# Generalized formulation for acoustics

Jean-Dominique Polack

Sorbonne Université, CNRS UMR 7190, Institut D'Alembert,
F-75005 Paris, France
jean-dominique.polack@sorbonne-universite.fr

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## 1 Introduction

This manuscript is an attempt to derive a generalized formulation for the wave equation in linear acoustics, including the conservation of energy. It makes use of the conservation of the stress-energy tensor, that is, the covariance of the stress-energy tensor is null in linear acoustics, as first proven by P.M. Morse and K.U. Ingard [MI68]. At stake is a novel approach to sound field computation that naturally accounts for losses, whereas current computational methods do not.

As a matter of fact, the stress-energy tensor and its covariant conservation are the basis of general relativity. The present paper therefore investigates to which extend general relativity has application in linear acoustics. Except for its derivation by Morse and Ingard [MI68], there has been no attempt to apply this formalism to acoustics. The motivation behind this investigation was the search for a natural setting for explaining the curvature of rays above absorbing surfaces experimentally proven by Janowsky and Spandöck in the 30s [JS37], and confirmed by Cremer and Müller [CM78].

## 2 The generalized wave equation

We consider a 4-dimensional time-space with its metric tensor $g_{ij}$ and the volume element $dV = \sqrt{|g|}dx^0 \dots dx^3$, where $g = \det g_{ij}$ [Lin 5]. The infinitesimal distance element is given by:

$$ds^2 = g_{ij}dx^i dx^j \tag{1}$$

and the generalized wave equation is:

$$\Box\Phi = \nabla_i g^{ij}\partial_j\Phi = 0 \tag{2}$$

where $\Phi$ is the velocity potential and $g^{ij}$ the inverse matrix of $g_{ij}$. Note that $\nabla_i$ is the covariant derivation with respect to $x^i$, which differs from the usual partial derivation $\partial_i$ in a way that depends on the tensor rank. For example, for a function $\Phi$:

$$\nabla_j\Phi = \partial_j\Phi = \Phi_j$$

but $\nabla_j \Phi_i = \partial_j \Phi_i - \Gamma^k_{ji} \Phi_k$ and $\nabla_j X^i = \partial_j X^i + \Gamma^i_{jk} X^k$, where $X^i$ are the (contravariant) components of vector $X$ and the $\Gamma^k_{ji}$ the Christoffel symbols linked to the derivatives of the elements of the metric tensor $g_{ij}$:

$$\Gamma^k_{ji} = \frac{1}{2} g^{kl} \left( \partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji} \right) \tag{3}$$

Note that, unlike ordinary differentiations, covariant derivations do not commute. Their commutators is given by the curvature of the time-space [Lin 5]:

$$\left( \nabla_i \nabla_j - \nabla_j \nabla_i \right) X^k = R^k_{mij} X^m \tag{4}$$

where the $R^k_{mij}$ are the elements of the Riemann tensor. However, for a function, $\nabla_i \partial_j \Phi = \nabla_j \partial_i \Phi$; and by construction, all covariant derivatives of the elements of the metric tensor are null. In other words, the contravariant derivation $\nabla^i$ is defined by:

$$\nabla^i = g^{ij} \nabla_j = \nabla_j g^{ij}$$

One calls vectors tensors with one upper index, such as $X^i$; and covectors tensors with one lower index, such as $\Phi_i$.

# 3 Volume deformation

In general, the velocity potential $\Phi$ is a complex function. So one can consider the product $\Phi^* \Box \Phi$, where $\Phi^*$ is the complex conjugate of $\Phi$. Differentiation rules lead to:

$$\Phi^* \Box \Phi = \Phi^* \nabla_i g^{ij} \partial_j \Phi = \nabla_i g^{ij} \left[ \Phi^* \partial_j \Phi \right] - \left[ \partial_i \Phi^* \right] g^{ij} \partial_j \Phi = 0 \tag{5}$$

that is, to:

$$\nabla_i g^{ij} \left[ \Phi^* \partial_j \Phi \right] = \left[ \partial_i \Phi^* \right] g^{ij} \partial_j \Phi \tag{6}$$

As the right member of the preceding equation is real, separating the real and imaginary parts leads to:

$$\nabla_i \left[ g^{ij} \frac{\Phi^* \partial_j \Phi + \Phi \partial_j \Phi^*}{2} \right] = \partial_i \Phi^* g^{ij} \partial_j \Phi = 2L \tag{7}$$

for the real part, where $L$ is the Lagrangian; and for the imaginary part, to:

$$\nabla_i \left[ g^{ij} \frac{\Phi^* \partial_j \Phi - \Phi \partial_j \Phi^*}{2i} \right] = \nabla^j \frac{\Phi^* \partial_j \Phi - \Phi \partial_j \Phi^*}{2i} = 0 \tag{8}$$

It should be noticed that $g^{ij} \frac{\Phi^* \partial_j \Phi + \Phi \partial_j \Phi^*}{2}$ can be rewritten as:

$$g^{ij} \frac{\Phi^* \partial_j \Phi + \Phi \partial_j \Phi^*}{2} = g^{ij} \partial_j \left( \frac{1}{2} |\Phi|^2 \right) \tag{9}$$

that is, as the 4-gradient of a real function. As a consequence, $g^{ij} \frac{\Phi^* \partial_j \Phi + \Phi \partial_j \Phi^*}{2}$ can be assimilated to a 4-velocity $-\mathfrak{V}^i$. Thus the real part of Eq. (6) simply reduces to:

$$\nabla_i \mathfrak{V}^i = -2L \tag{10}$$

meaning that the Lagrangian $L$ amounts to a volume deformation (divergence of a velocity). In a similar fashion, the imaginary part $\frac{\Phi^* \partial_j \Phi - \Phi \partial_j \Phi^*}{2i}$ can be assimilated to a "covector" potential $\mathfrak{A}_j$ with the gauge relation:

$$\nabla^i \mathfrak{A}_i = 0 \tag{11}$$

# 4 Conservation of the stress-energy tensor

We now consider the product $\partial_k \Phi^* \Box \Phi$. Once more, differentiation rules lead to:

$$\begin{aligned}\partial_k \Phi^* \Box \Phi = \partial_k \Phi^* \nabla_i g^{ij} \partial_j \Phi &= \nabla_i g^{ij} \left[\partial_k \Phi^* \partial_j \Phi\right] - \left[\nabla_i \partial_k \Phi^*\right] g^{ij} \partial_j \Phi \\ &= \nabla_i g^{ij} \left[\partial_k \Phi^* \partial_j \Phi\right] - \left[\nabla_k \partial_i \Phi^*\right] g^{ij} \partial_j \Phi \\ &= \nabla^j \left[\partial_k \Phi^* \partial_j \Phi\right] - \left[\nabla_k \partial_i \Phi^*\right] g^{ij} \partial_j \Phi = 0\end{aligned}$$

As $i$ and $j$ are mute indices, separating the real and imaginary parts of the preceding equation leads to:

- for the real part:

$$\nabla^j \left(\partial_j \Phi^* \partial_k \Phi + \partial_j \Phi \partial_k \Phi^*\right) = \nabla_k \left(\partial_i \Phi^* g^{ij} \partial_j \Phi\right) \tag{12}$$

  that is, to:

$$\nabla^i T_{ij} = 0 \tag{13}$$

  where $T_{ij}$ is the *symmetrical* stress-energy tensor, defined by:

$$T_{ij} = \frac{\partial_i \Phi^* \partial_j \Phi + \partial_i \Phi \partial_j \Phi^*}{2} - \frac{1}{2} g_{ij} \left(\partial_i \Phi^* g^{ij} \partial_j \Phi\right) \tag{14}$$

- for the imaginary part:

$$\begin{aligned}\nabla^j \left(\partial_j \Phi^* \partial_k \Phi - \partial_j \Phi \partial_k \Phi^*\right) = \left[\nabla_k \partial_i \Phi^*\right] g^{ij} \partial_j \Phi - \left[\nabla_k \partial_i \Phi\right] g^{ij} \partial_j \Phi^* \\ = -4\pi i \mathrm{j}_k\end{aligned} \tag{15}$$

  The covector $\mathrm{j}_i$ can be assimilated to a 4-current, as will become evident in next Section.

It is easy to recognise that the equation for the real part corresponds to the contravariant conservation of the stress-energy tensor, suggesting that $T_{ij}$ be proportional to the Ricci curvature tensor $R_{ij}$, and its reduction $T$ to the scalar curvature $R$, linked to the volume variation. This confirms the conclusion from Sec. 3 that $T$ measures a volume deformation.

# 5 Maxwell equations

In acoustics, the different coefficients of the stress-energy tensor can be interpreted as [MI68]:

- the total energy density $T_{00}$;
- the active acoustic intensity $T_{a0}, a \in [1,2,3]$, or $T_{0a}, a \in [1,2,3]$;
- the symmetrical wave-stress tensor $T_{ab}, (a,b) \in [1,2,3]$;

Conservation of the stress-energy tensor $T_{ij}$ therefore amounts to the conservation of energy.

On the other hand, let us introduce the antisymmetric tensor $F_{ij}$ defined by:

$$F_{ij} = \nabla_i \mathfrak{A}_j - \nabla_j \mathfrak{A}_i = -F_{ji} \tag{16}$$

In other words, $F_{ij}$ is the the external differential form (curl) of the covector potential $\mathfrak{A}$. Simple derivations lead to:

$$\begin{aligned}\nabla_i\mathfrak{A}_j &= \nabla_i\frac{\Phi^*\partial_j\Phi - \Phi\partial_j\Phi^*}{2i}\\ &= \frac{1}{2i}\left\{\partial_i\Phi^*\partial_j\Phi + \Phi^*\nabla_i\partial_j\Phi - \partial_i\Phi\partial_j\Phi^* - \Phi\nabla_i\partial_j\Phi^*\right\} \qquad (17)\\ \nabla_j\mathfrak{A}_i &= \frac{1}{2i}\left\{-\partial_j\Phi\partial_i\Phi^* + \Phi^*\nabla_j\partial_i\Phi + \partial_j\Phi^*\partial_i\Phi - \Phi\nabla_j\partial_i\Phi^*\right\} \qquad (18)\end{aligned}$$

and by subtracting the two equations, to:

$$F_{ij} = \nabla_i\mathfrak{A}_j - \nabla_j\mathfrak{A}_i = \frac{\partial_i\Phi^*\partial_j\Phi - \partial_j\Phi^*\partial_i\Phi}{2i} \qquad (19)$$

Due to its definition, $F_{ij}$ naturally satisfies the generalized Maxwell equations:

$$\begin{aligned}\nabla^i F_{ij} &= -4\pi\mathfrak{j}_j\\ \nabla_{[k}F_{ij]} &= \nabla_k F_{ij} + \nabla_i F_{jk} + \nabla_j F_{ki} = 0 \qquad (20)\end{aligned}$$

where the square brackets indicate sum over all cyclic permutations, that is, anti-symmetrization. The first equation is nothing else than Eq. (15), and leads to assimilate the covector $\mathfrak{j}_i$ to a 4-current. The second equation simply derives from the definition of tensor $F_{ij}$ and the commutative property of covariant derivations. Indeed:

$$\nabla_{[k}F_{ij]} = \nabla_k\left[\nabla_i\mathfrak{A}_j - \nabla_j\mathfrak{A}_i\right] + \nabla_i\left[\nabla_j\mathfrak{A}_k - \nabla_k\mathfrak{A}_j\right] + \nabla_j\left[\nabla_k\mathfrak{A}_i - \nabla_i\mathfrak{A}_k\right] = 0 \qquad (21)$$

Note that the first Eq. (20) implies current conservation. Indeed, form the definition of $\mathfrak{j}_i$:

$$\mathfrak{j}_i = \frac{\nabla_i\partial_k\Phi g^{kl}\partial_l\Phi^* - \nabla_i\partial_k\Phi^* g^{kl}\partial_l\Phi}{4\pi i} = \frac{1}{4\pi}\Im\left\{\nabla_i\partial_k\Phi g^{kl}\partial_l\Phi^*\right\} \qquad (22)$$

we obtain:

$$\begin{aligned}\nabla^i\mathfrak{j}_i &= \nabla_j g^{ji}\frac{\nabla_i\partial_k\Phi g^{kl}\partial_l\Phi^* - \nabla_i\partial_k\Phi^* g^{kl}\partial_l\Phi}{4\pi i}\\ &= \frac{1}{4\pi i}\left[\nabla_j g^{ji}\nabla_i\partial_k\Phi g^{kl}\partial_l\Phi^* + \nabla_i\partial_k\Phi g^{kl}g^{ij}\nabla_j\partial_l\Phi^*\right]\\ &- \frac{1}{4\pi i}\left[\nabla_j g^{ji}\nabla_i\partial_k\Phi^* g^{kl}\partial_l\Phi + \nabla_i\partial_k\Phi^* g^{kl}g^{ij}\nabla_j\partial_l\Phi\right]\\ &= \frac{1}{4\pi i}\left[\nabla_j g^{ji}\nabla_i\partial_k\Phi g^{kl}\partial_l\Phi^* - \nabla_j g^{ji}\nabla_i\partial_k\Phi^* g^{kl}\partial_l\Phi\right]\\ &= \frac{1}{4\pi i}\left[\nabla_j g^{ji}\nabla_k\partial_i\Phi g^{kl}\partial_l\Phi^* - \nabla_j g^{ji}\nabla_k\partial_i\Phi^* g^{kl}\partial_l\Phi\right]\end{aligned} \qquad (23)$$

The commutation relations for covariant derivatives leads to:

$$\nabla_j\nabla_k g^{ji}\partial_i\Phi = \nabla_k\left[\nabla_j g^{ji}\partial_i\Phi\right] + R^j_{mjk}g^{mi}\partial_i\Phi \qquad (24)$$

We recognize in the first term on the right hand side the generalized wave equation $\nabla_j g^{ji}\partial_i\Phi = 0$, and the term is therefore equal to 0. Thus:

$$\nabla_j g^{ji}\nabla_k\partial_i\Phi g^{kl}\partial_l\Phi^* = R^j_{mjk}g^{mi}\partial_i\Phi g^{kl}\partial_l\Phi^* = R_{mk}g^{mi}\partial_i\Phi g^{kl}\partial_l\Phi^*$$

where $R_{ik} = R^j_{ijk}$ is the Ricci tensor. In a similar fashion:

$$\nabla_j g^{ji}\nabla_k\partial_i\Phi^* g^{kl}\partial_l\Phi = R_{mk}g^{mi}\partial_i\Phi^* g^{kl}\partial_l\Phi$$

and Eq. (23) becomes:

$$\nabla^i \mathrm{j}_i = \frac{1}{4\pi i}\left[R_{mk}g^{mi}\partial_i\Phi g^{kl}\partial_l\Phi^* - R_{mk}g^{mi}\partial_i\Phi^* g^{kl}\partial_l\Phi\right]$$

Since all indices are mute and the Ricci tensor is symmetric, the two terms cancel out. Thus:

$$\nabla^i \mathrm{j}_i = 0 \tag{25}$$

There remains to interpret the antisymmetric tensor $F_{ij}$ in acoustical terms. By comparison with the stress-energy tensor, it is obvious that $F_{a0} = -F_{0a}, a \in [1,2,3]$ is the *reactive* acoustic intensity. Therefore, by analogy, we call the terms $F_{ab} = -F_{ba}, (a,b) \in [1,2,3]$ the *reactive* wave stress.

# 6 Solving the generalized wave equation

Let us now consider a 4-dimensional manifold $V$, with border $\partial V = S$, metric tensor $g_{ij}$ and volume element $dV = \sqrt{|g|}dx^0 \dots dx^3$. We consider boundary conditions of admittance type on $S$:

$$\partial_i\Phi\beta^i = 0 \tag{26}$$

where $\beta^i = \xi - i\sigma^i$ is the complex specific 4-admittance. Solving the wave equation in $V$ with boundary condition (26) on $S$ amounts to finding the extremum of the Lagrangian, that is, the extremum of the volume integral $\mathfrak{T} = \int_V T dV$ of the volume deformation $2L$ given by Eq. (10). Using Eq. (7), we thus obtain:

$$2\mathfrak{L} = 2\int_V L dV = \frac{1}{2}\int_V \nabla_i g^{ij}\left[\Phi^*\partial_j\Phi + \Phi\partial_j\Phi^*\right]\sqrt{|g|}dx^0\dots dx^n \tag{27}$$

Using the identity $\nabla_i X^i = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}X^i)$ [Lin 5] and applying Stokes theorem, the last expression leads to:

$$2\int_V L dV = \frac{1}{2}\int_S \left[\Phi^*\partial_j\Phi + \Phi\partial_j\Phi^*\right] g^{ji}n_i dS \tag{28}$$

Choosing the local coordinates on the boundary so that $n_i$ is the *outgoing* 4-vector, and that $g^{ji}n_i = \xi^j$ leads to:

$$\begin{aligned}\left[\Phi^*\partial_j\Phi + \partial_j\Phi^*\Phi\right] g^{ji}n_i &= \left[\Phi^*\partial_j\Phi + \partial_j\Phi^*\Phi\right]\xi^j \\ &= \left[\Phi^*\partial_j\Phi\beta^j + \partial_j\Phi^*\beta^{*j}\Phi\right] + \left[i\sigma^j\Phi^*\partial_j\Phi - i\sigma^\partial_j\Phi^*\Phi\right] \\ &= 0 + i\sigma^j\left[\Phi^*\partial_j\Phi - \partial_j\Phi^*\Phi\right] = 2\sigma^j\Im\left[\Phi^*\partial_j\Phi\right]\end{aligned} \tag{29}$$

Thus:

$$2\mathfrak{L} = 2\int_V T dV = \int_S \sigma^j \Im\left[\Phi^* \partial_j \Phi\right] dS \tag{30}$$

In a similar way, for the imaginary part, Eq. (8) implies:

$$0 = \int_V \nabla^j \left[\Phi^* \partial_j \Phi - \Phi \partial_j \Phi^*\right] dV = \int_S \left[\Phi^* \partial_j \Phi - \Phi \partial_j \Phi^*\right] g^{ji} n_i dS \tag{31}$$

with:

$$\begin{aligned}\left[\Phi^* \partial_j \Phi - \partial_j \Phi^* \Phi\right] g^{ji} n_i &= \left[\Phi^* \partial_j \Phi - \partial_j \Phi^* \Phi\right] \xi^j \\ &= \left[\Phi^* \partial_j \Phi \beta^j - \partial_j \Phi^* \beta^{*j} \Phi\right] + \left[i\sigma^j \Phi^* \partial_j \Phi + i\sigma_j^{\partial} \Phi^* \Phi\right] \\ &= 0 + i\sigma^j \left[\Phi^* \partial_j \Phi + \partial_j \Phi^* \Phi\right] = 2i\sigma^j \Re\left[\Phi^* \partial_j \Phi\right]\end{aligned} \tag{32}$$

We finally obtain:

$$\int_S \sigma^j \Re\left[\Phi^* \partial_j \Phi\right] dS = 0 \tag{33}$$

It should be noted that the metric tensor $g^{ij}$ is *normalized* by $|g| = c^2$, but its elements are not bounded. Similarly, the $\beta^i$ are not bounded.

# 7 Discussion

The striking aspect of these generalized equations is the presence of Maxwell equations equations to describe the evolution of the reactive intensity and wave stress. They are obtained as the external differential form of a covector potential $\mathfrak{A}_i$ that naturally satisfies the gauge relation $\nabla^i \mathfrak{A}_i = 0$.

There exist other attempts in the literature to associates Maxwell equations to the energy-stress tensor. Most notable is the Kaluza-Klein theory [Wik18], derived in the 1920s, and its successors. Unlike Kaluza-Klein theory, the present approach does not introduce an extra dimension to introduce Maxwell equations; nor is the metric tensor modified by the covector potential $\mathfrak{A}_i$.

The fact that thecovector potential and the Maxwell equations are associated to the reactive intensity field leads to a straightforward physical interpretation: they are associated to standing waves and the eigenmodes of the manifold. Indeed, its is well known that eigenmodes are characterized by a quadrature relation between pressure and particle velocity, that is, by a vanishing active intensity and a non-vanishing reactive intensity. Thus, compared to the local field described by the stress-energy tensor, the reactive field is the global signature due to the manifold geometry. Global and local field are coupled at the boundaries, as shown by Eq. (30).

# 8 Examples

We now give a few examples of metric tensors $g_{ij}$ that satisfy the admittance condition (26) on the boundary, with real specific admittance.

## 8.1 Propagation above constant absorbing plane

The case of sound propagation above a constant absorbing plane $x^1 = 0$ is addressed first. Boundary condition (26) then reduces to $\partial_0\Phi\beta^0 + \partial_1\Phi\beta^1 = 0$, with $\frac{\beta^0}{\beta^1} = -\xi$, the real specific admittance on the absorbing plane. As the outgoing 4-vector is $n_i = (0, -1, 0, 0)$, the local metric must satisfy $g^{j1}n_1 = \xi^j$, that is, $g^{10} = g^{01}$ is non-zero. In other words, the metric tensor and its inverse are no longer diagonal, and must be written as:

$$g^{ij} = \begin{pmatrix} -c & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; g_{ij} = \begin{pmatrix} -a & b & 0 & 0 \\ b & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{34}$$

where $a$, $b$, and $c$ only depend on the coordinate $x^1$ and not on the two other space coordinates. The boundary condition is then given $\partial_0\Phi b + \partial_1\Phi a = 0$, with $\frac{b}{a} = -\xi$, the real specific admittance on the absorbing plane. Normalizing $a$, $b$, and $c$ by $ac + b^2 = 1$, it is then obvious that $g = \det(g^{ij}) = -1$, $a > 0$, and $b \leq 0$.

**Lagrangian:** Simple calculation shows that the Lagrangian $L$ is given by:

$$L = \frac{1}{2}\left[-c|\Phi_0|^2 + b(\Phi_0^*\Phi_1 + \Phi_0\Phi_1^*) + a|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2\right] \tag{35}$$

**Christoffel symbols:** The Christoffel symbols are all equal to 0, except for:

$$\begin{aligned} &\Gamma^0_{00} = \Gamma^1_{00} = -\Gamma^1_{01} = \frac{1}{2}ba_1, \Gamma^0_{01} = \frac{1}{2}ca_1 \\ &\Gamma^0_{10} = -\Gamma^1_{11} = \frac{1}{2}ca_1, \Gamma^1_{10} = -\frac{1}{2}ba_1, \Gamma^0_{11} = -cb_1 + \frac{1}{2}bc_1 \end{aligned} \tag{36}$$

**Stress-energy tensor:** The conservation of the stress-energy tensor takes a simpler form for $T^{ij}$ than for $T_{ij}$, which still is symmetric. Indeed, Eq (13) can be written as:

$$g^{lj}\nabla^i T_{il} = g^{ik}\nabla_k T_{il} g^{lj} = \nabla_i T^{ij} = \partial_i T^{ij} + \Gamma^i_{in}T^{nj} + \Gamma^j_{in}T^{in} = 0 \tag{37}$$

that is ([Lin 5] p.54):

$$\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}T^i_j) + \Gamma^j_{in}T^{in} = 0 \tag{38}$$

or, after integration on a small 4-dimensional volume $V$ with border $\partial V$:

$$\int_{\partial V} n_i T^i_j dS + \int_V \Gamma^j_{in}T^{in}dV = 0 \tag{39}$$

In the last two equations, $T^{ij} = g^{ik}\frac{1}{2}(\Phi_k\Phi_l^* + \Phi_k^*\Phi_l)g^{lj} - g^{ij}L$ is given by:

$$\begin{pmatrix} \frac{1}{2}\left[|-c\Phi_0 + b\Phi_1|^2 + |\Phi_1|^2 + c|\Phi_2|^2 + c|\Phi_3|^2\right] \\ \frac{1}{2}\left[\Re\{(-c\Phi_0 + b\Phi_1)(b\Phi_0^* + a\Phi_1^*)\} - \Re(\Phi_0\Phi_1^*) - b|\Phi_2|^2 - b|\Phi_3|^2\right] \\ \Re\{(-c\Phi_0 + b\Phi_1)\Phi_2^*\} \\ \Re\{(-c\Phi_0 + b\Phi_1)\Phi_3^*\} \end{pmatrix}$$

$$\begin{array}{c}
\frac{1}{2}\left[\Re\{(-c\Phi_0+b\Phi_1)(b\Phi_0^*+a\Phi_1^*)\}-\Re(\Phi_0\Phi_1^*)-b|\Phi_2|^2-b|\Phi_3|^2\right]\\
\frac{1}{2}\left[|\Phi_1|^2+|b\Phi_0+a\Phi_1|^2-a|\Phi_2|^2-a|\Phi_3|^2\right]\\
\Re\{(b\Phi_0+a\Phi_1)\Phi_2^*\}\\
\Re\{(b\Phi_0+a\Phi_1)\Phi_3^*\}
\end{array}$$

$$\left.\begin{array}{c}
\Re\{(-c\Phi_0+b\Phi_1)\Phi_2^*\}\\
\Re\{(b\Phi_0+a\Phi_1)\Phi_2^*\}\\
\frac{1}{2}\left[c|\Phi_0|^2-2b\Re(\Phi_0\Phi_1^*)-a|\Phi_1|^2+|\Phi_2|^2-|\Phi_3|^2\right]\\
0
\end{array}\right.$$

$$\left.\begin{array}{c}
\Re\{(-c\Phi_0+b\Phi_1)\Phi_3^*\}\\
\Re\{(b\Phi_0+a\Phi_1)\Phi_3^*\}\\
0\\
\frac{1}{2}\left[c|\Phi_0|^2-2b\Re(\Phi_0\Phi_1^*)-a|\Phi_1|^2-|\Phi_2|^2+|\Phi_3|^2\right]
\end{array}\right)$$

$$\begin{array}{c}
\partial_0T^{00}+\partial_1T^{10}+\partial_2T^{20}+\partial_3T^{30}+[2\Gamma^0_{00}+\Gamma^1_{10}]T^{00}+[3\Gamma^0_{01}+\Gamma^1_{11}]T^{10}+\Gamma^0_{11}T^{11}\\
=0\\
\partial_0T^{01}+\partial_1T^{11}+\partial_2T^{21}+\partial_3T^{31}+\Gamma^1_{00}T^{00}+[\Gamma^0_{00}+3\Gamma^1_{10}]T^{01}+[\Gamma^0_{01}+2\Gamma^1_{11}]T^{11}\\
=0\\
\partial_0T^{02}+\partial_1T^{12}+\partial_2T^{22}+\partial_3T^{32}+\left[\Gamma^0_{00}+\Gamma^1_{10}\right]T^{02}+\left[\Gamma^0_{01}+\Gamma^1_{11}\right]T^{12}=0\\
\partial_0T^{03}+\partial_1T^{13}+\partial_2T^{23}+\partial_3T^{33}+\left[\Gamma^0_{00}+\Gamma^1_{10}\right]T^{03}+\left[\Gamma^0_{01}+\Gamma^1_{11}\right]T^{13}=0
\end{array}$$

that is to:

$$\begin{array}{l}
\partial_0T^{00}+\partial_1T^{10}+\partial_2T^{20}+\partial_3T^{30}+\frac{1}{2}ba_1T^{00}+ca_1T^{01}-(cb_1-\frac{1}{2}bc_1)T^{11}=0\\
\partial_0T^{01}+\partial_1T^{11}+\partial_2T^{21}+\partial_3T^{31}+\frac{1}{2}ba_1T^{00}-ba_1T^{01}-\frac{1}{2}ca_1T^{11}=0\\
\partial_0T^{02}+\partial_1T^{12}+\partial_2T^{22}+\partial_3T^{32}=0\\
\partial_0T^{03}+\partial_1T^{13}+\partial_2T^{23}+\partial_3T^{33}=0
\end{array}$$

**Ricci and scalar curvature:** The Ricci curvature tensor can be computed from the derivative of the Christoffel symbols, by contraction of the Riemann curvature tensor on two indices. One obtains successively:

- the Riemann tensor: $R^l_{kij}=\partial_i\Gamma^l_{jk}-\partial_j\Gamma^l_{ik}+\Gamma^m_{jk}\Gamma^l_{im}-\Gamma^m_{ik}\Gamma^l_{jm}$
- the Ricci tensor: $R_{ij}=R^l_{ilj}$
- the scalar curvature: $R=g^{ij}R_{ij}$

Lengthy calculations lead to the expression of the Ricci tensor and the scalar curvature:

$$R_{00}=\frac{1}{2}aa_{11},R_{01}=R_{10}=-\frac{1}{2}ba_{11},R_{11}=-\frac{1}{2}ca_{11},R=-a_{11}$$

all the other elements being null. Note that the last equation indicates that there always is volume expansion or contraction. However, the Einstein tensor $G^{ij}=R^{ij}-\frac{1}{2}g^{ij}R$ is identically null, in accordance with the fact that acoustical waves are not subject to gravitation.

**Ray equation:** Sound rays simply follow the equation $dx^i g_{ij} dx^j = 0$. With the notations:

$$x' = \frac{dx^1}{dx^0},\ y' = \frac{dx^2}{dx^0},\ z' = \frac{dx^3}{dx^0}$$

it reduces to Monge equation:

$$2bx' + c(x')^2 + (y')^2 + (z')^2 = a$$

Ray curvature can then be computed form the generalized acceleration equation:

$$\frac{dv^i}{d\tau} = -\Gamma^i_{kl} v^k v^l$$

where $\tau$ is the proper time defined by $d\tau^2 = -ds^2 = -g_{ij} dx^i dx^j$ and the $v^i$ are defined by:

$$v^i = \frac{dx^i}{d\tau}$$

We can then write:

$$\begin{aligned}
\frac{dv^i}{d\tau} &= \frac{d}{d\tau}\frac{dx^i}{d\tau} = -\Gamma^i_{kl} v^k v^l \\
&= \frac{d}{d\tau}\left(\frac{dx^i}{dx^0}\frac{dx^0}{d\tau}\right) = \frac{d^2x^i}{(dx^0)^2}\left(\frac{dx^0}{d\tau}\right)^2 + \frac{dx^i}{dx^0}\frac{d^2x^0}{(d\tau)^2} \\
&= \frac{d^2x^i}{(dx^0)^2}(v^0)^2 - \frac{dx^i}{dx^0}\Gamma^0_{kl} v^k v^l
\end{aligned}$$

Thus:

$$(x^i)'' = \frac{d^2x^i}{(dx^0)^2} = \left(\frac{dx^i}{dx^0}\Gamma^0_{kl} - \Gamma^i_{kl}\right)\frac{v^k}{v^0}\frac{v^l}{v^0} \tag{40}$$

We then obtain the system:

$$\begin{aligned}
x'' &= x'\left[\Gamma^0_{00} + 2\Gamma^0_{01}x' + \Gamma^0_{11}(x')^2\right] - \left[\Gamma^1_{00} + 2\Gamma^1_{01}x' + \Gamma^1_{11}(x')^2\right] \\
&= \frac{1}{2}\left[ba_1 + 3ba_1x' + 3ca_1(x')^2 - (2cb_1 - bc_1)(x')^3\right] \qquad (41) \\
y'' &= y'\left[\Gamma^0_{00} + 2\Gamma^0_{01}x' + \Gamma^0_{11}(x')^2\right] \\
&= \frac{1}{2}\left[ba_1 + 2ca_1x' + -(2cb_1 - bc_1)(x')^2\right]y' \qquad (42) \\
z'' &= z'\left[\Gamma^0_{00} + 2\Gamma^0_{01}x' + \Gamma^0_{11}(x')^2\right] \\
&= \frac{1}{2}\left[ba_1 + 2ca_1x' + -(2cb_1 - bc_1)(x')^2\right]z' \qquad (43)
\end{aligned}$$

Since $a$ must increase away from the boundary and $b$ is negative, $ba_1$ is negative and a ray parallel to the absorbing plane ($x' = 0$) is curved down toward the plane, as experimentally observed by Janowsky and Spandöck [JS37]. On the other hand, rays with normal incidence ($y' = z' = 0$) are not deviated and remain normal to the surface.

## 8.2 Propagation above plane with varying absorption

We still consider the plane $x^1 = 0$ with the same boundary condition as in the previous case: $\partial_0\Phi\beta^0 + \partial_1\Phi\beta^1 = 0$, with $\frac{\beta^0}{\beta^1} = -\xi$, the real specific admittance on the absorbing plane. The metric tensor and its inverse are still given by Eq. (34):

$$g^{ij} = \begin{pmatrix} -c & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; g_{ij} = \begin{pmatrix} -a & b & 0 & 0 \\ b & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{44}$$

with $ac + b^2 = 1$, $a > 0$, and $b \leq 0$. But now, $a$, $b$, and $c$ only depend on the 3 space coordinates $x^1$, $x^2$, and $x^3$.

**Lagrangian:** The Lagrangian $L$ is not changed:

$$L = \frac{1}{2}\left[-c|\Phi_0|^2 + b(\Phi_0^*\Phi_1 + \Phi_0\Phi_1^*) + a|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2\right] \tag{45}$$

**Christoffel symbols:** New non-zero Christoffel symbols must now be considered, besides the ones given by Eq. (36):

$$\begin{aligned}
&\Gamma^0_{02} = \Gamma^0_{20} = \frac{1}{2}(ca_2 + bb_2), \Gamma^0_{03} = \Gamma^0_{30} = \frac{1}{2}(ca_3 + bb_3) \\
&\Gamma^0_{12} = \Gamma^0_{21} = \frac{1}{2}(bc_2 - cb_2), \Gamma^0_{13} = \Gamma^0_{31} = \frac{1}{2}(bc_3 - cb_3) \\
&\Gamma^1_{02} = \Gamma^1_{20} = -\frac{1}{2}(ba_2 - ab_2), \Gamma^1_{03} = \Gamma^1_{30} = -\frac{1}{2}(ba_3 - ab_3) \\
&\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}(bb_2 + ac_2), \Gamma^1_{13} = \Gamma^1_{31} = \frac{1}{2}(bb_3 + ac_3) \\
&\Gamma^2_{00} = \frac{1}{2}a_2, \Gamma^2_{01} = \Gamma^2_{10} = -\frac{1}{2}b_2, \Gamma^2_{11} = -\frac{1}{2}c_2 \\
&\Gamma^3_{00} = \frac{1}{2}a_3, \Gamma^3_{01} = \Gamma^3_{10} = -\frac{1}{2}b_3, \Gamma^3_{11} = -\frac{1}{2}c_3
\end{aligned} \tag{46}$$

**Stress-energy tensor:** The conservation of the stress-energy tensor $T^{ij}$ now becomes:

$$\begin{aligned}
&\partial_0 T^{00} + \partial_1 T^{10} + \partial_2 T^{20} + \partial_3 T^{30} + [2\Gamma^0_{00} + \Gamma^1_{10}]T^{00} + [3\Gamma^0_{01} + \Gamma^1_{11}]T^{10} + \Gamma^0_{11}T^{11} \\
&+ [3\Gamma^0_{02} + \Gamma^1_{12}]T^{02} + [3\Gamma^0_{03} + \Gamma^1_{13}]T^{03} + 2\Gamma^0_{12}T^{12} + 2\Gamma^0_{13}T^{1"} = 0 \\
&\partial_0 T^{01} + \partial_1 T^{11} + \partial_2 T^{21} + \partial_3 T^{31} + \Gamma^1_{00}T^{00} + [\Gamma^0_{00} + 3\Gamma^1_{10}]T^{01} + [\Gamma^0_{01} + 2\Gamma^1_{11}]T^{11} \\
&+ 2\Gamma^1_{02}T^{02} + 2\Gamma^1_{03}T^{03} + [\Gamma^0_{02} + 3\Gamma^1_{12}]T^{12} + [\Gamma^0_{03} + 3\Gamma^1_{13}]T^{13} = 0 \\
&\partial_0 T^{02} + \partial_1 T^{12} + \partial_2 T^{22} + \partial_3 T^{32} + \Gamma^2_{00}T^{00} + 2\Gamma^2_{01}T^{01} + \Gamma^2_{11}T^{11} \\
&+ \left[\Gamma^0_{00} + \Gamma^1_{10}\right]T^{02} + \left[\Gamma^0_{01} + \Gamma^1_{11}\right]T^{12} = 0 \\
&\partial_0 T^{03} + \partial_1 T^{13} + \partial_2 T^{23} + \partial_3 T^{33} + \Gamma^3_{00}T^{00} + 2\Gamma^3_{01}T^{01} + \Gamma^3_{11}T^{11} \\
&+ \left[\Gamma^0_{00} + \Gamma^1_{10}\right]T^{03} + \left[\Gamma^0_{01} + \Gamma^1_{11}\right]T^{13} = 0
\end{aligned}$$

that is:

$$\partial_0 T^{00} + \partial_1 T^{10} + \partial_2 T^{20} + \partial_3 T^{30} + \frac{1}{2} b a_1 T^{00} + c a_1 T^{01} - (c b_1 - \frac{1}{2} b c_1) T^{11}$$
$$+ (c a_2 + b b_2) T^{02} + (c a_3 + b b_3) T^{03} + (b c_2 - c b_2) T^{12} + (b c_3 - c b_3) T^{13} = 0$$
$$\partial_0 T^{01} + \partial_1 T^{11} + \partial_2 T^{21} + \partial_3 T^{31} + \frac{1}{2} b a_1 T^{00} - b a_1 T^{01} - \frac{1}{2} c a_1 T^{11}$$
$$+ (a b_2 - b a_2) T^{02} + (a b_3 - b a_3) T^{03} + (b b_2 + a c_2) T^{12} + (b b_3 + a c_3) T^{13} = 0$$
$$\partial_0 T^{02} + \partial_1 T^{12} + \partial_2 T^{22} + \partial_3 T^{32} + \frac{1}{2} a_2 T^{00} - b_2 T^{01} - \frac{1}{2} c_2 T^{11} = 0$$
$$\partial_0 T^{03} + \partial_1 T^{13} + \partial_2 T^{23} + \partial_3 T^{33} + \frac{1}{2} a_3 T^{00} - b_3 T^{01} - \frac{1}{2} c_3 T^{11} = 0$$

# 9 Conclusion

Besides deriving the wave equation and the corresponding energy conservation in generalized coordinates, the present paper has proved the possibility to construct metric tensors $g^{ij}$ that satisfy the boundary condition $g^{ji} n_i = \beta^i$, with $\beta^i$ real, on the border $S$. However, the initial hypothesis that an analogue to Einstein equation holds in acoustics cannot be maintained. Even though the corresponding Ricci curvature tensor $R^{ij}$ satisfies the relation

$$\nabla_i \left[ R^{ij} - \frac{1}{2} g^{ij} R \right] = 0$$

with $R$ the scalar curvature, the stress-energy tensor $T^{ij}$ cannot be proportional to the Einstein tensor $G^{ij} = R^{ij} - \frac{1}{2} g^{ij} R$, as the metric tensor must be time invariant, and is in fact null in the present applications, as acoustical waves are not subject to gravitation. On the other hand, the stress-energy tensor is decaying with time and cannot be null. The solution of the generalized wave equation must therefore be numerically computed.

Beyond the falsification of the initial hypothesis, some new results have been obtained. Firstly, a generalized equation for energy conservation has been derived, that can now be numerically solved along the line of our previous papers [DPTP17, DPTP18]. Secondly, the thickness of "adaptation layer" between the surface admittance and the free space above is a free parameter in the theory; it can thus be adjusted to the case at hand, for example to a quarter wavelength as usually hypothesized in room acoustics. Thirdly, ray curvature above finite impedance surfaces has been proved, and a proper explanation to the experimental finding of Janowsky and Spandöck [JS37] can now be given, or to the computational artifice, amounting to multiple reflections of the rays on the boundaries, currently used in Boudary Element Methods (BEM) to compute propagation above flat lossy boundaries of varying admittance [Ras82]. Such results provide a validity check of the present generalized formulation for acoustics.

But totally unexpected is the discovery that reactive intensity and reactive wave stress satisfy Maxwell equations. It makes it possible to define some sort of "second order" stress-energy tensor that describes the conservation of the standing waves within the manifold where the acoustic field takes place. This discovery certainly deserves further investigation.

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