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Abstract

In this paper we establish the existence of solutions to a time-dependent problem for a linearly elastic body subjected to a confinement condition, expressing that all the points of the deformed reference configuration remain confined in a prescribed half space. This problem takes the form of a set of hyperbolic variational inequalities. The fact that any solution of the studied problem takes the form of a vector field instead of a real-valued function, the generality of the confinement condition under consideration, the fact that the integration domain is a subset of $\mathbb{R}^3$, and the choice of the function space where solutions are sought make the analysis substantially more complicated, thus requiring the adoption of new resolution strategies.

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1. Introduction

Hyperbolic models are used to describe many phenomena arising in classical mechanics like, for instance, the vibration of a string under the action of an

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external force. In this paper, we study the existence of solutions to an obstacle problem modelling the displacement of a three-dimensional linearly elastic body confined in a half space.

Obstacle problems arise in many applicative fields: For instance, the motion of three valves of the Aorta, that can be regarded as linearly elastic shells (cf., e.g., [1]), is governed by a mathematical model built up in a way such that each valve remains confined in a certain portion of space without colliding with the remaining two valves.

A substantial contribution to the theory of hyperbolic obstacle problems can be found in the seminal papers [2] and [3]. Other important contributions in this field can be found in the references [4], [5], [6], and [7], where the problems are set out as follows: the integration domain ω is a subset of $\mathbb{R}^2$, and the unknown function, at almost all time instants, is a real-valued function that belongs to $H_0^2(\omega)$. It is also worth mentioning the papers [8], [9], [10], [11], [12], [13], [14] and [15]. The nonlinear analysis tools used in this paper can be found in the recent monograph [16].

The first main novelty of this paper is that the unknown, represented by displacement of the linearly elastic body under consideration, is a vector field that, at almost all time instants, belongs to a nonempty, closed , and convex subset of the Sobolev space $H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$, where $\Omega \subset \mathbb{R}^3$ is a *domain*. This will require the implementation of a more general argument to recover the energy estimates in the Galerkin method.

The second main novelty is given by the generality of the confinement condition under consideration, which comprises *at once* all of the three components of the displacement vector field. Such a confinement conditions were first considered in the papers [17], [18], [19], and [20]. Other types of confinement conditions which are more amenable in the context of numerical simulations are discussed in the paper [21].

Finally, the method we propose here for recovering the initial condition for the first derivative in time of the displacement slightly differs from the one used in [4], [5], [6], and [7].

The paper is organised as follows. First, some notations and background are provided. Secondly, the main existence theorem for a dynamic linearly elastic body confined in a half space is established. Thirdly, and finally, some final comments about the uniqueness of the solution are made.

2. Geometrical preliminaries

For details about the classical notions of differential geometry recalled in this section, see, e.g. [1] or [22].

Latin indices, except when they are used for indexing sequences, take their values in the set $\{1, 2, 3\}$, and the summation convention with respect to repeated indices is systematically used in conjunction with this rule.

Given an open subset $\Omega \subset \mathbb{R}^3$, notations such as $L^2(\Omega)$ and $H^1(\Omega)$ denote the standard Lebesgue and Sobolev spaces. The notation $\mathcal{D}(\Omega)$ designates the space of functions that are infinitely differentiable over Ω and have a compact support in Ω. The notation $\|\cdot\|_X$ designates the norm of a vector space X. Spaces of vector-valued functions are denoted by boldface letters. Lebesgue-Bochner spaces (see, e.g., [23]) are designated by the notation $L^p(0, T; X)$, where $1 \le p \le \infty$, $T > 0$, and X is a Banach space satisfying the Radon-Nikodym property. The notation $\mathcal{M}([0, T]; X)$ designates the space of X-valued measures defined over the compact interval $[0, T]$ (see, e.g., [24] and [25]). The notation X^* designates the dual space of a vector space X and the notation ${}_{X^*}\langle \cdot, \cdot \rangle_X$ denotes the duality pair between X^* and X. The notation $\mathcal{C}^0([0, T]; X)$ denotes the space of X-valued continuous functions defined over the compact interval $[0, T]$ and the special notation $\langle\langle \cdot, \cdot \rangle\rangle_X$ denotes the duality pair between $(\mathcal{C}^0([0, T]; X))^*$ and $\mathcal{C}^0([0, T]; X)$. The notations $\dot{\eta}$ and $\ddot{\eta}$ denote the first weak derivative with respect to $t \in (0, T)$ and the second weak derivative with respect to $t \in (0, T)$ of a scalar function η defined over the interval $(0, T)$. The notations $\dot{\boldsymbol{\eta}}$ and $\ddot{\boldsymbol{\eta}}$ denote the first weak derivative with respect to $t \in (0, T)$ and the second weak derivative with respect to $t \in (0, T)$ of a vector field $\boldsymbol{\eta}$ defined over the interval $(0, T)$. A *domain* $\Omega \subset \mathbb{R}^3$ is a nonepmpty, open, bounded and connected subset with Lipschitz continuous boundary Γ, the set Ω being locally on the same side of Γ. The notation $\mathrm{d}x$ designates the *volume element* in Ω, the symbol $\mathrm{d}\Gamma$ designates the *area element* along Γ. Finally, let Γ_0 and Γ_1 be a $\mathrm{d}\Gamma$-measurable portion of the boundary such that $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma = \Gamma_0 \cup \Gamma_1$, and area $\Gamma_0 > 0$.

As a model of the three-dimensional "physical" space $\mathbb{R}^3$, we take a *real three-dimensional affine Euclidean space*, i.e., a set in which a point O has been chosen as the *origin* and with which a *real three-dimensional Euclidean space*, denoted $\mathbb{E}^3$, is associated. We equip $\mathbb{E}^3$ with an *orthonormal basis* consisting of three vectors $\boldsymbol{e}^i$. The Euclidean inner product of two elements $\boldsymbol{a}$ and $\boldsymbol{b}$ of $\mathbb{E}^3$ is denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$; the Euclidean norm of any $\boldsymbol{a} \in \mathbb{E}^3$ is denoted by $|\boldsymbol{a}|$; the Kronecker symbol is denoted by δ^{ij}.

The definition of $\mathbb{R}^3$ as an affine Euclidean space means that with any point $x \in \mathbb{R}^3$ is associated an uniquely defined vector $\boldsymbol{Ox} \in \mathbb{E}^3$. The origin $O \in \mathbb{R}^3$ and the orthonormal vectors $\boldsymbol{e}^i \in \mathbb{E}^3$ together constitute a *Cartesian frame* in $\mathbb{R}^3$ and the three components x_i of the vector $\boldsymbol{Ox}$ over the basis formed by $\boldsymbol{e}^i$ are called *Cartesian coordinates* of $x \in \mathbb{R}^3$, or the *Cartesian components* of $\boldsymbol{Ox} \in \mathbb{E}^3$. Once a Cartesian frame has been chosen, any point $x \in \mathbb{R}^3$ may be thus *identified* with the vector $\boldsymbol{Ox} = x_i \boldsymbol{e}^i \in \mathbb{E}^3$. We then denote $\partial_i = \partial/\partial x_i$.

The set $\overline{\Omega}$ is the *reference configuration* occupied by a *linearly elastic elastic body* in absence of applied body forces. We assume that $\overline{\Omega}$ is a natural state, i.e., that the body is stress-free in this configuration. We also assume, following [26], that the constituting material is *isotropic*, *homogeneous*, and *linearly elastic*. Under these assumptions, the behaviour of the linearly elastic material is governed by its two *Lamé constants* $\lambda \geq 0$ and $\mu > 0$. The positive constant ρ designates the *mass density* of the linearly elastic body per unit volume.

We also assume that the linearly elastic body to be subjected to *applied body forces* in its interior, whose density per unit volume is defined by means of its contravariant components $f^i \in L^\infty(0, T; L^2(\Omega))$ over the vectors $\boldsymbol{e}^i$.

In what follows, "a.e." stands for "almost everywhere". Define the space

$$\boldsymbol{V}(\Omega) := \{\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega); \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0\},$$

and equip it with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{V}(\Omega)} := \left(\sum_i \|v_i\|^2_{H^1(\Omega)} \right)^{1/2}.$$

Next, we define the three-dimensional elasticity tensor in Cartesian coordinates and we denote its components by A^{ijkl}. We recall that the contravariant components of this tensor are defined by (see, e.g., [26])

$$A^{ijkl} := \lambda \delta^{ij} \delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}),$$

and that $A^{ijkl} = A^{jikl} = A^{klij} \in \mathcal{C}^1(\overline{\Omega})$.

For each $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ we consider the *linearised change of metric tensor* $\boldsymbol{e}(\boldsymbol{v})$, whose components $e_{i\|j}(\boldsymbol{v})$ are defined by

$$e_{i\|j}(\boldsymbol{v}) := \frac{1}{2}(\partial_j v_i + \partial_i v_j) \in L^2(\Omega).$$

This tensor is symmetric, i.e., $e_{i\|j}(\boldsymbol{v}) = e_{j\|i}(\boldsymbol{v})$, for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$. Likewise, we can define the *time-dependent version of the linearised change of metric tensor* by considering the operator

$$\tilde{e}_{i\|j} : L^2(0,T;\boldsymbol{H}^1(\Omega)) \to L^2(0,T;L^2(\Omega))$$

defined by

$$\tilde{e}_{i\|j}(\boldsymbol{v})(t) = e_{i\|j}(\boldsymbol{v}(t)), \text{ for all } \boldsymbol{v} \in L^2(0,T;\boldsymbol{H}^1(\Omega)),$$

for almost all ("a.a." in what follows) $t \in (0,T)$. It is easy to see that such an operator is well-defined, linear and continuous. It can be easily verified (cf., e.g., [27]) that the continuity constant is independent of $t \in (0,T)$.

To begin with, we state *Korn's inequality in Cartesian coordinates* (see, e.g., Theorem 6.3-6 of [26]).

Theorem 1. *Let Ω be a domain in $\mathbb{R}^3$ and let Γ_0 be a nonzero area subset of the whole boundary Γ. Then, there exists a constant $C > 0$ such that*

$$C^{-1}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq \|\boldsymbol{e}(\boldsymbol{v})\|_{\boldsymbol{L}^2(\Omega)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)},$$

for all $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$. $\square$

Various proofs have been given of this delicate inequality; see in particular [28], [29], [30], [31], page 110 of [32], Sect. 6.3 of [33]; in [34], Korn's inequality is proved in the space $W^{1,p}(\Omega)$, $1 \leq p \leq \infty$; an elementary proof is given in [35] (see also Appendix (A) in [36]).

3. A natural formulation of the time-dependent obstacle problem for a linearly elastic body

In this paper, we consider a specific obstacle problem for a linearly elastic body subjected to a confinement condition, expressing that any admissible displacement vector field $v_i \boldsymbol{e}^i$, must be such that all the points of the corresponding deformed configuration remain in a half-space of the form

$$\mathbb{H} := \{x \in \mathbb{R}^3; \boldsymbol{Ox} \cdot \boldsymbol{q} \geq 0\},$$

where $\boldsymbol{q}$ is a *nonzero vector* given once and for all. Let us denote by $\boldsymbol{I}$ the identity mapping $\boldsymbol{I} : \overline{\Omega} \to \mathbb{E}^3$ and let us assume that the *undeformed* reference configuration satisfies

$$\boldsymbol{I}(x) \cdot \boldsymbol{q} > 0, \text{ for all } x \in \overline{\Omega},$$

or, in other words, *there is no contact between the obstacle and the reference configuration when no applied body forces are acting on the reference configuration.* Let us observe that this condition is assumed only for physical reasons and that it is not exploited in the forthcoming proofs.

The general confinement condition can be thus formulated as follows: any *admissible displacement vector field* must satisfy

$$(\boldsymbol{I}(x) + v_i(x)\boldsymbol{e}^i) \cdot \boldsymbol{q} \geq 0,$$

for all $x \in \overline{\Omega}$ or, possibly, only for a.a. $x \in \Omega$ when the covariant components v_i are required to belong to the Sobolev space $H^1(\Omega)$. The subset $\boldsymbol{U}(\Omega)$ of admissible displacements thus takes the form

$$\boldsymbol{U}(\Omega) := \{\boldsymbol{v} = (v^i) \in \boldsymbol{H}^1(\Omega); \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0 \\ \text{and } (\boldsymbol{I} + v_i\boldsymbol{e}^i) \cdot \boldsymbol{q} \geq 0 \text{ a.e. in } \Omega\}.$$

We emphasise that the *vectorial* confinement condition above, which was originally considered in [17], [18], [19] and [20], considerably departs from the *scalar* conditions favoured by many authors (see, e.g., [4], [5] and [7]). Such a confinement condition renders the analysis substantially more difficult, as the constraint now bears on a vector field, the displacement vector field of the reference configuration, instead of on only a single scalar-valued function.

A natural formulation of the corresponding time-dependent obstacle problem takes the form of a set of hyperbolic three-dimensional variational inequalities ("three-dimensional", in the sense that they are posed over the three-dimensional subset Ω), which can be derived by slightly modifying the model proposed by Xiao in the papers [37], [38] and [39].

Let us introduce the problem $\mathcal{P}(\Omega)$, which constitutes the point of departure of our analysis.

Problem $\mathcal{P}(\Omega)$**.** *Find* $\boldsymbol{u} = (u_i) : (0,T) \to \boldsymbol{V}(\Omega)$ *such that*

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{U}(\Omega)),\\
&\dot{\boldsymbol{u}} \in L^\infty(0,T;\boldsymbol{L}^2(\Omega)),\\
&\ddot{\boldsymbol{u}} \in \mathcal{M}([0,T];\boldsymbol{L}^2(\Omega)),
\end{aligned}$$

that satisfies the following variational inequalities

$$\begin{aligned}
&2\rho \left\langle\left\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t)\right\rangle\right\rangle_{\boldsymbol{L}^2(\Omega)}\\
&\quad + \int_0^T \int_\Omega A^{ijkl} e_{k\|l}(\boldsymbol{u}(t)) e_{i\|j}(\boldsymbol{v}(t) - \boldsymbol{u}(t)) \,\mathrm{d}x \,\mathrm{d}t\\
&\geq \int_0^T \int_\Omega f^i(t)(v_i(t) - u_i(t)) \,\mathrm{d}x \,\mathrm{d}t,
\end{aligned}$$

for all $\boldsymbol{v} \in \mathcal{D}(0,T;\boldsymbol{V}(\Omega))$ *such that* $\boldsymbol{v}(t) \in \boldsymbol{U}(\Omega)$ *for a.a.* $t \in (0,T)$*, and that satisfies the following initial conditions*

$$\begin{cases}
\boldsymbol{u}(0) = \boldsymbol{u}_0,\\
\dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1,
\end{cases} \tag{1}$$

where $\boldsymbol{u}_0 = (u_{i,0}) \in \boldsymbol{U}(\Omega)$*, and* $\boldsymbol{u}_1 = (u_{i,1}) \in \boldsymbol{L}^2(\Omega)$ *are prescribed.* ■

Observe that the "acceleration term" in Problem $\mathcal{P}(\Omega)$ is described in terms of a vector-valued measure. Note in passing that the concept of solution of Problem $\mathcal{P}(\Omega)$ is inspired by the one given on page 403 of [40]. The concept of *solution of Problem* $\mathcal{P}(\Omega)$ will be thoroughly explained in the proof of Theorem 6, which constitutes the main result of this paper.

We recall a very important inequality which is used to study evolutionary problems: Gronwall's inequality (see, e.g., the seminal paper [41] and Theorem 1.1 in Chapter III of [42]).

Theorem 2. *Let $T > 0$ and suppose that the function $y : [0, T] \to \mathbb{R}$ is absolutely continuous and such that*

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) \le a(t)y(t) + b(t), \text{ a.e. in } (0, T),$$

where $a, b \in L^1(0, T)$ and $a, b \ge 0$ a.e. in $(0, T)$. Then, it results

$$y(t) \le \left[y(0) + \int_0^t b(s)\,\mathrm{d}s\right] e^{\int_0^t a(s)\,\mathrm{d}s}, \text{ for all } t \in [0, T].$$

$\square$

4. Proof of existence of solutions to Problem $\mathcal{P}(\Omega)$

Let us recall a compactness result proved by Simons (see, e.g., Corollary 4 of [43]), which will be used in what follows to recover the initial conditions. In what follows, the symbol "$\hookrightarrow$" denotes a *continuous embedding*, whereas the symbol "$\hookrightarrow\hookrightarrow$" denotes a *compact embedding*.

Theorem 3. *Let $T > 0$ and let X, Y and Z be three Banach spaces such that*

$$X \hookrightarrow\hookrightarrow Y \hookrightarrow Z.$$

Let $(f_n)_{n=1}^\infty$ be a bounded sequence in $L^\infty(0, T; X)$ and assume that the sequence $(\dot{f}_n)_{n=1}^\infty$ is bounded in $L^\infty(0, T; Z)$. Then, there exists a subsequence, still denoted $(f_n)_{n=1}^\infty$, that converges in the space $\mathcal{C}^0([0, T]; Y)$. $\square$

In what follows we identify the spaces $L^2(\Omega)$ and $\boldsymbol{L}^2(\Omega)$ with their respective dual spaces, and we equip them with the following inner products

$$(v, w) \in L^2(\Omega) \times L^2(\Omega) \to \int_\Omega vw \, \mathrm{d}x,$$
$$(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega) \to \int_\Omega v_i w_i \, \mathrm{d}x.$$

We observe that the following chain of immersions holds

$$\boldsymbol{V}(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{L}^2(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{V}^*(\Omega),$$

viz., following the notation of [44], $\big(\boldsymbol{V}(\Omega), \boldsymbol{L}^2(\Omega), \boldsymbol{V}^*(\Omega)\big)$ is an *evolution triple* (or *Gelfand triple*).

Let us also recall a result on vector-valued measures proved by Zinger in the paper [45] (see also, e.g., page 182 of [24], and page 380 of [25]).

Theorem 4. *Let ω be a compact Hausdorff space and let X be a Banach space satisfying the Radon-Nikodym property. Let $\mathcal{F}$ be the collection of Borel sets of ω.*

There exists an isomorphism between $(\mathcal{C}^0(\omega; X))^*$ and the space of the regular Borel measures with finite variation taking values in X^*. In particular, for each $F \in (\mathcal{C}^0(\omega; X))^*$, there exists a unique regular Borel measure $\mu : \mathcal{F} \to X^*$ in $\mathcal{M}(\omega; X^*)$ with finite variation such that

$$\langle\langle \alpha, F \rangle\rangle_X = \int_\omega {}_{X^*}\langle \mathrm{d}\mu, \alpha \rangle_X,$$

for all $\alpha \in \mathcal{C}^0(\omega; X)$. □

Let us denote the *penalty parameter* by κ and let us introduce the corresponding *penalised problem* $\mathcal{P}(\kappa; \Omega)$.

Problem $\mathcal{P}(\kappa; \Omega)$. *Find $\boldsymbol{u}_\kappa = (u_{i,\kappa}) : (0, T) \to \boldsymbol{V}(\Omega)$ such that*

$$\boldsymbol{u}_\kappa \in L^\infty(0, T; \boldsymbol{V}(\Omega)),$$
$$\dot{\boldsymbol{u}}_\kappa \in L^\infty(0, T; \boldsymbol{L}^2(\Omega)),$$
$$\ddot{\boldsymbol{u}}_\kappa \in L^\infty(0, T; \boldsymbol{V}^*(\Omega)),$$

that satisifes the following nonlinear variational equations

$$
\begin{aligned}
&2\rho\,{}_{\boldsymbol{V}^*(\Omega)}\left\langle \ddot{u}_{i,\kappa}(t)\boldsymbol{e}^i, v_j\boldsymbol{e}^j\right\rangle_{\boldsymbol{V}(\Omega)}\\
&\quad+\int_\Omega A^{ijkl}e_{k\|l}(\boldsymbol{u}_\kappa(t))e_{i\|j}(\boldsymbol{v})\,\mathrm{d}x\\
&\quad-\frac{1}{\kappa}\int_\Omega \left(\{[\boldsymbol{I}+u_{i,\kappa}(t)\boldsymbol{e}^i]\cdot\boldsymbol{q}\}^-\right)(v_i\boldsymbol{e}^i\cdot\boldsymbol{q})\,\mathrm{d}x\\
&=\int_\Omega f^i(t)v_i\,\mathrm{d}x,
\end{aligned}
$$

for all $\boldsymbol{v}\in\boldsymbol{V}(\Omega)$*, in the sense of distributions in* $(0,T)$*, and that satisfies the initial conditions* (1)*.* ■

We first prove, by Galerkin method, that Problem $\mathcal{P}(\kappa;\Omega)$ admits a solution.

Theorem 5. *For each* $\kappa>0$*, Problem* $\mathcal{P}(\kappa;\Omega)$ *admits a solution.*

Proof. The proof is carried out via a Galerkin argument and is subdivided into three steps. To begin with, we fix $\kappa>0$.

(a) *Construction of a Galerkin approximation.* In order to construct such a scheme, we rely on the fact that $\boldsymbol{V}(\Omega)$ is an infinite dimensional separable Hilbert space which is also dense in $\boldsymbol{L}^2(\Omega)$, in order to infer the existence of an orthogonal basis $(\boldsymbol{w}^k)_{k=1}^\infty$ of the space $\boldsymbol{V}(\Omega)$, whose elements also constitute a Hilbert basis of the space $\boldsymbol{L}^2(\Omega)$.

The existence of such a basis is assured by the spectral theorem (Theorem 6.2-1 of [46]). For each positive integer $m\geq 1$, we denote by $\boldsymbol{E}^m$ the following m-dimensional linear hull

$$\boldsymbol{E}^m := \text{Span }(\boldsymbol{w}^k)_{k=1}^m \subset \boldsymbol{V}(\Omega)\subset \boldsymbol{L}^2(\Omega).$$

Since each element of this Hilbert basis is independent of the variable t, we have that $\boldsymbol{w}^k\in L^\infty(0,T;\boldsymbol{V}(\Omega))$ for each integer $1\leq k\leq m$.

We now discretise Problem $\mathcal{P}(\kappa;\Omega)$ and, in order to keep the notation simple, we drop the dependence of the vector fields entering the variational equations on the penalty parameter κ. Observe that the duality pair between $\boldsymbol{E}^m$ and its dual coincides with the inner product of $\boldsymbol{L}^2(\Omega)$ introduced beforehand.

Problem $\mathcal{P}_m(\kappa;\Omega)$**.** *Find functions* $c_k : [0,T] \to \mathbb{R}$, $1 \le k \le m$, *such that*

$$\boldsymbol{u}^m(t) := \sum_{k=1}^{m} c_k(t)\boldsymbol{w}^k, \ \textit{for a.a. } t \in (0,T),$$

and satisfying the following penalised variational equations *a.e. in* $(0,T)$

$$\begin{aligned}
&2\rho \int_\Omega \ddot{u}_i^m(t) w_i^p \,\mathrm{d}x + \int_\Omega A^{ijkl} e_{k\|l}(\boldsymbol{u}^m(t)) e_{i\|j}(\boldsymbol{w}^p)\,\mathrm{d}x \\
&\quad - \frac{1}{\kappa}\int_\Omega \left(\{[\boldsymbol{I} + u_i^m(t)\boldsymbol{e}^i]\cdot \boldsymbol{q}\}^-\right)(w_i^p \boldsymbol{e}^i \cdot \boldsymbol{q})\,\mathrm{d}x \\
&= \int_\Omega f^i(t) w_i^p \,\mathrm{d}x,
\end{aligned}$$

for each integer $1 \le p \le m$.

Such a function $\boldsymbol{u}^m$ *must satisfy, in addition, the following initial conditions*

$$\begin{aligned}
\boldsymbol{u}^m(0) &= \boldsymbol{u}_0^m, \\
\dot{\boldsymbol{u}}^m(0) &= \boldsymbol{u}_1^m, \\
\boldsymbol{u}_0^m \in \boldsymbol{V}(\Omega) \quad &\textit{and} \quad \boldsymbol{u}_0^m \to \boldsymbol{u}_0 \ \textit{in } \boldsymbol{V}(\Omega) \ \textit{as } m \to \infty, \\
\boldsymbol{u}_1^m \in \boldsymbol{L}^2(\Omega) \quad &\textit{and} \quad \boldsymbol{u}_1^m \to \boldsymbol{u}_1 \ \textit{in } \boldsymbol{L}^2(\Omega) \ \textit{as } m \to \infty,
\end{aligned} \tag{2}$$

where the initial data $\boldsymbol{u}_0^m$ *and* $\boldsymbol{u}_1^m$ *are, respectively, the projections of* $\boldsymbol{u}_0$ *and* $\boldsymbol{u}_1$ *onto the finite dimensional space* $\boldsymbol{E}^m$. ■

We immediately observe that the projections of $\boldsymbol{u}_0 = (u_{i,0})$ and $\boldsymbol{u}_1 = (u_{i,1})$ onto $\boldsymbol{E}^m$ can be expanded as follows (cf., e.g., Theorem 4.9-1 of [47])

$$\begin{aligned}
\boldsymbol{u}_0^m &= \sum_{k=1}^{m}\left(\int_\Omega u_{i,0} w_i^k \,\mathrm{d}x + \int_\Omega \partial_j u_{i,0} \partial_j w_i^k \,\mathrm{d}x\right)\boldsymbol{w}^k, \\
\boldsymbol{u}_1^m &= \sum_{k=1}^{m}\left(\int_\Omega u_{i,1} w_i^k \,\mathrm{d}x\right)\boldsymbol{w}^k.
\end{aligned}$$

Since the elements of the Hilbert basis do not depend on the time variable we can take the coefficients c_k as well as their derivatives outside the integral sign, getting a $m \times m$ nonlinear system of second order ordinary differential equations with respect to the variable t. Such a system can be rewritten in the form

$$
\begin{aligned}
2\rho\ddot{\boldsymbol{C}}(t) = &\left(-\int_\Omega A^{ijkl} e_{k\|l}(\boldsymbol{w}^r) e_{i\|j}(\boldsymbol{w}^p)\,\mathrm{d}x\right)_{p,r=1}^m \boldsymbol{C}(t)\\
&+\frac{1}{\kappa}\left(\int_\Omega \left(\{[\boldsymbol{I}+(\boldsymbol{C}(t)\cdot(w_i^1\dots w_i^m))\boldsymbol{e}^i]\cdot\boldsymbol{q}\}^-\right)(w_i^p\boldsymbol{e}^i\cdot\boldsymbol{q})\,\mathrm{d}x\right)_{p=1}^m \\
&+\left(\int_\Omega f^i(t) w_i^p\,\mathrm{d}x\right)_{p=1}^m
\end{aligned}
\tag{3}
$$

where $\boldsymbol{C}(t) := (c_1(t)\dots c_m(t))$, and satisfies the following initial conditions

$$
\begin{aligned}
c_k(0) &= \int_\Omega u_{i,0} w_i^k\,\mathrm{d}x + \int_\Omega \partial_j u_{i,0}\partial_j w_i^k\,\mathrm{d}x,\\
\dot{c}_k(0) &= \int_\Omega u_{i,1} w_i^k\,\mathrm{d}x.
\end{aligned}
$$

Observe that the *negative part operator* is a Lipschitz continuous function, i.e.,

$$
|b^- - a^-| \le |b-a|, \qquad \text{for all } a,b\in\mathbb{R}. \tag{4}
$$

By the Cauchy-Lipschitz theorem (cf., e.g., Theorem 3.8-1 of [47]), we deduce that for each integer $m\ge 1$ there exists a unique global solution $\boldsymbol{u}^m$ to Problem $\mathcal{P}_m(\kappa;\Omega)$, defined a.e. over the interval $(0,T)$, such that

$$
\begin{aligned}
\boldsymbol{u}^m &\in L^\infty(0,T;\boldsymbol{E}^m),\\
\dot{\boldsymbol{u}}^m &\in L^\infty(0,T;\boldsymbol{E}^m),\\
\ddot{\boldsymbol{u}}^m &\in L^\infty(0,T;\boldsymbol{E}^m).
\end{aligned}
\tag{5}
$$

(*b*) *Energy estimates for the Galerkin scheme.* Let us multiply the variational equations in Problem $\mathcal{P}_m(\kappa;\Omega)$ by $\dot{c}_k(t)$, with $0<t<T$, and sum with respect to k varying in the discrete set $\{1,\dots,m\}$. As a result, we obtain that the penalised variational equations in Problem $\mathcal{P}_m(\kappa;\Omega)$ take the form

$$
\begin{aligned}
&\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \dot{u}_i^m(t)\dot{u}_i^m(t)\,\mathrm{d}x\\
&\quad+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega A^{ijkl} e_{k\|l}(\boldsymbol{u}^m(t)) e_{i\|j}(\boldsymbol{u}^m(t))\,\mathrm{d}x\\
&\quad+\frac{1}{2\kappa}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \left(\{[\boldsymbol{I}+u_i^m(t)\boldsymbol{e}^i]\cdot\boldsymbol{q}\}^-\right)^2\,\mathrm{d}x\right)\\
&=\int_\Omega f^i(t)\dot{u}_i^m(t)\,\mathrm{d}x,
\end{aligned}
\tag{6}
$$

for a.a. $t \in (0,T)$.

Observe that the differentiation of the negative part is obtained as a result of the same computational steps as in Stampacchia's theorem (cf., e.g., [48]), together with an application of Theorem 8.28 of [23]. The change in sign of the penalty term is due to the properties of the *Heavyside function*.

Carrying out an integration over the interval $(0,t)$, where $0 < t \leq T$, changes (6) into

$$\begin{aligned}
&\rho \int_\Omega \dot{u}_i^m(t)\dot{u}_i^m(t)\,\mathrm{d}x + \frac{1}{2}\int_\Omega A^{ijkl} e_{k\|l}(\boldsymbol{u}^m(t)) e_{i\|j}(\boldsymbol{u}^m(t))\,\mathrm{d}x \\
&\quad + \frac{1}{2\kappa}\int_\Omega \left(\left\{[\boldsymbol{I} + u_i^m(t)\boldsymbol{e}^i]\cdot \boldsymbol{q}\right\}^-\right)^2 \mathrm{d}x \\
&= \rho \int_\Omega u_{i,1}^m u_{i,1}^m \,\mathrm{d}x + \frac{1}{2}\int_\Omega A^{ijkl} e_{k\|l}(\boldsymbol{u}_0^m) e_{i\|j}(\boldsymbol{u}_0^m)\,\mathrm{d}x \\
&\quad + \frac{1}{2\kappa}\int_\Omega \left(\left\{[\boldsymbol{I} + u_i^m(0)\boldsymbol{e}^i]\cdot \boldsymbol{q}\right\}^-\right)^2 \mathrm{d}x \\
&\quad + \int_0^t \int_\Omega f^i(\tau)\dot{u}_i^m(\tau)\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned} \tag{7}$$

By Cauchy-Schwarz inequality, there exists a constant $C > 0$ such that

$$\begin{aligned}
&\int_0^t \int_\Omega f^i(\tau)\dot{u}_i^m(\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leq \left(\int_0^T \|\boldsymbol{f}(t)\|^2_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\right)^{1/2} \left(\int_0^t \|\dot{\boldsymbol{u}}^m(\tau)\|^2_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}\tau\right)^{1/2} \\
&\leq \frac{1}{2}\left(\int_0^T \|\boldsymbol{f}(t)\|^2_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t + \int_0^t \|\dot{\boldsymbol{u}}^m(\tau)\|^2_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}\tau\right).
\end{aligned} \tag{8}$$

Since $\boldsymbol{u}_0 \in \boldsymbol{U}(\Omega)$, we have that

$$\int_\Omega \left(\left\{[\boldsymbol{I} + u_i^m(0)\boldsymbol{e}^i]\cdot \boldsymbol{q}\right\}^-\right)^2 \mathrm{d}x \to 0, \quad \text{as } m \to \infty.$$

As a result, there exists a positive integer $m(\kappa)$ such that

$$\int_\Omega \left(\left\{[\boldsymbol{I} + u_i^m(0)\boldsymbol{e}^i]\cdot \boldsymbol{q}\right\}^-\right)^2 \mathrm{d}x \leq \kappa, \quad \text{for all } m \geq m(\kappa). \tag{9}$$

By the uniform positive-definiteness of the elasticity tensor (A^{ijkl}), Korn's inequality (Theorem 1), (7), (8), and (9), we obtain that there exists a real constant $\tilde{C} > 0$ independent of $\boldsymbol{u}^m$ (and so independent of t, m and κ) for which the following estimate holds for all $m \geq m(\kappa)$

$$
\begin{aligned}
&\frac{1}{\tilde{C}} \left\{ \|\dot{\boldsymbol{u}}^m(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}^m(t)\|^2_{\boldsymbol{V}(\Omega)} \right\} \\
&\quad + \frac{1}{\kappa \tilde{C}} \left\| \left\{ [\boldsymbol{I} + u_i^m(t)\boldsymbol{e}^i] \cdot \boldsymbol{q} \right\}^- \right\|^2_{L^2(\Omega)} \\
&\leq \|\boldsymbol{u}_1\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}_0\|^2_{\boldsymbol{V}(\Omega)} + \|\boldsymbol{f}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \\
&\quad + \int_0^t \left\{ \|\dot{\boldsymbol{u}}^m(\tau)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}^m(\tau)\|^2_{\boldsymbol{V}(\Omega)} \right\} \mathrm{d}\tau \\
&\quad + \frac{1}{\kappa} \int_0^t \left\| \left\{ [\boldsymbol{I} + u_i^m(\tau)\boldsymbol{e}^i] \cdot \boldsymbol{q} \right\}^- \right\|^2_{L^2(\Omega)} \mathrm{d}\tau.
\end{aligned} \tag{10}
$$

An application of the Gronwall's inequality (Theorem 2) with $a \equiv \tilde{C} > 0$ and

$$
b \equiv \tilde{C} \left(\|\boldsymbol{u}_1\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}_0\|^2_{\boldsymbol{V}(\Omega)} + \|\boldsymbol{f}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \right) \geq 0
$$

gives the following upper bound

$$
\begin{aligned}
&\int_0^t \left\{ \|\dot{\boldsymbol{u}}^m(\tau)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}^m(\tau)\|^2_{\boldsymbol{V}(\Omega)} \right\} \mathrm{d}\tau \\
&\quad + \frac{1}{\kappa} \int_0^t \left\| \left\{ [\boldsymbol{I} + u_i^m(\tau)\boldsymbol{e}^i] \cdot \boldsymbol{q} \right\}^- \right\|^2_{L^2(\Omega)} \mathrm{d}\tau \\
&\leq \tilde{C} T e^{\tilde{C} T} \left\{ \|\boldsymbol{u}_1\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}_0\|^2_{\boldsymbol{V}(\Omega)} + \|\boldsymbol{f}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \right\},
\end{aligned} \tag{11}
$$

for all $t \in [0, T]$.

Therefore, we obtain that

$$
\begin{aligned}
&(\boldsymbol{u}^m)_{m=1}^\infty \text{ is uniformly bounded with respect to } m \text{ in } L^\infty(0,T;\boldsymbol{V}(\Omega)), \\
&(\dot{\boldsymbol{u}}^m)_{m=1}^\infty \text{ is uniformly bounded with respect to } m \text{ in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)),
\end{aligned} \tag{12}
$$

and, moreover, by (11), there exists a positive uniform constant L such that

$$
0 \leq \left\| \left\{ [\boldsymbol{I} + u_i^m \boldsymbol{e}^i] \cdot \boldsymbol{q} \right\}^- \right\|^2_{L^2(0,T;L^2(\Omega))} \leq L\kappa. \tag{13}
$$

Since the following direct sum decomposition holds true

$$
\boldsymbol{V}(\Omega) = \boldsymbol{E}^m \oplus (\boldsymbol{E}^m)^\perp,
$$

we get that for any $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$, with $\|\boldsymbol{v}\|_{\boldsymbol{V}(\Omega)} \leq 1$ and a.a. $t \in (0,T)$, the variational equations in Problem $\mathcal{P}_m(\kappa;\Omega)$ give

$$\begin{aligned}
&|_{\boldsymbol{V}^*(\Omega)}\langle \ddot{u}_i^m(t)\boldsymbol{e}^i, v_j\boldsymbol{e}^j\rangle_{\boldsymbol{V}(\Omega)}| \leq \|\boldsymbol{f}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + C\|\boldsymbol{u}^m\|_{L^\infty(0,T;\boldsymbol{V}(\Omega))} \\
&\quad + \frac{1}{\kappa}\left\|\left\{[\boldsymbol{I} + u_i^m\boldsymbol{e}^i]\cdot\boldsymbol{q}\right\}^-\right\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}$$

and, by (12) and (13), we thus infer that there exists a constant $C_\kappa > 0$, independent of m, such that

$$\|\ddot{\boldsymbol{u}}^m\|_{L^\infty(0,T;\boldsymbol{V}^*(\Omega))} \leq C_\kappa. \tag{14}$$

(c) *Passage to the limit and retrieval of Problem $\mathcal{P}(\kappa;\Omega)$.* By (12), (13) and (14) we can infer that there exist subsequences, still denoted $(\boldsymbol{u}^m)_{m=1}^\infty$, $(\dot{\boldsymbol{u}}^m)_{m=1}^\infty$ and $(\ddot{\boldsymbol{u}}^m)_{m=1}^\infty$ such that the following convergences take place

$$\begin{aligned}
\boldsymbol{u}^m &\overset{*}{\rightharpoonup} \boldsymbol{u}_\kappa, && \text{in } L^\infty(0,T;\boldsymbol{V}(\Omega)) \text{ as } m\to\infty,\\
\dot{\boldsymbol{u}}^m &\overset{*}{\rightharpoonup} \dot{\boldsymbol{u}}_\kappa, && \text{in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \text{ as } m\to\infty,\\
\ddot{\boldsymbol{u}}^m &\overset{*}{\rightharpoonup} \ddot{\boldsymbol{u}}_\kappa, && \text{in } L^\infty(0,T;\boldsymbol{V}^*(\Omega)) \text{ as } m\to\infty,\\
\kappa^{-1}\left\{[\boldsymbol{I} + u_i^m\boldsymbol{e}^i]\cdot\boldsymbol{q}\right\}^- &\rightharpoonup \chi_\kappa, && \text{in } L^2(0,T;L^2(\Omega)) \text{ as } m\to\infty.
\end{aligned} \tag{15}$$

By the Sobolev embedding theorem (Theorem 10.1.20 of [44]), we obtain

$$\begin{aligned}
\boldsymbol{u}^m &\rightharpoonup \boldsymbol{u}_\kappa, && \text{in } \mathcal{C}^0([0,T];\boldsymbol{L}^2(\Omega)) \text{ as } m\to\infty,\\
\dot{\boldsymbol{u}}^m &\rightharpoonup \dot{\boldsymbol{u}}_\kappa, && \text{in } \mathcal{C}^0([0,T];\boldsymbol{V}^*(\Omega)) \text{ as } m\to\infty,
\end{aligned} \tag{16}$$

An application of Theorem 8.28 of [23] to the fourth convergence of the process (15) gives

$$\kappa^{-1}\left\{[\boldsymbol{I} + u_i^m\boldsymbol{e}^i]\cdot\boldsymbol{q}\right\}^- \rightharpoonup \chi_\kappa, \quad \text{in } L^2((0,T)\times\Omega) \text{ as } m\to\infty. \tag{17}$$

By (4), the first convergence of (16) and the weak convergence (17), Theorem 8.28 of [23] and Theorem 8.62 of [23], we are in a position to apply Theorem 9.13-2 of [47] (where the involved monotone operator is nothing but the *negative part operator*) and, so, to obtain

$$\chi_\kappa = \kappa^{-1}\left\{[\boldsymbol{I} + u_{i,\kappa}\boldsymbol{e}^i]\cdot\boldsymbol{q}\right\}^- \in L^2((0,T)\times\Omega). \tag{18}$$

We now verify that $\boldsymbol{u}_\kappa$ is a solution to the penalised variational equations in Problem $(\mathcal{P}(\kappa;\Omega))$. Let $\psi \in \mathcal{D}(0,T)$ and let $\mu \geq 1$ be any integer. For each $m \geq \mu$, we have

$$\begin{aligned}
&2\rho \int_0^T \int_\Omega \ddot{u}_i^m(t) v_i \,\mathrm{d}x \psi(t) \,\mathrm{d}t \\
&\quad+ \int_0^T \int_\Omega A^{ijkl} e_{k\|l}(\boldsymbol{u}^m(t)) e_{i\|j}(\boldsymbol{v}) \,\mathrm{d}x \psi(t) \,\mathrm{d}t \\
&\quad- \frac{1}{\kappa} \int_0^T \int_\Omega \left(\left\{[\boldsymbol{I} + u_i^m(t)\boldsymbol{e}^i] \cdot \boldsymbol{q}\right\}^-\right) (v_i \boldsymbol{e}^i \cdot \boldsymbol{q}) \,\mathrm{d}x \psi(t) \,\mathrm{d}t \\
&= \int_0^T \int_\Omega f^i(t) v_i \,\mathrm{d}x \psi(t) \,\mathrm{d}t,
\end{aligned} \tag{19}$$

for all $\boldsymbol{v} \in \boldsymbol{E}^\mu$.

Keeping in mind (4), the convergence process (15), (18), the arbitrariness of $\psi \in \mathcal{D}(0,T)$, as well as the fact that

$$\overline{\bigcup_{\mu \geq 1} \boldsymbol{E}^\mu}^{\|\cdot\|_{\boldsymbol{V}(\Omega)}} = \boldsymbol{V}(\Omega),$$

a passage to the limit as $m \to \infty$ in (19) shows that $\boldsymbol{u}_\kappa$ is a solution to the penalised variational equations in Problem $\mathcal{P}(\kappa;\Omega)$.

The last thing that we have to check is the validity of the initial conditions for $\boldsymbol{u}_\kappa$. Let us introduce the operator $\boldsymbol{L}_0 : \mathcal{C}^0([0,T];\boldsymbol{L}^2(\Omega)) \to \boldsymbol{L}^2(\Omega)$ defined in a way such that $\boldsymbol{L}_0(\boldsymbol{v}) := \boldsymbol{v}(0)$. Such an operator $\boldsymbol{L}_0$ turns out to be linear and continuous and, therefore, by the first convergence of (16), we get that

$$\boldsymbol{u}_0^m \rightharpoonup \boldsymbol{u}_\kappa(0), \quad \text{ in } \boldsymbol{L}^2(\Omega).$$

Since $\boldsymbol{u}_0^m \to \boldsymbol{u}_0$ in $\boldsymbol{V}(\Omega)$, we deduce that $\boldsymbol{u}_\kappa(0) = \boldsymbol{u}_0$.

Similarly, let us introduce the operator $\boldsymbol{L}_1 : \mathcal{C}^0([0,T];\boldsymbol{V}^*(\Omega)) \to \boldsymbol{V}^*(\Omega)$ defined in a way such that $\boldsymbol{L}_1(\boldsymbol{v}) := \boldsymbol{v}(0)$. Such an operator $\boldsymbol{L}_1$ turns out to be linear and continuous and, therefore, by the second convergence of (16), we get that

$$\boldsymbol{u}_1^m \rightharpoonup \dot{\boldsymbol{u}}_\kappa(0), \quad \text{ in } \boldsymbol{V}^*(\Omega).$$

Since $\boldsymbol{u}_1^m \to \boldsymbol{u}_1$ in $\boldsymbol{L}^2(\Omega)$, we deduce that $\dot{\boldsymbol{u}}_\kappa(0) = \boldsymbol{u}_1$.

We have thus shown that $\boldsymbol{u}_\kappa$ is a solution of Problem $\mathcal{P}(\kappa;\Omega)$. This completes the proof. $\square$

We are now in a position to prove the existence of solutions of Problem $\mathcal{P}(\Omega)$, which constitutes the main result of this paper.

Theorem 6. *For each $\kappa > 0$, let $\boldsymbol{u}_\kappa$ denote a solution to Problem $\mathcal{P}(\Omega)$.*

Assume also that the following "uniformity property" holds: There exists a number $\bar{t}_0 > 0$, independent of κ, such that

$$[\boldsymbol{I} + u_{i,\kappa}(t)\boldsymbol{e}^i] \cdot \boldsymbol{q} \geq 0 \quad \text{ a.e. in } \Omega,$$

for a.a. $0 < t < \bar{t}_0$, for all $\kappa > 0$.

Then, Problem $\mathcal{P}(\Omega)$ admits a solution.

Proof. By the energy estimate (10) in Theorem 5, it can be easily observed that there exists a positive constant $c = c(\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{f})$ such that

$$\frac{1}{\tilde{C}} \left\{ \|\dot{\boldsymbol{u}}_\kappa\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{u}_\kappa\|^2_{L^\infty(0,T;\boldsymbol{V}(\Omega))} \right\} \leq c.$$

As a result, the sequences $(\boldsymbol{u}_\kappa)_{\kappa>0}$ and $(\dot{\boldsymbol{u}}_\kappa)_{\kappa>0}$ are uniformly bounded in $L^\infty(0,T;\boldsymbol{V}(\Omega))$ and $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$, respectively.

Let us consider, for a.a. $0 < t < T$, the partial differential equation associated with Problem $\mathcal{P}(\kappa;\Omega)$

$$2\rho\ddot{\boldsymbol{u}}_\kappa(t) + A\boldsymbol{u}_\kappa(t) - \frac{1}{\kappa} N\boldsymbol{u}_\kappa(t) = \boldsymbol{f}(t), \quad \text{ in } \boldsymbol{V}^*(\Omega), \tag{20}$$

where the operator $A : \boldsymbol{V}(\Omega) \to \boldsymbol{V}^*(\Omega)$ defined by

$${}_{\boldsymbol{V}^*(\Omega)}\langle A\boldsymbol{u}, \boldsymbol{v}\rangle_{\boldsymbol{V}(\Omega)} := \int_\Omega A^{ijkl} e_{k\|l}(\boldsymbol{u}) e_{i\|j}(\boldsymbol{v}) \,\mathrm{d}x, \quad \text{ for all } \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(\Omega),$$

is linear and continuous.

Similarly, we define the nonlinear operator $N : \boldsymbol{V}(\Omega) \to \boldsymbol{V}^*(\Omega)$ as

$${}_{\boldsymbol{V}^*(\Omega)}\langle N\boldsymbol{u}, \boldsymbol{v}\rangle_{\boldsymbol{V}(\Omega)} := \int_\Omega \left(\left\{ [\boldsymbol{I} + u_i\boldsymbol{e}^i] \cdot \boldsymbol{q} \right\}^- \right) \left(v_i\boldsymbol{e}^i \cdot \boldsymbol{q} \right) \,\mathrm{d}x, \quad \text{ for all } \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(\Omega).$$

Let us prove the uniform boundedness of the sequence $(N\boldsymbol{u}_\kappa)_{\kappa>0}$ by observing that

$$
\begin{aligned}
&\frac{1}{\kappa}\int_0^T \left(\sup_{\substack{\boldsymbol{v}\in \boldsymbol{V}(\Omega)\\ \|\boldsymbol{v}\|_{\boldsymbol{V}(\Omega)}\le 1}} \left|{}_{\boldsymbol{V}^*(\Omega)}\left\langle N\boldsymbol{u}_\kappa(t), \boldsymbol{v}\right\rangle_{\boldsymbol{V}(\Omega)}\right|\right) \,\mathrm{d}t\\
&\le \|\boldsymbol{f}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{u}_\kappa\|_{L^\infty(0,T;\boldsymbol{V}(\Omega))}\\
&\quad + \sup_{\substack{\boldsymbol{v}\in \boldsymbol{V}(\Omega)\\ \|\boldsymbol{v}\|_{\boldsymbol{V}(\Omega)}\le 1}} \left|{}_{\boldsymbol{V}^*(\Omega)}\langle \dot{\boldsymbol{u}}_\kappa(T) - \dot{\boldsymbol{u}}_\kappa(0), \boldsymbol{v}\rangle_{\boldsymbol{V}(\Omega)}\right|,
\end{aligned}
$$

where the last term in the right hand side derives from an application of Corollary 10.1.26 of [44]. We make use of this strategy to gain insight into a uniform bound for the nonlinear term, since nothing is known about the boundedness of the sequence $(\ddot{\boldsymbol{u}}_\kappa)_{\kappa>0}$ yet.

Observe that, by Theorem 4, the following chain of embeddings holds

$$
L^1(0,T;\boldsymbol{V}^*(\Omega)) \hookrightarrow \left(\mathcal{C}^0([0,T];\boldsymbol{V}^*(\Omega))\right)^* \hookrightarrow \left(\mathcal{C}^0([0,T];\boldsymbol{L}^2(\Omega))\right)^* \cong \mathcal{M}([0,T];\boldsymbol{L}^2(\Omega)).
$$

An application of (15) and (16) thus gives that the sequence $(N\boldsymbol{u}_\kappa)_{\kappa>0}$ is bounded in $L^1(0,T;\boldsymbol{V}(\Omega))$. Therefore, *a fortiori*, we have

$$
(\ddot{\boldsymbol{u}}_\kappa)_{\kappa>0} \text{ is bounded in } \left(\mathcal{C}^0([0,T];\boldsymbol{L}^2(\Omega))\right)^*.
$$

Hence, up to passing to a subsequence, we get that the following convergence process takes place

$$
\begin{aligned}
\boldsymbol{u}_\kappa &\overset{*}{\rightharpoonup} \boldsymbol{u}, &&\text{in } L^\infty(0,T;\boldsymbol{V}(\Omega)) \text{ as } \kappa\to 0,\\
\dot{\boldsymbol{u}}_\kappa &\overset{*}{\rightharpoonup} \dot{\boldsymbol{u}}, &&\text{in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \text{ as } \kappa\to 0,\\
\ddot{\boldsymbol{u}}_\kappa &\overset{*}{\rightharpoonup} \tilde{\boldsymbol{u}}, &&\text{in } (\mathcal{C}^0([0,T];\boldsymbol{L}^2(\Omega)))^* \text{ as } \kappa\to 0.
\end{aligned} \tag{21}
$$

We immediately deduce, by Theorem 4, that there exists a unique vector-valued measure $\boldsymbol{\mu}\in\mathcal{M}([0,T];\boldsymbol{L}^2(\Omega))$ such that

$$
\langle\langle \tilde{\boldsymbol{u}}(t), \boldsymbol{\sigma}(t)\rangle\rangle_{\boldsymbol{L}^2(\Omega)} = \int_0^T\int_\Omega \mathrm{d}\mu_i(t)\sigma_i(t)\,\mathrm{d}x\,\mathrm{d}t,
$$

for all $\boldsymbol{\sigma}\in\mathcal{C}^0([0,T];\boldsymbol{L}^2(\Omega))$. Clearly, the vector-valued measure $\boldsymbol{\mu}$ is regular (cf., e.g., [24]).

By Theorem 3, the following convergence holds, up to passing to a subsequence

$$\boldsymbol{u}_\kappa \to \boldsymbol{u}, \quad \text{in } \mathcal{C}^0([0,T]; \boldsymbol{L}^2(\Omega)). \tag{22}$$

Besides, by (13) we have

$$0 \le \left\| \left\{ [\boldsymbol{I} + u_{i,\kappa}\boldsymbol{e}^i] \cdot \boldsymbol{q} \right\}^- \right\|_{L^2(0,T;L^2(\Omega))} \le \sqrt{L\kappa}. \tag{23}$$

Consequently, by (4), (22) and (23), we get

$$\left\{ [\boldsymbol{I} + u_i(t)\boldsymbol{e}^i] \cdot \boldsymbol{q} \right\}^- = 0, \quad \text{a.e. in } \Omega, \text{ for a.a. } t \in (0,T), \tag{24}$$

i.e., $\boldsymbol{u}(t) \in \boldsymbol{U}(\Omega)$, for a.a. $t \in (0,T)$.

Given any $\boldsymbol{v} \in \mathcal{D}(0,T;\boldsymbol{V}(\Omega))$ such that $\boldsymbol{v}(t) \in \boldsymbol{U}(\Omega)$ for a.a. $t \in (0,T)$, we use $(\boldsymbol{v} - \boldsymbol{u}_\kappa)$ as a test function in the variational equations of Problem $\mathcal{P}(\kappa;\Omega)$, getting

$$\begin{aligned}
&2\rho \left\langle\left\langle \ddot{u}_{i,\kappa}(t)\boldsymbol{e}^i, (v_j(t) - u_{j,\kappa}(t))\boldsymbol{e}^j \right\rangle\right\rangle_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \int_0^T \int_\Omega A^{ijkl} e_{k\|l}(\boldsymbol{u}_\kappa(t)) e_{i\|j}(\boldsymbol{v}(t) - \boldsymbol{u}_\kappa(t)) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad - \frac{1}{\kappa} \int_0^T \int_\Omega \left(\{[\boldsymbol{I} + u_{i,\kappa}(t)\boldsymbol{e}^i] \cdot \boldsymbol{q}\}^- \right) ((v_j(t) - u_{j,\kappa}(t))\boldsymbol{e}^j \cdot \boldsymbol{q}) \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_0^T \int_\Omega f^i(t)(v_i(t) - u_{i,\kappa}(t)) \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned} \tag{25}$$

Besides, we observe that the third integral term of (25) is such that

$$\frac{1}{\kappa} \int_0^T \int_\Omega \left\{ [\boldsymbol{I} + u_{i,\kappa}(t)\boldsymbol{e}^i] \cdot \boldsymbol{q} \right\}^- ((v_j(t) - u_{j,\kappa}(t))\boldsymbol{e}^j \cdot \boldsymbol{q}) \,\mathrm{d}x \,\mathrm{d}t \ge 0, \tag{26}$$

for all $\kappa > 0$, since $\boldsymbol{v}(t) \in \boldsymbol{U}(\Omega)$ for a.a. $t \in (0,T)$.

By virtue of the continuity of the mappings $\tilde{e}_{i\|j}$ and the convergence process (21) we get that, for all $\boldsymbol{v} \in L^2(0,T;\boldsymbol{V}(\Omega))$, the mapping

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}(\Omega)) \to \int_0^T \int_\Omega A^{ijkl} \tilde{e}_{i\|j}(\boldsymbol{u})(t) \tilde{e}_{k\|l}(\boldsymbol{v})(t) \,\mathrm{d}x \,\mathrm{d}t$$

is linear and continuous. An application of the convergence process (21) thus gives

$$\begin{aligned}
&\int_0^T \int_\Omega A^{ijkl} \tilde{e}_{i\|j}(\boldsymbol{u}_\kappa)(t) \tilde{e}_{k\|l}(\boldsymbol{v})(t) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad \to \int_0^T \int_\Omega A^{ijkl} \tilde{e}_{i\|j}(\boldsymbol{u})(t) \tilde{e}_{k\|l}(\boldsymbol{v})(t) \,\mathrm{d}x \,\mathrm{d}t,
\end{aligned} \tag{27}$$

as $\kappa \to 0$.

By the convergence process (21), and (22) we infer the following convergence

$$\left\langle\left\langle \ddot{u}_{i,\kappa}(t)\boldsymbol{e}^i, (v_j(t) - u_{j,\kappa}(t))\boldsymbol{e}^j \right\rangle\right\rangle \to \left\langle\left\langle \tilde{u}_i(t)\boldsymbol{e}^i, (v_j(t) - u_j(t))\boldsymbol{e}^j \right\rangle\right\rangle, \tag{28}$$

as $\kappa \to 0$.

By the convergence process (21), and the continuity of the bilinear form

$$(\boldsymbol{u}, \boldsymbol{v}) \in L^2(0,T;\boldsymbol{V}(\Omega)) \times L^2(0,T;\boldsymbol{V}(\Omega)) \to \int_0^T \int_\Omega A^{ijkl} \tilde{e}_{i\|j}(\boldsymbol{u})(t) \tilde{e}_{k\|l}(\boldsymbol{v})(t) \,\mathrm{d}x \,\mathrm{d}t,$$

we obtain, in particular, that

$$\begin{aligned}
&\int_0^T \int_\Omega A^{ijkl} \tilde{e}_{i\|j}(\boldsymbol{u})(t) \tilde{e}_{k\|l}(\boldsymbol{u})(t) \,\mathrm{d}x \,\mathrm{d}t \\
&\le \liminf_{\kappa \to 0} \int_0^T \int_\Omega A^{ijkl} \tilde{e}_{i\|j}(\boldsymbol{u}_\kappa)(t) \tilde{e}_{k\|l}(\boldsymbol{u}_\kappa)(t) \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned} \tag{29}$$

Combining (26), (27), (28), and (29), we immediately deduce that the limit $\boldsymbol{u}$ is a solution to the variational inequalities in Problem $(\mathcal{P}(\Omega))$.

We can observe that, by the convergence process (21), the vector-valued measure $\boldsymbol{\mu} \in \mathcal{M}([0,T];\boldsymbol{L}^2(\Omega))$ can be interpreted as the acceleration of the limit displacement $\boldsymbol{u}$. Indeed, by the classical definition of weak derivative, we have that, for each i,

$$\int_0^T \dot{u}_{i,\kappa}(t)\varphi'(t) \,\mathrm{d}t = -\int_0^T \ddot{u}_{i,\kappa}(t)\varphi(t) \,\mathrm{d}t, \quad \text{for all } \varphi \in \mathcal{D}(0,T).$$

By the properties of Lebesgue-Bochner integrals we have that, for all $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ and all $\varphi \in \mathcal{D}(0,T)$, it results

$$\int_\Omega \int_0^T \dot{u}_{i,\kappa}(t)(\varphi'(t)v_i) \,\mathrm{d}t \,\mathrm{d}x = -\int_\Omega \int_0^T \ddot{u}_{i,\kappa}(t)(\varphi(t)v_i) \,\mathrm{d}t \,\mathrm{d}x,$$

so that, letting $\kappa \to 0$ (see Comment 3 of Chapter 4 of [49]) gives

$$\begin{aligned}
&\int_\Omega \left(\int_0^T \dot{u}_i(t)\varphi'(t) \,\mathrm{d}t\right) v_i \,\mathrm{d}x = \int_0^T \left(\int_\Omega \dot{u}_i(t) v_i \,\mathrm{d}x\right) \varphi'(t) \,\mathrm{d}t \\
&= -\left\langle\left\langle \tilde{\boldsymbol{u}}(t), \varphi(t)\boldsymbol{v} \right\rangle\right\rangle_{\boldsymbol{L}^2(\Omega)} = -\int_0^T \int_\Omega \mathrm{d}\mu_i(t)(v_i\varphi(t)) \,\mathrm{d}t,
\end{aligned}$$

where the first equality holds by Fubini's theorem, the second equality holds by Theorem 4, the third convergence of the process (21) and the definition of weak derivative, and, finally, the last equality holds true by Theorem 4.

To sum up, we have obtained that

$$\int_0^T \left(\int_\Omega \dot{u}_i(t) v_i \,\mathrm{d}x \right) \varphi'(t) \,\mathrm{d}t = - \int_0^T \int_\Omega \mathrm{d}\mu_i(t)(v_i \varphi(t)) \,\mathrm{d}t,$$

for all $\varphi \in \mathcal{D}(0,T)$ and all $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$.

We can thus regard the vector-valued measure $\boldsymbol{\mu}$ as the second weak derivative with respect to $t \in (0,T)$ of the limit displacement $\boldsymbol{u}$ obtained via the process (21). This justifies the following change in the notation

$$\boldsymbol{\mu} = \ddot{\boldsymbol{u}},$$

and the *symbol* $\ddot{\boldsymbol{u}}$ is now an element of $\mathcal{M}([0,T]; \boldsymbol{L}^2(\Omega))$.

In conclusion, we have shown that $\boldsymbol{u}$ is in the set $\boldsymbol{U}(\Omega)$ and that satisfies the variational inequalities in Problem $\mathcal{P}(\Omega)$, namely,

$$\begin{aligned}
&2\rho \left\langle\left\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t) \right\rangle\right\rangle_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \int_0^T \int_\Omega A^{ijkl} e_{k\|l}(\boldsymbol{u}(t)) e_{i\|j}(\boldsymbol{v}(t) - \boldsymbol{u}(t)) \,\mathrm{d}x \,\mathrm{d}t \\
&\geq \int_0^T \int_\Omega f^i(t)(v_i(t) - u_i(t)) \,\mathrm{d}x \,\mathrm{d}t,
\end{aligned}$$

for all $\boldsymbol{v} \in \mathcal{D}(0,T; \boldsymbol{V}(\Omega))$ such that $\boldsymbol{v}(t) \in \boldsymbol{U}(\Omega)$ for a.a. $t \in (0,T)$.

The last thing to check is the validity of the initial conditions for $\boldsymbol{u}$. Let us introduce the operator $\boldsymbol{L}_0 : \mathcal{C}^0([0,T]; \boldsymbol{L}^2(\Omega)) \to \boldsymbol{L}^2(\Omega)$ defined in a way such that $\boldsymbol{L}_0(\boldsymbol{v}) := \boldsymbol{v}(0)$. Such an operator turns out to be linear and continuous and, by the convergence (22), we get that

$$\boldsymbol{u}_\kappa(0) \to \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \text{ in } \boldsymbol{L}^2(\Omega).$$

For what concerns the initial condition for the first derivative of $\boldsymbol{u}$ with respect to t, we present an argument, making use of the assumed "uniformity property", that slightly differs from the ones used in [4], [5], and [7]. For sake of clarity, we present all the computations in detail. Observe that, by virtue of the "uniformity property", we have

$$[\boldsymbol{I} + u_{i,\kappa}(t)\boldsymbol{e}^i] \cdot \boldsymbol{q} \geq 0 \quad \text{a.e. in } \Omega,$$

for a.a. $0 < t < \bar{t}_0$, and for all $\kappa > 0$.

As a result, for a.a. $0 < t < \bar{t}_0$, equation (20) takes the simpler form

$$2\rho\ddot{\boldsymbol{u}}_\kappa(t) + A\boldsymbol{u}_\kappa(t) = \boldsymbol{f}(t), \quad \text{in } \boldsymbol{V}^*(\Omega), \tag{30}$$

since we have $N\boldsymbol{u}_\kappa(t) = \boldsymbol{0}$ in $\boldsymbol{V}^*(\Omega)$, for a.a. $0 < t < t_0$, for all $\kappa > 0$.

Since $\boldsymbol{f} = (f^i) \in L^\infty(0, T; \boldsymbol{L}^2(\Omega))$, we deduce that $(\ddot{\boldsymbol{u}}_\kappa)_{\kappa>0}$ is bounded in $L^\infty(0, \bar{t}_0; \boldsymbol{V}^*(\Omega))$ and, up to extracting a subsequence, we get that the following convergence takes place as $\kappa \to 0$

$$\ddot{\boldsymbol{u}}_\kappa \rightharpoonup \ddot{\boldsymbol{u}}, \quad \text{in } L^2(0, \bar{t}_0; \boldsymbol{V}^*(\Omega)). \tag{31}$$

Hence, by the convergence process (31) and the Sobolev embedding theorem (Theorem 10.1.20 of [44]), the following convergence holds

$$\dot{\boldsymbol{u}}_\kappa \rightharpoonup \dot{\boldsymbol{u}}, \quad \text{in } \mathcal{C}^0([0, \bar{t}_0]; \boldsymbol{V}^*(\Omega)). \tag{32}$$

Let us thus introduce the operator

$$\bar{\boldsymbol{L}}_1 : \mathcal{C}^0([0, \bar{t}_0]; \boldsymbol{V}^*(\Omega)) \to \boldsymbol{V}^*(\Omega)$$

defined in a way such that $\bar{\boldsymbol{L}}_1(\boldsymbol{v}) := \boldsymbol{v}(0)$, for all $\boldsymbol{v} \in \mathcal{C}^0([0, \bar{t}_0]; \boldsymbol{V}^*(\Omega))$. Such an operator $\bar{\boldsymbol{L}}_1$ is linear and continuous and, by the convergence (32) and the reflexivity of the space $\boldsymbol{V}^*(\Omega)$, we are in a position to recover the initial condition $\dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1$.

In conclusion, we have shown that $\boldsymbol{u}$ is a solution of Problem $\mathcal{P}(\Omega)$ and the proof is thus complete. □

5. About the uniqueness of the solution

To conclude the investigation, we observe that the following phenomenon that occurs in the *early stage*. We can indeed show that,

$$2\rho\ddot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) = \boldsymbol{f}(t), \quad \text{in } \boldsymbol{V}^*(\Omega), \tag{33}$$

admits a unique solution, for a.a. $0 < t < \bar{t}_0$. In this direction, we follow [50] (Theorem 4, Section 7.2).

To see this, let us show that the only solution to the initial value problem

$$\begin{aligned}
&2\rho\ddot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) = \boldsymbol{0}, \quad \text{ in } \boldsymbol{V}^*(\Omega), \text{ for a.a. } 0 < t < \bar{t}_0,\\
&\boldsymbol{u}(0) = \boldsymbol{0},\\
&\dot{\boldsymbol{u}}(0) = \boldsymbol{0},
\end{aligned} \tag{34}$$

is $\boldsymbol{u} \equiv \boldsymbol{0}$. To this aim, for any fixed $0 \le s \le \bar{t}_0$, let us define the function

$$\boldsymbol{v}(t) := \begin{cases} \int_t^s \boldsymbol{u}(\tau)\,\mathrm{d}\tau & , 0 \le t \le s,\\ \boldsymbol{0} & , s < t \le \bar{t}_0, \end{cases}$$

a.e. in Ω, with $\boldsymbol{v} \in \mathcal{C}^0([0,\bar{t}_0]; \boldsymbol{V}(\Omega))$. Since $\dot{\boldsymbol{u}}(0) = \boldsymbol{0} = \boldsymbol{v}(s)$, an application of the integration by parts formula (Corollary 10.1.26 of [44]) gives

$$\int_0^s \left\{ -2\rho \,{}_{\boldsymbol{V}^*(\Omega)}\langle \dot{u}_i(t)\boldsymbol{e}^i, \dot{v}_j(t)\boldsymbol{e}^j\rangle_{\boldsymbol{V}(\Omega)} + \int_\Omega A^{ijkl} e_{k\|l}(\boldsymbol{u}(t)) e_{i\|j}(\boldsymbol{v}(t))\,\mathrm{d}x \right\} \mathrm{d}t = 0.$$

Since $\dot{\boldsymbol{v}}(t) = -\boldsymbol{u}(t)$, for all $0 \le t \le s$, the latter formula becomes

$$\int_0^s \left\{ 2\rho \,{}_{\boldsymbol{V}^*(\Omega)}\langle \dot{u}_i(t)\boldsymbol{e}^i, u_j(t)\boldsymbol{e}^j\rangle_{\boldsymbol{V}(\Omega)} + \int_\Omega A^{ijkl} e_{k\|l}(\dot{\boldsymbol{v}}(t)) e_{i\|j}(\boldsymbol{v}(t))\,\mathrm{d}x \right\} \mathrm{d}t = 0.$$

Again, by integration by parts formula (Corollary 10.1.26 of [44]), we get

$$\begin{aligned}
&\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\|\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{1}{2}\int_\Omega A^{ijkl} e_{k\|l}(\boldsymbol{v}(t)) e_{i\|j}(\boldsymbol{v}(t))\,\mathrm{d}x \right) \mathrm{d}t\\
&= \rho\|\boldsymbol{u}(s)\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{2}\int_\Omega A^{ijkl} e_{k\|l}(\boldsymbol{v}(0)) e_{i\|j}(\boldsymbol{v}(0))\,\mathrm{d}x = 0.
\end{aligned}$$

We thus infer, $\|\boldsymbol{u}(s)\|_{\boldsymbol{L}^2(\Omega)} = 0$, for all $0 \le s \le \bar{t}_0$. By the arbitrariness of s, we conclude that the solution $\boldsymbol{u}$ is uniquely defined in the interval $[0, \bar{t}_0]$.

In conclusion, all the solutions to Problem $\mathcal{P}(\Omega)$ coincide in the interval $[0, \bar{t}_0]$.

6. A sufficient condition ensuring the "uniformity property"

Let us recall the "uniformity property" that we used to prove Theorem 6: *There exists a number $\bar{t}_0 > 0$, independent of κ, such that*

$$[\boldsymbol{I} + u_{i,\kappa}(t)\boldsymbol{e}^i] \cdot \boldsymbol{q} \ge 0 \quad \text{ a.e. in } \Omega,$$

for a.a. $0 < t < \bar{t}_0$*, for all* $\kappa > 0$*.*

In this section we identify a simple sufficient condition that insures the validity of the "uniformity property". Let us consider applied body forces $\boldsymbol{f}$ such that each one of their components f^i satisfies

$$f^i(t) = 0 \quad \text{a.e. in } \Omega,$$

for a.a. $0 < t < \tau_0$, for some $\tau_0 > 0$. As a result, almost all numbers $\bar{t}_0$ between 0 and τ_0 ensure the validity of the "uniformity property".

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References

[1] P. G. Ciarlet, Mathematical Elasticity. Vol. III: Theory of Shells, North-Holland, Amsterdam, 2000.

[2] M. Schatzman, A hyperbolic problem of second order with unilateral constraints: the vibrating string with a concave obstacle, J. Math. Anal. Appl. 73 (1) (1980) 138–191.

[3] J. U. Kim, A boundary thin obstacle problem for a wave equation, Comm. Partial Differential Equations 14 (8-9) (1989) 1011–1026.

[4] I. Bock, J. Jarušek, On hyperbolic contact problems, Tatra Mt. Math. Publ. 43 (2009) 25–40.

[5] I. Bock, J. Jarušek, Dynamic contact problem for viscoelastic von Kármán–Donnell shells, ZAMM Z. Angew. Math. Mech. 93 (2013) 733–744.

[6] I. Bock, J. Jarušek, A vibrating thermoelastic plate in a contact with an obstacle, Tatra Mt. Math. Publ. 63 (2015) 39–52.

[7] I. Bock, J. Jarušek, M. Šilhavý, On the solutions of a dynamic contact problem for a thermoelastic von Kármán plate, Nonlinear Anal. Real World Appl. 32 (2016) 111–135.

[8] M. Fundos, P. D. Panagiotopoulos, V. Rădulescu, Existence theorems of Hartmann-Stampacchia type for hemivariational inequalities and applications, J. Global Optimiz. 15 (1999) 41–54.

[9] M. Bocea, P. D. Panagiotopoulos, V. Rădulescu, A perturbation result for a double eigenvalue hemivariational inequality and applications, J. Global Optimiz. 14 (1999) 137–156.

[10] V. Rădulescu, Perturbations of hemivariational inequalities with constraints, Revue Roum. Math. Pures Appl. 44 (1999) 455–461.

[11] N. Papageorgiou, V. Rădulescu, D. Repovš, Nonlinear elliptic inclusions with unilateral constraint and dependence on the gradient, Applied Mathematics and Optimization 78 (2018) 1–23.

[12] N. Papageorgiou, V. Rădulescu, D. Repovš, Periodic solutions for implicit evolution equations, Evolution Equations and Control Theory 8 (2019) 621–631.

[13] V. Rădulescu, R. Xu, W. Lian, N. Zhao, Y. Yang, Global well-posedness for a class of fourth order nonlinear strongly damped wave equations, Advances in Calculus of Variations.

[14] N. S. Papageorgiou, V. D. Rădulescu, D. D. Repovš, Sensitivity analysis for optimal control problems governed by nonlinear evolution inclusions, Adv. Nonlinear Anal. 6 (2) (2017) 199–235.

[15] D. Mugnai, P. Pucci, Maximum principles for inhomogeneous elliptic inequalities on complete Riemannian manifolds, Adv. Nonlinear Stud. 9 (2009) 429–452.

[16] N. S. Papageorgiou, V. D. Rădulescu, D. D. Repovš, Nonlinear analysis—theory and methods, Springer Monographs in Mathematics, Springer, Cham, 2019.

[17] P. G. Ciarlet, C. Mardare, P. Piersanti, An obstacle problem for elliptic membrane shells, Math. Mech. Solids 24 (5) (2019) 1503–1529.

[18] P. G. Ciarlet, C. Mardare, P. Piersanti, Un problème de confinement pour une coque membranaire linéairement élastique de type elliptique, C.R. Acad. Sci. Paris, Sér. I 356 (10) (2018) 1040–1051.

[19] P. G. Ciarlet, P. Piersanti, An obstacle problem for Koiter's shells, Math. Mech. Solids 24 (10) (2019) 3061–3079.

[20] P. G. Ciarlet, P. Piersanti, A confinement problem for a linearly elastic Koiter's shell, C.R. Acad. Sci. Paris, Sér. I 357 (2019) 221–230.

[21] P. Piersanti, X. Shen, Numerical methods for static shallow shells lying over an obstacle, Num. Algorithms.

[22] P. G. Ciarlet, An Introduction to Differential Geometry with Applications to Elasticity, Springer, Dordrecht, 2005.

[23] G. Leoni, A First Course in Sobolev Spaces, Second Edition, Vol. 181 of Graduate Studies in Mathematics, American Mathematical Society, Providence, 2017.

[24] J. Diestel, J. J. Uhl, Vector measures, American Mathematical Society, Providence, R.I., 1977.

[25] N. Dinculeanu, Vector measures, Pergamon Press, Oxford-New York-Toronto, Ont.; VEB Deutscher Verlag der Wissenschaften, Berlin, 1967.

[26] P. G. Ciarlet, Mathematical Elasticity. Vol. I: Three-Dimensional Elasticity, North-Holland, Amsterdam, 1988.

[27] P. Piersanti, An existence and uniqueness theorem for the dynamics of flexural shells, Math. Mech. Solids.

[28] K. O. Friedrichs, On the boundary-value problems of the theory of elasticity and Korn's inequality, Annals of Math. 48 (1947) 441–471.

[29] J. Gobert, Une inégalité fondamentale de la théorie de l'élasticité, Bull. Soc. Roy. Sci. Liège 31 (1962) 182–191.

[30] I. Hlaváček, J. Nečas, On inequalities of Korn's type. I. Boundary-value problems for elliptic system of partial differential equations, Arch. Rational Mech. Anal. 36 (1970) 305–311.

[31] I. Hlaváček, J. Nečas, On inequalities of Korn's type. II. Applications to linear elasticity, Arch. Rational Mech. Anal. 36 (1970) 312–334.

[32] G. Duvaut, J. L. Lions, Inequalities in Mechanics and Physics, Springer, Berlin, 1976.

[33] J. Nečas, I. Hlaváček, Mathematical theory of elastic and elasto-plastic bodies: an introduction, Vol. 3 of Studies in Applied Mechanics, Elsevier Scientific Publishing Co., Amsterdam-New York, 1980.

[34] R. Temam, Problèmes mathématiques en plasticité, Vol. 12 of Méthodes Mathématiques de l'Informatique [Mathematical Methods of Information Science], Gauthier-Villars, Montrouge, 1983.

[35] J. A. Nitsche, On Korn's second inequality, RAIRO Anal. Numér. 15 (3) (1981) 237–248.

[36] T. Miyoshi, Foundations of the numerical analysis of plasticity, Vol. 107 of North-Holland Mathematics Studies, North-Holland Publishing Co., Amsterdam; Kinokuniya Company Ltd., Tokyo, 1985, lecture Notes in Numerical and Applied Analysis, 7.

[37] L. Xiao, Asymptotic analysis of dynamic problems for linearly elastic shells–justification of equations for dynamic flexural shells, Chinese Ann. Math. Ser. B 22 (3) (2001) 13–22.

[38] L. Xiao, Asymptotic analysis of dynamic problems for linearly elastic shells–justification of equations for dynamic membrane shells, Asymptot. Anal. 17 (2) (1998) 121–134.

[39] L. Xiao, Asymptotic analysis of dynamic problems for linearly elastic shells–justification of equations for dynamic Koiter shells, Chinese Ann. Math. Ser. B 22 (3) (2001) 267–274.

[40] J. L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod; Gauthier-Villars, Paris, 1969.

[41] T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Ann. of Math. 20 (4) (1919) 292–296.

[42] P. Hartman, Ordinary Differential Equations, Second Edition, Society for Industrial and Applied Mathematics, Philadelphia, 1982.

[43] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl. 146 (4) (1987) 65–96.

[44] S. Kyritsi-Yiallourou, N. S. Papageorgiou, Handbook of Applied Analysis, Springer, New York, 2009.

[45] I. Zinger, Linear functionals on the space of continuous mappings of a compact Hausdorff space into a Banach space, Rev. Math. Pures Appl. 2 (1957) 301–315.

[46] P. Raviart, J. Thomas, Introduction à l'Analyse Numérique des Équations aux Dérivées Partielles, Dunod, Paris, 1988.

[47] P. G. Ciarlet, Linear and Nonlinear Functional Analysis with Applications, Society for Industrial and Applied Mathematics, Philadelphia, 2013.

[48] G. Stampacchia, Èquations elliptiques du second ordre à coefficients discontinus, Séminaire de Mathématiques Supérieures, No. 16 (Été, 1965), Les Presses de l'Université de Montréal, Montreal, Que., 1966.

[49] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2011.

[50] L. C. Evans, Partial Differential Equations, Second Edition, American Mathematical Society, Providence, 2010.