Existence of solutions for scalar conservation laws with moving flux constraints

Thibault Liard, Benedetto Piccoli

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Thibault Liard[*†] Benedetto Piccoli[‡]

Abstract

We consider a coupled PDE-ODE model representing a slow moving vehicle immersed in vehicular traffic. The PDE consists of a scalar conservation law modeling the evolution of vehicular traffic and the trajectory of a slow moving vehicle is given by an ODE depending on the downstream traffic density. The slow moving vehicle may be regarded as a moving bottleneck influencing the bulk traffic flow via a moving flux pointwise constraint. We prove existence of solutions with respect to initial data of bounded variation. Approximate solutions are constructed via the wave-front tracking method and their limit are solutions of the Cauchy problem PDE-ODE.

Keywords: Scalar conservation laws with constraints; Wave-front tracking; Traffic flow modeling; non-classical shocks.

AMS classification: 35L65; 90B20

1 Introduction

1.1 Presentation of the Problem

The modeling of the impact of slow moving vehicles on the vehicular traffic has been studied by engineering communities [13, 15, 16] and in the applied mathematics [7, 8, 14] leading to a hybrid PDE-ODE model. The PDE models the evolution of vehicular traffic and the ODE represents the trajectory of slow moving vehicles. It is usual that a tractor or an Amish buggy produce a traffic jam when the density of cars is high enough. Thus, solutions of the PDE may be influenced by the ODE. Mathematically speaking, different approaches is used to model this impact; in [14], the authors multiply the usual flux function by a mollifier to represent the capacity drop of car flow due to the presence of a slow vehicle. They prove the existence of solutions in the sense of Fillipov ([11]) using a fractional step approach and assuming that the slow vehicle travels at maximal speed. In [7, 8], the slow moving vehicle is regarded as a moving constraint influencing solutions of the PDE via moving pointwise flux constraint. In [7], the authors defined the constrained Riemann

*DeustoTech, University of Deusto, 48007 Bilbao, Basque Country, Spain (`thibault.liard@deusto.es`).

†Facultad de Ingenieria, Universidad de Deusto, Avda.Universidades, 24, 48007, Bilbao - Basque Country-Spain.

‡Department of Mathematical Sciences and CCIB, Rutgers University-Camden, Camden, NJ, USA (`piccoli@camden.rutgers.edu`).

problem for the following hybrid PDE-ODE

$$
\begin{aligned}
&\partial_t \rho(t,x) + \partial_x(f(\rho(t,x))) = 0, && (t,x) \in \mathbb{R}_+ \times \mathbb{R}, && \text{(1a)}\\
&\rho(0,x) = \rho_0(x), && x \in \mathbb{R}, && \text{(1b)}\\
&f(\rho(t,y(t))) - \dot{y}(t)\rho(t,y(t)) \leqslant F_\alpha(\dot{y}) := \alpha \max_{\rho\in[0,\rho_{\max}]}(f(\rho) - \dot{y}\rho), && t \in \mathbb{R}_+, && \text{(1c)}\\
&\dot{y}(t) = \min(V_b, v(\rho(t,y(t)+))), && t \in \mathbb{R}_+, && \text{(1d)}\\
&y(0) = y_0. && && \text{(1e)}
\end{aligned}
$$

with $f(\rho) = 1-\rho$ and show that approximate solutions of (1) constructed by a wave-front tracking method converge to a weak solution of (1a)-(1b). This paper addresses the existence of solutions for the whole PDE-ODE systems (1). In [17], a proof of the stability of solutions for (1) is given using a wave-front tracking method and the notion of generalized tangent vectors. Some numerical methods have also been developed in [4, 5, 6, 9]; in [6], the algorithm used is based on Godunov schemes using reconstruction techniques to avoid diffusion effects and capture non-classical shocks. In [9], the authors use a wave-front tracking algorithm regarding a front-wave as a numerical object. An extension to second order model has been studied in [21]; they replace the Lighthill-Whitham-Richards (briefly LWR) model (1a) by the Aw-Rascle-Zhang (briefly ARZ) second order model [1, 22]. They define two different Riemann Solvers and they propose numerical methods.

1.2 A strongly coupled PDE-ODE system

We consider a stretch of road $\mathbb{R}$ where $\rho_{\max}$ and $V_{\max}$ stand for the maximum density and the maximum speed of cars allowed on the road respectively. Here we focus on the hybrid PDE-ODE model (1), proposed in [7], describing the impact of a slow moving vehicle on the evolution of vehicular traffic. The first order model (1a) with (1b) was proposed by Lighthill-Whitham-Richards [18, 19] and this model consists of a single conservation law for the traffic density. The function $\rho = \rho(t,x) \in [0,\rho_{\max}]$ denotes the macroscopic traffic density at time $t \geqslant 0$ and at the position $x \in \mathbb{R}$. The flux f is given by $f : \rho \in [0,\rho_{\max}] \to \rho v(\rho)$, where $v \in C^2([0,\rho_{\max}];[0,V_{\max}])$ is the average speed of cars. We assume that the flux satisfies the condition

(F) $f : C^2([0,\rho_{\max}];[0,+\infty))$, $\quad f(0) = f(\rho_{\max}) = 0$,
f is strictly concave: $-B \leqslant f''(\rho) \leqslant -\beta < 0$ for all $\rho \in [0,\rho_{\max}]$, for some $\beta, B > 0$.

In particular, the speed v is a strictly decreasing function and $V_{\max} := v(0)$ is the maximal speed of cars. The ODE (1d) with (1e) describes the trajectory of the slow moving vehicle starting at $(t,x) = (0,y_0)$: the slow moving vehicle moves at its maximum speed $V_b \in (0,V_{\max})$ as long as the downstream traffic moves faster, otherwise it has to adapt its velocity accordingly to the traffic density in front (see Figure 1 where we chose $v(\rho) = 1-\rho$).

The slow moving vehicle is regarded as a Moving Bottleneck (briefly **MB**), see Figure 2. It acts on the evolution of vehicular traffic through the moving constraint (1c). The left side of (1c) represents the flux of cars at the position of the MB in the MB reference frame. $F_\alpha(\dot{y}) := \alpha \max_{\rho\in[0,\rho_{\max}]}(f(\rho) - \dot{y}\rho)$ in the right side of (1c) is the reduced maximum flow due to the presence of the MB (see Figure 3 and Figure 4). For instance, if $v(\rho) = V_{\max}(1 - \frac{\rho}{\rho_{\max}})$ then we have $F_\alpha(\dot{y}) := \frac{\alpha\rho_{\max}}{4V_{\max}}(V_{\max} - \dot{y}(t))^2$.

For future use, $\check{\rho}_\alpha$ and $\hat{\rho}_\alpha$ with $\check{\rho}_\alpha < \hat{\rho}_\alpha$ denote the two solutions to the equation $F_\alpha(\dot{y}) + V_b\rho = f(\rho)$ and ρ^* is the solution to $V_b\rho = f(\rho)$ (see Figure 3 and Figure 4). Since f is strictly concave, $\check{\rho}_\alpha$, $\hat{\rho}_\alpha$ and ρ^* are well-defined. In the case where $v(\rho) = V_{\max}(1 - \frac{\rho}{\rho_{\max}})$, we have

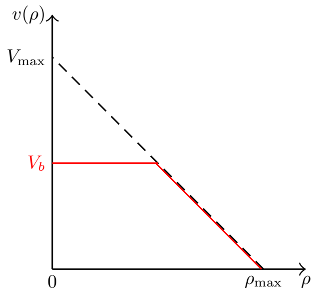


Figure 1: cars speed (--) and slow moving vehicle speed (−) with $v(\rho) = \rho(1-\rho)$.

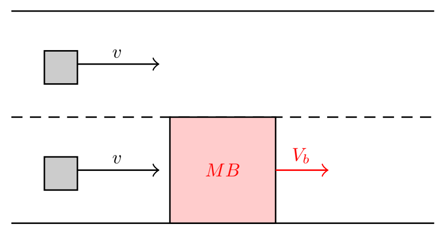


Figure 2: A slow moving vehicle regarded as a Moving Bottleneck (MB) blocking one lane.

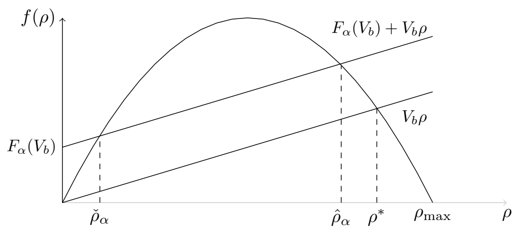


Figure 3: Flux function for $\dot{y} = V_b$ in a fixed reference frame.

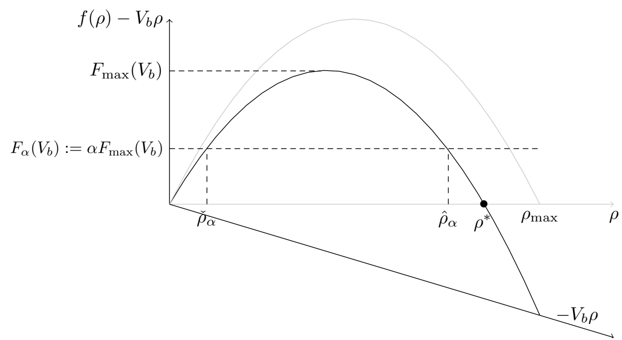


Figure 4: Flux function for $\dot{y} = V_b$ in the **MB** reference frame.

$\check{\rho}_\alpha = \rho_{\max}(V_{\max} - V)(\frac{1-\sqrt{1-\alpha}}{2V_{\max}})$, $\hat{\rho}_\alpha = \rho_{\max}(V_{\max} - V)(\frac{1+\sqrt{1-\alpha}}{2V_{\max}})$ and $\rho^* = \rho_{\max}(1 - \frac{V_b}{V_{\max}})$.

Notation: Given $\rho_1, \rho_2 \in [0, \rho_{\max}]$, we denote by $\sigma(\rho_1, \rho_2) := \frac{f(\rho_1) - f(\rho_2)}{\rho_1 - \rho_2}$ the Rankine-Hugoniot speed of the front-wave (ρ_1, ρ_2).

1.3 Main result

Let's introduce the definition of solutions to the constrained Cauchy problem (1) as in [7, Section 4].

Definition 1. *The couple*

$$(\rho, y) \in C^0\left([0, +\infty[; \boldsymbol{L}^1 \cap \mathbf{BV}(\mathbb{R}; [0, \rho_{\max}])\right) \times \mathbf{W}^{\mathbf{1,1}}\left([0, +\infty[; \mathbb{R}\right)$$

is a solution to (1) if

i The function ρ is a weak solution to the PDE in (1), for $(t, x) \in (0, +\infty) \times \mathbb{R}$, i.e for all $\varphi \in C_c^1(\mathbb{R}^2, \mathbb{R})$,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} (\rho \partial_t \varphi + f(\rho) \partial_x \varphi) dx dt + \int_{\mathbb{R}} \rho_0(x) \varphi(0, x) dx = 0.$$

ii *The function ρ satisfies Kružhkov entropy conditions on* $(0,+\infty)\times\mathbb{R}\setminus\{(t,y(t));t\in\mathbb{R}_+\}$, *i.e for every* $k\in[0,\rho_{\max}]$, *for all* $\varphi\in C_c^1(\mathbb{R}^2,\mathbb{R}_+)$ *such that* $\varphi(t,y(t))=0,\ t>0$,

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}}(|\rho-k|\partial_t\varphi+sgn(\rho-k)(f(\rho)-f(k))\partial_x\varphi)dxdt+\int_{\mathbb{R}}|\rho_0-k|\varphi(0,x)dx\geqslant 0;$$

iii *For a.e* $t\in\mathbb{R}_+$, $\dot{y}(t)=\min(V_b,v(\rho(t,y(t)+)))$ *or for every* $t\in\mathbb{R}^+$

$$y(t)=y_0+\int_0^t\min\left(V_b,v(\rho(s,y(s)+))\right)\,ds\,;$$

iv *The constraint* (1c) *is satisfied, in the sense that for a.e.* $t\in\mathbb{R}$

$$\lim_{x\to y(t)\pm}\left(f(\rho(t,x))-\dot{y}(t)\rho(t,x)\right)\leqslant F_\alpha(\dot{y})\,;$$

The goal of this paper is to prove the existence of solutions for the hybrid PDE-ODE system defined in (1).

Theorem 1. *Let* $\rho_0\in BV(\mathbb{R},[0,\rho_{\max}])$, *then the Cauchy problem* (1) *admits a solution in the sense of Definition 1.*

The proof of Theorem 1 is structured as follows: we contruct piecewise constant approximate solutions (ρ^n,y^n) of (1) via the wave-front tracking method described in Section 2.2. By introducing a suitable TV type functional $\Gamma(t)$ defined in (8), we show that there exists $C>0$ such that, for every $t\in\mathbb{R}_+$, $TV(\rho^n(t,\cdot))\leqslant C$ (see Section 3.1). Lemma 3 in Section 3.1 is devoted to prove the convergence of the approximate solution (ρ^n,y^n) to (ρ,y) as $n\to\infty$. In Section 3.2, we show that the limit ρ is a weak solution of (1) in the sense of Definition 1 and ρ is an entropy admissible solution in $(0,+\infty)\times\mathbb{R}\setminus\{(t,y(t));t\in\mathbb{R}_+\}$. Thus, the limit (ρ,y) verifies Definition 1 i and 1 ii. Moreover, we prove that the limit (ρ,y) verifies Definition 1 iv using that both ρ^n are ρ are weak solutions of (1a) on $\{(t,x)\in[0,T]\times\mathbb{R}/x<y(t)\}$ and on $\{(t,x)\in[0,T]\times\mathbb{R}/y(t)<x\}$. In section 3.3, we study the behavior of ρ^n around the point $(\bar{t},y^n(\bar{t}))$ in order to prove that the limit (ρ,y) verifies Definition 1 iii.

2 The Riemann problem of (1) and Wave-front tracking method

2.1 The Riemann problem with moving constraints

We consider (1) with Riemann type initial data

$$\rho_0(x)=\begin{cases}\rho_L & \text{if} \quad x<0\\ \rho_R & \text{if} \quad x>0\end{cases}\qquad\text{and}\quad y_0=0. \tag{2}$$

The definition of the Riemann solver for (1) and (2) is described in [7, Section 3]; we denote by $\mathcal{R}$ the standard Riemann solver for (1a)-(1b) where ρ_0 is defined in (2). We have the following:

Definition 2. *The constrained Riemann solver* $\mathcal{R}^\alpha:[0,\rho_{\max}]^2\mapsto \boldsymbol{L}^1_{loc}(\mathbb{R};[0,\rho_{\max}])$ *for* (1) *and* (2) *is defined as follows.*

i *If* $f(\mathcal{R}(\rho_L,\rho_R)(V_b))>F_\alpha(V_b)+V_b\mathcal{R}(\rho_L,\rho_R)(V_b)$, *then*

$$\mathcal{R}^\alpha(\rho_L,\rho_R)(x/t)=\begin{cases}\mathcal{R}(\rho_L,\hat{\rho}_\alpha)(x/t) & \text{if} \quad x<V_bt,\\ \mathcal{R}(\check{\rho}_\alpha,\rho_R)(x/t) & \text{if} \quad x\geqslant V_bt,\end{cases}\qquad\text{and}\quad y(t)=V_bt.$$

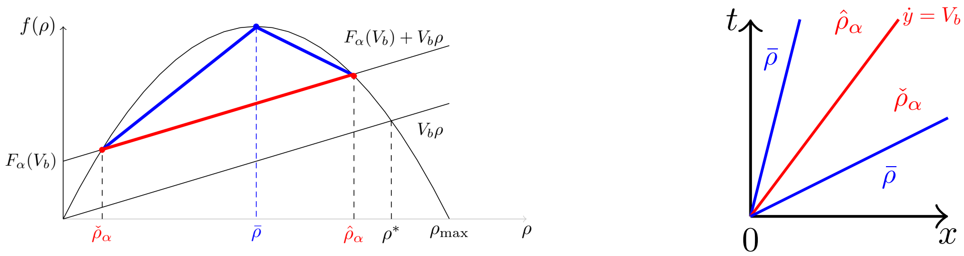
ii *If* $V_b\mathcal{R}(\rho_L, \rho_R)(V_b) \leqslant f(\mathcal{R}(\rho_L, \rho_R)(V_b)) \leqslant F_\alpha(V_b) + V_b\mathcal{R}(\rho_L, \rho_R)(V_b)$, *then*

$$\mathcal{R}^\alpha(\rho_L, \rho_R) = \mathcal{R}(\rho_L, \rho_R) \quad \textit{and} \quad y(t) = V_b t.$$

iii *If* $f(\mathcal{R}(\rho_L, \rho_R)(V_b)) < V_b\mathcal{R}(\rho_L, \rho_R)(V_b)$, *then*

$$\mathcal{R}^\alpha(\rho_L, \rho_R) = \mathcal{R}(\rho_L, \rho_R) \quad \textit{and} \quad y(t) = v(\rho_R)t.$$

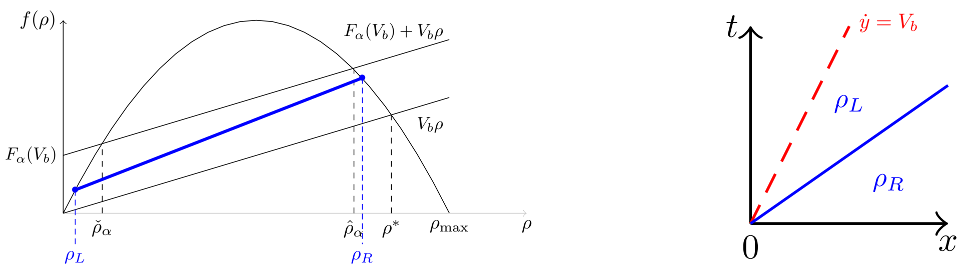
The three cases above are illustrated in Figure 5, Figure 6 and Figure 7.



Fundamental diagram representation

space-time diagram

Figure 5: The solution of the constrained Riemann problem of (1) with $\rho_L = \rho_R = \bar{\rho} \in (\check{\rho}_\alpha, \hat{\rho}_\alpha)$: case *i* of Definition 2.



Fundamental diagram representation

space-time diagram

Figure 6: The solution of the constrained Riemann problem of (1) with $0 < \rho_L < \check{\rho}_\alpha$ and $\hat{\rho}_\alpha < \rho_R \leqslant \rho_{\max}$: case *ii* of Definition 2.

2.2 Wave-front tracking method

We introduce on $[0, \rho_{\max}]$ the mesh $\widetilde{\mathcal{M}}_n = \{\tilde{\rho}_i^n\}_{i=0}^{2^n}$ defined by

$$\widetilde{\mathcal{M}}_n = \rho_{\max}(2^{-n}\mathbb{N} \cap [0, 1]).$$

We add the points $\check{\rho}_\alpha, \hat{\rho}_\alpha$ and ρ^* to the mesh $\widetilde{\mathcal{M}}_n$ as described in [7, Section 4.1]:

- if $\min_i |\check{\rho}_\alpha - \tilde{\rho}_i^n| = \rho_{\max}2^{-n-1}$ then we add the point $\check{\rho}_\alpha$ to the mesh

$$\mathcal{M}_n := \widetilde{\mathcal{M}}_n \cup \{\check{\rho}_\alpha\};$$

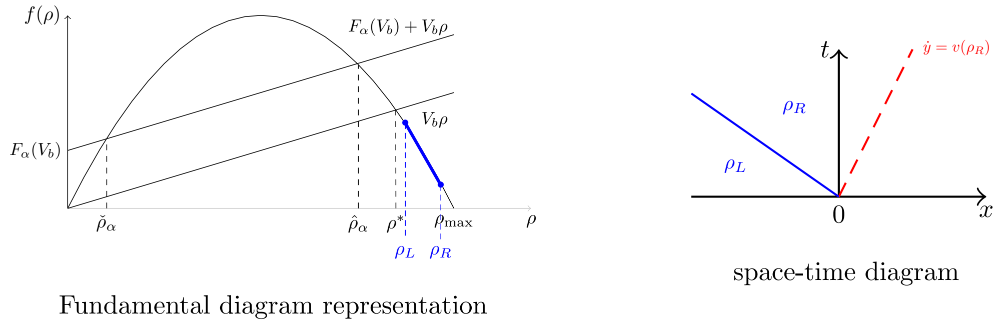


Figure 7: The solution of the constrained Riemann problem of (1) with $\rho^* < \rho_L < \rho_R$: case *iii* of Definition 2.

- if $|\check{\rho}_\alpha - \tilde{\rho}_l^n| = \min_i |\check{\rho}_\alpha - \tilde{\rho}_i^n| < \rho_{\max} 2^{-n-1}$ then we replace $\tilde{\rho}_l^n$ by $\check{\rho}_\alpha$

$$\mathcal{M}_n = \widetilde{\mathcal{M}_n} \cup \{\check{\rho}_\alpha\} \setminus \{\rho_l^n\};$$

- we perform the same operation for $\hat{\rho}_\alpha$ and for ρ^*.

We denote by $N := \text{card}(\mathcal{M}_n)$. We have $2^n \leqslant N \leqslant 2^n + 3$ and the constructed density mesh $\mathcal{M}_n := \{\rho_i^n\}_{i=0}^N$, sorted in ascending order, includes $\check{\rho}_\alpha$, $\hat{\rho}_\alpha$ and ρ^*. Moreover, for every $i, j \in \{0, \cdots, N\}$, we have

$$\rho_{\max} 2^{-n-1} \leqslant |\rho_i^n - \rho_j^n| \leqslant 3\rho_{\max}\, 2^{-n-1}. \tag{3}$$

Let $\rho_0 \in BV(\mathbb{R}, [0,1])$. Since our problem is scalar, we use the very first wave-front tracking algorithm proposed by Dafermos [3]; the initial density ρ_0 is approximated by piecewise constant functions ρ_0^n verifying $\rho_0^n(x) \in \mathcal{M}_n$ for a.e $x \in \mathbb{R}$. We denote by $(x_i^n)_{i=1,\cdots,M}$ the $M \in \mathbb{N}$ discontinuity points of ρ_0^n.

- If $\rho_0^n(x_i^n-) < \rho_0^n(x_i^n+)$, a shock wave $(\rho_0^n(x_i^n-), \rho_0^n(x_i^n+))$ is generated with speed given by the Rankine-Hugoniot condition
- If $\rho_0^n(x_i^n-) > \rho_0^n(x_i^n+)$, we split the rarefaction wave $(\rho_0^n(x_i^n-), \rho_0^n(x_i^n+))$ into a fan of rarefaction shocks; since, for every $x \in \mathbb{R}$, $\rho_0^n(x) \in \mathcal{M}_n = \{\rho_j^n\}_{j=0}^N$, there exists $j_0 < j_1$ such that $\rho_0^n(x_i-) = \rho_{j_1}^n$ and $\rho_0^n(x_i+) = \rho_{j_0}^n$. We create $j_1 - j_0$ rarefaction shocks $(\rho_j^n, \rho_{j+1}^n)_{j=j_0,\cdots,j_1-1}$ with speed prescribed by the Rankine-Hugoniot condition. The strength of each rarefaction shock is less than $3\rho_{\max} 2^{-n-1}$ and greater than $\rho_{\max} 2^{-n-1}$.

Thus, solving approximately the Riemann problem at each point of discontinuity of ρ_0^n as described above and piecing solutions together, we construct a solution ρ^n until two waves meet at time t_1. The approximate solution $\rho^n(t_1, \cdot)$ is a piecewise constant function verifying $\rho^n(t_1, x) \in \mathcal{M}_n$ for a.e $x \in \mathbb{R}$, the corresponding Riemann problems can again be approximately solved within the class of piecewise constant functions and so on. We define y^n to be the solution of

$$\begin{cases} \dot{y}(t) = \min(V_b, v(\rho^n(t, y(t)+))), & t \in \mathbb{R}_+, \\ y(0) = y_0, & x \in \mathbb{R}, \end{cases} \tag{4}$$

where $\rho^n(t, \cdot)$ is the wave-front tracking approximate solution at time t as described above with initial data ρ_0^n, see also [12, Section 2.6].

2.3 Structure of the approximate solution (ρ^n, y^n)

As soon as two discontinuity waves collide (see Figure 8), or a discontinuity wave hits the bus trajectory (Figure 9, Figure 10, Figure 11 and Figure 12) a new Riemann problem arises and its solution is obtained in the former case using the standard Riemann solver $\mathcal{R}$ and in the latter case using the constrained Riemann solver $\mathcal{R}^\alpha$, see Definition 2. There are no other possible interactions (for more details, we refer to [7]). The study of these interactions shows that no new rarefaction shock can arise at $t > 0$.

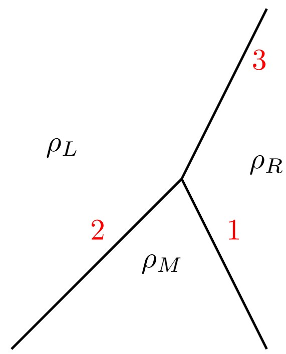
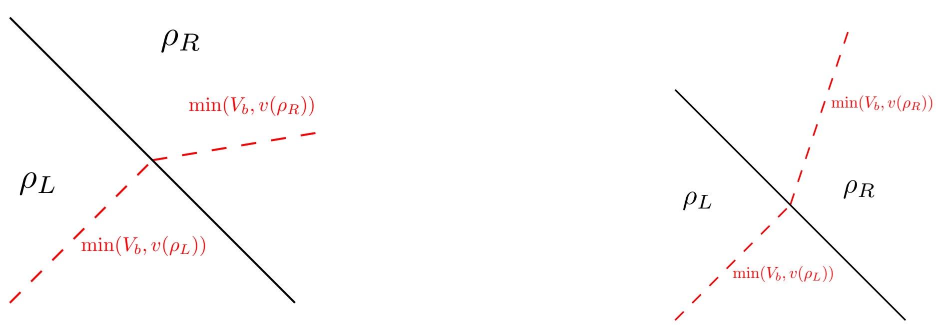


Figure 8: Two waves interact together producing a third wave

A wave-front (ρ_L, ρ_R) is called a shock if $\rho_L < \rho_R$, a rarefaction shock if $\rho_L > \rho_R$ and $3\rho_{\max} 2^{-n-1} \leqslant \rho_L - \rho_R \leqslant 3\rho_{\max} 2^{-n-1}$ or a non classical shock if $\rho_L = \hat{\rho}_\alpha$ and $\rho_R = \check{\rho}_\alpha$. Let $\rho_1, \rho_2 \in \mathcal{M}_n$ verifying that $\rho_2 < \rho_1$ and $\bar{t} > 0$, we introduce the set $\mathcal{A}(\rho_1, \rho_2, \bar{t}) \subset \mathcal{M}_n \times \mathbb{R} \times \mathbb{R}$ defined as follows:

$$(\rho_0^n, x_1, x_2) \in \mathcal{A}(\rho_1, \rho_2, \bar{t}) \quad \text{iff} \quad \left| \begin{array}{l} x_1 < x_2 \text{ with } \rho^n(\bar{t}, x_i) = \rho_i,\ i \in \{1, 2\}, \\ \forall x \in [x_1, x_2], \rho_{\max} 2^{-n-1} \leqslant \rho^n(t, x-) - \rho^n(t, x+) \leqslant 3\rho_{\max}\, 2^{-n-1}, \\ \text{or } \rho^n(t, x-) - \rho^n(t, x+) \leqslant 0, \end{array} \right.$$

where $\rho^n(\bar{t}, \cdot)$ is the wave-front tracking approximate solution at time $\bar{t}$ with initial data ρ_0^n. If $(\rho_0^n, x_1, x_2) \in \mathcal{A}(\rho_1, \rho_2, \bar{t})$ then $x \in [x_1, x_2] \to \rho^n(\bar{t}, x)$ may decrease by a jump of strength at most $3\rho_{\max}\, 2^{-n-1}$. Thus, the shocks or the rarefaction shocks are the only wave-fronts which are allowed over $\{\bar{t}\} \times [x_1, x_2]$.



Case a) $\rho^* \leqslant \rho_R < \rho_L$ and $\rho_{\max} 2^{-n-1} \leqslant \rho_L - \rho_R \leqslant 3\rho_{\max} 2^{-n-1}$.

Case b) $\rho^* < \rho_R$ and $\rho_L \in [0, \check{\rho}_\alpha] \cup [\hat{\rho}_\alpha, \rho_R)$.

Figure 9: Interaction coming from the right with the MB trajectory

Case a) $\rho_R \in (\hat\rho_\alpha, \rho_{\max}]$

Case b) $\rho_L = \hat\rho_\alpha$ and $\rho_{\max} 2^{-n-1} \leqslant \rho_L - \rho_R \leqslant 3\rho_{\max} 2^{-n-1}$

Figure 10: Interaction coming from the right with the MB trajectory cancelling (Case a)) or creating (Case b)) a non classical shock.

Case a) $\rho_L \in [0, \check\rho_\alpha)$

Case b) $\rho_R = \check\rho_\alpha$ and $\rho_{\max} 2^{-n-1} \leqslant \rho_L - \rho_R \leqslant 3\rho_{\max} 2^{-n-1}$

Figure 11: Interaction coming from the left with the MB trajectory cancelling (Case a)) or creating (Case b)) a non classical shock.

Lemma 1. *Let $\rho_1, \rho_2 \in \mathcal{M}_n$ verifying that $\rho_2 < \rho_1$ and $\bar t > 0$. We have*

$$\delta^n(\rho_1, \rho_2, \bar t) := \min_{(\rho_0^n, x_1, x_2) \in \mathcal{A}(\rho_1, \rho_2, \bar t)} x_2 - x_1 \geqslant \bar t \beta(\rho_1 - \rho_2 - \rho_{\max} 2^{-n+1}).$$

Remark 1. $\delta^n(\rho_1, \rho_2, \bar t)$ is the minimal length in space at time $\bar t$ to go from ρ_1 to ρ_2 only using shocks and rarefaction shocks.

Proof. Since $(\rho_0^n, x_1, x_2) \in \mathcal{A}(\rho_1, \rho_2, \bar t)$, the minimal length in space at time $\bar t$ to go from ρ_1 to ρ_2 is obtained by a fan of rarefaction shocks (ρ_1, ρ_2) coming from $(x, t) = (x_0, 0)$ (see Figure 13). Since $\rho_1, \rho_2 \in \mathcal{M}_n$, there exists $j_2 < j_1$ such that $\rho_1 = \rho_{j_1}^n$ and $\rho_2 = \rho_{j_2}^n$. Thus,

$$\begin{aligned} \delta^n(\rho_1, \rho_2, \bar t) &= \left(\bar t \sigma(\rho_{j_2+1}^n, \rho_{j_2}^n) + x_0\right) - \left(\bar t \sigma(\rho_{j_1}^n, \rho_{j_1-1}^n) + x_0\right) \\ &= \bar t \left(\sigma(\rho_{j_2+1}^n, \rho_{j_2}^n) - \sigma(\rho_{j_1}^n, \rho_{j_1-1}^n)\right) \end{aligned}$$

By definition of σ and using that f is strictly concave,

$$f'(\rho_{j_2+1}^n) < \sigma(\rho_{j_2+1}^n, \rho_{j_2}^n) < f'(\rho_{j_2}^n) \quad \text{and} \quad f'(\rho_{j_1}^n) < \sigma(\rho_{j_1}^n, \rho_{j_1-1}^n) < f'(\rho_{j_1-1}^n).$$

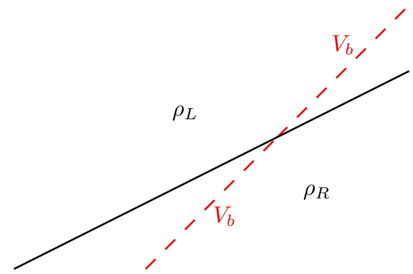


Figure 12: $\rho_L \in [0, \check{\rho}_\alpha]$, $\rho_R \in [0, \check{\rho}_\alpha] \cup [\hat{\rho}_\alpha, p^*]$ and $\rho_L + \rho_R < \rho^*$. Interaction coming from the left with the MB trajectory.

Using that $\rho_{\max} 2^{-n-1} \leqslant \rho^n_{j_2+1} - \rho^n_{j_2} \leqslant 3\rho_{\max} 2^{-n-1}$ and $\rho_{\max} 2^{-n-1} \leqslant \rho^n_{j_1} - \rho^n_{j_1-1} \leqslant 3\rho_{\max} 2^{-n-1}$, we conclude that

$$\begin{aligned} \delta^n(\rho_1, \rho_2, \bar{t}) \quad & > \bar{t}\left(f'(\rho^n_{j_2+1}) - f'(\rho^n_{j_1-1})\right), \\ & = \bar{t} f''(c)(\rho^n_{j_2+1} - \rho^n_{j_1-1}), \quad c \in (\rho^n_{j_2+1}, \rho^n_{j_1-1}), \\ & \geqslant \bar{t}\beta(\rho_1 - \rho_2 - \rho_{\max} 2^{-n+1}). \end{aligned}$$

□

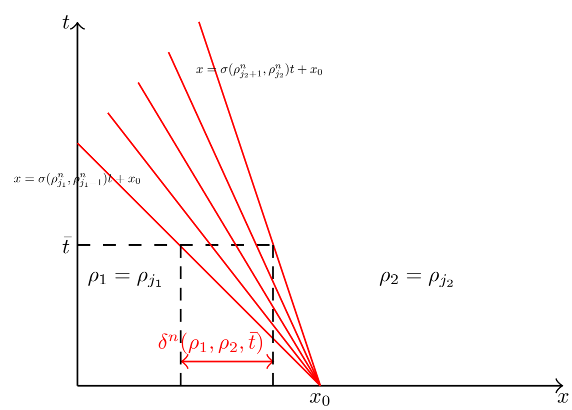


Figure 13: Illustration of the proof of Lemma 1

2.4 An instructive example

Assuming $f(\rho) = \rho v(\rho)$ with $v(\rho) = 1 - \rho$. Let $\rho_0(\cdot) = \hat{\rho}_\alpha \mathbb{1}_{(x_1, x_2)} + \mathbb{1}_{(x_2, +\infty)}$ and $y_0 = \frac{x_1 + x_2}{2}$ (see Figure 14a). We have $V_b = 1 - \check{\rho}_\alpha - \hat{\rho}_\alpha = v(\hat{\rho}_\alpha)$ and the solution (ρ, y) of (1) is

$$\rho(t,x) = \begin{cases} 0, & \text{if } (t,x) \in \left[\begin{array}{l} \{(t,x) \in [0, x_2 - x_1] \times \mathbb{R} \,/ x < (1 - \hat{\rho}_\alpha)t + x_1\}, \\ \{(t,x) \in [x_2 - x_1, \infty) \times \mathbb{R} \,/ x < (1 - \hat{\rho}_\alpha)(x_2 - x_1) + x_1\}, \end{array} \right. \\ \hat{\rho}_\alpha, & \text{if } (t,x) \in \{(t,x) \in [0, x_2 - x_1] \times \mathbb{R} \,/ (1 - \hat{\rho}_\alpha)t + x_1 < x < -\hat{\rho}_\alpha t + x_2\}, \\ 1, & \text{if } (t,x) \in \left[\begin{array}{l} \{(t,x) \in [0, x_2 - x_1] \times \mathbb{R} \,/ - \hat{\rho}_\alpha t + x_2 < x\}, \\ \{(t,x) \in [x_2 - x_1, \infty) \times \mathbb{R} \,/ (1 - \hat{\rho}_\alpha)(x_2 - x_1) + x_1 < x\}. \end{array} \right. \end{cases}$$

and

$$y(t) = \begin{cases} V_b t + y_0, & \text{if } t < \frac{x_2 - y_0}{1-\check{\rho}_\alpha}, \\ V_b(\frac{x_2-y_0}{1-\check{\rho}_\alpha}) + y_0, & \text{if } \frac{x_2-y_0}{1-\check{\rho}_\alpha} < t. \end{cases}$$

Since $\check{\rho}_\alpha \in \mathcal{M}_n$ and $\hat{\rho}_\alpha \in \mathcal{M}_n$, for n large enough, there exist $j_0, j_1 \in \{1, \cdots, N\}$ such that $\check{\rho}_\alpha = \rho^n_{j_0}$, $\hat{\rho}_\alpha = \rho^n_{j_1}$ and

$$\mathcal{M}_n = \{0, 2^{-n}, \cdots, \check{\rho}_\alpha := \rho^n_{j_0}, \rho^n_{j_0+1}, \cdots, \rho^n_{j_1-1}, \hat{\rho}_\alpha := \rho^n_{j_1}, \cdots, 1-2^{-n}, 1\}.$$

Let $\rho^n_0 = 2^{-n}\mathbb{1}_{(-\infty,x_1)} + \hat{\rho}_\alpha \mathbb{1}_{(x_1,y_0)} + \rho^n_{j_1-1}\mathbb{1}_{(y_0,x_2)} + (1-2^{-n})\mathbb{1}_{(x_2,+\infty)}$ (see Figure 14b) and $\hat{\rho}_\alpha - 3\,2^{-n-1} \leqslant \rho^n_{j_1-1} \leqslant \hat{\rho}_\alpha - 2^{-n-1}$. It is obvious that $\lim_{n\to\infty} \|\rho^n_0 - \rho_0\|_{L^1(\mathbb{R})} = 0$ and $TV(\rho^n_0) = TV(\rho_0)$. Since $\rho^n_{j_1-1} \in (\check{\rho}_\alpha, \hat{\rho}_\alpha)$, a non classical shock $(\hat{\rho}_\alpha, \check{\rho}_\alpha)$ and a shock wave $(\check{\rho}_\alpha, \rho^n_{j_1-1})$ are created at $(0, y_0)$. The shock wave $(\rho^n_{j_1-1}, 1-2^{-n})$ created at $(0, x_2)$ interacts with the shock wave $(\check{\rho}_\alpha, \rho^n_{j_1-1})$ at time $\bar{t}^n_1 = \frac{x_2-y_0}{1-\check{\rho}_\alpha - 2^{-n}}$. The resulting shock $(\check{\rho}_\alpha, 1-2^{-n})$ cancels the non classical shock at time $\bar{t}^n_2 := \left(\frac{\hat{\rho}_\alpha - \rho^n_{j_1-1}}{1-\hat{\rho}_\alpha - 2^{-n}} + 1\right)\bar{t}^n_1$. Moreover, we have $\bar{t} < \bar{t}^n_1 < \bar{t}^n_2$ and $\lim_{n\to\infty} \bar{t}^n_1 = \lim_{n\to\infty} \bar{t}^n_2 = \bar{t}$. We conclude that,

$$\rho(t, y(t)+) = \hat{\rho}_\alpha \text{ and } \rho^n(t, y^n(t)+) = \check{\rho}_\alpha, \quad t \in (0, \bar{t}),$$

$$\rho(t, y(t)+) = 1 \text{ and } \rho^n(t, y^n(t)+) = \check{\rho}_\alpha, \quad t \in [\bar{t}, t^n_2).$$

Thus, for every $t \in (0, \bar{t})$, we have $\lim_{n\to\infty} \rho^n(t, y^n(t)+) = \check{\rho}_\alpha \neq \hat{\rho}_\alpha = \rho(t, y(t)+)$. However, for every $t \in (0, \bar{t})$,

$$\lim_{n\to\infty} \min(V_b, v(\rho^n(t, y^n(t)+))) = V_b = \min(V_b, v(\rho(t, y(t)+))). \tag{5}$$

Morever, for every $t \in [\bar{t}, t^n_2)$,

$$\min(V_b, v(\rho^n(t, y^n(t)+))) = V_b \text{ and } \min(V_b, v(\rho(t, y(t)+))) = 0. \tag{6}$$

For every $t > t^n_2$,

$$\min(V_b, v(\rho^n(t, y^n(t)+))) = v(1-2^{-n}) \quad \text{and} \quad \min(V_b, v(\rho(t, y(t)+))) = 0. \tag{7}$$

Using that $t^n_2 \to \bar{t}$, (5), (6) and (7), we deduce that

$$\lim_{n\to\infty} \min(V_b, v(\rho^n(t, y^n(t)+))) = \min(V_b, v(\rho(t, y(t)+))), \quad \text{for a.e } t \in \mathbb{R}_+.$$

Example 2.4 shows that the equality $\lim_{n\to\infty} \rho^n(t, y^n(t)+) = \rho(t, y(t)+)$ for almost every $t \in \mathbb{R}_+$ doesn't hold since for every $t \in (0, \bar{t})$, $\rho(t, y(t)+) = \hat{\rho}_\alpha$ and $\rho^n(t, y^n(t)+) = \check{\rho}_\alpha$. To prove Definition 1 iii, we construct a measure-zero set $\mathcal{I}$ such that for every $t \in \mathcal{I}$

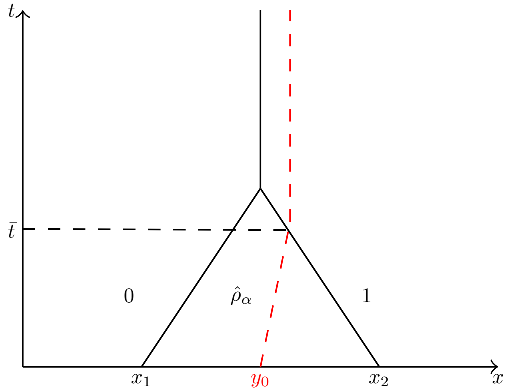
$$\lim_{n\to\infty} \min(V_b, v(\rho^n(t, y^n(t)+))) = \min(V_b, v(\rho(t, y(t)+))).$$

3 Proof of Theorem 1

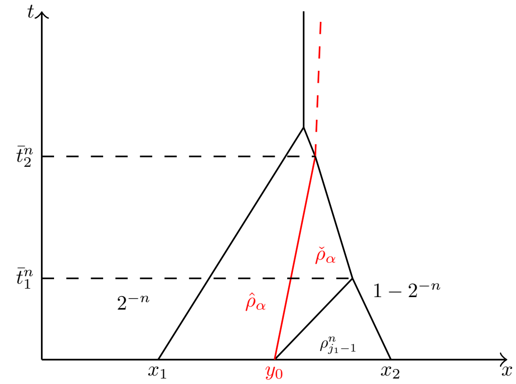
3.1 Convergence of the wave-front tracking approximate solutions (ρ^n, y^n)

The proof of convergence follows the same arguments as in [7]. For the sake of completeness, we write the proof in our case where f verifies **(F)**. For a.e $t \in \mathbb{R}$, we define the Total Variation functional

$$\Gamma(t) = \Gamma(\rho^n(t, \cdot)) = TV(\rho^n(t, \cdot)) + \gamma(t), \tag{8}$$



(a) Solution (ρ, y) of (1) with $(\rho_0(\cdot), y_0) = (\hat{\rho}_\alpha \mathbb{1}_{(x_1,x_2)}(\cdot) + \mathbb{1}_{(x_2,+\infty)}(\cdot), \frac{x_1+x_2}{2})$ and $\bar{t} = \frac{x_2-y_0}{1-\check{\rho}_\alpha}$.



(b) Approximate solution (ρ^n, y^n) of (1) with $n \in \mathbb{N}^*$, $(\rho_0^n(\cdot), y_0) = (2^{-n}\mathbb{1}_{(-\infty,x_1)} + \hat{\rho}_\alpha \mathbb{1}_{(x_1,y_0)} + \rho^n_{j_1-1}\mathbb{1}_{(y_0,x_2)} + (1-2^{-n})\mathbb{1}_{(x_2,+\infty)}, \frac{x_1+x_2}{2})$ with with $\hat{\rho}_\alpha - 3\,2^{-n-1} \leqslant \rho^n_{j_1-1} \leqslant \hat{\rho}_\alpha - 2^{-n-1}$ and $\bar{t}_1^n = \frac{x_2-y_0}{1-\check{\rho}_\alpha-2^{-n}}$.

Figure 14: Let $\bar{t} = \frac{x_2-y_0}{1-\check{\rho}_\alpha}$ and $n \in \mathbb{N}^*$. A case where $\rho(\bar{t}, y(\bar{t})+) \neq \rho^n(\bar{t}, y^n(\bar{t})+)$ over $(0, \bar{t})$.

where γ is given by

$$\gamma(t) = \begin{cases} -2|\hat{\rho}_\alpha - \check{\rho}_\alpha| & \text{if} \quad \rho^n(t, y^n(t)-) = \hat{\rho}_\alpha \text{ and } \rho^n(t, y^n(t)+) = \check{\rho}_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Above, $(\rho^n(t,\cdot), y^n(t))$ is the approximate solution of (1) at time t constructed by the wave-front tracking method described in Section 2.2.

Lemma 2. *[7, Lemma 2] For every $n \in \mathbb{N}$, at any interaction, the functional $\Gamma(t)$ either decreases by at least $\rho_{\max}2^{-n-1}$ or remains constant and the number of waves does not increase.*

Proof. If no interaction takes place at time $\bar{t}$, we immediately have $\Gamma(\bar{t}+) = \Gamma(\bar{t}-)$ and the number of wave-fronts remains constant. At any interaction time $t = \bar{t}$ either two wave-fronts interact or a wave-front hits the MB trajectory. All the possible interactions are described in Section 2.2.

- Case Figure 8; the wave-front (ρ_L, ρ_M) interacts with the wave-front (ρ_M, ρ_R) at time $\bar{t}$. We have
$$\Gamma(\bar{t}+) - \Gamma(\bar{t}-) = |\rho_R - \rho_L| - |\rho_R - \rho_M| - |\rho_M - \rho_L| \leqslant 0,$$
and the number of wave-fronts decreases by one.

- Case Figure 9 and Figure 12; a wave interacts at time $\bar{t}$ with a MB without creating or cancelling a non classical shock. We have
$$\Gamma(\bar{t}+) - \Gamma(\bar{t}-) = |\rho_R - \rho_L| - |\rho_R - \rho_L| = 0,$$
and the number of wave-fronts remains constant.

- Case Figure 10 a); a non classical shock $(\hat{\rho}_\alpha, \check{\rho}_\alpha)$ is cancelled at time $\bar{t}$ by a shock $(\check{\rho}_\alpha, \rho_R)$ coming from the right of the MB trajectory. Since $\rho_R > \hat{\rho}_\alpha$, we have
$$\Gamma(\bar{t}+) - \Gamma(\bar{t}-) = |\rho_R - \hat{\rho}_\alpha| - (|\rho_R - \check{\rho}_\alpha| + |\check{\rho}_\alpha - \hat{\rho}_\alpha| - 2|\check{\rho}_\alpha - \hat{\rho}_\alpha|) = 0,$$
and since a non classical shock is cancelled, the number of wave-fronts decreases by one.

- Case Figure 10 b); a non classical shock $(\hat{\rho}_\alpha, \check{\rho}_\alpha)$ is created at time $\bar{t}$ by a rarefaction shock (ρ_L, ρ_R) coming from the right of the MB trajectory. Since $\rho_L = \hat{\rho}_\alpha$ and $\rho_{\max} 2^{-n-1} \leqslant \rho_L - \rho_R \leqslant 3\rho_{\max} 2^{-n-1}$, we have

  $$\Gamma(\bar{t}+) - \Gamma(\bar{t}-) = (|\rho_R - \check{\rho}_\alpha| + |\check{\rho}_\alpha - \hat{\rho}_\alpha| - 2|\check{\rho}_\alpha - \hat{\rho}_\alpha|) - |\rho_R - \rho_L| \leqslant -2|\rho_R - \rho_L| \leqslant -\rho_{\max} 2^{-n-1},$$

  and since a non classical shock is created, the number of wave-fronts increases by one.

- Case Figure 11 a); a non classical shock $(\hat{\rho}_\alpha, \check{\rho}_\alpha)$ is cancelled at time $\bar{t}$ by a shock $(\rho_L, \hat{\rho}_\alpha)$ coming from the left of the MB trajectory. Since $\rho_L \in [0, \check{\rho}_\alpha)$, we have

  $$\Gamma(\bar{t}+) - \Gamma(\bar{t}-) = |\rho_L - \check{\rho}_\alpha| - (|\rho_L - \hat{\rho}_\alpha| + |\check{\rho}_\alpha - \hat{\rho}_\alpha| - 2|\check{\rho}_\alpha - \hat{\rho}_\alpha|) = 0,$$

  and since a non classical shock is cancelled, the number of wave-fronts decreases by one.

- Case Figure 11 b); a non classical shock $(\hat{\rho}_\alpha, \check{\rho}_\alpha)$ is created at time $\bar{t}$ by a rarefaction shock (ρ_L, ρ_R) coming from the left of the MB trajectory. Since $\rho_R = \check{\rho}_\alpha$ and $\rho_{\max} 2^{-n-1} \leqslant \rho_L - \rho_R \leqslant 3\rho_{\max} 2^{-n-1}$, we have

  $$\Gamma(\bar{t}+) - \Gamma(\bar{t}-) = (|\rho_L - \hat{\rho}_\alpha| + |\check{\rho}_\alpha - \hat{\rho}_\alpha| - 2|\check{\rho}_\alpha - \hat{\rho}_\alpha|) - |\rho_R - \rho_L| \leqslant -2|\rho_R - \rho_L| \leqslant -\rho_{\max} 2^{-n-1},$$

  and since a non classical shock is created, the number of wave-fronts increases by one.

□

From Lemma 2, we conclude that the wave front tracking procedure can be prolonged to any time $T > 0$ and for every $n \in \mathbb{N}$, for every $t \in \mathbb{R}_+$,

$$TV(\rho^n(t, \cdot)) \leqslant TV(\rho_0) + \gamma(0) - \gamma(t) \leqslant TV(\rho_0) + 2|\check{\rho}_\alpha - \hat{\rho}_\alpha|. \tag{9}$$

The inequality (9) is the key point to prove the convergence of the wave-front tracking approximate solution (ρ^n, y^n).

Lemma 3. *Let (ρ^n, y^n) be the approximate solution of (1) constructed by the wave-front tracking method described in Section 2.2. Assume $TV(\rho_0) \leqslant C$ with $C > 0$ and for every $x \in \mathbb{R}$ $0 \leqslant \rho_0(x) \leqslant \rho_{\max}$. Then, up to a subsequence, we have the following convergences*

$$\rho^n \to \rho, \quad \text{in } L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}; [0, \rho_{\max}]);$$

$$y^n(\cdot) \to y(\cdot), \quad \text{in } L^\infty([0, T]; \mathbb{R}) \text{ for all } T > 0;$$

$$\dot{y}^n(\cdot) \to \dot{y}(\cdot), \quad \text{in } L^1([0, T]; \mathbb{R}) \text{ for all } T > 0;$$

for some $\rho \in C^0(\mathbb{R}_+; L^1 \cap BV(\mathbb{R}; [0, \rho_{\max}]))$ and $y \in W^{1,1}([0, T]; \mathbb{R}) \cap C^0([0, T]; \mathbb{R})$ with Lipschitz constant V_b.

Proof. From (9) and using Helly's Theorem (see [2, Theorem 2.4]), there exists a function $\rho \in C^0\left([0, T]; \left(L^1 \cap BV\right)(\mathbb{R}; [0, \rho_{\max}])\right)$ and a subsequence of $(\rho^n)_n$, still denoted by $(\rho^n)_n$, such that $\rho^n \to \rho$ in $L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}; [0, \rho_{\max}])$. By construction of y^n (see Section 2.2), we deduce that

$$0 \leqslant \dot{y}^n(t) \leqslant V_b \tag{10}$$

for a.e. $t > 0$ and $n \in \mathbb{N} \setminus \{0\}$. Hence Ascoli Theorem [20, Theorem 7.25] implies that there exists a function $y \in C^0([0, T]; \mathbb{R})$ and a subsequence of $(y^n)_n$, still denoted by $(y^n)_n$, such that

y^n converges to y uniformly in $C^0([0,T];\mathbb{R})$. Moreover, y is a Lipschitz function with Lipschitz constant V_b. Thus, we have $y^n(\cdot) \to y(\cdot)$ in $L^\infty([0,T];\mathbb{R})$ for all $T>0$.

To prove that $\dot{y}^n(\cdot) \to \dot{y}(\cdot)$ in $L^1([0,T];\mathbb{R})$ for all $T>0$, we show that $TV(\dot{y}^n)$ is uniformly bounded. Since $\|\dot{y}^n\|_{L^\infty} \leqslant V_b$, it is sufficient to estimate the positive variation of $\dot{y}^n$ over $[0,T]$, denoted by $PV(\dot{y}^n;[0,T])$. More precisely, we have

$$TV(\dot{y}^n;[0,T]) \leqslant 2PV(\dot{y}^n;[0,T]) + \|\dot{y}^n\|_{L^\infty}. \tag{11}$$

From Figure 8, Figure 9, Figure 10, Figure 11 and Figure 12 in Section 2.3, the speed of the MB is increasing only by interactions with rarefaction waves coming from the right of the MB trajectory. Since all rarefaction shocks start at $t=0$, we have $PV(\dot{y}^n;[0,T]) \leqslant TV(\rho_0)$. From (11), we deduce that

$$TV(\dot{y}^n;[0,T]) \leqslant 2TV(\rho_0) + V_b,$$

which concludes the proof of Lemma 3. □

3.2 The limit (ρ, y) verifies the points i-ii-iv of Definition 1

From (9), for every $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}$, there exist $\lim\limits_{x\to x_0, x>x_0} \rho^n(t_0,x) := \rho^n(t_0,x_0+)$ and $\lim\limits_{x\to x_0, x<x_0} \rho^n(t_0,x) := \rho^n(t_0,x_0-)$ and from Lemma 3 there exist $\lim\limits_{x\to x_0, x>x_0} \rho(t_0,x) := \rho(t_0,x_0+)$ and $\lim\limits_{x\to x_0, x<x_0} \rho(t_0,x) := \rho(t_0,x_0-)$.

We start by proving that the limit (ρ,y) defined in Lemma 3 verifies Definition 1 i-ii. Since ρ^n is a weak solution of (1a) with initial density ρ_0^n, then, for every $\varphi \in C_c^1(\mathbb{R}^2;\mathbb{R})$,

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(\rho^n \partial_t \varphi + f(\rho^n)\partial_x \varphi\right) dx\, dt + \int_{\mathbb{R}} \rho_0^n(x)\varphi(0,x)\, dx = 0. \tag{12}$$

From Lemma 3, by passing to the limit in (12) as $n \to +\infty$, we conclude that ρ is a weak solution of (1a) and (1b). Similarly, we prove that the limit ρ is an entropy admissible solution both in $\mathbb{R}_+ \times]-\infty, y(t)[$ and in $\mathbb{R}_+ \times]y(t), +\infty[$: points i and ii of Definition 1 hold.

Let $T>0$. To prove point iv of Definition 1, as in [10, Section 5], we use the fact that both ρ^n and ρ are weak solutions of (1a) in $\{(t,x) \in [0,T]\times\mathbb{R}/x<y(t)\}$ and $\{(t,x)\in[0,T]\times\mathbb{R}/y(t)<x\}$. Since the speed of y^n and y are finite and for every $t \in (0,T]$, $y^n(t) \to y(t)$ as $n \to \infty$, there exists a compact and connected set $K \subset (0,T]\times\mathbb{R}$ with smooth boundary such that $(t,y^n(t)) \in K$ and $(t,y(t)) \in K$ for every $n \in \mathbb{N}$ and for every $t \in (0,T]$. Let $\psi : (0,T] \to \mathbb{R}$ be a C^1 function with compact support in K. We introduce the vector fields $g^n : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined respectively by

$$g^n(t,x) = (\rho^n(t,x)\psi(t,x), f(\rho^n(t,x))\psi(t,x)),$$

and

$$g(t,x) = (\rho(t,x)\psi(t,x), f(\rho(t,x))\psi(t,x)).$$

Applying the divergence theorem to g^n on $\{(t,x)\in[0,T]\times\mathbb{R}/x<y^n(t)\}$ and on $\{(t,x)\in[0,T]\times\mathbb{R}/y^n(t)<x\}$, we have, for every $\psi \in C_c^1((0,T]\times\mathbb{R},\mathbb{R})$ with compact support in K,

$$\int_{\{(t,x)\in(0,T]\times\mathbb{R}/y^n(t)<x\}} \operatorname{div} g^n(t,x)dtdx = \int_0^T (f(\rho^n(t,y^n(t)+)) - \rho^n(t,y^n(t)+)\dot{y}^n(t))\psi(t,y^n(t)+)dt, \tag{13}$$

$$\int_{\{(t,x)\in(0,T]\times\mathbb{R}/y^n(t)>x\}} \operatorname{div} g^n(t,x)dtdx = \int_0^T (f(\rho^n(t,y^n(t)-)) - \rho^n(t,y^n(t)-)\dot{y}^n(t))\psi(t,y^n(t)-)dt. \tag{14}$$

and applying the divergence theorem to g on $\{(t,x) \in [0,T] \times \mathbb{R}/x < y(t)\}$ and on $\{(t,x) \in [0,T] \times \mathbb{R}/y(t) < x\}$, we have, for every $\psi \in C_c^1((0,T] \times \mathbb{R}, \mathbb{R})$ with compact support in K,

$$\int_{\{(t,x)\in(0,T]\times\mathbb{R}/y(t)<x\}} \operatorname{div} g(t,x)dtdx = \int_0^T (f(\rho(t,y(t)+)) - \rho(t,y(t)+)\dot{y}(t))\psi(t,y(t)+)dt. \quad (15)$$

$$\int_{\{(t,x)\in(0,T]\times\mathbb{R}/y(t)>x\}} \operatorname{div} g(t,x)dtdx = \int_0^T (f(\rho(t,y(t)-)) - \rho(t,y(t)-)\dot{y}(t))\psi(t,y(t)-)dt. \quad (16)$$

From Lemma 3 and using dominated convergence theorem, we deduce that

$$\lim_{n\to\infty} \int_{\{(t,x)\in(0,T]\times\mathbb{R}/y^n(t)<x\}} \operatorname{div} g^n(t,x)dtdx = \int_{\{(t,x)\in(0,T]\times\mathbb{R}/y(t)<x\}} \operatorname{div} g(t,x)dtdx. \quad (17)$$

$$\lim_{n\to\infty} \int_{\{(t,x)\in(0,T]\times\mathbb{R}/y^n(t)>x\}} \operatorname{div} g^n(t,x)dtdx = \int_{\{(t,x)\in(0,T]\times\mathbb{R}/y(t)>x\}} \operatorname{div} g(t,x)dtdx. \quad (18)$$

Since (ρ^n, y^n) verifies the point iv of Definition 1, we have

$$\int_0^T (f(\rho^n(t,y^n(t)+)) - \rho^n(t,y^n(t)+)\dot{y}^n(t))\psi(t,y^n(t)+)dt \leqslant \int_0^T F_\alpha(\dot{y}^n(t))\psi(t,y^n(t)+)dt. \quad (19)$$

$$\int_0^T (f(\rho^n(t,y^n(t)-)) - \rho^n(t,y^n(t)-)\dot{y}^n(t))\psi(t,y^n(t)-)dt \leqslant \int_0^T F_\alpha(\dot{y}^n(t))\psi(t,y^n(t)-)dt. \quad (20)$$

with $F_\alpha(\dot{y}^n(t)) := \alpha \max_{\rho\in[0,\rho_{\max}]}(f(\rho) - \dot{y}^n(t)\rho)$. From (13), (14), (15) (16), (17), (18) and Lemma 3, by passing to the limit in (19) and (20) we have for every $\psi \in C_c^1 \leqslant ((0,T] \times \mathbb{R}, \mathbb{R})$ with compact support in K,

$$\int_0^T (f(\rho(t,y(t)+)) - \rho(t,y(t)+)\dot{y}(t))\psi(t,y(t)+)dt \leqslant \int_0^T F_\alpha(y(t))\psi(t,y(t))dt,$$

$$\int_0^T (f(\rho(t,y(t)-)) - \rho(t,y(t)-)\dot{y}(t))\psi(t,y(t)-)dt \leqslant \int_0^T F_\alpha(y(t))\psi(t,y(t))dt,$$

with $F_\alpha(\dot{y}(t)) := \alpha \max_{\rho\in[0,\rho\max]}(f(\rho) - \dot{y}(t)\rho)$, whence the point iv of Definition 1

3.3 The limit (ρ, y) verifies the point iii of Definition 1

Let $\epsilon > 0$, from Lemma 3 and using the fact that (ρ^n, y^n) satisfies (4), there exists a measure-zero set $\mathcal{N}$ such that, for every $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$,

- $\lim_{n\to\infty} \rho^n(\bar{t}, x) = \rho(\bar{t}, x)$ for almost every $x \in \mathbb{R}$,
- $y(\cdot)$ is a differentiable function at $t = \bar{t}$,
- $\lim_{n\to\infty} \dot{y}^n(\bar{t}) = \dot{y}(\bar{t})$. In particular, for n large enough, $|y^n(\bar{t}) - y(\bar{t})| \leqslant \frac{\beta\bar{t}\epsilon}{2}$.
- For every $n \in \mathbb{N}$, $\dot{y}^n(\bar{t}) = \min(V_b, v(\rho^n(\bar{t}, y^n(\bar{t}))))$.

We will prove that for every $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$,

$$\lim_{n\to\infty} \min(V_b, v(\rho^n(\bar{t}, y^n(\bar{t})+))) = \min(V_b, v(\rho(\bar{t}, y(\bar{t})+))).$$

We denote by $\rho_+ := \lim_{x\to y(\bar{t}),x>y(\bar{t})} \rho(\bar{t}, x)$ and $\rho_- := \lim_{x\to y(\bar{t}),x<y(\bar{t})} \rho(\bar{t}, x)$. The following Lemma gives the range of ρ^n and ρ in a neighbourhood of $(\bar{t}, y(\bar{t}))$, see Figure 15.

Lemma 4. *Fix $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\epsilon > 0$. Assume that $\rho_-, \rho_+ \in [0, \rho_{\max}]$. There exists $\delta > 0$ such that*

$$\rho(\bar{t}, x) \in \begin{cases} (\max(\rho_- - \frac{\epsilon}{2}, 0), \min(\rho_- + \frac{\epsilon}{2}, \rho_{\max})) & \forall\, x \in (y(\bar{t}) - \delta, y(\bar{t})), \\ (\max(\rho_+ - \frac{\epsilon}{2}, 0), \min(\rho_+ + \frac{\epsilon}{2}, \rho_{\max})) & \forall\, x \in (y(\bar{t}), y(\bar{t}) + \delta), \end{cases} \tag{21}$$

and there exits $0 < \tilde{\delta} < \delta$ such that, for $n \in \mathbb{N}$ large enough,

$$\rho^n(\bar{t}, x) \in \begin{cases} (\max(\rho_- - \epsilon, 0), \min(\rho_- + \epsilon, \rho_{\max})) & \forall\, x \in (\min(y(\bar{t}), y^n(\bar{t})) - \tilde{\delta}, \min(y(\bar{t}), y^n(\bar{t}))), \\ (\max(\rho_+ - \epsilon, 0), \min(\rho_+ + \epsilon, \rho_{\max})) & \forall\, x \in (\max(y(\bar{t}), y^n(\bar{t})), \max(y(\bar{t}), y^n(\bar{t})) + \tilde{\delta}). \end{cases} \tag{22}$$

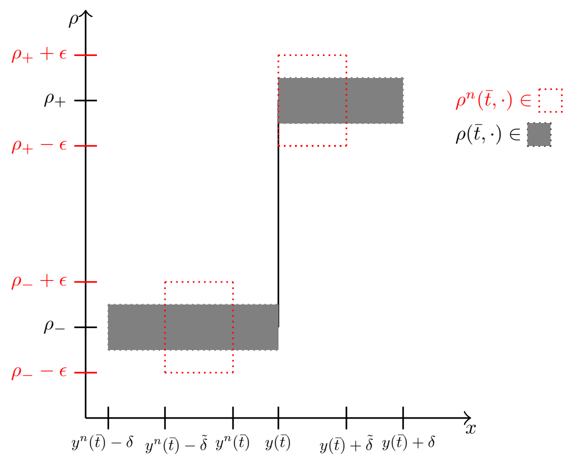


Figure 15: Illustration of Lemma 4; $\rho_-, \rho_+ \in [0, \rho_{\max}]$ with $y^n(\bar{t}) < y(\bar{t})$. The approximate density $\rho^n(\bar{t}, \cdot)$ over $[y^n(\bar{t}) - \tilde{\delta}, y^n(\bar{t})] \cup [y(\bar{t}), y(\bar{t}) + \tilde{\delta}]$ belongs to the area surrounded by the dotted lines (...) and $\rho(\bar{t}, \cdot)$ over $[y(\bar{t}) - \delta, y(\bar{t}) + \delta]$ belongs to the shaded zone.

Proof. From Lemma 3, there exists $C > 0$ such that $TV(\rho(\bar{t}, \cdot)) < C$. Thus, we have for every $\epsilon > 0$, there exists $\delta > 0$ such that $TV(\rho_{|(y(\bar{t}), y(\bar{t})+\delta)}) < \frac{\epsilon}{2}$ and $TV(\rho_{|(y(\bar{t})-\delta, y(\bar{t}))}) < \frac{\epsilon}{2}$. This implies (21). We argue by contradiction to prove that there exists $\tilde{\delta}$ verifying $0 < \tilde{\delta} < \delta$ such that, for n large enough,

$$\rho^n(\bar{t}, x) \in (\rho_+ - \epsilon, \rho_+ + \epsilon),$$

for every $x \in (\max(y^n(\bar{t}), y(\bar{t})), \max(y^n(\bar{t}), y(\bar{t})) + \tilde{\delta})$; we assume that for every $\tilde{\delta} > 0$ with $0 < \tilde{\delta} < \delta$, for every $n_0 \in \mathbb{N}$, there exists $n \geqslant n_0$ and $x_n \in (\max(y^n(\bar{t}), y(\bar{t})), \max(y^n(\bar{t}), y(\bar{t})) + \tilde{\delta})$ such that $\rho^n(\bar{t}, x_n) \in [0, \rho_+ - \epsilon] \cup [\rho_+ + \epsilon, \rho_{\max}]$. In particular, choosing $\tilde{\delta} = \frac{\delta}{n}$, we construct a sequence $(x_n)_{n \in \mathbb{N}}$ such that

* $\lim_{n \to \infty} x_n = y(\bar{t})$,
* $x_n > \max(y^n(\bar{t}), y(\bar{t}))$,
* $\rho^n(\bar{t}, x_n) \in [0, \rho_+ - \epsilon] \cup [\rho_+ + \epsilon, \rho_{\max}]$.

From Lemma 3, there exists a sequence $(z_m)_{m \in \mathbb{N}}$ such that $z_m > y(\bar{t})$, $\lim_{m \to \infty} z_m = y(\bar{t})$ and $\lim_{n \to \infty} \rho^n(\bar{t}, z_m) = \rho(\bar{t}, z_m) \in (\rho_+ - \frac{\epsilon}{2}, \rho_+ + \frac{\epsilon}{2})$. Thus, for n large enough, $\rho^n(\bar{t}, z_m) \in (\rho_+ - \frac{3\epsilon}{4}, \rho_+ - \frac{3\epsilon}{4})$.

- If $\rho^n(\bar{t}, x_n) \in [0, \rho_+ - \epsilon]$, by diagonal method, we construct $(z_n)_{n \in \mathbb{N}}$ such that

- $\max(y^n(\bar{t}), y(\bar{t})) < z_n < x_n$,
- $\lim_{n\to\infty} z_n = y(\bar{t})$,
- $\rho^n(t, z_n) \in (\rho_+ - \frac{3\epsilon}{4}, \rho_+ + \frac{3\epsilon}{4})$.

Since $\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n = y(\bar{t})$, for n large enough, we have $x_n - z_n \leqslant \frac{\bar{t}\beta\epsilon}{8}$. Since $z_n > y^n(\bar{t})$ and $x_n > y^n(\bar{t})$, to go from $\rho^n(\bar{t}, z_n)$ to $\rho^n(\bar{t}, x_n)$, we only have shocks or rarefaction shocks. From Lemma 1, the minimal length in space at time $\bar{t}$ to go from $\rho^n(\bar{t}, z_n)$ to $\rho^n(\bar{t}, x_n)$ is

$$\delta^n(\rho^n(\bar{t}, z_n), \rho^n(\bar{t}, x_n), \bar{t}) \geqslant \bar{t}\beta(\rho^n(\bar{t}, z_n) - \rho^n(\bar{t}, x_n) - \rho_{\max}2^{-n+1}).$$

Therefore, for n large enough, $\delta^n(\rho^n(\bar{t}, z_n), \rho^n(\bar{t}, x_n), \bar{t}) > \frac{\bar{t}\beta\epsilon}{8}$. Since $x_n - z_n \leqslant \frac{\beta\bar{t}\epsilon}{8}$ we have $\delta^n(\rho^n(\bar{t}, z_n), \rho^n(\bar{t}, x_n), \bar{t}) > x_n - z_n$, whence the contradiction.

- If $\rho^n(\bar{t}, x_n) \in [\rho_+ + \epsilon, \rho_{\max}]$, by diagonal method, we construct $(z_n)_{n\in\mathbb{N}}$ such that
  - $\max(y^n(\bar{t}), y(\bar{t})) < x_n < z_n$,
  - $\lim_{n\to\infty} z_n = y(\bar{t})$,
  - $\rho^n(t, z_n) \in (\rho_+ - \frac{3\epsilon}{4}, \rho_+ + \frac{3\epsilon}{4})$.

  Since $\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n = y(\bar{t})$, for n large enough, $z_n - x_n \leqslant \frac{\bar{t}\beta\epsilon}{8}$. Since $z_n > y^n(\bar{t})$ and $x_n > y^n(\bar{t})$, to go from $\rho^n(\bar{t}, x_n)$ to $\rho^n(\bar{t}, z_n)$, we only have shocks or rarefaction shocks. From Lemma 1, the minimal length in space at time $\bar{t}$ to go from $\rho^n(\bar{t}, x_n)$ to $\rho^n(\bar{t}, z_n)$ is

$$\delta^n(\rho^n(\bar{t}, x_n), \rho^n(\bar{t}, z_n), \bar{t}) \geqslant \bar{t}\beta(\rho^n(\bar{t}, x_n) - \rho^n(\bar{t}, z_n) - \rho_{\max}2^{-n+1}).$$

  Therefore, for n large enough, $\delta^n(\rho^n(\bar{t}, x_n), \rho^n(\bar{t}, z_n), \bar{t}) > \frac{\bar{t}\beta\epsilon}{8}$. Since $z_n - x_n \leqslant \frac{\beta\bar{t}\epsilon}{8}$ we have $\delta(\rho^n(\bar{t}, x_n), \rho^n(\bar{t}, z_n), \bar{t}) > z_n - x_n$, whence the contradiction.

Using the same strategy as above, we also show that there exists $\tilde{\delta}$ verifying $0 < \tilde{\delta} < \delta$ such that, for n large enough and for every $x \in (\min(y^n(\bar{t}), y(\bar{t})) - \tilde{\delta}, \min(y^n(\bar{t}), y(\bar{t})))$, $\rho^n(\bar{t}, x) \in (\rho_- - \epsilon, \rho_- + \epsilon)$. □

3.3.1 Point iii of Definition 1 when $(\rho_-, \rho_+) \in [\rho^*, \rho_{\max}]$

Lemma 5. *Fix $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\epsilon > 0$. If $(\rho_-, \rho_+) \in [\rho^*, \rho_{\max}]$, the only possible case is $\rho_- \leqslant \rho_+$.*

Proof. Assume that $\rho^* \leqslant \rho_+ < \rho_-$. We have $\hat{\rho}_\alpha < \rho^*$ and for ϵ small enough, $\rho_+ < \rho_- - 3\epsilon$. From Lemma 4, we have

$$\rho^n(\bar{t}, \min(y^n(\bar{t}), y(\bar{t}))-) \in (\rho_- - \epsilon, \min(\rho_- + \epsilon, \rho_{\max})) \subset (\rho^*, \rho_{\max}] \tag{23}$$

and

$$\rho^n(\bar{t}, \max(y^n(\bar{t}), y(\bar{t}))+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon) \subset (\rho^* - \epsilon, \rho_{\max}]. \tag{24}$$

Since $\hat{\rho}_\alpha < \rho_+ + \epsilon$, to go from $\rho_- - \epsilon$ to $\rho_+ + \epsilon$ in ρ^n we only have shocks and rarefaction shocks. Therefore, from Lemma 1 and for n large enough

$$\delta^n\left(\rho_- - \epsilon, \rho_+ + \epsilon, \bar{t}\right) > \frac{\beta\bar{t}\epsilon}{2}. \tag{25}$$

Using that $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$, we have $|y^n(\bar{t}) - y(\bar{t})| \leqslant \frac{\bar{t}\beta\epsilon}{2}$. Therefore, from (23), (24) and (25), we conclude that, for n large enough,

$$\delta^n\left(\rho^n(\bar{t}, \min(y^n(\bar{t}), y(\bar{t}))-), \rho^n(\bar{t}, \max(y^n(\bar{t}), y(\bar{t}))+), \bar{t}\right) > |y^n(\bar{t}) - y(\bar{t})|,$$

whence the contradiction. □

Lemma 6. *Fix $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\epsilon > 0$. Assume that $\rho^* \leqslant \rho_- \leqslant \rho_+$. Then for $n \in \mathbb{N}^*$ large enough,*

$$\rho^n(\bar{t}, x) \in (\rho_- - 2\epsilon, \min(\rho_+ + 2\epsilon, \rho_{\max})),$$

for every $x \in (\min(y(\bar{t}), y^n(\bar{t})), \max(y(\bar{t}), y^n(\bar{t})))$.

An illustration of Lemma 6 is given in Figure 16.

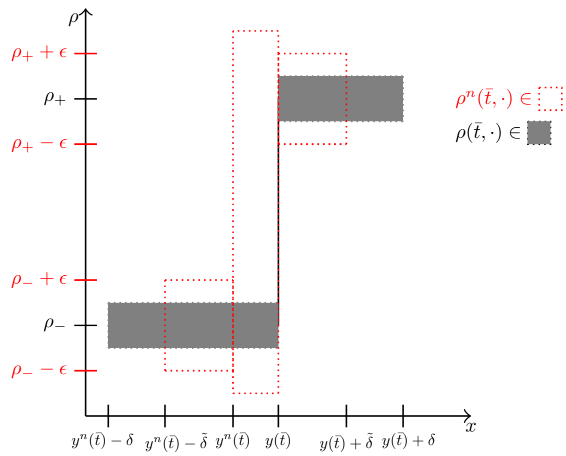


Figure 16: Illustration of Lemma 6; $\rho^* \leqslant \rho_- \leqslant \rho_+$ with $y^n(\bar{t}) < y(\bar{t})$. The approximate density $\rho^n(\bar{t}, \cdot)$ over $(y^n(\bar{t}) - \tilde{\delta}, y(\bar{t}) + \tilde{\delta})$ belongs to the area surrounded by the dotted lines (...) and $\rho(\bar{t}, \cdot)$ over $(y(\bar{t}) - \delta, y(\bar{t}) + \delta)$ belongs to the shaded zone.

Proof. We argue by contradiction; in the same spirit of Proof of Lemma 4, we construct a sequence $(x_n)_{n\in\mathbf{N}}$ such that

* $\lim_{n\to\infty} x_n = y(\bar{t})$,
* $\min(y^n(\bar{t}), y(\bar{t})) < x_n < \max(y^n(\bar{t}), y(\bar{t}))$,
* $\rho^n(\bar{t}, x_n) \in [0, \rho_- - 2\epsilon] \cup [\min(\rho_+ + 2\epsilon, \rho_{\max}), \rho_{\max}]$.

From Lemma 4, $\rho^n(t, \min(y^n(\bar{t}), y(\bar{t}))-) \in (\rho_- - \epsilon, \min(\rho_- + \epsilon, \rho_{\max})$ and $\rho^n(t, \max(y^n(\bar{t}), y(\bar{t}))+) \in (\rho_+ - \epsilon, \min(\rho_+ + \epsilon, \rho_{\max}))$. By construction of $(x_n)_{n\in\mathbf{N}}$ and using that $\bar{t} \in \mathbb{R}\backslash\mathcal{N}$, we have

$$x_n - \min(y^n(\bar{t}), y(\bar{t})) \leqslant |y^n(\bar{t}) - y(\bar{t})| \leqslant \frac{\beta\bar{t}\epsilon}{2}. \tag{26}$$

and

$$\max(y^n(\bar{t}), y(\bar{t})) - x_n \leqslant |y^n(\bar{t}) - y(\bar{t})| \leqslant \frac{\beta\bar{t}\epsilon}{2}. \tag{27}$$

- Assuming that $\rho^n(\bar{t}, x_n) \in [0, \rho_- - 2\epsilon]$. Since $\hat{\rho}_\alpha < \rho_- - 2\epsilon$, to go from $\rho_- - \epsilon$ to $\rho_- - 2\epsilon$ in ρ^n we only have shocks or rarefaction shocks. Therefore, from Lemma 1, for n large enough,

$$\delta^n(\rho_- - \epsilon, \rho_- - 2\epsilon, \bar{t}) > \frac{\bar{t}\beta\epsilon}{2}. \tag{28}$$

  From (26) and (28), for n large enough, we have $\delta^n(\rho_- - \epsilon, \rho_- - 2\epsilon, \bar{t}) > x_n - \min(y^n(\bar{t}), y(\bar{t}))$. Using that $\rho^n(\bar{t}, x_n) \in [0, \rho_- - 2\epsilon]$ and from Lemma 4, $\rho^n(t, \min(y^n(\bar{t}), y(\bar{t}))-) \in (\rho_- - \epsilon, \min(\rho_- + \epsilon, \rho_{\max})$, we have a contradiction.

- Assuming that $\rho_+ < \rho_{\max}$ and $\rho^n(\bar{t}, x_n) \in [\rho_+ + 2\epsilon, \rho_{\max}]$. Since $\hat{\rho}_\alpha < \rho_+ + \epsilon$, to go from $\rho_+ + 2\epsilon$ to $\rho_+ + \epsilon$ in ρ^n we only have shocks or rarefaction shocks. Therefore, from Lemma 1, for n large enough
$$\delta^n(\rho_+ + 2\epsilon, \rho_+ + \epsilon, \bar{t}) > \frac{\bar{t}\beta\epsilon}{2}. \tag{29}$$
From (27) and (29), for n large enough, we have $\delta^n(\rho_+ + 2\epsilon, \rho_+ + \epsilon, \bar{t}) > \max(y^n(\bar{t}), y(\bar{t})) - x_n$. Using that $\rho^n(\bar{t}, x_n) \in [\rho_+ + 2\epsilon, \rho_{\max}]$ and from Lemma 4 $\rho^n(t, \max(y^n(\bar{t}), y(\bar{t}))+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon)$, we have a contradiction.

□

Proof of point iii of Definition 1 when $(\rho_-, \rho_+) \in [\rho^*, \rho_{\max}]$: From Lemma 5, the only possible case is $\rho^* \leqslant \rho_- \leqslant \rho_+$.

- If $\rho_+ = \rho_-$; using Lemma 4 and Lemma 6, we have
$$v(\min(\rho_+ + 2\epsilon, \rho_{\max})) \leqslant \min(V_b, v(\rho^n(\bar{t}, y^n(\bar{t})+))) := \dot{y}^n(\bar{t}) \leqslant \min(V_b, v(\rho_+ - 2\epsilon)). \tag{30}$$
Since $\bar{t} \in \mathbb{R}_+^* \setminus \mathcal{N}$, by passing to the limit in (30) as $n \to \infty$, we deduce that for the arbitrarily of ϵ
$$\dot{y}(\bar{t}) = v(\rho_+) := \min(V_b, v(\rho(\bar{t}, y(\bar{t})+))). \tag{31}$$

- If $\rho_+ \neq \rho_-$ and $y(\bar{t}) \leqslant y^n(\bar{t})$ for an infinite set of indices n; from Lemma 4 we have
$$v(\min(\rho_+ + \epsilon, \rho_{\max})) \leqslant \min(V_b, v(\rho^n(t, y^n(\bar{t})+))) := \dot{y}^n(\bar{t}) \leqslant v(\rho_+ - \epsilon). \tag{32}$$
Since $\bar{t} \in \mathbb{R}_+^* \setminus \mathcal{N}$, the equality (31) holds by passing to the limit in (32) as $n \to \infty$.

- If $\rho_+ \neq \rho_-$ and $y^n(\bar{t}) < y(\bar{t})$ for an infinite set of indices n; in this case, from Lemma 4 and Lemma 6, $\rho^n(\bar{t}, y^n(\bar{t})+) \in (\rho_- - 2\epsilon, \rho_+ + 2\epsilon)$. We study the behavior of the approximate solution (ρ^n, y^n) in the triangle $\mathcal{T}_0$ defined by
$$\mathcal{T}_0 := \left\{(t, x) \in [\bar{t}, t_f[\times]v(\rho_- - 2\epsilon)(t - \bar{t}) + y^n(\bar{t}) - \tilde{\delta}, f'(\rho_+ + 2\epsilon)(t - \bar{t}) + y(\bar{t}) + \tilde{\delta}[\right\}, \tag{33}$$
with $t_f = \frac{y(\bar{t}) - y^n(\bar{t}) + 2\tilde{\delta}}{v(\rho_- - 2\epsilon) - f'(\rho_+ + 2\epsilon)}$. The structure of the proof is illustrated in Figure 17.

Lemma 7. *Fix $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\epsilon > 0$. Assume that $\rho^* \leqslant \rho_- < \rho_+$ and $y^n(\bar{t}) < y(\bar{t})$ for an infinite set of indices n. There exists a piecewise constant function $\xi^n(\cdot)$ such that for every $t \in [\bar{t}, t_f^\xi)$,*
$$(t, \xi^n(t)) \in \mathcal{T}_0, \tag{34}$$
and extending $\xi^n(\cdot)$ to $\mathbb{R}_+$ by imposing that $\xi^n(t) = \xi^n(t_f^\xi)$ for every $t \in [t_f^\xi, \infty)$, we have
$$\rho^n(t, x+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon), \quad \forall (t, x) \in \{(t, x) \in [\bar{t}, +\infty) \times \mathbb{R},\ x > \xi^n(t)\} \cap \mathcal{T}_0. \tag{35}$$
We denote by t_f^ξ and $t_f^{y^n}$ the time when $\xi^n(\cdot)$ and $y^n(\cdot)$ exit the triangle $\mathcal{T}_0$ respectively. Then we have $\min(t_f^{y^n}, t_f^\xi) \geqslant \bar{t} + c$ with $c > 0$ independent of n and there exists $t_n \in [\bar{t}, \min(t_f^{y^n}, t_f^\xi))$ such that $y^n(t_n) = \xi^n(t_n)$ and $\lim_{n \to \infty} t_n = \bar{t}$.

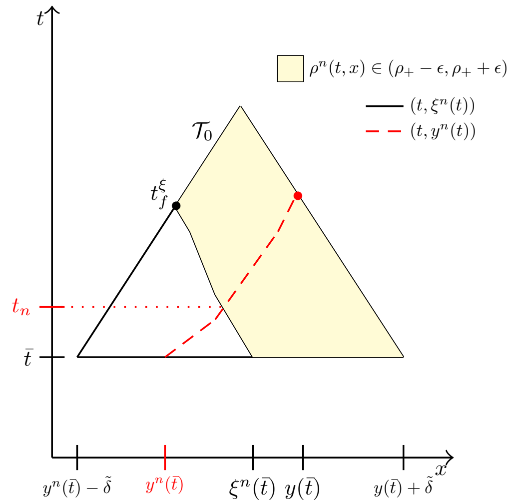


Figure 17: $\rho^* \leqslant \rho_- < \rho_+ \leqslant \rho_{\max}$ with $y^n(\bar{t}) < y(\bar{t})$, $n \in \mathbb{N}$.

The proof of Lemma 7 is postponed in Appendix A. From Lemma 3, for a.e $t > \bar{t}$

$$y^n(t) - y^n(\bar{t}) = \int_{\bar{t}}^{t} \dot{y}^n(s) ds \tag{36}$$

and

$$\lim_{n\to\infty} y^n(t) = y(t). \tag{37}$$

We fix $t \in (\bar{t}, \bar{t} + c]$ with c defined in Lemma 7 such that (36) and (37) hold. For n large enough, $t > t_n$ and $\dot{y}^n(s) \in (\rho_+ - \epsilon, \rho_+ + \epsilon)$ for every $s \in [t_n, t]$. By passing to the limit in (36), we have for a.e $t \in (\bar{t}, \bar{t} + c]$

$$\frac{y(t) - y(\bar{t})}{t - \bar{t}} \in [v(\rho_+ + \epsilon), v(\rho_+ - \epsilon)]. \tag{38}$$

Using that y is differentiable at time $\bar{t}$ and the arbitrarily of ϵ, we have

$$\dot{y}(\bar{t}) = v(\rho_+) = \min(V_b, v(\rho(\bar{t}, y(\bar{t}) +))).$$

3.3.2 Point iii of Definition 1 when $(\rho_-, \rho_+) \in [0, \rho^*]$

Lemma 8. *Fix $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\epsilon > 0$. Assume that $(\rho_-, \rho_+) \in [0, \rho^*]$. For $n \in \mathbb{N}^*$ large enough, for every $x \in (\min(y^n(\bar{t}), y(\bar{t})) - \tilde{\delta}), \max(y^n(\bar{t}), y(\bar{t})) + \tilde{\delta}))$,*

$$\rho^n(\bar{t}, x) \in (0, \rho^* + 2\epsilon).$$

Proof. From Lemma 4 and using that $(\rho_-, \rho_+) \in [0, \rho^*]$,

$$0 \leqslant \rho^n(t, x) < \rho^* + \epsilon, \tag{39}$$

for every $x \in \Big(\min(y^n(\bar{t}), y(\bar{t})) - \tilde{\delta}), \min(y^n(\bar{t}), y(\bar{t}))\Big) \cup \Big(\max(y^n(\bar{t}), y(\bar{t}))), \max(y^n(\bar{t}), y(\bar{t})) + \tilde{\delta})\Big)$. To prove Lemma 8, we argue by contradiction: assuming that there exits a sequence $(x_n)_{n\in\mathbb{N}^*}$ such that, for every $n \in \mathbb{N}$,

$$x_n \in [\min(y^n(\bar{t}), y(\bar{t})), \max(y^n(\bar{t}), y(\bar{t}))] \quad \text{and} \quad \rho^n(\bar{t}, x_n) \in [\rho^* + 2\epsilon, \rho_{\max}]. \tag{40}$$

Since $\hat{\rho}_\alpha < \rho^* + \epsilon$, to go from $\rho^* + 2\epsilon$ to $\rho^* + \epsilon$ in ρ^n we can only have shocks or rarefaction shocks. Therefore, from Lemma 1, for n large enough,

$$\delta^n(\rho^* + 2\epsilon, \rho^* + \epsilon, \bar{t}) > \frac{t\beta\epsilon}{2}. \tag{41}$$

From (40) and (41) and $\bar{t} \in \mathbb{R}_+^* \setminus \mathcal{N}$, we have $\delta^n(\rho_+ + 2\epsilon, \rho_+ + \epsilon, \bar{t}) > \max(y^n(\bar{t}), y(\bar{t})) - x_n$ and $\delta^n(\rho_+ + 2\epsilon, \rho_+ + \epsilon, \bar{t}) > x_n - \min(y^n(\bar{t}), y(\bar{t}))$. Using that $\rho^n(\bar{t}, x_n) \in [\rho^* + 2\epsilon, \rho_{\max}]$ and (39), we have a contradiction.

□

Proof of point iii of Definition 1 when $(\rho_-, \rho_+) \in [0, \rho^*]$**:** Since $\bar{t} \in \mathbb{R}_+^* \setminus \mathcal{N}$,

$$\dot{y}(\bar{t}) = \lim_{n\to\infty} \dot{y}^n(\bar{t}) = \lim_{n\to\infty} \min(V_b, v(\rho^n(\bar{t}, y^n(\bar{t})+)).$$

From Lemma 8, $v(\rho^* + \epsilon) \leqslant \min(V_b, v(\rho^n(\bar{t}, y^n(\bar{t}))) \leqslant V_b$. Since $\rho_+ \in [0, \rho^*]$, for arbitrarily of ϵ we conclude that

$$\dot{y}(\bar{t}) = V_b = \min(V_b, v(\rho(\bar{t}, y(\bar{t})+))).$$

3.3.3 Point iii of Definition 1 when $\rho_- < \rho^* < \rho_+$ or $\rho_+ < \rho^* < \rho_-$

Lemma 9. *The only possible case is* $\rho_- < \rho^* < \rho_+$

Proof. Assuming that $\rho_+ < \rho^* < \rho_-$. From Lemma 4, we have $\rho^n(\bar{t}, \min(y^n(\bar{t}), y(\bar{t}))-) \in (\min(\rho_- - \epsilon, 0), \rho_- + \epsilon) \subset (\rho^* + \frac{\epsilon}{2}, \rho_{\max})$ and $\rho^n(\bar{t}, \max(y^n(\bar{t}), y(\bar{t}))+) \in [\rho_+ - \epsilon, \rho_+ + \epsilon] \subset (0, \rho^* - \frac{\epsilon}{2})$. Since $\hat{\rho}_\alpha < \rho^* - \frac{\epsilon}{2}$, to go from $\rho^* + \frac{\epsilon}{2}$ to $\rho^* - \frac{\epsilon}{2}$ in ρ^n we only have shocks and rarefaction shocks. Therefore, from Lemma 1, for n large enough,

$$\delta^n\left(\rho^* + \frac{\epsilon}{2}, \rho^* - \frac{\epsilon}{2}, \bar{t}\right) > \frac{\bar{t}\beta\epsilon}{2}. \tag{42}$$

Using that $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$, we have $|y^n(\bar{t}) - y(\bar{t})| \leqslant \frac{\bar{t}\beta\epsilon}{2}$. Therefore, from (42), we conclude that

$$\delta^n\left(\rho^n(\bar{t}, \min(y^n(\bar{t}), y(\bar{t})-)), \rho^n(\bar{t}, \max(y^n(\bar{t}), y(\bar{t})+)), \bar{t}\right) > |y^n(\bar{t}) - y(\bar{t})|,$$

whence the contradiction. □

Proof of point iii of Definition 1 when $\rho_- < \rho^* < \rho_+$ **or** $\rho_+ < \rho^* < \rho_-$**:**
From Lemma 9, the only possible case is $\rho_- < \rho^* < \rho_+$.

- If $y(\bar{t}) \leqslant y^n(\bar{t})$ for an infinite set of indices n; from Lemma 4 we have

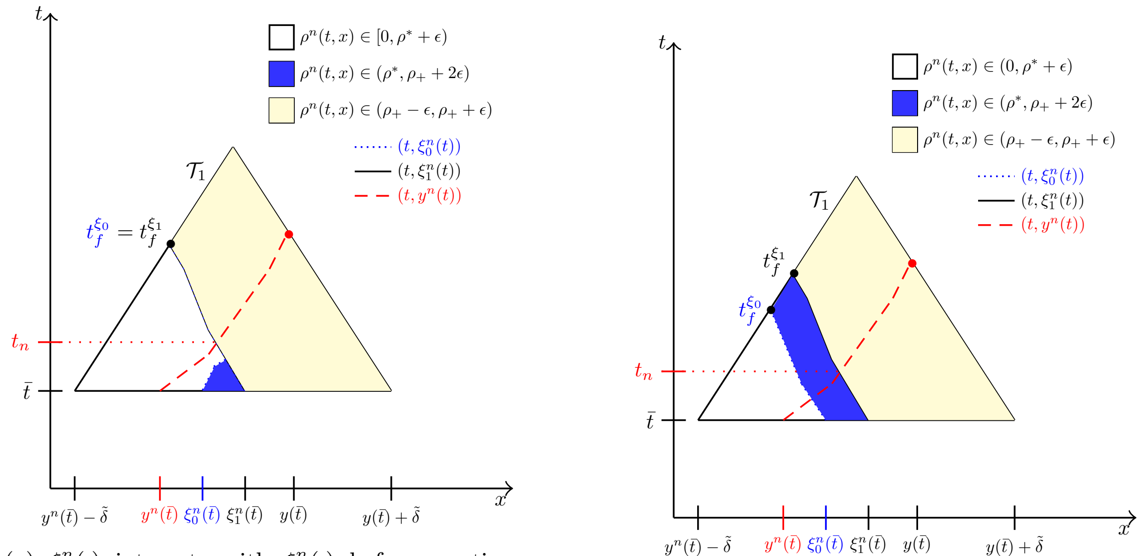
$$v(\min(\rho_+ + \epsilon, \rho_{\max})) \leqslant \min(V_b, v(\rho^n(\bar{t}, y^n(\bar{t})+))) := \dot{y}^n(\bar{t}) \leqslant v(\rho_+ - \epsilon). \tag{43}$$

  Since $\bar{t} \in \mathbb{R}_+^* \setminus \mathcal{N}$, the equality (31) holds by passing to the limit in (43) as $n \to \infty$ and using the arbitrarily of ϵ.

- If $y^n(\bar{t}) < y(\bar{t})$ for an infinite set of indices n; we study the behavior of the approximate solution (ρ^n, y^n) in the triangle $\mathcal{T}_1$ defined by

$$\mathcal{T}_1 := \left\{(t,x) \in [\bar{t}, t_f[\times]v(0)(t-\bar{t}) + y^n(\bar{t}) - \tilde{\delta}, f'(\rho_+ + 2\epsilon)(t-\bar{t}) + y(\bar{t}) + \tilde{\delta}[\right\}, \tag{44}$$

  with $t_f = \frac{y(\bar{t}) - y^n(\bar{t}) + 2\tilde{\delta}}{v(0) - f'(\rho_+ + 2\epsilon)}$. The structure of the proof is illustrated in Figure 18.



(a) $\xi_0^n(\cdot)$ interacts with $\xi_1^n(\cdot)$ before meeting $y^n(\cdot)$.

(b) $\xi_0^n(\cdot)$ doesn't interact with $\xi_1^n(\cdot)$.

Figure 18: Illustration of Lemma 10; $\rho_- < \rho^* < \rho_+ < \rho_{\max}$ and $y^n(\bar{t}) < y(\bar{t})$.

Lemma 10. *Fix $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\epsilon > 0$. Assume that $\rho_- < \rho^* < \rho_+$ and $y^n(\bar{t}) < y(\bar{t})$ for an infinite set of indices n. There exists a piecewise constant function $\xi_1^n(\cdot)$ such that for every $t \in [\bar{t}, t_f^{\xi_1})$,*

$$(t, \xi_1^n(t)) \in \mathcal{T}_1, \tag{45}$$

and extending $\xi_1^n(\cdot)$ to $\mathbb{R}_+$ by imposing that $\xi_1^n(t) = \xi_1^n(t_f^{\xi_1})$ for every $t \in [t_f^{\xi_1}, \infty)$, we have

$$\rho^n(\bar{t}, x+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon), \quad \forall (t,x) \in \{(t,x) \in [\bar{t}, +\infty) \times \mathbb{R},\, x > \xi_1^n(t)\} \cap \mathcal{T}_1. \tag{46}$$

We denote by $t_f^{\xi_1}$ and $t_f^{y^n}$ the time when $\xi_1^n(\cdot)$ and $y^n(\cdot)$ exit the triangle $\mathcal{T}_1$ respectively. Then we have $\min(t_f^{y^n}, t_f^{\xi_1}) > \bar{t} + c$ with $c > 0$ independent of n and there exists $t_n \in [\bar{t}, \min(t_f^{y^n}, t_f^{\xi_1})]$ such that $y^n(t_n) = \xi_1^n(t_n)$ and $\lim_{n\to\infty} t_n = \bar{t}$.

The proof of Lemma 10 is postponed in Appendix B. Following the same argument as Section 3.3.2, (36), (37) and (38) hold. Using that y is differentiable at time $\bar{t}$ and the arbitrarily of ϵ, we have

$$\dot{y}(\bar{t}) = v(\rho_+) = \min(V_b, v(\rho(\bar{t}, y(\bar{t})+))).$$

A Proof of Lemma 7

We have $\rho_-, \rho_+ \in [\rho^*, \rho_{\max}]$, $\rho_- < \rho_+$ and $y^n(\bar{t}) < y(\bar{t})$ for an infinite set of indices n. There exists a subsequence of $(y^n)_{n\in\mathbb{N}}$, still denoted by $(y^n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $y^n(\bar{t}) < y(\bar{t})$. The construction of $\xi^n(\cdot)$ is based on the three following lemmas:

Lemma 11. *For every $(t,x) \in \mathcal{T}_0$, $\rho^n(t,x) \in [\rho_- - 2\epsilon, \min(\rho_+ + 2\epsilon, \rho_{\max})]$.*

Proof. From Lemma 4 and Lemma 6, for every $x \in (y^n(\bar{t}) - \tilde{\delta}, y(\bar{t}) + \tilde{\delta})$, we have $\rho^n(\bar{t}, x) \in [\rho_- - 2\epsilon, \rho_+ + 2\epsilon]$. Since for every $\rho \in [0, \rho_{\max}]$, $\sigma(\rho, \rho_- - 2\epsilon) \leqslant v(\rho_- - 2\epsilon)$ and $f'(\rho_+ + 2\epsilon) \leqslant \sigma(\rho, \rho_- - 2\epsilon)$, an outside front-wave of $\mathcal{T}_0$ cannot enter in the triangle $\mathcal{T}_0$. Thus all discontinuity waves in $\mathcal{T}_0$ are coming from the segment $\{\bar{t}\} \times [y^n(\bar{t}) - \tilde{\delta}, y(\bar{t}) + \tilde{\delta}]$. Since, $\hat{\rho}_\alpha < \rho_- - 2\epsilon$, we deduce that we have $\rho^n(t,x) \in [\rho_- - 2\epsilon, \rho_+ + 2\epsilon]$ for every $(t,x) \in \mathcal{T}_0$ and a non-classical shock cannot appear along the trajectory of y^n in the triangle $\mathcal{T}_0$. □

By construction of ρ^n via the wave-front tracking method, $\rho^n(\bar{t},\cdot)$ has $N(\bar{t},n)$ points of discontinuity $x_1^n < \cdots < x_j^n < \cdots < x_{N(\bar{t},n)}^n$ such that for every $j \in \{1,\cdots,N(\bar{t},n)\}$, $\rho^n(\bar{t},x_j^n-) \in \mathcal{M}_n$ and $\rho^n(\bar{t},x_j^n+) \in \mathcal{M}_n$.

Lemma 12. *There exists $j_0 \in \{1,\cdots,N(\bar{t},n)\}$ such that $x_{j_0}^n \in [y^n(\bar{t}), y(\bar{t})]$ and for every $j \geqslant j_0$*

$$\rho^n(\bar{t},x_j^n+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon). \tag{47}$$

with $x_j^n < y(\bar{t}) + \tilde{\delta}$.

Proof. From Lemma 4 we have $\rho^n(\bar{t},\cdot) \in (\rho_+ - \epsilon, \rho_+ + \epsilon)$ over $(y(\bar{t}), y(\bar{t}) + \tilde{\delta})$. In particular, we have $\rho^n(\bar{t}, y(\bar{t})+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon)$. Moreover, there exists $j_0 \in \{1,\cdots,N(\bar{t},n)\}$ such that $x_{j_0}^n \leqslant y(\bar{t}) < x_{j_0+1}^n$. Thus, $\rho^n(\bar{t},x_{j_0}^n+) = \rho^n(\bar{t},y(\bar{t})+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon)$ and for every $j \geqslant j_0$, $x_{j_0}^n \leqslant x_j^n \leqslant y(\bar{t}) + \tilde{\delta}$, whence $\rho^n(\bar{t},x_j^n+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon)$. From Lemma 4 and using $\rho_- < \rho_+$, $\rho^n(\bar{t},y^n(\bar{t})-) \in (\rho_- - \epsilon, \rho_- + \epsilon)$. Thus, $y^n(\bar{t}) \leqslant x_{j_0}^n$. □

The proof of Lemma 7 is illustrated in Figure 17. We track forward in time the wave-front denoted by $\xi^n(\cdot)$ constructed by a wave front tracking method and starting at $\xi^n(0) = x_{j_0}^n$; for every $t \in [\bar{t}, t_1]$,

$$\xi^n(t) = x_{j_0}^n + (t - \bar{t})\sigma(\rho^n(\bar{t},x_{j_0}^n-), \rho^n(\bar{t},x_{j_0}^n+)),$$

where t_1 is defined as follows:

- if $\xi^n(\cdot)$ never interacts with a front-wave in the triangle $\mathcal{T}_0$ then t_1 is the time when $\xi^n(\cdot)$ exits the triangle $\mathcal{T}_0$.
- otherwise, t_1 is the first time when $\xi^n(\cdot)$ interacts with a front-wave. By construction of ρ^n, two waves interacting together produces a third one (see Figure 8). Thus, for every $t \in [t_1, t_2]$,

  $$\xi^n(t) = \xi^n(t_1) + (t - t_1)\sigma(\rho^n(t_1, \xi^n(t_1)-), \rho^n(t_1, \xi^n(t_1)+)),$$

  where t_2 is defined as follows:

  - if $t \in (t_1, \infty] \mapsto \xi^n(t)$ never interacts with a front-wave in the triangle $\mathcal{T}_0$, t_2 is the time when $\xi^n(\cdot)$ exits the triangle $\mathcal{T}_0$.
  - otherwise, t_2 is the first time where $\xi^n : (t_1, \infty) \to \mathbb{R}$ interacts with a front-wave and so on.

By induction, we construct a piecewise constant function $\xi^n(\cdot)$ such that for every $t \in [\bar{t}, t_f^\xi)$, $(t, \xi^n(t)) \in \mathcal{T}_0$ with $t_f^\xi = \sup_{t\in[\bar{t},\infty],\,(t,\xi^n(t))\in\mathcal{T}} t$. We extend $\xi^n(\cdot)$ to $\mathbb{R}_+$ by imposing that, for every $t \in [t_f^\xi, \infty)$, $\xi^n(t) = \xi^n(t_f^\xi)$. Since an outside wavefront of $\mathcal{T}_0$ cannot enter in $\mathcal{T}_0$ and from Lemma 12, we conclude that for every $(t,x) \in \{(t,x) \in [\bar{t}, +\infty) \times \mathbb{R},\, x > \xi^n(t)\} \cap \mathcal{T}_0$

$$\rho^n(t, x+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon). \tag{48}$$

From Lemma 11 and (48), we have for a.e $t \in (\bar{t}, t_f^\xi)$

$$\sigma(\rho_+ + \epsilon, \rho_+ + 2\epsilon) \leqslant \dot{\xi}^n(t) \leqslant \sigma(\rho_- - 2\epsilon, \rho_+ - \epsilon). \tag{49}$$

Let $t_f^{y^n} := \sup_{t\in[\bar{t},\infty],(t,y^n(t))\in\mathcal{T}_0} t$ be the time when $y^n(\cdot)$ exits the triangle $\mathcal{T}_0$. From Lemma 11, for every $t \in [\bar{t}, t_f^{y^n})$, we have

$$(t, y^n(t)) \in \mathcal{T}_0 \quad \text{and} \quad v(\rho_+ + 2\epsilon) \leqslant \dot{y}^n(t) \leqslant v(\rho_- - 2\epsilon). \tag{50}$$

Using (49), we have

$$t_f^{\xi} > \bar{t} + \min\left(\frac{\tilde{\delta}}{v(\rho_- - 2\epsilon) - \sigma(\rho_+ + \epsilon, \rho_+ + 2\epsilon)}, \frac{\tilde{\delta}}{\sigma(\rho_- - 2\epsilon, \rho_+ - \epsilon) - f'(\rho_+ + 2\epsilon)}\right) \tag{51}$$

and using (50)

$$t_f^{y^n} \geqslant \bar{t} + \min\left(\frac{\tilde{\delta}}{v(\rho_- - 2\epsilon) - f'(\rho_+ + 2\epsilon)}, \frac{\tilde{\delta}}{v(\rho_- - 2\epsilon) - v(\rho_+ + 2\epsilon)}\right) \tag{52}$$

From (51) and (52), ther exists $c > 0$ independent of n such that

$$\min(t_f^{\xi}, t_f^{y^n}) \geqslant \bar{t} + c.$$

From (49) and (50),

$$\dot{y}^n(t) - \dot{\xi}^n(t) \geqslant v(\rho_+ + 2\epsilon) - \sigma(\rho_- - 2\epsilon, \rho_+ - \epsilon) > 0 \tag{53}$$

Using (53), $y^n(\cdot)$ interacts with $\xi^n(\cdot)$ at time $t_n > \bar{t}$ and

$$t_n \leqslant \frac{\xi^n(\bar{t}) - y^n(\bar{t})}{v(\rho_+ + 2\epsilon) - \sigma(\rho_- - 2\epsilon, \rho_+ - \epsilon)}. \tag{54}$$

Using that $\lim_{n\to\infty} y^n(\bar{t}) = y(\bar{t})$ and $y^n(\bar{t}) \leqslant \xi^n(\bar{t}) \leqslant y(\bar{t})$ and (54), we have $\lim_{n\to\infty} t_n = 0$.

B Proof of Lemma 10

We have $\rho_- < \rho^* < \rho_+$ and $y^n(\bar{t}) < y(\bar{t})$ for an infinite set of indices n. There exists a subsequence of $(y^n)_{n\in\mathbb{N}}$, still denoted by $(y^n)_{n\in\mathbb{N}}$, such that for every $n \in \mathbb{N}$ $y^n(\bar{t}) < y(\bar{t})$. By construction of ρ^n in Section 2.2, $\rho^n(\bar{t}, \cdot)$ has $N(\bar{t}, n)$ points of discontinuity $x_1^n < \cdots < x_j^n < \cdots < x_{N(\bar{t},n)}^n$ such that for every $j \in \{1, \cdots, N(\bar{t}, n)\}$, $\rho^n(\bar{t}, x_j^n-) \in \mathcal{M}_n$ and $\rho^n(\bar{t}, x_j^n+) \in \mathcal{M}_n$.

Lemma 13. *There exists $j_1 \in \{1, \cdots, N(\bar{t}, n)\}$ such that*

$$x_{j_1}^n \in [y^n(\bar{t}), y(\bar{t})] \quad \textit{and} \quad \rho^n(\bar{t}, x_j^n+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon),$$

for $j \geqslant j_1$ such that $x_j^n < y(\bar{t}) + \tilde{\delta}$.

Proof. From Lemma 4, we have $\rho^n(\bar{t}, \cdot) \in (\rho_+ - \epsilon, \rho_+ + \epsilon)$ over $(y(\bar{t}), y(\bar{t}) + \tilde{\delta})$. In particular, we have $\rho^n(\bar{t}, y(t)+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon)$. Moreover, there exists $j_1 \in \{1, \cdots, N(\bar{t}, n)\}$ such that $x_{j_1}^n \leqslant y(\bar{t}) < x_{j_1+1}^n$ and $\rho^n(\bar{t}, x_{j_1}+) = \rho^n(\bar{t}, y(\bar{t})+)$. For every $j \geqslant j_1$, $x_{j_1}^n \leqslant x_j^n \leqslant y(\bar{t}) + \tilde{\delta}$ and $\rho^n(\bar{t}, x_j^n+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon)$. From Lemma 4 and using $\rho_- < \rho^* < \rho_+$, $\rho^n(\bar{t}, y^n(\bar{t})-) \in (\max(0, \rho_- - \epsilon), \rho_- + \epsilon)$. Thus, $y^n(\bar{t}) \leqslant x_{j_1}^n$. □

Lemma 14. *There exists $j_0 \in \{1, \cdots, N(\bar{t}, n)\}$ such that*

$$x_{j_0}^n \in [y^n(\bar{t}), y(\bar{t})] \quad \textit{and} \quad \rho^n(\bar{t}, x_j^n+) \in (\rho^*, \rho_+ + 2\epsilon),$$

for $j \geqslant j_0$ such that $x_j^n < y(\bar{t}) + \tilde{\delta}$.

Proof. From Lemma 4, there exits $j_0 \in \{1, \cdots, N(\bar{t}, n)\}$ such that $\rho^n(\bar{t}, x^n_{j_0}-) \leqslant \rho^*$ and $\rho^n(\bar{t}, x^n_{j_0}+) > \rho^*$ with $x^n_{j_0} \in [y^n(\bar{t}), y(\bar{t})]$ and for every $j > j_0$, $\rho^n(\bar{t}, x^n_j+) > \rho^*$. We assume that there exists $k > j_0$ such that $\rho^n(\bar{t}, x^n_k+) \geqslant \rho_+ + 2\epsilon$. Using $\rho^* < \rho_+$ and Lemma 13, we have $\rho^* < \rho^n(\bar{t}, x^n_{j_1}+)$. Thus, we only have shocks and rarefaction shocks to go from $\rho^n(\bar{t}, x^n_k+)$ to $\rho^n(\bar{t}, x^n_{j_1}+)$. From Lemma 1 and Lemma 14, for n large enough,

$$\delta^n(\rho^n(\bar{t}, x^n_k+), \rho^n(\bar{t}, x^n_{j_1}+)) > \frac{\beta \bar{t} \epsilon}{2}.$$

Using that $x^n_k, x^n_{j_1} \in [y^n(\bar{t}), y(\bar{t})]$ and $|y^n(\bar{t}) - y(\bar{t})| \leqslant \frac{\beta \bar{t} \epsilon}{2}$, we have a contradiction. □

The proof of Lemma 10 is illustrated in Figure 18. We track forward in time two wavefronts denoted by $\xi^n_0(\cdot)$ and $\xi^n_1(\cdot)$ constructed by a wave front tracking method and starting at $\xi^n_0(0) = x^n_{j_0}$ and $\xi^n_1(0) = x^n_{j_1}$; for $i \in \{0, 1\}$, since $x^n_{j_i}$ is a discontinuity point of $\rho^n(\bar{t}, \cdot)$, a wave-front $\xi^n_i(\cdot)$ such that $\xi^n_i(0) = x^n_{j_i}$ is constructed via the wave-front tracking method and we follow it until it interacts with an other wave-front or $y^n(\cdot)$ at time t^1_i. By construction of $\mathcal{T}_1$ defined in (44), other wave-fronts out of the triangle $\mathcal{T}_1$ cannot interact with a wave-front in the triangle $\mathcal{T}_1$. Thus, from Lemma 14, for every $t \in [0, t^1_0]$, for every $x \in [\xi^n_0(t), f'(\rho_+ + 2\epsilon)(t - \bar{t}) + y(\bar{t}) + \tilde{\delta}]$,

$$\rho^n(t, x+) \in (\rho^*, \rho_+ + 2\epsilon) \tag{55}$$

and from Lemma 13, for every $t \in [0, t^1_1]$, for every $x \in [\xi^n_1(t), f'(\rho_+ + 2\epsilon)(t - \bar{t}) + y(\bar{t}) + \tilde{\delta}]$,

$$\rho^n(t, x+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon). \tag{56}$$

- If $\xi^n_i(\cdot)$ interacts with a shock or a rarefaction shock at time t^1_i; we follow the unique front-wave produced (see Figure 8). Moreover, $\rho^n(t^1_0, x+) \in (\rho^*, \rho_+ + 2\epsilon)$ for every $x \in [\xi^n_0(t^1_0), f'(\rho_+ + 2\epsilon)(t - \bar{t}) + y(\bar{t}) + \tilde{\delta}]$ and $\rho^n(t^1_1, x+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon)$ for every $x \in [\xi^n_1(t), f'(\rho_+ + 2\epsilon)(t - \bar{t}) + y(\bar{t}) + \tilde{\delta}]$.
- If $\xi^n_i(\cdot)$ interacts with $y^n(\cdot)$ at time t^1_i; from (55), (56) and using that all the possible interaction between a front-wave and $y^n(\cdot)$ is described in Figure 9, Figure 10, Figure 11 and Figure 12, we deduce that only the cases illustrated in Figure 9 and Figure 10 a) are possible. Thus, a unique front-wave is produced. Moreover, $\rho^n(t^1_0, \xi^n_0(t^1_0)+) \in (\rho^*, \rho_+ + 2\epsilon)$, $\rho^n(t^1_1, \xi^n_1(t^1_1)+) \in (\rho_+ - 2\epsilon, \rho_+ + \epsilon)$ and $\dot{y}^n(t^1_i) = v(\rho(t^1_i, y^n(t^1_i)+))$.

By an iteration procedure, we construct $\xi^n_0(\cdot)$ and $\xi^n_1(\cdot)$ over $[\bar{t}, t^{n,\xi^n_0}_f)$ and $[\bar{t}, t^{\xi_1}_f)$ respectively. For $i = 1, 2$, we extend $\xi^n_i(\cdot)$ to $\mathbb{R}_+$ by imposing that, for every $t \in [t^{n,\xi^n_i}_f, \infty)$, $\xi^n_i(t) = \xi^n_i(t^{n,\xi^n_i}_f)$. We conclude that, for every $(t, x) \in \{(t, x) \in [\bar{t}, +\infty) \times \mathbb{R},\ x < \xi^n_0(t)\} \cap \mathcal{T}_1$

$$\rho^n(t, x+) \in (0, \rho^* + \epsilon), \tag{57}$$

for every $(t, x) \in \{(t, x) \in [\bar{t}, +\infty) \times \mathbb{R},\ x > \xi^n_0(t)\} \cap \mathcal{T}_1$

$$\rho^n(t, x+) \in (\rho^*, \rho_+ + 2\epsilon), \tag{58}$$

and for every $(t, x) \in \{(t, x) \in [\bar{t}, +\infty) \times \mathbb{R},\ x > \xi^n_1(t)\} \cap \mathcal{T}_1$,

$$\rho^n(t, x+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon). \tag{59}$$

For $i = 1, 2$, we denote by $t^{\xi_i}_f$ and $t^{y^n}_f$ the time when $\xi^n_i(\cdot)$ and $y^n(\cdot)$ exits the triangle $\mathcal{T}_1$. We notice that for every $t \in \mathbb{R}$, $\xi^n_0(t) \leqslant \xi^n_1(t)$ and as soon as there exists $t_1 \geqslant \bar{t}$ such that $\xi^n_0(t_1) = \xi^n_1(t_1)$, we have for every $t \in [t_1, +\infty]$ $\xi^n_0(t) = \xi^n_1(t)$. From (57), (58) and (59), we have

$$t^{n,\xi^n_0}_f \geqslant \bar{t} + \min\left(\frac{\tilde{\delta}}{v(0) - \sigma(\rho^* + \epsilon, \rho_+ + 2\epsilon)}, \frac{\tilde{\delta}}{v(\rho_+ - \epsilon) - f'(\rho_+ + 2\epsilon)}\right) \tag{60}$$

and

$$t_f^{n,\xi_1^n} \geqslant \bar{t} + \min\left(\frac{\tilde{\delta}}{v(0)-\sigma(\rho_+ + 2\epsilon, \rho_+ + \epsilon)}, \frac{\tilde{\delta}}{v(\rho_+ - \epsilon) - f'(\rho_+ + 2\epsilon)}\right) \tag{61}$$

Therefore, using that $\lim_{n\to\infty} y^n(\bar{t}) = y(\bar{t})$, $y^n(\bar{t}) \in [y^n(\bar{t}) - \tilde{\delta}, y(\bar{t}) + \tilde{\delta}]$ and the finite speed of y^n, there exists $c > 0$ independent of n such that

$$\min(t_f^{y_0^n}, t_f^{n,\xi_0^n}, t_f^{n,\xi_1^n}) \geqslant \bar{t} + c.$$

From (57), (58) and (59), for every $t > \bar{t}$ such that $(t, \xi_0^n(t)) \in \mathcal{T}_1$, $(t, \xi_1^n(t)) \in \mathcal{T}_1$ and $(t, y^n(t)) \in \mathcal{T}_1$, if $y^n(\cdot)$ belongs to the area A_1 defined by for every $(t,x) \in A_1$, $\rho^n(t,x) \in (\rho^*, \rho_+ + 2\epsilon)$ (see the shaded zone in Figure 18) then $v(\rho_+ + 2\epsilon) \leqslant \dot{y}^n(t) \leqslant v(\rho^*)$ and we have

$$\sigma(\rho_+ + 2\epsilon, \rho_+ + \epsilon) \leqslant \dot{\xi}_1^n(t) \leqslant \sigma(\rho^*, \rho_+ - \epsilon) \tag{62}$$

and if $y^n(\cdot)$ belongs to the area A_2 defined by for every $(t,x) \in A_2$, $\rho^n(t,x) \in (0, \rho^* + \epsilon)$ (see white zone in Figure 18) then $v(\rho^* + \epsilon) \leqslant \dot{y}^n(t) \leqslant V_b$ then either (62) holds or

$$\sigma(\rho^* + \epsilon, \rho_+ + \epsilon) \leqslant \dot{\xi}_1^n(t) \leqslant v(\rho_+ - \epsilon) \tag{63}$$

From (62), (63) and using that f is strictly concave

$$\dot{y}^n(t) - \dot{\xi}_1^n(t) \geqslant v(\rho^* + \epsilon) - \sigma(\rho^*, \rho_+ - \epsilon) > 0 \tag{64}$$

Using (64), $y^n(\cdot)$ interacts with $\xi_1^n(\cdot)$ at time $t_n > \bar{t}$ and

$$t_n \leqslant \frac{\xi_1^n(\bar{t}) - y^n(\bar{t})}{v(\rho^* + \epsilon) - \sigma(\rho^*, \rho_+ - \epsilon)}. \tag{65}$$

Using that $\lim_{n\to\infty} y^n(\bar{t}) = y(\bar{t})$ and $y^n(\bar{t}) \leqslant \xi_1^n(\bar{t}) \leqslant y(\bar{t})$ and (65), $\lim_{n\to\infty} t_n = 0$.

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