefore $\alpha > 1$ and $\beta = -\alpha$. Therefore $G \neq 0$.

Remark. *In the case of Riemann Zeta, the function χ is not a Dirichlet character but it is in line with our study as it respects the conditions of the lemmas 3.2 and 3.3. Therefore this proof is also proves the Riemann Hypothesis.*

Conclusion. *In both cases above we have $\alpha > 1$, $\beta = -\alpha$ and the function G is nonzero: $G \neq 0$.*

Concerning the term G_{N_0}. We have

$$G_{N_0} = \frac{v_{N_0}}{(N_0)^{a_0}} B_{N_0} = \frac{v_{N_0}\lambda_{N_0}}{(N_0)^{2a_0}} \tag{214}$$

Where the sequence $(v_n\lambda_n)$ is bounded.

Therefore we have the asymptotic expansion of $G_{N_0} + \sum_{n=N_0+1}^{+\infty} g(k)$ as follows:

$$G_{N_0} + \sum_{n=N_0+1}^{+\infty} g(k) = -G(N_0+1) + \mathcal{O}\Big(\frac{1}{(N_0)^{2a_0}}\Big) \tag{215}$$

Remark. *The remaining terms in the expression of R_{N_0} are all of the order of $\frac{1}{(N_0)^{2a_0}}$ and above. So the dominant term in the expression R_{N_0} is the term $-G(N_0+1)$. The function G is a nonzero function. Therefore this term is the dominant term in the expression of $\frac{1}{N}\sum_{n=N_0}^{N-1} G_{n+1}$. Hence based on the sign of $G(N_0+1)$ we can show that the limit of $\sum_{n=1}^{N-1} G_{n+1}$ can be both $+\infty$ and $-\infty$.*

Let's now study the function f:

$$f(x) = -\frac{\alpha}{2a_0-1} + \frac{\big(2b_0\beta+\gamma(1-2a_0)\big)\sin(2b_0x)+\big(\beta(1-2a_0)-2\gamma b_0\big)\cos(2b_0x)}{(2b_0)^2+(1-2a_0)_0^2} \quad (216)$$

$$= \alpha_2+\beta_2\sin(2b_0x)+\gamma_2\cos(2b_0x) \quad (217)$$

The function f is also nonzero function. Otherwise we will have $\alpha = 0$ which is in contradiction with the fact that $\alpha > 1$.
The function f is a linear combination of the functions sin and cos. So the function f is differentiable and bounded. The function f is periodic of period $\frac{\pi}{b_0}$.
Let's calculate the function f values at the following points:

$$f(0) = \alpha_2+\gamma_2 \quad (218)$$

$$f(\frac{\pi}{b_0}) = \alpha_2-\gamma_2 \quad (219)$$

We have either $f(0) \neq 0$ or $f(\frac{\pi}{b_0}) \neq 0$, otherwise we will have $\alpha = 0$ which is again in contradiction with the fact that $\alpha > 1$. Let's assume $f(0) \neq 0$. From the values above we have the following:

- If $\Big(\alpha_2+\gamma_2\Big) > 0$ then $f(0) > 0$.
- If $\Big(\alpha_2+\gamma_2\Big) < 0$ then $f(0) < 0$.

In case of $f(0) > 0$ we can prove that the limit of $\sum_{n=1}^{N-1} G_{n+1}$ goes to $-\infty$ and in the case of $f(0) < 0$ we can prove that the limit of $\sum_{n=1}^{N-1} G_{n+1}$ goes to $+\infty$. Therefore, in all cases we have the series $\sum_{n\geq 1} G_{n+1}$ diverges. So let's assume from now on that $f(0) < 0$ as the same proof can be done in both cases.
From the lemma 3.3 in [1], there exist $N_1 \geq N_0$ such that $|\cos(2b_0\ln(N_1+1))-1| \leq \epsilon$ and $|\sin(2b_0\ln(N_1+1))| \leq \epsilon$ for any $1 > \epsilon > 0$.
Let's define $\beta_0 = -f(0)$. $\beta_0 > 0$.
Let's fix $\epsilon > 0$ to be very small such that $0 < \epsilon < \min\Big(1, \frac{\beta_0}{10000}\Big)$.

Let's define $\gamma_0 = |\beta_2| + |\gamma_2|$. We have the same proof steps as in the case of F_n.

Let's N_1 be such that $|\cos(2b_0\ln(N_1+1))-1| \leq \epsilon/\gamma_0$ and also $|\sin(2b_0\ln(N_1+1))-0| \leq \epsilon/\gamma_0$. Therefore:

$$\left|G(N_1+1) - \frac{f(0)}{(N_1+1)^{2a_0-1}}\right| \leq \left|\left\{\frac{\Big((\beta_2)\sin(2b_0\ln(N_1+1)) + (\gamma_2)\cos(2b_0\ln(N_1+1))\Big)}{(N_1+1)^{2a_0-1}}\right.\right. \tag{220}$$

$$\left.\left. - \frac{\Big((\beta_2)\sin(2b_0 0) + (\gamma_2)\cos(2b_0 0)\Big)}{(N_1+1)^{2a_0-1}}\right\}\right| \tag{221}$$

$$\leq \frac{\Big(|\beta_2|\,|\sin(2b_0\ln(N_1+1))| + |\gamma_2|\,|\cos(2b_0\ln(N_1+1)) - 1|\Big)}{(N_1+1)^{2a_0-1}} \tag{222}$$

$$\leq \frac{\Big(|\beta_2|\,\epsilon/\gamma_0 + |\gamma_2|\,\epsilon/\gamma_0\Big)}{(N_1+1)^{2a_0-1}} \tag{223}$$

$$\leq \frac{\epsilon}{(N_1+1)^{2a_0-1}} \tag{224}$$

Therefore

$$\left|\sum_{n=N_0+1}^{+\infty} g(n) - \frac{\beta_0}{(N_0+1)^{2a_0-1}}\right| \leq \left\{\left|\sum_{n=N_0+1}^{+\infty} g(n) + G(N_1+1)\right|\right. \tag{225}$$

$$\left. + \left|-G(N_1+1) - \frac{-f(0)}{(N_1+1)^{2a_0-1}}\right|\right\} \leq \frac{\epsilon}{(N_1+1)^{2a_0-1}} + \frac{K}{2a_0(N_1)^{2a_0}} \tag{226}$$

Hence

$$\left|\sum_{n=N_1+1}^{+\infty} g(n) - \frac{\beta_0}{(N_1+1)^{2a_0-1}}\right| \leq \frac{\epsilon}{(N_1+1)^{2a_0-1}} + \frac{K}{2a_0(N_1)^{2a_0}} \tag{227}$$

Therefore

$$\sum_{n=N_1+1}^{+\infty} g(n) \geq \frac{\beta_0 - \epsilon}{(N_1+1)^{2a_0-1}} - \frac{K}{2a_0(N_1)^{2a_0}} \tag{228}$$

Therefore

$$R_{N_1} = \sum_{k=qN_1+1}^{+\infty} g(k) + F_{N_0} + \lim_{N\to+\infty}\left[\sum_{n=N_1}^{N} \frac{\xi_n}{(qn)^{2a_0+1}} + \sum_{n=qN_1+1}^{Nq} \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}}\right] \tag{229}$$

$$\geq \frac{\beta_0 - \epsilon}{(N_0+1)^{2a_0-1}} - \frac{K}{2a_0(N_1)^{2a_0}} + F_{N_1} + \lim_{N\to+\infty}\left[\sum_{n=N_1}^{N} \frac{\xi_n}{(qn)^{2a_0+1}} + \sum_{n=qN_1+1}^{Nq} \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}}\right] \tag{230}$$

We have the sequences (ξ_k), (D_k) and (ε_k) are bounded. Plus we have $a_0 + 2 \geq 2a_0 + 1 > a_0 + 1 \geq 2a_0 > 1$, therefore the series $\sum_{k\geq 1} \frac{D_k}{k^{2a_0+1}}$, $\sum_{k\geq 1} \frac{\varepsilon_k}{k^{a_0+2}}$ are converging absolutely.

Let the positive constant M such that:
For each $k \geq N_1$ and $N \geq N_1$:

$$|c_k| \leq M \tag{231}$$
$$|D_k| \leq M \tag{232}$$
$$|\varepsilon_k| \leq M \tag{233}$$

Therefore, we have $a_0 > 0$:

$$\left| \sum_{n=N_1}^{+\infty} \frac{\xi_n}{(qn)^{2a_0+1}} + \sum_{n=qN_1+1}^{+\infty} \frac{D_n}{n^{2a_0+1}} + \frac{\varepsilon_n}{n^{a_0+2}} \right| \tag{234}$$
$$\leq \frac{M}{2a_0(qN_1)^{a_0}} + \frac{M}{2a_0(N_1)^{2a_0}} + \frac{M}{(a_0+1)(N_1)^{a_0+1}} \tag{235}$$

Therefore

$$R_{N_1} \geq \frac{\beta_0 - \epsilon}{(N_1+1)^{2a_0-1}} - \frac{K}{2a_0(N_1)^{2a_0}} - \frac{M}{2a_0(qN_1)^{2a_0}} - \frac{M}{2a_0(N_1)^{2a_0}} - \frac{M}{(a_0+1)(N_1)^{a_0+1}} \tag{236}$$

Let's define the sequence (δ_n):

$$\delta_n = \frac{v_n\lambda_n(n+1)^{2a_0-1}}{n^{2a_0}} - \frac{K(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} - \frac{M(n+1)^{2a_0-1}}{2a_0(qn)^{2a_0}} - \frac{M(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} - \frac{M(n+1)^{2a_0-1}}{(a_0+1)(n)^{a_0+1}} \tag{237}$$

As we have $\frac{1}{2} < a_0 < 1$, therefore $2a_0 - 1 < a_0 < 2a_0 < a_0 + 1$
Therefore

$$\lim_{n\to+\infty} -\frac{v_n\lambda_n(n+1)^{2a_0-1}}{n^{2a_0}} + \frac{K(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{2a_0(qn)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{2a_0(n)^{2a_0}} + \frac{M(n+1)^{2a_0-1}}{(a_0+1)(n)^{a_0+1}} = 0 \tag{238}$$

Therefore the $\lim_{n\to+\infty} \delta_n = 0$.
So we can choose N_1 such that $|\delta_{N_1}| < \epsilon$, i.e $\delta_{N_1} \leq \epsilon$.
Therefore

$$R_{N_1} \geq \frac{1}{(N_1+1)^{2a_0-1}} \left(\beta_0 + 2\epsilon \right) > 0 \tag{239}$$

Therefore $\lim_{N\to+\infty} \sum_{n=N_0}^{N} G_n = +\infty$.

□

□

Conclusion. *We have the serie $\sum_{n\geq 1} V_n^2$ is converging absolutely thanks to $2a_0 > 1$. We have from the lemmas 3.1 that:*

$$\lim_{N\to\infty} \sum_{n=1}^{N} G_n = +\infty \tag{240}$$

Therefore

$$\lim_{N\to\infty} B_N^2 = +\infty \tag{241}$$

This result is in contradiction with the fact that s is a $L(s,\chi)$ zero therefore the limit $\lim_{N\to\infty} B_N = 0$ should be zero. Therefore s with $\frac{1}{2} < a_0 = \Re(s) < 1$ cannot be a zero for $L(s,\chi)$.

3.2 Case One: $\frac{1}{2} < a_0 \leq 1$ and χ trivial

For the trivial character χ_0 modulus q we have:

$$\zeta(s) = L(s, \chi_0) \prod_{p \, \mathrm{Prime}, p/q} \left(1 - \frac{1}{p^s}\right)^{-1} \tag{242}$$

Where ζ is the Riemann Zeta function.
The product $\prod_{p \, \mathrm{Prime}, p/q} \left(1 - \frac{1}{p^s}\right)^{-1}$ is finite, bounded and nonzero as $\frac{1}{2} < \Re(s) = a_0 \leq 1$. From the equation above we have: if s is a zero for $L(s, \chi_0)$, then it is also a zero for the Riemann $zeta(s)$. We saw in the previous case that if $\frac{1}{2} < \Re(s) = a_0 \leq 1$, it is not possible for the Riemann zeta function to have such a zero. Therefore $L(s, \chi_0)$ cannot have a zero where $\frac{1}{2} < \Re(s) = a_0 \leq 1$.

3.3 Conclusion

We saw that if s is a $L(s, \chi)$ zero, then real part $\Re(s)$ can only be $\frac{1}{2}$ as all other possibilities can be discarded using the functional equation like in [1]. Therefore the Generalized Riemann hypothesis is true: *The non-trivial zeros of $L(s, \chi)$ have real part equal to $\frac{1}{2}$.*

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