Linear dynamical systems on Hilbert spaces: typical properties and explicit examples

Sophie Grivaux, Etienne Matheron, Quentin Menet

Linear dynamical systems on Hilbert spaces: typical properties and explicit examples

S. Grivaux

É. Matheron

Q. Menet

Author address:

CNRS, Laboratoire Paul Painlevé, UMR 8524, Université de Lille, Cité Scientifique, Bâtiment M2, 59655 Villeneuve d'Ascq Cedex, France
E-mail address: sophie.grivaux@math.univ-lille1.fr

Laboratoire de Mathématiques de Lens, Université d'Artois, Rue Jean Souvraz SP 18, 62307 Lens, France
E-mail address: etienne.matheron@univ-artois.fr

Laboratoire de Mathématiques de Lens, Université d'Artois, Rue Jean Souvraz SP 18, 62307 Lens, France
E-mail address: quentin.menet@univ-artois.fr

Contents

Abstract

We solve a number of questions pertaining to the dynamics of linear operators on Hilbert spaces, sometimes by using Baire category arguments and sometimes by constructing explicit examples. In particular, we prove the following results.

(i) A typical hypercyclic operator is not topologically mixing, has no eigenvalues and admits no non-trivial invariant measure, but is densely distributionally chaotic.

(ii) A typical upper-triangular operator with coefficients of modulus 1 on the diagonal is ergodic in the Gaussian sense, whereas a typical operator of the form "diagonal with coefficients of modulus 1 on the diagonal plus backward unilateral weighted shift" is ergodic but has only countably many unimodular eigenvalues; in particular, it is ergodic but not ergodic in the Gaussian sense.

(iii) There exist Hilbert space operators which are chaotic and $\mathcal{U}$-frequently hypercyclic but not frequently hypercyclic, Hilbert space operators which are chaotic and frequently hypercyclic but not ergodic, and Hilbert space operators which are chaotic and topologically mixing but not $\mathcal{U}$-frequently hypercyclic.

We complement our results by investigating the descriptive complexity of some natural classes of operators defined by dynamical properties.

Received by the editor January 29, 2018.

1991 *Mathematics Subject Classification.* 47A16, 37A05, 54E52, 54H05.

Key words and phrases. Linear dynamical systems; Hilbert spaces; Baire category; frequent and $\mathcal{U}$-frequent hypercyclicity; ergodicity; chaos; topological mixing.

This work was supported in part by the grant ANR-17-CE40-0021 of the French National Research Agency ANR (project Front), by the EU IRSES grant AOS (PIRSES-GA-2012-318910) and by the Labex CEMPI (ANR-11-LABX-0007-01).

We are grateful to Alfred Peris for interesting discussions, and also for kindly allowing us to reproduce here his proof of Theorem 5.39. We also would like to thank the anonymous referee for valuable suggestions leading to an improvement of the presentation of our results.

CHAPTER 1

Introduction

1. General overview

This monograph is a contribution to the study of linear dynamical systems on Hilbert spaces. In other words, we are interested in the behavior of orbits of bounded linear operators defined on a Hilbert space. The symbol $\mathcal{H}$ will always designate a complex separable infinite-dimensional Hilbert space, and we denote by $\mathfrak{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$.

We refer to the books [**8**] or [**32**] for background on linear dynamics, and to the papers [**26**] or [**29**] for a glimpse at the richness of the class of linear dynamical systems and its potential usefulness in general ergodic theory. A quick review of a number of definitions will be given in the next section. Let us just recall here that an operator $T \in \mathfrak{B}(\mathcal{H})$ is said to be *hypercyclic* if it admits at least one dense orbit (and hence a dense G_δ set of such orbits). The class of all hypercyclic operators on $\mathcal{H}$ will be denoted by $\mathrm{HC}(\mathcal{H})$. Recall also that an operator $T \in \mathfrak{B}(\mathcal{H})$ is said to be *chaotic* if it is hypercyclic and admits a dense set of periodic vectors, and *frequently hypercyclic* (resp. *$\mathcal{U}$-frequently hypercyclic*) if there exists at least one vector $x \in \mathcal{H}$ whose orbit visits frequently every non-empty open set $V \subseteq \mathcal{H}$, in the sense that the set of integers $i \in \mathbb{N}$ such that $T^i x$ belongs to V has positive lower (resp. upper) density.

These definitions have, of course, nothing to do with the Hilbert space structure. The dynamical properties of linear operators have been studied in arbitrary Banach spaces, and even arbitrary topological vector spaces. However, there are good reasons for focusing on Hilbert spaces. The first that may come to mind is perhaps that "this is the natural setting for doing operator theory" – which is of course a highly questionable statement. Less subjectively, the richness of the Hilbert space structure allows for the construction of many interesting examples. Also, in some parts of linear dynamics, especially all that concerns *ergodic-theoretic* properties of linear operators, the general picture is neater on Hilbert spaces than on arbitrary Banach spaces. Finally, some natural questions in the area have been solved recently for operators on general Banach space but *not* in the Hilbertian case. This should not seem paradoxical, if one compares for example with the current state of affairs regarding the famous Invariant Subspace Problem.

The questions we are considering in this work are quite basic. In very general terms, they are of the following type: given two interesting dynamical properties, do there exist linear operators on $\mathcal{H}$ satisfying one of them but not the other?

Perhaps the most famous question of this type is Herrero's "$T \oplus T$ problem", stated in [**36**] and asking whether there exist hypercyclic operators which are not *topologically weakly mixing*. As should be expected since this is trivially true for dynamical systems on compact spaces, the answer is "Yes": this was shown by De La Rosa and Read in [**19**], and then in [**7**] for operators on ℓ_p spaces, in particular for Hilbert space operators. However, the proof is surprisingly non-trivial. To give just one more example, the third named author was recently able to solve another well-known problem in the area by constructing in [**40**] operators on ℓ_p spaces which are chaotic but not frequently hypercyclic – actually, not even $\mathcal{U}$-frequently hypercyclic. In this monograph, we will be especially interested in the following questions:

- are there operators which are frequently hypercyclic but not *ergodic*, *i.e.* do not admit an ergodic measure with full support?
- are there operators which are $\mathcal{U}$-frequently hypercyclic but not frequently hypercyclic?

- are there operators which are ergodic but do not admit a *Gaussian* ergodic measure with full support?

Note that the answer to the first two questions is known to be "Yes" for operators on the Banach space c_0 (see [**10**] and [**30**], as well as [**15**] for further developments); but the Hilbertian case had not been settled. As for the third question, we are not aware of any previous answer, on any Banach space.

One can attack questions of this type from two complementary points of view (of course, there are other strategies as well).

- "Collective" point of view: among the properties one is interested in, one may try to determine which ones are *generic* and which ones are not, in a Baire category sense. Once this is done, one may be able to distinguish two properties because one of them is generic and the other one is not.
- "Individual" point of view: when it is unclear how a Baire category approach could work, one may still try to construct explicit examples of operators satisfying (or not) such or such a property.

Note that it is quite natural to present the two viewpoints in this order, because the indirect, Baire category approach is in some sense simpler. Indeed, it is usually not too difficult to guess when it should work, and in this case the technical details are likely to be rather soft. On the other hand, one may reasonably expect to face technical difficulties when attempting a direct construction (and perhaps quite serious ones if Baire category is inefficient).

To give meaning to the "collective" viewpoint, one has to fix an appropriate topological setting. We will in fact consider *two* natural topologies on $\mathfrak{B}(\mathcal{H})$: the *Strong Operator Topology* (denoted by $\mathtt{SOT}$), and the *Strong* Operator Topology* (denoted by $\mathtt{SOT}^*$). Restricted to any closed ball $\mathfrak{B}_M(\mathcal{H}) := \{T \in \mathfrak{B}(\mathcal{H});\ \|T\| \leq M\}$, these topologies are *Polish* (separable and completely metrizable). Moreover, $\mathrm{HC}(\mathcal{H})$ is easily seen to be $\mathtt{SOT}$-G_δ in $\mathfrak{B}(\mathcal{H})$, so that for any $M > 1$, the family $\mathrm{HC}_M(\mathcal{H}) := \{T \in \mathrm{HC}(\mathcal{H});\ \|T\| \leq M\}$ is itself a Polish space in its own right with respect to both $\mathtt{SOT}$ and $\mathtt{SOT}^*$. This opens the way to Baire category arguments in $\mathfrak{B}_M(\mathcal{H})$ or $\mathrm{HC}_M(\mathcal{H})$. Of course, there is nothing new in this observation: Baire category methods have already proved as useful in operator theory as anywhere else; see in particular [**22**].

Recall that a subset of a Polish space X is said to be *meager* in X if it can be covered by countably many closed sets with empty interior, and *comeager* if its complement is meager (equivalently, if it contains a dense G_δ subset of X). Following a well-established terminology, we will say that a property of elements of X is *typical* if the set of all $x \in X$ satisfying it is comeager in X, and that a property is *atypical* if its negation is typical. In this work, we obtain among other things the following results, for any $M > 1$.

- An $\mathtt{SOT}^*$-typical $T \in \mathfrak{B}_M(\mathcal{H})$ is (topologically) weakly mixing but not mixing.
- An $\mathtt{SOT}^*$-typical $T \in \mathfrak{B}_M(\mathcal{H})$ is densely distributionally chaotic.
- An $\mathtt{SOT}^*$-typical $T \in \mathrm{HC}_M(\mathcal{H})$ has no eigenvalues and admits no non-trivial invariant measure. In particular, chaoticity and $\mathcal{U}$-frequent hypercyclicity are atypical properties of hypercyclic operators (in the $\mathtt{SOT}^*$ sense).

Admittedly, there results are not that surprising (except, perhaps, the one concerning $\mathcal{U}$-frequent hypercyclicity). Still, they do sketch the landscape, and they explain in some sense why some results in the area turn out to be (or are likely to be) harder to prove than some others. For example, the fact that topological weak mixing is a typical property seems to prevent the use of "soft" arguments for establishing the existence of hypercyclic operators which are not weakly mixing; and indeed, the proofs in [**19**] and [**7**] rely on rather technical arguments.

On the other hand, something more unexpected happens if one restricts oneself to the class of operators of the form $T = D + B$, where D is a diagonal operator and B is a weighted unilateral backward shift (with respect to some fixed orthonormal basis of $\mathcal{H}$):

- in the SOT sense, most "diagonal plus shift" hypercyclic operators are ergodic and yet have only countably many unimodular eigenvalues. In particular, there exist operators on $\mathcal{H}$ which are ergodic but admit no Gaussian ergodic measure with full support.

This solves a rather intriguing question, which goes back to Flytzanis' paper [**23**, p. 8] and has been very much in the air in the last few years.

As for the "individual" viewpoint, we will elaborate on the kind of operators constructed by the third named author in [**40**]. Recall that the main result of [**40**] is the existence of chaotic operators on ℓ_p spaces which are not $\mathcal{U}$-frequently hypercyclic. In the present work, we describe a general scheme allowing to produce, among other things,

- operators on $\mathcal{H}$ which are chaotic and frequently hypercyclic but not ergodic;
- operators on $\mathcal{H}$ which are chaotic and $\mathcal{U}$-frequently hypercyclic but not frequently hypercyclic.

These are the first examples of such kinds of operators living on a Hilbert space, even if one dispenses with the chaoticity assumption: no examples of frequently hypercyclic non-ergodic operators, nor of $\mathcal{U}$-frequently hypercyclic non frequently hypercyclic operators living on a Hilbert space were known before. As already mentioned, up to now the only available examples were operators acting on c_0.

Moreover, our constructions will also enable us to improve the main result of [**40**] by showing that there exist

- operators on $\mathcal{H}$ which are chaotic and *topologically mixing* but not $\mathcal{U}$-frequently hypercyclic.

One point is especially worth mentioning regarding these results: it is not at all accidental that all the operators we construct turn out to be chaotic. On the contrary, our proofs rely heavily on new criteria for frequent hypercyclicity and $\mathcal{U}$-frequent hypercyclicity (stronger than the usual ones), in which the periodic points play a central role.

In this part of the monograph, our constructions and arguments are rather technical, and it is not at all clear that they could be by-passed by suitable Baire category arguments. In fact, as already suggested above, there are reasons to believe that technicalities are unavoidable here: since $\mathcal{U}$-frequent hypercyclicity, frequent hypercyclicity and ergodicity are atypical properties, it seems difficult to distinguish them using simply the Baire category theorem; unless one restricts oneself to some cleverly chosen special class of operators, which has yet to be found.

The basic questions we address in this work can also be considered from a third point of view, which is that of *descriptive set theory*. Indeed, once it is known (by any argument) that two properties of linear operators are not the same, *i.e.* that two classes of operators are distinct, it is natural to wonder if a stronger conclusion might hold true, namely that these classes do not have the same *complexity* in the sense of descriptive set theory. More generally, it can be a quite interesting problem to determine the exact descriptive complexity of a given class of operators. We do so for chaotic operators and for topologically mixing operators. We also obtain a partial result for $\mathcal{U}$-frequently hypercyclic operators, whose proof relies on our general scheme for constructing operators with special properties.

We finish this overview by pointing out that the difficulty of the existence results presented in this work is specifically connected with the linear setting, and that the corresponding results are essentially trivial if one moves over to the broader setting of *Polish dynamical systems* (*i.e.* dynamical systems of the form (X, T), where T is a continuous self-map of a Polish space X; see for instance the book [**25**], as well as [**18**] for more on linear systems as special cases of Polish systems). Indeed, extending the definitions of frequent and $\mathcal{U}$-frequent hypercyclicity to the Polish setting in the obvious way, it is not difficult to see that frequently hypercyclic non-ergodic Polish dynamical systems do exist, as well as $\mathcal{U}$-frequently hypercyclic Polish dynamical systems which are not frequently hypercyclic. Here are two simple examples.

First, consider an irrational rotation R of the unit circle $\mathbb{T}$, and denote by m the normalized Lebesgue measure on $\mathbb{T}$. Then $(\mathbb{T}, \mathcal{B}, m; R)$ is an ergodic dynamical system for which all points are frequently hypercyclic. Let C be a compact subset of $\mathbb{T}$ which has empty interior and is such that $m(C) > 0$, and consider the set $X := \mathbb{T} \setminus \bigcup_{n\in\mathbb{N}} R^{-n}(C)$. Then X is a dense G_δ subset of $\mathbb{T}$ which is R-invariant, and (X, R) is thus a frequently hypercyclic Polish dynamical system. But, as $m(X) = 0$ (by ergodicity) and R is uniquely ergodic, (X, R) admits no invariant measure at all. This shows in particular the existence of frequently hypercyclic Polish dynamical systems which are not ergodic. This example is due to B. Weiss (private communication).

Now, consider a frequently hypercyclic operator T on $\mathcal{H}$. Then, the set $\mathrm{UFHC}(T)$ of all $\mathcal{U}$-frequently hypercyclic vectors for T is comeager in $\mathcal{H}$ ([**10**], see also [**15**]), whereas the set $\mathrm{FHC}(T)$ of frequently hypercyclic vectors for T is meager ([**41**], [**10**]). Hence $\mathrm{UFHC}(T) \setminus \mathrm{FHC}(T)$ is comeager in $\mathcal{H}$. Let G be a dense G_δ subset of $\mathcal{H}$ contained in $\mathrm{UFHC}(T) \setminus \mathrm{FHC}(T)$, and set $X := \bigcap_{n\geq 0} T^{-n}(G)$. Then X is a dense G_δ subset of $\mathcal{H}$ which is T-invariant, so that (X, T) is a Polish dynamical system. Since X is still contained in $\mathrm{UFHC}(T) \setminus \mathrm{FHC}(T)$, all points of X are $\mathcal{U}$-frequently hypercyclic for T but none of them is frequently hypercyclic. In particular, the Polish dynamical system (X, T) is $\mathcal{U}$-frequently hypercyclic but not frequently hypercyclic.

2. Background and notations

In this section, we recall some well-known definitions, referring to [**8**] or [**32**] for more details. We also fix some notations that will be used throughout the monograph. As we will consider quite a few classes of operators, for the reader's convenience we give in the very last chapter the list of the abbreviations we will be using to denote these classes.

2.1. Transitivity, mixing and weak mixing. If $T \in \mathfrak{B}(\mathcal{H})$, we set, for any subsets A, B of $\mathcal{H}$,

$$\mathcal{N}_T(A, B) := \{i \in \mathbb{N};\ T^i(A) \cap B \neq \emptyset\}.$$

It is well-known that T is hypercyclic if and only if it is *topologically transitive*, *i.e.* $\mathcal{N}_T(U, V) \neq \emptyset$ for all open sets $U, V \neq \emptyset$. A stronger property is *topological mixing*: T is topologically mixing if all sets $\mathcal{N}_T(U, V)$ are *cofinite* rather than just non-empty; that is, for any pair (U, V) of non-empty open sets in $\mathcal{H}$, one has $T^i(U) \cap V \neq \emptyset$ for all but finitely many $i \in \mathbb{N}$. In between transitivity and mixing is *topological weak mixing*: an operator $T \in \mathfrak{B}(\mathcal{H})$ is topologically weakly mixing if $T \times T$ is hypercyclic on $\mathcal{H} \times \mathcal{H}$; in other words, if $\mathcal{N}_T(U_1, V_1) \cap \mathcal{N}_T(U_2, V_2) \neq \emptyset$ for all non-empty open sets $U_1, V_1, U_2, V_2 \subseteq \mathcal{H}$. We set

$$\mathrm{TWMIX}(\mathcal{H}) := \{\text{topologically weakly mixing operators on } \mathcal{H}\},$$
$$\mathrm{TMIX}(\mathcal{H}) := \{\text{topologically mixing operators on } \mathcal{H}\};$$

so that

$$\mathrm{TMIX}(\mathcal{H}) \subseteq \mathrm{TWMIX}(\mathcal{H}) \subseteq \mathrm{HC}(\mathcal{H}).$$

It is easy to see (for example by considering weighted backward shifts) that the inclusion $\mathrm{TMIX}(\mathcal{H}) \subseteq \mathrm{TWMIX}(\mathcal{H})$ is proper. As already mentioned, the inclusion $\mathrm{TWMIX} \subseteq \mathrm{HC}$ is also proper: this was first proved by De La Rosa and Read in [**19**] for operators living on a suitably manufactured Banach space, and then in [**7**] for Hilbert space operators. Recall also that, according to a nice result of Bès and Peris [**12**], the topologically weakly mixing operators are exactly those satisfying the so-called *Hypercyclicity Criterion*.

2.2. Chaos. An operator $T \in \mathfrak{B}(\mathcal{H})$ is said to be *chaotic* in the sense of Devaney if it is hypercyclic and its periodic points are dense in $\mathcal{H}$ (in the linear setting, hypercyclicity automatically implies sensitive dependence on the initial conditions). We set

$$\mathrm{CH}(\mathcal{H}) := \{\text{chaotic operators on } \mathcal{H}\}.$$

It is not completely obvious, but nonetheless true, that chaotic operators are topologically weakly mixing (see e.g. [**8**, Ch. 4]). In other words,

$$\mathrm{CH}(\mathcal{H}) \subseteq \mathrm{TWMIX}(\mathcal{H}).$$

2.3. Ergodic-theoretic properties. In this work, the word "measure" will always mean "Borel probability measure". A measure μ on $\mathcal{H}$ is *invariant* for some operator $T \in \mathfrak{B}(\mathcal{H})$ if $\mu \circ T^{-1} = \mu$; and an invariant measure μ for T is *non-trivial* if $\mu \neq \delta_0$ (note that the point mass δ_0 is invariant for any $T \in \mathfrak{B}(\mathcal{H})$ since $T(0) = 0$). Also, a measure μ on $\mathcal{H}$ is said to have *full support* if $\mu(V) > 0$ for every non-empty open set V. This means exactly that the topological support of μ is equal to the whole space $\mathcal{H}$.

An operator $T \in \mathfrak{B}(\mathcal{H})$ is said to be *ergodic* if it admits an invariant measure μ with full support with respect to which it is ergodic, *i.e.* such that for every Borel subset B of $\mathcal{H}$ satisfying $T^{-1}B = B$, we have $\mu(B) = 0$ or 1. If the measure μ can be taken to be Gaussian, we say that T is *ergodic in the Gaussian sense*, or *G-ergodic*. The operator T is said to be *mixing* if it admits an invariant measure μ with full support with respect to which it is strongly mixing, *i.e.* such that $\mu(A \cap T^{-n}(B)) \to \mu(A)\mu(B)$ for any Borel sets $A, B \subseteq \mathcal{H}$. Mixing *in the Gaussian sense* is defined as expected. The corresponding notations are the following:

$$\begin{aligned}
\mathrm{ERG}(\mathcal{H}) &:= \big\{\text{ergodic operators on } \mathcal{H}\big\};\\
\text{G-ERG}(\mathcal{H}) &:= \big\{\text{operators on } \mathcal{H} \text{ which are ergodic in the Gaussian sense}\big\};\\
\mathrm{MIX}(\mathcal{H}) &:= \big\{\text{mixing operators on } \mathcal{H}\big\};\\
\text{G-MIX}(\mathcal{H}) &:= \big\{\text{operators on } \mathcal{H} \text{ which are mixing in the Gaussian sense}\big\}.
\end{aligned}$$

Since the measures involved are required to have full support, it is obvious by definition that ergodic operators are hypercyclic and that mixing operators are topologically mixing. That is,

$$\mathrm{ERG}(\mathcal{H}) \subseteq \mathrm{HC}(\mathcal{H}) \qquad \text{and} \qquad \mathrm{MIX}(\mathcal{H}) \subseteq \mathrm{TMIX}(\mathcal{H}).$$

These inclusions are proper: for example, if B is any compact weighted backward shift on the Hilbert space $\ell_2(\mathbb{N})$, the operator $T = I + B$ is topologically mixing (see [**8**] or [**32**]) but not ergodic. Indeed, its spectrum is reduced to the point $\{1\}$, so that it is by [**49**] not even $\mathcal{U}$-frequently hypercyclic. We also set

$$\begin{aligned}
\mathrm{INV}(\mathcal{H}) &:= \big\{T \in \mathfrak{B}(\mathcal{H}) \text{ admitting a non-trivial invariant measure}\big\};\\
\mathrm{INV}_f(\mathcal{H}) &:= \big\{T \in \mathfrak{B}(\mathcal{H}) \text{ admitting an invariant measure with full support}\big\};\\
\text{G-INV}_f(\mathcal{H}) &:= \big\{T \in \mathfrak{B}(\mathcal{H}) \text{ admitting a Gaussian invariant measure with full support}\big\}.
\end{aligned}$$

Thus, for example, any operator admitting a non-zero periodic point belongs to $\mathrm{INV}(\mathcal{H})$, and any chaotic operator lies in $\mathrm{INV}_f(\mathcal{H})$. So we have

$$\mathrm{CH}(\mathcal{H}) \subseteq \mathrm{INV}_f(\mathcal{H}) \cap \mathrm{HC}(\mathcal{H}).$$

This inclusion cannot be reversed: there exist even G-ergodic operators on $\mathcal{H}$ which are not chaotic (see [**3**] or [**8**, Section 6.5]).

At this point, we would like to stress that in the present monograph, we will mostly focus on invariant measures which are *not required to be Gaussian*. This is a true change of point of view compared with earlier works on the ergodic theory of linear dynamical systems (see for instance, among other works, [**23**], [**5**], [**6**], [**8**] ...), where Gaussian measures play a central role. But in retrospect, this is in fact quite natural: as we shall see in Chapter 4, it turns out that within a certain natural class of upper-triangular operators on $\mathcal{H}$, the ergodic operators which are not ergodic in the Gaussian sense are typical. Note also that non-Gaussian measures already played an essential role in such works as [**42**], [**26**] or [**29**] for instance.

2.4. Frequent hypercyclicity and $\mathcal{U}$-frequent hypercyclicity. Let $T \in \mathfrak{B}(\mathcal{H})$. For any $x \in \mathcal{H}$ and $B \subseteq \mathcal{H}$, set

$$\mathcal{N}_T(x, B) := \{i \in \mathbb{N};\ T^i x \in B\}.$$

The operator T is said to be *frequently hypercyclic* if there exists some $x \in \mathcal{H}$ such that, for any open set $V \neq \emptyset$, the set $\mathcal{N}_T(x, V)$ has positive lower density; and T is *$\mathcal{U}$-frequently hypercyclic* if there is some $x \in \mathcal{H}$ such that all sets $\mathcal{N}_T(x, V)$ has positive upper density.

Frequent hypercyclicity was introduced in [**5**] and rather extensively studied since then (for instance in [**10**], [**14**], [**28**], [**30**], [**40**], [**49**] ...). The study of upper-frequent hypercyclicity is more recent. This notion was introduced by Shkarin in [**49**]. Until the last few years, it has perhaps been unfairly considered as somehow "less interesting", probably because of a lack of examples or results exhibiting truly different behaviors between frequently and $\mathcal{U}$-frequently hypercyclic operators. However, despite a formal similarity in the definitions, there are some important differences between frequent and $\mathcal{U}$-frequent hypercyclicity. For example, the set of frequently hypercyclic vectors for a frequently hypercyclic operator T on $\mathcal{H}$ is always *meager* in $\mathcal{H}$ ([**41**], [**10**]) whereas if T is a $\mathcal{U}$-frequently hypercyclic operator on $\mathcal{H}$, the set of all $\mathcal{U}$-frequently hypercyclic vectors for T is *comeager* in $\mathcal{H}$ ([**10**]; see also [**15**] for more along these lines). This may lead to believe that $\mathcal{U}$-frequent hypercyclicity is a typical property while frequent hypercyclicity is not; which is in fact a wrong intuition: as we shall see, both properties are atypical. Yet, these properties are not equivalent: Bayart and Rusza exhibited in [**10**] examples of $\mathcal{U}$-frequently hypercyclic operators on c_0 which are not frequently hypercyclic. The notion of $\mathcal{U}$-frequent hypercyclicity was further studied by Bonilla and Grosse-Erdmann in [**15**], and it plays an important role in the present work.

One can go one step further and study *$\mathcal{F}$-hypercyclic* operators, for a given family $\mathcal{F}$ of subsets of $\mathbb{N}$ (see in particular [**13**] and [**15**]); but we will not follow this quite interesting route here. As for the notations, we set

$$\mathrm{FHC}(\mathcal{H}) := \{\text{frequently hypercyclic operators on } \mathcal{H}\},$$
$$\mathrm{UFHC}(\mathcal{H}) := \{\mathcal{U}\text{-frequently hypercyclic operators on } \mathcal{H}\}.$$

Although the definitions of frequent hypercyclicity and $\mathcal{U}$-frequent hypercyclicity make no explicit reference to measures, they have a partly "metrical" flavor; and indeed, invariant measures are quite relevant here. First, it follows easily from the pointwise ergodic theorem that ergodic operators are frequently hypercyclic. Moreover, it is shown in [**30**] that $\mathcal{U}$-frequently hypercyclic operators on $\mathcal{H}$ admit invariant measures with full support (this is in fact true for operators living on any reflexive Banach space). So we have

$$\mathrm{ERG}(\mathcal{H}) \subseteq \mathrm{FHC}(\mathcal{H}) \subseteq \mathrm{UFHC}(\mathcal{H}) \subseteq \mathrm{HC}(\mathcal{H}) \cap \mathrm{INV}_f(\mathcal{H}).$$

The inclusion $\mathrm{UFHC}(\mathcal{H}) \subseteq \mathrm{HC}(\mathcal{H}) \cap \mathrm{INV}_f(\mathcal{H})$ is proper: as already mentioned, it is proved in [**40**] that there exist even chaotic operators on $\mathcal{H}$ which are not $\mathcal{U}$-frequently hypercyclic. As we shall see in Chapter 7 (Theorems 7.6 and 7.10), the other two inclusions are also proper.

2.5. Properties related to eigenvalues. If $T \in \mathfrak{B}(\mathcal{H})$, we denote by $\mathcal{E}(T)$ the set of all *unimodular eigenvectors* of T, *i.e.* eigenvectors x of T whose associated eigenvalue $\lambda(x)$ has modulus 1. We say that T has a *spanning set of unimodular eigenvectors* if $\overline{\mathrm{span}}\, \mathcal{E}(T) = \mathcal{H}$; and that T has a *perfectly spanning set of unimodular eigenvectors* if for any countable set $D \subseteq \mathbb{T}$, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$, we have $\overline{\mathrm{span}}\, \{x \in \mathcal{E}(T);\ \lambda(x) \notin D\} = \mathcal{H}$. The notations are as follows:

$$\mathrm{SPAN}(\mathcal{H}) := \big\{T \in \mathfrak{B}(\mathcal{H}) \text{ with spanning unimodular eigenvectors}\big\};$$
$$\mathrm{PSPAN}(\mathcal{H}) := \big\{T \in \mathfrak{B}(\mathcal{H}) \text{ with perfectly spanning unimodular eigenvectors}\big\}.$$

We also define

$$\mathrm{NEV}(\mathcal{H}) := \big\{T \in \mathfrak{B}(\mathcal{H}) \text{ with no eigenvalues}\big\};$$
$$\mathrm{CEV}(\mathcal{H}) := \big\{T \in \mathfrak{B}(\mathcal{H}) \text{ with only countably many eigenvalues}\big\}.$$

Properties of unimodular eigenvectors of an operator $T \in \mathfrak{B}(\mathcal{H})$ are closely related to those of invariant *Gaussian* measures for T (see for instance [**5**], [**6**], [**8**]): indeed, it turns out that in fact

$$\mathrm{SPAN}(\mathcal{H}) = \text{G-INV}_f(\mathcal{H}) \qquad \text{and} \qquad \mathrm{PSPAN}(\mathcal{H}) = \text{G-ERG}(\mathcal{H});$$

so that there was in fact no need for introducing new notations. However, things are not that neat on arbitrary Banach spaces: see [**6**], [**8**] or [**9**]. This is one of the advantages of working on Hilbert spaces, or at least on "nice" Banach spaces.

2.6. Distributional chaos. Distributional chaos was first defined by Schweitzer and Smítal in [**50**], under the name *strong chaos*, for self-maps of a compact interval, and several definitions have been proposed afterwards in the context of general metric spaces. There is no need to recall these definitions here, because things simplify greatly in the linear setting: as shown in [**11**], an operator $T \in \mathfrak{B}(\mathcal{H})$ is distributionally chaotic if and only if it admits a *distributionally irregular vector*, *i.e.* a vector $x \in \mathcal{H}$ for which there exist two sets A and B of integers, both of upper density 1 in $\mathbb{N}$, such that

$$\lim_{\substack{i\to\infty \\ i\in A}} \|T^i x\| = 0 \quad \text{and} \quad \lim_{\substack{i\to\infty \\ i\in B}} \|T^i x\| = \infty.$$

The corresponding notation is the following one:

$$\mathrm{DCH}(\mathcal{H}) := \big\{\text{distributionally chaotic operators on } \mathcal{H}\big\}.$$

A related class is that of *densely distributionally chaotic operators*, *i.e.* of operators which admit a dense set of distributionally irregular vectors. We use the following notation for this set:

$$\mathrm{DDCH}(\mathcal{H}) := \big\{\text{densely distributionally chaotic operators on } \mathcal{H}\big\}.$$

Obviously, $\mathrm{DDCH}(\mathcal{H}) \subseteq \mathrm{DCH}(\mathcal{H})$.

2.7. The parameter $c(T)$. Ergodicity and distributional chaos are closely related to a natural parameter introduced in [**30**]. This quantity $c(T) \in [0,1]$ is associated to any hypercyclic operator T, and essentially represents the maximal frequency with which the orbit of a hypercyclic vector for T can visit a ball in $\mathcal{H}$ centered at 0. The precise definition is as follows: for any $\alpha > 0$, we have

$$c(T) = \sup_{x\in HC(T)} \overline{\mathrm{dens}}\, \mathcal{N}_T\big(x, B(0,\alpha)\big).$$

It is shown in [**30**] that in fact $\overline{\mathrm{dens}}\mathcal{N}_T\big(x, B(0,\alpha)\big) = c(T)$ for a comeager set of vectors $x \in HC(T)$; so we have in particular (for any $\alpha > 0$)

$$c(T) = \sup\left\{c \geq 0;\ \overline{\mathrm{dens}}\Big(\mathcal{N}_T\big(x, B(0,\alpha)\big)\Big) \geq c \quad \text{for a comeager set of } x \in HC(T)\right\}.$$

Note also that $c(T) > 0$ if the operator T is $\mathcal{U}$-frequently hypercyclic.

The last class of operators which we introduce is that of operators $T \in \mathrm{HC}(\mathcal{H})$ such that $c(T)$ is maximal:

$$\mathrm{c}^1(\mathcal{H}) := \big\{T \in \mathrm{HC}(\mathcal{H})\,;\, c(T) = 1\big\}.$$

Since, as proved in [**11**], the set of all distributionally irregular vectors for a given operator $T \in \mathrm{DDCH}(\mathcal{H})$ is comeager in $\mathcal{H}$, it is clear that $\mathrm{DDCH}(\mathcal{H}) \cap \mathrm{HC}(\mathcal{H}) \subseteq \mathrm{c}^1(\mathcal{H})$. Moreover, it is shown in [**30**] (more accurately, half in [**10**] and half in [**30**]) that ergodic operators are densely distributionally chaotic: $\mathrm{ERG}(\mathcal{H}) \subseteq \mathrm{DDCH}(\mathcal{H})$. So we have

$$\mathrm{ERG}(\mathcal{H}) \subseteq \mathrm{DDCH}(\mathcal{H}) \cap \mathrm{HC}(\mathcal{H}) \subseteq \mathrm{c}^1(\mathcal{H}).$$

It is left as an open question in [**30**] to determine whether $\mathrm{HC}(\mathcal{H}) \cap \mathrm{INV}_f(\mathcal{H}) \subseteq \mathrm{c}^1(\mathcal{H})$, or whether at least $\mathrm{FHC}(\mathcal{H}) \subseteq \mathrm{c}^1(\mathcal{H})$. It follows from the proof of the main result of [**40**] that the first inclusion does not hold true: the chaotic operator T constructed there belongs to $\mathrm{HC}(\mathcal{H}) \cap \mathrm{INV}_f(\mathcal{H})$ (as does any chaotic operator on $\mathcal{H}$) but satisfies $c(T) = 0$. However this operator is not frequently hypercyclic, and so the examples of [**40**] do not disprove the second inclusion. We will nonetheless show in the present work that this inclusion does not hold true either (Theorem 7.6).

2.8. A last notation. If $\Gamma(\mathcal{H})$ is a class of operators on $\mathcal{H}$, then, for any $M > 0$, we set

$$\Gamma_M(\mathcal{H}) := \{T \in \Gamma(\mathcal{H});\ \|T\| \leq M\}.$$

3. Organization of the monograph

In Chapter 2, we recall a few basic facts concerning the topologies $\mathtt{SOT}$ and $\mathtt{SOT}^*$, and then prove some "typicality" results in the spaces $\mathfrak{B}_M(\mathcal{H})$ and $\mathrm{HC}_M(\mathcal{H})$, $M > 1$ with respect to the Strong* Operator Topology $\mathtt{SOT}^*$. These results can be summarized as follows: for any $M > 1$, an $\mathtt{SOT}^*$-typical $T \in \mathrm{HC}_M(\mathcal{H})$ is topologically weakly mixing but not mixing (Propositions 2.16 and 2.19), has no eigenvalues (Corollary 2.25), admits no non-trivial invariant measure (Theorem 2.36), but is densely distributionally chaotic (Proposition 2.40).

Chapter 3 is a digression, in which we discuss the descriptive complexity of some of the families of operators introduced above. We show that for any $M > 1$, $\mathrm{TMIX}_M(\mathcal{H})$ and $\mathrm{CH}_M(\mathcal{H})$ are Borel subsets of $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$ of class exactly $\mathbf{\Pi}^0_3$, aka $F_{\sigma\delta}$ (Propositions 3.2 and 3.4), whereas $\mathrm{UFHC}_M(\mathcal{H})$ is Borel of class at most $\mathbf{\Pi}^0_4$, and neither $\mathbf{\Pi}^0_3$ nor $\mathbf{\Sigma}^0_3$ (Corollary 3.24). We also show that some rather natural classes of operators defined by dynamical properties are *non-Borel* in $\mathfrak{B}_M(\mathcal{H})$; for example, the family of all operators $T \in \mathfrak{B}_M(\mathcal{H})$ admitting a hypercyclic restriction to an invariant subspace is non-Borel (Proposition 3.29), as well as the family of distributionally chaotic operators $T \in \mathfrak{B}_M(\mathcal{H})$ (Proposition 3.36). In contrast, the class of densely distributionally chaotic operators $T \in \mathfrak{B}_M(\mathcal{H})$ is G_δ (Proposition 2.40).

In Chapter 4, we consider ergodicity properties of *upper triangular* operators with coefficients of modulus 1 on the diagonal, this time with respect to the Strong Operator Topology $\mathtt{SOT}$. We show first that for any $M > 1$, an $\mathtt{SOT}$-typical upper triangular operator $T \in \mathfrak{B}_M(\mathcal{H})$ is ergodic in the Gaussian sense (Proposition 4.4). On the other hand, we essentially show in Theorem 4.9 that an $\mathtt{SOT}$-typical operator $T \in \mathfrak{B}_M(\mathcal{H})$ of the form "diagonal + backward shift" (where the diagonal operator has coefficients of modulus 1 on the diagonal) is ergodic but admits only countably many eigenvalues (and hence is ergodic but not ergodic in the Gaussian sense).

In Chapter 5, we prove several criteria for an operator T to be $\mathcal{U}$-frequently hypercyclic or frequently hypercyclic. These criteria are rather different in spirit from the by now classical Frequent Hypercyclicity Criterion ([**14**]) or from the more recent criteria which can be found in [**13**] or [**15**], since they rely explicitly on the existence of many *periodic* (or almost periodic) vectors for the operator T. However, our criterion for frequent hypercyclicity turns out to be stronger than the classical one: indeed, any operator satisfying the so-called *Operator Specification Property* also satisfies the assumptions of our criterion, whereas it is known [**2**] that operators satisfying the assumptions of the classical Frequent Hypercyclicity Criterion have the Operator Specification Property (but not conversely, see [**2**]). In the present work, these criteria for $\mathcal{U}$-frequent hypercyclicity and frequent hypercyclicity are instrumental: we use them in order to simplify the proofs of several later results (more precisely, Corollary 5.11, Theorem 5.14 and Theorem 5.35 will be used in the proofs of the main results of Chapter 6). However, we believe that these criteria might be useful in other situations as well; and for this reason we have stated them in the setting of general Banach spaces.

In Chapter 6, which is by far the the most difficult part of this work, we develop a general machinery for producing hypercyclic operators with special properties. This machinery is very much inspired from the construction of [**40**], but things are done here in greater generality. Again, we hope that this approach could be useful to solve other questions as well. The operators we construct depend on a number of parameters, and we are able to determine in a rather precise way how the parameters influence on $\mathcal{U}$-frequent or frequent hypercyclicity, ergodicity or topological mixing.

In Chapter 7, we use the machinery developed in Chapter 6 to produce the examples we are looking for. The main results we obtain are the existence of chaotic and frequently hypercyclic operators which are not in $\mathrm{c}^1(\mathcal{H})$ and hence are not ergodic (Theorem 7.6), the existence of chaotic

and $\mathcal{U}$-frequently hypercyclic operators which are not frequently hypercyclic (Theorem 7.10), and the existence of chaotic and topologically mixing operators which are not $\mathcal{U}$-frequently hypercyclic (Theorem 7.15).

We conclude the monograph (Chapter 8) with a short list of possibly interesting questions.

CHAPTER 2

Typical properties of hypercyclic operators

1. The strong and strong* topologies

The *Strong Operator Topology* (SOT) on $\mathfrak{B}(\mathcal{H})$ is defined as follows: any $T_0 \in \mathfrak{B}(\mathcal{H})$ has a neighborhood basis consisting of sets of the form

$$\mathfrak{U}_{T_0;x_1,\dots,x_s,\varepsilon} := \big\{T \in \mathfrak{B}(\mathcal{H})\,;\, \|(T-T_0)x_i\| < \varepsilon \quad \text{for } i = 1,\dots,s\big\}$$

where $x_1,\dots,x_s \in \mathcal{H}$ and $\varepsilon > 0$. Thus, a net $(T_i) \subseteq \mathfrak{B}(\mathcal{H})$ tends to $T_0 \in \mathfrak{B}(\mathcal{H})$ with respect to SOT if and only if $T_i x \to T_0 x$ in norm for every $x \in H$.

The second (and perhaps a little less well-known) topology we will use is the *Strong* Operator Topology* (SOT*), which is the "self-adjoint" version of SOT. A basis of SOT*-neighborhoods of $T_0 \in \mathfrak{B}(\mathcal{H})$ is provided by the sets of the form

$$\mathfrak{V}_{T_0;x_1,\dots,x_s,\varepsilon} := \big\{T \in \mathfrak{B}(\mathcal{H})\,;\, \|(T-T_0)x_i\| < \varepsilon \text{ and } \|(T-T_0)^*x_i\| < \varepsilon \quad \text{for } i = 1,\dots,s\big\}$$

where $x_1,\dots,x_s \in \mathcal{H}$ and $\varepsilon > 0$. In other words, a net (T_i) tends to $T_0 \in \mathfrak{B}(\mathcal{H})$ with respect to SOT* if and only if $T_i \xrightarrow{\text{SOT}} T_0$ and $T_i^* \xrightarrow{\text{SOT}} T_0^*$. Obviously, SOT is coarser than SOT*, which is in turn coarser than the norm topology. The topologies induced by SOT and SOT* on any closed ball $\mathfrak{B}_M(\mathcal{H})$ are easily seen to be Polish, *i.e.* separable and completely metrizable (see e.g. [**44**, Section 4.6.2]). This will be of primary importance for us.

The following simple lemma will be used repeatedly, sometimes without explicit mention.

Lemma 2.1. *Let $M > 0$. If $\mathfrak{B}(\mathcal{H})$ is endowed with either* SOT *or* SOT**, then the map $(T,S) \mapsto TS$ is continuous from $\mathfrak{B}_M(\mathcal{H}) \times \mathfrak{B}_M(\mathcal{H})$ into $\mathfrak{B}(\mathcal{H})$. Consequently, for any fixed integer $n \geq 1$, the map $T \mapsto T^n$ is* (SOT, SOT)*-continuous and* (SOT*, SOT*)*-continuous from $\mathfrak{B}_M(\mathcal{H})$ into $\mathfrak{B}(\mathcal{H})$.*

Proof. It is enough to check the statement involving the SOT topology; so we have to show that for any fixed vector $x \in \mathcal{H}$, the map $(T,S) \mapsto TSx$ is continuous from $\mathfrak{B}_M(\mathcal{H}) \times \mathfrak{B}_M(\mathcal{H})$ into $\mathcal{H}$. Since the map $(T,S) \mapsto (T,Sx)$ is continuous from $\mathfrak{B}_M(\mathcal{H}) \times \mathfrak{B}_M(\mathcal{H})$ into $\mathfrak{B}_M(\mathcal{H}) \times \mathcal{H}$, it suffices to check that the map $(T,u) \mapsto Tu$ is continuous from $\mathfrak{B}_M(\mathcal{H}) \times \mathcal{H}$ into $\mathcal{H}$. Now, for any operators $T, T_0 \in \mathfrak{B}_M(\mathcal{H})$ and any vectors $u, u_0 \in \mathcal{H}$, we have

$$\|Tu - T_0u_0\| = \|T(u-u_0) + (T-T_0)u_0\| \leq M\|u-u_0\| + \|(T-T_0)u_0\|.$$

The result follows immediately. □

It is easy to deduce from Lemma 2.1 that $\mathrm{HC}_M(\mathcal{H})$ is SOT-G_δ in $\mathfrak{B}_M(\mathcal{H})$ for any $M > 1$. We state this as

Corollary 2.2. *For any $M > 1$, $\mathrm{HC}_M(\mathcal{H})$ is* SOT*-G_δ and* SOT**-G_δ in $\mathfrak{B}_M(\mathcal{H})$. Hence $(\mathrm{HC}_M(\mathcal{H}), \text{SOT})$ and $(\mathrm{HC}_M(\mathcal{H}), \text{SOT}^*)$ are Polish spaces.*

Proof. It follows from Lemma 2.1 that if A and V are two non-empty subsets of $\mathcal{H}$ with V open, then, for any fixed $n \geq 1$, the set

$$\mathfrak{O}_{n;A,V} := \{T \in \mathfrak{B}_M(\mathcal{H});\ T^n(A) \cap V \neq \emptyset\}$$

is SOT-open in $\mathfrak{B}_M(\mathcal{H})$. Now, fix a countable basis of non-empty open sets $(V_p)_{p\geq 1}$ of $\mathcal{H}$, and observe that

$$\mathrm{HC}_M(\mathcal{H}) = \bigcap_{p,\,q\geq 1} \bigcup_{n\geq 1} \big\{T \in \mathfrak{B}_M(\mathcal{H})\,;\, T^n(V_p) \cap V_q \neq \emptyset\big\} = \bigcap_{p,\,q\geq 1} \bigcup_{n\geq 1} \mathfrak{O}_{n;V_p,V_q}.$$

This proves that $\mathrm{HC}_M(\mathcal{H})$ is G_δ in $\mathfrak{B}_M(\mathcal{H})$ with respect to $\mathtt{SOT}$. Since the topology $\mathtt{SOT}^*$ is finer than the topology $\mathtt{SOT}$, $\mathrm{HC}_M(\mathcal{H})$ is also G_δ in $\mathfrak{B}_M(\mathcal{H})$ with respect to $\mathtt{SOT}^*$. □

We now state a less immediate fact, which will be proved in Corollary 2.12 below. The corresponding (weaker) $\mathtt{SOT}$ statement can be found in [**17**], and its $\mathtt{SOT}^*$ analogue undoubtedly would have been proved there if there had been any need to do so.

PROPOSITION 2.3. *For any $M > 1$, $\mathrm{HC}_M(\mathcal{H})$ is dense in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. Hence, $\mathrm{HC}_M(\mathcal{H})$ is comeager in $\mathfrak{B}_M(\mathcal{H})$, both for the $\mathtt{SOT}$ and the $\mathtt{SOT}^*$ topology.*

Note that this result, although not very difficult, is not trivial either. For example, since the map $T \mapsto T^*$ is a homeomorphism of $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$, it immediately implies that a typical operator $T \in \mathfrak{B}_M(\mathcal{H})$ is hypercyclic and has a hypercyclic adjoint. In particular, this argument proves that *there exist* hypercyclic operators whose adjoint is also hypercyclic; which is a classical result of Salas obtained via an explicit construction in [**45**].

We deduce immediately from Proposition 2.3 that the word "typical" has the same meaning in the whole of $\mathfrak{B}_M(\mathcal{H})$ or in the subclass $\mathrm{HC}_M(\mathcal{H})$ of hypercyclic operators in $\mathfrak{B}_M(\mathcal{H})$. This will be used repeatedly below.

COROLLARY 2.4. *Let $\Gamma(\mathcal{H})$ be a class of operators on $\mathcal{H}$, and let $M > 1$. If $\Gamma_M(\mathcal{H})$ is a dense G_δ subset of $(\mathfrak{B}_M(\mathcal{H}), \tau)$, where τ is either $\mathtt{SOT}$ or $\mathtt{SOT}^*$, then $\Gamma_M(\mathcal{H}) \cap \mathrm{HC}(\mathcal{H})$ is a dense G_δ subset of $(\mathrm{HC}_M(\mathcal{H}), \tau)$. Conversely, if $\Gamma(\mathcal{H}) \subseteq \mathrm{HC}(\mathcal{H})$ and if $\Gamma_M(\mathcal{H})$ is dense G_δ (resp. comeager) in $(\mathrm{HC}_M(\mathcal{H}), \tau)$, then $\Gamma_M(\mathcal{H})$ is dense G_δ (resp. comeager) in $(\mathfrak{B}_M(\mathcal{H}), \tau)$.*

PROOF. This is obvious: if $\Gamma_M(\mathcal{H})$ is dense G_δ in $(\mathfrak{B}_M(\mathcal{H}), \tau)$ then, by Corollary 2.2, Proposition 2.3 and the Baire category theorem, $\Gamma_M(\mathcal{H}) \cap \mathrm{HC}(\mathcal{H})$ is dense G_δ in $\mathfrak{B}_M(\mathcal{H})$, and hence in $\mathrm{HC}_M(\mathcal{H})$; and likewise for the converse. The "comeager" case follows from the "G_δ" case. □

Here is a last fact concerning the topology $\mathtt{SOT}^*$ that will be quite useful for us. Note that the corresponding statement for $\mathtt{SOT}$ is *false*. This is an important difference between the two topologies, which explains in particular why we will encounter some subsets of $\mathfrak{B}_M(\mathcal{H})$ which are G_δ with respect to $\mathtt{SOT}^*$ and not with respect to $\mathtt{SOT}$.

LEMMA 2.5. *Let us denote by w the weak topology of $\mathcal{H}$. If B is a bounded subset of $\mathcal{H}$, then the map $(T, x) \mapsto Tx$ is continuous from $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*) \times (B, w)$ into $(\mathcal{H}, w)$.*

PROOF. We have to show that for any fixed vector $e \in \mathcal{H}$, the map $(T, x) \mapsto \langle Tx, e\rangle$ is continuous on $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*) \times (B, w)$. The key point is that one can separate T from x by writing $\langle Tx, e\rangle = \langle x, T^*e\rangle$ (this trick would be useless, of course, if we were working with the $\mathtt{SOT}$ topology).

Let (T_i, x_i) be a net in $\mathfrak{B}(\mathcal{H}) \times B$ converging to some element $(T, x) \in \mathfrak{B}(\mathcal{H}) \times B$; that is, $T_i \xrightarrow{\mathtt{SOT}^*} T$ and $x_i \xrightarrow{w} x$. Then

$$\begin{aligned}\big|\langle T_i x_i, e\rangle - \langle Tx, e\rangle\big| &= \big|\langle x_i, T_i^* e\rangle - \langle x, T^* e\rangle\big| \\ &\le \big|\langle x_i, T_i^* e - T^* e\rangle\big| + \big|\langle x_i - x, T^* e\rangle\big|.\end{aligned}$$

Since the net (x_i) is bounded in norm and $x_i \xrightarrow{w} x$, this shows that $\langle T_i x_i, e\rangle \to \langle Tx, e\rangle$. □

1.1. Why $\mathtt{SOT}$ and $\mathtt{SOT}^*$? There are other natural topologies on $\mathfrak{B}(\mathcal{H})$, most notably the operator norm topology, of course, and the *Weak Operator Topology* ($\mathtt{WOT}$). The norm topology is not very well-suited for Baire category arguments, mainly because it is much too strong; in particular, the lack of separability seems unacceptable. The topology $\mathtt{WOT}$ is better behaved in this respect, being Polish on any closed ball $\mathfrak{B}_M(\mathcal{H})$. However, since we are interested in typical properties of *hypercyclic* operators, it seems better to consider topologies with respect to which $\mathrm{HC}_M(\mathcal{H})$ is comeager in $\mathfrak{B}_M(\mathcal{H})$ for any $M > 1$. This is definitely not the case for $\mathtt{WOT}$. Indeed,

it is proved in [**20**] that a typical element of $(\mathfrak{B}_1(\mathcal{H}), \mathtt{WOT})$ is *unitary*. It follows that a $\mathtt{WOT}$-typical $T \in \mathfrak{B}_M(\mathcal{H})$ is a multiple of a unitary operator and hence not hypercyclic. Incidentally, the operator norm topology has the same "drawback", in an even stronger way: the hypercyclic operators (actually, even the cyclic operators) are in fact *nowhere dense* in $\mathfrak{B}(\mathcal{H})$; see [**8**, Section 2.5].

So we will consider neither the operator norm topology nor the Weak Operator Topology in this work. Actually, when working in the whole of $\mathfrak{B}_M(\mathcal{H})$ or $\mathrm{HC}_M(\mathcal{H})$, $M > 1$, we will always use $\mathtt{SOT}^*$ rather than $\mathtt{SOT}$. The reason is that, with respect to $\mathtt{SOT}$, a result from [**22**] gives a rather complete picture as far as typical properties are concerned: *a typical element of* $(\mathfrak{B}_1(\mathcal{H}), \mathtt{SOT})$ *is unitarily equivalent to the operator* $B^{(\infty)}$*, the countable direct* ℓ_2*-sum of the unilateral backward shift* B *on* $\ell_2(\mathbb{N})$. It follows that for every $M > 1$, the class of operators $T \in \mathrm{HC}_M(\mathcal{H})$ which are unitarily equivalent to $MB^{(\infty)}$ is comeager in $(\mathrm{HC}_M(\mathcal{H}), \mathtt{SOT})$. Now, the dynamical properties of the operators $MB^{(\infty)}$, $M > 1$ are quite strong and very well understood: these operators are mixing in the Gaussian sense (and hence ergodic and topologically mixing), densely distributionally chaotic, and they have nearly any other strong dynamical property one might think of. This explains why, when trying to determine which properties are typical within the class of *all* hypercyclic operators, we will use $\mathtt{SOT}^*$ rather than $\mathtt{SOT}$. The situation in this setting is more involved, and thus leads to more interesting results.

On the other hand, we will see that within specific subclasses of $\mathrm{HC}(\mathcal{H})$ consisting of upper triangular operators, the topology $\mathtt{SOT}$ becomes much more useful. This is not really surprising, since triangularity is not exactly a self-adjoint property.

2. How to prove density results

For future reference, we state here a simple criterion for a class of operators to be dense in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT})$ or $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$, $M > 0$. We will use it repeatedly in the sequel.

Lemma 2.6. *Let* $\Gamma(\mathcal{H})$ *be a class of operators on* $\mathcal{H}$*, and let* $M > 0$*. Let also* $(e_k)_{k\geq 1}$ *be an orthonormal basis of* $\mathcal{H}$*, and for each* $r \geq 1$*, denote by* $\mathcal{H}_r$ *the finite-dimensional subspace* $\operatorname{span}[e_k\,;\, 1 \leq k \leq r]$ *of* $\mathcal{H}$*. Assume that for every* $r \geq 1$*, for any operator* $A \in \mathfrak{B}(\mathcal{H}_r)$ *with* $\|A\| < M$ *and for every* $\varepsilon > 0$*, there exists an operator* $T \in \Gamma_M(\mathcal{H})$ *such that*

$$(*) \qquad \|(T - A)e_k\| < \varepsilon \qquad \text{for } k = 1, \dots, r.$$

Then $\Gamma_M(\mathcal{H})$ *is dense in* $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT})$*. If* $(*)$ *is replaced by its self-adjoint version*

$$(**) \qquad \|(T - A)e_k\| < \varepsilon \quad \text{and} \quad \|(T - A)^* e_k\| < \varepsilon \qquad \text{for } k = 1, \dots, r,$$

then $\Gamma_M(\mathcal{H})$ *is dense in* $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$*.*

Note that there is a slight abuse of notation in the statement of Lemma 2.6: we consider the operator $A \in \mathfrak{B}(\mathcal{H}_r)$ as an operator on $\mathcal{H}$ by identifying it with $P_r^* A P_r$, where $P_r : \mathcal{H} \to \mathcal{H}_r$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_r$.

Proof. We will prove the assertion concerning the $\mathtt{SOT}^*$-topology, the proof of the $\mathtt{SOT}$ statement being exactly the same. Fix $T_0 \in \mathfrak{B}_M(\mathcal{H})$, $\varepsilon > 0$, and $x_1, \dots, x_s \in \mathcal{H}$. Without loss of generality, we can suppose that $\|T_0\| < M$. We are looking for an operator $T \in \Gamma_M(\mathcal{H})$ such that

$$(1) \qquad \max_{1\leq j\leq s} \max\big(\|(T - T_0)x_j\|, \|(T - T_0)^* x_j\|\big) < \varepsilon.$$

Since $\|T\| \leq M$ for every $T \in \Gamma_M(\mathcal{H})$, and since every vector x_j can be approximated by a finite linear combination of the basis vectors e_k, there exists an integer $r_0 \geq 1$ sufficiently large such that (1) above holds true as soon as

$$\max_{1\leq k\leq r_0} \max\big(\|(T - T_0)e_k\|, \|(T - T_0)^* e_k\|\big) < \frac{\varepsilon}{2}\cdot$$

For each $r \geq r_0$, consider the operator $A_r = P_r T_0 P_r$ Observe that $\|A_r\| \leq \|T_0\| < M$. By our assumption, there exists $T \in \Gamma(\mathcal{H})$ such that

$$\max_{1\leq k\leq r} \max\big(\|(T-A_r)e_k\|, \|(T-A_r)^* e_k\|\big) < \frac{\varepsilon}{2}\cdot$$

Now

$$\|(T-T_0)e_k\| \leq \|(T-A_r)e_k\| + \|(P_rT_0P_r - T_0)e_k\| < \frac{\varepsilon}{2} + \|(P_r - I)T_0 e_k\|$$

and

$$\|(T-T_0)^* e_k\| < \frac{\varepsilon}{2} + \|(P_r - I)T_0^* e_k\|$$

for every $1 \leq k \leq r$. Since $\lim_{r\to\infty} P_r = I$ for the $\mathtt{SOT}$ topology, one can choose r so large that

$$\max_{1\leq k\leq r_0} \max\big(\|(P_r - I)T_0 e_k\|, \|(P_r - I)T_0^* e_k\|\big) < \varepsilon,$$

from which the conclusion of Lemma 2.6 follows. □

Remark 2.7. It is sometimes more convenient to endow $\mathcal{H}$ with an orthonormal basis $(f_k)_{k\in\mathbb{Z}}$ indexed by $\mathbb{Z}$ rather than by $\mathbb{N}$. In this case, the corresponding version of Lemma 2.6 reads as follows.

For each $r \geq 0$, denote by $\mathcal{H}_r$ the finite-dimensional subspace span $[f_k\,;\, -r \leq k \leq r]$ of $\mathcal{H}$. Assume that for every $r \geq 0$, for any operator $A \in \mathfrak{B}(\mathcal{H}_r)$ with $\|A\| < M$ and every $\varepsilon > 0$, there exists an operator $T \in \Gamma_M(\mathcal{H})$ such that

$$(*) \qquad \|(T-A)f_k\| < \varepsilon \qquad \text{for } k = -r, \dots, r.$$

Then $\Gamma_M(\mathcal{H})$ is dense in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT})$. If $(*)$ is replaced by

$$(**) \qquad \|(T-A)f_k\| < \varepsilon \quad \text{and} \quad \|(T-A)^* f_k\| < \varepsilon \qquad \text{for } k = -r, \dots, r,$$

then $\Gamma_M(\mathcal{H})$ is dense in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.

3. Construction of mixing operators, and density of G-MIX$(\mathcal{H})$

In order to show that a property is typical, we first need to prove the density of the set of operators satisfying it. In this section, we show that the class G-MIX$_M(\mathcal{H})$ is dense in $(\mathrm{HC}_M(\mathcal{H}), \mathtt{SOT}^*)$ for any $M > 1$. This will be achieved by considering perturbations of weighted unilateral or bilateral weighted shifts with respect to some orthonormal basis of $\mathcal{H}$. We show that these operators admit spanning eigenvector fields which are analytic in a neighborhood of the unit circle, and hence are mixing in the Gaussian sense. The precise statement we will use is the following.

Lemma 2.8. *Let $T \in \mathfrak{B}(\mathcal{H})$. Assume that there exists a connected open set $\Omega \subseteq \mathbb{C}$ with $\Omega \cap \mathbb{T} \neq \emptyset$ and a family $(E_i)_{i\in I}$ of holomorphic maps, $E_i : \Omega \to \mathcal{H}$, such that $TE_i(\lambda) = \lambda E_i(\lambda)$ for every $i \in I$ and every $\lambda \in \Omega$, and $\overline{\mathrm{span}}\,\{E_i(\lambda);\ i \in I\,,\ \lambda \in \Omega\} = \mathcal{H}$. Then T is mixing in the Gaussian sense.*

Proof. Recall that we denote by $\mathcal{E}(T)$ the set of all unimodular eigenvectors of T, and by $\lambda(x)$ the eigenvalue associated to $x \in \mathcal{E}(T)$. By [**5**, Th. 3.29], it is enough to show that for any Borel set $D \subseteq \mathbb{T}$ of Lebesgue measure 0, we have $\overline{\mathrm{span}}\,\{x \in \mathcal{E}(T);\ \lambda(x) \notin D\} = \mathcal{H}$. Let $y \in \mathcal{H}$ be orthogonal to the set $\{x \in \mathcal{E}(T);\ \lambda(x) \notin D\}$. Then $\langle y, E_i(\lambda)\rangle = 0$ for every $i \in I$ and every $\lambda \in (\Omega \cap \mathbb{T}) \setminus D$. Since the functions $\langle y, E_i(\,\cdot\,)\rangle$ are holomorphic on Ω, and since $(\Omega \cap \mathbb{T}) \setminus D$ certainly has an accumulation point in Ω (because D has Lebesgue measure 0), it follows that y is orthogonal to all vectors $E_i(\lambda)$, $i \in I$, $\lambda \in \Omega$, and hence that $y = 0$. This concludes the proof. □

Remark 2.9. The assumptions of Lemma 2.8 imply that the operator T is also chaotic: since the roots of unity contained in Ω have an accumulation point in Ω, this follows as above from the identity principle for holomorphic functions.

Let us first consider perturbations of unilateral weighted shifts. Let $(e_k)_{k\geq 1}$ be an orthonormal basis of the Hilbert space $\mathcal{H}$, and let $\boldsymbol{\omega} = (\omega_k)_{k\geq 1}$ be a *unilateral weight sequence*, *i.e.* a bounded sequence of positive real numbers. Let $r \geq 1$ be an integer, and let A be an operator acting on the finite-dimensional space $\mathcal{H}_r = \mathrm{span}[e_k\,;\, 1 \leq k \leq r]$. We define a bounded operator $B_{A,\boldsymbol{\omega}}$ on $\mathcal{H}$ by setting

$$B_{A,\boldsymbol{\omega}} e_k = \begin{cases} Ae_k & \text{for every } 1 \leq k \leq r \\ \omega_{k-r} e_{k-r} & \text{for every } k > r. \end{cases}$$

Proposition 2.10. *Let $\boldsymbol{\omega}$ be a unilateral weight sequence, $r \geq 1$, and $A \in \mathfrak{B}(\mathcal{H}_r)$. Suppose that for every $1 \leq l \leq r$,*

$$R_l := \liminf_{p\to\infty} \big(\omega_{pr+l} \cdots \omega_{r+l}\omega_l\big)^{1/p} > \max(1, \|A\|).$$

Then the operator $B_{A,\boldsymbol{\omega}}$ is mixing in the Gaussian sense. Besides, $B_{A,\boldsymbol{\omega}}$ is also chaotic.

Proof. Solving formally the equation $B_{A,\boldsymbol{\omega}} x = \lambda x$, where $\lambda \in \mathbb{C}$ and $x = (x_k)_{k\geq 1} \in \mathbb{C}^{\mathbb{N}}$, one gets the following identities:

$$(\lambda - A)P_r x = \sum_{l=1}^{r} \omega_l x_{l+r} e_l \qquad \text{and} \qquad \omega_k x_{k+r} = \lambda x_k \quad \text{for every } k > r.$$

From this (setting $y := P_r x$) we infer that the eigenvectors of $B_{A,\boldsymbol{\omega}}$ associated to the eigenvalue λ must be given by the formula

$$E_y(\lambda) = y + \sum_{l=1}^{r} \frac{1}{\omega_l} \langle (\lambda - A)y, e_l\rangle \left(e_{r+l} + \sum_{p\geq 2} \frac{\lambda^{p-1}}{\omega_{(p-1)r+l} \cdots \omega_{r+l}} e_{pr+l} \right) \tag{2}$$

where y is a non-zero vector of $\mathcal{H}_r$. Conversely, if y belongs to $\mathcal{H}_r \setminus \{0\}$ is such that the above formula makes sense, then $E_y(\lambda)$ is an eigenvector of $B_{A,\boldsymbol{\omega}}$ with associated eigenvalue λ. It follows that the complex number λ is an eigenvalue of $B_{A,\boldsymbol{\omega}}$ as soon as

$$\sum_{p\geq 2} \left| \frac{\lambda^{p-1}}{\omega_{(p-1)r+l} \cdots \omega_l} \right|^2 < \infty \qquad \text{for all } 1 \leq l \leq r,$$

which holds true whenever $|\lambda| < R := \min_{1\leq l\leq r} R_l$. In this case, the eigenvector field E_y is well-defined and holomorphic on the open disk $D(0, R)$ for every $y \in \mathcal{H}_r$. Note that our assumption implies that $R > 1$, so that the disk $D(0, R)$ contains $\mathbb{T}$.

By Lemma 2.8 and Remark 2.9, in order to show that $B_{A,\boldsymbol{\omega}}$ belongs to G-MIX($\mathcal{H}$) and is chaotic, it suffices to check that the eigenvectors $E_y(\lambda)$, $y \in \mathcal{H}_r$, $|\lambda| < R$, span a dense subspace of $\mathcal{H}$. So let $u \in \mathcal{H}$ be such that $\langle E_y(\lambda), u\rangle = 0$ for every $y \in \mathcal{H}_r$ and every $|\lambda| < R$. Writing u as $u = \sum_{k\geq 1} u_k e_k$, this means that

$$\langle y, u\rangle + \sum_{l=1}^{r} \frac{1}{\omega_l} \langle y, (\lambda - A)^* e_l\rangle \left(\overline{u}_{r+l} + \sum_{p\geq 2} \overline{u}_{pr+l} \frac{\lambda^{p-1}}{\omega_{(p-1)r+l} \cdots \omega_{r+l}} \right) = 0$$

for every $y \in \mathcal{H}_r$. It follows that each vector

$$u + \sum_{l=1}^{r} \frac{1}{\omega_l} \left(u_{r+l} + \sum_{p\geq 2} u_{pr+l} \frac{\overline{\lambda}^{p-1}}{\omega_{(p-1)r+l} \cdots \omega_{r+l}} \right) (\lambda - A)^* e_l, \qquad |\lambda| < R$$

is orthogonal to $\mathcal{H}_r$, *i.e.* belongs to the closed linear span of the vectors e_k, $k > r$. In other words,

$$P_r u + \sum_{l=1}^{r} \frac{1}{\omega_l} \left(u_{r+l} + \sum_{p\geq 2} u_{pr+l} \frac{\overline{\lambda}^{p-1}}{\omega_{(p-1)r+l} \cdots \omega_{r+l}} \right) (\lambda - A)^* e_l = 0.$$

Consider now the open subset $\Omega = D(0,R) \setminus \sigma(A)$ of $\mathbb{C}$, where $\sigma(A)$ denotes the spectrum of A. Applying the operator $(\lambda - A)^{*-1}$ to the previous equation, we obtain that for every $\lambda \in \Omega$:

$$(\lambda - A)^{*-1} P_r u = -\sum_{l=1}^{r} \frac{1}{\omega_l} \left(u_{r+l} + \sum_{p \geq 2} u_{pr+l} \frac{\overline{\lambda}^{p-1}}{\omega_{(p-1)r+l} \dots \omega_{r+l}} \right) e_l. \tag{3}$$

Since the expression on the right hand side of (3) defines an antiholomorphic map on $D(0,R)$ and since the disk $D(0,R)$ contains $\sigma(A)$ (recall that $R > \|A\|$ by assumption), it follows that the map $\lambda \mapsto (\lambda - A)^{*-1} P_r u$ extends antiholomorphically to the whole complex plane. But $(\lambda - A)^{*-1} P_r u \to 0$ as $|\lambda| \to \infty$, so the function $\lambda \mapsto (\lambda - A)^{*-1} P_r u$ must vanish identically on Ω by Liouville's Theorem, which is possible only if $P_r u = 0$.

Going back to (3), the fact that $P_r u = 0$ yields that $u_{pr+l} = 0$ for every $1 \leq l \leq r$ and every $p \geq 2$, and that $u_{r+l} = 0$ for every $1 \leq l \leq r$. Thus $u = 0$, and this concludes the proof of Proposition 2.10. □

Remark 2.11. The definition of the operator $B_{A,\boldsymbol{\omega}}$ shows that every eigenvalue of A is also an eigenvalue of $B_{A,\boldsymbol{\omega}}$. This explains why the conditions $R_l > 1$ for every $l = 1, \dots, r$ are not sufficient to ensure that $B_{A,\boldsymbol{\omega}}$ be mixing in the Gaussian sense. Indeed, if $\lambda \in \sigma(A)$ is such that $|\lambda| > \max_{1 \leq l \leq r} R_l$, then λ is an isolated eigenvalue of $B_{A,\boldsymbol{\omega}}$ with $|\lambda| > 1$, and this prevents $B_{A,\boldsymbol{\omega}}$ from being hypercyclic. This is to be compared with Remark 2.15 below.

From Proposition 2.10 we easily deduce a basic density result, which gives in particular the promised proof of Proposition 2.3 above.

Corollary 2.12. *For every $M > 1$, the class of operators* $\text{G-MIX}_M(\mathcal{H}) \cap \text{CH}_M(\mathcal{H})$ *is dense in* $(\text{HC}_M(\mathcal{H}), \texttt{SOT}^*)$.

Proof. We are going to apply Lemma 2.6. So, let $(e_k)_{k \geq 1}$ be an orthonormal basis of $\mathcal{H}$, and let us fix $r \geq 1$, an operator $A \in \mathfrak{B}(\mathcal{H}_r)$ with $\|A\| < M$ and $\varepsilon > 0$. Let also $\delta > 0$ be a small positive number to be specified below. We define a weight sequence $\boldsymbol{\omega} = (\omega_k)_{k \geq 1}$ as follows:

$$\omega_k = \begin{cases} \delta & \text{for every } 1 \leq k \leq r \\ M & \text{for every } k > r; \end{cases}$$

and we consider the associated operator $B_{A,\boldsymbol{\omega}}$ acting on $\mathcal{H}$. Identifying A with $P_r^* A P_r \in \mathfrak{B}(\mathcal{H})$, we have for every $1 \leq k \leq r$:

$$B_{A,\boldsymbol{\omega}} e_k = A e_k \qquad \text{and} \qquad B_{A,\boldsymbol{\omega}}^* e_k = A^* e_k + \delta e_{k+r}\,,$$

so that $\|(B_{A,\boldsymbol{\omega}} - A)^* e_k\| = \delta$. It follows that if $\delta < \varepsilon$, then

$$\max_{1 \leq k \leq r} \max\left(\|\big(B_{A,\boldsymbol{\omega}} - A)e_k\|, \|(B_{A,\boldsymbol{\omega}} - A)^* e_k\|\right) < \varepsilon.$$

The assumption of Proposition 2.10 is clearly satisfied, so that $B_{A,\boldsymbol{\omega}}$ belongs to $\text{G-MIX}(\mathcal{H}) \cap \text{CH}(\mathcal{H})$.

To estimate the norm of $B_{A,\boldsymbol{\omega}}$, note that for every $x = \sum_{j \geq 1} x_j e_j \in H$, we have

$$B_{A,\boldsymbol{\omega}} x = A P_r x + \sum_{k=1}^{r} x_{k+r}\, \delta e_k + \sum_{k > r} x_{k+r}\, M e_k$$

so that

$$\| B_{A,\boldsymbol{\omega}} x \|^2 = \left\| A P_r x + \delta \sum_{k=1}^{r} x_{k+r} e_k \right\|^2 + M^2 \sum_{k > r} |x_{k+r}|^2.$$

Since $\|A\| < M$, it follows that $\|B_{A,\boldsymbol{\omega}}\| = M$ if $\delta > 0$ is sufficiently small. So $B_{A,\boldsymbol{\omega}}$ belongs to $\text{G-MIX}_M(\mathcal{H}) \cap \text{CH}_M(\mathcal{H})$, and Lemma 2.6 now allows us to conclude the proof of Corollary 2.12. □

REMARK 2.13. Corollary 2.12 does not state that $\text{G-MIX}_M(\mathcal{H}) \cap \text{CH}_M(\mathcal{H})$ is comeager in $(\text{HC}_M(\mathcal{H}), \texttt{SOT}^*)$. Indeed, we will prove below that $\text{G-MIX}_M(\mathcal{H})$ and $\text{CH}_M(\mathcal{H})$ are actually meager in $(\text{HC}_M(\mathcal{H}), \texttt{SOT}^*)$.

Let us now turn to a bilateral analogue of Proposition 2.10, which we state as Proposition 2.14 below. Let $(f_k)_{k\in\mathbb{Z}}$ be an orthonormal basis of the space $\mathcal{H}$, and let $\boldsymbol{\omega} = (\omega_k)_{k\in\mathbb{Z}}$ be a *bilateral weight sequence*, *i.e.* a bounded sequence of positive real numbers indexed by $\mathbb{Z}$. For any integer $r \geq 0$, we write $\mathcal{H}_r = \text{span}[e_k\,;\,|k| \leq r]$, and let $A \in \mathfrak{B}(\mathcal{H}_r)$ be a bounded operator on $\mathcal{H}_r$. We define a bounded operator $S_{A,\boldsymbol{\omega}}$ on $\mathcal{H}$ by setting

$$S_{A,\boldsymbol{\omega}} f_k = \begin{cases} Af_k + \omega_{k-(2r+1)} f_{k-(2r+1)} & \text{for every } |k| \leq r \\ \omega_{k-(2r+1)} f_{k-(2r+1)} & \text{for every } |k| > r. \end{cases}$$

PROPOSITION 2.14. *Let $\boldsymbol{\omega}$ be a bilateral weight sequence, $r \geq 1$ and $A \in \mathfrak{B}(\mathcal{H}_r)$. Suppose that for every $-r \leq l \leq r$,*

$$R_l := \liminf_{p\to\infty} \big(\omega_{p(2r+1)+l} \cdots \omega_{(2r+1)+l}\big)^{1/p} > 1$$

and

$$r_l := \limsup_{p\to\infty} \big(\omega_{l-p(2r+1)} \cdots \omega_{l-(2r+1)}\big)^{1/p} < 1.$$

Then $S_{A,\boldsymbol{\omega}}$ is mixing in the Gaussian sense and chaotic.

PROOF. The proof is so similar to that of Proposition 2.10 that we will not give it in detail. A complex number λ is an eigenvalue of $S_{A,\boldsymbol{\omega}}$ as soon as the series

$$\sum_{p\geq 2} \left| \frac{\lambda^{p-1}}{\omega_{(p-1)(2r+1)+l} \cdots \omega_{(2r+1)+l}} \right|^2 \quad \text{and} \quad \sum_{p\geq 1} \left| \frac{\omega_{l-p(2r+1)} \cdots \omega_{l-(2r+1)}}{\lambda^p} \right|^2$$

are convergent for all $-r \leq l \leq r$. If we define $R := \min_{-r\leq l\leq r} R_l$ and $r := \max_{-r\leq l\leq r} r_l$, our assumption implies that $r < 1 < R$. Any complex number λ belonging to the annulus $\{r < |\lambda| < R\}$ is an eigenvalue of $S_{A,\boldsymbol{\omega}}$, and the eigenvectors of $S_{A,\boldsymbol{\omega}}$ associated to λ have the form

$$\begin{aligned} E_y(\lambda) = \sum_{l=-r}^{r} \frac{1}{\omega_l} \langle (\lambda - A)y, e_l \rangle \Bigg(& e_{(2r+1)+l} + \sum_{p\geq 2} \frac{\lambda^{p-1}}{\omega_{(p-1)(2r+1)+l} \cdots \omega_{(2r+1)+l}} e_{p(2r+1)+l} \\ & + \sum_{p\geq 1} \frac{\omega_{l-p(2r+1)} \cdots \omega_{l-(2r+1)}}{\lambda^p} e_{-p(2r+1)+l} \Bigg) \end{aligned} \tag{4}$$

where $y \in \mathcal{H}_r$. Since $r < 1 < R$, the annulus $\{r < |\lambda| < R\}$ contains $\mathbb{T}$, and an argument similar to the one given in the proof of Proposition 2.10 show that $S_{A,\boldsymbol{\omega}}$ belongs to $\text{G-MIX}(H) \cap \text{CH}_M(\mathcal{H})$. Indeed, if $u \in \mathcal{H}$ is such that $\langle E_y(\lambda), u\rangle = 0$ for every $y \in \mathcal{H}_r$ and every $r < |\lambda| < R$, then

$$\begin{aligned} \sum_{l=-r}^{r} \frac{1}{\omega_l} \Bigg(& u_{(2r+1)+l} + \sum_{p\geq 2} \frac{\overline{\lambda}^{p-1}}{\omega_{(p-1)(2r+1)+l} \cdots \omega_{(2r+1)+l}} u_{p(2r+1)+l} \\ & + \sum_{p\geq 1} \frac{\omega_{l-p(2r+1)} \cdots \omega_{l-(2r+1)}}{\overline{\lambda}^p} u_{-p(2r+1)+l} \Bigg) (\lambda - A)^* e_l = 0 \end{aligned}$$

for every $r < |\lambda| < R$ and every $l = -r, \ldots, r$, from which it follows that

$$\begin{aligned} \sum_{l=-r}^{r} \frac{1}{\omega_l} \Bigg(& u_{(2r+1)+l} + \sum_{p\geq 2} \frac{\overline{\lambda}^{p-1}}{\omega_{(p-1)(2r+1)+l} \cdots \omega_{(2r+1)+l}} u_{p(2r+1)+l} \\ & + \sum_{p\geq 1} \frac{\omega_{l-p(2r+1)} \cdots \omega_{l-(2r+1)}}{\overline{\lambda}^p} u_{-p(2r+1)+l} \Bigg) = 0 \end{aligned}$$

for every $r < |\lambda| < R$ and every $l = -r, \ldots, r$. Hence $u = 0$, and Proposition 2.14 is proved. □

REMARK 2.15. The description of the eigenvectors of $S_{A,\boldsymbol{\omega}}$ given in (4) above shows that if the two series

$$\sum_{p\geq 1}\big(\omega_{p(2r+1)+l}\cdots\omega_{(2r+1)+l}\big)^{-2} \quad \text{and} \quad \sum_{p\geq 1}\big(\omega_{l-p(2r+1)}\cdots\omega_{l-(2r+1)}\big)^{-2}$$

are divergent for every $|l|\leq r$, then the operator $S_{A,\boldsymbol{\omega}}$ has no eigenvalue, whatever the choice of $A\in\mathfrak{B}(\mathcal{H}_r)$. This observation will be useful for the proof of Proposition 2.24 below. It is also interesting to observe that the assumptions on R_l and r_l in Proposition 2.14 do not involve the operator A, contrary to what happens in Proposition 2.10. This is coherent with the fact that the eigenvalues of A do not necessarily appear as eigenvalues of $S_{A,\boldsymbol{\omega}}$, while they do appear as eigenvalues of $B_{A,\boldsymbol{\omega}}$.

4. Topological weak mixing and topological mixing

In this section, we show that topological weak mixing is a typical property, whereas topological mixing is atypical. In view of the corresponding well-known analogues in ergodic theory due to Halmos and Rohlin (see e.g. [**34**, pp. 77-80], and [**21**] for a more general result), this should not be surprising at all.

PROPOSITION 2.16. *For every $M>1$, the class $\mathrm{TWMIX}_M(\mathcal{H})$ is a dense G_δ (and hence comeager) subset of $(\mathrm{HC}_M(\mathcal{H}),\mathtt{SOT}^*)$.*

PROOF. That $\mathrm{TWMIX}_M(\mathcal{H})$ is G_δ follows from the fact that $\mathrm{HC}_M(\mathcal{H}\times\mathcal{H})$ is G_δ in $\mathfrak{B}_M(\mathcal{H}\times\mathcal{H})$, and the $\mathtt{SOT}^*$-continuity of the map $T\mapsto T\times T$. Since operators in G-MIX$(\mathcal{H})$ are topologically mixing, density follows from Corollary 2.12. □

Our next proposition (Proposition 2.19) states that topologically mixing operators form a meager class in $(\mathrm{HC}_M(\mathcal{H}),\mathtt{SOT}^*)$. Its proof relies on Lemmas 2.32 and 2.33, which belong to the forthcoming Section 6. We nonetheless prefer to present things in this order, because the purposes of Lemmas 2.32 and 2.33 will appear more clearly in our proof of the typicality of operators without non-trivial invariant measures. But for the reader's convenience, we state these two lemmas here as Lemmas 2.17 and 2.18 respectively. Here is the first one (corresponding to Lemma 2.32 in Section 6):

LEMMA 2.17. *Let B be a closed ball of $\mathcal{H}$ not containing the point 0. Let also $M>0$. For any integer $n\geq 1$, the set*

$$\mathfrak{O}_{n,B}=\big\{T\in\mathfrak{B}_M(\mathcal{H})\,;\ \textit{there exist } n \textit{ distinct integers } p_1,\dots,p_n \textit{ such that } T^{p_i}(B)\cap T^{p_j}(B)=\emptyset \textit{ for every } i\neq j,\ 1\leq i,j\leq n\big\}$$

is open in $(\mathfrak{B}_M(\mathcal{H}),\mathtt{SOT}^*)$.

And then the second one (corresponding to Lemma 2.33 in Section 6):

LEMMA 2.18. *Let $e\in\mathcal{H}$ with $\|e\|=1$, and let $0<\rho<1$. Denote by B the closed ball $\overline{B}(e,\rho)$. Let also $M>1$. For any $n\geq 1$, the open set $\mathfrak{O}_{n,B}$ is dense in $(\mathfrak{B}_M(\mathcal{H}),\mathtt{SOT}^*)$.*

We now have:

PROPOSITION 2.19. *For every $M>1$, the class $\mathrm{TMIX}_M(\mathcal{H})$ is meager in $(\mathrm{HC}_M(\mathcal{H}),\mathtt{SOT}^*)$.*

PROOF. By Corollary 2.4, it is enough to show that the class $\mathrm{TMIX}_M(\mathcal{H})$ is meager in $(\mathfrak{B}_M(\mathcal{H}),\mathtt{SOT}^*)$. Let B be a non-trivial closed ball in $\mathcal{H}$. We certainly have

$$\mathrm{TMIX}_M(\mathcal{H})\subseteq\bigcup_{N\geq 1}\mathfrak{F}_N,$$

where

$$\mathfrak{F}_N:=\bigcap_{n\geq N}\big\{T\in\mathfrak{B}_M(\mathcal{H})\,;\,T^n(B)\cap B\neq\emptyset\big\}\qquad\text{for every } N\geq 1.$$

Each set $\mathfrak{F}_N$ is closed in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. Indeed, we may write

$$T \in \mathfrak{F}_N \iff \forall n \geq N \; \exists x \in B \; : \; T^n x \in B.$$

Since B is weakly closed in $\mathcal{H}$, the condition "$T^n x \in B$" defines a closed subset of $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*) \times (B, w)$ by Lemmas 2.1 and 2.5; and since B is weakly compact, this shows that $\mathfrak{F}_N$ is closed in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. To conclude the proof, it is enough to show that for some suitable choice of the ball B, the closed sets $\mathfrak{F}_N$ have empty interior in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$; or, equivalently, that the open sets

$$\mathfrak{O}_N := \{T \in \mathfrak{B}_M(\mathcal{H}); \; \exists n \geq N \; : \; T^n(B) \cap B = \emptyset\}$$

are dense in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. We choose for B the ball $\overline{B}(e, 1/2)$, where $e \in \mathcal{H}$ satisfies $\|e\| = 1$. Then $\mathfrak{O}_N$ contains the set $\mathfrak{O}_{N+1,B}$ of Lemma 2.17, so it is dense in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$ by Lemma 2.18. □

Remark 2.20. The same proofs would show that topological weak mixing is typical and topological mixing is atypical for operators on ℓ_p spaces. It would be interesting to know if this is still true on every Banach space with separable dual. In this respect, it is worth mentioning that there exist on *any* separable Banach space hypercyclic operators which are not topologically mixing ([**31**]).

4.1. Some illustrations. In this section, we present some consequences of Propositions 2.16 and 2.19.

First, we have the following amusing fact: a typical operator $T \in \mathrm{HC}_M(\mathcal{H})$, $M > 1$, satisfies the Hypercyclicity Criterion but not Kitai's criterion.

In the same spirit, it follows from Proposition 2.16 and the Baire Category Theorem that a typical $T \in \mathfrak{B}_M(\mathcal{H})$ is such that T and T^* are both topologically weakly mixing; but no operator with this property can be topologically mixing since otherwise $T \times T^*$ would be hypercyclic on $\mathcal{H} \times \mathcal{H}$, which can never happen. Note that this argument provides an alternative proof of Proposition 2.19.

Here is now a less immediate consequence of the comeagerness of $\mathrm{TWMIX}_M(\mathcal{H})$, which is a partial strengthening of the main theorem of [**54**]. This result could also be easily deduced from [**27**, Th. 4.1], the proof of which is, however, quite different.

Proposition 2.21. *Let Z be a linear subspace of $\mathcal{H}$ with countable algebraic dimension. For any $M > 1$, the set of all $T \in \mathfrak{B}_M(\mathcal{H})$ such that every vector $z \in Z \setminus \{0\}$ is hypercyclic for T is comeager in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.*

The proof of Proposition 2.21 relies on two lemmas (Lemmas 2.22 and 2.23 below) concerning the existence of particular hypercyclic N-tuples for typical N-fold products of operators.

We begin by fixing some notation. For any operator $T \in \mathfrak{B}(\mathcal{H})$ and every $N \in \mathbb{N}$, we denote by T_N the N-fold product operator $T \times \cdots \times T$ acting on $\mathcal{H}^N = \mathcal{H} \times \cdots \times \mathcal{H}$. Also, for any finite sequence $\mathbf{f} = (f_1, \ldots, f_N)$ of vectors of $\mathcal{H}$, we denote by $\mathrm{Gram}(\mathbf{f})$ the associated *Gram matrix*,

$$\mathrm{Gram}(\mathbf{f}) := \big(\langle f_i, f_j \rangle\big)_{1 \leq i,j \leq N}$$

and we define $\mathcal{H}_{\mathbf{f}} := \big\{\mathbf{x} = (x_1, \ldots, x_N) \in \mathcal{H}^N; \; \mathrm{Gram}(\mathbf{x}) = \mathrm{Gram}(\mathbf{f})\big\}$. Note that $\mathcal{H}_{\mathbf{f}}$ is a closed subset of $\mathcal{H}^N$, and hence a Polish space. Our first lemma is

Lemma 2.22. *Let $\mathbf{f} = (f_1, \ldots, f_N)$ be a finite sequence of linearly independent vectors in $\mathcal{H}$. For any operator $T \in \mathrm{TWMIX}(\mathcal{H})$, the set $\mathcal{H}_{\mathbf{f}} \cap HC(T_N)$ is dense in $\mathcal{H}_{\mathbf{f}}$.*

Proof. Let us first recall that any topologically weakly mixing operator T on $\mathcal{H}$ is in fact *totally hypercyclic*, which means that for any $N \geq 1$, the operator T_N is hypercyclic on $\mathcal{H}^N$. This is a classical result, which has nothing to do with linearity; see e.g [**25**], or [**8**, Th. 4.6].

Let $(u_1, \dots, u_N) \in \mathcal{H}_{\mathbf{f}}$ be arbitrary. We are looking for some N-tuple $(x_1, \dots, x_N) \in \mathcal{H}_{\mathbf{f}}$ such that $(x_1, \dots, x_N)$ lies in $\mathrm{HC}(T_N)$ and $(x_1, \dots, x_N)$ is very close to $(u_1, \dots, u_N)$. Since $\mathrm{HC}(T_N)$ is dense in $\mathcal{H}^N$, one can first choose $(z_1, \dots, z_N) \in \mathrm{HC}(T_N)$ very close to $(u_1, \dots, u_N)$.

Note that the N-tuple $\mathbf{u} = (u_1, \dots, u_N)$ consists of linearly independent vectors since the Gram matrix $\mathrm{Gram}(\mathbf{u}) = \mathrm{Gram}(\mathbf{f})$ is invertible; and that $(z_1, \dots, z_N)$ is linearly independent as well since it belongs to $\mathrm{HC}(T_N)$. Let us denote by $(\widetilde{u}_1, \dots, \widetilde{u}_N)$ and $(\widetilde{z}_1, \dots, \widetilde{z}_N)$ the sequences obtained by applying the Gram-Schmidt orthonormalization process to $(u_1, \dots, u_N)$ and $(z_1, \dots, z_N)$ respectively. Then $(\widetilde{u}_1, \dots, \widetilde{u}_N)$ and $(\widetilde{z}_1, \dots, \widetilde{z}_N)$ are very close to each other, provided $(u_1, \dots, u_N)$ and $(z_1, \dots, z_N)$ are sufficiently close. Now, define $(x_1, \dots, x_N) \in \mathcal{H}^N$ as follows:

$$x_i := \sum_{l=1}^{N} \langle u_i, \widetilde{u}_l \rangle \, \widetilde{z}_l \qquad \text{for } i = 1, \dots, N.$$

We have by definition

$$\langle x_i, x_j \rangle = \sum_{l=1}^{N} \langle u_i, \widetilde{u}_l \rangle \, \overline{\langle u_j, \widetilde{u}_l \rangle} = \langle u_i, u_j \rangle = \langle f_i, f_j \rangle \quad \text{for } i, j = 1, \dots, N,$$

so that $(x_1, \dots, x_N)$ belongs to $\mathcal{H}_{\mathbf{f}}$. Moreover, each vector x_i, $1 \le i \le N$, is very close to $\sum_{l=1}^{N} \langle u_i, \widetilde{u}_l \rangle \, \widetilde{u}_l = u_i$, so $(x_1, \dots, x_N)$ is very close to $(u_1, \dots, u_N)$. It remains to show that $(x_1, \dots, x_N)$ belongs to $\mathrm{HC}(T_N)$.

Since the vectors x_i are linearly independent (because $(x_1, \dots, x_N) \in \mathcal{H}_{\mathbf{f}}$) and belong to $\mathrm{span}\,[\widetilde{z}_1, \dots, \widetilde{z}_N] = \mathrm{span}\,[z_1, \dots, z_N]$, they form a basis of $\mathrm{span}\,[z_1, \dots, z_N]$. So we may write each vector x_i as

$$x_i = \sum_{j=1}^{N} c_{i,j} \, z_j,$$

where the matrix $(c_{i,j})_{1 \le i,j \le N}$ is invertible. Now, let $V_1, \dots, V_N$ be non-empty open sets in $\mathcal{H}$, and define

$$\mathcal{V} := \Big\{ (y_1, \dots, y_N) \in \mathcal{H}^N ; \; \sum_{j=1}^{N} c_{i,j} \, y_j \in V_i \quad \text{for } i = 1, \dots, N \Big\}.$$

Since the matrix $(c_{ij})_{1 \le i,j \le N}$ is invertible, this is a *non-empty* open subset of $\mathcal{H}^N$. As $(z_1, \dots, z_N) \in \mathrm{HC}(T_N)$, one can find an integer n such that $(T^n z_1, \dots, T^n z_N)$ belongs to $\mathcal{V}$, which means that $(T^n x_1, \dots, T^n x_N)$ belongs to $V_1 \times \cdots \times V_N$ by the definition of $\mathcal{V}$. This shows that $(x_1, \dots, x_N)$ belongs to $\mathrm{HC}(T_N)$. □

Having established Lemma 2.22, we can now state and prove Lemma 2.23, from which Proposition 2.21 will easily follow.

Lemma 2.23. *Let $M > 1$. For any finite family $\mathbf{f} = (f_1, \dots, f_N)$ of linearly independent vectors in $\mathcal{H}$, the set of all $T \in \mathfrak{B}_M(\mathcal{H})$ such that $(f_1, \dots, f_N)$ belongs to $\mathrm{HC}(T_N)$ is comeager in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.*

Proof. Let us consider the set

$$\mathcal{G} := \Big\{ (x_1, \dots, x_N, T) \in \mathcal{H}_{\mathbf{f}} \times \mathfrak{B}_M(\mathcal{H}) ; \; (x_1, \dots, x_N) \in \mathrm{HC}(T_N) \Big\}.$$

This is a G_δ subset of $\mathcal{H}_{\mathbf{f}} \times (\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. Moreover, Claim 2.22 asserts that for any $T \in \mathrm{TWMIX}_M(\mathcal{H})$, the T-section of $\mathcal{G}$ is dense in $\mathcal{H}_{\mathbf{f}}$, and hence comeager in $\mathcal{H}_{\mathbf{f}}$ since this is a G_δ set. Since $\mathrm{TWMIX}_M(\mathcal{H})$ is comeager in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$, it follows, by the Kuratowski-Ulam Theorem (see for instance [**38**, Section 8.K]), that there exists at least one (in fact, comeager many) $\mathbf{x} = (x_1, \dots, x_N) \in \mathcal{H}_{\mathbf{f}}$ such that the set

$$\mathfrak{G}_{\mathbf{x}} := \{ T \in \mathfrak{B}_M(\mathcal{H}) ; \; (x_1, \dots, x_N) \in \mathrm{HC}(T_N) \}$$

is comeager in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.

Now, since $(x_1, \dots, x_N)$ belongs to $\mathcal{H}_{\mathbf{f}}$, *i.e.* $\langle x_i, x_j \rangle = \langle f_i, f_j \rangle$ for every $i, j = 1 \dots, N$, one can find a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that $Uf_i = x_i$ for every $i = 1, \dots, N$. Then the map $T \mapsto U^{-1}TU$ maps $\mathfrak{B}_M(\mathcal{H})$ bijectively onto itself because U is unitary, and is a homeomorphism with respect to the topology $\mathtt{SOT}^*$. Therefore, the set $\mathfrak{G}_{\mathbf{f}} := U^{-1}\mathfrak{G}_{\mathbf{x}}U$ is comeager in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. Since by definition $(f_1, \dots, f_N)$ belongs to $\mathrm{HC}(T_N)$ if T belongs to $\mathfrak{G}_{\mathbf{f}}$, this concludes the proof of Lemma 2.23. □

With Lemma 2.23 at hand, we can now prove Proposition 2.21.

Proof of Proposition 2.21. Let $(f_i)_{i \geq 1}$ be an algebraic basis of Z. By Lemma 2.23, the set

$$\mathfrak{G} := \{T \in \mathfrak{B}_M(\mathcal{H});\ \forall N \geq 1\ :\ (f_1, \dots, f_N) \in \mathrm{HC}(T_N)\}$$

is comeager in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. So it is enough to show that that if T belongs to $\mathfrak{G}$, then every non-zero vector $z \in Z$ is a hypercyclic vector for T. Let V be a non-empty open set in $\mathcal{H}$. Write z as $z = \sum_{i=1}^N z_i f_i$, and consider the set

$$\mathcal{V}_z := \Big\{(x_1, \dots, x_N) \in \mathcal{H}^N;\ \sum_{i=1}^N z_i x_i \in V\Big\}.$$

Since $z \neq 0$, this is a non-empty open set in $\mathcal{H}^N$. As T belongs to $\mathfrak{G}$, it follows that there exists $n \geq 1$ such that $(T^n f_1, \dots, T^n f_N)$ belongs to $\mathcal{V}_z$, which means exactly that $T^n z$ belongs to V. Hence z is indeed a hypercyclic vector for T. □

5. Hypercyclic operators without eigenvalues

The following result shows that operators without eigenvalues are typical.

Proposition 2.24. *For any $M > 0$, the class $\mathrm{NEV}_M(\mathcal{H})$ is a dense G_δ subset of the space $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.*

From this and Corollary 2.4, we obtain

Corollary 2.25. *For any $M > 1$, a typical operator $T \in (\mathrm{HC}_M(\mathcal{H}), \mathtt{SOT}^*)$ has no eigenvalues.*

As a matter of fact, Proposition 2.24 is already proved in [**22**], where typical properties of contraction operators are studied for various topologies (see also [**20**]). However, since the proof is not that complicated, and in order to keep this work as self-contained as possible, we outline it below.

Proof of Proposition 2.24. We divide the proof into two steps. In what follows, we fix $M > 0$.

Claim 2.26. The set $\mathrm{NEV}_M(\mathcal{H})$ is a G_δ subset of $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.

Proof of Claim 2.26. To any closed ball $B \subseteq \mathcal{H}$, we associate the following subset of $\mathfrak{B}_M(\mathcal{H})$:

$$\mathfrak{M}_B = \big\{T \in \mathfrak{B}_M(\mathcal{H})\,;\, \exists\, \lambda \in \mathbb{C},\ \exists\, x \in B \text{ with } Tx = \lambda x\big\}.$$

Let us show that this set $\mathfrak{M}_B$ is F_σ in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. To this aim, we endow B with the weak topology, and introduce the set

$$\mathcal{F}_B = \big\{(T, \lambda, x) \in \mathfrak{B}_M(\mathcal{H}) \times \mathbb{C} \times B\,;\, Tx = \lambda x\big\}.$$

Then $\mathfrak{M}_B$ is the projection of $\mathcal{F}_B$ on the first coordinate. Moreover, the set $\mathcal{F}_B$ is closed in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*) \times \mathbb{C} \times (B, w)$ by Lemma 2.5. Since the space $\mathbb{C} \times (B, w)$ is K_σ (because (B, w) is compact), it follows that $\mathfrak{M}_B$ is F_σ.

Let now $(B_q)_{q \geq 1}$ be a sequence of closed balls of $\mathcal{H}$ not containing the point 0, such that $\bigcup_{q \geq 1} B_q = \mathcal{H} \setminus \{0\}$. Then $\mathfrak{B}_M(\mathcal{H}) \setminus \mathrm{NEV}_M(\mathcal{H}) = \bigcup_{q \geq 1} \mathfrak{M}_{B_q}$ is an F_σ set, so that $\mathrm{NEV}_M(\mathcal{H})$ is a G_δ set in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. □

Claim 2.27. The set $\mathrm{NEV}_M(\mathcal{H})$ is dense in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.

Proof of Claim 2.27. Let $(f_k)_{k\in\mathbb{Z}}$ be an orthonormal basis of $\mathcal{H}$. For each $r \geq 0$, we set $\mathcal{H}_r := \mathrm{span}[f_k, ; |k| \leq r]$ and we denote by P_r the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_r$. By Lemma 2.6 and Remark 2.7, it suffices to show that for every $r \geq 0$, every $A \in \mathfrak{B}(\mathcal{H}_r)$, with $\|A\| < M$, and every $\varepsilon > 0$, there exists an operator $T \in \mathrm{NEV}_M(\mathcal{H})$ such that

$$\|(T - A)f_k\| < \varepsilon \quad \text{and} \quad \|(T - A)^* f_k\| < \varepsilon \qquad \text{for every } k = -r, \dots, r.$$

Let $\boldsymbol{\omega} = (\omega_k)_{k\in\mathbb{Z}}$ be a bilateral weight sequence with $0 < \omega_k \leq M$ for every $k \in \mathbb{Z}$ and $\omega_k = \delta$ for every index k with $|k + r| \leq 2r + 1$. As in the proof of Corollary 2.12, one easily checks that if $\delta > 0$ is sufficiently small, the bilateral weighted shift operator $S_{A,\boldsymbol{\omega}}$ (defined with respect to the basis $(f_k)_{k\in\mathbb{Z}}$) satisfies

$$\|(S_{A,\boldsymbol{\omega}} - A)f_k\| < \varepsilon \quad \text{and} \quad \|(S_{A,\boldsymbol{\omega}} - A)^* f_k\| < \varepsilon \qquad \text{for every } k = -r, \dots, r.$$

Moreover, if the weight sequence $\boldsymbol{\omega}$ is chosen in such a way that the series

$$\sum_{p\geq 1} \left(\omega_{p(2r+1)+l} \cdots \omega_{(2r+1)+l}\right)^{-2} \quad \text{and} \quad \sum_{p\geq 1} \left(\omega_{l-p(2r+1)} \cdots \omega_{l-(2r+1)}\right)^{-2}$$

are divergent for every $|l| \leq r$, then $S_{A,\boldsymbol{\omega}}$ has no eigenvalue by Remark 2.15. So $T := S_{A,\boldsymbol{\omega}}$ satisfies the required assumptions for a suitable choice of the weight $\boldsymbol{\omega}$. □

The two claims above complete the proof of Proposition 2.24. □

Remark 2.28. The fact that we are using the topology $\mathtt{SOT}^*$ is crucial in the proof of Claim 2.26 in order to obtain that the sets $\mathcal{F}_B$ above are closed in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*) \times \mathbb{C} \times (B, w)$. The situation turns out to be completely different if one considers the topology $\mathtt{SOT}$ instead of $\mathtt{SOT}^*$. Indeed, it is proved in [**22**] that an $\mathtt{SOT}$-typical $T \in \mathfrak{B}_1(\mathcal{H})$ has the property that *every* $\lambda \in \mathbb{C}$ *with* $|\lambda| < 1$ *is an eigenvalue of* T. More precisely, a typical $T \in \mathfrak{B}_1(\mathcal{H})$ is unitarily equivalent to the infinite-dimensional backward unilateral shift operator. So a typical $T \in (\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT})$ has the whole disk $D(0, M)$ within the set of its eigenvalues.

Remark 2.29. The proof of Corollary 2.25 is a good example of the usefulness of Corollary 2.4 for simplifying arguments of this kind: although it is easy to construct the weight sequence $\boldsymbol{\omega}$ above in such a way that the operator $S_{A,\boldsymbol{\omega}}$ is $\mathtt{SOT}^*$-close to A and has no eigenvalue, it is technically much less obvious to ensure that $S_{A,\boldsymbol{\omega}}$ is additionally hypercyclic. In other words, it would be less easy to prove Corollary 2.25 directly, without using Corollary 2.4.

Remark 2.30. Here is a very short alternative proof of Proposition 2.24. First, we may assume without loss of generality that $M > 1$ because the property of having (or not having) eigenvalues is invariant under scalar multiplication. Then, since hypercyclic operators are typical in $\mathfrak{B}_M(\mathcal{H})$ and since the map $T \mapsto T^*$ is a homeomorphism of $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$, we know that a typical $T \in \mathfrak{B}_M(\mathcal{H})$ has the property that T^* is hypercyclic. Since the adjoint of a hypercyclic operator can have no eigenvalues, this concludes the proof.

The following consequence of Corollary 2.25 is worth mentioning.

Corollary 2.31. *For every* $M > 1$, $\mathrm{CH}_M(\mathcal{H})$ *is meager in* $(\mathrm{HC}_M(\mathcal{H}), \mathtt{SOT}^*)$. *In other words, a typical hypercyclic operator on* $\mathcal{H}$ *is not chaotic.*

This is indeed obvious since chaotic operators have plenty of eigenvalues. Note that we are using here the fact that $\mathcal{H}$ is a *complex* Hilbert space, so that periodic points are linear combinations of eigenvectors whose associated eigenvalues are roots of unity. Nonetheless, Corollary 2.31 holds true on real Hilbert spaces as well; see Remark 2.37 below.

6. Hypercyclic operators without invariant measures

It follows easily from Proposition 2.24 that for any $M > 1$, the operators in $\mathrm{HC}_M(\mathcal{H})$ admitting a non-trivial invariant measure with a second-order moment form a meager class in $(\mathrm{HC}_M(\mathcal{H}), \mathtt{SOT}^*)$. Indeed, if $T \in \mathrm{HC}_M(\mathcal{H})$ admits an invariant measure $\mu \neq \delta_0$ such that $\int_H \|x\|^2 \, d\mu(x) < \infty$, then the Gaussian measure m whose covariance operator is given by the formula

$$\langle Rx, y\rangle = \int_H \langle x, z\rangle \, \overline{\langle y, z\rangle} \, d\mu(z), \quad x, y \in H$$

is T-invariant. Its support is the closed linear span of the support of μ, and hence is non-trivial. This closed subspace is spanned by unimodular eigenvectors of T (see [**5**] or [**8**] for details), from which it follows that T does not belong to $\mathrm{NEV}_M(\mathcal{H})$.

The main result of this section (Theorem 2.36 below) is that the $\mathtt{SOT}^*$-typical operator $T \in \mathrm{HC}_M(\mathcal{H})$ actually admits no non-trivial invariant measure at all. Its proof relies on the next two lemmas.

Lemma 2.32. *Let B be a closed ball of $\mathcal{H}$ not containing the point 0. Let also $M > 0$. For any integer $n \geq 1$, the set*

$$\mathfrak{O}_{n,B} = \big\{T \in \mathfrak{B}_M(\mathcal{H})\,;\ \textit{there exist } n \textit{ distinct integers } p_1, \dots, p_n \textit{ such that } T^{p_i}(B) \cap T^{p_j}(B) = \emptyset \textit{ for every } i \neq j,\ 1 \leq i, j \leq n\big\}$$

is open in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. Consequently, the set

$$\mathfrak{G}_B = \big\{T \in \mathfrak{B}_M(\mathcal{H})\,;\ \textit{for every } n \geq 1,\ \textit{there exist } n \textit{ iterates of } B \textit{ under the action of } T \textit{ which are pairwise disjoint}\big\}$$

is G_δ in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.

Proof. The second part of the lemma follows immediately from the first, since

$$\mathfrak{G}_B = \bigcap_{n \geq 1} \mathfrak{O}_{n,B}.$$

To derive the first part, it is enough to show that if we fix $p, q \geq 1$, then the set

$$\mathfrak{O} := \{T \in \mathfrak{B}_M(\mathcal{H});\ T^p(B) \cap T^q(B) = \emptyset\}$$

is $\mathtt{SOT}^*$-open in $\mathfrak{B}_M(\mathcal{H})$. If $T \in \mathfrak{B}_M(\mathcal{H})$, we may write

$$T \in \mathfrak{O} \iff \forall x, y \in B \ :\ T^p x \neq T^q y.$$

Since the map $(T, u) \mapsto T^n u$ is continuous on $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*) \times (B, w)$ for any $n \geq 1$ by Lemmas 2.1 and 2.5, the condition "$T^p x \neq T^q y$" defines an open subset of $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*) \times (B, w) \times (B, w)$. Since B is weakly compact, it follows that $\mathfrak{O}$ is indeed $\mathtt{SOT}^*$-open in $\mathfrak{B}_M(\mathcal{H})$, its complement being the projection of a closed subset of $\mathfrak{B}_M(\mathcal{H}) \times B \times B$ along the compact factor $B \times B$. □

Lemma 2.33. *Let $e \in \mathcal{H}$ with $\|e\| = 1$, and let $0 < \rho < 1$. Denote by B the closed ball $\overline{B}(e, \rho)$. Let also $M > 1$. For any $n \geq 1$, the open set $\mathfrak{O}_{n,B}$ is dense in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.*

Proof. Let us fix an orthonormal basis $(e_k)_{k \geq 1}$ of $\mathcal{H}$ with $e_1 = e$. Our aim being to apply Lemma 2.6, we fix $r \geq 1$, an operator $A \in \mathfrak{B}(\mathcal{H}_r)$ such that $\|A\| < M$, and $\varepsilon > 0$. We are looking for an operator $T \in \mathfrak{O}_{n,B}$ such that

$$\|(T - A)e_k\| < \varepsilon \quad \text{and} \quad \|(T - A)^* e_k\| < \varepsilon \qquad \text{for } k = 1, \dots, r.$$

We will define a sequence $(C_N)_{N > 2r}$ of operators, with $C_N \in \mathfrak{B}(\mathcal{H}_N)$ for every $N > 2r$, and show that if N is sufficiently large, the operator $P_N^* C_N P_N \in \mathfrak{B}(\mathcal{H})$ belongs to $\mathfrak{O}_{n,B}$ and satisfies the above estimates. Here P_N denotes as usual the canonical projection of $\mathcal{H}$ onto $\mathcal{H}_N$.

Let $\delta > 0$ and $\gamma > 1$, to be fixed later on in the proof. For each $N > 2r$, consider the operator $C_N \in \mathfrak{B}(\mathcal{H}_N)$ defined in the following way:

$$C_N e_k = \begin{cases} Ae_k + \delta e_{k+r} & \text{for every } 1 \le k \le r \\ \gamma e_{k+r} & \text{for every } r < k \le N - r \\ 0 & \text{for every } N - r < k \le N. \end{cases}$$

Thus, in matrix form,

$$C_N = \begin{pmatrix} \boxed{\begin{array}{c} A \\ \hline \begin{matrix} \delta & & \\ & \ddots & \\ & & \delta \end{matrix} \end{array}} & 0 & 0 \\ 0 & \boxed{\begin{matrix} \gamma & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \gamma \end{matrix}} & 0 \end{pmatrix}$$

We note that if $\|A\| < \gamma < M$ and if δ is sufficiently small, then $\|C_N\| = \gamma < M$. Moreover, if δ is sufficiently small, then

$$\|(C_N - A)e_k\| < \varepsilon \quad \text{and} \quad \|(C_N - A)^* e_k\| < \varepsilon \qquad \text{for every } k = 1, \dots, r \tag{5}$$

whatever the choice of the integer $N > 2r$. The key of the proof lies in the following simple computation.

Claim 2.34. For every $N > 2r$, every $p \ge 1$ such that $pr + 1 \le N$, and every $x \in \mathcal{H}$,

$$\langle C_N^p x, e_{pr+1} \rangle = \gamma^{p-1} \delta \langle x, e_1 \rangle.$$

Proof of Claim 2.34. Clearly $\langle C_N x, e_{r+1} \rangle = \delta \langle x, e_1 \rangle$; so we may write $C_N x$ as

$$C_N x = \delta \langle x, e_1 \rangle e_{r+1} + \sum_{\substack{k=1 \\ k \ne r+1}}^{N} \langle C_N x, e_k \rangle e_k.$$

Hence

$$C_N^{p-1} x = \delta \gamma^{p-1} \langle x, e_1 \rangle e_{pr+1} + \sum_{\substack{k=1 \\ k \ne r+1}}^{N} \langle C_N x, e_k \rangle C_N^{p-1} e_k.$$

Now $C_N^{p-1} e_k$ belongs to the closed linear span of the vectors e_j, $1 \le j \le N$, $j \ne pr + 1$, for every $1 \le k \le N$ with $k \ne r + 1$. Indeed,

- if $1 \le k \le r$, a straightforward induction shows that $C_N^{p-1} e_k \in \operatorname{span}[e_j \,;\, 1 \le j \le pr]$;
- if $r + 1 < k \le N - (p-1)r$, then $C_N^{p-1} e_k = \gamma^{p-1} e_{k+(p-1)r}$;
- if $N - (p-1)r < k \le N$, we have $C_N^{p-1} e_k = 0$.

Thus $\langle C_N^{p-1} x, e_{pr+1} \rangle = \delta \gamma^{p-1} \langle x, e_1 \rangle$, which is the claim of Claim 2.34. □

From Claim 2.34, it is not hard to deduce

Claim 2.35. Let $1 \le p, q \le N$ be such that $pr + 1 \le N$. If $\gamma^{p-q-1} \delta > \frac{1+\rho}{1-\rho}$, then $C_N^p(P_N B) \cap C_N^q(P_N B) = \emptyset$.

Proof of Claim 2.35. If $x, y \in P_N B$, then by Claim 2.34

$$\|C_N^p y - C_N^q x\| \geq \left|\langle C_N^p x - C_N^q y, e_{pr+1}\rangle\right| \geq \gamma^{p-1}\delta|\langle x, e_1\rangle| - \|C_N^q\|\,\|y\|.$$

Moreover, since x and y belong to $P_N B \subseteq B$, we have $|\langle x, e_1\rangle| \geq 1-\rho$ and $\|y\| \leq 1+\rho$. Hence

$$\|C_N^p y - C_N^q x\| \geq \gamma^{p-1}\delta(1-\rho) - \gamma^q(1+\rho).$$

Thus $C_N^q(P_N B) \cap C_N^p(P_N(B))$ is empty as soon as $\gamma^{p-1}\delta(1-\rho) - \gamma^q(1+\rho) > 0$, which proves our claim. □

We now choose the various parameters in this construction in the following order: first we choose γ such that $\max(1, \|A\|) < \gamma < M$. Then we choose $\delta > 0$ so small that (5) holds true for every $N \geq 1$. Lastly, we choose $N > 2r$ so large that there exist n distinct integers $1 \leq p_1 < p_2 < \cdots < p_n \leq N$ with $p_n r + 1 \leq N$, such that the gaps between two consecutive integers p_j are so large that $\gamma^{p_j - p_i - 1}\delta > (1-\rho)/(1+\rho)$ for every $1 \leq i < j \leq n$. By Claim 2.35, the operator C_N then satisfies $C_N^{p_j}(P_N B) \cap C_N^{p_i}(P_N B) = \emptyset$ for every $1 \leq i < j \leq n$. So the operator $T := P_N^* C_N P_N \in \mathfrak{B}_M(\mathcal{H})$ is such that $T^{p_j}(B) \cap T^{p_i}(B) = \emptyset$ for every $1 \leq i < j \leq n$, that is, T belongs to $\mathfrak{O}_{n,B}$; and by (5) we also have $\|(T-A)e_k\| < \varepsilon$ and $\|(T-A)^* e_k\| < \varepsilon$ for $k = 1, \ldots, r$. This concludes the proof of Lemma 2.33. □

Using Lemmas 2.32 and 2.33, it is now not difficult to prove that $\mathtt{SOT}^*$-typical operators admit no non-trivial invariant measure.

Theorem 2.36. *For every $M > 1$, the set $\mathrm{HC}_M(\mathcal{H}) \setminus \mathrm{INV}(\mathcal{H})$ is comeager in the space $(\mathrm{HC}_M(\mathcal{H}), \mathtt{SOT}^*)$.*

Proof. Combining Lemmas 2.32 and 2.33, we obtain that $\mathfrak{G}_B$ is a dense G_δ subset of the space $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$ for every ball $B = \overline{B}(e, \rho)$, where $\|e\| = 1$ and $0 < \rho < 1$. By Corollary 2.4, it follows that $\mathfrak{G}_B \cap \mathrm{HC}_M(\mathcal{H})$ is a dense G_δ subset of $(\mathrm{HC}_M(\mathcal{H}), \mathtt{SOT}^*)$ for each such ball B. Let $(B_q)_{q\geq 1}$ be a countable family of such balls with the property that

$$\bigcup_{q\geq 1} B_q = B(0,2) \setminus \{0\}.$$

Then $\mathfrak{G} := \left(\bigcap_{q\geq 1} \mathfrak{G}_{B_q}\right) \cap \mathrm{HC}_M(\mathcal{H})$ is a dense G_δ subset of $(\mathrm{HC}_M(\mathcal{H}), \mathtt{SOT}^*)$.

Every element T of $\mathfrak{G}$ enjoys the property that for any $q \geq 1$ and any $n \geq 1$, there exist n iterates $T^j(B_q)$ of B_q which are distinct. It follows that if m is any invariant (probability) measure for T, then $m(B_q) = 0$ for every $q \geq 1$ and hence $m(B(0,2) \setminus \{0\}) = 0$.

Suppose now that $T \in \mathfrak{G}$ admits an invariant measure $m \neq \delta_0$. Then one can find a closed ball B' not containing 0 such that $m(B') > 0$. Consider for any $R > 0$ the measure m_R on $\mathcal{H}$ defined by $m_R(C) = m(RC)$ for any Borel subset C of $\mathcal{H}$. Each such measure m_R is an invariant probability measure on $\mathcal{H}$. Moreover, if R is sufficiently large then $R^{-1}B' \subseteq B(0,2) \setminus \{0\}$, so that $m_R(B(0,2) \setminus \{0\}) > 0$, which is a contradiction. We have thus proved that any operator $T \in \mathfrak{G}$ admits no invariant measure except δ_0, and hence that $\mathrm{HC}_M(\mathcal{H}) \setminus \mathrm{INV}(\mathcal{H})$ is comeager in $(\mathrm{HC}_M(\mathcal{H}), \mathtt{SOT}^*)$. □

Remark 2.37. The above proof does not use the fact that $\mathcal{H}$ is a *complex* Hilbert space; so Theorem 2.36 holds true as well for real Hilbert spaces. This shows in particular that in the real setting also, the chaotic operators form a meager subset of $\mathfrak{B}_M(\mathcal{H})$; more precisely, a typical operator $T \in \mathfrak{B}_M(\mathcal{H})$ has no periodic point except 0. Indeed, any periodic point $x \neq 0$ for an operator T gives rise to a "canonical" invariant measure supported on the orbit of x, namely $m := \frac{1}{N}\sum_{i=0}^{N-1} \delta_{T^i x}$, where $N \geq 1$ is a period of x.

Remark 2.38. Theorem 2.36 implies a weak form of Proposition 2.24, namely that operators without any eigenvalue *of modulus* 1 are typical. Indeed, if T admits a unimodular eigenvalue and if x is an associated eigenvector, then there is a canonical invariant measure $m \neq \delta_0$ supported on $\mathbb{T} \cdot x$, namely the image of the Lebesgue measure on $\mathbb{T}$ under the map $\lambda \mapsto \lambda x$.

Regarding the eigenvalues, one may also note that if m is any non-trivial invariant measure for an operator $T \in \mathfrak{B}(\mathcal{H})$, then $m\big(\ker(T-\lambda)\big) = 0$ for every complex number λ such that $|\lambda| \neq 1$. Indeed, if $m \neq \delta_0$ is a measure such that $m\big(\ker(T-\lambda)\big) > 0$, one can find a ball B not containing 0 such that $m(B_\lambda) \neq 0$, where $B_\lambda = B \cap \ker(T-\lambda)$. Since $|\lambda| \neq 1$, it is easily checked that B_λ has infinitely many pairwise disjoint iterates under T, which is not possible since the measure m is T-invariant.

Since $\mathcal{U}$-frequently hypercyclic operators on a Hilbert space always admit an invariant measure with full support by [**30**], Theorem 2.36 immediately implies:

Corollary 2.39. *For every $M > 1$, $\mathrm{UFHC}_M(\mathcal{H})$ is meager in $(\mathrm{HC}_M(\mathcal{H}), \mathtt{SOT}^*)$. In other words, a typical hypercyclic operator on $\mathcal{H}$ is not $\mathcal{U}$-frequently hypercyclic.*

7. Densely distributionally chaotic operators

In this section, our aim is to show that, for any $M > 1$, the class $\mathrm{DDCH}_M(\mathcal{H})$ of densely distributionally chaotic operators in $\mathfrak{B}_M(\mathcal{H})$ is a dense G_δ subset of $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$, from which it follows that the class $\mathrm{DCH}_M(\mathcal{H})$ of distributionally chaotic operators in $\mathfrak{B}_M(\mathcal{H})$ is a comeager subset of $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.

Proposition 2.40. *For any $M > 1$, the set $\mathrm{DDCH}_M(\mathcal{H})$ of densely distributionally chaotic operators in $\mathfrak{B}_M(\mathcal{H})$ is a dense G_δ subset of $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.*

Proof. Recall that an operator $T \in \mathfrak{B}(\mathcal{H})$ is densely distributionally chaotic if and only if it admits a dense set of distributionally irregular vectors, *i.e.* of vectors $x \in \mathcal{H}$ for which there exist two sets of integers $A, B \subseteq \mathbb{N}$ with upper density 1 such that $T^i x \to 0$ as $i \to \infty$ along A and $\|T^i x\| \to \infty$ as $i \to \infty$ along B. Since the set of distributionally irregular vectors for T can be written as

$$G_T = \bigcap_{\varepsilon \in \mathbb{Q}^{+*}} \bigcap_{N \geq 1} G_{T,\varepsilon,N}$$

where

$$G_{T,\varepsilon,N} := \Big\{ x \in \mathcal{H}\,;\, \exists\, m, n \geq N \ : \ \#\{1 \leq i \leq m\,;\, \|T^i x\| < \varepsilon\} \geq m(1-\varepsilon) \\ \text{and } \#\{1 \leq i \leq n\,;\, \|T^i x\| > 1/\varepsilon\} \geq n(1-\varepsilon) \Big\},$$

it follows that G_T is a dense G_δ subset of $\mathcal{H}$ whenever T is densely distributionally chaotic. Denoting by $(V_p)_{p \geq 1}$ a countable basis of non-empty open subsets of $\mathcal{H}$, we infer from this observation that an operator $T \in \mathfrak{B}_M(\mathcal{H})$ belongs to $\mathrm{DDCH}_M(\mathcal{H})$ if and only if

$$\forall\, \varepsilon \in \mathbb{Q}^{+*}\ \forall\, N \geq 1\ \forall\, p \geq 1\ \ \exists x \in V_p\ \ \exists n, m \geq N\ \#\{1 \leq i \leq m\,;\, \|T^i x\| < \varepsilon\} \geq m(1-\varepsilon) \\ \text{and } \#\{1 \leq i \leq n\,;\, \|T^i x\| > 1/\varepsilon\} \geq n(1-\varepsilon).$$

Using this, we can prove

Claim 2.41. The set $\mathrm{DDCH}_M(\mathcal{H})$ is a G_δ subset of $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT})$.

Proof of Claim 2.41. It suffices to show that for every $\varepsilon \in \mathbb{Q}^{+*}$, $N \geq 1$, $p \geq 1$, $m, n \geq N$ and $x \in \mathcal{H}$, the set

$$\Big\{ T \in \mathfrak{B}_M(\mathcal{H})\,;\, \#\{1 \leq i \leq m\,;\, \|T^i x\| < \varepsilon\} \geq m(1-\varepsilon) \\ \text{and } \#\{1 \leq i \leq n\,;\, \|T^i x\| > 1/\varepsilon\} \geq n(1-\varepsilon) \Big\}$$

is SOT-open. So let $T_0 \in \mathfrak{B}_M(\mathcal{H})$ belong to this set, and let $r \geq m(1-\varepsilon)$ be an integer such that there exist r indices $1 \leq i_1 < \cdots < i_r \leq m$ such that $\|T^{i_j} x\| < \varepsilon$ for every $1 \leq j \leq r$. If $T \in \mathfrak{B}_M(\mathcal{H})$ is sufficiently close to T_0 for the SOT-topology, we still have $\|T^{i_j} x\| < \varepsilon$ for every $1 \leq j \leq r$. Hence $\#\{1 \leq i \leq m\,;\, \|T^i x\| < \varepsilon\} \geq m(1-\varepsilon)$. In the same way, the set of operators

T such that $\#\{1 \leq i \leq n\,;\, \|T^i x\| > 1/\varepsilon\} \geq n(1-\varepsilon)$ is SOT-open in $\mathfrak{B}_M(\mathcal{H})$. This proves Claim 2.41. □

Claim 2.42. The set $\mathrm{DDCH}_M(\mathcal{H})$ is dense in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.

Proof of Claim 2.42. By a result of [**30**], every ergodic operator on $\mathcal{H}$ is densely distributionally chaotic. Since G-MIX$(\mathcal{H})$ is dense in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$ by Corollary 2.12, it immediately follows that $\mathrm{DDCH}_M(\mathcal{H})$ is dense in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. □

By Claims 2.41 and 2.42, the proof of Proposition 2.40 is now complete. □

Since $\mathrm{DDCH}(\mathcal{H}) \subseteq \mathrm{DCH}(\mathcal{H})$, and since any operator $T \in \mathrm{DDCH}(\mathcal{H}) \cap \mathrm{HC}(\mathcal{H})$ satisfies $c(T) = 1$, Proposition 2.40 has the following immediate consequences:

Corollary 2.43. *For any $M > 1$, the set $\mathrm{DCH}_M(\mathcal{H})$ of distributionally chaotic operators in $\mathfrak{B}_M(\mathcal{H})$ is comeager in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.*

Corollary 2.44. *For any $M > 1$, the set $\mathrm{c}^1_M(\mathcal{H})$ of operators $T \in \mathrm{HC}_M(\mathcal{H})$ with $c(T) = 1$ is comeager in $(\mathrm{HC}_M(\mathcal{H}), \mathtt{SOT}^*)$.*

There is an alternative approach for proving the comeagerness of $\mathrm{DDCH}_M(\mathcal{H})$. It makes use of the following lemma, which is interesting in itself.

Lemma 2.45. *Let T be an ergodic operator on $\mathcal{H}$, and let m be a T-invariant ergodic measure with full support for T. For every ε, $R > 0$, one can find two other ergodic measures μ and ν for T, both with full support, such that $\mu(\overline{B}(0,R)) > 1-\varepsilon$ and $\nu(\overline{B}(0,R)) < \varepsilon$ respectively.*

Proof. The proof relies on a dilation argument already used several times in [**30**]. This argument has been essentially given at the end of the proof of Theorem 2.36, but we repeat it anyway. For any $\rho > 0$, let m_ρ be the measure on $\mathcal{H}$ defined by $m_\rho(C) = m(\rho C)$ for any Borel subset C of $\mathcal{H}$. We have $m_\rho(\overline{B}(0,R)) = m(\overline{B}(0,\rho R))$. All these measures m_ρ are ergodic for T and have full support. Moreover, $m(\overline{B}(0,\rho R)) \to m(\mathcal{H}) = 1$ as $\rho \to \infty$ and $m(\overline{B}(0,\rho R)) \to m(\{0\}) = 0$ as $\rho \to 0$ (that $m(\{0\})$ is necessarily equal to 0 follows from the ergodicity of m with respect to T and the fact that $m \neq \delta_0$; recall that $T(0) = 0$). So there exist ρ_1, $\rho_2 > 0$ such that $\mu = m_{\rho_1}$ and $\nu = m_{\rho_2}$ satisfy $\mu(\overline{B}(0,R)) > 1-\varepsilon$ and $\nu(\overline{B}(0,R)) < \varepsilon$. □

We deduce from Lemma 2.45 the following proposition:

Proposition 2.46. *Let $M > 1$. For every vector $x \in \mathcal{H} \setminus \{0\}$, the set*

$$\mathfrak{G}^x = \big\{T \in \mathfrak{B}_M(\mathcal{H})\,;\, x \text{ is a distributionally irregular vector for } T\big\}$$

is comeager in $(\mathrm{HC}_M(\mathcal{H}), \mathtt{SOT}^*)$.

Proof. Let us consider the following two subsets of $\mathfrak{B}_M(\mathcal{H})$:

$$\mathfrak{G}^x_\infty := \big\{T \in \mathfrak{B}_M(\mathcal{H})\,;\, \forall\, R > 0 \ : \underline{\mathrm{dens}}\, \mathcal{N}_T(x, \overline{B}(0,R)) = 0\big\}, \quad \text{and}$$
$$\mathfrak{G}^x_0 := \big\{T \in \mathfrak{B}_M(\mathcal{H})\,;\, \forall\, r > 0 \ : \ \underline{\mathrm{dens}}\, \mathcal{N}_T(x, \mathcal{H} \setminus \overline{B}(0,r)) = 0\big\}.$$

It is not difficult to see that

$$\mathfrak{G}^x_\infty \cap \mathfrak{G}^x_0 \subseteq \mathfrak{G}^x.$$

Indeed, if $T \in \mathfrak{G}^x_\infty$, then one can find a set $A \subseteq \mathbb{N}$ with $\overline{\mathrm{dens}}(A) = 1$ such that $\|T^i x\| \to \infty$ as $i \to \infty$ along A; whereas if $T \in \mathfrak{G}^x_0$, one can find a set $B \subseteq \mathbb{N}$ with $\overline{\mathrm{dens}}(B) = 1$ such that $\|T^i x\| \to 0$ as $i \to \infty$ along B. So it is enough to show that $\mathfrak{G}^x_\infty$ and $\mathfrak{G}^x_0$ are both comeager in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. We will actually concentrate on $\mathfrak{G}^x_\infty$ only, the proof for $\mathfrak{G}^x_0$ being completely similar.

For any ε, $R > 0$, we introduce the set

$$\mathfrak{H}^x_{\varepsilon,R} = \bigcap_{k \geq 1} \bigcup_{n \geq k} \Big\{T \in \mathfrak{B}_M(\mathcal{H})\,;\, \#\{1 \leq i \leq n\,;\, \|T^i x\| < R\} < n\varepsilon\Big\}.$$

Claim 2.47. Each set $\mathfrak{H}^x_{\varepsilon,R}$ is $\mathtt{SOT}$-G_δ.

Proof of Claim 2.47. Proceed as in the proof of Claim 2.41. □

Claim 2.48. Each set $\mathfrak{H}^x_{\varepsilon,R}$ is $\mathtt{SOT}^*$-dense in $\mathfrak{B}_M(\mathcal{H})$.

Proof of Claim 2.48. We will use the density of G-MIX$_M(\mathcal{H})$ in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$, proved in Corollary 2.12 above.

Let $\mathfrak{U}$ be a non-empty open subset of $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. By Corollary 2.12, we know that G-MIX$_{M'}(\mathcal{H})$ is dense in $(\mathfrak{B}_{M'}(\mathcal{H}), \mathtt{SOT}^*)$ for every $1 < M' < M$. Since $\bigcup_{M'<M} \mathfrak{B}_{M'}(\mathcal{H})$ is obviously dense in $\mathfrak{B}_M(\mathcal{H})$, it follows that $\mathfrak{U}$ contains an operator T which is mixing in the Gaussian sense and satisfies $\|T\| < M$. The operator T is in particular ergodic. By Lemma 2.45, T admits an ergodic measure with full support ν such that $\nu(\overline{B}(0, 2R)) < \varepsilon$. Birkhoff's ergodic theorem then implies that the set

$$\mathcal{E} := \Big\{ y \in \mathcal{H} \,;\, \limsup_{n\to\infty} \frac{1}{n} \#\{1 \le i \le n \,;\, \|T^i y\| \le 2R\} < \varepsilon \Big\}$$

is dense in $\mathcal{H}$.

Let now $\delta > 0$ be a small positive number, to be fixed later on in the proof. Since $x \neq 0$, the density of $\mathcal{E}$ in $\mathcal{H}$ implies the existence of an isomorphism L of $\mathcal{H}$ with the following properties: $Lx \in \mathcal{E}$, $\|I - L\| < \delta$, and $\|I - L^{-1}\| < \delta$. Consider now the operator $S = L^{-1}TL$. Since $\|T\| < M$, we have $\|S\| < M$ if δ is sufficiently small. Also, S belongs to $\mathfrak{U}$ as soon as δ is sufficiently small. We thus fix $\delta > 0$ such that these two conditions are satisfied. It now remains to prove that S belongs to $\mathfrak{H}^x_{\varepsilon,R}$, which will conclude our proof that $\mathfrak{H}^x_{\varepsilon,R}$ is dense in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.

For every $n \ge 1$, we have $LS^n x = T^n Lx$. Since Lx belongs to $\mathcal{E}$, it follows that

$$\limsup_{n\to\infty} \frac{1}{n} \#\big\{1 \le i \le n \,;\, \|LS^i x\| \le 2R\big\} < \varepsilon.$$

Observe now that if $\|S^i x\| \le R$, then $\|LS^i x\| \le \|L\| \, \|S^i x\| \le 2R$ (as soon as $\delta \le 2$, of course). It follows that

$$\#\big\{1 \le i \le n \,;\, \|S^i x\| \le R\big\} \le \#\big\{1 \le i \le n \,;\, \|LS^i x\| \le 2R\big\} < n\varepsilon$$

for all sufficiently large n. Hence S belongs to $\mathfrak{H}^x_{\varepsilon,R}$, and this concludes the proof of Claim 2.48. □

The two claims above imply that all the sets $\mathfrak{H}^x_{\varepsilon,R}$ are comeager (in fact, dense G_δ) in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. Since

$$\mathfrak{G}^x_\infty = \bigcap_{\varepsilon \in \mathbb{Q}^*_+} \bigcap_{R \in \mathbb{Q}^*_+} \mathfrak{H}^x_{\varepsilon,R},$$

it follows that $\mathfrak{G}^x_\infty$ is comeager as well. The case of $\mathfrak{G}^x_0$ being exactly similar, the proof of Proposition 2.46 is now complete. □

We can now deduce from Proposition 2.46 that the set DDCH$_M(\mathcal{H})$ is comeager in the space $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. By the Kuratowski-Ulam Theorem, Proposition 2.46 implies that the set

$$\mathfrak{G} := \big\{ (T, x) \in \mathfrak{B}_M(\mathcal{H}) \times \mathcal{H} \,;\, x \text{ is a distributionally irregular vector for } T \big\}$$

is comeager in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*) \times \mathcal{H}$; and this, in turn, implies that the set

$$\big\{ T \in \mathfrak{B}_M(\mathcal{H}) \,;\, \forall^* x \in \mathcal{H} \text{ is distributionally irregular for } T \big\}$$

is comeager in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. Here, "$\forall^* x \in \mathcal{H}$" means "for quasi-all $x \in \mathcal{H}$ in the Baire category sense". This shows that DDCH$_M(\mathcal{H})$ is comeager in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.

8. Summary

Let us summarize the results obtained so far: for any $M > 1$, an $\mathtt{SOT}^*$-typical hypercyclic operator T

- is topologically weakly mixing but not topologically mixing;
- has empty point spectrum, and hence is not chaotic;
- has no non-trivial invariant measure, hence is not $\mathcal{U}$-frequently hypercyclic, and *a fortiori* not ergodic;
- but is densely distributionally chaotic and hence satisfies $c(T) = 1$.

We shall see in Chapter 4 below that the picture changes drastically when we consider $\mathtt{SOT}$-typical elements of some natural classes of upper-triangular operators with respect to a given orthonormal basis of $\mathcal{H}$.

CHAPTER 3

Descriptive set-theoretic issues

In what follows, we fix $M > 1$. We have seen in the previous section that $\mathrm{CH}_M(\mathcal{H})$, $\mathrm{TMIX}_M(\mathcal{H})$ and $\mathrm{UFHC}_M(\mathcal{H})$ are meager in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. In this chapter, we are going to show that these classes of operators are also *Borel* in $\mathfrak{B}_M(\mathcal{H})$ with respect to $\mathtt{SOT}$ and $\mathtt{SOT}^*$, and we will discuss their exact descriptive complexity in some details. Moreover, we will show that some natural classes of operators defined by dynamical properties are *non-Borel* in $\mathfrak{B}_M(\mathcal{H})$.

Recall the standard notations for Borel classes: $\mathbf{\Sigma}^0_1 = \text{open}$, $\mathbf{\Pi}^0_1 = \text{closed}$, $\mathbf{\Sigma}^0_2 = F_\sigma$, $\mathbf{\Pi}^0_2 = G_\delta$, and in general, for any countable ordinal $\xi \geq 2$:

$$\mathbf{\Sigma}^0_\xi = \Big(\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta\Big)_\sigma \qquad \text{and} \qquad \mathbf{\Pi}^0_\xi = \Big(\bigcup_{\eta<\xi} \mathbf{\Sigma}^0_\eta\Big)_\delta,$$

where $\mathfrak{A}_\sigma$ denotes the family of all countable unions of sets from the family $\mathfrak{A}$ and $\mathfrak{A}_\delta$ is the family of all countable intersections of sets from $\mathfrak{A}$. We refer the reader to [**38**] for (much) more information on the Borel hierarchy.

1. Complexity of the families $\mathbf{TMIX}_M(\mathcal{H})$, $\mathbf{CH}_M(\mathcal{H})$, $\mathbf{UFHC}_M(\mathcal{H}) \cap \mathbf{CH}_M(\mathcal{H})$ and $\mathbf{UFHC}_M(\mathcal{H})$

The following fact will allow us to concentrate mainly on the topology $\mathtt{SOT}^*$.

Lemma 3.1. *The identity map id* : $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}) \to (\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$ *is* Baire 1*; in other words, any* $\mathtt{SOT}^*$*-open subset of* $\mathfrak{B}_M(\mathcal{H})$ *is* $\mathtt{SOT}$-$\mathbf{\Sigma}^0_2$*. Therefore, for every countable ordinal* ξ*, any* $\mathtt{SOT}^*$-$\mathbf{\Sigma}^0_\xi$ *subset of* $\mathfrak{B}_M(\mathcal{H})$ *is* $\mathtt{SOT}$-$\mathbf{\Sigma}^0_{\xi+1}$ *and any* $\mathtt{SOT}^*$-$\mathbf{\Pi}^0_\xi$ *set is* $\mathtt{SOT}$-$\mathbf{\Pi}^0_{\xi+1}$*.*

Proof. The second part of the lemma follows from the first one by an easy transfinite induction argument. So we concentrate on the first part.

Since $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$ is second-countable, it is enough to show that any basic $\mathtt{SOT}^*$-open set is $\mathtt{SOT}$-$\mathbf{\Sigma}^0_2$. Therefore, we just have to check that if $x, a \in \mathcal{H}$ and $\varepsilon > 0$, then the set $\mathfrak{U} := \{T \in \mathfrak{B}_M(\mathcal{H});\ \|T^*x - a\| < \varepsilon\}$ is $\mathtt{SOT}$-F_σ. But this is clear since

$$T \in \mathfrak{U} \iff \exists k \in \mathbb{N} \left(\forall h \in \mathcal{H},\ \|h\| \leq 1\ :\ |\langle x, Th\rangle - \langle a, h\rangle| \leq \varepsilon - \frac{1}{k}\right)$$

and the condition under brackets is $\mathtt{SOT}$-closed. □

1.1. Complexity of $\mathbf{TMIX}_M(\mathcal{H})$.

The complexity of $\mathrm{TMIX}_M(\mathcal{H})$ is given by the following proposition:

Proposition 3.2. *The set* $\mathrm{TMIX}_M(\mathcal{H})$ *is a* $\mathbf{\Pi}^0_4$ *subset of* $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT})$*, and a "true"* $\mathbf{\Pi}^0_3$ *subset of* $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$*, i.e. a* $\mathbf{\Pi}^0_3$ *set which is not* $\mathbf{\Sigma}^0_3$*.*

Proof. Let us first show that $\mathrm{TMIX}_M(\mathcal{H})$ is $\mathbf{\Pi}^0_3$ with respect to $\mathtt{SOT}^*$, and hence (by Lemma 3.1) $\mathbf{\Pi}^0_4$ with respect to $\mathtt{SOT}$. Let $(B_p)_{p\geq 1}$ be a countable family of closed balls of $\mathcal{H}$ whose interiors form a basis of open sets for $\mathcal{H}$. Then an operator $T \in \mathfrak{B}_M(\mathcal{H})$ is topologically mixing if and only if

$$\forall p, q \geq 1\ \exists N \in \mathbb{N}\ \forall n \geq N\ :\ T^n(B_p) \cap B_q \neq \emptyset.$$

For each fixed data (p,q,n), the condition "$T^n(B_p)\cap B_q\neq\emptyset$" is $\mathtt{SOT}^*$-closed, by weak compactness of B_p, weak closedness of B_q and continuity of the map $(T,x)\mapsto T^nx$ from $(\mathfrak{B}_M(\mathcal{H}),\mathtt{SOT}^*)\times(B_p,w)$ into $(\mathcal{H},w)$. This shows that $\mathrm{TMIX}_M(\mathcal{H})$ is $\mathbf{\Pi}^0_3$ in $(\mathfrak{B}_M(\mathcal{H}),\mathtt{SOT}^*)$.

In order to show that $\mathrm{TMIX}_M(\mathcal{H})$ is a true $\mathbf{\Pi}^0_3$ subset of $(\mathfrak{B}_M(\mathcal{H}),\mathtt{SOT}^*)$, we assume that $\mathcal{H}=\ell^2(\mathbb{N})$ and we use weighted backward shifts on $\mathcal{H}$. It is well-known that a weighted backward shift $B_{\boldsymbol{\omega}}$ on $\mathcal{H}$ is topologically mixing if and only if the weight sequence $\boldsymbol{\omega}=(\omega_k)_{k\geq1}$ satisfies

$$\lim_{n\to\infty}|\omega_1\cdots\omega_n|=\infty.$$

Let us denote by $\mathcal{S}$ the set of all sequences of positive integers $s=(s_k)_{k\geq1}$ such that $s_{k+1}\leq Ms_k$ for all $k\geq1$. This is a closed subset of the Polish space $\mathbb{N}^{\mathbb{N}}$, and hence a Polish space as well. We need the following fact.

Claim 3.3. The set $\mathcal{S}_\infty:=\{s\in\mathcal{S};\ s_k\to\infty \text{ as } k\to\infty\}$ is a true $\mathbf{\Pi}^0_3$ set in $\mathcal{S}$.

Proof of Claim 3.3. It is well known (see e.g. [**38**, Section 23.A]) that the set $\mathcal{N}_\infty:=\{\alpha\in\mathbb{N}^{\mathbb{N}};\ \alpha_k\to\infty\}$ is a true $\mathbf{\Pi}^0_3$ set in $\mathbb{N}^{\mathbb{N}}$. So we just need to find a continuous map $\Phi:\mathbb{N}^{\mathbb{N}}\to\mathcal{S}$ such that $\Phi^{-1}(\mathcal{S}_\infty)=\mathcal{N}_\infty$. In other words, our goal is to associate to each $\alpha\in\mathbb{N}^{\mathbb{N}}$ another sequence $s\in\mathbb{N}^{\mathbb{N}}$ in such a way that $s_k\to\infty$ exactly when $\alpha_k\to\infty$ and, additionally, $s_{k+1}\leq Ms_k$ for all $k\geq1$; and this needs to be done in a continuous fashion. For any $\alpha\in\mathbb{N}^{\mathbb{N}}$, we define $s=s(\alpha)$ as follows: $s_1=\alpha_1$ and, for every $k\geq1$,

$$s_{k+1}=\begin{cases}\alpha_{k+1} & \text{if } \alpha_{k+1}\leq Ms_k\\ Ms_k & \text{if } \alpha_{k+1}>Ms_k.\end{cases}$$

It is obvious that $s_{k+1}\leq Ms_k$ for all $k\geq1$ and that the map $\alpha\mapsto s(\alpha)$ is continuous. Moreover, since $M>1$, it is straightforward to check that $\alpha_k\to\infty$ if and only if $s_k\to\infty$. □

Going back to the proof of Proposition 3.2, we associate to each $s\in\mathcal{S}$ a weight sequence $\boldsymbol{\omega}(s)=(\omega_k(s))_{k\geq1}$ defined as follows:

$$\omega_1(s)=1\qquad\text{and}\qquad\omega_{k+1}(s)=\frac{s_{k+1}}{s_k}\qquad\text{for every } k\geq1.$$

Since $s\in\mathcal{S}$, we have $0<\omega_k(s)\leq M$ for all $k\geq1$, so that the weighted shift $B_{\boldsymbol{\omega}(s)}$ on $\mathcal{H}$ satisfies $\|B_{\boldsymbol{\omega}(s)}\|\leq M$. Moreover, the map $s\mapsto\boldsymbol{\omega}(s)$ is clearly continuous from $\mathcal{S}$ into $\mathbb{R}^{\mathbb{N}}$, and hence the map $s\mapsto B_{\boldsymbol{\omega}(s)}$ is continuous from $\mathcal{S}$ into $(\mathfrak{B}_M(\mathcal{H}),\mathtt{SOT}^*)$. Finally, since $\omega_1(s)\cdots\omega_k(s)=s_k/s_1$ for all $k\geq1$, the shift $B_{\boldsymbol{\omega}(s)}$ is topologically mixing if and only if $s_k\to\infty$. We have thus constructed a continuous map $\Phi:\mathcal{S}\to\mathfrak{B}_M(\mathcal{H})$ such that $\Phi^{-1}(\mathrm{TMIX}_M(\mathcal{H}))=\mathcal{S}_\infty$, which proves that $\mathrm{TMIX}_M(\mathcal{H})$ is a true $\mathbf{\Pi}^0_3$ subset of $(\mathfrak{B}_M(\mathcal{H}),\mathtt{SOT}^*)$ by Claim 3.3. □

1.2. Complexity of $\mathbf{CH}_M(\mathcal{H})$. We now consider the class of chaotic operators on $\mathcal{H}$. Our aim in this section is to prove the following result:

Proposition 3.4. *The set $\mathrm{CH}_M(\mathcal{H})$ is a $\mathbf{\Pi}^0_4$ subset of $(\mathfrak{B}_M(\mathcal{H}),\mathtt{SOT})$, and a true $\mathbf{\Pi}^0_3$ subset of $(\mathfrak{B}_M(\mathcal{H}),\mathtt{SOT}^*)$.*

To prove this result, we will make use of the so-called *Kalisch operators*. These operators, which were introduced by Kalisch in [**37**], display interesting dynamical properties (see for instance [**8**, Section 5.5.3]). They are defined as follows. Let $T:L^2(0,2\pi)\to L^2(0,2\pi)$ be the operator defined for every $f\in L^2(0,2\pi)$ by

$$Tf(\theta)=e^{i\theta}f(\theta)-\int_0^\theta ie^{it}f(t)\,dt,\quad\theta\in(0,2\pi).$$

A simple computation shows that for any $\lambda=e^{i\alpha}\in\mathbb{T}\setminus\{1\}$, the function $f_\lambda:=\mathbf{1}_{(\alpha,2\pi)}$ is an eigenvector of T associated to the eigenvalue λ, and that $\ker(T-\lambda)=\mathrm{span}\,[f_\lambda]$. Note also that the map $\lambda\mapsto f_\lambda$ is continuous from $\mathbb{T}\setminus\{1\}$ into $L^2(0,2\pi)$. In particular, if $\Lambda\neq\emptyset$ is any compact subset of $\mathbb{T}\setminus\{1\}$, the closed subspace $\mathcal{H}_\Lambda$ of $L^2(0,2\pi)$ spanned by the functions $f_\lambda,\lambda\in\Lambda$, is T-invariant.

The Kalisch operator associated to Λ is the operator $T_\Lambda : \mathcal{H}_\Lambda \to \mathcal{H}_\Lambda$ induced by T on $\mathcal{H}_\Lambda$. The following lemma summarizes the known facts that we will need regarding these operators.

LEMMA 3.5. *For any compact set $\Lambda \subseteq \mathbb{T} \setminus \{1\}$, the Kalisch operator T_Λ has the following properties.*

(a) *The spectrum of T_Λ is exactly equal to Λ.*
(b) *The operator T_Λ is hypercyclic if and only if Λ is a perfect set, in which case T_Λ is actually ergodic in the Gaussian sense.*

PROOF. (a) This is proved in [**37**] (see also [**4**, Lemma 2.12]).

(b) Recall that a *perfect set* is a non-empty compact set without isolated points. If Λ is a perfect set, then T_Λ belongs to $\mathrm{PSPAN}(\mathcal{H}_\Lambda)$ because the map $\lambda \mapsto f_\lambda$ is continuous; and hence T_Λ is ergodic in the Gaussian sense. Conversely, if Λ has an isolated point λ_0, then T_Λ cannot be hypercyclic, otherwise the restriction of T_Λ to the one-dimensional space $\mathrm{span}\,[f_{\lambda_0}]$ would be hypercyclic as well by the Riesz decomposition theorem, which is of course not possible. □

As the proof of Proposition 3.4 is somewhat involved, we split it into several sub-statements. First, it is rather straightforward to show:

LEMMA 3.6. *The set $\mathrm{CH}_M(\mathcal{H})$ is $\mathbf{\Pi}^0_3$ with respect to $\mathtt{SOT}^*$, and hence $\mathbf{\Pi}^0_4$ with respect to $\mathtt{SOT}$.*

PROOF. Let $(B_p)_{p\geq 1}$ be a countable family of closed balls whose interiors form a basis of open sets for $\mathcal{H}$. By definition, an operator $T \in \mathfrak{B}_M(\mathcal{H})$ is chaotic if and only if it is hypercyclic and each ball B_p contains a periodic point for T. In other words:

$$T \in \mathrm{CH}_M(\mathcal{H}) \iff T \in \mathrm{HC}_M(\mathcal{H}) \quad \text{and} \quad \forall p \in \mathbb{N}\ \exists N \in \mathbb{N}\ :\ \Big(\exists x \in B_p\ :\ T^N x = x\Big).$$

For each fixed pair (p, N), the condition under brackets is $\mathtt{SOT}^*$-closed by continuity of the map $(T, x) \mapsto T^N x$ and weak compactness of the ball B_p. Therefore, the second half of the condition on the right hand side of the above display defines a $\mathbf{\Pi}^0_3$ set; and since $\mathrm{HC}_M(\mathcal{H})$ is G_δ, it follows that $\mathrm{CH}_M(\mathcal{H})$ is $\mathbf{\Pi}^0_3$ in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. □

The proof that $\mathrm{CH}_M(\mathcal{H})$ is a true $\mathbf{\Pi}^0_3$ set is more involved.

First, we deduce from Lemma 3.5 the following characterization of those compact subsets Λ of $\mathbb{T} \setminus \{1\}$ for which T_Λ is chaotic. Let us denote by Ω the subset of $\mathbb{T}$ consisting of all roots of unity.

CLAIM 3.7. Let Λ be a perfect subset of $\mathbb{T} \setminus \{1\}$. Then the Kalisch operator T_Λ is chaotic if and only if $\Omega \cap \Lambda$ is dense in Λ.

PROOF. Since the map $\lambda \mapsto f_\lambda$ is continuous and f_λ is a periodic point of T_Λ if $\lambda \in \Omega \cap \Lambda$, it is clear that $\mathrm{Per}(T_\Lambda)$ is dense in $\mathcal{H}_\Lambda$ if $\Omega \cap \Lambda$ is dense in Λ. Since Λ is assumed to be perfect, T_Λ is hypercyclic, and hence chaotic. Conversely, assume that $\Omega \cap \Lambda$ is not dense in Λ and choose $\lambda \in \Lambda \setminus \Lambda_0$, where $\Lambda_0 = \overline{\Omega \cap \Lambda}$. Then f_λ is not an eigenvector of T_{Λ_0} since $\sigma(T_{\Lambda_0}) = \Lambda_0$. Since however f_λ is an eigenvector of T, this means that f_λ does not belong to $\mathcal{H}_{\Lambda_0} = \overline{\mathrm{span}}\,[f_\xi;\ \xi \in \Lambda_0]$, *i.e.* that f_λ does not belong to $\overline{\mathrm{span}}\,[f_\xi;\ \xi \in \Omega \cap \Lambda]$. But since $\sigma(T_\Lambda) = \Lambda$ and $\ker(T - \xi) = \mathrm{span}\,[f_\xi]$ for every $\xi \in \mathbb{T} \setminus \{1\}$, we have

$$\mathrm{span}\,[f_\xi;\ \xi \in \Omega \cap \Lambda] = \mathrm{span}\Big[\bigcup_{\xi \in \Omega \cap \Lambda} \ker(T_\Lambda - \xi)\Big] = \mathrm{Per}(T_\Lambda).$$

So $\mathrm{Per}(T_\Lambda)$ is not dense in $\mathcal{H}_\Lambda$ and T_Λ is not chaotic. □

Here is now a purely descriptive set-theoretic fact, which is certainly well-known but for which we were unable to locate a reference. Here and afterwards, given a compact metric space E, we denote by $\mathcal{K}(E)$ the space of all non-empty compact subsets of E endowed with its usual topology, and by $\mathcal{K}_{\mathrm{perf}}(E)$ the set of all perfect subsets of E. Recall that $\mathcal{K}(E)$ is compact metrizable, and that $\mathcal{K}_{\mathrm{perf}}(E)$ is a G_δ subset of $\mathcal{K}(E)$. So $\mathcal{K}_{\mathrm{perf}}(E)$ is a Polish space. See [**38**] or [**52**] for more details.

Claim 3.8. Let $E \subseteq \mathbb{T}$ be a perfect set such that $\Omega \cap E$ is dense in E. Then the set $\mathcal{W} := \{\Lambda \in \mathcal{K}_{\rm perf}(E);\ \Omega \cap \Lambda \text{ is dense in } \Lambda\}$ is a true $\mathbf{\Pi}^0_3$ set in $\mathcal{K}_{\rm perf}(E)$.

Proof. Since $\mathcal{K}_{\rm perf}(E)$ is G_δ in $\mathcal{K}(E)$, it is in fact enough to show that $\mathcal{W}$ is a true $\mathbf{\Pi}^0_3$ set in $\mathcal{K}(E)$. That $\mathcal{W}$ is $\mathbf{\Pi}^0_3$ is easy to check. In order to show that it is a true $\mathbf{\Pi}^0_3$ set, we proceed as follows.

Let us denote by $\mathbf{C}$ the Cantor space $\{0,1\}^{\mathbb{N}}$, and by $\mathbf{Q}$ the set of all "rationals" of $\mathbf{C}$, *i.e.* $\mathbf{Q} = \{\alpha = (\alpha(i))_{i\geq 1} \in \mathbf{C};\ \alpha(i) \text{ is eventually } 0\}$. It is known (see [**38**, Section 23.A]) that the set

$$W := \{\bar\alpha = (\alpha_n)_{n\geq 1} \in \mathbf{C}^{\mathbb{N}};\ \forall n \geq 1\ :\ \alpha_n \in \mathbf{Q}\}$$

is a true $\mathbf{\Pi}^0_3$ set in $\mathbf{C}^{\mathbb{N}}$. So it is enough to find a continuous map $\Phi : \mathbf{C}^{\mathbb{N}} \to \mathcal{K}(E)$ such that $\Phi^{-1}(\mathcal{W}) = W$.

We first note that if $M \subseteq E$ is a perfect set such that $\Omega \cap M$ is dense in M, then $\mathcal{W} \cap \mathcal{K}(M)$ is not G_δ in $\mathcal{K}(M)$. Indeed, on the one hand $\mathcal{W} \cap \mathcal{K}(M)$ is easily seen to be dense in $\mathcal{K}(M)$ because M is perfect and $\Omega \cap M$ is dense in M; and on the other hand $\mathcal{W} \cap \mathcal{K}(M)$ is meager in $\mathcal{K}(M)$ because it is disjoint from the G_δ set $\mathcal{K}(M \setminus \Omega)$, which is dense in $\mathcal{K}(M)$ because M is perfect and Ω is countable. Hence $\mathcal{W} \cap \mathcal{K}(M)$ cannot be G_δ in $\mathcal{K}(M)$ by the Baire Category Theorem. By Wadge's Lemma (see [**38**, Th. 21.14]) applied to the two Borel sets $\mathcal{W} \cap \mathcal{K}(M)$ and $\mathbf{C} \setminus \mathbf{Q}$, it follows that for any perfect set M as above, $\mathcal{W} \cap \mathcal{K}(M)$ is "$\mathbf{\Sigma}^0_2$-hard", *i.e.* one can find a continuous map $\varphi : \mathbf{C} \to \mathcal{K}(M)$ such that $\varphi^{-1}\big(\mathcal{W} \cap \mathcal{K}(M)\big) = \mathbf{Q}$.

Now let us choose a sequence $(M_n)_{n\geq 1}$ of pairwise disjoint perfect subsets of E such that $\Omega \cap M_n$ is dense in M_n for every $n \geq 1$ and the sets M_n accumulate to some point $a \in E$, which means that every neighborhood of a contains all but finitely many of the sets M_n. This is possible because E is perfect and $\Omega \cap E$ is dense in E. For each $n \geq 1$, let $\varphi_n : \mathbf{C} \to \mathcal{K}(M_n)$ be a continuous map such that $\varphi_n^{-1}\big(\mathcal{W} \cap \mathcal{K}(M_n)\big) = \mathbf{Q}$. One can then define a map $\Phi : \mathbf{C}^{\mathbb{N}} \to \mathcal{K}(E)$ by setting

$$\Phi(\bar\alpha) := \{a\} \cup \bigcup_{n=1}^{\infty} \varphi_n(\alpha_n) \qquad \text{for every } \bar\alpha = (\alpha_n) \in \mathbf{C}^{\mathbb{N}}.$$

Since the sets M_n accumulate to a, it is clear that each $\Phi(\bar\alpha)$ is indeed a compact subset of $\mathbb{T}$ and that the map Φ is continuous. Moreover, it is equally clear that $\Phi(\bar\alpha)$ is perfect if and only if all the sets $\varphi_n(\alpha_n)$ are perfect, and that $\Omega \cap \Phi(\bar\alpha)$ is dense in $\Phi(\bar\alpha)$ if and only if $\Omega \cap \varphi_n(\alpha_n)$ is dense in $\varphi_n(\alpha_n)$ for every $n \geq 1$. Hence $\Phi(\bar\alpha)$ belongs to $\mathcal{W}$ if and only if α_n belongs to $\varphi_n^{-1}\big(\mathcal{W} \cap \mathcal{K}(M_n)\big) = \mathbf{Q}$ for all $n \geq 1$. We have thus proved that

$$\Phi^{-1}(\mathcal{W}) = \big\{\bar\alpha;\ \forall n \geq 1\ :\ \alpha_n \in \mathbf{Q}\big\} = W,$$

which concludes the proof of Claim 3.8. □

We need yet one more fact, which is a simple and certainly well-known consequence of the Michael Selection Theorem. The version of this theorem which we use here runs as follows (see [**52**, Section III.19]).

Let X be a zero-dimensional compact space, and let Y be a complete metric space. Let $\Phi : X \to \mathcal{F}(Y)$ be a lower semi-continuous map, where $\mathcal{F}(Y)$ denotes the set of all non-empty closed subsets of Y. Then Φ admits a continuous selection*, i.e. there exists a continuous map $f : X \to Y$ such that $f(x)$ belongs to $\Phi(x)$ for every $x \in X$.*

Claim 3.9. Let E be a zero-dimensional compact metric space. There exists a sequence $(\xi_n)_{n\geq 1}$ of continuous maps from $\mathcal{K}_{\rm perf}(E)$ into E such that for each $\Lambda \in \mathcal{K}_{\rm perf}(E)$, the points $\xi_n(\Lambda)$, $n \geq 1$, belong to Λ, are distinct, and the countable set $\{\xi_n(\Lambda);\ n \geq 1\}$ is dense in Λ.

Proof. Since E is zero-dimensional, one can choose a countable basis $(V_i)_{i\geq 1}$ for the topology of E consisting of (non-empty) clopen sets. For each $\Lambda \in \mathcal{K}_{\rm perf}(E)$, let us denote by $i_1(\Lambda) < i_2(\Lambda) < \dots$ the integers i such that $\Lambda \cap V_i \neq \emptyset$ (since Λ is perfect, there are infinitely many such integers i). As the V_i are clopen sets, it is not hard to see that the functions i_n, $n \geq 1$, are locally constant.

In particular, the map $\Phi : \Lambda \mapsto \Lambda \cap V_{i_1(\Lambda)}$ from $\mathcal{K}_{\mathrm{perf}}(E)$ into $\mathcal{K}(E)$ is continuous. Since E is zero-dimensional, $\mathcal{K}_{\mathrm{perf}}(E)$ is zero-dimensional as well, and one can apply the version of Michael's Selection Theorem quoted above to the map Φ. This yields a continuous map $\Lambda \mapsto \xi_1(\Lambda)$ from $\mathcal{K}_{\mathrm{perf}}(E)$ into E such that $\xi_1(\Lambda)$ belongs to $\Lambda \cap V_{i_1(\Lambda)}$ for every $\Lambda \in \mathcal{K}_{\mathrm{perf}}(E)$. Now, any perfect set $\Lambda \in \mathcal{K}_{\mathrm{perf}}(E)$ satisfies $\Lambda \cap (V_{i_2(\Lambda)} \setminus \{\xi_1(\Lambda)\}) \neq \emptyset$; so there exists an integer j such that $\xi_1(\Lambda) \in V_j$ and $V_{i_2(\Lambda)} \setminus V_j \neq \emptyset$. Moreover, if we denote by $j_2(\Lambda)$ the smallest such integer j with these two properties, the map $\Lambda \mapsto j_2(\Lambda)$ is locally constant, and hence the map $\Lambda \mapsto \Lambda \cap \big(V_{i_2(\Lambda)} \setminus V_{j_2(\Lambda)}\big)$ is continuous from $\mathcal{K}_{\mathrm{perf}}(E)$ into $\mathcal{K}(E)$. Applying Michael's Selection Theorem a second time, we obtain a continuous map $\xi_2 : \mathcal{K}_{\mathrm{perf}}(E) \to E$ such that $\xi_2(\Lambda) \in \Lambda \cap \big(V_{i_2(\Lambda)} \setminus V_{j_2(\Lambda)}\big)$ for every $\Lambda \in \mathcal{K}_{\mathrm{perf}}(E)$; in particular, $\xi_2(\Lambda) \in \Lambda \cap V_{i_2(\Lambda)}$ and $\xi_2(\Lambda) \neq \xi_1(\Lambda)$. Continuing in this fashion, we obtain the required sequence $(\xi_n)_{n\geq 1}$. □

We are now in position to prove:

LEMMA 3.10. *The set* $\mathrm{CH}_M(\mathcal{H})$ *is a true* $\mathbf{\Pi}^0_3$ *set in* $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.

PROOF. Let us first choose a compact set $E \subseteq \mathbb{T} \setminus \{1\}$ with the following properties: E is perfect and zero-dimensional, $\Omega \cap E$ is dense in E and, moreover, $\|T_E\| \leq M$. Such a compact set does exist. Indeed, the definition of the operator T shows that the norm of the restriction of T to the subspace $L^2(u, 2\pi) \subseteq L^2(0, 2\pi)$ tends to 1 as $u \to 2\pi$. Therefore, if $\Lambda_0 \subseteq \mathbb{T} \setminus \{1\}$ is a non-trivial closed arc sufficiently close to 1 then $\|T_{\Lambda_0}\| \leq M$; so it is enough to take as E any perfect set with empty interior contained in Λ_0 and such that $\Omega \cap E$ is dense in E.

Having fixed E in this way, let $(\xi_n)_{n\geq 1}$ be a sequence of continuous maps given by Claim 3.9, selecting a dense sequence of distinct points in each perfect set $\Lambda \subseteq E$. Note that if $\Lambda \in \mathcal{K}_{\mathrm{perf}}(E)$, the points $\xi_n(\Lambda)$ are distinct and form a dense subset of Λ, so that the functions $f_{\xi_n(\Lambda)}$ are linearly independent and span a dense subspace of $\mathcal{H}_\Lambda$. Applying the Gram-Schmidt orthonormalization process to the sequence $(f_{\xi_n(\Lambda)})_{n\geq 1}$, we obtain an orthonormal basis $(e_n(\Lambda))_{n\geq 1}$ of $\mathcal{H}_\Lambda$ which depends continuously on Λ, that is, each map $\Lambda \mapsto e_n(\Lambda)$ is continuous from $\mathcal{K}_{\mathrm{perf}}(E)$ into $L^2(0, 2\pi)$.

Let us now fix an orthonormal basis $(e_n)_{n\geq 1}$ of $\mathcal{H}$. For each $\Lambda \in \mathcal{K}_{\mathrm{perf}}(E)$, denote by $U_\Lambda : \mathcal{H}_\Lambda \to \mathcal{H}$ the unitary operator defined by setting $U_\Lambda e_n(\Lambda) = e_n$ for every $n \geq 1$. Since $\|T_E\| \leq M$, one can define a map $\Phi : \mathcal{K}_{\mathrm{perf}}(E) \to \mathfrak{B}_M(\mathcal{H})$ by setting

$$\Phi(\Lambda) := U_\Lambda T_\Lambda U_\lambda^{-1} \qquad \text{for every } \Lambda \in \mathcal{K}_{\mathrm{perf}}(E).$$

Since $\Phi(\Lambda)$ and T_Λ are unitarily isomorphic, $\Phi(\Lambda)$ is chaotic if and only if T_Λ is, which holds true exactly when $\Omega \cap \Lambda$ is dense in Λ. Thus, we have

$$\Phi^{-1}(\mathrm{CH}_M(\mathcal{H})) = \mathcal{W}.$$

Claim 3.8 will allow us to conclude the proof, provided that we are able to show that the map Φ is continuous from $\mathcal{K}_{\mathrm{perf}}(E)$ into $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. This relies on the following observation. For every $\Lambda \in \mathcal{K}_{\mathrm{perf}}(E)$, let us denote by $P_\Lambda : L^2(0, 2\pi) \to \mathcal{H}_\Lambda$ the orthogonal projection of $L^2(0, 2\pi)$ onto $\mathcal{H}_\Lambda$, and by $J_\Lambda : \mathcal{H}_\Lambda \to L^2(0, 2\pi)$ the canonical embedding of $\mathcal{H}_\Lambda$ into $L^2(0, 2\pi)$. In other words, $J_\Lambda = P_\Lambda^*$.

CLAIM 3.11. For any $f \in L^2(0, 2\pi)$, the map $\Lambda \mapsto U_\Lambda P_\Lambda f$ is continuous from $\mathcal{K}_{\mathrm{perf}}(E)$ into $\mathcal{H}$; and for every $x \in \mathcal{H}$, the map $\Lambda \mapsto J_\Lambda U_\Lambda^{-1} x$ is continuous from $\mathcal{K}_{\mathrm{perf}}(E)$ into $L^2(0, 2\pi)$.

PROOF OF CLAIM 3.11. Fix $f \in L^2(0, 2\pi)$. Since $P_\Lambda f = \sum\limits_{n=1}^{\infty} \langle f, e_n(\Lambda)\rangle\, e_n(\Lambda)$, we have

$$U_\Lambda P_\Lambda f = \sum_{n=1}^{\infty} \langle f, e_n(\Lambda)\rangle\, e_n.$$

The maps $\Lambda \mapsto e_n(\Lambda)$, $n \geq 1$, being continuous, it suffices to show that the convergence of the above series is uniform (with respect to Λ) on compact subsets of $\mathcal{K}_{\mathrm{perf}}(E)$ in order to derive the

continuity of the map $\Lambda \mapsto U_\Lambda P_\Lambda f$. Now, we have for any $N \geq 1$

$$R_N(\Lambda) := \Big\| \sum_{n>N} \langle f, e_n(\Lambda)\rangle\, e_n \Big\|^2 = \sum_{n>N} |\langle f, e_n(\Lambda)\rangle|^2 = \Big\| f - \sum_{n=1}^{N} \langle f, e_n(\Lambda)\rangle\, e_n(\Lambda) \Big\|^2.$$

In particular, the maps $\Lambda \mapsto R_N(\Lambda)$ are continuous. Since the sequence $(R_N)_{N\geq 1}$ is non-increasing, it converges uniformly to 0 on compact subsets of $\mathcal{K}_{\mathrm{perf}}(E)$ by Dini's Theorem. This proves the first part of Claim 3.11. The proof of the second part is exactly similar. □

Let us now prove that Φ is continuous from $\mathcal{K}_{\mathrm{perf}}(E)$ into $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$, which amounts to showing that for any $x \in \mathcal{H}$, the maps $\Lambda \mapsto \Phi(\Lambda)x$ and $\Lambda \mapsto \Phi(\Lambda)^* x$ are continuous. Since $T_\Lambda = P_\Lambda T J_\Lambda$, we have

$$\Phi(\Lambda)x = U_\Lambda P_\Lambda\, T\, J_\Lambda U_\Lambda^{-1} x \qquad \text{and} \qquad \Phi(\Lambda)^* x = U_\Lambda P_\Lambda\, T^*\, J_\Lambda U_\Lambda^{-1} x.$$

By Claim 3.11, the map $\Lambda \mapsto f_\Lambda := T J_\Lambda U_\Lambda^{-1} x$ is continuous from $\mathcal{K}_{\mathrm{perf}}(E)$ into $L^2(0, 2\pi)$. By Claim 3.11 again, and since the map $\Lambda \mapsto U_\Lambda P_\Lambda$ takes values in a bounded subset of $\mathfrak{B}(L^2(0,2\pi), \mathcal{H})$, the map $\Lambda \mapsto \Phi(\Lambda)x = U_\Lambda P_\Lambda f_\Lambda$ is continuous. Likewise, the map $\Lambda \mapsto \Phi(\Lambda)^* x$ is continuous. The equality $\Phi^{-1}(\mathrm{CH}_M(\mathcal{H})) = \mathcal{W}$, the continuity of Φ and Claim 3.8 now imply that $\mathrm{CH}_M(\mathcal{H}))$ is a true $\mathbf{\Pi}^0_3$ set in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. □

Combining Lemmas 3.6 and Lemma 3.10, we see that we have finally shown that $\mathrm{CH}_M(\mathcal{H})$ is a $\mathbf{\Pi}^0_4$ subset of $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT})$, and a true $\mathbf{\Pi}^0_3$ subset of $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$, and thus proved Proposition 3.4.

1.3. Complexity of $\mathbf{UFHC}_M(\mathcal{H})$ and $\mathbf{UFHC}_M(\mathcal{H}) \cap \mathbf{CH}_M(\mathcal{H})$. In this section, we study the descriptive complexity of the sets $\mathrm{UFHC}_M(\mathcal{H})$ and $\mathrm{UFHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$ in the space $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$, for any $M > 1$. Although we have been unable to determine the exact complexity of these sets, we obtain some simple upper bounds and some rather non-trivial lower bounds. Our first result relies on the following observation, first made in [**10**] and then developed in [**15**].

Lemma 3.12. *Let $(V_q)_{q\geq 1}$ be a countable basis of (non-empty) open subsets of $\mathcal{H}$, and let T be a bounded operator on $\mathcal{H}$.*

(1) Assume that T belongs to UFHC($\mathcal{H}$), and let $x_0 \in H$ be a $\mathcal{U}$-frequently hypercyclic vector for T. For each $q \geq 1$, denote by $2\delta_q$ the upper density of the set $\mathcal{N}_T(x_0, V_q)$. For any integers $q, N \geq 1$, define

$$G_{q,N} := \Big\{ x \in \mathcal{H};\ \exists n \geq N\ :\ \#\{1 \leq i \leq n\,;\, T^i x \in V_q\} \geq n\delta_q \Big\}.$$

Then

$$G := \bigcap_{q\geq 1} \bigcap_{N \geq 1} G_{q,N}$$

is a dense G_δ subset of $\mathcal{H}$ which consists entirely of $\mathcal{U}$-frequently hypercyclic vectors for T.

(2) Conversely, assume that there exists a sequence of positive numbers $(\delta_q)_{q\geq 1}$ such that for any $q, N \geq 1$, the open set $G_{q,N}$ defined by the formula above is dense in $\mathcal{H}$. Then T is $\mathcal{U}$-frequently hypercyclic and admits a dense G_δ set of $\mathcal{U}$-frequently hypercyclic vectors x which satisfy

$$\overline{dens}\, \mathcal{N}_T(x, V_q) \geq \delta_q \qquad \text{for every } q \geq 1.$$

We deduce from this the following proposition:

Proposition 3.13. *For any $M > 1$, the set $\mathrm{UFHC}_M(\mathcal{H})$ is $\mathbf{\Pi}^0_4$ in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT})$, hence also in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.*

Proof. Let us denote by $\mathbb{Q}_+^*$ the set of all positive rational numbers. It follows immediately from Lemma 3.12 that an operator $T \in \mathfrak{B}_M(\mathcal{H})$ belongs to $\mathrm{UFHC}_M(\mathcal{H})$ if and only if

$$\forall q \geq 1\ \exists \delta \in \mathbb{Q}_+^*\ \forall N \geq 1\ \forall p \geq 1\ \exists x \in V_p\ \exists n \geq N\ :\ \#\{1 \leq i \leq n\,;\, T^i x \in V_q\} \geq n\delta.$$

In order to prove that $\mathrm{UFHC}_M(\mathcal{H})$ is $\mathbf{\Pi}_4^0$ in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT})$, it is enough to check that each set

$$\Big\{T \in \mathfrak{B}_M(\mathcal{H})\,;\, \#\{1 \leq i \leq n\,;\, T^i x \in V_q\} \geq n\delta\Big\}$$

is $\mathtt{SOT}$-open in $\mathfrak{B}_M(\mathcal{H})$. This is easy: if T_0 belong to this set, one can find $k \geq n\delta$ distinct integers $1 \leq i_1, \dots i_k \leq n$ such that $T_0^{i_s} x$ belongs to V_q for every $s = 1, \dots, k$. If $T \in \mathfrak{B}_M(\mathcal{H})$ is sufficiently close to T_0 for the $\mathtt{SOT}$-topology, it still satisfies $T^{i_s} x \in V_q$ for every $1 \leq s \leq k$, and thus belongs to our set. Hence $\mathrm{UFHC}_M(\mathcal{H})$ is $\mathbf{\Pi}_4^0$ in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT})$, and thus in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$ as well. □

Remark 3.14. An "alternative" proof of the Borelness of $\mathrm{UFHC}_M(\mathcal{H})$ could run as follows. According to Lemma 3.12, an operator T belongs to $\mathrm{UFHC}(\mathcal{H})$ if and only if quasi-all vectors $x \in H$ (in the Baire category sense) are $\mathcal{U}$-frequently hypercyclic for T. So, for $T \in \mathfrak{B}_M(\mathcal{H})$, we may write

$$T \in \mathrm{UFHC}_M(\mathcal{H}) \iff \forall^* x \in \mathcal{H}\ \big(x \text{ is } \mathcal{U}\text{-frequently hypercyclic for } T\big).$$

Since the relation $B(T, x) \longleftrightarrow (x$ is $\mathcal{U}$-frequently hypercyclic for $T)$ is Borel in the product space $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}) \times \mathcal{H}$, and since the category quantifier $\forall^*$ preserves Borelness (see for instance [**38**, Section 16.A]), it follows that $\mathrm{UFHC}_M(\mathcal{H})$ is Borel in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT})$.

Remark 3.15. In view of Proposition 3.13, it is natural to wonder whether the class of frequently hypercyclic operators $\mathrm{FHC}_M(\mathcal{H})$ is also Borel in $(\mathrm{HC}_M(\mathcal{H}), \mathtt{SOT})$. Let $(V_q)_{q \geq 1}$ be a countable basis of open sets for $\mathcal{H}$. An operator $T \in \mathfrak{B}_M(\mathcal{H})$ is frequently hypercyclic if and only if

$$\exists x \in \mathcal{H}\ \Big(\forall q \geq 1\ \exists \delta \in \mathbb{Q}_+^*\ \exists N \geq 1\ \forall n \geq N\ :\ \#\{1 \leq i \leq n\,;\, T^i x \in V_q\} > n\delta\Big);$$

and since the relation $R(T, x)$ defined by the expression between brackets is Borel in $\mathfrak{B}_M(\mathcal{H}) \times \mathcal{H}$, we deduce that $\mathrm{FHC}_M(\mathcal{H})$ is a $\mathbf{\Sigma}_1^1$ (that is: analytic) subset of $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT})$. It is quite tempting to conjecture that it is in fact non-Borel in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT})$.

We are now interested in the complexity of the class $\mathrm{UFHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$. Our first result gives an upper bound:

Proposition 3.16. *The set $\mathrm{UFHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$ is a difference of $\mathbf{\Sigma}_3^0$ sets in the space $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$, i.e. it can be written as $A \setminus B$, where A and B are both $\mathtt{SOT}^*$-$\mathbf{\Sigma}_3^0$ sets.*

The proof of this relies on the following lemma, which characterizes in a rather surprising way the $\mathcal{U}$-frequently hypercyclic operators within the class of chaotic operators. We state it without proof as it follows from Theorem 5.23, to be proved below.

Lemma 3.17. *Let T be a chaotic operator on $\mathcal{H}$. Then T is $\mathcal{U}$-frequently hypercyclic if and only if $c(T) > 0$.*

Proof Proposition 3.16. By Lemma 3.17, we have $\mathrm{UFCH}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H}) = \mathrm{c}_M^+(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$, where

$$\mathrm{c}^+(\mathcal{H}) = \{T \in \mathrm{HC}(\mathcal{H})\,;\, c(T) > 0\}.$$

Hence, since $\mathrm{CH}_M(\mathcal{H})$ is $\mathtt{SOT}^*$-$\mathbf{\Pi}_3^0$ by Proposition 3.4, it suffices to show that $\mathrm{c}_M^+(\mathcal{H})$ is $\mathtt{SOT}^*$-$\mathbf{\Sigma}_3^0$. Let $(V_q)_{q \geq 1}$ be a countable basis of non-empty open subsets of $\mathcal{H}$, and observe that, by the definition of $c(T)$, an operator $T \in \mathrm{HC}_M(\mathcal{H})$ belongs to $\mathrm{c}^+(\mathcal{H})$ if and only if

$$\exists\, \varepsilon \in \mathbb{Q}^{+*}\ \forall q \geq 1\ \forall N \geq 1\ \exists x \in V_q\ \exists n \geq N :\ \#\{1 \leq i \leq n\,;\, \|T^i x\| < 1\} \geq n\varepsilon.$$

This condition is easily seen to define an $\mathtt{SOT}^*$-$\mathbf{\Sigma}_3^0$ subset of $\mathfrak{B}_M(\mathcal{H})$; and since $\mathrm{HC}_M(\mathcal{H})$ is G_δ in $\mathfrak{B}_M(\mathcal{H})$, it follows that $\mathrm{c}_M^+(\mathcal{H})$ is $\mathtt{SOT}^*$-$\mathbf{\Sigma}_3^0$. This concludes the proof of Proposition 3.16. □

Our second step in the study of the complexity of $\mathrm{UFHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$ is

PROPOSITION 3.18. *There is no* $\mathtt{SOT}^*$-$\mathbf{\Sigma}_3^0$ *set* A *such that* $\text{G-MIX}_M(\mathcal{H}) \cap \text{CH}_M(\mathcal{H}) \subseteq A \subseteq \text{UFHC}_M(\mathcal{H}) \cup \text{CH}_M(\mathcal{H})$. *In particular,* $\text{UFHC}_M(\mathcal{H}) \cap \text{CH}_M(\mathcal{H})$ *is not* $\mathtt{SOT}^*$-$\mathbf{\Sigma}_3^0$ *in* $\mathfrak{B}_M(\mathcal{H})$.

The proof relies on two lemmas. The first lemma says that there is no G_δ set A such that $\text{G-MIX}_M(\mathcal{H}) \cap \text{CH}_M(\mathcal{H}) \subseteq A \subseteq \text{UFHC}_M(\mathcal{H}) \cup \text{CH}_M(\mathcal{H})$, whereas the second lemma will allow us to use a direct sum argument to show that in fact, there is no $\mathbf{\Sigma}_3^0$ set A with this property.

Our first lemma reads as follows. Recall that we denote by $\mathbf{Q}$ the set of all "rationals" of the Cantor space $\mathbf{C}$,

$$\mathbf{Q} = \{\alpha = (\alpha(i))_{i\geq 1} \in \mathbf{C};\ \alpha(i) = 0 \text{ for all but finitely many } i\}.$$

LEMMA 3.19. *There exists a continuous map* $\alpha \mapsto T_\alpha$ *from* $\mathbf{C}$ *into* $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$ *with the following two properties:*

- *if* $\alpha \in \mathbf{Q}$, *then* T_α *is chaotic and mixing in the Gaussian sense, and hence chaotic and* $\mathcal{U}$*-frequently hypercyclic;*
- *if* $\alpha \notin \mathbf{Q}$, *then the spectrum of* T_α *is reduced to the point* $\{1\}$, *and hence* T_α *is not chaotic and not* $\mathcal{U}$*-frequently hypercyclic.*

PROOF. The very last part of the second assertion follows from the fact that the spectrum of a chaotic or $\mathcal{U}$-frequently hypercyclic operator has no isolated points ([**49**]).

We assume that $\mathcal{H} = \ell^2(\mathbb{N})$, endowed with its canonical basis. For any unilateral weight sequence $\boldsymbol{\omega} = (\omega_j)_{j\geq 1}$, let us denote as usual by $B_{\boldsymbol{\omega}}$ the associated weighted shift acting on $\mathcal{H}$. If $\inf_{j\geq 1} |\omega_j| > 0$, the operator $T_{\boldsymbol{\omega}} = I + B_{\boldsymbol{\omega}}$ is chaotic and mixing in the Gaussian sense since it admits a spanning holomorphic eigenvector field defined in a neighborhood of the point 1 (see Lemma 2.8). On the other hand, if $\omega_j \to 0$ as $j \to \infty$, then $\sigma(B_{\boldsymbol{\omega}}) = \{0\}$, so that $\sigma(T_{\boldsymbol{\omega}}) = \{1\}$. So, setting $c := M - 1 > 0$, it is enough to find a continuous map $\alpha \mapsto \boldsymbol{\omega}(\alpha)$ from $\mathbf{C}$ into $(0, c]^{\mathbb{N}}$ such that $\inf_{j\geq 1} \omega_j(\alpha) > 0$ if $\alpha \in \mathbf{Q}$ while $\omega_j(\alpha) \to 0$ as $j \to \infty$ if $\alpha \notin \mathbf{Q}$. Once this is done, the map $\alpha \mapsto T_\alpha := T_{\boldsymbol{\omega}(\alpha)}$ will enjoy the two properties stated in Lemma 3.19. Now, it is quite easy to define such a map $\alpha \mapsto \boldsymbol{\omega}(\alpha)$: just set, for every $j \geq 1$,

$$\omega_j(\alpha) := \frac{c}{1 + n_j(\alpha)}, \quad \text{where} \quad n_j(\alpha) = \#\{i \leq j;\ \alpha(i) = 1\}.$$

Lemma 3.19 is thus proved. □

To state our second lemma, let us choose a sequence $(\mathcal{H}_n)_{n\geq 1}$ of infinite-dimensional closed subspaces of $\mathcal{H}$ such that $\mathcal{H}$ can be decomposed as the orthogonal direct sum of the spaces $\mathcal{H}_n$, *i.e.* $\mathcal{H} = \bigoplus_{n\geq 1} \mathcal{H}_n$.

LEMMA 3.20. *Let* $T \in \mathfrak{B}(\mathcal{H})$ *have the form* $T = \bigoplus_{n\geq 1} T_n$ *with respect to the decomposition* $\mathcal{H} = \bigoplus_{n\geq 1} \mathcal{H}_n$, *where* $T_n \in \mathfrak{B}(\mathcal{H}_n)$ *for each* $n \geq 1$.

(i) *If all operators* T_n *are chaotic and mixing in the Gaussian sense, then so is* T.
(ii) *If* T *is* $\mathcal{U}$*-frequently hypercyclic or chaotic, then so are all operators* T_n.

PROOF. (i) Assume that all operators T_n are chaotic and mixing in the Gaussian sense, and fix for each $n \geq 1$ a mixing Gaussian measure with full support μ_n for T_n. Each second-order moment $\int_{\mathcal{H}_n} \|x_n\|^2 d\mu((x_n))$ is finite; and by rescaling (*i.e.* replacing each measure μ_n by a measure $\tilde{\mu}_n$ defined by setting $\tilde{\mu}_n(B) = \mu_n(\varepsilon_n^{-1} B)$ for every Borel subset B of $\mathcal{H}_n$, for a suitably small $\varepsilon_n > 0$), we may assume without loss of generality that the series $\sum_{n\geq 1} \int_{\mathcal{H}_n} \|x_n\|^2 d\mu_n(x_n)$ is convergent.

Let us denote by $\boldsymbol{\mu}$ the infinite product measure $\bigotimes_{n\geq 1} \mu_n$ on the product space $\prod_{n\geq 1} \mathcal{H}_n$. Then

$$\int_{\prod_{n\geq 1} \mathcal{H}_n} \Big(\sum_{n=1}^{\infty} \|x_n\|^2\Big)\, d\boldsymbol{\mu}((x_n)) = \sum_{n=1}^{\infty} \int_{\mathcal{H}_n} \|x_n\|^2 d\mu_n(x_n) < \infty,$$

from which it follows that $\boldsymbol{\mu}$ is in fact concentrated on the set

$$\mathbf{H} := \Big\{(x_n)_{n\geq 1} \in \prod_{n\geq 1} \mathcal{H}_n;\ \sum_{n=1}^{\infty} \|x_n\|^2 < \infty\Big\}.$$

We may therefore define a probability measure μ on $\mathcal{H}$ as follows: for any Borel subset C of $\mathcal{H}$,

$$\mu(C) = \boldsymbol{\mu}\Big(\Big\{(x_n)_{n\geq 1} \in \mathbf{H};\ \sum_{n=1}^{\infty} x_n \in C\Big\}\Big).$$

This measure μ is obviously Gaussian, and it is T-invariant because each measure μ_n is T_n-invariant. Indeed, for any bounded Borel function $f : \mathcal{H} \to \mathbb{R}$, we have

$$\int_{\mathcal{H}} (f \circ T)\, d\mu = \int_{\mathbf{H}} f\Big(\sum_{n=1}^{\infty} T_n x_n\Big)\, d\boldsymbol{\mu}((x_n)) = \int_{\mathbf{H}} f\Big(\sum_{n=1}^{\infty} x_n\Big)\, d\boldsymbol{\mu}((x_n)) = \int_{\mathcal{H}} f\, d\mu.$$

The measure μ has full support. Let indeed V be a non-empty open set in $\mathcal{H}$. Choose a point $a \in V$, and $\varepsilon > 0$ such that $B(a, 2\varepsilon) \subseteq V$. We may assume that a belongs to $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_N$ for some $N \geq 1$, where N is so large that

$$\int_{\mathbf{H}^N} \sum_{n>N} \|x_n\|^2\, d\boldsymbol{\mu}^N((x_n)) < \varepsilon^2, \tag{6}$$

where $\mathbf{H}^N := \Big\{(x_n)_{n\geq N} \in \prod_{n\geq N} \mathcal{H}_n;\ \sum_{n=1}^{\infty} \|x_n\|^2 < \infty\Big\}$ and $\boldsymbol{\mu}^N := \bigotimes_{n>N} \mu_n$. Then the set $\{(x_n)_{n\geq 1} \in \mathbf{H};\ \sum_{n\geq 1} x_n \in V\}$ contains

$$\mathbf{A} := \Big\{(x_n)_{n\geq 1} \in \mathbf{H};\ x_1 + \cdots + x_N \in B(a,\varepsilon) \quad \text{and} \quad \sum_{n>N} \|x_n\|^2 < \varepsilon^2\Big\}.$$

Moreover, if we set $\mathbf{A}^N := \big\{(x_n)_{n>N};\ \sum_{n>N} \|x_n\| < \varepsilon^2\big\}$, then $\boldsymbol{\mu}^N(\mathbf{A}^N) > 0$ by (6). Also, setting $\boldsymbol{\mu}_N := \bigotimes_{n\leq N} \mu_n$ and $\mathbf{A}_N := \{(x_1, \ldots, x_N);\ x_1 + \cdots + x_N \in B(a,\varepsilon)\}$, the fact that the measures $\mu_1, \ldots, \mu_N$ have full support implies that $\boldsymbol{\mu}_N(\mathbf{A}_N) > 0$. Hence $\mu(V) \geq \mu(A) = \boldsymbol{\mu}_N(\mathbf{A}_N)\, \boldsymbol{\mu}^N(\mathbf{A}^N) > 0$.

Finally, T is mixing with respect to μ because each operator T_n, $n \geq 1$, is mixing with respect to μ_n: this can be shown easily by checking that $\mu(A \cap T^{-k}(B)) \to \mu(A)\mu(B)$ as $k \to \infty$ for any Borel sets $A, B \subseteq \mathcal{H}$ whose definition depends on finitely many coordinates only with respect to the decomposition $\mathcal{H} = \bigoplus_{n\geq 1} \mathcal{H}_n$.

Thus, we have shown that T is mixing in the Gaussian sense. Finally, since all operators T_n are chaotic, it is easily checked that the periodic points of T are dense in $\mathcal{H}$; hence T is chaotic.

(ii) Fix $n \geq 1$. If we denote by $\pi_n : \mathcal{H} \to \mathcal{H}_n$ the canonical projection of $\mathcal{H}$ onto $\mathcal{H}_n$, then $T_n \pi_n = \pi_n T$. Since π_n is continuous and onto (that it has dense range would suffice for our argument), (ii) follows easily: if $x \in \mathcal{H}$ is a $\mathcal{U}$-frequently hypercyclic vector for T, then $x_n := \pi_n x$ is a $\mathcal{U}$-frequently hypercyclic vector for T_n; and if T is chaotic, then $\pi_n(\mathrm{HC}(T)) \subseteq \mathrm{HC}(T_n)$ and $\pi_n(\mathrm{Per}(T))$ is a dense subset of $\mathcal{H}_n$ consisting of periodic vectors for T_n. □

It is now easy to deduce from Lemmas 3.20 and 3.19 that no subset A of $\mathfrak{B}_M(\mathcal{H})$ with the property that $\text{G-MIX}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H}) \subseteq A \subseteq \mathrm{UFHC}_M(\mathcal{H}) \cup \mathrm{CH}_M(\mathcal{H})$ can be $\mathtt{SOT}^*$-$\mathbf{\Sigma}^0_3$, and thus to prove Proposition 3.18.

Proof of Proposition 3.18. Let us fix a set A as above. Decompose $\mathcal{H}$ as $\mathcal{H} = \bigoplus_{n\geq 1} \mathcal{H}_n$, where the spaces $\mathcal{H}_n$, $n \geq 1$, are infinite-dimensional. Let us define a map $\Phi : \mathbf{C}^{\mathbb{N}} \to \mathfrak{B}_M(\mathcal{H})$ as follows: for every $\bar{\alpha} = (\alpha_n)_{n\geq 1} \in \mathbf{C}^{\mathbb{N}}$,

$$\Phi(\bar{\alpha}) := \bigoplus_{n\geq 1} T_{\alpha_n},$$

where the operators $T_{\alpha_n} \in \mathfrak{B}_M(\mathcal{H}_n)$, $n \geq 1$, are given by Lemma 3.19. Then Φ is continuous, and Lemmas 3.20 and 3.19 imply that

$$\Phi^{-1}(A) = W = \{\bar{\alpha} = (\alpha_n)_{n\geq 1} \in \mathbf{C}^{\mathbb{N}};\ \forall n \geq 1\ :\ \alpha_n \in \mathbf{Q}\}.$$

Since W is a true $\mathbf{\Pi}^0_3$ subset of $\mathbf{C}^{\mathbb{N}}$, hence not a $\mathbf{\Sigma}^0_3$ set, it follows that A cannot be $\mathbf{\Sigma}^0_3$. This concludes the proof of Proposition 3.18. □

Our last result along these lines is

PROPOSITION 3.21. *There is no* $\mathtt{SOT}^*$-$\mathbf{\Pi}^0_3$ *set* B *such that* $\mathrm{FHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H}) \subseteq B \subseteq \mathrm{UFHC}_M(\mathcal{H})$. *In particular,* $\mathrm{UFHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$ *is not* $\mathtt{SOT}^*$-$\mathbf{\Pi}^0_3$ *in* $\mathfrak{B}_M(\mathcal{H})$.

The proof of Proposition 3.21 make use of the machinery developed in Chapter 6 below. The reader may therefore prefer to skip this proof and return to it after reading Chapter 6.

PROOF. In what follows, we assume that $\mathcal{H} = \ell_2(\mathbb{N})$, endowed with its canonical basis. Let M_0 be such that $1 < M_0 < M$, and choose an even integer $C \geq 3$ large enough to have

$$M_0 + \sum_{k=1}^{\infty} 2^{k-1} M_0^{-\frac{1}{2}C^k} \leq M \quad \text{and} \quad \sum_{k=1}^{\infty} 2^k M_0^{-\frac{1}{12}C^k} C^k \leq 1. \tag{7}$$

Now, let us denote by $\mathcal{D}$ the set of all infinite sequence of integers $\delta = (\delta^{(k)})_{k\geq 0}$ with $\delta^{(0)} = 1$ such that $C \leq \delta^{(k)} \leq C^{4k}$ and $\delta^{(k)} \geq C\,\delta^{(k-1)}$ for every $k \geq 1$. This is a closed subset of $\mathbb{N}^{\mathbb{N}}$ and hence a Polish space. For any element δ of $\mathcal{D}$, we denote by T_δ the operator of C_+-type on $\mathcal{H}$ defined as follows: for any $k \geq 1$

$$\Delta^{(k)} = C^{4k}, \quad v^{(k)} = M_0^{-\frac{1}{2}\delta^{(k)}} \quad \text{and} \quad w_i^{(k)} = \begin{cases} M_0 & \text{if} \quad 1 \leq i \leq \delta^{(k)}, \\ 1 & \text{if} \quad \delta^{(k)} < i < \Delta^{(k)}. \end{cases}$$

Note that in the terminology of Chapter 6, T_δ looks exactly like an *operator of* $C_{+,1}$*-type* with $\tau^{(k)} = \frac{1}{2}\delta^{(k)}$, except that in the definition of the weights $v^{(k)}$ and $w_i^{(k)}$ the constant 2 has been replaced by M_0. (See Section 2 for the definition of C_+-type operators, and Section 2 for the definition of $\mathrm{C}_{+,1}$-type operators.) Observe that if $\delta = (\delta^{(k)})_{k\geq 0}$ belongs to $\mathcal{D}$, then $\delta^{(k)} \geq C^k$ for all $k \geq 1$. By (7), it follows that $\|T_\delta\| \leq M$ for every $\delta \in \mathcal{D}$. Moreover, it is clear that the map $\delta \mapsto T_\delta$ is continuous from $\mathcal{D}$ into $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.

The key point of the proof that $\mathrm{UFHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$ is not $\mathtt{SOT}^*$-$\mathbf{\Pi}^0_3$ is the following lemma, which is actually nothing but a reformulation of the forthcoming Theorem 7.2, in a slightly different setting.

LEMMA 3.22. *Let* $\delta = (\delta^{(k)})_{k\geq 0}$ *belong to* $\mathcal{D}$. *Then the operator* T_δ *is always chaotic, and* T_δ *is* $\mathcal{U}$*-frequently hypercyclic if and only if* $\delta^{(k)}/C^{4k}$ *does not tend to* 0 *as* $k \to \infty$*; in which case* T_δ *is in fact frequently hypercyclic.*

PROOF OF LEMMA 3.22. We apply a modified version of Theorem 7.2, with $p = 2$. Accordingly, we set for every $k \geq 1$

$$\gamma_k := M_0^{\delta^{(k-1)} - \frac{1}{2}\delta^{(k)}} \sqrt{\Delta^{(k)}} = M_0^{\delta^{(k-1)} - \frac{1}{2}\delta^{(k)}} C^{2k}.$$

Since $\delta^{(k)} \leq C^{-1}\delta^{(k+1)} \leq \frac{1}{3}\delta^{(k+1)}$, we have $\delta^{(k)} - \frac{1}{2}\delta^{(k+1)} \leq -\frac{1}{2}\delta^{(k)} \leq \delta^{(k-1)} - \frac{1}{2}\delta^{(k)}$, from which it follows easily that the sequence $(\gamma_k)_{k\geq 1}$ is non-increasing. Moreover, we have $\gamma_k \leq M_0^{-\frac{1}{6}C^k} C^{2k}$ for all $k \geq 1$. Hence condition (7) above yields that

$$\sum_{k=1}^{\infty} 2^k \gamma_k^{1/2} \leq 1.$$

The analogues of the "additional assumptions" in Theorem 7.2 are thus satisfied, and it follows that T_δ is $\mathcal{U}$-frequently hypercyclic on $\mathcal{H}$ if and only if $\delta^{(k)}/C^{4k}$ does not tend to 0 as $k \to \infty$. □

Now, let $B \subseteq \mathfrak{B}_M(\mathcal{H})$ have the property that $\mathrm{FHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H}) \subseteq B \subseteq \mathrm{UFHC}_M(\mathcal{H})$. Consider the map $\Phi : \mathcal{D} \to \mathfrak{B}_M(\mathcal{H})$ defined by setting $\Phi(\delta) := T_\delta$ for every $\delta \in \mathcal{D}$. As already mentioned, the map Φ is continuous from $\mathcal{D}$ into $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. By Lemma 3.22 we have

$$\Phi^{-1}(B) = \mathcal{D} \setminus \mathcal{D}_0, \quad \text{where} \quad \mathcal{D}_0 = \{\delta \in \mathcal{D};\ C^{-4k}\delta^{(k)} \to 0 \text{ as } k \to \infty\}.$$

Now, a proof quite similar to that of Claim 3.3 shows that $\mathcal{D}_0$ is a true $\mathbf{\Pi}^0_3$ set in $\mathcal{D}$, so that $\mathcal{D} \setminus \mathcal{D}_0$ is a true $\mathbf{\Sigma}^0_3$ set in $\mathcal{D}$. Hence B cannot be $\mathbf{\Pi}^0_3$ in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. This concludes the proof of Proposition 3.21. □

We summarize the statements of Propositions 3.16, 3.18 and 3.21 in the single Proposition 3.23.

Proposition 3.23. *The following statements hold true.*

(1) *The set* $\mathrm{UFHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$ *is a difference of* $\mathbf{\Sigma}^0_3$ *sets in* $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.
(2) *There is no* $\mathtt{SOT}^*$-$\mathbf{\Sigma}^0_3$ *set* A *such that* $\text{G-MIX}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H}) \subseteq A \subseteq \mathrm{UFHC}_M(\mathcal{H}) \cup \mathrm{CH}_M(\mathcal{H})$. *In particular,* $\mathrm{UFHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$ *is not* $\mathtt{SOT}^*$-$\mathbf{\Sigma}^0_3$ *in* $\mathfrak{B}_M(\mathcal{H})$.
(3) *There is no* $\mathtt{SOT}^*$-$\mathbf{\Pi}^0_3$ *set* B *such that* $\mathrm{FHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H}) \subseteq B \subseteq \mathrm{UFHC}_M(\mathcal{H})$. *In particular,* $\mathrm{UFHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$ *is not* $\mathtt{SOT}^*$-$\mathbf{\Pi}^0_3$ *in* $\mathfrak{B}_M(\mathcal{H})$.

As a straightforward consequence of Proposition 3.23, we obtain:

Corollary 3.24. *The set* $\mathrm{UFHC}_M(\mathcal{H})$ *is neither* $\mathbf{\Sigma}^0_3$, *nor* $\mathbf{\Pi}^0_3$ *in* $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.

Proof. By Proposition 3.23, $\mathrm{UFHC}_M(\mathcal{H})$ is not $\mathbf{\Sigma}^0_3$, and it is not $\mathbf{\Pi}^0_3$ because $\mathrm{CH}_M(\mathcal{H})$ is $\mathbf{\Pi}^0_3$ and $\mathrm{UFHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$ is not. □

Another immediate consequence is

Corollary 3.25. *The set* $\mathrm{TMIX}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$ *is a true* $\mathbf{\Pi}^0_3$ *set in* $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.

Proof. By Propositions 3.2 and 3.4, $\mathrm{TMIX}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$ is $\mathbf{\Pi}^0_3$; and by Proposition 3.23, it is not $\mathbf{\Sigma}^0_3$. □

Remark 3.26. Propositions 3.4 and 3.23 together formally yield the statement that, since $\mathrm{CH}_M(\mathcal{H})$ is $\mathbf{\Pi}^0_3$ in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$ while $\mathrm{UFHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$ is not $\mathbf{\Pi}^0_3$, these two sets cannot be equal; in other words, there exist chaotic operators in $\mathfrak{B}_M(\mathcal{H})$ which are not $\mathcal{U}$-frequently hypercyclic. However, there is nothing magic here: the proof of Proposition 3.23 relies heavily on a construction, carried out in Chapter 6, of explicit chaotic operators which are not $\mathcal{U}$-frequently hypercyclic.

Remark 3.27. The proof of Proposition 3.23 has established that the set $\mathrm{c}^+_M(\mathcal{H}) = \{T \in \mathrm{HC}_M(\mathcal{H});\ c(T) > 0\}$ is a true $\mathtt{SOT}^*$-$\mathbf{\Sigma}^0_3$ subset of $\mathfrak{B}_M(\mathcal{H})$.

Remark 3.28. It is likely that $\mathrm{TMIX}_M(\mathcal{H})$ and $\mathrm{CH}_M(\mathcal{H})$ are both true $\mathbf{\Pi}^0_4$ subsets of the space $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT})$, *i.e.* $\mathbf{\Pi}^0_4$ but not $\mathbf{\Sigma}^0_4$, that $\mathrm{UFHC}_M(\mathcal{H})$ is a true $\mathbf{\Pi}^0_4$ set in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$, and that $\mathrm{UFHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$ is a true difference of $\mathbf{\Sigma}^0_3$ sets in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. However, we have been unable to prove any of these facts.

2. Some non-Borel sets in $\mathfrak{B}_M(\mathcal{H})$

We conclude this section by showing that some natural classes of operators defined by dynamical properties are non-Borel in the space $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$, for any $M > 1$.

If Γ is any class of Hilbert space operators, we will say that an operator T *has a restriction in* Γ if there is a closed T-invariant (infinite-dimensional) subspace $\mathcal{E}$ of $\mathcal{H}$ such that the operator $T_{|\mathcal{E}}$ induced by T on $\mathcal{E}$ belongs to $\Gamma(\mathcal{E})$. We denote by $\widehat{\Gamma}(\mathcal{H})$ the class of all operators $T \in \mathfrak{B}(\mathcal{H})$ admitting a restriction in Γ.

Proposition 3.29. *Let* $\Gamma(\mathcal{H})$ *be a class of operators on* $\mathcal{H}$ *such that* $\text{G-ERG}(\mathcal{H}) \subseteq \Gamma(\mathcal{H}) \subseteq \mathrm{HC}(\mathcal{H})$. *For any* $M > 1$, *the set* $\widehat{\Gamma}_M(\mathcal{H})$ *is non-Borel in* $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.

Before proving Proposition 3.29, we recall that for every compact subset Λ of $\mathbb{T} \setminus \{1\}$, $T_\Lambda : \mathcal{H}_\Lambda \to \mathcal{H}_\Lambda$ denotes the Kalisch operator associated to Λ (this notation has been introduced in the proof of Proposition 3.4). For any compact subset E of $\mathbb{T} \setminus \{1\}$, we denote by $\mathcal{K}_\infty(E)$ the set of all infinite compact subsets of E.

LEMMA 3.30. *Let E be a compact subset of $\mathbb{T} \setminus \{1\}$. There exists a sequence $(e_n)_{n\geq 1}$ of Borel maps from $\mathcal{K}_\infty(E)$ into $L^2(0, 2\pi)$ such that $(e_n(\Lambda))_{n\geq 1}$ is an orthonormal basis of $\mathcal{H}_\Lambda$ for every $\Lambda \in \mathcal{K}_\infty(E)$.*

PROOF. By the Kuratowski-Ryll-Nardzewski Selection Theorem (see [**38**, Section 12.C]), there exists a sequence $(\xi_n)_{n\geq 1}$ of Borel maps $\xi_n : \mathcal{K}(E) \to E$ such that $\{\xi_n(\Lambda);\ n \geq 1\}$ is a dense subset of Λ for every $\Lambda \in \mathcal{K}(E)$. If Λ belongs to $\mathcal{K}_\infty(E)$, there exists a sequence $(\zeta_n(\Lambda))_{n\geq 1}$ of distinct elements of Λ such that $\{\zeta_n(\Lambda);\ n \geq 1\}$ is dense in Λ. Indeed, it suffices to set $\zeta_1(\Lambda) = \xi_1(\Lambda)$ and, for every $n \geq 1$, $\zeta_n(\Lambda) = \xi_{k_n}(\Lambda)$ where $k_n := \min\{j \geq n\ ;\ \text{ for all } 1 \leq i < j,\ \xi_i(\Lambda) \neq \xi_j(\Lambda)\}$. Then the maps $\Lambda \mapsto \zeta_n(\Lambda)$, $n \geq 1$, are Borel on $\mathcal{K}_\infty(E)$, and by construction the elements $\zeta_n(\Lambda)$, $n \geq 1$, are distinct and form a dense subset of Λ. Applying the Gram-Schmidt orthonormalization process to the sequence $(f_{\zeta_n(\Lambda)})_{n\geq 1}$, which is linearly independent, we get the required orthonormal basis $(e_n(\Lambda))_{n\geq 1}$ of $\mathcal{H}_\Lambda$. □

The key step in the proof of Proposition 3.29 is the following result of Waterman [**53**].

LEMMA 3.31. *If Λ is a compact subset of $\mathbb{T} \setminus \{1\}$, the operator T_Λ* admits spectral synthesis, *which means that every closed T_Λ-invariant subspace of $\mathcal{H}_\Lambda$ is spanned by the eigenvectors belonging to it. In other words, any invariant subspace $\mathcal{E} \subseteq \mathcal{H}_\Lambda$ for T_Λ has the form $\mathcal{E} = \mathcal{H}_K$ for some compact set $K \subseteq \Lambda$.*

We are now ready to prove Proposition 3.29.

PROOF OF PROPOSITION 3.29. Let us fix a non-trivial arc $E \subseteq \mathbb{T} \setminus \{1\}$ such that $\|T_E\| \leq M$. As in the proof of Proposition 3.4, let $(e_n)_{n\geq 1}$ be an orthonormal basis of $\mathcal{H}$, and for any $\Lambda \in \mathcal{K}_\infty(E)$, denote by $U_\Lambda : \mathcal{H}_\Lambda \to \mathcal{H}$ the unitary operator sending $e_n(\Lambda)$ to e_n for all $n \geq 1$, where $e_n(\Lambda)$ is given by Lemma 3.30. Then define a map $\Phi : \mathcal{K}_\infty(E) \to \mathfrak{B}_M(\mathcal{H})$ by setting

$$\Phi(\Lambda) = U_\Lambda T_\Lambda U_\Lambda^{-1} \quad \text{ for every } \Lambda \in \mathcal{K}_\infty(E).$$

Proceeding as in the proof of Proposition 3.4, and using the fact that the maps $\Lambda \mapsto e_n(\Lambda)$, $n \geq 1$, are Borel on $\mathcal{K}_\infty(E)$, we deduce that the map Φ is Borel from $\mathcal{K}_\infty(E)$ into $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.

If $\Lambda \in \mathcal{K}_\infty(E)$ is uncountable, it admits a perfect subset K, so the restriction of T_Λ to the invariant subspace $\mathcal{H}_K$ (which is nothing else but T_K) is ergodic in the Gaussian sense. Thus $\Phi(\Lambda)$ belongs to $\widehat{G\text{-}ERG}(\mathcal{H}) \subseteq \widehat{\Gamma}(\mathcal{H})$. On the other hand, if Λ is countable then, by Lemma 3.31, T_Λ has no hypercyclic restriction. Hence $\Phi(\Lambda)$ belongs to $\mathfrak{B}(\mathcal{H}) \setminus \widehat{HC}(\mathcal{H}) \subseteq \mathfrak{B}(\mathcal{H}) \setminus \widehat{\Gamma}(\mathcal{H})$. So we have proved that

$$\Phi^{-1}\Big(\widehat{\Gamma}_M(\mathcal{H})\Big) = \mathcal{K}_\infty(E) \setminus \mathcal{K}_\omega(E).$$

Since $\mathcal{K}_\omega(E)$ is notoriously non-Borel in $\mathcal{K}(E)$ (see [**38**, Section 27.B]) and hence also in $\mathcal{K}_\infty(E)$ because the latter is Borel in $\mathcal{K}(E)$, this concludes the proof. □

REMARK 3.32. The same proof shows that $\widehat{\text{G-MIX}}_M(\mathcal{H})$, the family of all operators $T \in \mathfrak{B}_M(\mathcal{H})$ admitting a restriction which is mixing in the Gaussian sense, is non-Borel in the space $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. Instead of working with the class $\mathcal{K}_\omega(E)$ of countable subsets of E, one has to consider the class $U_0 \cap \mathcal{K}(E)$ of compact subsets of E which are *sets of extended uniqueness*, also called U_0*-sets*. By definition, a compact set $\Lambda \subseteq \mathbb{T}$ is a U_0-set if it is negligible for every probability measure σ on $\mathbb{T}$ whose Fourier coefficients vanish at infinity. It follows from the main result of [**9**] that if $\Lambda \subseteq \mathbb{T} \setminus \{1\}$ is not a U_0-set, then the Kalisch operator T_Λ has a restriction which is mixing in the Gaussian sense, whereas if Λ is a U_0-set, T_Λ is not mixing in the Gaussian sense. Moreover, it is well-known that the class $U_0 \cap \mathcal{K}(E)$ is non-Borel in $\mathcal{K}(E)$ for any non-trivial arc $E \subseteq \mathbb{T}$ (see

[**39**, Th. VIII.2.6]). Using these facts, one can prove the aforementioned result in exactly the same way as above: the key identity

$$\Phi^{-1}\Big(\widehat{\Gamma}_M(\mathcal{H})\Big) = \mathcal{K}_\infty(E) \setminus \mathcal{K}_\omega$$

at the end of the proof of Proposition 3.29 is replaced by

$$\Phi^{-1}\Big(\widehat{\Gamma}_M(\mathcal{H})\Big) = \mathcal{K}_\infty(E) \setminus U_0.$$

Remark 3.33. Let $\Gamma_0(\mathcal{H})$ be a class of operators of $\mathcal{H}$ with the following two properties:

(i) G-ERG$(\mathcal{H}) \subseteq \Gamma_0(\mathcal{H})$;
(ii) if Λ is any countable compact subset of $\mathbb{T} \setminus \{1\}$, the Kalisch operator T_Λ does not belong to $\Gamma_0(\mathcal{H})$.

Then, exactly the same proof as that of Proposition 3.29 shows the following: *if $\Gamma(\mathcal{H})$ is a class of operators on $\mathcal{H}$ such that* G-ERG$(H) \subseteq \Gamma(\mathcal{H}) \subseteq \Gamma_0(\mathcal{H})$ *then, for any $M > 1$, the set $\widehat{\Gamma}_M(\mathcal{H})$ is non-Borel in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.*

Our aim is now to prove that for any $M > 1$, the class $\mathrm{DCH}_M(\mathcal{H})$ of distributionally chaotic operators in $\mathfrak{B}_M(\mathcal{H})$ is non-Borel in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. Note that this is in strong contrast with Proposition 2.40, according to which the class of *densely* distributionally chaotic operators in $\mathfrak{B}_M(\mathcal{H})$ is G_δ. The proof of this statement relies on the following observation:

Lemma 3.34. *An operator $T \in \mathfrak{B}(\mathcal{H})$ is distributionally chaotic if and only if it has a restriction which is densely distributionally chaotic.*

Proof. If there exists a closed subspace $\mathcal{E}$ of $\mathcal{H}$ such that $T|_\mathcal{E}$ is densely distributionally chaotic, T admits in particular a distributionally irregular vector belonging to $\mathcal{E}$, and hence T is distributionally chaotic. Conversely, assume that T is distributionally chaotic, and let $x \in \mathcal{H}$ be a distributionally irregular vector. Let $\mathcal{E}$ be the closed T-invariant subspace spanned by the orbit of x, and denote by S the operator induced by T on $\mathcal{E}$. As the vector $x \in \mathcal{E}$ has a distributionally unbounded orbit under the action of S (*i.e.* $\|S^n x\| \to \infty$ as $n \to \infty$ along a subset B of $\mathbb{N}$, with $\overline{\mathrm{dens}}(B) = 1$), S admits by [**11**, Prop. 8] a comeager set of vectors with distributionally unbounded orbit. Also, the orbit of x under the action of S is distributionally near to 0 (*i.e.* $\|S^n x\| \to 0$ as $n \to \infty$ along a subset A of $\mathbb{N}$, with $\overline{\mathrm{dens}}(A) = 1$). It follows that $\|S^n y\| \to 0$ as $n \to \infty$ along A for every vector y belonging to the linear span of the vectors $S^p x$, $p \geq 0$. Hence S admits a dense set of vectors whose orbit is distributionally near to 0. By [**11**, Prop. 9], it follows that the set of vectors whose orbit under the action of S is distributionally near to 0 is comeager in $\mathcal{E}$. By the Baire Category Theorem, we conclude that S admits a comeager set of distributionally irregular vectors, *i.e.* S is densely distributionally chaotic. □

We will also need the following characterizations of (densely) distributionally chaotic Kalisch operators:

Lemma 3.35. *Let Λ be a compact subset of $\mathbb{T}\setminus\{1\}$, and let T_Λ be the associated Kalisch operator on H_Λ.*

(1) *The operator T_Λ is densely distributionally chaotic if and only if Λ is a perfect set.*
(2) *The operator T_Λ is distributionally chaotic if and only if Λ is uncountable.*

Proof. Let us start by proving assertion (1). If Λ is a perfect set then T_Λ is an ergodic operator, so that it is in particular densely distributionally chaotic by [**30**]. Conversely, suppose that Λ is not a perfect set, and let λ_0 be an isolated point of Λ. Then the eigenfunction f_{λ_0} does not belong to $\mathcal{H}_{\Lambda\setminus\{\lambda_0\}}$, and $\mathcal{H}_\Lambda$ can be written as a topological direct sum $\mathcal{H}_\Lambda = \mathrm{span}[f_{\lambda_0}] \oplus \mathcal{H}_{\Lambda\setminus\{\lambda_0\}}$. Hence, there exists a non-zero vector g_0 of $\mathcal{H}_\Lambda$ with $\langle g_0, f_{\lambda_0}\rangle = 1$ and $g \perp \mathcal{H}_{\Lambda\setminus\{\lambda_0\}}$, and a bounded projection Q of $\mathcal{H}_\Lambda$ onto $\mathcal{H}_{\Lambda\setminus\{\lambda_0\}}$, such that $f = \langle g_0, f\rangle f_{\lambda_0} + Qf$ for every $f \in \mathcal{H}_\Lambda$. It follows that $T_\Lambda^n f = \langle g_0, f\rangle \lambda_0^n f_{\lambda_0} + T_\Lambda^n Qf$ for every $n \geq 1$, so that $\langle g_0, T_\Lambda^n f\rangle = \langle g_0, f\rangle \lambda_0^n$. Hence $\|T_\Lambda^n f\| \geq |\langle g_0, f\rangle|/\|g_0\|$ for every $n \geq 1$. Thus, we see that the orbit under T_Λ of any vector $f \in \mathcal{H}_\Lambda$ with

$\langle g_0, f\rangle \neq 0$ is bounded away from 0, and hence that no such vector can be distributionally irregular. So T_Λ is not densely distributionally chaotic.

The proof of assertion (2) relies on Lemmas 3.31 and 3.34 above. By Lemma 3.34, T_Λ is distributionally chaotic if and only if it admits a restriction which is densely distributionally chaotic. Since, by Lemma 3.31, any such restriction is of the form $T_K \in \mathfrak{B}(\mathcal{H}_K)$, where K is a compact subset of Λ, it follows from assertion (1) that T_Λ is distributionally chaotic if and only if Λ contains a perfect set, which happens exactly when Λ is uncountable. This proves (2). □

With these tools at hand, we are ready to prove

Proposition 3.36. *For any $M > 1$, the class $\mathrm{DCH}_M(\mathcal{H})$ of distributionally chaotic operators in $\mathfrak{B}_M(\mathcal{H})$ is non-Borel in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.*

Proof. Consider the class $\Gamma(\mathcal{H}) = \Gamma_0(\mathcal{H}) := \mathrm{DDCH}(\mathcal{H})$. This class $\Gamma_0(\mathcal{H})$ satisfies the assumptions of Remark 3.33: G-ERG$(\mathcal{H})$ is contained in $\Gamma_0(\mathcal{H})$ by a result of [**30**], and it follows from Lemma 3.35 that whenever Λ is a countable subset of $\mathbb{T} \setminus \{1\}$, the Kalisch operator T_Λ is not (densely) distributionally chaotic. Since G-ERG$(\mathcal{H}) \subseteq \Gamma(\mathcal{H}) \subseteq \Gamma_0(\mathcal{H})$, we conclude that for any $M > 1$, $\widehat{\Gamma}_M(\mathcal{H})$ is non-Borel in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. But $\widehat{\Gamma}_M(\mathcal{H}) = \mathrm{DCH}_M(\mathcal{H})$ by Lemma 3.34, and hence $\mathrm{DCH}_M(\mathcal{H})$ is non-Borel in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. □

Corollary 3.37. *For any $M > 1$, the class $\mathrm{DCH}_M(\mathcal{H}) \setminus \mathrm{DDCH}_M(\mathcal{H})$ of operators in $\mathfrak{B}_M(\mathcal{H})$ which are distributionally chaotic but not densely distributionally chaotic is non-Borel in the space $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$.*

Proof. Since $\mathrm{DDCH}_M(H)$ is a G_δ subset of $(\mathrm{HC}_M(\mathcal{H}), \mathtt{SOT}^*)$ by Proposition 2.40, it is in particular Borel. So one immediately deduces from Proposition 3.36 that $\mathrm{DCH}_M(\mathcal{H}) \setminus \mathrm{DDCH}_M(\mathcal{H})$ is non-Borel in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$. □

Remark 3.38. Corollary 3.37 formally yields the existence of distributionally chaotic operators which are not densely distributionally chaotic. But the existence of such operators can of course be deduced directly from Lemma 3.35: a Kalisch operator T_Λ is distributionally chaotic but not densely distributionally chaotic if and only if Λ is uncountable but not perfect.

CHAPTER 4

Ergodicity for upper-triangular operators

1. Definitions and setting

Let us fix an orthonormal basis $(e_k)_{k\geq 1}$ of the complex separable infinite-dimensional Hilbert space $\mathcal{H}$. We will denote by $\mathcal{H}_{00}$ the linear subspace of $\mathcal{H}$ consisting of finitely supported vectors with respect to the basis $(e_k)_{k\geq 1}$.

In this section, we concentrate ourselves on the study of the typical properties of operators on $\mathcal{H}$ which are *upper-triangular* with respect to the basis $(e_k)_{k\geq 1}$, with diagonal coefficients *of modulus* 1 and *distinct*. This class of operators will be denoted by $\mathfrak{T}_0(\mathcal{H})$:

$$\mathfrak{T}_0(\mathcal{H}) := \big\{T \in \mathfrak{B}(\mathcal{H})\,;\, \forall\, k \geq 1 \ : \ Te_k \in \mathrm{span}[e_1, \ldots, e_k] \text{ and } |\langle Te_k, e_k\rangle| = 1\big\}.$$

For each $T \in \mathfrak{T}_0(\mathcal{H})$ and $k \geq 1$, we write $\lambda_k(T) := \langle Te_k, e_k\rangle$. We also define

$$\mathfrak{T}(\mathcal{H}) := \big\{T \in \mathfrak{T}_0(\mathcal{H})\,;\, \forall\, j, k \geq 1 \text{ with } j \neq k,\ \lambda_j(T) \neq \lambda_k(T)\big\}$$

and

$$\mathfrak{T}_{\mathrm{ind}}(\mathcal{H}) := \big\{T \in \mathfrak{T}(\mathcal{H})\,;\, \text{the diagonal coefficients } \lambda_k(T) \text{ are rationally independent}\big\}.$$

Recall that if $(\lambda_k)_{k\in I}$ is a finite or infinite family of elements of $\mathbb{T}$ with $\lambda_k = e^{2i\pi\theta_k}$ for every $k \in I$, it is said to be *rationally independent* if for any finite subset F of I, the family $(\theta_k)_{k\in F}$ consists of $\mathbb{Q}$-independent numbers.

For any $M > 0$, we denote by $\mathfrak{T}_{0,M}(\mathcal{H})$, $\mathfrak{T}_M(\mathcal{H})$, and $\mathfrak{T}_{\mathrm{ind},M}(\mathcal{H})$ the set of operators in $\mathfrak{B}_M(\mathcal{H})$ which belong to $\mathfrak{T}_0(\mathcal{H})$, $\mathfrak{T}(\mathcal{H})$, and $\mathfrak{T}_{\mathrm{ind}}(\mathcal{H})$ respectively. It is not difficult to check that these sets are G_δ in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT})$. So we may state:

Fact 4.1. For any $M > 0$, the sets $\mathfrak{T}_{0,M}(\mathcal{H})$, $\mathfrak{T}_M(\mathcal{H})$, and $\mathfrak{T}_{\mathrm{ind},M}(\mathcal{H})$ are Polish spaces when endowed with the topology $\mathtt{SOT}$.

Note that (by the Gram-Schmidt orthonormalization process) any bounded operator on $\mathcal{H}$ with spanning unimodular eigenvectors associated to distinct eigenvalues is upper triangular with respect to some orthonormal basis $(e_k)_{k\geq 1}$, and belongs to the associated class $\mathfrak{T}(\mathcal{H})$. Hence, given the importance in linear dynamics of unimodular eigenvectors, it is quite natural to investigate typical properties of elements of the spaces $(\mathfrak{T}_M(\mathcal{H}), \mathtt{SOT})$, $M > 1$. Some interesting and unexpected phenomena occur in this setting, which we will describe shortly. Meanwhile, we state for future reference two elementary facts concerning the $\mathtt{SOT}$-continuity of certain maps naturally associated to operators in $\mathfrak{T}(\mathcal{H})$.

When T belongs to $\mathfrak{T}(\mathcal{H})$, there exists for each $k \geq 1$ a unique vector $u \in \mathrm{span}[e_1, \ldots, e_k]$ with $\langle u, e_k\rangle = 1$ such that $Tu = \lambda_k(T)u$. We denote this vector by $u_k(T)$, so that $Tu_k(T) = \lambda_k(T)u_k(T)$. We also denote by $\widetilde{u}_k(T)$ the normalized eigenvector $\widetilde{u}_k(T) = u_k(T)/\|u_k(T)\|$ of T. Here are some useful observations concerning the functions λ_k and u_k:

Lemma 4.2. *For each $k \geq 1$, the functions $T \mapsto \lambda_k(T)$ and $T \mapsto u_k(T)$ are continuous from $(\mathfrak{T}(\mathcal{H}), \mathtt{SOT})$ into $\mathbb{T}$ and $\mathcal{H}$ respectively.*

PROOF. The continuity of the function $T \mapsto \lambda_k(T)$ is obvious from the definition of $\lambda_k(T)$. As to the function $T \mapsto u_k(T)$, the fact that $\langle u_k(T), e_k\rangle = 1$ implies that

$$\langle u_k(T), e_i\rangle = \frac{1}{\lambda_k(T) - \lambda_i(T)} \sum_{j=i+1}^{k} \langle Te_j, e_i\rangle \langle u_k(T), e_j\rangle$$

for every $1 \leq i \leq k-1$. A straightforward induction then shows that the scalar functions $T \mapsto \langle u_k(T), e_i\rangle$, $1 \leq i \leq k-1$, are $\mathtt{SOT}$-continuous on $\mathfrak{T}(\mathcal{H})$, from which Observation 4.2 follows. □

LEMMA 4.3. *Let $u \in \mathcal{H}_{00}$, and choose $r \geq 1$ such that $u \in \operatorname{span}[e_1, \ldots, e_r]$. For any $T \in \mathfrak{T}(\mathcal{H})$, the vector u can be written as $u = \sum_{k=1}^{r} a_k(T) u_k(T)$, where the functions $T \mapsto a_k(T)$, $1 \leq k \leq r$, are continuous from $(\mathfrak{T}(\mathcal{H}), \mathtt{SOT})$ into $\mathbb{C}$.*

PROOF. This follows directly from Cramer's formulas and the continuity of the maps $T \mapsto u_k(T)$, $k \geq 1$. □

2. Perfect spanning is typical

The following result shows that within any class $\mathfrak{T}_M(\mathcal{H})$, $M > 1$, the perfect spanning property (equivalently, ergodicity in the Gaussian sense) is a typical property with respect to the topology $\mathtt{SOT}$.

PROPOSITION 4.4. *For every $M > 1$, the set $\mathrm{PSPAN}_M(H) = \text{G-ERG}_M(\mathcal{H})$ is comeager in $(\mathfrak{T}_M(\mathcal{H}), \mathtt{SOT})$. Consequently, $\mathrm{ERG}_M(H)$, $\mathrm{FHC}_M(H)$ and $\mathrm{UFHC}_M(H)$ are all comeager in $(\mathfrak{T}_M(\mathcal{H}), \mathtt{SOT})$.*

For the proof, we will make use of the following useful criterion for perfect spanning, which goes back to [**28**, Th. 4.1] (see also [**30**, Prop. 6.1]).

LEMMA 4.5. *An operator $T \in \mathfrak{T}(\mathcal{H})$ has a perfectly spanning set of unimodular eigenvectors as soon as the following property is satisfied: for every $\varepsilon > 0$ and every $k \geq 1$, there exists $l \geq 1$ with $l \neq k$ and $\alpha \in \mathbb{C}$ such that $\|u_k(T) - \alpha u_l(T)\| < \varepsilon$.*

PROOF OF PROPOSITION 4.4. From Lemma 4.5, we deduce that the set

$$\mathfrak{G} = \bigcap_{i \geq 1} \bigcap_{k \geq 1} \bigcup_{l \neq k} \bigcup_{\alpha \in \mathbb{C}} \left\{T \in \mathfrak{T}_M(\mathcal{H})\,;\, \|u_k(T) - \alpha u_l(T)\| < 2^{-i}\right\}$$

is contained in $\mathrm{PSPAN}_M(H)$. Moreover, it follows from Lemma 4.2 that $\mathfrak{G}$ is also a G_δ subset of $(\mathfrak{T}_M(\mathcal{H}), \mathtt{SOT})$. So we just have to prove that $\mathfrak{G}$ is dense in $\mathfrak{T}_M(\mathcal{H})$; and for this, it is enough to show that each open set

$$\mathfrak{O}_{i,k} = \bigcup_{l \neq k} \bigcup_{\alpha \in \mathbb{C}} \left\{T \in \mathfrak{T}_M(\mathcal{H})\,;\, \|u_k(T) - \alpha u_l(T)\| < 2^{-i}\right\}, \quad i,\, k \geq 1$$

is dense in $(\mathfrak{T}_M(\mathcal{H}), \mathtt{SOT})$.

Let us fix $i, k \geq 1$ and $A \in \mathfrak{T}_M(\mathcal{H})$. We have to show that for any $r \geq 1$, there exists an operator $T \in \mathfrak{O}_{i,k}$ such that Te_j is arbitrarily close to Ae_j for every $j = 1 \ldots, r$. Without loss of generality, we assume that $r \geq k$. Without loss of generality, we can suppose also that $\|A\| < M$. Indeed, if $\|A\| = M$, consider the function $f:\, t \mapsto \|D + t(A - D)\|$ defined on the compact set $[0, 1]$, where D is the diagonal operator given by the diagonal coefficients of A. The function f is convex on $[0, 1]$, so that its maximum is attained at either 0 or 1. Since $f(0) = 1 < M = f(1)$, $f(t) < M$ for every $t \in [0, 1)$. If $t \in [0, 1)$ is sufficiently close to 1, the operator $A' := D + t(A - D)$ is upper-triangular with respect to $(e_k)_{k \geq 1}$, has the same diagonal coefficients as A, satisfies $\|A'\| < M$, and is as close as we wish to A for the $\mathtt{SOT}$ topology. We thus suppose to begin with that $\|A\| < M$.

We set, for every $r \geq 1$, $\mathcal{H}_r := \operatorname{span}[e_1, \ldots, e_r]$, and denote by P_r the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_r$.

For every $1 \le j \le r$, we denote by λ_j the j-th diagonal coefficient of A: $\lambda_j = \langle Ae_j, e_j\rangle$. We recall that $|\lambda_j| = 1$. Let us denote by $\boldsymbol{\Lambda}_r$ the set of all sequences $\boldsymbol{\lambda} = (\lambda_j)_{j>r}$ with $|\lambda_j| = 1$ such that all the elements λ_j, $j > r$ are distinct and distinct from $1, \lambda_1, \dots, \lambda_r$, and $\lambda_j \to 1$ as $j \to \infty$. For any element $\boldsymbol{\lambda}$ of $\boldsymbol{\Lambda}_r$ and any unilateral weight sequence $\boldsymbol{\omega} = (\omega_j)_{j\ge1}$, we consider the operator $A_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ on $\mathcal{H}$ defined by

$$A_{\boldsymbol{\lambda},\boldsymbol{\omega}} e_j = \begin{cases} Ae_j & \text{for every } 1 \le j \le r \\ \omega_{j-r} e_{j-r} + \lambda_j e_j & \text{for every } j > r. \end{cases}$$

Note that since $\|A\| < M$, $\|A_{\boldsymbol{\lambda},\boldsymbol{\omega}}\| \le M$ for every $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_r$ provided $\|\boldsymbol{\omega}\|_\infty := \sup_{j\ge1} |\omega_j|$ is small enough. Hence in this case $A_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ belongs to $\mathfrak{T}_M(\mathcal{H})$. Moreover, we have $A_{\boldsymbol{\lambda},\boldsymbol{\omega}} e_j = Ae_j$ for every $j = 1, \dots, r$. Let us now fix a weight sequence $\boldsymbol{\omega}$ such that $\omega_j \to 0$ as $j \to \infty$ and $\|\boldsymbol{\omega}\|_\infty$ is small enough. We are going to show that $A_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ belongs to $\mathfrak{O}_{i,k}$ provided the sequence $\boldsymbol{\lambda}$ is well-chosen. This will prove the density of $\mathfrak{O}_{i,k}$ in $(\mathfrak{T}_M(\mathcal{H}), \mathtt{SOT})$.

We first note the following fact concerning the eigenvalues of the operator $A_{\boldsymbol{\lambda},\boldsymbol{\omega}}$.

Claim 4.6. For any $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_r$, the only eigenvalues of $A_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ are $\lambda_1, \dots, \lambda_r$, the terms of the sequence $\boldsymbol{\lambda}$, and possibly the point 1. Moreover, for any $i \ge 1$, the eigenspace associated to λ_i is spanned by the vector $u_i(A_{\boldsymbol{\lambda},\boldsymbol{\omega}})$.

Proof of Claim 4.6. If λ is any complex number, solving formally the equation $A_{\boldsymbol{\lambda},\boldsymbol{\omega}} x = \lambda x$ with $x = \sum_{j\ge1} x_j e_j$, yields that

$$(A - \lambda) P_r x = \sum_{j=1}^{r} \omega_j x_{j+r} e_j \qquad \text{and} \qquad x_{j+r} = \frac{\lambda - \lambda_j}{\omega_j}\, x_j \quad \text{for every } j > r.$$

Since $\omega_j \to 0$ and $\lambda_j \to 1$ as $j \to \infty$, we infer from these equations that there is no solution $x \in \mathcal{H} \setminus \{0\}$ to these equations if λ does not belong to the set $\{\lambda_j;\ j \ge 1\} \cup \{1\}$. This proves the first part of Claim 4.6.

For the same reasons combined with the fact that the λ_j are distinct and distinct from 1, we also observe that if a vector $x \in \mathcal{H}$ satisfies $A_{\boldsymbol{\lambda},\boldsymbol{\omega}} x = \lambda_i x$ for some $i \ge 1$, then $x_j = 0$ for all $j > i$. Hence $\ker(A_{\boldsymbol{\lambda},\boldsymbol{\omega}} - \lambda_i)$ is contained in $\mathcal{H}_i$ for every $i \ge 1$, which proves the second part of Claim 4.6 since $A_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ belongs to $\mathfrak{T}(\mathcal{H})$ and the eigenvalues λ_j of $A_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ are all distinct. □

The key point is now the following observation. Recall that $\lambda_1, \dots, \lambda_r$ are fixed, that $1 \le k \le r$, and that we are especially interested in λ_k. Recall also that

$$\widetilde{u}_{k+r}(A_{\boldsymbol{\lambda},\boldsymbol{\omega}}) = u_{k+r}(A_{\boldsymbol{\lambda},\boldsymbol{\omega}}) / \|u_{k+r}(A_{\boldsymbol{\lambda},\boldsymbol{\omega}})\|.$$

Claim 4.7. There exists a positive constant γ such that for any $\boldsymbol{\lambda} = (\lambda_j)_{j>r} \in \boldsymbol{\Lambda}_r$,

$$|\lambda_{k+r} - \lambda_k| \ge \gamma \operatorname{dist}\big(\widetilde{u}_{k+r}(A_{\boldsymbol{\lambda},\boldsymbol{\omega}}), \operatorname{span}[u_k(A_{\boldsymbol{\lambda},\boldsymbol{\omega}})]\big).$$

Proof of Claim 4.7. For any $\boldsymbol{\lambda} = (\lambda_j)_{j>r} \in \boldsymbol{\Lambda}_r$, the restriction of the operator $A_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ to $\mathcal{H}_r$ is equal to $P_r A P_r$, which does not depend on $\boldsymbol{\lambda}$. Also, $\ker(A_{\boldsymbol{\lambda},\boldsymbol{\omega}} - \lambda_k) = \operatorname{span}[u_k(A_{\boldsymbol{\lambda},\boldsymbol{\omega}})] \subseteq \mathcal{H}_r$, and the r eigenvalues $\lambda_1, \dots, \lambda_r$ of $P_r A P_r$ are distinct. These observations imply that there exist two positive constants δ and C such that, on the one hand:

$$\forall \boldsymbol{\lambda} \in \boldsymbol{\Lambda}_r \ \ \forall x \in \mathcal{H}_r \ : \ \|(A_{\boldsymbol{\lambda},\boldsymbol{\omega}} - \lambda_k)x\| \ge \delta \operatorname{dist}(x, \operatorname{span}[u_k(A_{\boldsymbol{\lambda},\boldsymbol{\omega}})]) \tag{8}$$

and, on the other hand:

$$\forall \lambda \in \mathbb{C} \setminus \{\lambda_1, \dots, \lambda_r\} \ \ \forall x \in \mathcal{H}_r \ : \ \|(A_{\boldsymbol{\lambda},\boldsymbol{\omega}} - \lambda)^{-1} x\| \ge \frac{C}{\operatorname{dist}(\lambda, \{\lambda_1, \dots, \lambda_r\})}\, \|x\|. \tag{9}$$

Now, a simple computation yields that

$$\widetilde{u}_{k+r}(A_{\boldsymbol{\lambda},\boldsymbol{\omega}}) = \frac{v_k + e_{k+r}}{\big(\|v_k\|^2 + 1\big)^{1/2}}$$

where $v_k \in \mathcal{H}_r$ satisfies the equation $(A_{\boldsymbol{\lambda},\boldsymbol{\omega}} - \lambda_{k+r})v_k + \omega_k e_k = 0$. Since λ_{k+r} does not belong to the set $\{\lambda_1, \dots, \lambda_r\}$, we may write $v_k = -\omega_k (A_{\boldsymbol{\lambda},\boldsymbol{\omega}} - \lambda_{k+r})^{-1} e_k$, and (9) implies that

$$\|v_k\| \geq C \, \frac{|\omega_k|}{\operatorname{dist}\bigl(\lambda_{k+r}, \{\lambda_1, \dots, \lambda_r\}\bigr)} \geq C \, \frac{|\omega_k|}{|\lambda_{k+r} - \lambda_k|}. \tag{10}$$

On the other hand, $(A_{\boldsymbol{\lambda},\boldsymbol{\omega}} - \lambda_k)v_k = (\lambda_{k+r} - \lambda_k)v_k - \omega_k e_k$, so that by (8)

$$|\lambda_{k+r} - \lambda_k| \; \|v_k\| + |\omega_k| \geq \delta \operatorname{dist}\bigl(v_k, \operatorname{span}[u_k(A_{\boldsymbol{\lambda},\boldsymbol{\omega}})]\bigr).$$

Hence

$$\operatorname{dist}\bigl(v_k + e_{k+r}, \operatorname{span}[u_k(A_{\boldsymbol{\lambda},\boldsymbol{\omega}})]\bigr) \leq \frac{1}{\delta}\bigl(|\lambda_{k+r} - \lambda_k| \, \|v_k\| + |\omega_k|\bigr) + 1$$

and

$$\begin{aligned}
&\operatorname{dist}\bigl(\widetilde{u}_{k+r}(A_{\boldsymbol{\lambda},\boldsymbol{\omega}}), \operatorname{span}[u_k(A_{\boldsymbol{\lambda},\boldsymbol{\omega}})]\bigr) \\
&\qquad \leq \frac{1}{\delta}\biggl(|\lambda_{k+r} - \lambda_k| \frac{\|v_k\|}{\bigl(\|v_k\|^2 + 1\bigr)^{1/2}} + \frac{|\omega_k|}{\bigl(\|v_k\|^2 + 1\bigr)^{1/2}}\biggr) + \frac{1}{\bigl(\|v_k\|^2 + 1\bigr)^{1/2}} \\
&\qquad \leq \frac{1}{\delta}\biggl(|\lambda_{k+r} - \lambda_k| + \frac{|\omega_k|}{\|v_k\|}\biggr) + \frac{1}{\|v_k\|} \\
&\qquad \leq |\lambda_{k+r} - \lambda_k| \biggl(\frac{1}{\delta} + \frac{1}{C} + \frac{1}{C|\omega_k|}\biggr) \qquad \text{by (10).}
\end{aligned}$$

Setting $1/\gamma := 1/\delta + 1/C + 1/(C \min_{1 \leq j \leq r} |\omega_j|)$ yields the desired inequality. □

We now come back to our main proof: recall that our aim is to show that $A_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ belongs to $\mathfrak{O}_{i,k}$ for some suitable choice of $\lambda \in \boldsymbol{\Lambda}_r$. By Claim 4.7, we have

$$\operatorname{dist}\left(\widetilde{u}_{k+r}(A_{\boldsymbol{\lambda},\boldsymbol{\omega}}), \operatorname{span}[u_k(A_{\boldsymbol{\lambda},\boldsymbol{\omega}})]\right) \leq \frac{|\lambda_{k+r} - \lambda_k|}{\gamma} \quad \text{for every } \boldsymbol{\lambda} \in \boldsymbol{\Lambda}_r. \tag{11}$$

We now choose $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_r$ such that λ_{k+r} is so close to λ_k that the quantity on the right hand side of the inequality (11) is less that $2^{-i}\eta_k$, where $\eta_k = \min\bigl(1, (2\|u_k(A_{\boldsymbol{\lambda},\boldsymbol{\omega}})\|)^{-1}\bigr)$. Note that η_k does not depend on $\boldsymbol{\lambda}$ since the restriction of $A_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ to $\mathcal{H}_r$ does not. With this choice of $\boldsymbol{\lambda}$, there exists a scalar $\beta \in \mathbb{C}$ such that $\|\widetilde{u}_{k+r}(A_{\boldsymbol{\lambda},\boldsymbol{\omega}}) - \beta u_k(A_{\boldsymbol{\lambda},\boldsymbol{\omega}})\| < 2^{-i}\eta_k$. Since $\|\widetilde{u}_{k+r}(A_{\boldsymbol{\lambda},\boldsymbol{\omega}})\| = 1$ and $2^{-i}\eta_k \leq 1/2$, we have $\|\beta u_k(A_{\boldsymbol{\lambda},\boldsymbol{\omega}})\| \geq 1/2$, so that $|\beta| \geq \eta_k$. It follows that $\|\alpha u_{k+r}(A_{\boldsymbol{\lambda},\boldsymbol{\omega}}) - u_k(A_{\boldsymbol{\lambda},\boldsymbol{\omega}})\| < 2^{-i}$, where $\alpha := 1/(\beta\|u_{k+r}(A_{\boldsymbol{\lambda},\boldsymbol{\omega}})\|)$, and this shows that $A_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ belongs to $\mathfrak{O}_{i,k}$.

Thus we have proved that $\mathfrak{O}_{i,k}$ is indeed dense in $(\mathfrak{T}_M(\mathcal{H}), \mathtt{SOT})$ for every $i, k \geq 1$, which concludes the proof of Proposition 4.4. □

Remark 4.8. It is natural to wonder whether the class of *chaotic* operators is also comeager in $\mathfrak{T}_M(\mathcal{H})$. This does not look quite clear from the above proof.

3. Ergodicity vs ergodicity in the Gaussian sense

In this section, our aim is to show that ergodicity and ergodicity in the Gaussian sense are not the same property. More precisely, we will show that there exist ergodic operators on $\mathcal{H}$ which have only countably many eigenvalues, and hence are not ergodic in the Gaussian sense since G-ERG$(\mathcal{H})$ = PSPAN$(\mathcal{H})$. This result will follow from a Baire category argument performed in a suitable class of upper-triangular operators on $\mathcal{H}$, namely operators which are the sum of a diagonal operator and a weighted backward shift operator (with respect to some fixed orthonormal basis of $\mathcal{H}$).

3.1. Statement of the main result. Let $(e_k)_{k\geq 1}$ be as usual a fixed orthonormal basis of the Hilbert space $\mathcal{H}$. To every pair $(\boldsymbol{\lambda},\boldsymbol{\omega}) = \big((\lambda_k)_{k\geq 1}, (\omega_k)_{k\geq 1}\big)$ of bounded sequences of complex numbers, we associate the operator $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ defined by

$$T_{\boldsymbol{\lambda},\boldsymbol{\omega}} = D_{\boldsymbol{\lambda}} + B_{\boldsymbol{\omega}},$$

where $D_{\boldsymbol{\lambda}}$ is the diagonal operator with diagonal coefficients λ_k, $k \geq 1$, associated to the basis $(e_k)_{k\geq 1}$, and $B_{\boldsymbol{\omega}}$ is the backward shift operator with respect to $(e_k)_{k\geq 1}$ with weights ω_k, $k \geq 1$. Each operator $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ belongs to $\mathfrak{T}(\mathcal{H})$. Now, let us introduce the following notation: we denote by $\boldsymbol{\Lambda}$ the set of sequences $\boldsymbol{\lambda} = (\lambda_k)_{k\geq 1}$ of distinct complex numbers of modulus 1 such that $(\lambda_k)_{k\geq 1}$ tends to 1 as k tends to infinity, and for every $M > 0$, by $\boldsymbol{\Omega}_M$ the set of all weight sequences $\boldsymbol{\omega} = (\omega_k)_{k\geq 1}$ such that $0 < \omega_k \leq M$ for every $k \geq 1$. We also set $\boldsymbol{\Omega} := \bigcup_{M>0} \boldsymbol{\Omega}_M$.

We endow the set $\boldsymbol{\Lambda}$ with the topology induced by $\ell^\infty(\mathbb{N})$. Since $\boldsymbol{\Lambda}$ is contained in the separable closed subspace $c(\mathbb{N})$ of $\ell^\infty(\mathbb{N})$ consisting of all convergent sequences, and since $\boldsymbol{\Lambda}$ is easily seen to be a G_δ subset of $\ell^\infty(\mathbb{N})$, it follows that $\boldsymbol{\Lambda}$ is a Polish space. As for the spaces $\boldsymbol{\Omega}_M$, $M > 0$, we endow them with the product topology. Each $\boldsymbol{\Omega}_M$ being a G_δ subset of $\mathbb{R}^{\mathbb{N}}$, it is thus a Polish space as well.

Our aim is to investigate the properties of the operator $T_{\boldsymbol{\lambda}\boldsymbol{\omega}}$ for a typical choice of parameters $(\boldsymbol{\lambda},\boldsymbol{\omega}) \in \boldsymbol{\Lambda} \times \boldsymbol{\Omega}$. For every $M > 0$, we consider the following two sets of parameters:

$$\boldsymbol{\mathcal{E}}_M := \big\{(\boldsymbol{\lambda},\boldsymbol{\omega}) \in \boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M \,;\, T_{\boldsymbol{\lambda},\boldsymbol{\omega}} \text{ is ergodic}\big\}$$

and

$$\boldsymbol{\mathcal{D}}_M := \big\{(\boldsymbol{\lambda},\boldsymbol{\omega}) \in \boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M \,;\, \sigma_p(T_{\boldsymbol{\lambda},\boldsymbol{\omega}}) \subseteq \{\lambda_k,\, k \geq 1\} \cup \{1\}\big\},$$

where $\sigma_p(T)$ denotes the point spectrum (*i.e.* the set of eigenvalues) of an operator $T \in \mathfrak{B}(\mathcal{H})$. The operators belonging to $\boldsymbol{\mathcal{D}}_M$ are those which have the smallest possible set of eigenvalues among the operators $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$, $(\boldsymbol{\lambda},\boldsymbol{\omega}) \in \boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$. In particular, they have countable unimodular point spectrum.

Our main result can now be stated as follows:

THEOREM 4.9. *For any $M > 2$, the two sets $\boldsymbol{\mathcal{E}}_M$ and $\boldsymbol{\mathcal{D}}_M$ are comeager in $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$.*

As an immediate consequence, we obtain

COROLLARY 4.10. *There exist ergodic operators on $\mathcal{H}$, of the form $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$, $(\boldsymbol{\lambda},\boldsymbol{\omega}) \in \boldsymbol{\Lambda} \times \boldsymbol{\Omega}$, which have countable unimodular point spectrum. In particular, these operators are ergodic but not ergodic in the Gaussian sense.*

Theorem 4.9 consists of two statements (that $\boldsymbol{\mathcal{E}}_M$ is comeager, and that $\boldsymbol{\mathcal{D}}_M$ is also comeager) whose proofs turn out to completely independent of each other. The remainder of this section is organized as follows. We first prove in Subsection 3.2 some simple facts concerning the eigenvalues and the eigenvectors of the operators $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$. Then we prove in Subsection 3.3 that $\boldsymbol{\mathcal{D}}_M$ is comeager in $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$; this proof is actually rather straightforward. Subsection 3.4 contains an auxiliary proposition which will be crucial for showing the comeagerness of $\boldsymbol{\mathcal{E}}_M$. And finally, we prove in Subsection 3.5 that $\boldsymbol{\mathcal{E}}_M$ is comeager in $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$.

3.2. Preliminaries on the eigenvalues of $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$. The following simple lemma gives a complete description of the eigenvectors of the operators $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$.

LEMMA 4.11. *Fix $(\boldsymbol{\lambda},\boldsymbol{\omega}) \in \boldsymbol{\Lambda} \times \boldsymbol{\Omega}$, and $\lambda \in \mathbb{C}$. Then λ is an eigenvalue of $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ if and only if the vector*

$$E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda) := e_1 + \sum_{n\geq 2} \Big(\prod_{j=1}^{n-1} \frac{\lambda - \lambda_j}{\omega_j} \Big) e_n$$

is a well-defined vector of $\mathcal{H}$. In this case, $\ker(T_{\boldsymbol{\lambda},\boldsymbol{\omega}} - \lambda)$ *=span* $[E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda)]$.

PROOF. It suffices to solve formally the equation $T_{\boldsymbol{\lambda},\boldsymbol{\omega}} x = \lambda x$ in $\mathbb{C}^{\mathbb{N}}$. □

REMARK 4.12. From Lemma 4.11, we see in particular that for every $k \geq 1$,

$$E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda_k) = e_1 + \sum_{n=2}^{k} \Big(\prod_{j=1}^{n-1} \frac{\lambda_k - \lambda_j}{\omega_j} \Big) e_n$$

is an eigenvector of $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ with eigenvalue λ_k, and hence is proportional to $u_k(T_{\boldsymbol{\lambda},\boldsymbol{\omega}})$.

The next lemma provides necessary conditions for the point spectrum of $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ to be either "maximal" or "minimal". For any element $\boldsymbol{\omega}$ of $\boldsymbol{\Omega}$, we denote by $R_{\boldsymbol{\omega}}$ the radius of convergence of the series

$$\sum_{j \geq 1} \frac{z^j}{\omega_1 \cdots \omega_j}.$$

LEMMA 4.13. *Let $(\boldsymbol{\lambda},\boldsymbol{\omega})$ be an element of $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}$.*

(1) *If $R_{\boldsymbol{\omega}} > 2$, the map $\lambda \mapsto E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda)$ is well-defined and analytic on a neighborhood of the unit circle $\mathbb{T}$. Consequently, $\overline{\operatorname{span}}\,[E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda_k)\ ;\ k > k_0] = \mathcal{H}$ for every $k_0 \geq 1$.*
(2) *If $R_{\boldsymbol{\omega}} = 0$, the eigenvalues of $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ are contained in the set $\{\lambda_k\ ;\ k \geq 1\} \cup \{1\}$.*

PROOF. The first part of assertion (1) is a direct consequence of Lemma 4.11: since $R_{\boldsymbol{\omega}} > 2$, the series defining $E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda)$ is convergent in $\mathcal{H}$ for every λ belonging to the open disk $D(0, R_{\boldsymbol{\omega}} - 1)$, and the map $\lambda \mapsto E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda)$ is analytic there. In particular, this map is analytic in a neighborhood of $\mathbb{T}$. Since the sequence $(\lambda_k)_{k>k_0}$ has an accumulation point in $D(0, R_{\boldsymbol{\omega}} - 1)$ for every $k_0 \geq 1$, it follows from the analyticity of $E_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ that the closed linear span of the vectors $E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda_k)$, $k > k_0$, coincides with the closed linear span of all the eigenvectors $E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda)$, $\lambda \in \mathbb{T}$. Since $E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda_k)$ is proportional to $u_k(T_{\boldsymbol{\lambda},\boldsymbol{\omega}})$ and since the vectors $u_k(T_{\boldsymbol{\lambda},\boldsymbol{\omega}})$, $k \geq 1$, span a dense subspace of $\mathcal{H}$, this proves the second part of assertion (1).

As to assertion (2), suppose that $R_{\boldsymbol{\omega}} = 0$. Let us show that the series defining $E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda)$ does not converge when $\lambda \in \mathbb{C}$ does not belong to $\{\lambda_k\ ;\ k \geq 1\} \cup \{1\}$. So let fix such a complex number λ. Since λ_k tends to 1 as k tends to infinity, there exists a number $\delta > 0$ such that $|\lambda - \lambda_k| \geq \delta$ for every $k \geq 1$. Hence

$$\Big| \prod_{j=1}^{n-1} \frac{\lambda - \lambda_j}{\omega_j} \Big|^2 \geq \frac{\delta^{\,2(n-1)}}{(\omega_1 \ldots \omega_{n-1})^2} \qquad \text{for every } n \geq 2.$$

Since $R_{\boldsymbol{\omega}} = 0$, it follows that

$$\sum_{n \geq 2} \Big| \prod_{j=1}^{n-1} \frac{\lambda - \lambda_j}{\omega_j} \Big|^2 = \infty,$$

and hence $E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda)$ is not defined as a vector of $\mathcal{H}$. □

3.3. Comeagerness of $\boldsymbol{\mathcal{D}}_M$. Observe that the set of all $\boldsymbol{\omega} \in \boldsymbol{\Omega}_M$ such that

$$R_{\boldsymbol{\omega}} = \liminf_{j \to \infty} (\omega_1 \ldots \omega_j)^{1/j} = 0$$

is G_δ in $\boldsymbol{\Omega}_M$, and that this set is dense in $\boldsymbol{\Omega}_M$ because it contains all sequences $\boldsymbol{\omega} \in \boldsymbol{\Omega}_M$ such that ω_j tends to 0 as j tends to infinity. Hence the set $\{(\boldsymbol{\lambda}, \boldsymbol{\omega}) \in \boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M\ ;\ R_{\boldsymbol{\omega}} = 0\}$ is G_δ and dense in $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$. By part (2) of Lemma 4.13, it follows that $\boldsymbol{\mathcal{D}}_M$ is comeager in $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$.

3.4. An auxiliary result. The next proposition will be a key step to prove that ergodic elements are typical in $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$ for any $M > 2$.

In what follows, we fix $M > 2$. For any $\boldsymbol{\omega} \in \boldsymbol{\Omega}_M$, any open set $U \neq \emptyset$ in $\mathcal{H}$ and any $\alpha > 0$, we set

$$\boldsymbol{\mathcal{O}}^{\boldsymbol{\omega}}_{U,\,\alpha} := \Big\{ \boldsymbol{\lambda} \in \boldsymbol{\Lambda} \; ; \; \exists\, r \geq 1 \;\; \exists\, (a_1, \ldots, a_r) \in \mathbb{C}^r \;\; : \\ \sum_{k=1}^{r} a_k u_k\big(T_{\boldsymbol{\lambda},\boldsymbol{\omega}}\big) \in U \quad \text{and} \quad \sum_{k=1}^{r} |a_k|^2 \|u_k(T_{\boldsymbol{\lambda},\boldsymbol{\omega}})\|^2 < \alpha \Big\}.$$

Note that $\boldsymbol{\mathcal{O}}^{\boldsymbol{\omega}}_{U,\,\alpha}$ is an *open* subset of $\boldsymbol{\Lambda}$: this follows from the continuity of the map $(\boldsymbol{\lambda}, \boldsymbol{\omega}) \mapsto T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ from $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$ into $(\mathfrak{T}_{M+1}(\mathcal{H}), \mathtt{SOT})$, combined with Lemma 4.2.

PROPOSITION 4.14. *Let U be a non-empty open set in $\mathcal{H}$, and let $\alpha > 0$. If $\boldsymbol{\omega} \in \boldsymbol{\Omega}_M$ is such that $R_{\boldsymbol{\omega}} > 2$, then the open set $\boldsymbol{\mathcal{O}}^{\boldsymbol{\omega}}_{U,\,\alpha}$ is dense in $\boldsymbol{\Lambda}$.*

PROOF. This relies on some arguments used in [**30**], more precisely in the proof of [**30**, Th. 5.12].

Let us fix $\boldsymbol{\theta} \in \boldsymbol{\Lambda}$ and $\varepsilon > 0$. We are looking for an element $\boldsymbol{\lambda}$ of $\boldsymbol{\mathcal{O}}^{\boldsymbol{\omega}}_{U,\,\alpha}$ such that $\|\boldsymbol{\lambda} - \boldsymbol{\theta}\|_\infty < \varepsilon$.

Writing $\boldsymbol{\theta}$ as $\boldsymbol{\theta} = (\theta_k)_{k \geq 1}$, we fix $k_0 \geq 1$ such that $|\theta_k - 1| < \varepsilon/3$ for every $k > k_0$. By assertion (1) of Lemma 4.13, the linear span of the vectors $E_{\boldsymbol{\theta},\boldsymbol{\omega}}(\theta_s)$, $s > k_0$, is dense in $\mathcal{H}$, so there exists an index $Q \geq 1$ and complex numbers $b_{k_0+1}, \ldots, b_{k_0+q}$ such that the vector

$$z := \sum_{s=k_0+1}^{k_0+Q} b_s \, E_{\boldsymbol{\theta},\boldsymbol{\omega}}(\theta_s) \text{ belongs to } U.$$

We now proceed as in [**28**] or [**30**], and replace each coefficient b_s by a certain sum of scalars of the form $\sum_{j=0}^{N-1} c_{s+jQ}$ with $\sum_{j=0}^{N-1} |c_{s+jQ}|^2$ sufficiently small. More precisely, we define

$$c_{s+jQ} = \frac{1}{N} \, b_s \quad \text{and} \quad v_{s+jQ} = E_{\boldsymbol{\theta},\boldsymbol{\omega}}(\theta_s)$$

for every $k_0 + 1 \leq s \leq k_0 + Q$ and every $0 \leq j \leq N - 1$, where $N \geq 1$ is an integer so large that

$$\sum_{k=k_0+1}^{k_0+NQ} |c_k|^2 \|v_k\|^2 = \frac{1}{N} \sum_{s=k_0+1}^{k_0+Q} |b_s|^2 \; \|E_{\boldsymbol{\theta},\boldsymbol{\omega}}(\theta_s)\|^2 < \alpha. \tag{12}$$

Observe that $\sum\limits_{k=k_0+1}^{k_0+NQ} c_k v_k = z$ by construction, so that we have

$$\sum_{k=k_0+1}^{k_0+NQ} c_k v_k \in U. \tag{13}$$

The next step in the proof is to define $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ with $\|\boldsymbol{\lambda} - \boldsymbol{\theta}\|_\infty < \varepsilon$, in such a way that each vector v_k, $k_0 + 1 \leq k \leq k_0 + NQ$, can be approximated by the eigenvector $E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda_k)$ of $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ associated to the eigenvalue λ_k, and the λ_k, $k_0 + 1 \leq k \leq k_0 + NQ$, are all distinct. The sequence $\boldsymbol{\lambda}$ is defined as follows: we keep $\lambda_k = \theta_k$ for every $1 \leq k \leq k_0 + Q$ and every $k > k_0 + NQ$, so that in particular $\lambda_k \to 1$ as $k \to \infty$. For $k_0 + 1 \leq s \leq k_0 + Q$ and $1 \leq j \leq N - 1$, we choose λ_{s+jQ} extremely close to θ_s, in such a way that these new coefficients λ_k are all distinct and distinct from all the θ_k. Since we already know that $\lambda_k \to 1$ as $k \to \infty$, the sequence $\boldsymbol{\lambda}$ defined in this way belongs to $\boldsymbol{\Lambda}$. We can certainly ensure that $|\lambda_{s+j+1} - \theta_s| < \varepsilon$ for all $k_0 + 1 \leq s \leq k_0 + Q$ and $1 \leq j \leq N - 1$, and thus that

$$|\lambda_{s+jQ} - \theta_{s+jQ}| < \varepsilon/3 + |\theta_{s+jQ} - 1| + |\theta_s - 1| < \varepsilon$$

since $s > k_0$. All the remaining coefficients of $\boldsymbol{\lambda}$ and $\boldsymbol{\theta}$ coincide, and hence $\|\boldsymbol{\lambda} - \boldsymbol{\theta}\|_\infty < \varepsilon$.

Let us now show that the quantities

$$\|E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda_{s+jQ}) - v_{s+jQ}\| = \|E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda_{s+jQ}) - E_{\boldsymbol{\theta},\boldsymbol{\omega}}(\theta_s)\|,$$

where $k_0+1 \le s \le k_0+Q$ and $0 \le j \le N-1$, can be made as small as we wish, provided each coefficient λ_{s+jQ} is close enough to θ_s. We consider separately two cases.

- Assume that $k_0+1 \le s \le k_0+Q$ and $j=0$. In this case $\lambda_s = \theta_s$ by definition, so that $E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda_s) = E_{\boldsymbol{\theta},\boldsymbol{\omega}}(\theta_s)$.

- Assume now that $k_0+1 \le s \le k_0+Q$ and $1 \le j \le N-1$. In this case, we use the explicit expression of the eigenvectors provided by Lemma 4.11 and write

$$E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda_{s+jQ}) - E_{\boldsymbol{\theta},\boldsymbol{\omega}}(\theta_s) = \sum_{n=2}^{s} \Bigg[\prod_{i=1}^{n-1}\Big(\frac{\lambda_{s+jQ}-\lambda_i}{\omega_i}\Big) - \prod_{i=1}^{n-1}\Big(\frac{\theta_s-\theta_i}{\omega_i}\Big)\Bigg] e_n + \sum_{n=s+1}^{s+jQ} \prod_{i=1}^{n-1}\Big(\frac{\lambda_{s+jQ}-\lambda_i}{\omega_i}\Big) e_n.$$

Fix an arbitrarily small number $\delta > 0$. The integers s and j being fixed, and the numbers λ_i and θ_i being by definition equal for every $1 \le i \le k_0+Q$, the norm of the first sum in the above expression can be made less that $\delta/2$, provided that the difference $|\lambda_{s+jQ}-\theta_s|$ is sufficiently small. As for the second term, observe that s belongs to the set $\{1,\dots,n-1\}$ for every $s+1 \le n \le s+jQ$, so that the term $\lambda_{s+jQ}-\lambda_s = \lambda_{s+jQ}-\theta_s$ always appears in the product $\prod_{i=1}^{n-1}(\lambda_{s+jQ}-\lambda_i)/\omega_i$. Thus if $|\lambda_{s+jQ}-\theta_s|$ is sufficiently small, the norm of the second term is less than $\delta/2$ too. Hence one can ensure that $\|E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda_{s+jQ}) - E_{\boldsymbol{\theta},\boldsymbol{\omega}}(\theta_s)\| < \delta$ for all $k_0+1 \le s \le k_0+Q$ and $1 \le j \le N-1$.

So we have proved that for any $\delta > 0$, one can construct $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ such that $\|\boldsymbol{\lambda}-\boldsymbol{\theta}\|_\infty < \varepsilon$ and $\|E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda_k) - v_k\| < \delta$ for every $k_0+1 \le k \le k_0+NQ$. If δ is now chosen small enough, the two conditions

$$\sum_{k=k_0+1}^{k_0+NQ} c_k\, E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda_k) \in U \quad \text{and} \quad \sum_{k=k_0+1}^{k_0+NQ} |c_k|^2\, \|E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda_k)\|^2 < \alpha$$

simultaneously hold true, by (13) and (12) respectively. Remembering that each vector $u_k(T_{\boldsymbol{\lambda},\boldsymbol{\omega}})$ is proportional to $E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda_k)$, we eventually obtain that there exist complex coefficients a_k, $k_0+1 \le k \le k_0+NQ$, such that

$$\sum_{k=k_0+1}^{k_0+NQ} a_k\, u_k(T_{\boldsymbol{\lambda},\boldsymbol{\omega}}) \in U \quad \text{and} \quad \sum_{k=k_0+1}^{k_0+NQ} |a_k|^2\, \|u_k(T_{\boldsymbol{\lambda},\boldsymbol{\omega}})\|^2 < \alpha.$$

Hence $\boldsymbol{\lambda}$ belongs to $\boldsymbol{\mathcal{O}}^{\boldsymbol{\omega}}_{U,\alpha}$, and this concludes the proof of Proposition 4.14. □

3.5. Comeagerness of $\boldsymbol{\mathcal{E}}_M$. The proof that $\boldsymbol{\mathcal{E}}_M$ is comeager in $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$ ultimately relies on the the following ergodicity criterion proved in [**30**, Cor. 5.5].

PROPOSITION 4.15. *Let X be a separable Banach space, and let $T \in \mathfrak{B}(X)$. Assume that for any open set $\Omega \neq \emptyset$ of X with $T^{-1}(\Omega) \subseteq \Omega$, any neighborhood W of 0, and any $\varepsilon > 0$, there exists a T-invariant probability measure μ on X with compact support such that $\mu(\Omega) = 1$ and $\mu(W) > 1-\varepsilon$. Then T is ergodic.*

In order to apply this criterion, we need the following lemma, which makes the link with Proposition 4.14.

LEMMA 4.16. *Let $T \in \mathfrak{T}(\mathcal{H})$, let U be a non-empty open set in $\mathcal{H}$, let $\varepsilon > 0$, and let W be a neighborhood of 0 in $\mathcal{H}$. Let also $r \ge 1$, and assume that the eigenvalues $\lambda_1(T),\dots,\lambda_r(T)$ of T are rationally independent. Finally, let $a_1,\dots,a_r$ be r complex numbers and assume that the vector $\sum_{k=1}^r a_k u_k(T)$ belongs to U. If the quantity $\sum_{k=1}^r |a_k|^2 \|u_k(T)\|^2$ is small enough, there exists a compactly supported T-invariant measure μ on $\mathcal{H}$ such that*

- *$\mu(\Omega_U) = 1$, where $\Omega_U = \bigcup_{n\ge 0} T^{-n}(U)$, and*
- *$\mu(W) > 1-\varepsilon$.*

PROOF. We only sketch the proof, since the argument is already essentially given in [**30**]. Let $(\chi_k)_{k=1}^r$ be a sequence of independent random Steinhaus variables defined on some standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let μ be the measure on $\mathcal{H}$ given by the distribution of the $\mathcal{H}$-valued random variable $\sum_{k=1}^r \chi_k a_k u_k(T)$. This measure μ, which may be called the *Steinhaus measure associated with the vector* $u = \sum_{k=1}^r a_k u_k(T)$, is T-invariant and has compact support. As observed in [**30**, Fact 5.16], the independence of the unimodular numbers $\lambda_1(T), \dots, \lambda_r(T)$ and the condition $\sum_{k=1}^r a_k u_k(T) \in U$ imply that $\mu(\Omega_U) = 1$. Also, we have by orthogonality of the Steinhaus variables χ_k, $1 \le k \le r$, that

$$\int_H \|x\|^2 d\mu(x) = \mathbb{E}\left(\sum_{k=1}^r \chi_k a_k u_k(T)\right) = \sum_{k=1}^r |a_k|^2 \, \|u_k(T)\|^2.$$

It then follows from Markov's inequality that $\mu(W) > 1 - \varepsilon$ provided that the quantity

$$\sum_{k=1}^r |a_k|^2 \, \|u_k(T)\|^2$$

is sufficiently small. □

We can now proceed to the

PROOF THAT $\boldsymbol{\mathcal{E}}_M$ IS COMEAGER IN $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$. Let $(U_p)_{p \ge 1}$ be a countable basis of non-empty open subsets for $\mathcal{H}$. For each $\boldsymbol{\omega} \in \boldsymbol{\Omega}_M$, define the following G_δ subset of $\boldsymbol{\Lambda}$:

$$\boldsymbol{G}_{\boldsymbol{\omega}} := \bigcap_{p \ge 1} \bigcap_{q \ge 1} \boldsymbol{\mathcal{O}}^{\boldsymbol{\omega}}_{U_p, 2^{-q}},$$

and set

$$\boldsymbol{G} := \big\{(\boldsymbol{\lambda}, \boldsymbol{\omega}) \in \boldsymbol{\Lambda} \times \boldsymbol{\Omega} \, ; \, \boldsymbol{\lambda} \in \boldsymbol{G}_{\boldsymbol{\omega}}\big\}.$$

CLAIM 4.17. The set $\boldsymbol{G}$ is a dense G_δ subset of $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$.

PROOF OF CLAIM 4.17. The fact that $\boldsymbol{G}$ is G_δ in $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$ is immediate. As to its density, we first observe that since $M > 2$, Proposition 4.14 implies that $\boldsymbol{G}_{\boldsymbol{\omega}}$ is dense in $\boldsymbol{\Lambda}$ as soon as $\omega_j = M$ for j sufficiently large. It follows that the set $\{\boldsymbol{\omega} \in \boldsymbol{\Omega}_M \, ; \, \boldsymbol{G}_{\boldsymbol{\omega}} \text{ is dense in } \boldsymbol{\Lambda}\}$ is dense in $\boldsymbol{\Omega}_M$. Since this set is also clearly G_δ in $\boldsymbol{\Omega}_M$, it is thus comeager in $\boldsymbol{\Omega}_M$. The Kuratowski-Ulam theorem then implies that the G_δ set $\boldsymbol{G}$ is comeager and hence dense in $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$. □

It is now easy to show that the set $\boldsymbol{\mathcal{E}}_M$ is comeager in $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$. Indeed, it follows from Proposition 4.15 and Lemma 4.16 that $T_{\boldsymbol{\lambda}, \boldsymbol{\omega}}$ is ergodic as soon as $\boldsymbol{\lambda}$ belongs to $\boldsymbol{G}_{\boldsymbol{\omega}}$ and the terms λ_k, $k \ge 1$ of the sequence $\boldsymbol{\lambda}$ are rationally independent. Hence the set

$$\boldsymbol{G}_{\text{ind}} = \big\{(\boldsymbol{\lambda}, \boldsymbol{\omega}) \in \boldsymbol{G} \, ; \text{ the unimodular numbers } \lambda_k, \ k \ge 1 \text{ are rationally independent}\big\}$$

is contained in $\boldsymbol{\mathcal{E}}_M$. Since the conditions that the numbers λ_k should be rationally independent is easily seen to define a dense G_δ subset of $\boldsymbol{\Lambda}$, Claim 4.17 implies that $\boldsymbol{G}_{\text{ind}}$ is a dense G_δ subset of $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$, and it follows that $\boldsymbol{\mathcal{E}}_M$ is indeed comeager in $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$. □

REMARK 4.18. To show that the ergodic operators are comeager in $\boldsymbol{\Lambda} \times \boldsymbol{\Omega}_M$, we have used in a crucial way the fact that the terms λ_k, $k \ge 1$ of the sequences $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ involved in the proof are rationally independent. This is not so surprising in view, for instance, of [**30**, Fact 5.16]. Yet the role of independence in these issues remains rather mysterious. We develop this a little bit in Section 4 below.

3.6. Some comments on ergodicity and unimodular eigenvalues. Corollary 4.10 provides examples of hypercyclic operators on a Hilbert space with a spanning set of unimodular eigenvectors but only countably many unimodular eigenvalues. The question of the existence of such operators was raised by Flytzanis in [**23**, p. 8], and answered recently in [**40**]. Indeed, the chaotic non-frequently hypercyclic operators constructed there have only countably many eigenvalues, which are all roots of unity, and the associated eigenvectors span the space. Corollary 4.10 strengthens this by showing that there even *ergodic* operators with the above properties, and that such examples are in some sense much less exotic than suggested by the rather technical construction of [**40**].

However, since all the operators we have been considering here have infinitely many eigenvalues, Corollary 4.10 leaves open the question of the existence of ergodic operators on $\mathcal{H}$ *without any eigenvalue*.

Such operators are known to exist on the Banach space $\mathcal{C}_0([0,2\pi])$ of continuous functions on $[0,2\pi]$ vanishing at the point 0. One example is the Kalisch operator T, again defined by the formula

$$Tf(\theta) = e^{i\theta} f(\theta) - \int_0^\theta ie^{it} f(t)\, dt,$$

but now considered as acting on $\mathcal{C}_0([0,2\pi])$. It is shown in [**6**] that T is ergodic in the Gaussian sense (see also [**8**, Section 5.5.4]), but T does not admit any unimodular eigenvalue since its L^2 eigenfunctions are not continuous.

On the other hand, an operator on a Hilbert space which is ergodic in the Gaussian sense definitely has a lot of unimodular eigenvalues (recall that G-ERG$(\mathcal{H})$ = PSPAN$(\mathcal{H})$); but a general ergodic operator might possibly have no eigenvalue at all.

4. Additional remarks

In this section, we present a short discussion of some questions motivated by the results obtained above, as well as some further results concerning dynamical properties of the "diagonal plus shift" operators $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$.

4.1. Some natural questions. Two questions concerning the existence of operators on a Hilbert space with particular ergodic-theoretic-like properties remain unanswered at this stage of our work. The first one is

Question 4.19. Do there exist $\mathcal{U}$-frequently hypercyclic operators on $\mathcal{H}$ which are not frequently hypercyclic?

It was proved by Bayart and Rusza in [**10**] that such operators do exist on the space c_0, but the question was left open for Hilbert (or even reflexive) spaces. In the light of the discussion carried out in Chapter 3, it seems natural to conjecture that the two classes UFHC$(\mathcal{H})$ and FHC$(\mathcal{H})$ should have different descriptive complexity, and *hence* should be distinct; but we have been unable to solve the question using this approach. The second question runs as follows:

Question 4.20. Do there exist frequently hypercyclic operators on $\mathcal{H}$ which are not ergodic?

This question comes from [**30**], where it is proved that frequently hypercyclic non-ergodic operators do exist on the space c_0. The proof of this result again relies on a construction of [**10**]: it is proved in [**30**] that the frequently hypercyclic bilateral weighted shifts T on $c_0(\mathbb{Z})$ defined in [**10**] satisfy $c(T) < 1$, and thus cannot be ergodic. For some reasons, we found it rather tempting to try to attack this question by Baire category methods; but again, we did not succeed in this way.

We *do* solve Questions 4.19 and 4.20 in Chapter 7 below, but using widely different methods. The operators we will use are generalizations of those introduced by the third named author in [**40**] in order to solve the question of the existence of a chaotic non-frequently hypercyclic operator. The two main results we will obtain are that indeed, there exist $\mathcal{U}$-frequently hypercyclic operators

on $\mathcal{H}$ which are not frequently hypercyclic, as well as frequently hypercyclic operators which are not ergodic. All our examples turn out to be chaotic; and we will complement these results by showing that there also exist on $\mathcal{H}$ operators which are chaotic and topologically mixing but not $\mathcal{U}$-frequently hypercyclic.

In another direction, the role of *rational independence* in all that concerns the links between unimodular eigenvalues and ergodicity properties of operators needs to be clarified. To be a little more specific, let us consider the class $\mathfrak{T}_{\text{ind}}(\mathcal{H})$ of operators in $\mathfrak{T}(\mathcal{H})$ whose diagonal coefficients are rationally independent. Rather surprisingly, it seems that very little is known concerning such operators. For example, to our knowledge the following question is open:

QUESTION 4.21. Let T be a hypercyclic operator belonging to $\mathfrak{T}_{\text{ind}}(\mathcal{H})$. Is T necessarily ergodic? frequently hypercyclic? $\mathcal{U}$-frequently hypercyclic?

The only currently known examples of hypercyclic operators with spanning unimodular eigenvectors which are not frequently hypercyclic or not ergodic are those constructed in [**40**] and the ones that will be considered in Chapter 6. As already mentioned, all these operators are chaotic. Moreover, for many of these operators, the only unimodular eigenvalues are roots of unity and for some of those, each eigenvalue has multiplicity one. So these operators belong to $\mathfrak{T}(\mathcal{H})$ for some suitably chosen orthonormal basis $(e_k)_{k\geq 1}$, but not to $\mathfrak{T}_{\text{ind}}(\mathcal{H})$ for any basis $(e_k)_{k\geq 1}$. A positive answer to Question 4.21 would show that this is not accidental, *i.e.* that there must be a strong amount of dependence between the eigenvalues of *any* of the operators constructed in [**40**] or in Chapter 6 of the present work.

As a matter of fact, Question 4.21 seems to be open even for the operators $T_{\boldsymbol{\lambda},\boldsymbol{\omega}} = D_{\boldsymbol{\lambda}} + B_{\boldsymbol{\omega}}$ considered in Section 3. Even more prosaically, it seems quite desirable (and perhaps not too difficult) to determine when exactly an operator of the form $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ is hypercyclic.

4.2. More on the operators $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$. We finish this section by collecting some simple facts that we do know concerning operators of the form $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$. Let us introduce the following notations.

- If $\boldsymbol{\omega}$ is a unilateral weight sequence, we set $a_1(\boldsymbol{\omega}) := 1$ and

$$a_j(\boldsymbol{\omega}) := \frac{1}{\omega_1 \cdots \omega_{j-1}} \quad \text{for every } j \geq 2.$$

Also, using the same notation as in Section 3 above, we denote by $R_{\boldsymbol{\omega}}$ the radius of convergence of the series $\sum a_j(\boldsymbol{\omega}) z^j$:

$$R_{\boldsymbol{\omega}} = \liminf_{j\to\infty} |\omega_1 \cdots \omega_{j-1}|^{1/j}.$$

- To each $\boldsymbol{\lambda} \in \mathbb{T}^{\mathbb{N}}$, we associate polynomials $P_{\boldsymbol{\lambda},j}$, $j \geq 1$, defined as follows: $P_{\boldsymbol{\lambda},1}(\lambda) \equiv 1$ and

$$P_{\boldsymbol{\lambda},j}(\lambda) = \prod_{k=1}^{j-1} (\lambda - \lambda_k) \quad \text{for every } j \geq 2.$$

PROPOSITION 4.22. *Let $\boldsymbol{\omega}$ be a unilateral weight sequence, and let $\boldsymbol{\lambda} \in \mathbb{T}^{\mathbb{N}}$.*

(0) *A complex number λ is an eigenvalue of $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ if and only if*

$$\sum_{j=1}^{\infty} |a_j(\boldsymbol{\omega})|^2 \, |P_j(\lambda)|^2 < \infty.$$

In this case, $\ker(T - \lambda) = \operatorname{span}[E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda)]$ *where*

$$E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda) = \sum_{j=1}^{\infty} a_j(\boldsymbol{\omega}) P_{\boldsymbol{\lambda},j}(\lambda)\, e_j.$$

(1) *If $R_{\boldsymbol{\omega}} = 0$, then $\sigma_p(T_{\boldsymbol{\lambda},\boldsymbol{\omega}}) \subseteq \overline{\{\lambda_k;\ k \geq 1\}}$.*

(2) *If* $\sigma(B_{\boldsymbol{\omega}}) = \{0\}$ *(e.g. if* $\omega_j \to 0$ *as* $j \to \infty$*), then* $\sigma(T_{\boldsymbol{\lambda},\boldsymbol{\omega}}) \subseteq \overline{\{\lambda_k;\ k \geq 1\}}$*. More generally,* $\sigma(T_{\boldsymbol{\lambda},\boldsymbol{\omega}})$ *is contained in the set* $\{\lambda \in \mathbb{C};\ \operatorname{dist}(\lambda, \{\lambda_k\}) \leq r(B_{\boldsymbol{\omega}})\}$*, where* $r(B_{\boldsymbol{\omega}})$ *denotes the spectral radius of* $B_{\boldsymbol{\omega}}$*.*

Assume now that $\boldsymbol{\lambda}$ *belongs to* $\boldsymbol{\Lambda}$*, i.e. that the elements* λ_k*,* $k \geq 1$ *are distinct and that* $\lambda_k \to 1$ *as* $k \to \infty$*.*

(3) *If* $R_{\boldsymbol{\omega}} > 0$*, then* $\sigma_p(T_{\boldsymbol{\lambda},\boldsymbol{\omega}})$ *contains* $\{\lambda_k;\ k \geq 1\} \cup D(1, R_{\boldsymbol{\omega}})$ *and is contained in* $\{\lambda_k;\ k \geq 1\} \cup \overline{D}(1, R_{\boldsymbol{\omega}})$*.*

(4) *If* $R_{\boldsymbol{\omega}} > \sup_{k\geq 1} |\lambda_k - 1|$*, then* $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ *is ergodic in the Gaussian sense. Moreover, the map* $\lambda \mapsto E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda)$ *is analytic on the open disk* $D(0, R_{\boldsymbol{\omega}} - 1)$*.*

(5) *If* $R_{\boldsymbol{\omega}} < \sup_{k\geq 1} |\lambda_k - 1|$*, then* $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ *is not hypercyclic.*

(6) *If* $R_{\boldsymbol{\omega}} = \sup_{k\geq 1} |\lambda_k - 1|$*, let* k_0 *be the largest integer* $k \geq 1$ *such that* $|\lambda_k - 1| \geq R_{\boldsymbol{\omega}}$*. If*

$$\sum_{j\geq k_0+1} \left| \frac{\omega_1 \cdots \omega_{j-1}}{(\lambda_{k_0} - \lambda_{k_0+1}) \cdots (\lambda_{k_0} - \lambda_j)} \right|^2 < \infty,$$

then $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ *is not ergodic in the Gaussian sense.*

Proof. (0) This is the content of Lemma 4.11.

(1) If $\lambda \in \mathbb{C}$ does not belong to the closure of $\{\lambda_k;\ \ k \geq 1\}$, then $\inf_{k\geq 1} |\lambda - \lambda_k| = \delta > 0$, and thus $|P_j(\lambda)| \geq \delta^{j-1}$ for all $j \geq 1$. If $R_{\boldsymbol{\omega}} = 0$, assertion (0) implies that λ is not an eigenvalue of $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$.

(2) Let $\lambda \in \mathbb{C}$. We have $T_{\boldsymbol{\lambda},\boldsymbol{\omega}} - \lambda I = D_{\boldsymbol{\xi}} + B_{\boldsymbol{\omega}}$, where $\boldsymbol{\xi} = (\xi_k)_{k\geq 1}$ is defined by $\xi_k = \lambda_k - \lambda$ for every $k \geq 1$. If $\delta := \operatorname{dist}(\lambda, \{\lambda_k\}) > 0$, then $D_{\boldsymbol{\xi}}$ is invertible and $T_{\boldsymbol{\lambda},\boldsymbol{\omega}} - \lambda I$ can be written as

$$T_{\boldsymbol{\lambda},\boldsymbol{\omega}} - \lambda I = D_{\boldsymbol{\xi}}(I + D_{\boldsymbol{\xi}}^{-1} B_{\boldsymbol{\omega}}) = D_{\boldsymbol{\xi}}(I + B_{\boldsymbol{\xi}^{-1}\boldsymbol{\omega}}),$$

where $\boldsymbol{\xi}^{-1}\boldsymbol{\omega} = (\xi_k^{-1}\boldsymbol{\omega}_k)_{k\geq 1}$. Moreover, since $\sup_{k\geq 1} \xi_k^{-1} = 1/\delta$, the spectral radius of $B_{\boldsymbol{\xi}^{-1}\boldsymbol{\omega}}$ is at most $r(B_{\boldsymbol{\omega}})/\delta$. It follows that $T_{\boldsymbol{\lambda},\boldsymbol{\omega}} - \lambda I$ is invertible as soon as $\delta > r(B_{\boldsymbol{\omega}})$.

(3) We already know by assertion (0) that $\sigma_p(T_{\boldsymbol{\lambda},\boldsymbol{\omega}})$ contains $\{\lambda_k\ ;\ k \geq 1\}$. Now, let λ belong to the open disk $D(1, R_{\boldsymbol{\omega}})$, and choose r such that $|\lambda - 1| < r < R_{\boldsymbol{\omega}}$. Since $\lambda_k \to 1$, there exists $k_0 \geq 1$ such that $|\lambda - \lambda_k| < r$ for all $k \geq k_0$. It follows that there exists a constant $C > 0$ such that $|P_j(\lambda)| \leq C\, r^j$ for every $j \geq 1$, so that $E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda)$ is well-defined and analytic on $D(1, R_{\boldsymbol{\omega}})$.

Conversely, assume that λ does not belong to the set $\{\lambda_k\ ;\ k \geq 1\} \cup \overline{D}(1, R_{\boldsymbol{\omega}})$. Since $\lambda_k \to 1$, one can find $r > R_{\boldsymbol{\omega}}$ such that $|\lambda - \lambda_k| \geq r$ for k sufficiently large; and since $\lambda \neq \lambda_k$ for all $k \geq 1$, it follows that there exists a constant $c > 0$ such that $|P_j(\lambda)| \geq c\, r^j$ for every $j \geq 1$. Hence $E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda)$ is not well-defined.

(4) By assertion (3), the map $\lambda \mapsto E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda)$ is analytic on the disk $D(1, R_{\boldsymbol{\omega}})$; so it is enough to show that the vectors $E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda)$, $\lambda \in D(1, R_{\boldsymbol{\omega}})$, span a dense subspace of $\mathcal{H}$. Let $y = \sum_{j\geq 1} y_j e_j$ be a vector of $\mathcal{H}$ which is orthogonal to all the vectors $E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda)$, $\lambda \in D(1, R_{\boldsymbol{\omega}})$. We thus have

$$\sum_{j=1}^{\infty} \bar{y}_j a_j(\boldsymbol{\omega}) P_{\boldsymbol{\lambda},j}(\lambda) = 0 \qquad \text{for every } \lambda \in D(1, R_{\boldsymbol{\omega}}).$$

The series above is convergent on the disk $D(1, R_{\boldsymbol{\omega}})$, which contains all the points λ_k by assumption. So

$$\sum_{j=1}^{\infty} \bar{y}_j a_j(\boldsymbol{\omega})\, P_{\boldsymbol{\lambda},j}(\lambda_k) = 0 \qquad \text{for every } k \geq 1.$$

Since $P_{\boldsymbol{\lambda},j}(\lambda_k) = 0$ whenever $j > k$ and $P_{\boldsymbol{\lambda},k}(\lambda_k) \neq 0$, it follows that $\bar{y}_j a_j(\boldsymbol{\omega}) = 0$ for every $j \geq 1$, so that $y = 0$.

(5) Let $k_0 \geq 1$ be such that $|\lambda_{k_0} - 1| > R_{\boldsymbol{\omega}}$. Then λ_{k_0} is an isolated eigenvalue of $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ by assertion (3), which is easily seen not to belong to the essential spectrum of $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$. It then follows from [**35**] that $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ cannot be hypercyclic.

(6) By assertion (1), we can assume that $R_{\boldsymbol{\omega}} > 0$, since otherwise the point spectrum $\sigma_p(T_{\boldsymbol{\lambda},\boldsymbol{\omega}})$ of $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ is countable, and $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ is certainly not ergodic in the Gaussian sense in this case. It suffices to show that the unimodular eigenvectors of $T_{\boldsymbol{\lambda},\boldsymbol{\omega}}$ are not perfectly spanning. Since $(\sigma_p(T) \cap \mathbb{T}) \setminus D(1, R_{\boldsymbol{\omega}})$ is a finite set by assertion (3), it is in turn enough to show that the vectors $E_{\boldsymbol{\lambda},\boldsymbol{\omega}}(\lambda)$, $\lambda \in D(1, R_{\boldsymbol{\omega}})$, do not span a dense subspace of $\mathcal{H}$. So we have to find coefficients c_j, $j \geq 1$, with the property that

$$\sum_{j=1}^{\infty} c_j P_{\boldsymbol{\lambda},j}(\lambda) = 0 \qquad \text{for every } \lambda \in D(1, R_{\boldsymbol{\omega}}),$$

with the additional requirement that

$$\sum_{j=1}^{\infty} \left| \frac{c_j}{a_j(\boldsymbol{\omega})} \right|^2 < \infty, \quad \textit{i.e.} \text{ that} \quad \sum_{j \geq 2} |c_j\, \omega_1 \cdots \omega_{j-1}|^2 < \infty. \tag{14}$$

We define the coefficients c_j by setting $c_j = 0$ for every $1 \leq j < k_0$, $c_{k_0} = 1$ and

$$c_j = \prod_{i=k_0+1}^{j} (\lambda_{k_0} - \lambda_i)^{-1} \quad \text{for every } j > k_0.$$

Then condition (14) is satisfied by assumption. Moreover, the coefficients c_j have been defined in such a way that

$$\sum_{j=1}^{N} c_j P_{\boldsymbol{\lambda},j}(\lambda) = P_{\boldsymbol{\lambda},k_0}(\lambda) \cdot \prod_{k=k_0+1}^{N} \left(\frac{\lambda - \lambda_k}{\lambda_{k_0} - \lambda_k} \right) \qquad \text{for all } N \geq k_0 + 1.$$

Since $|\lambda_{k_0} - 1| \geq R_{\boldsymbol{\omega}}$ and $\lambda_k \to 1$ as $k \to \infty$, it follows that $\sum_{j=1}^{\infty} c_j P_{\boldsymbol{\lambda},j}(\lambda) = 0$ for every λ such that $|\lambda - 1| < R_{\boldsymbol{\omega}}$. This completes the proof of assertion (6). $\square$

CHAPTER 5

Periodic points at the service of hypercyclicity

In this chapter, we depart from our standing assumption and work in the general context of Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, not restricting ourselves to the Hilbertian setting. Actually, most of our results hold in the framework of arbitrary Polish topological vector spaces, as should be clear from the proofs.

Let thus X be a real or complex separable Banach space. Our aim is to obtain general sufficient conditions for an operator $T \in \mathfrak{B}(X)$ to be frequently hypercyclic or $\mathcal{U}$-frequently hypercyclic. These criteria are rather different from the classical ones, since they involve the *periodic points* of the operator T. They will greatly contribute to simplify some proofs in the next section, and they might hopefully be useful elsewhere also. For any bounded operator T on X, we denote by $\mathrm{Per}(T)$ the set of its periodic points, and by $\mathrm{per}_T(x)$, or simply $\mathrm{per}(x)$, the period of a periodic point x of T.

1. Precompact orbits and topological mixing

We start by giving a very simple criterion for an operator $T \in \mathfrak{B}(X)$ to be topologically mixing, which involves the points of X with *precompact orbit* under the action of T. We denote by $\mathrm{Prec}(T)$ the set of points with precompact orbit.

Proposition 5.1. *Let $T \in \mathfrak{B}(X)$. Assume that $\mathrm{Prec}(T)$ is dense in X and that for every neighborhood W of 0 in X and every non-empty open subset V of X, the set $\mathcal{N}_T(W, V)$ is cofinite. Then T is topologically mixing.*

Proof. It suffices to show that for any vectors $y \in \mathrm{Prec}(T)$, $x \in X$, and any $\varepsilon > 0$, the set $\mathcal{N}_T\big(B(y,\varepsilon), B(x,\varepsilon)\big)$ is cofinite. Since y is a vector with precompact orbit, there exists a finite subset $\{y_1, \dots, y_d\}$ of X such that any point of $\mathrm{Orb}(T, y)$ lies within distance less than $\varepsilon/2$ of the set $\{y_1, \dots, y_d\}$. In other words, there exists for any $n \geq 1$ an index $i_n \in \{1, \dots, d\}$ such that $\|T^n y - y_{i_n}\| < \varepsilon/2$.

By assumption, each set $\mathcal{N}_T\big(B(0,\varepsilon), B(x - y_i, \varepsilon/2)\big)$ is cofinite. Hence, one can find an integer N such that for every $n \geq N$ and every $1 \leq i \leq d$, there exists $z_{n,i} \in X$ such that

$$\|z_{n,i}\| < \varepsilon \qquad \text{and} \qquad \|T^n z_{n,i} - (x - y_i)\| < \varepsilon/2.$$

For every $n \geq N$, we then have

$$\|T^n(y + z_{n,i_n}) - x\| \leq \|T^n y - y_{i_n}\| + \|T^n z_{n,i_n} - (x - y_{i_n})\| < \varepsilon.$$

Hence $y + z_{n,i_n}$ belongs to $B(y,\varepsilon)$ and $T^n(y + z_{n,i_n})$ belongs to $B(x,\varepsilon)$, which shows that every integer $n \geq N$ lies in the set $\mathcal{N}_T\big((B(y,\varepsilon), B(x,\varepsilon)\big)$. □

Since periodic points have a finite orbit, Proposition 5.1 immediately implies the following result.

Corollary 5.2. *Let $T \in \mathfrak{B}(X)$. If $\mathrm{Per}(T)$ is dense in X and if for every neighborhood W of 0 in X and every non-empty open subset V of X, the set $\mathcal{N}_T(W, V)$ is cofinite, then T is chaotic and topologically mixing.*

We also obtain in a similar fashion.

COROLLARY 5.3. *Let $T \in \mathfrak{B}(X)$. If there exists a dense set of points $x \in X$ such that $T^i x \to 0$ as $i \to \infty$ and if for every neighborhood W of 0 in X and every non-empty open subset V of X, the set $\mathcal{N}_T(W, V)$ is cofinite, then T is topologically mixing.*

Finally, the linearity assumption on the system (X, T) implies the following strengthened version of Proposition 5.1.

COROLLARY 5.4. *Let $T \in \mathfrak{B}(X)$. Assume that* $\mathrm{Prec}(T)$ *is dense in X and that there exists a subset X_0 of X with dense linear span such that the following property holds true: for any $x_0 \in X_0$ and every $\varepsilon > 0$, there exists $N \geq 1$ such that for every $n \geq N$, one can find $z \in X$ satisfying $\|z\| < \varepsilon$ and $\|T^n z - x_0\| < \varepsilon$. Then T is topologically mixing.*

PROOF. Let $V \subseteq X$ be a non-empty open set. Since X_0 spans a dense subspace of X, there exist $\varepsilon > 0$ and $x = \sum_{k=1}^{K} \alpha_k x_k \in V$ with $x_k \in X_0$ and $\alpha_k \neq 0$ for every $1 \leq k \leq K$ such that $B(x, \varepsilon)$ is contained in V. Moreover, by assumption, there exists $N \geq 1$ such that for every $1 \leq k \leq K$, every $n \geq N$, one can find $z_{n,k} \in X$ such that

$$\|z_{n,k}\| < \frac{\varepsilon}{K|\alpha_k|} \quad \text{and} \quad \|T^n z_{n,k} - x_k\| < \frac{\varepsilon}{K|\alpha_k|}.$$

The vector $z := \sum_{k=1}^{K} \alpha_k z_{n,k}$ then satisfies $\|z\| < \varepsilon$ and $\|T^n z - x\| < \varepsilon$, and the desired result follows from Proposition 5.1. □

REMARK 5.5. Unlike in Corollary 5.8 below, it is not possible, in Proposition 5.1, to replace the assumption that all the sets $\mathcal{N}_T(W, V)$ are cofinite (W and V non-empty open subsets of X with $0 \in W$) by the assumption that all the sets $\mathcal{N}_T(U, W)$ are cofinite (W and U non-empty open subsets of X with $0 \in W$), even if one assumes additionally that T is hypercyclic. Indeed, every unilateral weighted backward shift has a dense set of points with finite orbit and is such that $\mathcal{N}_T(U, W)$ is cofinite for every neighborhood W of 0 and every non-empty open set U; but there exist hypercyclic weighted shifts which are not topologically mixing.

2. Uniform recurrence and topological weak mixing

In this section, we give a simple criterion for an operator $T \in \mathfrak{B}(X)$ to be weakly topologically mixing. It involves the *uniformly recurrent* points of T. Recall that a point $x \in X$ is said to be *recurrent* for T if, for any neighborhood O of x, the set $\mathcal{N}_T(x, O) = \{i \in \mathbb{N};\ T^i x \in O\}$ is non-empty (or, equivalently, infinite). A point $x \in X$ is said to be *uniformly recurrent* for T if, for any neighborhood O of x, the set $\mathcal{N}_T(x, O)$ has bounded gaps. For example, every periodic point is obviously uniformly recurrent. The following more general fact will be useful. Recall first that a compact, T-invariant subset K of X is said to be *minimal* if it has no proper (non-empty) closed T-invariant subset. For example, if x is a periodic point of T with period N, then $K := \{x, Tx, \ldots, T^{N-1}x\}$ is minimal. Recall also that we denote by $\mathcal{E}(T)$ the set of all unimodular eigenvectors of T.

LEMMA 5.6. *Let $T \in \mathfrak{B}(X)$. If K is a compact minimal T-invariant subset of X, then every point of K is uniformly recurrent for T. In particular, every point $z \in \operatorname{span} \mathcal{E}(T)$ is uniformly recurrent for T.*

PROOF. The first part of the statement is well-known (and has nothing to do with linearity; see e.g. [**24**, Theorem 1.15]). As for the second part, it is enough to show that if z belongs to $\operatorname{span} \mathcal{E}(T)$, then its closed T-orbit $K = \overline{\{T^n z;\ n \geq 0\}}$ is a compact minimal T-invariant set.

Write z as $z = \sum_{j=1}^{N} u_j$, where for every $1 \leq j \leq N$, $Tu_j = \gamma_j u_j$ for some $\gamma_j \in \mathbb{T}$. Let Γ be the closed subgroup of $\mathbb{T}^N$ generated by the N-tuple $\gamma := (\gamma_1, \ldots, \gamma_N)$. Then $K = \{\sum_{j=1}^{N} \omega_j u_j;\ \omega = (\omega_1, \ldots, \omega_N) \in \Gamma\}$, so that K is compact (and, of course, T-invariant). So we just have to check that K is minimal for the action of T. Let $a \in K$ be arbitrary. We have to show that any point $y \in K$ can be approximated as close as we wish by points of the form $T^n a$, $n \geq 0$.

Write a and y as $a = \sum_{j=1}^{N} \omega_j u_j$ and $y = \sum_{j=1}^{N} \eta_j u_j$ respectively, where $\omega = (\omega_1, \dots, \omega_N)$ and $\eta = (\eta_1, \dots, \eta_N)$ belong to Γ. For any integer $n \geq 0$, we have $T^n a = \sum_{j=1}^{N} \gamma_j^n \omega_j u_j$. Since the N-tuples $\gamma^n = (\gamma_1^n, \dots, \gamma_N^n)$, $n \geq 0$, are dense in Γ (this follows from the fact that since the group $\mathbb{T}^N$ is compact, the closed semigroup generated by γ is in fact a group), there exists an integer $n \geq 0$ such that γ^n is as close as we wish to $\omega^{-1}\eta = (\omega_1^{-1}\eta_1, \dots, \omega_N^{-1}\eta_N)$. Then $T^n a$ is as close as we wish to y, as required. □

We can now state our criterion for topological weak mixing.

PROPOSITION 5.7. *Let $T \in \mathfrak{B}(X)$, and let Z be a T-invariant subset of X consisting of uniformly recurrent points. Assume that for every non-empty open subset V of X and every neighborhood W of 0 in X, one can find a vector $z \in Z$ and an integer $n \geq 0$ such that $z \in W$ and $T^n z \in V$. Then T is topologically weakly mixing and Z is dense in X.*

PROOF. That Z is dense in X is clear from the assumption since Z is T-invariant. In order to prove that T is topologically weakly mixing, we are going to show that $\mathcal{N}_T(U, W) \cap \mathcal{N}_T(W, V)$ is non-empty for any non-empty open subsets U, V of X and any neighborhood W of 0 in X. The so-called *three open sets condition* (see for example [**8**, Ch. 4]) will then imply that T is topologically weakly mixing.

Let us first show that for any neighborhood W of 0 and any open set $V \neq \emptyset$, the set $\mathcal{N}_T(W, V)$ has bounded gaps. Choose a vector $z \in Z$ and an integer $n \geq 1$ such that z belongs to W and $T^n z$ belongs to V, and then an open neighborhood O of z such that $O \subseteq W$ and $T^n(O) \subseteq V$. Since z is uniformly recurrent, the set $\mathcal{N}_T(z, O)$ has bounded gaps. Then $n + \mathcal{N}_T(z, O)$ has bounded gaps as well, and the result follows since $n + \mathcal{N}_T(z, O) \subseteq \mathcal{N}_T(z, V) \subseteq \mathcal{N}_T(W, V)$.

We show next that $\mathcal{N}_T(U, W)$ is non-empty for any non-empty open subset U of X and any neighborhood W of 0 in X. By assumption, one can find a point $x \in W$ and an integer $n \geq 1$ such that x is a (uniformly) recurrent point of T and $u := T^n x$ lies in U. Since x is recurrent, one can find an integer $p > n$ such that $T^p x$ lies in W. Then $T^{p-n} u = T^p x$ belongs to W, and hence $\mathcal{N}_T(U, W)$ is non-empty.

It is now easy to conclude the proof: all the sets $\mathcal{N}_T(W, V)$ have bounded gaps, and all the sets $\mathcal{N}_T(U, W)$ contain arbitrarily large intervals, because they are all non-empty and the sets W are neighborhoods of the fixed point 0. Hence any set of the form $\mathcal{N}_T(W, V)$ meets any other set of the form $\mathcal{N}_T(U, W)$. □

As a first consequence of Proposition 5.7, we show that for operators having a dense set of uniformly recurrent points, topological weak mixing is equivalent to some formally much weaker "transitivity-like" properties.

COROLLARY 5.8. *Let $T \in \mathfrak{B}(X)$, and assume that uniformly recurrent points of T are dense in X. The following assertions are then equivalent:*

(a) *T is topologically weakly mixing;*
(b) *for every non-empty open subset V of X and every neighborhood W of 0 in X, the set $\mathcal{N}_T(W, V)$ is non-empty;*
(c) *for every non-empty open subset U of X and every neighborhood W of 0 in X, the set $\mathcal{N}_T(U, W)$ is non-empty.*

PROOF. Obviously, (a) implies both (b) and (c). Moreover, since the uniformly recurrent points of T are assumed to be dense in X, it follows at once from Proposition 5.7 (taking as Z the set of all uniformly recurrent points for T) that (b) implies (a). So it remains to show that (c) implies (b).

Suppose that (c) holds true, and let U and W be two non-empty open subsets of X with $0 \in W$. By (c) and since uniformly recurrent points of T are dense in X, one can find a uniformly recurrent point $u \in U$ and an integer $n \geq 0$ such that $z := T^n u$ belongs to W. Since u is in

particular recurrent, one can find $p > n$ such that $T^p u$ belongs to U. Then $T^{p-n} z = T^p u$ lies in U, so that $\mathcal{N}_T(W, U)$ is non-empty. □

As another consequence of Proposition 5.7, we now state a criterion for topological weak mixing which formally resembles the ergodicity criterion stated as Proposition 4.15 above.

Corollary 5.9. *Let $T \in \mathfrak{B}(X)$. Assume that for each non-empty open subset Ω of X such that $T^{-1}(\Omega) \subseteq \Omega$ and each neighborhood W of 0, one can find a minimal T-invariant compact set K such that $K \cap \Omega \cap W$ is non-empty. Then T is topologically weakly mixing.*

Proof. By Lemma 5.6, every point z in the above compact set K is uniformly recurrent for T. We apply Proposition 5.7, taking as Z the set of all uniformly recurrent points for T. Let V be a non-empty open subset of X. Applying the assumption of Corollary 5.9 to $\Omega := \bigcup_{n \geq 0} T^{-n}(V)$, we immediately see that the assumption of Proposition 5.7 is satisfied, and hence that T is topologically weakly mixing. □

Corollary 5.10. *Let $T \in \mathfrak{B}(X)$, where X is a* complex *Banach space. Assume that for every $x \in X$ and $\varepsilon > 0$, there exist $z \in \operatorname{span} \mathcal{E}(T)$ and $n \geq 0$ such that $\|z\| < \varepsilon$ and $\|T^n z - x\| < \varepsilon$. Then T is topologically weakly mixing and $\operatorname{span} \mathcal{E}(T)$ is dense in X.*

Proof. This follows from Lemma 5.6 and Proposition 5.7 applied with $Z := \operatorname{span} \mathcal{E}(T)$. □

Our last corollary is the result on which we will elaborate to state our $\mathcal{U}$-frequent hypercyclicity and frequent hypercyclicity criteria.

Corollary 5.11. *Let $T \in \mathfrak{B}(X)$. Assume that for every $x \in X$ and $\varepsilon > 0$, there exist $z \in \operatorname{Per}(T)$ and $n \geq 1$ such that $\|z\| < \varepsilon$ and $\|T^n z - x\| < \varepsilon$. Then T is chaotic.*

Proof. Since periodic points are uniformly recurrent, exactly the same proof as that of the previous corollary shows that T is topologically weakly mixing and has a dense set of periodic points. □

Corollary 5.11 can also be proved by a more constructive argument. Since the flexibility of this alternative proof will prove extremely important below for the proofs of some ($\mathcal{U}$-) frequent hypercyclicity criteria, we present it here.

Direct proof of Corollary 5.11. Before starting the proof, we note that the assumption of Corollary 5.11 obviously implies that $\|T\| > 1$ and that $\operatorname{Per}(T)$ is dense in X.

The first step of the proof is to observe that the assumption of Corollary 5.11 can be reinforced as follows:

Claim 5.12. For any $x \in X$, any $\varepsilon > 0$, and any integers $N, M \geq 1$, there exist $z \in \operatorname{Per}(T)$ and $n \geq 1$ with $n = M \pmod N$ such that $\|z\| < \varepsilon$ and $\|T^n z - x\| < \varepsilon$.

Proof of Claim 5.12. Assuming (as we may) that x is non-zero, choose $z' \in \operatorname{Per}(T)$ and $n' \geq 1$ such that $\|z'\| < \varepsilon'$ and $\|T^{n'} z' - x\| < \varepsilon$, where $\varepsilon' > 0$ has to be specified. Then $\|T\|^{n'} \|z'\| \geq \|x\| - \varepsilon'$, so that $\|T\|^{n'} \geq \frac{\|x\| - \varepsilon'}{\varepsilon'}$. It follows that if ε' is small enough, then $n' > N$. Let then $0 \leq r < N$ be an integer such that $n := n' - r$ is equal to $M \pmod N$, and set $z = T^r z'$. Then $\|z\| < \|T\|^r \varepsilon' \leq \|T\|^N \varepsilon'$, so $\|z\| < \varepsilon$ if ε' is small enough. Also, $T^n z = T^{n'-r} T^r z' = T^{n'} z'$, so that $\|T^n z - x\| < \varepsilon' \leq \varepsilon$. □

Let now $(x_j)_{j \geq 1}$ be a dense sequence of vectors of X belonging to $\operatorname{Per}(T)$. A straightforward induction shows that there exists a sequence $(z_j)_{j \geq 1}$ of elements of $\operatorname{Per}(T)$, as well as a strictly increasing sequence $(n_j)_{j \geq 1}$ of integers such that, for every $j \geq 1$,

(i) $\|z_j\| < 2^{-j} \|T\|^{-n_{j-1}}$;
(ii) $\|T^{n_j} z_j - (x_j - \sum_{i<j} z_i)\| < 2^{-j}$;
(iii) n_j is a multiple of the period of the vector $\sum_{i<j} z_i$.

Now, set

$$z := \sum_{i=1}^{\infty} z_i,$$

which is well-defined by (i). Let us now show that z is a hypercyclic vector for T. For every $j \geq 1$, we have $T^{n_j}\Big(\sum_{i<j} z_i\Big) = \sum_{i<j} z_i$ by (iii), so that

$$T^{n_j} z - x_j = \sum_{i<j} z_i + T^{n_j} z_j + \sum_{i>j} T^{n_j} z_i - x_j.$$

It follows that

$$\begin{aligned}\|T^{n_j} z - x_j\| &\leq \Big\|T^{n_j} z_j - \Big(x_j - \sum_{i<j} z_i\Big)\Big\| + \sum_{i>j} \|T\|^{n_j} \|z_i\| \\ &< 2^{-j} + \sum_{i>j} \|T\|^{n_j} 2^{-i} \|T\|^{-n_{i-1}} \leq 2^{-(j-1)} \qquad \text{by (i) and (ii).}\end{aligned}$$

This terminates our direct proof of Corollary 5.11. □

To conclude this section, we prove a slightly stronger version of Corollary 5.11.

Corollary 5.13. *Let X_0 be a subset of X with dense linear span. Suppose that for every $x_0 \in X_0$ and every $\varepsilon > 0$, there exist $z \in \mathrm{Per}(T)$ and $n \geq 1$ such that $\|z\| < \varepsilon$ and $\|T^n z - x_0\| < \varepsilon$. Then T is chaotic.*

Proof. It is suffices to show that the assumption of Corollary 5.13 can be extended from vectors x_0 of X_0 to arbitrary vectors x of X: once this is done, Corollary 5.11 applies.

Since the linear span of X_0 is dense in X, it suffices to consider vectors x of the form $x = \sum_{k=1}^{r} a_k x_k$, where $x_k \in X_0$ and $a_k \in \mathbb{K} \setminus \{0\}$ for every $1 \leq k \leq r$. An induction on k, $1 \leq k \leq r$, allows us to construct, using (an obvious modification of) Claim 5.12, vectors $z_k \in \mathrm{Per}(T)$ and integers l_k, $1 \leq k \leq r$, with the following properties:

(i) $\|z_k\| < \dfrac{\varepsilon}{r\,|a_k|}$;

(ii) $\|T^{l_1+\cdots+l_k} z_k - x_k\| < \dfrac{\varepsilon}{r\,|a_k|}$;

(iii) l_k is a multiple of $d_{k-1} = \displaystyle\prod_{i=1}^{k-1} \mathrm{per}_T(z_i)$.

Now, set $z := \sum_{k=1}^{r} a_k z_k$: this z is clearly a periodic vector for T, with $\|z\| < \varepsilon$ by (i). Set also $n := l_1 + \cdots + l_r$. We have

$$T^n z = \sum_{k=1}^{r} a_k T^n z_k = \sum_{k=1}^{r} a_k T^{l_1+\cdots+l_k} z_k \qquad \text{by (iii),}$$

so that

$$\|T^n z - x\| \leq \sum_{k=1}^{r} |a_k|\, \|T^{l_1+\cdots+l_k} z_k - x_k\| < \varepsilon \qquad \text{by (ii).}$$

This concludes the proof of Corollary 5.13. □

In the next two sections, we will give two "variations" of Corollary 5.11, where we show that if the assumption $\|T^n z - x\| < \varepsilon$ is replaced by the requirement that the orbit of z approximates that of x during a sufficiently large time, *i.e.* $\|T^{n+k} z - T^k x\| < \varepsilon$ for a large number of indices k, then T is $\mathcal{U}$-frequently hypercyclic, and even sometimes frequently hypercyclic.

3. A criterion for $\mathcal{U}$-frequent hypercyclicity

Here is the version of our criterion for $\mathcal{U}$-frequent hypercyclicity that we will use in Chapter 6.

THEOREM 5.14. *Let $T \in \mathfrak{B}(X)$. Assume that there exist a dense linear subspace X_0 of X with $T(X_0) \subseteq X_0$ and $X_0 \subseteq \mathrm{Per}(T)$, and a constant $\alpha \in (0,1)$ such that the following property holds true: for every $x \in X_0$ and every $\varepsilon > 0$, there exist $z \in X_0$ and $n \geq 1$ such that*

(1) $\|z\| < \varepsilon$;
(2) $\|T^{n+k}z - T^k x\| < \varepsilon$ *for every* $0 \leq k \leq n\alpha$.

Then T is chaotic and $\mathcal{U}$-frequently hypercyclic.

PROOF. Since X_0 is dense in X, the periodic points of T are dense in X. So we only have to show that T is $\mathcal{U}$-frequently hypercyclic.

As in the direct proof of Corollary 5.11 given above, we first show that given $N \geq 1$, the integer n appearing in the assumption of Theorem 5.14 can be supposed to be a multiple of N. Given $x \in X_0$ and $\varepsilon > 0$, there exist $z' \in X_0$ and $n' \geq 1$ such that $\|z'\| < \varepsilon'$ and $\|T^{n'+k}z' - T^k x\| < \varepsilon'$ for every $0 \leq k \leq \alpha n'$, where $\varepsilon' > 0$ has to be specified. Taking ε' small enough, we can assume (see the proof of Claim 5.12) that $n' > N$. Let then $0 \leq r < N$ be such that $n := n' - r$ is a multiple of N, and set $z = T^r z'$. Then z belongs to X_0 because $T(X_0) \subseteq X_0$. Also $\|z\| < \|T\|^N \varepsilon' < \varepsilon$ if ε' is small enough; and $T^{n+k}z = T^{n'+k}z'$, so that $\|T^{n+k}z - T^k x\| < \varepsilon$ for every $0 \leq k \leq \alpha n'$. Since $n \leq n'$, the inequality holds true *a fortiori* for every $0 \leq k \leq \alpha n$.

Let now $(x_l)_{l\geq 1}$ be a dense sequence of vectors of X_0, and let $(I_l)_{l\geq 1}$ be a partition of $\mathbb{N}$ into infinite sets. We define a sequence $(y_j)_{j\geq 1}$ of vectors of X_0 by setting $y_j = x_l$ for every $j \in I_l$. In other words, the sequence $(y_j)_{j\geq 1}$ enumerates infinitely many times each point of the dense sequence $(x_l)_{l\geq 1}$, and I_l denotes the set of all indices $j \geq 1$ such that $y_j = x_l$.

By induction on $j \geq 1$, we construct a sequence $(z_j)_{j\geq 1}$ of vectors of X_0 and a strictly increasing sequence $(n_j)_{j\geq 1}$ of integers such that

(i) $\|T^k z_j\| < 2^{-j}$ for every $0 \leq k \leq (1+\alpha)n_{j-1}$;
(ii) $\|T^{n_j+k}z_j - T^k(y_j - \sum_{i<j} z_i)\| < 2^{-j}$ for every $0 \leq k \leq \alpha n_j$;
(iii) n_j is a multiple of the period of the vector $\sum_{i<j} z_i$.

We set $z := \sum_{i\geq 1} z_i$, which is well-defined by (i), and prove that z is a $\mathcal{U}$-frequently hypercyclic vector for T. Fix $l \geq 1$. For every $j \in I_l$ and every $k \geq 0$, we have by (iii)

$$T^{n_j+k}z - x_l = T^k\Big(\sum_{i<j} z_i\Big) + T^{n_j+k}z_j - y_j + \sum_{i>j} T^{n_j+k}z_i.$$

Moreover, if k is a multiple of $\mathrm{per}(x_l) = \mathrm{per}(y_j)$, say $k = m\,\mathrm{per}(x_l)$, we can write

$$T^{m\,\mathrm{per}(x_l)}\Big(\sum_{i<j} z_i\Big) - y_j = T^{m\,\mathrm{per}(x_l)}\Big(\sum_{i<j} z_i - y_j\Big).$$

Hence we obtain that for any $m \geq 0$,

$$\|T^{n_j+m\,\mathrm{per}(x_l)}z - x_l\| \leq \Big\|T^{n_j+m\,\mathrm{per}(x_l)}z_j - T^{m\,\mathrm{per}(x_l)}\Big(y_j - \sum_{i<j} z_i\Big)\Big\| + \sum_{i>j} \|T^{n_j+m\,\mathrm{per}(x_l)}z_i\|.$$

By (i) and (ii), it follows that for every $j \geq 1$ and every $0 \leq m \leq \alpha n_j/\mathrm{per}(x_l)$, we have

$$\|T^{n_j+m\,\mathrm{per}(x_l)}z - x_l\| < 2^{-j} + \sum_{i>j} 2^{-i} = 2^{-(j-1)}.$$

We deduce from this inequality that for any $\varepsilon > 0$,

$$\begin{aligned}\overline{\mathrm{dens}}\,\mathcal{N}_T(z, B(x_l, \varepsilon)) &\geq \limsup_{j \in I_l} \frac{1}{(1+\alpha)n_j} \#\big\{n_j + m\,\mathrm{per}(x_l)\,;\, 0 \leq m \leq \alpha n_j/\mathrm{per}(x_l)\big\} \\ &\geq \frac{\alpha}{(1+\alpha)\mathrm{per}(x_l)} > 0.\end{aligned}$$

Hence z is a $\mathcal{U}$-frequently hypercyclic vector for T, and Theorem 5.14 follows. □

Remark 5.15. The proof of Theorem 5.14 shows that T admits $\mathcal{U}$-frequently hypercyclic vectors $z \in X$ with the property that for every $l \geq 1$, there exists $\delta_l > 0$ such that

$$\overline{\mathrm{dens}}\,\mathcal{N}_T(z, B(x_l, \varepsilon)) \geq \delta_l \quad \text{for every } \varepsilon > 0$$

(the point being that δ_l does not depend on ε). Hence there exists, for every $l \geq 1$, a sequence $(n_{k,l})_{k\geq 1}$ of integers of positive upper density such that $T^{n_{k,l}}z$ tends to x_l as k tends to infinity. As x_l is a periodic point for T, there is no contradiction in this (see [**30**, Rem. 4.8]).

3.1. Uniform recurrence, almost periodic points and $\mathcal{U}$-frequent hypercyclicity. Even if this is not quite clear at first sight, the following result generalizes Theorem 5.14 (see Corollary 5.19 below).

Theorem 5.16. *Let $T \in \mathfrak{B}(X)$. Assume that the uniformly recurrent points of T are dense in X, and that there exists $\alpha > 0$ such that the following property holds true: for any $\varepsilon > 0$ and any non-empty open subset O of X, one can find $x \in O$ and an arbitrarily large integer n such that $\|T^{n+k}x\| < \varepsilon$ for every $0 \leq k \leq \alpha n$. Then T is $\mathcal{U}$-frequently hypercyclic.*

Proof. It is enough to show that for any non-empty open subset V of X, there exists a constant $\alpha_V > 0$ such that, for every $N \geq 1$, the open set

$$G_{V,N} := \Big\{u \in X;\ \exists m \geq N\ :\ \#\{i \leq m;\ T^i u \in V\} \geq \alpha_V\, m\Big\}$$

is dense in X. Indeed, if $(V_q)_{q\geq 1}$ is a countable basis of open subsets of X, the density of each of the sets $G_{V_q,N}$ implies that $G := \bigcap_{N,q} G_{V_q,N}$ is a dense G_δ subset of X consisting of $\mathcal{U}$-frequently hypercyclic vectors for T.

So let V be a non-empty open subset of X. Choose a uniformly recurrent point $v \in V$, and $\varepsilon > 0$ such that $B(v, \varepsilon) \subseteq V$. Since v is uniformly recurrent, the set

$$D_V := \{k \geq 1;\ \|T^k v - v\| < \varepsilon/2\}$$

has bounded gaps. Observe that we may call this set D_V since ε depends on v and V, and v depends on V. So there exist a constant $c_V > 0$ and an integer $M_V \geq 1$ such that $\#(D_V \cap J) \geq c_V\, \#J$ for all intervals J of $\mathbb{N}$ of length at least M_V. We set

$$\alpha_V := \frac{c_V \alpha}{1+\alpha}\,,$$

and we show that with this choice of α_V, all the open sets $G_{V,N}$, $N \geq 1$, are dense in X. Let us fix $N \geq 1$, and a non-empty open subset U of X. We have to show that $G_{V,N} \cap U$ is non-empty. Set $O := U - v$, which is a non-empty open subset of X. By assumption, one can find $x \in O$ and an integer $n \geq N$ such that $\alpha n \geq M_V$ and $\|T^{n+k}x\| < \varepsilon/2$ for every $0 \leq k \leq \alpha n$. Then $u := x + v$ belongs to U and our aim is to show that u belongs to $G_{V,N}$. For any $0 \leq k \leq \alpha n$, we have

$$\|T^{n+k}u - v\| \leq \|T^{n+k}x\| + \|T^{n+k}v - v\| < \varepsilon/2 + \|T^{n+k}v - v\|\,;$$

in other words,

$$\forall i \in [n, (1+\alpha)n]\ :\ \|T^i u - v\| < \varepsilon/2 + \|T^i v - v\|.$$

Moreover, since the interval $J = [n, (1+\alpha)n]$ has length at least M_V, we know that $\#\,(J \cap D_V) \geq c_V \#J$, *i.e.*

$$\#\,\big\{i \in [n, (1+\alpha)n];\ \|T^i v - v\| < \varepsilon/2\big\} \geq c_V\, \alpha n.$$

It follows that

$$\#\{1 \le i \le (1+\alpha)n;\ \|T^i u - v\| < \varepsilon\} \ge c_V\, \alpha n\,,$$

and hence that u belongs to $G_{V,N}$. □

We now state and prove a few consequences of Theorem 5.16. For the first one, we need a definition: if T is a bounded operator on X, a point $x \in X$ is said to be *almost periodic* for T if, for every $\varepsilon > 0$, the set

$$D_{x,\varepsilon} := \{n \ge 1;\ \forall k \ge 1\ :\ \|T^{n+k}x - T^k x\| < \varepsilon\}$$

has bounded gaps. Thus, periodic points are almost periodic and almost periodic points are uniformly recurrent. The following fact is more interesting.

Example 5.17. Let $T \in \mathfrak{B}(X)$. Then any vector x belonging to $\operatorname{span}\mathcal{E}(T)$ is almost periodic for T.

Proof. We first recall that since x belongs to $\operatorname{span}\mathcal{E}(T)$, it is uniformly recurrent for T (Lemma 5.6); so for any $\eta > 0$, the set $D^\eta := \{n \ge 1;\ \|T^n x - x\| < \eta\}$ has bounded gaps. Now, assuming that x is non-zero, we write x as $x = \sum_{i=1}^r u_i$, where $u_1, \dots, u_r$ belong to $\mathcal{E}(T)$ and are linearly independent. Then the restriction of T to the finite-dimensional subspace $E := \operatorname{span}(u_1, \dots, u_r)$ is *power-bounded*, being diagonalizable with only unimodular eigenvalues. Since E contains the T-orbit of x, it follows that there exists a finite constant C such that, for every $k \ge 0$,

$$\|T^{n+k}x - T^k x\| = \|T^k(T^n x - x)\| \le C\,\|T^n x - x\|.$$

So the set D^η with $\eta := \varepsilon/C$ is contained in $D_{x,\varepsilon}$, which concludes the proof. □

From Theorem 5.16, we now deduce

Corollary 5.18. *Let $T \in \mathfrak{B}(X)$. Assume that there exists some constant $\alpha > 0$ such that, for every non-empty open subset O of X and every $\varepsilon > 0$, there exists an almost periodic point $x \in O$ with the following property: for any $\eta > 0$, one can find $z \in X$ and $n \ge 1$ such that $\|z\| < \eta$ and $\|T^{n+k}z - T^k x\| < \varepsilon$ for all $0 \le k \le \alpha n$. Then T is $\mathcal{U}$-frequently hypercyclic.*

Proof. By assumption, the uniformly recurrent points are certainly dense in X. In order to check the second assumption in Theorem 5.16, let us fix a non-empty open subset O of X and $\varepsilon > 0$. Let us choose a non-zero almost periodic point $x_0 \in O$ satisfying the assumption of Corollary 5.18, and then an integer $M \ge 1$ such that any interval of $\mathbb{N}$ of length M contains a point of the set

$$D := \{n \ge 1;\ \forall k \ge 1\ :\ \|T^{n+k}x_0 - T^k x_0\| < \varepsilon/2\}.$$

Now, choose $z' \in X$ with $\|z'\|$ arbitrarily small and an arbitrarily large integer $n' > M$ such that $\|T^{n'+k}z' - T^k x_0\| < \varepsilon/2$ for every $0 \le k \le \alpha n'$ (since x_0 is non-zero, one can ensure that n' be arbitrarily large by taking $\|z'\|$ small enough.) Having fixed z' and n' in this way, we can pick $0 \le p \le M$ such that $n := n' - p$ belongs to D. Then $z := T^p z'$ has arbitrarily small norm, so $x := x_0 - z$ belongs to O. Moreover, since $n \le n'$ and $T^n z = T^{n'} z'$, we have for every $0 \le k \le \alpha n$:

$$\|T^{n+k}x\| \le \|T^{n+k}x_0 - T^k x_0\| + \|T^k x_0 - T^{n'+k}z'\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

The assumptions of Theorem 5.16 are thus satisfied. □

Here is an immediate consequence of Corollary 5.18, whose statement is a bit less convoluted. This result shows in particular that in Theorem 5.14, one can replace the assumption that the vectors of X_0 are periodic by the assumption that they are almost periodic. Moreover, it also shows that it was in fact unnecessary to assume that X_0 was a *linear subspace* of X.

Corollary 5.19. *Let $T \in \mathfrak{B}(X)$. Assume that there exist a dense subset X_0 of X consisting of almost periodic points for T and a constant $\alpha \in (0,1)$ such that, for every $x \in X_0$ and every $\varepsilon > 0$, one can find $z \in X_0$ and $n \ge 1$ satisfying $\|z\| < \varepsilon$ and $\|T^{n+k}z - T^k x\| < \varepsilon$ for all $0 \le k \le \alpha n$. Then T is $\mathcal{U}$-frequently hypercyclic.*

Since Corollary 5.19 is so similar to Theorem 5.14, it is natural to ask whether one can provide a "constructive" proof of it resembling that of Theorem 5.14. We do so now, with the additional assumption that the above dense set X_0 is a linear subspace of X.

Constructive proof of Corollary 5.19. Define sequences $(x_l)_{l\geq 1}$, $(I_l)_{l\geq 1}$ and $(y_j)_{j\geq 1}$ as in the beginning of the proof of Theorem 5.14. By induction, we construct a sequence $(z_j)_{j\geq 1}$ of vectors of X_0 and an increasing sequence of integers $(n_j)_{j\geq 1}$ such that

(i) $\|T^k z_j\| < 2^{-j}$ for every $0 \leq k \leq (1+\alpha)n_{j-1}$;
(ii) $\|T^{n_j+k} z_j - T^k(y_j - \sum_{i<j} z_i)\| < 2^{-j}$ for every $0 \leq k \leq \alpha n_j$;
(iii') $\left\|T^{n_j+k}\big(\sum_{i<j} z_i\big) - T^k\big(\sum_{i<j} z_i\big)\right\| < 2^{-j}$ for all $k \geq 0$.

Let $j \geq 1$. Assume that the construction has been carried out up to the step $j-1$, and let us construct n_j and z_j. Since the vector $\sum_{i<j} z_i$ lies in X_0, it is almost periodic; so one can find an integer $M \geq 1$ such that every interval of $\mathbb{N}$ of length M contains a point of the set

$$D := \left\{ n \geq 1;\ \forall k \geq 1\ :\ \left\|T^{n+k}\Big(\sum_{i<j} z_i\Big) - T^k\Big(\sum_{i<j} z_i\Big)\right\| < 2^{-j} \right\}.$$

By assumption on T, there exists $n' \in \mathbb{N}$ with $n' > n_{j-1} + M$ and $z' \in X_0$ such that

- $\|T^k z'\| < 2^{-j}$ for every $0 \leq k \leq (1+\alpha)n_{j-1} + M$;
- $\|T^{n'+k} z' - T^k(y_j - \sum_{i<j} z_i)\| < 2^{-j}$ for every $0 \leq k \leq \alpha n'$.

Choose now $0 \leq p \leq M$ such that $n_j := n' - p$ belongs to the set D, and let $z_j := T^p z'$. Then (iii') holds true by the choice of n_j, (i) clearly holds true, and (ii) holds true as well because $T^{n_j+k} z_j = T^{n'+k} z'$ for every $k \geq 0$ and every $n' \geq n_j$. This concludes the inductive step.

Let us prove that the vector $z := \sum_{i\geq 1} z_i$ is a $\mathcal{U}$-frequently hypercyclic vector for T. This time, we write for every $l \geq 1$, $j \in I_l$ and $k \geq 0$:

$$\begin{aligned} T^{n_j+k} z - x_l &= T^{n_j+k}\Big(\sum_{i<j} z_i\Big) - T^k\Big(\sum_{i<j} z_i\Big) + T^{n_j+k} z_j - T^k\Big(y_j - \sum_{i<j} z_i\Big) \\ &\quad + T^k x_l - x_l + \sum_{i>j} T^{n_j+k} z_i. \end{aligned}$$

Since n_j belongs to D, we deduce that if $0 \leq k \leq \alpha n_j$, then

$$\|T^{n_j+k} z - x_l\| \leq 2^{-j} + 2^{-j} + \|T^k x_l - x_l\| + 2^{-j} = 3 \cdot 2^{-j} + \|T^k x_l - x_l\|.$$

Now, since x_l is uniformly recurrent for T, the set

$$D_{l,\varepsilon} := \{k \geq 1;\ \|T^k x_l - x_l\| < \varepsilon/2\}$$

has bounded gaps. So one can find a constant $c_{l,\varepsilon} > 0$ such that $\#(D_{l,\varepsilon} \cap J) \geq c_{l,\varepsilon}\, \#J$ for all sufficiently large intervals $J \subseteq \mathbb{N}$. From this, it follows that if j is large enough, then

$$\#\big\{n_j \leq i \leq (1+\alpha)n_j;\ \|T^i z - x_l\| < \varepsilon\big\} \geq c_{l,\varepsilon}\, \alpha n_j,$$

and hence that

$$\overline{\mathrm{dens}}\, \mathcal{N}_T(z, B(x_l, \varepsilon)) \geq c_{l,\varepsilon} \limsup_{j\to\infty} \frac{\alpha n_j}{(1+\alpha)n_j} = \frac{c_{l,\varepsilon}\alpha}{1+\alpha}.$$

□

Finally, here is a somewhat unexpected consequence of Theorem 5.16.

Corollary 5.20. *Let $T \in \mathfrak{B}(X)$. Assume that the uniformly recurrent points of T are dense in X, and that there also exists a dense set of points $x \in X$ such that $T^i x \to 0$ as $i \to \infty$. Then T is $\mathcal{U}$-frequently hypercyclic.*

Proof. This follows immediately from Theorem 5.16. □

REMARK 5.21. Corollary 5.20 shows in particular that if a unilateral weighted backward shift on $\ell^p(\mathbb{N})$ or $c_0(\mathbb{N})$ has a dense set of uniformly recurrent points, then it is $\mathcal{U}$-frequently hypercyclic. However, a much stronger result is true: *if a weighted backward shift has just a single non-zero uniformly recurrent point, then it is in fact chaotic and frequently hypercyclic.*

PROOF. Let $B_{\boldsymbol{\omega}}$ be a weighted backward shift on $X = c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$, $1 \leq p < \infty$, associated to the weight sequence $\boldsymbol{\omega} = (\omega_k)_{k\geq 1}$, and assume that $B_{\boldsymbol{\omega}}$ has a non-zero uniformly recurrent point $x \in X$. By the classical Frequent Hypercyclicity/Chaoticity Criterion (see [**8**] or [**32**]), it is enough to show that the sequence $\left(\prod_{k=1}^n \omega_k^{-1}\right)$ belongs to the space X. We check this in the case where $X = c_0(\mathbb{N})$, the ℓ_p case being similar. So we have to show that $\prod_{k=1}^n \omega_k \to \infty$ as $n \to \infty$. Since $x = \sum_{k\geq 1} x_k e_k$ is non-zero, there exists $k_0 \geq 1$ such that $|x_{k_0}| > 0$. Let $0 < \varepsilon < |x_{k_0}|/2$. Since x is uniformly recurrent for $B_{\boldsymbol{\omega}}$, there exists a strictly increasing sequence of integers $(n_j)_{j\geq 1}$ and a positive integer M such that $n_{j+1} - n_j \leq M$ and $\|B_{\boldsymbol{\omega}}^{n_j} x - x\| < \varepsilon$ for every $j \geq 1$. In particular, we have

$$\left| \left(\prod_{k=k_0+1}^{k_0+n_j} \omega_k \right) x_{k_0+n_j} - x_{k_0} \right| < \varepsilon$$

and hence

$$\prod_{k=k_0+1}^{k_0+n_j} |\omega_k| > \frac{\varepsilon}{|x_{k_0+n_j}|}.$$

Since x belongs to $c_0(\mathbb{N})$, we deduce that

$$\prod_{k=k_0+1}^{k_0+n_j} |\omega_k| \to \infty \quad \text{as } j \to \infty.$$

Moreover, if n is a sufficiently large integer, there exists an integer $j \geq 1$ such that $k_0 + n_{j-1} \leq n < k_0 + n_j$. Since $n_{j+1} - n_j \leq M$, this implies that

$$\prod_{k=k_0+1}^{n} |\omega_k| \geq \frac{\prod_{k=k_0+1}^{k_0+n_j} |\omega_k|}{\|w\|_\infty^M},$$

and this concludes the proof. □

REMARK 5.22. A natural question that comes to mind in view of Corollary 5.20 is whether any *chaotic and topologically mixing* operator has to be $\mathcal{U}$-frequently hypercyclic. We will prove in Section 4 that it is not the case.

3.2. More about $\mathcal{U}$-frequent hypercyclicity and $c(T)$. Corollary 5.20 says in essence that lots of uniformly recurrent points plus lots of orbits tending to 0 imply $\mathcal{U}$-frequent hypercyclicity. In the same spirit, we now prove the following result, which again may look rather surprising at first sight.

THEOREM 5.23. *Let $T \in \mathfrak{B}(X)$ be hypercyclic, and assume that T has a dense set of uniformly recurrent points. Then T is $\mathcal{U}$-frequently hypercyclic if and only if $c(T) > 0$.*

PROOF. One implication is clear: if T is $\mathcal{U}$-frequently hypercyclic, then $c(T) > 0$ by the very definition of $c(T)$.

Conversely, assume that $c(T) > 0$. Then there is a comeager set $G \subseteq X$ such that, for every $x \in G$, one can find a subset D_x of $\mathbb{N}$ with $\overline{\operatorname{dens}}\, D_x \geq c(T)$ such that $\|T^n x\| \to 0$ as $n \to \infty$ along D_x (see [**30**, Prop. 4.7]). The key step of the proof lies in the next fact, where we use for convenience the following notation: if V is an open subset of X and $B = B(u, \varepsilon)$ is an open ball of X, we write $B \prec V$ if there exists $\varepsilon' > \varepsilon$ such that $B(u, \varepsilon') \subseteq V$.

CLAIM 5.24. Let $x_0 \in X$ be a uniformly recurrent point for T, and let V be an open neighborhood of x_0. Let also B be an open ball with center x_0 such that $B \prec V$, and choose an integer N such that every interval $I \subseteq \mathbb{N}$ of cardinality at least N intersects $\mathcal{N}_T(x_0, B)$. Then, for every $y \in x_0 + G$, we have $\overline{\text{dens}}\,\mathcal{N}_T(y, V) \geq c(T)/N$.

PROOF OF CLAIM 5.24. Write $y = x_0 + x$, with $x \in G$. Since $\overline{\text{dens}}\, D_x \geq c(T)$ and

$$\bigcup_{k=0}^{N-1} \big(\mathcal{N}_T(x_0, B) - k\big) = \mathbb{N}$$

by the choice of N, one can find $k \in \{0, \ldots, N-1\}$ such that $\overline{\text{dens}}\,\big(D_x \cap (\mathcal{N}_T(x_0, B) - k)\big) \geq c(T)/N$; equivalently, $\overline{\text{dens}}\,\big((D_x + k) \cap \mathcal{N}_T(x_0, B)\big) \geq c(T)/N$. Now, we have $T^n y = T^n x_0 + T^n x$ for all $n \in \mathbb{N}$, and $T^n x \to 0$ as $n \to \infty$ along $D_x + k$. Since $B \prec V$, it follows that all but finitely many integers $n \in (D_x + k) \cap \mathcal{N}_T(x_0, B)$ belong to $\mathcal{N}_T(y, V)$, and hence that $\mathcal{N}_T(y, V) \geq c(T)/N$. □

It is now easy to conclude the proof of Theorem 5.23. Let $(V_p)_{p \geq 1}$ be a countable basis of (non-empty) open sets for X. Choosing for each $p \geq 1$ a uniformly recurrent point $x_p \in V_p$ and an open ball B_p with center x_p such that $B_p \prec V_p$, we see that the following holds true: for each $p \geq 1$, there is an integer N_p and a comeager set $G_p \subseteq X$ (namely, $G_p = x_p + G$) such that $\overline{\text{dens}}\,\mathcal{N}_T(z, V_p) \geq c(T)/N_p$ for every $z \in G_p$. Then every vector z in the comeager set $G_\infty := \bigcap_{p \geq 1} G_p$ is a $\mathcal{U}$-frequently hypercyclic vector for T. □

REMARK 5.25. In Theorem 5.23, one cannot replace the assumption of $\mathcal{U}$-frequent hypercyclicity with that of frequent hypercyclicity. Indeed, as we shall see in Theorem 7.10, there exist operators which are chaotic and $\mathcal{U}$-frequently hypercyclic (hence, hypercyclic with a dense set of uniformly recurrent points and such that $c(T) > 0$) but not frequently hypercyclic.

When the operator T is *chaotic*, the proof of Theorem 5.23 gives a more precise statement, which says that if $c(T) > 0$, then T is "$\mathcal{U}$-frequently hypercyclic with estimates".

COROLLARY 5.26. *Let T be a chaotic operator on X with $c(T) > 0$. For any open set $V \neq \emptyset$ in X, let us denote by $N(V)$ the smallest period of all periodic vectors of T belonging to V. Then there is a comeager set of vectors $z \in X$ such that*

$$\overline{dens}\,\mathcal{N}_T(z, V) \geq \frac{1}{N(V)}\, c(T) \qquad \textit{for every open set } V \neq \emptyset.$$

PROOF. Note that if the point x_0 in Claim 5.24 is a periodic point of T, then one can take as N the period of x_0. Then follow the proof of Theorem 5.23. □

From Corollary 5.26, we immediately deduce

COROLLARY 5.27. *If $T \in \mathfrak{B}(X)$ is chaotic and* ergodic*, then there is a comeager set of vectors $z \in X$ such that $\overline{dens}\,\mathcal{N}_T(z, V) \geq 1/N(V)$ for every non-empty open subset V of X.*

PROOF. This is clear since $c(T) = 1$ for any ergodic operator T. □

REMARK 5.28. All vectors $z \in X$ satisfying the conclusion of Corollary 5.27 are $\mathcal{U}$-frequently hypercyclic for T, but none of them is frequently hypercyclic. Indeed, if V_0 is any open ball centered at 0, then $N(V_0) = 1$, so that $\overline{\text{dens}}\,\mathcal{N}_T(z, V_0) = 1$. Hence, one must have $\underline{\text{dens}}\,\mathcal{N}_T(z, V) = 0$ for any open set V disjoint from V_0.

To conclude this section, we now proceed to prove a generalization of Corollary 5.27 to ergodic operators $T \in \mathfrak{B}(X)$ which are not necessarily chaotic: we require that T belongs to $\text{SPAN}(X)$, *i.e.* that the unimodular eigenvectors of T span a dense subspace of X.

We first need to introduce some notation. To every $\lambda \in \mathbb{T}$, we associate the rotation-invariant measure ν_λ on $\mathbb{T}$ defined as follows:

$$\nu_\lambda = \begin{cases} \dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1} \delta_{\{\lambda^k\}} & \text{if } \lambda^N = 1 \text{ for some } N \geq 1, \\ \text{the normalized Lebesgue measure on } \mathbb{T} \text{ otherwise.} \end{cases}$$

Suppose now that $T \in \mathfrak{B}(X)$, and that $\mathbf{u} = (u_1, u_2, \dots)$ is a finite or infinite sequence of linearly independent eigenvectors associated to unimodular eigenvalues $\lambda_1, \lambda_2, \dots$. Let us denote by $\mathcal{E}^{\mathbf{u}}$ the linear span of the vectors u_j, $j \geq 1$. Any vector $u \in \mathcal{E}^{\mathbf{u}}$ may be written in a unique way as

$$u = \sum_{k=1}^{r} a_k^{\mathbf{u}}(u) u_k,$$

where the coefficients $a_k^{\mathbf{u}}(u)$, $1 \leq k \leq r$, are complex scalars. To any such vector u, we associate the T-invariant measure $\nu_{u,T}^{\mathbf{u}}$ on X defined by

$$\nu_{u,T}^{\mathbf{u}}(A) = \prod_{k=1}^{r} \nu_{\lambda_k}\Big(\big\{(\mu_1, \dots, \mu_r) \in \mathbb{T}^r \,;\, \sum_{k=1}^{r} a_k^{\mathbf{u}}(u)\mu_k u_k \in A\big\}\Big)$$

for every Borel subset A of X. Note that if the unimodular numbers λ_k are not roots of unity (and X is a Hilbert space $\mathcal{H}$), then $\nu_{u,T}^{\mathbf{u}}$ is nothing but the Steinhaus measure associated to u considered in the proof of Lemma 4.16.

Finally, for any open set $V \neq \emptyset$, we define

$$\delta_{V,T}^{\mathbf{u}} := \sup\big\{\nu_{u,T}^{\mathbf{u}}(B(u,\varepsilon)) \,;\, \varepsilon > 0,\ u \in \mathcal{E}^{\mathbf{u}},\ B(u,\varepsilon) \prec V\big\}.$$

As in Fact 5.24, the notation $B(u,\varepsilon) \prec V$ means that $B(u,\varepsilon') \subseteq V$ for some $\varepsilon' > \varepsilon$.

We may now state

Proposition 5.29. *Let T be an ergodic operator on X belonging to* $\mathrm{SPAN}(X)$*, and let $\mathbf{u} = (u_k)_{k\geq 1}$ be an infinite linearly independent sequence of unimodular eigenvectors whose linear span $\mathcal{E}^{\mathbf{u}}$ is dense in X. Then, there exists a comeager subset G of X such that every vector $z \in G$ satisfies*

$$\overline{\mathrm{dens}}\,\mathcal{N}_T(z,V) \geq \delta_{V,T}^{\mathbf{u}} \quad \textit{for every open set } V \neq \emptyset.$$

The key step in the proof of Proposition 5.29 is Lemma 5.30 below.

Lemma 5.30. *Let T be an ergodic operator on X, and let $\mathbf{u} = (u_1, \dots, u_r)$ be a finite sequence of linearly independent unimodular eigenvectors for T. Let also u be a vector belonging to $\mathcal{E}^{\mathbf{u}}$. Given any ε, $\gamma > 0$, there exists a T-invariant measure m on X, ergodic for T and with full support, such that*

$$m\big(B(u,\varepsilon)\big) \geq (1-\gamma)\,\nu_{u,T}^{\mathbf{u}}\big(B(u,(1-\gamma)\varepsilon)\big).$$

Since the sequence $\mathbf{u}$ is fixed, we will remove any reference to it in the proofs of Proposition 5.29 and Lemma 5.30; so we will write for instance $a_k(u)$ instead of $a_k^{\mathbf{u}}(u)$, and so on.

Proof. Let m_0 be an ergodic measure with full support for T. For each $R > 0$, consider the probability measure m_R defined on X by setting

$$m_R(A) = \int_{K_u} m_0\big(R(A-x)\big)\, d\nu_{u,T}(x) \quad \text{for every Borel subset } A \text{ of } X,$$

where $K_u := \big\{\sum_{k=1}^{r} a_k(u)\mu_k u_k,\ \mu_k \in \mathbb{T},\ 1 \leq k \leq r\big\}$. Note that K_u is a compact subset of X and contains the support of the measure $\nu_{u,T}$. It clearly satisfies $T(K_u) = K_u$. The measure m_R thus

defined is T-invariant. Indeed, we have for every Borel subset A of X

$$\begin{aligned} m_R(T^{-1}A) &= \int_{K_u} m_0\big(R(T^{-1}A - x)\big)\, d\nu_{u,T}(x) = \int_{K_u} m_0\big(R(T^{-1}(A - Tx))\big)\, d\nu_{u,T}(x) \\ &= \int_{T(K_u)} m_0\big(R(A - x)\big)\, d\nu_{u,T}(x) = m_R(A). \end{aligned}$$

The next step of the proof is to show that $m_R(\mathrm{HC}(T)) = 1$. Since

$$m_R(\mathrm{HC}(T)) = \int_{K_u} m_0\big(R(\mathrm{HC}(T) - x)\big)\, d\nu_{u,T}(x)$$

and $m_0(\mathrm{HC}(T)) = 1$, it suffices to show that $\mathrm{HC}(T) \subseteq \mathrm{HC}(T) - x$ for every vector $x \in K_u$, *i.e.* that $\mathrm{HC}(T) + x \subseteq \mathrm{HC}(T)$ for every such x. Write x as $x = \sum_{k=1}^{r} a_k \mu_k u_k$, where $a_k = a_k(u)$ and μ_k belongs to $\mathbb{T}$ for every $1 \le k \le r$, and fix a vector $z \in \mathrm{HC}(T)$. In order to show that $z + x$ belongs to $\mathrm{HC}(T)$, we need to show that for every $y \in X$ and every $\delta > 0$, there exists an integer $n \ge 1$ such that $\|T^n(z + x) - y\| < \delta$. Now, a result of Shkarin [**48**] states the following: given a hypercyclic operator S on a Banach space Z, and a compact topological group G generated by an element $g \in G$, the set

$$\big\{(S^n z, g^n),\ n \ge 1\big\}$$

is dense in $Z \times G$ for every vector $z \in Z$ which is hypercyclic for S. We apply this result to the operator T and to the subgroup G of $\mathbb{T}^r$ generated by $g := (\lambda_1, \dots, \lambda_r)$. Since the r-tuple $(1, \dots, 1)$ belongs to G, and $z \in X$ is a hypercyclic vector for T, there exists for every $\delta > 0$ an integer $n \ge 1$ such that $\|T^n z - (y - x)\| < \delta/2$ and $\max_{1 \le k \le r} |a_k| \,.\, \|u_k\| \,.\, |\lambda_k^n - 1| < \delta/(2r)$. It follows that

$$\|T^n x - x\| = \Big\| \sum_{k=1}^{r} a_k \big(\lambda_k^n - 1\big) u_k \Big\| < \frac{\delta}{2},$$

so that $\|T^n(z + x) - y\| < \delta$. This proves our claim, and shows that $m_R(\mathrm{HC}(T)) = 1$ for every $R > 0$.

Let us now estimate from below the quantities $m_R(B(u, \varepsilon))$. By the definition of m_R, we have

$$m_R(B(u, \varepsilon)) \ge \int_{K_u \cap B(u, (1-\gamma)\varepsilon)} m_0(R(B(u, \varepsilon) - x))\, d\nu_{u,T}(x).$$

Now, observe that for every $x \in B(u, (1-\gamma)\varepsilon)$, the set $B(u, \varepsilon) - x$ contains the ball $B(0, \gamma\varepsilon)$. Indeed, every $y \in X$ with $\|y\| < \gamma\varepsilon$ can be written as $y = u + (y - u + x) - x$ with $\|y - u + x\| < \gamma\varepsilon + (1-\gamma)\varepsilon = \varepsilon$. It follows that

$$\begin{aligned} m_R(B(u, \varepsilon)) &\ge \int_{K_u \cap B(0, (1-\gamma)\varepsilon)} m_0(B(0, R\gamma\varepsilon))\, d\nu_{u,T}(x) \\ &= m_0(B(0, R\gamma\varepsilon)) \cdot \nu_{u,T}(B(u, (1-\gamma)\varepsilon)). \end{aligned}$$

Since $m_0(B(0, R\gamma\varepsilon))$ tends to 1 as R tends to infinity, there exists $R_0 > 0$ such that

$$m_{R_0}(\mathrm{HC}(T) \cap B(u, \varepsilon)) = m_{R_0}(B(u, \varepsilon)) > (1 - \gamma)\, \nu_{u,T}(B(u, (1-\gamma)\varepsilon)).$$

Applying the *Ergodic Decomposition Theorem* (see e.g. [**46**, Th. 2.5]) to the measure m_{R_0}, we obtain that there exists an ergodic measure m for T such that

$$m(\mathrm{HC}(T) \cap B(u, \varepsilon)) > (1 - \gamma)\, \nu_{u,T}(B(u, (1-\gamma)\varepsilon)).$$

Since $m(\mathrm{HC}(T)) > 0$, the measure m has full support, and this concludes the proof of Lemma 5.30. □

We can now prove Proposition 5.29.

PROOF. Let $(V_p)_{p\geq 1}$ be a countable basis of (non-empty) open sets for X with the following property: for any open set V and any open ball B such that $B \prec V$, one can find $p \geq 1$ such that $B \prec V_p \subseteq V$. This additional property implies that for any open set $V \neq \emptyset$, we have

$$\delta_{V,T} = \sup\{\delta_{V_p,T};\ V_p \subseteq V\}. \tag{15}$$

For each $p \geq 1$, one may choose a sequence $(u_{p,k})_{k\geq 1}$ of vectors of $\mathcal{E}^{\mathbf{u}}$, a sequence of positive numbers $(\varepsilon_{p,k})_{k\geq 1}$ and a sequence of positive numbers $(\gamma_{p,k})_{k\geq 1}$ tending to 0 as k tends to infinity such that

$$B(u_{p,k},\varepsilon_{p,k}) \prec V_p \quad \text{for every } k \geq 1 \text{ and } \alpha_{p,k} := \nu_{u_{p,k},T}\big(B(u_{p,k},(1-\gamma_{p,k})\varepsilon_{p,k})\big) \underset{k\to\infty}{\longrightarrow} \delta_{V_p,T}.$$

By Lemma 5.30 and the pointwise ergodic theorem, we see that for each fixed pair (p,k) of integers, the set of all vectors $z \in X$ such that

$$\underline{\text{dens}}\,\mathcal{N}_T\big(z, B(u_{p,k},\varepsilon_{p,k})\big) \geq (1-\gamma_{p,k})\alpha_{p,k}$$

is dense in X. In particular, the set

$$G_{p,k} := \bigcap_{N\geq 1}\ \bigcup_{n\geq N}\Big\{z \in X;\ \#\{1 \leq i \leq n\,;\, T^i z \in B(u_{p,k},\varepsilon_{p,k})\} \geq (1-\gamma_{p,k})\,\alpha_{p,k}\Big\}$$

is a dense G_δ subset of X. Hence $G := \bigcap_{p,k\geq 1} G_{p,k}$ is a dense G_δ subset of X too, and by the definition of G, every vector $z \in G$ satisfies

$$\overline{\text{dens}}\,\mathcal{N}_T(z,V_p) \geq \delta_{V_p,T} \qquad \text{for every } p \geq 1.$$

Property (15) then allows us to conclude the proof of Proposition 5.29. □

4. A criterion for frequent hypercyclicity

In this section, our aim is to prove a criterion of the same kind as Theorem 5.14 for frequent hypercyclicity.

Before stating the result, we isolate a purely combinatorial lemma showing that some sets of integers constructed in a rather complicated way have positive lower density.

LEMMA 5.31. *Let $(d_j)_{j\geq 1}$ and $(n_j)_{j\geq 1}$ be two strictly increasing sequences of integers and $0 < \alpha < 1$ such that for all $j \geq 1$:*

$$4d_{j-1} < \alpha d_j < n_j \leq d_j,$$

where $d_0 = 1$. Let also $(j_m)_{m\geq 1}$ be a strictly increasing sequence of integers with bounded gaps, i.e. $\sup_{m\geq 1}(j_{m+1} - j_m) := r < \infty$. Finally, let $p \in \mathbb{N}$. For each $m \geq 1$, define a family of sets $(A_{m,j})_{0\leq j<j_{m+1}-j_m}$ as follows:

$$A_{m,0} := \Big\{n_{j_m} + kd_{j_m} + k'p\,;\, 0 \leq k' \leq \frac{\alpha d_{j_m}}{p},\ 0 \leq k \leq \frac{\alpha d_{j_m+1}}{d_{j_m}} - 2\Big\},$$

and, for $1 \leq j < j_{m+1} - j_m$,

$$A_{m,j} := \bigcup_{1\leq k\leq \frac{\alpha d_{j_m+j+1}}{d_{j_m+j}}-1} \Big(A_{m,j-1} + kd_{j_m+j}\Big).$$

Then, the set $A := \bigcup_{m\geq 1}\ \bigcup_{j=0}^{j_{m+1}-j_m-1} A_{m,j}$ has positive lower density.

PROOF. In what follows, we use the notation

$$A_m := \bigcup_{j=0}^{j_{m+1}-j_m-1} A_{m,j}.$$

We start by listing a few remarks on the structure of the sets $A_{m,j}$ and A_m.

CLAIM 5.32. The sets $A_{m,j}$ and A_m have the following properties.

(a) $A_{m,0} \subseteq [n_{j_m}, \alpha d_{j_m+1}]$ and $A_{m,j} \subseteq [d_{j_m+j}, \alpha d_{j_m+j+1}]$ for $1 \le j < j_{m+1} - j_m$. In particular, the sets $A_{m,j}$ are contained in successive (disjoint) subintervals of $\mathbb{N}$.
(b) Inside each set $A_{m,j}$, $j \ge 1$, the translates $A_{m,j-1} + kd_{j_m+j}$ are pairwise disjoint.
(c) $A_m \subseteq [n_{j_m}, \alpha d_{j_m+1}]$. In particular, the sets A_m, $m \ge 1$ are contained in successive subintervals of $\mathbb{N}$.

PROOF OF CLAIM 5.32. (a) The second assertion follows from the first because $\alpha < 1$. As for the first assertion, since obviously $\min A_{m,0} \ge n_{j_m}$ and $\min A_{m,j} \ge d_{j_m+j}$ if $j \ge 1$, it is enough to check that

$$\max\ A_{m,j} \le \alpha d_{j_m+j+1} \qquad \text{for } 0 \le j \le j_{m+1} - j_m. \tag{16}$$

For $j = 0$ we have $\max A_{m,0} \le n_{j_m} + \alpha d_{j_m+1} - 2d_{j_m} + \alpha d_{j_m} \le \alpha d_{j_m+1}$ because $n_{j_m} \le d_{j_m}$ and $\alpha < 1$. Then (16) follows by a straightforward induction on $j < j_{m+1} - j_m$.

(b) This is clear since $\max(A_{m,j-1} + kd_{j_m+j}) \le \alpha d_{j_m+j} + kd_{j_m+j} < (k+1)d_{j_m+j}$ by (a), whereas $\min(A_{m,j-1} + (k+1)d_{j_m+j}) \ge (k+1)d_{j_m+j}$.

(c) This follows from (a) and since $\alpha d_{j_{m+1}} < n_{j_{m+1}}$. □

CLAIM 5.33. There is no redundancy in the definition of $A_{m,0}$. In other words, any $n \in A_{m,0}$ can be written in only one way as $n = n_{j_m} + kd_{j_m} + k'p$ with $0 \le k \le \alpha d_{j_m+1}/d_{j_m} - 2$ and $0 \le k' \le \alpha d_{j_m}/p$.

PROOF OF CLAIM 5.33. Assume that $n = n_{j_m} + kd_{j_m} + k'p = n_{j_m} + \widetilde{k}d_{j_m} + \widetilde{k}'p$. Then $(k - \widetilde{k})d_{j_m} = (k' - \widetilde{k}')p$, so that $|k - \widetilde{k}|d_{j_m} \le \alpha d_{j_m}$ because $0 \le k', \widetilde{k}' \le \alpha d_{j_m}/p$. Hence $k = \widetilde{k}$ because $\alpha < 1$, and hence $k' = \widetilde{k}'$. □

Next, we estimate from below the size of the sets $A_{m,j}$. Recall that r is the maximal gap in the sequence (j_m).

CLAIM 5.34. For all $m \ge 1$, we have $\#A_{m,0} \ge \dfrac{\alpha^2 d_{j_m+1}}{2p}$ and

$$\#A_{m,j} \ge \frac{\alpha^{r+2}}{2^{r+1}} \cdot \frac{d_{j_m\ +j+1}}{p} \quad , \quad 1 \le j < j_{m+1} - j_m.$$

PROOF OF CLAIM 5.34. Let us fix $m \ge 1$. By Claim 5.33 and since $\alpha d_{j_m+1} > 4d_{j_m}$, we have

$$\#A_{m,0} \ge \frac{\alpha d_{j_m}}{p}\Big(\frac{\alpha d_{j_m+1}}{d_{j_m}} - 1\Big) \ge \frac{\alpha d_{j_m}}{p} \cdot \frac{\alpha d_{j_m+1}}{2d_{j_m}} = \frac{\alpha^2 d_{j_m+1}}{2p}.$$

Moreover, if $1 \le j < j_{m+1} - j_m$ then, by Claim 5.32 (b) and since $\alpha d_{j_m+j+1} > 4d_{j_m+j}$, we see that

$$\#A_{m,j} \ge \Big(\frac{\alpha d_{j_m+j+1}}{d_{j_m+j}} - 1\Big)\#A_{m,j-1} \ge \frac{\alpha d_{j_m+j+1}}{2d_{j_m+j}}\#A_{m,j-1}.$$

It follows that for all $1 \le j < j_{m+1} - j_m$, we have

$$\#A_{m,j} \ge \Big(\frac{\alpha}{2}\Big)^j \frac{d_{j_m+j+1}}{d_{j_m+1}}\#A_{m,0} \ge \frac{\alpha^{j+2}}{2^{j+1}} \cdot \frac{d_{j_m+j+1}}{p};$$

and since $j_{m+1} - j_m \le r$ and $\alpha/2 < 1$, this eventually gives

$$\#A_{m,j} \ge \frac{\alpha^{r+2}}{2^{r+1}} \cdot \frac{d_{j_m+j+1}}{p}.$$

□

We are now going to show that there exists some positive number δ such that

$$\frac{1}{n}\,\#([1,n]\cap A)\geq\delta \qquad \text{for every } n\in A. \tag{17}$$

This will be enough to ensure that A has positive lower density. Indeed, if we denote by $A=\{a_q\,;\,q\geq 1\}$ the increasing enumeration of A, then (17) means that $q/a_q\geq\delta$ for every $q\geq 1$, *i.e.* $a_q\leq q/\delta$; and this is easily seen to imply that A has positive lower density.

Let us fix $n\in A$, so that $n\in A_m$ for some $m\geq 1$. We consider separately two cases.

Case 1. $n\in A_{m,0}$.

By the definition of $A_{m,0}$, we may write

$$n=n_{j_m}+kd_{j_m}+k'p,\quad \text{where } 0\leq k\leq\frac{\alpha d_{j_m+1}}{d_{j_m}}-2 \text{ and } 0\leq k'\leq\frac{\alpha d_{j_m}}{p}.$$

Note that, by Claim 5.32, the set $[1,n]$ contains A_{m-1} because $n\geq n_{j_m}>\alpha d_{j_m}$. Since A_{m-1} is disjoint from $A_{m,0}$ (again by Claim 5.32), it follows that we have in particular

$$\#\big([1,n]\cap A\big)\geq\#A_{m-1,j_m-j_{m-1}-1}+\#\big([1,n]\cap A_{m,0}\big).$$

Moreover, the set $[1,n]\cap A_{m,0}$ contains all integers of the form $n_{j_m}+sd_{j_m}+s'p$, where $0\leq s<k$ and $0\leq s'\leq\frac{\alpha d_{j_m}}{p}$ (since all of them are in $A_{m,0}$ and the largest is not greater than $n_{j_m}+(k-1)d_{j_m}+\alpha d_{j_m}\leq n_{j_m}+kd_{j_m}\leq n$). Hence, by Claim 5.33,

$$\#\big([1,n]\cap A_{m,0}\big)\geq k\,\cdot\,\frac{\alpha d_{j_m}}{p}.$$

By Claim 5.34, it follows that

$$\#\big([1,n]\cap A\big)\geq\frac{\alpha^{r+2}d_{j_m}}{2^{r+1}p}+k\frac{\alpha d_{j_m}}{p}\geq\frac{\alpha^{r+2}}{2^{r+1}}\cdot\frac{(k+1)d_{j_m}}{p}.$$

Since $n\leq n_{j_m}+(k+\alpha)d_{j_m}\leq 2(k+1)d_{j_m}$, we conclude that

$$\#\big([1,n]\cap A\big)\geq\frac{\alpha^{r+2}}{2^{r+2}p}\,n.$$

Case 2. $n\in A_{m,j}$ for some $1\leq j<j_{m+1}-j_m$.

In this case, we may write

$$n=kd_{j_m+j}+i,\quad \text{with } 1\leq k\leq\frac{\alpha d_{j_m+j+1}}{d_{j_m+j}}-1 \text{ and } i\in A_{m,j-1}.$$

By Claim 5.32, we know that $\max A_{m,j-1}<\min A_{m,j}\leq n$. Hence $[1,n]$ contains $A_{m,j-1}$ and

$$\#\big([1,n]\cap A\big)\geq\#A_{m,j-1}+\#\big([1,n]\cap A_{m,j}\big).$$

Moreover, since $n\geq kd_{j_m+j}\geq(k-1)d_{j_m+j}+\max A_{m,j-1}$ and the translates $A_{m,j-1}+ld_{j_m+j}$, $l<k$ are pairwise disjoint (and contained in $A_{m,j}$) by Claim 5.32, we also have

$$\#\big([1,n]\cap A_{m,j}\big)\geq(k-1)\,\#A_{m,j-1}.$$

By Claim 5.34, it follows that

$$\#\big([1,n]\cap A\big)\geq k\,\frac{\alpha^{r+2}d_{j_m+j}}{2^{r+1}p};$$

and since $n\leq(k+1)d_{j_m+j}$ (because $n=kd_{j_m+j}+i$ and $\max A_{m,j-1}\leq d_{j_m+j}$), we conclude that

$$\#\big([1,n]\cap A\big)\geq\frac{\alpha^{r+2}}{2^{r+2}p}\,n.$$

We have thus shown that

$$\#\big([1,n]\cap A\big)\geq \frac{\alpha^{r+2}}{2^{r+2}}\,n \qquad \text{for every } n\in A,$$

which concludes the proof of Lemma 5.31. □

We now state our criterion for frequent hypercyclicity. At first sight, this criterion looks very much like Theorem 5.14. The main difference is the following: instead of requiring that $\|T^{n+k}z-T^kx\|$ should be small for a duration proportional to the "time" n needed for the orbit of z to come close to x, we have to require that these quantities should be small for a duration proportional to a certain multiple d of the period of z. (This is indeed a much stronger requirement, see the beginning of the proof of Theorem 5.35). The criterion reads as follows.

THEOREM 5.35. *Let $T\in\mathfrak{B}(X)$. Assume that there exist a dense linear subspace X_0 of X with $T(X_0)\subseteq X_0$ and $X_0\subseteq \mathrm{Per}(T)$, and a constant $\alpha\in(0,1/2)$ such that the following property holds true: for every $x\in X_0$, every $\varepsilon>0$ and every integer $d_0\geq 1$ which is the period of some vector y of X_0, there exist $z\in X_0$ and integers n, $d\geq 1$ such that*

(0) *d is a multiple of d_0 and of $\mathrm{per}(z)$;*
(1) *$\|T^kz\|<\varepsilon$ for every $0\leq k\leq \alpha d$;*
(2) *$\|T^{n+k}z-T^kx\|<\varepsilon$ for every $0\leq k\leq\alpha d$.*

Then T is chaotic and frequently hypercyclic.

PROOF. We first note that upon subtracting some multiple of d to n (where d and n are given by the assumptions of Theorem 5.35), we can always assume that $d\geq n$. The assumption of Theorem 5.35 is thus seen to be stronger than that of Theorem 5.14. Also, we can require that $n>\alpha d$ and the same argument as in the beginning of the proof of Theorem 5.14 (taking ε very small) shows that, given $N\geq 1$, we can always add to the assumption of Theorem 5.35 the additional hypothesis that the integer n is a multiple of N.

Let now $(x_l)_{l\geq 1}$ be a dense sequence of vectors of X contained in X_0, and let $(I_l)_{l\geq 1}$ be a partition of $\mathbb{N}$ such that each set I_l is infinite and has bounded gaps. We denote by r_l the maximum size of a gap between two successive elements of I_l. As usual, we define the vectors y_j, $j\geq 1$, by setting $y_j=x_l$ for every $j\in I_l$. By induction on $j\geq 1$, we construct a sequence $(z_j)_{j\geq 1}$ of vectors of X_0 and two strictly increasing sequences of integers $(d_j)_{j\geq 1}$ and $(n_j)_{j\geq 1}$ such that

(i) d_j is a multiple of $\mathrm{per}\,(\sum_{i=1}^{j-1}z_i)$ and of $\mathrm{per}(z_j)$;
(ii) $\|T^kz_j\|<2^{-j}$ for every $0\leq k\leq\alpha d_j$;
(iii) $\|T^{n_j+k}z_j-T^k(y_j-\sum_{i=1}^{j-1}z_i)\|<2^{-j}$ for every $0\leq k\leq\alpha d_j$;
(iv) n_j is a multiple of $\mathrm{per}(\sum_{i=1}^{j-1}z_i)$ and $\alpha d_j<n_j\leq d_j$;
(v) $\alpha d_j>4d_{j-1}$.

We set $z=\sum\limits_{i\geq 1}z_i$, and show that z is a frequently hypercyclic vector for T.

Let us fix $l\geq 1$, and write I_l as $I_l=\{j_m\,;\,m\geq 1\}$, where $(j_m)_{m\geq 1}$ is strictly increasing and $j_{m+1}-j_m\leq r_l$ for every $m\geq 1$. For each $m\geq 1$, we define a family of sets $(A_{m,j})_{0\leq j<j_{m+1}-j_m}$ as in Lemma 5.31, with $p:=\mathrm{per}(x_l)$:

$$A_{m,0}:=\Big\{n_{j_m}+kd_{j_m}+k'\mathrm{per}(x_l)\,;\,0\leq k'\leq\frac{\alpha d_{j_m}}{\mathrm{per}(x_l)},\ 0\leq k\leq\frac{\alpha d_{j_m+1}}{d_{j_m}}-2\Big\},$$

and, for $1\leq j<j_{m+1}-j_m$,

$$A_{m,j}:=\bigcup_{1\leq k\leq\frac{\alpha d_{j_m+j+1}}{d_{j_m+j}}-1}\Big(A_{m,j-1}+kd_{j_m+j}\Big).$$

Since we will need this below, we recall from the proof of Lemma 5.31 that for all m and $0 \leq j < j_{m+1} - j_m$, we have

$$\max\ A_{m,j} \leq \alpha d_{j_m+j+1}. \tag{18}$$

Claim 5.36. For every $m \geq 1$ and every $n \in A_{m,j}$, $0 \leq j < j_{m+1} - j_m$, we have

$$\|T^n z - x_l\| \leq 2^{-(j_m-1)}.$$

Proof of Claim 5.36. For any $n \in A_{m,j}$, we have

$$\|T^n z - x_l\| \leq \Big\| T^n\Big(\sum_{s=1}^{j_m+j} z_s\Big) - x_l \Big\| + \sum_{s>j_m+j} \|T^n z_s\|. \tag{19}$$

Since $n \leq \alpha d_{j_m+j+1} \leq \alpha d_s$ for every $s > j_m + j$ by (18), the second term in (19) is easy to control:

$$\sum_{s>j_m+j} \|T^n z_s\| < \sum_{s>j_m+j} 2^{-s}.$$

We now have to estimate the term $\Big\| T^n\Big(\sum_{s=1}^{j_m+j} z_s\Big) - x_l \Big\|$. The index m being fixed, we show by induction on $0 \leq j < j_{m+1} - j_m$ that

$$\Big\| T^n\Big(\sum_{s=1}^{j_m+j} z_s\Big) - x_l \Big\| < \sum_{u=0}^{j} 2^{-(j_m+u)}. \tag{20}$$

- Suppose first that n belongs to $A_{m,0}$, so that

$$n = n_{j_m} + kd_{j_m} + k' \operatorname{per}(x_l) \quad \text{with} \quad 0 \leq k \leq \frac{\alpha d_{j_m+1}}{d_{j_m}} - 2 \quad \text{and} \quad 0 \leq k' \leq \frac{\alpha d_{j_m}}{\operatorname{per} x_l}.$$

Then

$$\begin{aligned} T^n\Big(\sum_{s=1}^{j_m} z_s\Big) - x_l &= T^{n_{j_m}+k'\operatorname{per}(x_l)}\Big(\sum_{s=1}^{j_m} z_s\Big) - x_l \qquad \text{by (i)} \\ &= T^{n_{j_m}+k'\operatorname{per}(x_l)} z_{j_m} - T^{k'\operatorname{per}(x_l)}\Big(x_l - \sum_{s=1}^{j_m-1} z_s\Big) \qquad \text{by (iv)}. \end{aligned}$$

Since $k'\operatorname{per}(x_l) \leq \alpha d_{j_m}$, we deduce from (iii) that $\Big\| T^n\Big(\sum_{s=1}^{j_m} z_s\Big) - x_l \Big\| \leq 2^{-j_m}$.

- Suppose now that (20) has been proved up to the index $j-1$ for some $1 \leq j < j_{m+1} - j_m$, and that n belongs to $A_{m,j}$. Write

$$n = kd_{j_m+j} + i \qquad \text{with} \quad i \in A_{m,j-1} \quad \text{and} \quad 0 \leq k \leq \frac{\alpha d_{j_m+j+1}}{d_{j_m+j}} - 1.$$

Then

$$\begin{aligned} T^n\Big(\sum_{s=1}^{j_m+j} z_s\Big) - x_l &= T^{kd_{j_m+j}+i}\Big(\sum_{s=1}^{j_m+j} z_s\Big) - x_l = T^i\Big(\sum_{s=1}^{j_m+j} z_s\Big) - x_l \qquad \text{by (i)} \\ &= T^i\Big(\sum_{s=1}^{j_m+j-1} z_s\Big) - x_l + T^i z_{j_m+j}. \end{aligned}$$

Thus

$$\Big\| T^n\Big(\sum_{s=1}^{j_m+j} z_s\Big) - x_l \Big\| < \sum_{u=0}^{j-1} 2^{-(j_m+u)} + \|T^i z_{j_m+j}\|$$

by the induction hypothesis. Now, $i \leq \alpha d_{j_m+j}$ by (18) since i belongs to $A_{m,j-1}$, and thus $\|T^i z_{j_m+j}\| < 2^{-(j_m+j)}$ by (i). Hence

$$\Big\|T^n\Big(\sum_{s=1}^{j_m+j} z_s\Big) - x_l\Big\| < \sum_{u=0}^{j} 2^{-(j_m+u)},$$

which concludes our induction. The inequality (20) being proved, we deduce that

$$\|T^n z - x_l\| < \sum_{u\geq 0} 2^{-(j_m+u)} = 2^{-(j_m-1)}$$

for every $m \geq 1$ and every $n \in A_{m,j}$, $0 \leq j < j_{m+1} - j_m$. □

It follows from Claim 5.36 that given $\varepsilon > 0$, there exists $m_0 \geq 1$ such that $\mathcal{N}_T(z, B(x_l, \varepsilon))$ contains the set

$$\bigcup_{m\geq m_0} \bigcup_{j=0}^{j_{m+1}-j_m-1} A_{m,j}.$$

By Lemma 5.31, the latter set has positive lower density. Hence $\mathcal{N}_T(z, B(x_l, \varepsilon))$ has positive lower density for each l and every $\varepsilon > 0$, which concludes the proof of Theorem 5.35. □

Remark 5.37. The restriction that the integer d_0 above should be the period of some vector $y \in X_0$ may seem rather artificial. The reason for stating Theorem 5.35 as we did is to be found in the proof of Theorem 6.9 below: we will use Theorem 5.35 as stated to prove the frequent hypercyclicity of the operators involved there without having to impose certain divisibility conditions on the numbers $\Delta^{(k)}$, $k \geq 1$.

Remark 5.38. The constant α involved in the statement of Theorem 5.35 cannot be greater than 1/2. Indeed, otherwise one could find a periodic vector z such that more than half of the points in the orbit of z are close to 0 and more than half of these points are close to some periodic orbit far away from 0. So at least one point in the orbit of z would have to be both close to 0 and far away from 0, which is impossible.

4.1. Link with the Operator Specification Property. We conclude this section by a result which shows that many operators which are known to be frequently hypercyclic satisfy the assumption of Theorem 5.35. This is the case for all operators which satisfy the Frequent Hypercyclicity Criterion and, more generally, for operators with the so-called *Operator Specification Property*.

This last property, which has been recently introduced and studied by Bartoll, Martínez-Giménez, and Peris ([**1**], [**2**]), is the linear version of the classical *Specification Property* for compact dynamical systems introduced by Bowen in [**16**]. The definition reads as follows: an operator $T \in \mathfrak{B}(X)$ has the Operator Specification Property (OSP) if there exists an increasing sequence $(K_m)_{m\geq 1}$ of T-invariant subsets of X with $0 \in K_1$, the union of which is dense in X, such that for each $m \geq 1$, the restriction of T to K_m has the Specification Property in the sense of [**16**], which means that the following holds true:

$(*)$ for every $\delta > 0$, there exists an integer $N_{\delta,m} \geq 1$ such that for every finite family $y_1, \ldots, y_s$ of points of K_m, and any integers $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ with $j_{r+1} - k_r \geq N_{\delta,m}$ for every $1 \leq r \leq s-1$, there is a point $x \in K_m$ such that for every $1 \leq r \leq s$,

$$\sup_{j_r \leq i \leq k_r} \|T^i x - T^i y_r\| < \delta \quad \text{while} \quad T^{N_{\delta,m}+k_s} x = x.$$

Stated in an informal way, $(*)$ means that arbitrary large pieces of the orbits of finitely many points of K_m can be approximated by the orbits of a single periodic point of K_m, provided that the gaps between the different sets of indices where we require this approximation are sufficiently large.

It is proved in [**2**] that operators with the OSP are chaotic, topologically mixing, and frequently hypercyclic. Moreover, any operator satisfying the general version of the Frequent Hypercyclicity Criterion given in [**14**] has the OSP. We prove in Theorem 5.39 below that operators with the OSP satisfy the assumption of Theorem 5.35. This was pointed out to us by Alfred Peris, who kindly allowed us to reproduce his proof here. This improves on a previous observation (proved in a preliminary version of this work) according to which operators satisfying the Frequent Hypercyclicity Criterion also satisfy the assumptions of Theorem 5.35.

THEOREM 5.39. *Let $T \in \mathfrak{B}(X)$ be an operator with the* OSP*. There exists a dense subspace X_0 of X with $T(X_0) \subseteq X_0$ and $X_0 \subseteq \mathrm{Per}(T)$, such that, for every $\alpha \in (0, 1/2)$, every $x \in X_0$, every $\varepsilon > 0$, and every integer $d_0 \geq 1$, there exist $z \in X_0$ and integers n, $d \geq 1$ such that properties* (0)*,* (1)*, and* (2) *of Theorem 5.35 hold true.*

PROOF. Let $(K_m)_{m\geq 1}$ be an increasing sequence of T-invariant subsets of X with $0 \in K_1$, the union of which is dense in X, such that $(*)$ above holds true for every $m \geq 1$. We define $X_0 := \mathrm{span}\big[\bigcup_{m\geq 1} K_m \cap \mathrm{Per}(T)\big]$, which is clearly T-invariant and dense in X.

Let us fix $x \in X_0$, $\varepsilon > 0$ and $d_0 \geq 1$. The vector x can be written as $x = \sum_{m=1}^{m_0} a_m x_m$, where a_m is a scalar and x_m is a vector belonging to K_m for every $1 \leq m \leq m_0$. By [**2**, Prop. 10], the map induced by T on the set $K = \sum_{m=1}^{m_0} a_m K_m$ has the Specification Property. So we can assume without loss of generality that x belongs to K_m for some $m \geq 1$. Let now $N_{\varepsilon,m}$ be such that property $(*)$ above holds true, and let $d \geq 1$ be a multiple of d_0 so large that $\alpha d + N_{\varepsilon,m} < d/2$. We then define d' to be the integer part of αd, $n = N_{\varepsilon,m} + d'$, $y_1 = 0$, and $y_2 = T^{l-n}x$, where l is any multiple of $\mathrm{per}(x)$ such that $l > n$. We also set $j_1 = 0$, $k_1 = d'$, $j_2 = n$, and $k_2 = d - N_{\varepsilon,m}$. As $j_2 - k_1 = n - d' = N_{\varepsilon,m}$, the Specification Property on K_m implies that there exists a point $z \in K_m$ such that

$$\begin{aligned} \|T^k z - T^k y_1\| &= \|T^k z\| < \varepsilon \quad \text{for every } 0 \leq k \leq d'; \\ \|T^k z - T^k y_2\| &< \varepsilon \quad \text{for every } n \leq k \leq k_2; \\ T^{N_{\varepsilon,m}+k_2} z &= T^d z = z. \end{aligned}$$

In other words,

$$\begin{aligned} \|T^k z\| &< \varepsilon \quad \text{for every } 0 \leq k \leq \alpha d\,; \\ \|T^{n+k} z - T^k x\| &< \varepsilon \quad \text{for every } 0 \leq k \leq k_2 - n \end{aligned}$$

since $T^{n+k} y_2 = T^{n+k+l-n} x = T^k x$; and d is a multiple of the period of z. Since $k_2 - n = d - 2N_{\varepsilon,m} - d' \geq \alpha d$, this shows that assumptions (0), (1), and (2) of Theorem 5.35 are satisfied. □

REMARK 5.40. The converse of Theorem 5.39 is not true; that is, operators satisfying the assumptions of Theorem 5.35 need not have the OSP. Indeed, we will construct in Chapter 6 (more precisely, in Example 7.7) some operators which satisfy the assumptions of Theorem 5.35 and yet are not topologically mixing.

To conclude this section, we essentially show that the OSP implies ergodicity. This is coherent with what happens in the non-linear setting: it is proved in [**51**] that compact dynamical systems with the Specification Property are ergodic. They actually enjoy a much stronger property: given such a system (X, T), the ergodic measures with full support for (X, T) are comeager in the set $\mathcal{P}_T(X)$ of T-invariant probability measures endowed with its natural Polish topology. On the other hand, we do not know whether operators with the OSP are mixing (*i.e.* admit mixing measures with full support), whereas it is known that the Frequent Hypercyclicity Criterion does imply mixing, and in fact mixing in the Gaussian sense (see [**42**] and [**9**]).

PROPOSITION 5.41. *If $T \in \mathfrak{B}(X)$ satisfies the* OSP*, and if the sequence $(K_m)_{m\geq 1}$ of T-invariant subsets of X appearing in the definition of the* OSP *consists of* compact *sets, then T is ergodic.*

The assumption of Proposition 5.41 that the sets K_m, $m \geq 1$, be compact seems to be no real restriction. As mentioned in [**2**], all known examples of operators with the OSP do satisfy the OSP with respect to a sequence of T-invariant compact subsets $(K_m)_{m\geq 1}$ of X.

Proof. Let us denote by $\mathcal{P}(X)$ the space of all Borel probability measures on X endowed with its natural Polish topology (a sequence $(\mu_n)_{n\geq 1}$ of elements of $\mathcal{P}(X)$ converges to $\mu \in \mathcal{P}(X)$ if and only if $\int_X f d\mu_n$ tends to $\int_X f d\mu$ as n tends to infinity for every bounded continuous function f from X into $\mathbb{R}$), and by $\mathcal{P}_T(X) \subseteq \mathcal{P}(X)$ the set of all T-invariant measures. Then $\mathcal{P}_T(X)$ is a Polish space, being a closed subset of $\mathcal{P}(X)$. We denote by $\mathcal{E}_T(X) \subseteq \mathcal{P}_T(X)$ the set of all ergodic measures for T. For any family of measures $\mathcal{M} \subseteq \mathcal{P}(X)$, we denote by $\mathcal{M}^*$ the family of all measures $\mu \in \mathcal{M}$ with full support. Finally, for any subset $\mathcal{M}$ of $\mathcal{P}(X)$ and any Borel subset A of X, we set $\mathcal{M}(A) := \{\mu \in \mathcal{M};\ \mu(A) = 1\}$.

Let $(K_m)_{m\geq 1}$ be an increasing sequence of *compact* T-invariant subsets of X with $\overline{\bigcup_{m\geq 1} K_m} = X$ such that $T_{|K_m}$ has the Specification Property for every $m \geq 1$. By a result of [**51**], we know that for each $m \geq 1$, the set $\mathcal{E}_T^*(K_m)$ of all measures $\mu \in \mathcal{E}_T(X)$ with support equal to K_m is dense in $\mathcal{P}_T(K_m)$. Let us now denote by $\mathcal{M}$ the closure of $\bigcup_{m\geq 1} \mathcal{P}_T(K_m)$ in $\mathcal{P}_T(X)$.

Claim 5.42. The set $\mathcal{E}_T(X) \cap \mathcal{M}$ is a dense G_δ subset of $\mathcal{M}$.

Proof of Claim 5.42. It is known that $\mathcal{E}_T(X)$ is a G_δ subset of $\mathcal{P}_T(X)$ (see [**43**] for a detailed proof), so that $\mathcal{E}_T(X) \cap \mathcal{M}$ is G_δ in $\mathcal{M}$. Moreover, since $\mathcal{E}_T(K_m)$ is dense in $\mathcal{P}_T(K_m)$ for each m by [**51**], it is clear that $\mathcal{E}_T(X) \cap \mathcal{M}$ is dense in $\mathcal{M}$. □

Claim 5.43. The set $\mathcal{M}^*$ is a dense G_δ subset of $\mathcal{M}$.

Proof of Claim 5.43. Let $(O_p)_{p\geq 1}$ be a countable basis of non-empty open subsets of X. Set, for each $p \geq 1$, $\mathcal{O}_p := \{\mu \in \mathcal{P}(X);\ \mu(O_p) > 0\}$. Then each set $\mathcal{O}_p$ is open in $\mathcal{P}(X)$, and moreover

$$\mathcal{M}^* = \mathcal{M} \cap \Big(\bigcap_{p\geq 1} \mathcal{O}_p\Big).$$

So we just have to show that each set $\mathcal{M} \cap \mathcal{O}_p$ is dense in $\mathcal{M}$. Let us fix $p \geq 1$, and let $\mathcal{U}$ be a non-empty open subset of $\mathcal{M}$. Since the sequence $(K_m)_{m\geq 1}$ is increasing, the definition of $\mathcal{M}$ implies that $\mathcal{U} \cap \mathcal{P}_T(K_m)$ is non-empty for all m sufficiently large. Moreover, since $\bigcup_{m\geq 1} K_m$ is dense in X and $(K_m)_{m\geq 1}$ is increasing, $K_m \cap O_p$ is non-empty too for all m sufficiently large. So there exists $m \geq 1$ such that $K_m \cap O_p$ and $\mathcal{U} \cap \mathcal{P}_T(K_m)$ are both non-empty. Since $\mathcal{E}_T^*(K_m)$ is dense in $\mathcal{P}_T(K_m)$ and $\mathcal{U} \cap \mathcal{P}_T(K_m)$ is open in $\mathcal{P}_T(K_m)$, it follows that there exists a measure $\mu \in \mathcal{E}_T(X)$ with support equal to K_m such that μ belongs to $\mathcal{U}$ and $\mu(O_p) > 0$. Hence, we have shown that $\mathcal{U} \cap \mathcal{O}_p$ is non-empty. □

The two facts above combined with the Baire Category Theorem applied in $\mathcal{M}$ imply that $\mathcal{E}_T(X) \cap \mathcal{M}^*$ is non-empty. In particular $\mathcal{E}_T(X)^*$ is non-empty, *i.e.* T is ergodic. □

Remark 5.44. We will see in Chapter 6 that the assumptions of Theorem 5.35 (which are sufficient for frequent hypercyclicity) do *not* imply ergodicity. So Proposition 5.41 makes the difference between the OSP and our criterion for frequent hypercyclicity all the more tangible.

CHAPTER 6

Operators of C-type and of C_+-type

In this chapter, we introduce some particular classes of operators, which are defined on any space $\ell_p(\mathbb{N})$, $1 \leq p < \infty$. We do not restrict ourselves to the Hilbertian case $p = 2$, because the general case adds no extra complication. It is within these classes that we will exhibit operators which are chaotic and frequently hypercyclic but not ergodic, operators which are chaotic and $\mathcal{U}$-frequently hypercyclic but not frequently hypercyclic and operators which are chaotic and topologically mixing but not $\mathcal{U}$-frequently hypercyclic (see Chapter 7).

A word of caution: for technical reasons, we have decided that $\mathbb{N}$ *starts at* 0; that is, $\mathbb{N} = \{0, 1, 2, \dots\}$. Accordingly, the canonical basis of $\ell_p(\mathbb{N})$ will be denoted by $(e_k)_{k\geq 0}$. Also, we denote by c_{00} the linear span of the vectors e_k, $k \geq 0$, *i.e.* the subspace of $\ell_p(\mathbb{N})$ consisting of all finitely supported vectors.

1. Operators of C-type: basic facts

In view of our criteria for $\mathcal{U}$-frequent hypercyclicity and frequent hypercyclicity relying on the existence of periodic points, we would like to find a rich family of operators for which we can easily find a large supply of periodic points. For example, we could consider operators $T_{w,b}$ defined by

$$T_{w,b}e_k = \begin{cases} w_{k+1}e_{k+1} & \text{if } k \in [b_n, b_{n+1}-1) \\ \left(\prod_{j=b_n+1}^{b_{n+1}-1} w_j\right)^{-1} e_{b_n} & \text{if } k = b_{n+1}-1 \text{ with } n \geq 0 \end{cases}$$

where $w = (w_j)_{j\geq 1}$ is a weight sequence and $b = (b_n)_{n\geq 1}$ is a strictly increasing sequence of integers with $b_0 = 0$. Indeed, with this definition it is clear that (whatever the choice of w and b) every basis vector e_k and hence every vector $x \in c_{00}$ is periodic for $T_{w,b}$.

However, none of these operators is hypercyclic since they are direct sums of finite-dimensional operators, and there exist no hypercyclic operators in finite dimension. It is thus necessary to perturb the operators $T_{w,b}$ in order to obtain an interesting family of *hypercyclic* operators in which finite sequences are still periodic. For reasons that will be explained below, the operators in this family will be called *operators of C-type*. Any operator of C-type will be associated to four parameters v, w, φ, and b, where

- $v = (v_n)_{n\geq 1}$ is a sequence of non-zero complex numbers such that $\sum_{n\geq 1} |v_n| < \infty$;
- $w = (w_j)_{j\geq 1}$ is a sequence of complex numbers which is both bounded and bounded below, *i.e.* $0 < \inf_{k\geq 1} |w_k| \leq \sup_{k\geq 1} |w_k| < \infty$;
- φ is a map from $\mathbb{N}$ into itself, such that $\varphi(0) = 0$, $\varphi(n) < n$ for every $n \geq 1$, and the set $\varphi^{-1}(l) = \{n \geq 0\,;\, \varphi(n) = l\}$ is infinite for every $l \geq 0$;
- $b = (b_n)_{n\geq 0}$ is a strictly increasing sequence of positive integers such that $b_0 = 0$ and $b_{n+1} - b_n$ is a multiple of $2(b_{\varphi(n)+1} - b_{\varphi(n)})$ for every $n \geq 1$.

DEFINITION 6.1. The *operator of C-type* $T_{v,\,w,\,\varphi,\,b}$ on $\ell_p(\mathbb{N})$ associated to the data v, w, φ, and b given as above is defined by

$$T_{v,\,w,\,\varphi,\,b}\ e_k = \begin{cases} w_{k+1}\, e_{k+1} & \text{if } k \in [b_n, b_{n+1}-1),\ n \geq 0, \\ v_n\, e_{b_{\varphi(n)}} - \Big(\displaystyle\prod_{j=b_n+1}^{b_{n+1}-1} w_j\Big)^{-1} e_{b_n} & \text{if } k = b_{n+1}-1,\ n \geq 1, \\ -\Big(\displaystyle\prod_{j=b_0+1}^{b_1-1} w_j\Big)^{-1} e_0 & \text{if } k = b_1 - 1. \end{cases}$$

Note that by convention, an empty product is declared to be equal to 1; more precisely,

$$\prod_{j=b_n+1}^{b_{n+1}-1} w_j = 1 \quad \text{when } b_{n+1} = b_n + 1$$

(which can happen only for $n = 0$).

Without any additional assumption on the parameters, these formulas define a linear map on c_{00} only. The first issue is of course the *boundedness* of $T_{v,\,w,\,\varphi,\,b}$.

LEMMA 6.2. *The operator $T_{v,\,w,\,\varphi,\,b}$ is well-defined and bounded on $\ell_p(\mathbb{N})$ as soon as*

$$\inf_{n\geq 0} \prod_{b_n<j<b_{n+1}} |w_j| > 0. \tag{21}$$

PROOF. This is rather clear. Indeed, it appears that $T_{v,\,w,\,\varphi,\,b}$ can be written as

$$T_{v,\,w,\,\varphi,\,b} = \bigoplus_{n\geq 0} C_{w,\,b,\,n} + R_{v,b},$$

where $R_{v,b}$ is the operator defined by

$$R_{v,\,b}x = \sum_{n\geq 1} v_n x_{b_{n+1}-1}\, e_{b_{\varphi(n)}}, \quad x \in \ell_p(\mathbb{N})$$

which is clearly bounded (and even compact) because $\sum_{n\geq 1} |v_n| < \infty$, and, for each $n \geq 1$, $C_{w,\,b,\,n}$ is a finite-dimensional cyclic operator acting on $E_n := \operatorname{span}[e_k\,;\, b_n \leq k \leq b_{n+1}-1]$. Condition (21) implies that $\sup_{n\geq 1} \|C_{w,\,b,\,n}\|$ is finite, and $T_{v,\,w,\,\varphi,\,b}$ is thus bounded. □

From now on, we will always assume that Condition (21) is satisfied. Also, when no confusion arises, we will write simply T instead of $T_{v,\,w,\,\varphi,\,b}$.

REMARK 6.3. We call such operators *operators of C-type* for two different reasons. On the one hand, "C" stands for "cyclic": as we have just explained, each operator $T_{v,\,w,\,\varphi,\,b}$ is a compact perturbation of an infinite direct sum of cyclic finite-dimensional operators $\bigoplus_{n\geq 0} C_{w,\,b,\,n}$, where $C_{w,\,b,\,n}$ is defined on $E_n = \operatorname{span}[e_k\,;\, b_n \leq k \leq b_{n+1}-1]$. On the other hand, we will see in a moment that, as a consequence of their particular structure, the operators $T_{v,\,w,\,\varphi,\,b}$ happen to be chaotic under a very mild restriction on the parameters; so, "C" stands for "chaotic" as well.

The following identity will be used repeatedly: if $T = T_{v,\,w,\,\varphi,\,b}$ is an operator of C-type on $\ell_p(\mathbb{N})$, then

$$T^{\,b_{n+1}-b_n}\, e_{b_n} = v_n \Big(\prod_{j=b_n+1}^{b_{n+1}-1} w_j\Big)\, e_{b_{\varphi(n)}} - e_{b_n} \quad \text{for every } n \geq 1. \tag{22}$$

This allows to show that operators of C-type always have plenty of periodic points:

LEMMA 6.4. *If $T = T_{v,\,w,\,\varphi,\,b}$ is an operator of C-type on $\ell_p(\mathbb{N})$, then every basis vector e_k is periodic for $T_{v,\,w,\,\varphi,\,b}$; more precisely,*

$$T^{\,2(b_{n+1}-b_n)} e_k = e_k \qquad \textit{if } k \in [b_n, b_{n+1}),\ n \geq 0.$$

Consequently, every vector $x \in c_{00}$ is periodic for $T_{v,\,w,\,\varphi,\,b}$, and hence $T_{v,\,w,\,\varphi,\,b}$ has a dense set of periodic points.

PROOF. Since e_k is a non-zero multiple of $T^{\,k-b_n} e_{b_n}$ for every $b_n \le k < b_{n+1}$, it suffices to prove that $T^{\,2(b_{n+1}-b_n)}\, e_{b_n} = e_{b_n}$ for every $n \ge 0$. We prove this by induction on n.

– If $n = 0$, then $T^{\,b_1-b_0}\, e_0 = -e_0$ by definition of Te_{b_1-1}, and thus $T^{\,2(b_1-b_0)}\, e_0 = e_0$.

– Let $n \ge 1$, and assume that the result has been proved for all $m < n$. We know that $b_{n+1} - b_n$ is a multiple of $2(b_{\varphi(n)+1} - b_{\varphi(n)})$ and since $\varphi(n) < n$, it follows by (22) and the induction hypothesis that

$$T^{\,2(b_{n+1}-b_n)}\, e_{b_n} = v_n \Big(\prod_{j=b_n+1}^{b_{n+1}-1} w_j \Big) e_{b_{\varphi(n)}} - T^{\,b_{n+1}-b_n}\, e_{b_n} = e_{b_n}.$$

□

Using the above result and Corollary 5.13, we can now obtain the following sufficient condition for an operator of C-type to be chaotic.

PROPOSITION 6.5. *Suppose that*

$$\limsup_{\substack{N\to\infty \\ N\in\,\varphi^{-1}(n)}} |v_N| \prod_{j=b_N+1}^{b_{N+1}-1} |w_j| = \infty \quad \text{for every } n \ge 0.$$

Then the operator of C-type $T = T_{v,\,w,\,\varphi,\,b}$ on $\ell_p(\mathbb{N})$ is chaotic.

PROOF. We apply Corollary 5.13 with $X_0 := \{e_k\,;\, k \ge 0\}$. Fix $k \ge 0$ and $\varepsilon > 0$. We are looking for a vector $z \in \mathrm{Per}(T)$ and an integer $m \ge 1$ such that $\|z\| < \varepsilon$ and $\|T^m z - e_k\| < \varepsilon$. Let $n \ge 0$ be such that k belongs to $[b_n, b_{n+1})$. By assumption, there exists $N \in \varphi^{-1}(n)$ such that

$$|v_N| \prod_{j=b_N+1}^{b_{N+1}-1} |w_j| > \frac{1}{\varepsilon} \Big(\prod_{j=b_n+1}^{k} |w_j| \Big)^{-1} \max\{1, \|w\|_\infty\}^{k-b_n}. \tag{23}$$

The vector

$$z := v_N^{-1} \Big(\prod_{j=b_N+1}^{b_{N+1}-1} w_j \Big)^{-1} \Big(\prod_{j=b_n+1}^{k} w_j \Big)^{-1} e_{b_N}$$

is periodic by Lemma 6.4, and satisfies $\|z\| < \varepsilon$ by (23). Moreover, since $\varphi(N) = n$, we have by (22):

$$\begin{aligned} T^{\,b_{N+1}-b_N+k-b_n}\, e_{b_N} &= v_N \Big(\prod_{j=b_N+1}^{b_{N+1}-1} w_j \Big) T^{\,k-b_n}\, e_{b_n} - T^{\,k-b_n}\, e_{b_N} \\ &= v_N \Big(\prod_{j=b_N+1}^{b_{N+1}-1} w_j \Big) \Big(\prod_{j=b_n+1}^{k} w_j \Big) e_k - \Big(\prod_{j=b_N+1}^{b_N+k-b_n} w_j \Big) e_{b_N+k-b_n}. \end{aligned}$$

By definition of z, this implies that

$$T^{\,b_{N+1}-b_N+k-b_n}\, z = e_k - \Big(\prod_{j=b_N+1}^{b_N+k-b_n} w_j \Big) v_N^{-1} \Big(\prod_{j=b_N+1}^{b_{N+1}-1} w_j \Big)^{-1} \Big(\prod_{j=b_N+1}^{k} w_j \Big)^{-1} e_{b_N+k-b_n};$$

and by (23), it follows that $\|T^{\,b_{N+1}-b_N+k-b_n}\, z - e_k\| < \varepsilon$. The assumptions of Corollary 5.13 are thus satisfied, and T is chaotic. □

Recall that every chaotic operator is topologically weakly mixing [**8**, Ch. 4] and reiteratively hypercyclic [**13**]. In the following sections, we will be interested in frequent hypercyclicity, $\mathcal{U}$-frequent hypercyclicity and topological mixing for operators of C-type.

To this end, we will often consider operators of C-type for which the data v, w, φ and b have a special structure.

Definition 6.6. An operator of C-type $T_{v,\,w,\,\varphi,\,b}$ is said to be an *operator of C_+-type* if for every integer $k \geq 1$,

- $\varphi(n) = n - 2^{k-1}$ for every $n \in [2^{k-1}, 2^k)$, so that $\varphi([2^{k-1}, 2^k)) = [0, 2^{k-1})$;
- the blocks $[b_n, b_{n+1})$, $n \in [2^{k-1}, 2^k)$, all have the same size, which we denote by $\Delta^{(k)}$:
$$b_{n+1} - b_n = \Delta^{(k)} \qquad \text{for every } n \in [2^{k-1}, 2^k);$$
- the sequence v is constant on the interval $[2^{k-1}, 2^k)$: there exists $v^{(k)}$ such that
$$v_n = v^{(k)} \qquad \text{for every } n \in [2^{k-1}, 2^k);$$
- the sequences of weights $(w_{b_n+i})_{1 \leq i < \Delta^{(k)}}$ are independent of $n \in [2^{k-1}, 2^k)$: there exists a sequence $(w_i^{(k)})_{1 \leq i < \Delta^{(k)}}$ such that
$$w_{b_n+i} = w_i^{(k)} \quad \text{for every } 1 \leq i < \Delta^{(k)} \text{ and every } n \in [2^{k-1}, 2^k).$$

Remark 6.7. By definition, the map φ is the same for all operators of C_+-type, so it is no longer a "parameter". However, we will continue using the notation $T_{v,\,w,\,\varphi,\,b}$.

2. Operators of C_+-type: how to be FHC or UFHC

Our first result concerning operators of C_+-type gives a sufficient set of conditions for a C_+-type operator to be $\mathcal{U}$-frequently hypercyclic. This will be deduced from Theorem 5.14. In the proof, we will make use of the following formula: for every $k \geq 1$, every $l < 2^{k-1}$ and every $1 \leq s \leq \Delta^{(k)}$, we have

$$T^s e_{b_{2^{k-1}+l+1}-s} = v^{(k)} \Big(\prod_{i=\Delta^{(k)}-s+1}^{\Delta^{(k)}-1} w_i^{(k)} \Big) e_{b_l} - \Big(\prod_{i=1}^{\Delta^{(k)}-s} w_i^{(k)} \Big)^{-1} e_{b_{2^{k-1}+l}} \tag{24}$$

This holds true because for every $l < 2^{k-1}$, we have $\varphi(2^{k-1} + l) = l$.

Theorem 6.8. *Let $T = T_{v,\,w,\,\varphi,\,b}$ be an operator of C_+-type on $\ell_p(\mathbb{N})$. Suppose that there exists a constant $\alpha > 0$ such that the following property holds true : for every $C \geq 1$ and every integer $k_0 \geq 1$, there exist two integers $k > k_0$ and $1 \leq m < \Delta^{(k)}$ such that*

$$|v^{(k)}| \prod_{i=\Delta^{(k)}-m}^{\Delta^{(k)}-1} |w_i^{(k)}| \geq C \qquad \textit{and} \qquad |v^{(k)}| \prod_{i=m'+1}^{\Delta^{(k)}-1} |w_i^{(k)}| > C \quad \textit{for every } 0 \leq m' \leq \alpha m.$$

Then T is chaotic and $\mathcal{U}$-frequently hypercyclic.

Proof. We are going to show that the assumption of Theorem 5.14 is satisfied with $X_0 = \operatorname{span}[e_k\,;\, k \geq 0]$. So let us fix $x \in X_0$ and $\varepsilon > 0$. We choose $k_0 \geq 1$ such that x may be written as

$$x = \sum_{l < 2^{k_0}} \sum_{j=b_l}^{b_{l+1}-1} x_j e_j.$$

Let $C > 0$ be a very large number, to be specified later on in the proof. By assumption, there exist $k > k_0$ and $1 \leq m < \Delta^{(k)}$ such that

$$|v^{(k)}| \prod_{i=\Delta^{(k)}-m}^{\Delta^{(k)}-1} |w_i^{(k)}| > C \tag{25}$$

and

$$|v^{(k)}| \prod_{i=m'+1}^{\Delta^{(k)}-1} |w_i^{(k)}| > C \qquad \text{for every } 0 \leq m' \leq \alpha m. \tag{26}$$

Note that since the sequences v and w are bounded, it follows from (25) that the integer m can be chosen as large as we please, provided that C is large enough. So we may assume from the beginning that $m > 2\Delta^{(k_0)}$. We will also assume that $\alpha < 1$, which will be useful below.

We set

$$z := \sum_{l<2^{k_0}} \sum_{j=b_l}^{b_{l+1}-1} x_j \left(v^{(k)} \prod_{i=\Delta^{(k)}-m+j-b_l+1}^{\Delta^{(k)}-1} w_i^{(k)} \right)^{-1} \Big(\prod_{i=1}^{j-b_l} w_{b_l+i} \Big)^{-1} e_{b_{2^{k-1}+l+1}-m+j-b_l}.$$

Our aim is to prove that if C has been suitably chosen, then

$$\|z\| < \varepsilon \qquad \text{and} \qquad \|T^{m+m'}z - T^{m'}x\| < \varepsilon \quad \text{for every } 0 \le m' \le \alpha m/2.$$

Theorem 5.14 will then conclude the proof.

The first of these two claims is the easiest one to prove. Indeed, since the weight sequence w is bounded and bounded from below, it follows from (25) that

$$\|z\| \le \|x\| \cdot C^{-1} \cdot A^{\Delta^{(k_0)}},$$

where A is an absolute constant. So $\|z\| < \varepsilon$ if C is large enough.

Let us now estimate the norm of the vector $T^{m+m'}z - T^{m'}x$ for every $0 \le m' \le \alpha m/2$. Note that if $0 \le l \le 2^{k_0}-1$ and $b_l \le j < b_{l+1}$, then $m-(j-b_l) \ge 1$ since $0 \le j-b_l < \Delta^{(k_0)}$ and $m > \Delta^{(k_0)}$. Applying (24) with $s := m-(j-b_l)$, we get

$$\begin{aligned} T^{m-(j-b_l)} e_{b_{2^{k-1}+l+1}-m+j-b_l} = v^{(k)} \Big(\prod_{i=\Delta^{(k)}-m+j-b_l+1}^{\Delta^{(k)}-1} w_i^{(k)} \Big) e_{b_l} \\ - \left(\prod_{i=1}^{\Delta^{(k)}-m+j-b_l} w_i^{(k)} \right)^{-1} e_{b_{2^{k-1}+l}}, \end{aligned}$$

and hence

$$\begin{aligned} T^m e_{b_{2^{k-1}+l+1}-m+j-b_l} = \quad & \left(v^{(k)} \prod_{i=\Delta^{(k)}-m+j-b_l+1}^{\Delta^{(k)}-1} w_i^{(k)} \right) \Big(\prod_{i=1}^{j-b_l} w_{b_l+i} \Big) e_j \\ & - \left(\prod_{i=j-b_l+1}^{\Delta^{(k)}-m+j-b_l} w_i^{(k)} \right)^{-1} e_{b_{2^{k-1}+l}+j-b_l} \end{aligned}$$

because $T^{j-b_l} e_{b_l} = \Big(\prod_{i=1}^{j-b_l} w_{b_l+i} \Big) e_j$ and $T^{j-b_l} e_{b_{2^{k-1}+l}} = \Big(\prod_{i=1}^{j-b_l} w_i^{(k)} \Big) e_{b_{2^{k-1}+l}+j-b_l}$.

Moreover, if $0 \le m' \le \alpha m/2$, then

$$T^{m'} e_{b_{2^{k-1}+l}+j-b_l} = \left(\prod_{i=j-b_l+1}^{j-b_l+m'} w_i^{(k)} \right) e_{b_{2^{k-1}+l}+j-b_l+m'},$$

because $j-b_l+m' < b_{l+1}-b_l+\alpha m/2 < \Delta^{(k_0)}+m/2 < m < \Delta^{(k)}$. So we get

$$\begin{aligned} T^{m+m'} e_{b_{2^{k-1}+l+1}-m+j-b_l} = & \left(v^{(k)} \prod_{i=\Delta^{(k)}-m+j-b_l+1}^{\Delta^{(k)}-1} w_i^{(k)} \right) \Big(\prod_{i=1}^{j-b_l} w_{b_l+i} \Big) T^{m'} e_j \\ & - \left(\prod_{i=j-b_l+1}^{\Delta^{(k)}-m+j-b_l} w_i^{(k)} \right)^{-1} \left(\prod_{i=j-b_l+1}^{j-b_l+m'} w_i^{(k)} \right) e_{b_{2^{k-1}+l}+j-b_l+m'}. \end{aligned}$$

By definition of z, it follows that for any $0 \le m' \le \alpha m/2$, we have

$$T^{m+m'}z = T^{m'}x - \sum_{l<2^{k_0}} \sum_{j=b_l}^{b_{l+1}-1} \left(v^{(k)} \prod_{i=j-b_l+m'+1}^{\Delta^{(k)}-1} w_i^{(k)} \right)^{-1} \cdot \left(\prod_{i=1}^{j-b_l} w_{b_l+i} \right)^{-1} x_j e_{b_{2^{k-1}+l}+j-b_l+m'}.$$

By (26) and since the weight sequence w is bounded and bounded below, this implies that

$$\|T^{m+m'}z - T^{m'}x\| < \|x\| \cdot C^{-1} \cdot A^{\Delta^{(k_0)}},$$

where (as above) A is an absolute constant. So we have $\|T^{m+m'}z - T^{m'}x\| < \varepsilon$ for every $0 \le m' \le \alpha m/2$ if C is large enough.

The assumptions of Theorem 5.14 are thus satisfied, and this concludes the proof of Theorem 6.8. □

Using Theorem 5.35 instead of Theorem 5.14, we can obtain sufficient conditions of the same kind as above for operators of C_+-type to be frequently hypercyclic.

THEOREM 6.9. *Let $T = T_{v,w,\varphi,b}$ be an operator of C_+-type on $\ell_p(\mathbb{N})$. Suppose that there exists a constant $\alpha > 0$ such that the following property holds true: for every $C \ge 1$ and every $k_0 \ge 1$, there exist two integers $k \ge k_0$ and $1 \le m < \Delta^{(k)}$ such that*

$$|v^{(k)}| \prod_{j=\Delta^{(k)}-m}^{\Delta^{(k)}-1} |w_j^{(k)}| \ge C \tag{27}$$

and

$$|v^{(k)}| \prod_{j=m'+1}^{\Delta^{(k)}-1} |w_j^{(k)}| > C \quad \text{for every } \ 0 \le m' \le \alpha\Delta^{(k)}. \tag{28}$$

Then T is chaotic and frequently hypercyclic.

PROOF. The proof of Theorem 6.9 is so similar to that of Theorem 6.8 that we only sketch it very briefly. The role of the integer d in the assumption of Theorem 5.35 is played by the integer $2\Delta^{(k)}$, which is a period of z. If k is chosen sufficiently large at the beginning of the proof, $\Delta^{(k)}$ can be supposed to be a multiple of the period of any fixed vector $y \in X_0$: indeed, such a vector always has a period of the form $\Delta^{(k_1)}$ for some integer $k_1 \ge 1$, and $2\Delta^{(k_1)}$ divides $\Delta^{(k)}$ as soon as $k > k_1$ since $\Delta^{(k)} = b_{2^{k-1}+2^{k_1-1}+1} - b_{2^{k-1}+2^{k_1-1}}$, $\Delta^{(k_1)} = b_{2^{k_1-1}+1} - b_{2^{k_1-1}}$ and $\varphi(2^{k-1} + 2^{k_1-1}) = 2^{k_1-1}$. □

Now that we have obtained sufficient conditions for the frequent and $\mathcal{U}$-frequent hypercyclicity of operators of C_+-type, we need to find *necessary* conditions as well. This we do in the next section.

3. Operators of C-type: how *not* to be FHC or UFHC

Since $\mathcal{U}$-frequent hypercyclicity and frequent hypercyclicity are strong notions of hypercyclicity, one might think that it is easier to prove that an operator $T \in \mathfrak{B}(\mathcal{H})$ does *not* have one of these properties, than to prove that an operator has it. However, instead of exhibiting a single $\mathcal{U}$-frequent or frequent hypercyclic vector, we now need to prove that no vector of $\mathcal{H}$ whatsoever can be $\mathcal{U}$-frequent or frequent hypercyclic; and put in this way, this no longer looks that easy.

In this section, we are going to single out some conditions ensuring that an operator of C-type T is not $\mathcal{U}$-frequently hypercyclic or not frequently hypercyclic. As suggested in the few lines above, the arguments will be rather more technical than in the previous section. However, we can give the basic idea immediately: if for every hypercyclic vector x for T, we are able to find some

$\varepsilon > 0$ such that the set $\mathcal{N}_T(x, B(0,\varepsilon))$ has upper density (resp. lower density) equal to 0, then T will not be $\mathcal{U}$-frequently hypercyclic (resp. frequently hypercyclic).

3.1. The main criterion, in abstract form. The following notation will be used throughout this section: given an operator of C-type $T = T_{v,\,w,\,\varphi,\,b}$, we denote for each $n \geq 0$ by P_n the canonical projection of $\ell_p(\mathbb{N})$ onto $E_n = \mathrm{span}[e_k,\ b_n \leq k < b_{n+1}]$: if $x = \sum_{k\geq 0} x_k e_k \in \ell_p(\mathbb{N})$, then

$$P_n x = \sum_{k=b_n}^{b_{n+1}-1} x_k e_k.$$

The following theorem provides sufficient conditions for an operator of C-type to be non-$\mathcal{U}$-frequently hypercyclic or non-frequently hypercyclic. These conditions are stated in terms of the projections P_n.

THEOREM 6.10. *Let T be an operator of C-type on $\ell_p(\mathbb{N})$. Suppose that for every hypercyclic vector $x \in \ell_p(\mathbb{N})$ for T, there exist*

- *a positive constant C,*
- *a non-increasing sequence $(\beta_l)_{l\geq 1}$ of positive real numbers with $\sum_{l\geq 1}\sqrt{\beta_l} \leq 1$,*
- *a sequence $(X_l)_{l\geq 0}$ of non-negative real numbers,*
- *a non-decreasing sequence $(N_l)_{l\geq 1}$ of integers tending to infinity,*

such that the following conditions are satisfied:

(1) $\|P_n x\| \leq X_n$ *for every* $n \geq 0$*;*
(2) $\sup_{j\geq 0} \|P_n T^j P_l\, x\| \leq C\beta_l X_l$ *for every* $l \geq 1$ *and every* $0 \leq n < l$*;*
(3) $\sup_{0\leq j\leq N_l} \|P_n T^j P_l\, x\| \leq C\beta_l \|P_l x\|$ *for every* $l \geq 1$ *and every* $0 \leq n < l$*;*
(C) $\liminf_{l\to\infty} \inf_{k\geq N_l} \dfrac{\#\{0 \leq j \leq k\,;\, \|P_l T^j P_l\, x\| \geq 2CX_l\}}{k+1} = 1.$

Then T is not $\mathcal{U}$-frequently hypercyclic.

If Condition (C) *is replaced by*

(C') $\liminf_{l\to\infty} \inf_{k\geq N_{\min(\varphi^{-1}(l))}} \dfrac{\#\{0 \leq j \leq k\,;\, \|P_l T^j P_l\, x\| \geq 2CX_l\}}{k+1} = 1,$

then T is not frequently hypercyclic.

The usefulness of this result lies in the fact that the lower bounds for the densities are given in terms of norms $\|P_n T^j P_n x\|$ which are easily computable. Moreover, since $P_n x$ is periodic, we can determine the cardinality of the sets $\{0 \leq j \leq k; \|P_n T^j P_n x\| \geq 2CX_n\}$ by examining only a fixed finite number of iterates (independent of k).

Theorem 6.10 will follow very easily from the next lemma, which provides, under conditions (1), (2), and (3) above on the projections P_n, lower bounds for the upper and lower density of some sets $\mathcal{N}_T(x, B(0,\varepsilon)^c)$, where x is any non-zero vector of $\ell_p(\mathbb{N})$ and ε is a positive number depending on x. Here, of course, $B(0,\varepsilon)^c$ denotes the complement of the the ball $B(0,\varepsilon)$ in $\ell_p(\mathbb{N})$.

LEMMA 6.11. *Let T be an operator of C-type on $\ell_p(\mathbb{N})$. Fix $x \in \ell_p(\mathbb{N}) \setminus \{0\}$ and suppose that there exist*

- *a positive constant C,*
- *a non-increasing sequence $(\beta_l)_{l\geq 1}$ of positive real numbers with $\sum_{l\geq 1}\sqrt{\beta_l} \leq 1$,*
- *a sequence $(X_l)_{l\geq 0}$ of non-negative real numbers,*
- *a non-decreasing sequence $(N_l)_{l\geq 1}$ of integers tending to infinity,*

such that the following conditions are satisfied:

(1) $\|P_n x\| \leq X_n$ *for every* $n \geq 0$*;*
(2) $\sup_{j\geq 0} \|P_n T^j P_l\, x\| \leq C\beta_l X_l$ *for every* $l \geq 1$ *and every* $0 \leq n < l$*;*
(3) $\sup_{0\leq j\leq N_l} \|P_n T^j P_l\, x\| \leq C\beta_l \|P_l x\|$ *for every* $l \geq 1$ *and every* $0 \leq n < l$*;*
(4) $\sup_{j\geq 0} \sum_{l>n} \|P_n T^j P_l x\| > CX_n$ *for every* $n \geq 0$.

Then there exists $\varepsilon > 0$ *such that*

$$\underline{\text{dens}}\, \mathcal{N}_T\big(x, B(0,\varepsilon)^c\big) \geq \liminf_{l\to\infty} \inf_{k\geq N_l} \frac{\#\big\{0 \leq j \leq k\,;\, \|P_l T^j P_l\, x\| \geq 2CX_l\big\}}{k+1}$$

and

$$\overline{\text{dens}}\, \mathcal{N}_T\big(x, B(0,\varepsilon)^c\big) \geq \liminf_{l\to\infty} \inf_{k\geq N_{\min(\varphi^{-1}(l))}} \frac{\#\big\{0 \leq j \leq k\,;\, \|P_l T^j P_l\, x\| \geq 2CX_l\big\}}{k+1}.$$

Before giving the proof of Lemma 6.11, let us first show how it implies Theorem 6.10.

Proof of Theorem 6.10. For every $n \geq 0$ and every $x \in \ell_p(\mathbb{N})$, the vector $\sum_{0\leq l\leq n} P_l\, x$ is a periodic vector for T. Hence, if x is hypercyclic for T then

$$\sup_{j\geq 0} \sum_{l>n} \|P_n T^j P_l\, x\| = \infty. \tag{29}$$

Indeed, otherwise we would have

$$\sup_{j\geq 0} \|P_n T^j\, x\| \leq \sup_{j\geq 0} \Big\|T^j\Big(\sum_{0\leq l\leq n} P_l\, x\Big)\Big\| + \sup_{j\geq 0} \sum_{l>n} \|P_n T^j P_l\, x\| < \infty,$$

a contradiction with the hypercyclicity of x.

By (29), condition (4) of Lemma 6.11 is satisfied for any choice of the sequence $(X_n)_{n\geq 0}$. It follows that if condition (C) in Theorem 6.10 is satisfied, then, for any $x \in HC(T)$, one can find $\varepsilon > 0$ such that $\overline{\text{dens}}\,\big(\mathcal{N}_T(x, B(0,\varepsilon)\big) = 0$; whereas if (C') is satisfied, then, for any $x \in HC(T)$, one can find $\varepsilon > 0$ such that $\underline{\text{dens}}\,\big(\mathcal{N}_T(x, B(0,\varepsilon)\big) = 0$. This concludes the proof. □

Let us now go back to the proof of Lemma 6.11. Before giving it, let us first explain the general idea. The starting point is given by the inequalities

$$\|T^j x\| \geq \|P_n T^j x\| \geq \|P_n T^j P_n x\| - \sum_{l>n} \|P_n T^j P_l x\|.$$

In other words, we get a lower bound for $\|T^j x\|$ as soon as we have a lower bound for some quantity $\|P_n T^j P_n x\|$ and an upper bound for $\|P_n T^j P_l x\|$, $l > n$. The role of the lower bound for $\|P_n T^j P_n x\|$ will be played by $2CX_n$, which will be bigger than 2ε. Assumption (3) will provide us with an upper bound for $\|P_n T^j P_l x\|$ when j is smaller than N_l. However, condition (4) tells us that $\|P_n T^j P_l x\|$ will be large for some $j \geq 0$ and some $l > n$. Assumption (2) will then be used to deduce that $X_l \geq X_n$, which will allow us to deduce that $2CX_l \geq 2\varepsilon$ and to repeat the above arguments for $\|P_l T^j x\|$, and so on.

Proof of Lemma 6.11. If x is a non-zero periodic vector for T, one can obviously choose $\varepsilon > 0$ such that

$$\underline{\text{dens}}\, \mathcal{N}_T(x, B(0,\varepsilon)^c) = 1$$

and the conclusion of Lemma 6.11 is satisfied. So we henceforward suppose that x is not a periodic vector for T.

We first note that there exists an integer $l_0 \geq 1$ such that

$$\|P_{l_0}\, x\| \geq \sqrt{\beta_{l_0}}\, \|x - P_0\, x\|.$$

Indeed, since $\sum_{l\geq 1}\sqrt{\beta_l}\leq 1$ it is impossible to have $\|P_l\,x\|<\sqrt{\beta_l}\;\|x-P_0\,x\|$ for every $l\geq 1$. Since x is not periodic, $x-P_0\,x$ is non-zero. Assumption (1) of Lemma 6.11 implies thus that X_{l_0} is non-zero and we set $\varepsilon=CX_{l_0}$.

We now construct, by induction on $n\geq 1$, a strictly increasing sequence of integers $(l_n)_{n\geq 1}$ such that if we set

$$j_{n-1}:=\min\Big\{j\geq 0\,;\,\sum_{l>l_{n-1}}\|P_{l_{n-1}}T^jP_l\,x\|>CX_{l_{n-1}}\Big\},$$

then

$$j_{n-1}>N_{l_n}\quad\text{and}\quad X_{l_n}\geq\frac{1}{\sqrt{\beta_{l_n}}}\,X_{l_{n-1}}\quad\text{for every } n\geq 1.$$

Observe that this does make sense: the set involved in the definition of j_{n-1} is non-empty by assumption (4).

Suppose that the integers $l_1,\dots,l_{n-1}$ have already been constructed. Then there exists an integer $l_n>l_{n-1}$ with the property that

$$\|P_{l_{n-1}}T^{j_{n-1}}P_{l_n}\,x\|>C\sqrt{\beta_{l_n}}X_{l_{n-1}}. \tag{30}$$

Indeed, if we had $\|P_{l_{n-1}}T^{j_{n-1}}P_l\,x\|\leq C\sqrt{\beta_l}X_{l_{n-1}}$ for every $l>l_{n-1}$, this would imply that $\sum_{l>l_{n-1}}\|P_{l_{n-1}}T^{j_{n-1}}P_l\,x\|\leq CX_{l_{n-1}}$, violating the definition of j_{n-1}.

By assumption (2), we have

$$X_{l_n}\geq\frac{1}{C\beta_{l_n}}\|P_{l_{n-1}}T^{j_{n-1}}P_{l_n}\,x\|\geq\frac{1}{\sqrt{\beta_{l_n}}}\,X_{l_{n-1}}.$$

In order to show that $j_{n-1}>N_{l_n}$, we observe that for every $0\leq j\leq N_{l_n}$,

$$\|P_{l_{n-1}}T^jP_{l_n}\,x\|\leq C\beta_{l_n}\,\|P_{l_n}\,x\|\leq C\beta_{l_n}\,\|x-P_0x\|\leq\frac{C\beta_{l_n}}{\sqrt{\beta_{l_0}}}\,\|P_{l_0}\,x\|\leq\frac{C\beta_{l_n}}{\sqrt{\beta_{l_0}}}\,X_{l_0},$$

by assumption (3) and the definition of l_0. Since the sequence $(\beta_l)_{l\geq 0}$ is decreasing, this yields that $\|P_{l_{n-1}}T^jP_{l_n}\,x\|\leq C\sqrt{\beta_{l_n}}\,X_{l_0}$. Also, the induction hypothesis implies that $X_{l_0}\leq X_{l_1}\leq\cdots\leq X_{l_n}$, so that $\|P_{l_{n-1}}T^jP_{l_n}\,x\|\leq C\sqrt{\beta_{l_n}}\,X_{l_{n-1}}$ for every $0\leq j\leq N_{l_n}$. Inequality (30) thus implies that $j_{n-1}>N_{l_n}$ and this finishes the induction. Note that the sequence $(j_n)_{n\geq 0}$ tends to infinity as n tends to infinity.

For any integer $s\geq 1$, let us denote by n_s be the smallest integer such that s belongs to $[j_{n_s-1},j_{n_s})$. Then n_s tends to infinity as s tends to infinity. Since

$$\|T^j\,x\|\geq\|P_n\,T^j\,x\|\geq\|P_n\,T^jP_n\,x\|-\sum_{l>n}\|P_n\,T^jP_l\,x\|$$

for every $j\geq 0$ and $n\geq 0$, we have

$$\|T^jx\|\geq\|P_{l_n}T^jP_{l_n}\,x\|-C\,X_{l_n}\quad\text{for every } 0\leq j<j_n.$$

It follows from this inequality and the fact that $j<j_{n_s}$ for every $j<s$ that

$$\begin{aligned}\big\{0\leq j\leq s\,;\,\|P_{l_{n_s}}T^jP_{l_{n_s}}\,x\|\geq 2CX_{l_{n_s}}\big\}&\subseteq\big\{0\leq j\leq s\,;\,\|T^jx\|\geq CX_{l_{n_s}}\big\}\\&\subseteq\big\{0\leq j\leq s\,;\,\|T^jx\|\geq CX_{l_0}\big\}.\end{aligned}$$

Hence, since $\varepsilon=CX_{l_0}$, we have

$$\underline{\text{dens}}\;\mathcal{N}_T\big(x,B(0,\varepsilon)^c\big)\geq\liminf_{s\to\infty}\;\frac{\#\big\{0\leq j\leq s\,;\,\|P_{l_{n_s}}T^jP_{l_{n_s}}\,x\|\geq 2CX_{l_{n_s}}\big\}}{s+1}.$$

Now $N_{l_s} < j_{n_s-1} \le s$, so that s belongs to the interval $[N_{l_s}, \infty)$ for each $s \ge 1$. Thus

$$\frac{\#\{0 \le j \le s\,;\, \|P_{l_{n_s}} T^j P_{l_{n_s}} x\| \ge 2CX_{l_{n_s}}\}}{s+1} \ge \inf_{k \ge N_{l_s}} \frac{\#\{0 \le j \le k\,;\, \|P_{l_{n_s}} T^j P_{l_{n_s}} x\| \ge 2CX_{l_{n_s}}\}}{k+1}$$

and

$$\begin{aligned}\underline{\text{dens}}\, \mathcal{N}_T\big(x, B(0,\varepsilon)^c\big) &\ge \liminf_{s\to\infty} \inf_{k \ge N_{l_s}} \frac{\#\{0 \le j \le k\,;\, \|P_{l_{n_s}} T^j P_{l_{n_s}} x\| \ge 2CX_{l_{n_s}}\}}{k+1} \\ &\ge \liminf_{l\to\infty} \inf_{k \ge N_l} \frac{\#\{0 \le j \le k\,;\, \|P_l T^j P_l x\| \ge 2CX_l\}}{k+1}\end{aligned}$$

since $l_{n_s} \to \infty$ as $s \to \infty$. This proves the first part of Lemma 6.11.

As for the second part, we proceed in the same way, starting from the inequality

$$\overline{\text{dens}}\, \mathcal{N}_T\big(x, B(0,\varepsilon)^c\big) \ge \limsup_{s\to\infty} \frac{\#\{0 \le j \le s\,;\, \|P_{l_{n_s}} T^j P_{l_{n_s}} x\| \ge 2CX_{l_{n_s}}\}}{s+1}.$$

Our first observation is that $\min(\varphi^{-1}(l_n)) \le l_{n+1}$ for every $n \ge 1$. This follows, in a slightly roundabout way, from the fact that $P_{l_n} T^{j_n} P_{l_{n+1}}$ is non-zero. Indeed, the definition of T implies that there exists an integer $i \ge 1$ such that $\varphi^i(l_{n+1}) = l_n$. Then $\varphi^{i-1}(l_{n+1})$ belongs to $\varphi^{-1}(l_n)$, and since $\varphi^{i-1}(l_{n+1}) \le l_{n+1}$, it follows that $\min(\varphi^{-1}(l_n)) \le l_{n+1}$.

We now claim that there exists a strictly increasing sequence of integers $(s_i)_{i\ge1}$ such that $s_i \ge N_{\min(\varphi^{-1}(l_{n_{s_i}}))}$ for every $i \ge 1$. Recall that $N_{l_{n_i+1}} < j_{n_i}$ for all $i \ge 1$. Thus there exists $s_i \in [j_{n_i-1}, j_{n_i})$ such that $s_i \ge N_{l_{n_i+1}}$. As s_i belongs to the interval $[j_{n_i-1}, j_{n_i})$, we have $n_{s_i} \le n_i$, and thus $s_i \ge N_{l_{n_{s_i}+1}} \ge N_{\min(\varphi^{-1}(l_{n_{s_i}}))}$. Extracting if necessary in order to make the sequence $(s_i)_{i\ge1}$ strictly increasing, we obtain the sequence we are looking for.

We now have

$$\begin{aligned}\overline{\text{dens}}\, \mathcal{N}_T\big(x, B(0,\varepsilon)^c\big) &\ge \liminf_{i\to\infty} \frac{\#\{0 \le j \le s_i\,;\, \|P_{l_{n_{s_i}}} T^j P_{l_{n_{s_i}}} x\| \ge 2CX_{l_{n_{s_i}}}\}}{s_i+1} \\ &\ge \liminf_{n\to\infty} \inf_{k \ge N_{\min(\varphi^{-1}(l_n))}} \frac{\#\{0 \le j \le k\,;\, \|P_{l_n} T^j P_{l_n} x\| \ge 2CX_{l_n}\}}{k+1} \\ &\ge \liminf_{l\to\infty} \inf_{k \ge N_{\min(\varphi^{-1}(l))}} \frac{\#\{0 \le j \le k\,;\, \|P_l T^j P_l x\| \ge 2CX_l\}}{k+1}.\end{aligned}$$

This concludes the proof of Lemma 6.11. □

3.2. How to check the assumptions. The assumptions of Theorem 6.10 are partly of an "abstract" nature, since they involve the projections P_n and not explicitly the parameters v, w, φ, and b defining T. We now provide concrete conditions on v, w, φ, and b which imply that the operator of C-type $T_{v,w,\varphi,b}$ satisfies the assumptions of Theorem 6.10.

As a rule, conditions (1), (2), and (3) will be obtained by requiring that the products of weights

$$|v_n| \cdot \sup_{j \in [b_{\varphi(n)}, b_{\varphi(n)+1})} \prod_{s=b_{\varphi(n)}+1}^{j} |w_s|$$

decrease sufficiently rapidly and by setting, given a hypercyclic vector x for $T_{v,w,\varphi,b}$,

$$X_l := \Big\| \sum_{k=b_l}^{b_{l+1}-1} \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s \Big) x_k e_k \Big\|, \quad l \ge 0.$$

More precisely, we have the following result, which generalizes Claims 5 and 6 in [**40**].

PROPOSITION 6.12. *Let $T = T_{v,\,w,\,\varphi,\,b}$ be an operator of C-type on $\ell_p(\mathbb{N})$, and let $(C_n)_{n\geq 0}$ be a sequence of positive numbers with $0 < C_n < 1$. Assume that*

$$|v_n| \,.\, \sup_{j\in[b_{\varphi(n)},b_{\varphi(n)+1})} \Big(\prod_{s=b_{\varphi(n)}+1}^{j} |w_s|\Big) \leq C_n \quad \textit{for every } n \geq 0.$$

Then, for any $x \in \ell_p(\mathbb{N})$, we have for every $l \geq 1$ and every $0 \leq n < l$,

$$(1) \qquad \sup_{j\geq 0} \|P_n T^j P_l\, x\| \leq C_l\,(b_{l+1}-b_l)^{\frac{p-1}{p}} \Big\| \sum_{k=b_l}^{b_{l+1}-1} \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s\Big) x_k e_k \Big\|$$

and

$$(2) \qquad \sup_{j\leq N} \|P_n T^j P_l\, x\| \leq C_l\,(b_{l+1}-b_l)^{\frac{p-1}{p}} \Big(\sup_{b_{l+1}-N\leq k<b_{l+1}} \prod_{s=k+1}^{b_{l+1}-1} |w_s| \Big) \|P_l\, x\|$$

for every $1 \leq N \leq b_{l+1}-b_l$.

PROOF. Fix $l \geq 1$ and $0 \leq n < l$. We first remark that if $n \neq \varphi^M(l)$ for all $M \geq 1$, the definition of T implies that $P_n T^j P_l\, x = 0$ for every $j \geq 0$ (this argument was already used at the end of the proof of Lemma 6.11). So the required inequalities are obvious in this case.

Suppose now that $n = \varphi^M(l)$ for some integer $M \geq 1$. In this case, we claim that

$$(31) \qquad \sup_{j\geq 0} \|P_n T^j\, e_k\| \leq |v_l| \Big(\prod_{s=k+1}^{b_{l+1}-1} |w_s|\Big) \sup_{j\geq 0} \|P_n T^j e_{b_{\varphi(l)}}\|$$

for every $k \in [b_l, b_{l+1})$. This inequality will allow us to run an induction procedure in order to estimate the quantity $\sup_{j\geq 0} \|P_n T^j P_l\, x\|$.

In order to prove (31), we fix $j_0 \geq 0$ and we consider separately several cases.

- Suppose first that $0 \leq j_0 < b_{l+1}-k$. Then $T^{j_0}\, e_k$ is a scalar multiple of e_{k+j_0}, and $P_n T^{j_0}\, e_k = 0$.
- Now, suppose that $b_{l+1}-k \leq j_0 < b_{l+1}-k+b_{l+1}-b_l$. Then

$$P_n T^{j_0}\, e_k = P_n T^{j_0-(b_{l+1}-k)} \Big(v_l \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s\Big) e_{b_{\varphi(l)}}\Big),$$

so that

$$\|P_n T^{j_0}\, e_k\| \leq |v_l| \Big(\prod_{s=k+1}^{b_{l+1}-1} |w_s|\Big) \sup_{j\geq 0} \|P_n T^j\, e_{b_{\varphi(l)}}\|.$$

- Finally, suppose that $b_{l+1}-k+b_{l+1}-b_l \leq j_0 < 2(b_{l+1}-b_l)$. In this case, we write

$$T^{j_0} e_k = \Big(\prod_{s=b_l+1}^{k} w_s\Big)^{-1} T^{j_0+k-b_l} e_{b_l}.$$

By assumption, we have $2(b_{l+1}-b_l) \leq j_0+k-b_l < 2(b_{l+1}-b_l)+(b_{l+1}-b_l)$; that is,

$$j_0+k-b_l = i+2(b_{l+1}-b_l) \qquad \text{with } 0 \leq i < b_{l+1}-b_l.$$

Since $T^{2(b_{l+1}-b_l)}\, e_{b_l} = e_{b_l}$, it follows that

$$T^{j_0} e_k = \Big(\prod_{s=b_l+1}^{k} w_s\Big)^{-1} T^i e_{b_l} = \Big(\prod_{s=b_l+1}^{k} w_s\Big)^{-1} \Big(\prod_{s=b_l+1}^{l+i} w_s\Big) e_{b_l+i}$$

and hence that $P_n T^{j_0}\, e_k = 0$.

We have thus proved the required inequality for every $0 \leq j_0 < 2(b_{l+1}-b_l)$. Since

$$T^{2(b_{l+1}-b_l)}\, e_k = e_k,$$

it holds true in fact for every $j \geq 0$, and this proves (31).

Let now M_0 be the minimum of the integers $M \geq 1$ such that $\varphi^M(l) = n$. We start from the straightforward estimate

$$\sup_{j\geq 0} \|P_n T^j P_l\, x\| \leq \sum_{k=b_l}^{b_{l+1}-1} |x_k| \sup_{j\geq 0} \|P_n T^j\, e_k\|$$

and apply (31). This gives

$$\begin{aligned}\sup_{j\geq 0} \|P_n T^j P_l\, x\| &\leq \sum_{k=b_l}^{b_{l+1}-1} |x_k| \cdot |v_l| \cdot \Big(\prod_{s=k+1}^{b_{l+1}-1} |w_s|\Big) \sup_{j\geq 0} \|P_n T^j e_{b_{\varphi(l)}}\| \\ &= |v_l| \cdot \sup_{j\geq 0} \|P_n T^j\, e_{b_{\varphi(l)}}\| \cdot \Big\| \sum_{k=b_l}^{b_{l+1}-1} \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s\Big) x_k e_k \Big\|_1,\end{aligned}$$

where $\|z\|_1$ denotes the ℓ^1 norm of a vector $z \in c_{00}$.

By induction, we obtain

$$\begin{aligned}\sup_{j\geq 0} \|P_n T^j P_l\, x\| &\leq |v_l| \cdot \prod_{r=1}^{M_0-1} \Big(|v_{\varphi^r(l)}| \cdot \prod_{s=b_{\varphi^r(l)}+1}^{b_{\varphi^r(l)+1}-1} |w_s|\Big) \cdot \\ &\qquad \sup_{j\geq 0} \|P_n T^j\, e_{b_n}\| \cdot \Big\| \sum_{k=b_l}^{b_{l+1}-1} \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s\Big) x_k e_k \Big\|_1 \\ &= \prod_{r=1}^{M_0-1} \Big(|v_{\varphi^{r-1}(l)}| \prod_{s=b_{\varphi^r(l)}+1}^{b_{\varphi^r(l)+1}-1} |w_s|\Big) \cdot |v_{\varphi^{M_0-1}(l)}| \\ &\qquad \sup_{j\geq 0} \|P_n T^j e_{b_n}\| \cdot \Big\| \sum_{k=b_l}^{b_{l+1}-1} \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s\Big) x_k e_k \Big\|_1.\end{aligned}$$

Applying the assumption of Proposition 6.12, it follows that

$$\begin{aligned}\sup_{j\geq 0} \|P_n T^j P_l\, x\| \leq \Big(\prod_{r=1}^{M_0-1} C_{\varphi^{r-1}(l)}\Big) \cdot |v_{\varphi^{M_0-1}(l)}| \cdot \sup_{j\geq 0} \|P_n T^j\, e_{b_n}\| \cdot \\ \Big\| \sum_{k=b_l}^{b_{l+1}-1} \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s\Big) x_k e_k \Big\|_1.\end{aligned}$$

Now, the definition of T and the fact that $T^{2(b_{n+1}-b_n)}\, e_{b_n} = e_{b_n}$ show that

$$\sup_{j\geq 0} \|P_n T^j\, e_{b_n}\| \leq \sup_{j\in[b_n, b_{n+1})} \prod_{s=b_n+1}^{j} |w_s|.$$

Since $n = \varphi^{M_0}(l)$, it follows (by the assumption of Proposition 6.12), that

$$\begin{aligned}\sup_{j\geq 0} \|P_n T^j P_l\, x\| &\leq \Big(\prod_{r=1}^{M_0-1} C_{\varphi^{r-1}(l)}\Big) \cdot C_{\varphi^{M_0-1}(l)} \cdot \Big\| \sum_{k=b_l}^{b_{l+1}-1} \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s\Big) x_k e_k \Big\|_1 \\ &\leq C_l \cdot \Big\| \sum_{k=b_l}^{b_{l+1}-1} \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s\Big) x_k e_k \Big\|_1,\end{aligned}$$

because all the constants C_n are supposed to belong to $(0,1)$ and $\varphi^0(l) = l$. By Hölder's inequality, we conclude that

$$\sup_{j\geq 0} \|P_n T^j P_l\, x\| \leq C_l \,\cdot\, (b_{l+1}-b_l)^{1-\frac{1}{p}} \,\cdot\, \Big\| \sum_{k=b_l}^{b_{l+1}-1} \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s \Big) x_k e_k \Big\|,$$

which proves assertion (1) of Proposition 6.12.

The proof of assertion (2) is now straightforward. Indeed, we have for every $1 \leq N \leq b_{l+1}-b_l$ and every $0 \leq j \leq N$:

$$T^j P_l\, x = T^j \Big(\sum_{k=b_l}^{b_{l+1}-N-1} x_k e_k \Big) + T^j \Big(\sum_{k=b_{l+1}-N}^{b_{l+1}-1} x_k e_k \Big).$$

Since the first term belongs to the linear span of the vectors e_i, $b_l \leq i < b_{l+1}$, it follows that

$$P_n T^j P_l\, x = P_n T^j \Big(\sum_{k=b_{l+1}-N}^{b_{l+1}-1} x_k e_k \Big),$$

and so, by the already proved assertion (1) of Proposition 6.12,

$$\begin{aligned} \|P_n T^j P_l\, x\| &\leq C_l \,\cdot\, (b_{l+1}-b_l)^{1-\frac{1}{p}} \,\cdot\, \Big\| \sum_{k=b_{l+1}-N}^{b_{l+1}-1} \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s \Big) x_k e_k \Big\| \\ &\leq C_l \,\cdot\, (b_{l+1}-b_l)^{1-\frac{1}{p}} \,\cdot\, \sup_{b_{l+1}-N \leq k < b_{l+1}} \Big(\prod_{s=k+1}^{b_{l+1}-1} |w_s\,| \Big) \,\cdot\, \|P_l\, x\|. \end{aligned}$$

This concludes the proof of Proposition 6.12. □

The next result provides conditions on the parameters of an operator of C-type ensuring that assumption (C) or (C') of Theorem 6.10 is satisfied. This generalizes Claim 7 of [**40**].

Proposition 6.13. *Let $T = T_{v,\,w,\,\varphi,\,b}$ be an operator of C-type on $\ell_p(\mathbb{N})$, and let $x \in \ell_p(\mathbb{N})$. Fix $l \geq 0$, and define*

$$X_l = \Big\| \sum_{k=b_l}^{b_{l+1}-1} \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s \Big) x_k e_k \Big\|.$$

Suppose that there exist two integers $0 \leq k_0 < k_1 \leq b_{l+1}-b_l$ such that

$$|w_{b_l+k}| = 1 \quad \textit{for every } k \in (k_0, k_1) \quad \textit{and} \quad \prod_{s=b_l+k_0+1}^{b_{l+1}-1} |w_s| = 1.$$

Then we have for every $J \geq 0$

$$\begin{aligned} \frac{1}{J+1}\, \#\Big\{ 0 \leq j \leq J\,;\, \|P_l T^j P_l\, x\| \geq X_l/2 \Big\} \\ \geq 1 - 2\big(b_{l+1}-b_l-(k_1-k_0)\big) \,\cdot\, \Big(\frac{1}{J+1} + \frac{1}{b_{l+1}-b_l} \Big). \end{aligned}$$

Proof. Fix $j \geq 0$ and $k \in [b_l, b_{l+1})$, and set $n := j+k-b_l \mod (b_{l+1}-b_l)$. Then

$$\begin{aligned} P_l T^j\, e_k &= \pm \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s \Big) \Big(\prod_{s=b_l+1}^{b_l+n} w_s \Big) \Big(\prod_{s=b_l+1}^{b_{l+1}-1} w_s \Big)^{-1} e_{b_l+n} \\ &= \pm \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s \Big) \Big(\prod_{s=b_l+n+1}^{b_{l+1}-1} w_s \Big)^{-1} e_{b_l+n} \end{aligned}$$

(the plus or minus sign appearing in these equalities depends on the parity of the unique integer $s \geq 0$ such that n belongs to the interval $[s(b_{l+1}-b_l), (s+1)(b_{l+1}-b_l))$).

If we suppose that n belongs to $[k_0, k_1)$, we have

$$\prod_{s=b_l+n+1}^{b_{l+1}-1} |w_s| = \prod_{s=b_l+k_0+1}^{b_{l+1}-1} |w_s| = 1,$$

and thus there exists $\zeta_{j,k} \in \mathbb{T}$ such that

$$P_l T^j\, e_k = \zeta_{j,k} \cdot \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s \Big)\, e_{b_l+n}.$$

Setting, for every $j \geq 0$,

$$I_j := \Big\{ k \in [b_l, b_{l+1})\,;\, j + k - b_l \mod (b_{l+1} - b_l) \text{ does not belong to } [k_0, k_1) \Big\},$$

we obtain that

$$P_l T^j P_l\, x = \sum_{k=b_l}^{b_{l+1}-1} x_k\, P_l T^j\, e_k = \sum_{k\,\in\,[b_l, b_{l+1})\setminus I_j} x_k\, P_l T^j\, e_k + \sum_{k\,\in\, I_j} x_k\, P_l T^j\, e_k$$

satisfies

$$\begin{aligned} \|P_l T^j P_l\, x\| &\geq \Big\| \sum_{k\,\in\,[b_l, b_{l+1})\setminus I_j} x_k\, P_l T^j\, e_k \Big\| = \Big\| \sum_{k\,\in\,[b_l, b_{l+1})\setminus I_j} x_k\, \zeta_{j,k} \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s \Big)\, e_k \Big\| \\ &= \Big\| \sum_{k\,\in\,[b_l, b_{l+1})\setminus I_j} x_k \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s \Big)\, e_k \Big\| \geq X_l - \Big\| \sum_{k\,\in\, I_j} x_k \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s \Big)\, e_k \Big\|. \end{aligned}$$

We now consider separately two cases:

- Suppose first that $\Big\| \sum_{k\,\in\, I_j} x_k \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s \Big)\, e_k \Big\| < X_l/2$ for every $j \geq 0$. Then we have

$$\|P_l T^j P_l\, x\| \geq X_l/2 \quad \text{for every } j \geq 0,$$

and thus

$$\frac{1}{J+1}\, \#\big\{ 0 \leq j \leq J\,;\, \|P_l T^j P_l\, x\| \geq X_l/2 \big\} = 1 \quad \text{for every } J \geq 0.$$

- Now, suppose that there exists $j_0 \geq 0$ such that

$$\Big\| \sum_{k\,\in\, I_{j_0}} x_k \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s \Big)\, e_k \Big\| \geq \frac{X_l}{2}.$$

Then we have for every integer $j \geq 0$ such that $I_j \cap I_{j_0}$ is empty:

$$\|P_l T^j P_l\, x\| \geq \Big\| \sum_{k\,\in\,[b_l, b_{l+1})\setminus I_j} \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s \Big) x_k e_k \Big\| \geq \Big\| \sum_{k\,\in\, I_{j_0}} \Big(\prod_{s=k+1}^{b_{l+1}-1} w_s \Big) x_k e_k \Big\| \geq \frac{X_l}{2}.$$

It follows that, for every $J \geq 0$,

$$\begin{aligned} &\frac{1}{J+1}\, \#\Big\{ 0 \leq j \leq J\,;\, \|P_l T^j P_l\, x\| \geq X_l/2 \Big\} \\ &\qquad\qquad \geq 1 - \frac{1}{J+1}\, \#\Big\{ 0 \leq j \leq J\,;\, I_j \cap I_{j_0} \neq \emptyset \Big\}. \end{aligned} \tag{32}$$

Now, we remark that if we set $i_j := j \mod (b_{l+1} - b_j)$ for every $j \geq 0$, we have

$$I_j = \begin{cases} [b_l + k_1 - i_j, b_{l+1}) \cup [b_l, k_0 - i_j) & \text{if } 0 \leq i_j < k_0 \\ [b_l + k_1 - i_j, b_{l+1} + k_0 - i_j) & \text{if } k_0 \leq i_j \leq k_1 \\ [b_l, b_{l+1} + k_0 - i_j) \cup [b_{l+1} + k_1 - i_j, b_{l+1}) & \text{if } k_1 < i_j < b_{l+1} - b_l \end{cases}$$

and $\#I_j = b_{l+1} - b_l - (k_1 - k_0)$ for every $j \geq 0$. This particular structure of the sets I_j implies that

$$\# \left\{0 \leq j \leq J\,;\, I_j \cap I_{j_0} \neq \emptyset\right\} \leq 2(b_{l+1} - b_l - (k_1 - k_0)) \cdot \left(\left\lfloor \frac{J}{b_{l+1} - b_l} \right\rfloor + 1 \right).$$

Plugging this into (32) yields that

$$\frac{1}{J+1}\, \#\Big\{0 \leq j \leq J\,;\, \|P_l T^j P_l\, x\| \geq X_l/2\Big\} \geq 1 - 2(b_{l+1} - b_l - (k_1 - k_0))\Big(\frac{1}{J+1} + \frac{1}{b_{l+1} - b_l}\Big),$$

which is the inequality we were looking for. □

Propositions 6.12 and 6.13 will be repeatedly used in Sections 2, 3 and 4 in order to exhibit the desired counterexamples.

4. Operators of C-type: how to be mixing or not mixing

In this section, we give some conditions ensuring that an operator of C-type is or is not topologically mixing. This will allow us to show in particular in Section 2.2 that the assumptions of our criterion for frequent hypercyclicity (Theorem 5.35) do not imply the Operator Specification Property.

Proposition 6.14. *Let $T = T_{v,\,w,\,\varphi,\,b}$ be an operator of C-type on $\ell_p(\mathbb{N})$. Suppose that for every $\eta > 0$ and every $n \geq 0$, the set*

$$S_{\eta,n} := \bigcup_{m \in \mathcal{N}_{\eta,n}} \Big\{ s \in [1, b_{m+1} - b_m) \;:\; |v_m| \prod_{i=b_{m+1}-s}^{b_{m+1}-1} |w_i| > \frac{1}{\eta} \Big\} \quad \textit{is cofinite,}$$

where $\mathcal{N}_{\eta,n} = \{m \in \varphi^{-1}(n) \;:\; |v_m| \prod_{i=b_m+1}^{b_{m+1}-1} |w_i| > \frac{1}{\eta}\}$. Then T is topologically mixing.

Proof. Since T has a dense set of periodic points, Corollary 5.4 implies that it suffices to show that for every $k \geq 0$ and every $\varepsilon > 0$, there exists an integer $M \geq 1$ such that for every $m \geq M$, there exists a vector $z \in \ell_p(\mathbb{N})$ such that $\|z\| < \varepsilon$ and $\|T^m z - e_k\| < \varepsilon$.

Fix thus $k \geq 0$ and $\varepsilon > 0$. Let also $n \geq 0$ be such that k belongs to $[b_n, b_{n+1})$, and let $\eta > 0$ be a sufficiently small positive number, to be fixed later on in the proof. By assumption, the set $S_{\eta,n}$ is cofinite. So it suffices to show that if η is small enough, then, for every $s \in S_{\eta,n}$, one can find a vector $z_s \in \ell_p(\mathbb{N})$ such that

$$\|z_s\| < \varepsilon \quad \text{and} \quad \|T^{s+1+k-b_n} z_s - e_k\| < \varepsilon.$$

Fix $s \in S_{\eta,n}$, so that s belongs to $[1, b_{m+1} - b_m)$ for some integer m of $\mathcal{N}_{\eta,n}$ and

$$|v_m| \prod_{i=b_{m+1}-s}^{b_{m+1}-1} |w_i| > \frac{1}{\eta}.$$

We define a vector z_s by setting

$$z_s := \frac{1}{v_m \prod_{j=b_n+1}^{k} w_j \prod_{j=b_{m+1}-s}^{b_{m+1}-1} w_j}\, e_{b_{m+1}-s-1}.$$

Then

$$\|z_s\| = \frac{1}{|v_m| \prod_{j=b_n+1}^{k} |w_j| \prod_{j=b_{m+1}-s}^{b_{m+1}-1} |w_j|} < \frac{\eta}{\prod_{j=b_n+1}^{k} |w_j|};$$

so that $\|z_s\| < \varepsilon$ for every $s \in S_{\eta,n}$ if η is small enough. Moreover, since $\varphi(m) = n$ we also have

$$T^{s+1+k-b_n} z_s = e_k - \frac{\prod_{j=b_m+1}^{b_m+k-b_n} w_j}{v_m \prod_{j=b_m+1}^{b_{m+1}-1} w_j \prod_{j=b_n+1}^{k} w_j} e_{b_m+k-b_n},$$

and hence (since m belongs to $\mathcal{N}_{\eta,n}$)

$$\|T^{s+1+k-b_n} z_s - e_k\| = \Big| \frac{\prod_{j=b_m+1}^{b_m+k-b_n} w_j}{v_m \prod_{j=b_m+1}^{b_{m+1}-1} w_j \prod_{j=b_n+1}^{k} w_j} \Big| < \frac{\prod_{j=b_m+1}^{b_m+k-b_n} |w_j|}{\prod_{j=b_n+1}^{k} |w_j|}\, \eta.$$

Since the sequence of weights $(w_j)_{j\geq 1}$ is bounded by a positive constant M,

$$\|T^{s+1+k-b_n} z_s - e_k\| \leq \frac{M^{b_{n+1}-b_n}}{\prod_{j=b_n+1}^{k} |w_j|}\, \eta,$$

and the quantity on the right hand side does not depend on $s \in S_{\eta,n}$. So we also have

$$\|T^{s+1+k-b_n} z_s - e_k\| < \varepsilon$$

for every $s \in S_{\eta,n}$ provided η is small enough. □

For operators of C_+-type, the statement of Proposition 6.14 can be slightly simplified.

Corollary 6.15. *Let $T = T_{v,\,w,\,\varphi,\,b}$ be an operator of C_+-type on $\ell_p(\mathbb{N})$. Suppose that for every $\varepsilon > 0$, the set*

$$S_\varepsilon := \bigcup_{k\in\mathcal{K}_\varepsilon} \Big\{ n \in [1, \Delta^{(k)}[\ ;\ |v^{(k)}| \prod_{i=\Delta^{(k)}-n}^{\Delta^{(k)}-1} |w_i^{(k)}| > \frac{1}{\varepsilon} \Big\} \quad \textit{is cofinite,}$$

where $\mathcal{K}_\varepsilon = \{k \geq 1\ :\ |v_k| \prod_{i=1}^{\Delta^{(k)}-1} |w_i^{(k)}| > \frac{1}{\varepsilon}\}$. Then T is topologically mixing.

Proof. This follows immediately from the special structure of C_+-type operators and the definition of the map φ in this case. □

We now use Proposition 6.12 in order to formulate sufficient conditions for an operator of C-type to be *not* topologically mixing.

Proposition 6.16. *Let $T = T_{v,\,w,\,\varphi,\,b}$ be a C-type operator on $\ell_p(\mathbb{N})$. Let $(C_n)_{n\geq 0}$ be a sequence of positive numbers with $0 < C_n < 1$ for every $n \geq 0$. Assume that*

$$|v_n| \cdot \sup_{j\in[b_{\varphi(n)}, b_{\varphi(n)+1})} \Big(\prod_{s=b_{\varphi(n)}+1}^{j} |w_s| \Big) \leq C_n \quad \textit{for every } n \geq 0,$$

and that there exists a positive constant K such that for infinitely many integers $n \geq 0$, we have

$$\sum_{l>n} C_l\, (b_{l+1} - b_l)^{1-\frac{1}{p}} \Big(\sup_{b_{l+1}-2(b_{n+1}-b_n)\leq i<b_{l+1}} \prod_{s=i+1}^{b_{l+1}-1} |w_s| \Big) \leq K. \tag{33}$$

Then T is not topologically mixing.

Proof. Recall that we denote for each $l \geq 0$ by P_l the canonical projection of $\ell_p(\mathbb{N})$ onto the finite-dimensional space span $[e_j;\ b_l \leq j < b_{l+1}]$. It is enough to show that if $n \geq 0$ satisfies the assumption (33), then

$$\big\| P_0 T^{2(b_{n+1}-b_n)} x \big\| \leq 1 \quad \text{for every } x \in B\big(0, \frac{1}{1+K}\big).$$

Indeed, this will imply that $2(b_{n+1} - b_n)$ does not belong to $\mathcal{N}_T\big(B(0, \frac{1}{1+K}), B(3e_0, 1)\big)$ for every such n, and hence that T is not topologically mixing.

Let us fix a vector $x \in B\big(0, \frac{1}{1+K}\big)$ and an integer $n \geq 0$ satisfying (33). Recalling that $T^{2(b_{n+1}-b_n)} e_k = e_k$ for every $0 \leq k < b_{n+1}$, we have

$$P_0 T^{2(b_{n+1}-b_n)} x = P_0 \sum_{l=0}^{n} P_l x + P_0 \sum_{l>n} T^{2(b_{n+1}-b_n)} P_l x = P_0 x + \sum_{l>n} P_0 T^{2(b_{n+1}-b_n)} P_l x.$$

Moreover, if $l > n$, it follows from Proposition 6.12 that

$$\left\|P_0 T^{2(b_{n+1}-b_n)} P_l x\right\| \leq C_l \left(b_{l+1} - b_l\right)^{1-\frac{1}{p}} \Big(\sup_{b_{l+1}-2(b_{n+1}-b_n)\leq i<b_{l+1}} \prod_{s=i+1}^{b_{l+1}-1} |w_s| \Big) \|P_l x\|.$$

So we get that

$$\left\|P_0 T^{2(b_{n+1}-b_n)} x\right\| \leq \|P_0 x\| + \sum_{l>n} \|P_0 T^{2(b_{n+1}-b_n)} P_l x\| \leq \|x\| + K\|x\| \leq 1,$$

which proves Proposition 6.16. □

CHAPTER 7

Explicit counterexamples

1. Summary

In this chapter, we show how the general conditions for frequent hypercyclicity, $\mathcal{U}$-frequent hypercyclicity and topological mixing obtained in Chapter 6 allow us to exhibit interesting counterexamples, namely:

- operators which are chaotic and frequently hypercyclic but not ergodic (Examples 7.7 and 7.19);
- operators which are chaotic and $\mathcal{U}$-frequently hypercyclic but not frequently hypercyclic (Example 7.11);
- operators which are chaotic and topologically mixing but not $\mathcal{U}$-frequently hypercyclic (Example 7.16).

We also show that, as far as the unimodular eigenvalues are concerned, C-type operators can exhibit quite various behaviors: under rather general conditions on the parameters, a C-type operator has only countably many eigenvalues, but at the other extreme, C-type operators can also be mixing in the Gaussian sense (Theorems 7.22 and 7.30).

Finally, we use C-type operators to show that an infinite direct of frequently hypercyclic can fail to be $\mathcal{U}$-frequently hypercyclic (Theorem 7.33).

2. Operators of $\mathrm{C}_{+,1}$-type: FHC does not imply ergodic

In this section, we restrict ourselves to operators of C_+-type for which the parameters v, w, and b satisfy the following conditions: for every $k \geq 1$,

$$v^{(k)} = 2^{-\tau^{(k)}} \quad \text{and} \quad w_i^{(k)} = \begin{cases} 2 & \text{if } \ 1 \leq i \leq \delta^{(k)} \\ 1 & \text{if } \ \delta^{(k)} < i < \Delta^{(k)} \end{cases}$$

where $(\tau^{(k)})_{k\geq 1}$ and $(\delta^{(k)})_{k\geq 1}$ are two strictly increasing sequences of integers with $\delta^{(k)} < \Delta^{(k)}$ for every $k \geq 1$. We call such operators *operators of $C_{+,1}$-type.*

If, in order to simplify matters, we assume that $\delta^{(k)} = 2\tau^{(k)}$ for every $k \geq 1$, then the key parameter of our counterexamples will be the quantity $\frac{\delta^{(k)}}{\Delta^{(k)}}$, *i.e.* the proportion of weights equal to 2 in each block $[b_n, b_{n+1})$. For instance, if we consider the vector $x = 2^{-\frac{\delta^{(k)}}{2}} e_{b_{2^{k-1}}}$, we remark that $T^{\Delta^{(k)}} x$ is close to e_0, and that the orbit of x follows the orbit of e_0 during an interval of time proportional to $\delta^{(k)}$. Since the period of x is equal to $2\Delta^{(k)}$, it seems plausible in view of Theorem 5.35 that T will be frequently hypercyclic as soon as $\limsup_{k\to\infty} \frac{\delta^{(k)}}{\Delta^{(k)}} > 0$. On the other hand, we will show that if $\limsup_{k\to\infty} \frac{\delta^{(k)}}{\Delta^{(k)}}$ is too small, T cannot be ergodic.

2.1. How to be FHC or UFHC. Our first result gives a readable sufficient condition for an operator of $\mathrm{C}_{+,1}$-type to be chaotic and frequently hypercyclic.

THEOREM 7.1. *Let $T = T_{v,\,w,\,\varphi,\,b}$ be an operator of $C_{+,1}$-type on $\ell_p(\mathbb{N})$.*

(1) *If* $\limsup\limits_{k\to\infty} (\delta^{(k)} - \tau^{(k)}) = \infty$, *then T is chaotic.*

(2) *If* $\limsup_{k\to\infty} \dfrac{\delta^{(k)}-\tau^{(k)}}{\Delta^{(k)}} > 0$, *then* T *is chaotic and frequently hypercyclic.*

Proof. The first statement is a direct consequence of Proposition 6.5: for every $n \geq 0$, the expressions of $v^{(k)}$ and $(w_j^{(k)})_{1\leq j<\Delta^{(k)}}$ combined with the fact that $\varphi([2^k, 2^{k+1})) = [0, 2^{k-1})$ for every $k \geq 1$ yield that

$$\limsup_{\substack{N\to\infty \\ N\in\varphi^{-1}(n)}} |v_N| \cdot \prod_{j=b_N+1}^{b_{N+1}-1} |w_j| = \limsup_{k\to\infty} 2^{\delta^{(k)}-\tau^{(k)}} = \infty.$$

As to the second statement, it is a consequence of Theorem 6.9. Let us fix a real number α with

$$0 < \alpha < \frac{1}{2} \limsup_{k\to\infty} \frac{\delta^{(k)}-\tau^{(k)}}{\Delta^{(k)}}.$$

Fix also $C \geq 1$ and $k_0 \geq 1$. By our assumption and the fact that $\Delta^{(k)}$ tends to infinity as k tends to infinity, there exists an integer $k \geq k_0$ such that

$$\frac{\delta^{(k)}-\tau^{(k)}}{\Delta^{(k)}} > 2\alpha \quad \text{and} \quad \alpha\Delta^{(k)} > \log_2 C.$$

We apply Theorem 6.9 with $m = \Delta^{(k)} - 1$. We have

$$|v^{(k)}| \cdot \prod_{i=1}^{\Delta^{(k)}-1} |w_i^{(k)}| = 2^{\delta^{(k)}-\tau^{(k)}} > 2^{2\alpha\Delta^{(k)}} \geq C,$$

which gives condition (27) of Theorem 6.9. As for condition (28), we have for every $0 \leq m' \leq \alpha\Delta^{(k)}$:

$$|v^{(k)}| \cdot \prod_{i=m'+1}^{\Delta^{(k)}-1} |w_i^{(k)}| \geq 2^{\delta^{(k)}-m'-\tau^{(k)}} \geq 2^{\delta^{(k)}-\alpha\Delta^{(k)}-\tau^{(k)}} > C.$$

It thus follows from Theorem 6.9 that T is frequently hypercyclic. □

The next result gives, under some additional assumptions, *necessary and sufficient* conditions for an operator of $\mathrm{C}_{+,1}$-type to be $\mathcal{U}$-frequently hypercyclic or frequently hypercyclic (which turn out to be the same under the additional assumptions).

Theorem 7.2. *Let* $T = T_{v,w,\varphi,b}$ *be an operator of* $C_{+,1}$*-type on* $\ell_p(\mathbb{N})$. *We additionally assume that the sequence* $(\gamma_k)_{k\geq 1}$ *defined by* $\gamma_k := 2^{\delta^{(k-1)}-\tau^{(k)}}\big(\Delta^{(k)}\big)^{1-\frac{1}{p}}$ *is non-increasing, and that the following two conditions are satisfied:*

$$\limsup_{k\to\infty} \frac{\tau^{(k)}}{\delta^{(k)}} < 1 \quad \textit{and} \quad \sum_{k\geq 1} 2^k \gamma_k^{1/2} \leq 1.$$

Then, the following assertions are equivalent:

(1) T *is* $\mathcal{U}$*-frequently hypercyclic;*
(2) T *is frequently hypercyclic;*
(3) $\limsup_{k\to\infty} \delta^{(k)}/\Delta^{(k)} > 0$.

Proof. Since $\limsup_{k\to\infty} \frac{\tau^{(k)}}{\delta^{(k)}} < 1$, Condition (3) is equivalent to $\limsup_{k\to\infty} \frac{\delta^{(k)}-\tau^{(k)}}{\Delta^{(k)}} > 0$. It follows from this observation and Theorem 7.1 that (3) implies (2), which of course implies (1). It remains to prove that if $\limsup_{k\to\infty} \frac{\delta^{(k)}}{\Delta^{(k)}} = 0$, T is not $\mathcal{U}$-frequently hypercyclic. Given a hypercyclic vector x for T, we need to show that x cannot be a $\mathcal{U}$-frequently hypercyclic vector for T, and for this it suffices to find C, $(\beta_l)_{l\geq 1}$, $(X_l)_{l\geq 1}$, and $(N_l)_{l\geq 1}$ satisfying assumptions (1), (2), (3) and (C) of

Theorem 6.10. This is done thanks to Propositions 6.12 and 6.13. We first fix the sequence $(\beta_l)_{l\geq 1}$ by setting

$$\beta_l = 4\,\gamma_k \quad \text{for every} \;\; l \in [2^{k-1}, 2^k),\; k \geq 1.$$

Since the sequence $(\gamma_k)_{k\geq 1}$ is non-increasing, the sequence $(\beta_l)_{l\geq 1}$ is non-increasing. Also

$$\sum_{l\geq 1} \sqrt{\beta_l} = \sum_{k\geq 1} 2^{k-1}\,(4\,\gamma_k)^{1/2} \leq 1$$

by assumption. We have for every $k \geq 1$ and every $n \in [2^{k-1}, 2^k)$

$$|v_n| \cdot \sup_{j\,\in\,[b_{\varphi(n)}, b_{\varphi(n)+1})} \prod_{s=b_{\varphi(n)+1}}^{j} |w_s| \leq 2^{\,\delta^{(k-1)}-\tau^{(k)}}.$$

If we set $C_n = 2^{\,\delta^{(k-1)}-\tau^{(k)}}$ for every $n \in [2^{k-1}, 2^k)$, Proposition 6.12 implies that for every $k \geq 0$, every $l \in [2^{k-1}, 2^k)$ and every $0 \leq n < l$,

$$\begin{aligned}\sup_{j\geq 0} \|P_n T^j P_l\, x\| &\leq 2^{\,\delta^{(k-1)}-\tau^{(k)}} \cdot \big(\Delta^{(k)}\big)^{1-\frac{1}{p}} \cdot \Big\| \sum_{i=b_l}^{b_{l+1}-1} \Big(\prod_{s=i+1}^{b_{l+1}-1} w_s \Big)\, x_i e_i \Big\| \\ &\leq \frac{\beta_l}{4} \cdot \Big\| \sum_{i=b_l}^{b_{l+1}-1} \Big(\prod_{s=i+1}^{b_{l+1}-1} w_s \Big)\, x_i e_i \Big\|\end{aligned}$$

and, for every $1 \leq N \leq \Delta^{(k)}$,

$$\sup_{0\leq j\leq N} \|P_n T^j P_l\, x\| \leq \frac{\beta_l}{4} \cdot \Big(\prod_{i=\Delta^{(k)}-N+1}^{\Delta^{(k)}-1} |w_i^{(k)}| \Big) \cdot \|P_l\, x\|.$$

We now set $N_l := \Delta^{(k)} - \delta^{(k)}$ for every $l \in [2^{k-1}, 2^k)$ and every $k \geq 1$. Remembering that $w_i^{(k)} = 1$ if $\delta^k < i < \Delta^{(k)}$, we obtain that

$$\sup_{0\leq j\leq N_l} \|P_n T^j P_l\, x\| \leq \frac{\beta_l}{4} \|P_l\, x\| \quad \text{for every } l \geq 1.$$

Finally, we set for every $k \geq 1$ and every $l \in [2^{k-1}, 2^k)$

$$X_l := \Big\| \sum_{i=b_l}^{b_{l+1}-1} \Big(\prod_{s=i+1}^{b_{l+1}-1} w_s \Big)\, x_i e_i \Big\|.$$

Obviously, $\|P_l\, x\| \leq X_l$, and we have proved that for every $0 \leq n < l$,

$$\sup_{j\geq 0} \|P_n T^j P_l\, x\| \leq \frac{\beta_l}{4}\, X_l \quad \text{and} \quad \sup_{0\leq j\leq N_l} \|P_n T^j P_l\, x\| \leq \frac{\beta_l}{4}\, \|P_l\, x\|.$$

It remains to check that condition (C) of Theorem 6.10 holds true, and for this we use Proposition 6.13.

We have $w_i^{(k)} = 1$ for every $i \in (\delta^{(k)}, \Delta^{(k)})$, and $\prod_{i=\delta^{(k)}+1}^{\Delta^{(k)}-1} w_i^{(k)} = 1$. The assumptions of Proposition 6.13 are thus satisfied for $k_0 = \delta^{(k)}$ and $k_1 = \Delta^{(k)}$ if $l \in [2^{k-1}, 2^k)$, and we get

$$\frac{1}{J+1}\, \#\Big\{0 \leq j \leq J\,;\, \|P_l T^j P_l\, x\| \geq X_l/2\Big\} \geq 1 - 2\delta^{(k)} \Big(\frac{1}{J+1} + \frac{1}{\Delta^{(k)}} \Big)$$

for every $J \geq 0$, every $k \geq 1$, and every $l \in [2^{k-1}, 2^k)$. Therefore,

$$\inf_{J\geq N_l} \frac{1}{J+1}\, \#\Big\{0 \leq j \leq J\,;\, \|P_l T^j P_l\, x\| \geq X_l/2\Big\} \geq 1 - 2\delta^{(k)} \Big(\frac{1}{\Delta^{(k)} - \delta^{(k)} + 1} + \frac{1}{\Delta^{(k)}} \Big)$$

for every $k \geq 1$ and every $l \in [2^{k-1}, 2^k)$, and it follows that

$$\liminf_{l\to\infty} \inf_{J\geq N_l} \frac{1}{J+1} \#\Big\{0 \leq j \leq J\,;\, \|P_l T^j P_l\, x\| \geq X_l/2\Big\}$$
$$\geq \liminf_{k\to\infty} \left(1 - 2\delta^{(k)}\Big(\frac{1}{\Delta^{(k)} - \delta^{(k)} + 1} + \frac{1}{\Delta^{(k)}}\Big)\right) = 1,$$

since $\limsup_{k\to\infty} \frac{\delta^{(k)}}{\Delta^{(k)}} = 0$ by assumption. Theorem 6.10 eventually yields that T is not $\mathcal{U}$-frequently hypercyclic, and this concludes the proof. □

Remark 7.3. Theorem 7.2 implies the main result of [**40**], *i.e.* that there exist chaotic operators on $\ell_p(\mathbb{N})$ which are not $\mathcal{U}$-frequently hypercyclic.

2.2. A word about the OSP. In this short section, we show that there exist $\mathrm{C}_{+,1}$-type operators on $\ell_2(\mathbb{N})$ which satisfy the assumptions of the criterion for frequent hypercyclicity stated in Theorem 5.35, and yet do not have the Operator Specification Property (recall that we have proved in Section 4.1 that operators with the OSP do satisfy the assumptions of Theorem 5.35). The following corollary is a simple consequence of Proposition 6.16 applied to $\mathrm{C}_{+,1}$-type operators.

Corollary 7.4. *Let $T = T_{v,\,w,\,\varphi,\,b}$ be an operator of $C_{+,1}$-type on $\ell_p(\mathbb{N})$ and let $\gamma_k = 2^{\delta^{(k-1)} - \tau^{(k)}} \big(\Delta^{(k)}\big)^{1-\frac{1}{p}}$ for every $k \geq 1$. Assume that $\tau^{(k)} > \delta^{(k-1)}$ for every $k \geq 1$ and that there exists a positive constant K such that, for every $k_0 \geq 1$,*

$$\sum_{k>k_0} 2^k \gamma_k \leq K 4^{-\Delta^{(k_0)}}.$$

Then T is not topologically mixing.

Proof. Let $n \geq 1$ be an integer, and let k_n be the unique integer such that $2^{k_n - 1} \leq n < 2^{k_n}$. Then $v_n = v^{(k_n)} = 2^{-\tau^{(k_n)}}$ and $\varphi(n) < 2^{k_n - 1}$, and setting

$$C_n := |v_n| \sup_{j\in[b_{\varphi(n)}, b_{\varphi(n)+1})} \Big(\prod_{s=b_{\varphi(n)}+1}^{j} |w_s|\Big),$$

we have $0 < C_n \leq 2^{-\tau^{(k_n)}}\, 2^{\delta^{(k_n - 1)}} < 1$. Moreover, if n is an integer of the form $n = 2^k - 1$ for some integer $k \geq 1$, then $b_{n+1} - b_n = \Delta^{(k)}$, and moreover $k_l > k$ for every $l > n$. It follows that

$$\sum_{l>n} C_l\, (b_{l+1} - b_l)^{1-\frac{1}{p}} \Big(\sup_{b_{l+1} - 2(b_{n+1} - b_n) \leq i < b_{l+1}} \prod_{s=i+1}^{b_{l+1}-1} |w_s|\Big)$$

is less than

$$\leq \sum_{r>k} 2^{r-1} 2^{\delta^{(r-1)} - \tau^{(r)}} (\Delta^{(r)})^{1-\frac{1}{p}} \,.\, 2^{2\Delta^{(k)}} \leq K.$$

Hence Proposition 6.16 applies, and T is not topologically mixing. □

Since the condition appearing in Corollary 7.4 is compatible with those of Theorem 7.2 above (see Example 7.7), and since operators with the OSP are topologically mixing, we immediately deduce:

Corollary 7.5. *There exist operators on $\ell_p(\mathbb{N})$ which satisfy the assumptions of Theorem 7.2 and yet do not have the Operator Specification Property.*

Since the assumptions of Theorem 7.2 imply those of Theorem 5.35, it follows that there exist operators on $\ell_p(\mathbb{N})$ which satisfy the criterion for frequent hypercyclicity stated in Theorem 5.35 without having the OSP. In particular, this criterion for frequent hypercyclicity is strictly stronger than the "classical" one.

2.3. FHC but not ergodic. We are now ready to give examples of operators of $\mathrm{C}_{+,1}$-type which are frequently hypercyclic but not ergodic.

THEOREM 7.6. *Let T be an operator of $C_{+,1}$-type on $\ell_p(\mathbb{N})$, and set, for every $k \geq 1$, $\gamma_k := 2^{\delta^{(k-1)}-\tau^{(k)}}\big(\Delta^{(k)}\big)^{1-\frac{1}{p}}$. Suppose that the sequence $(\gamma_k)_{k\geq 1}$ is non-increasing, and that the following three conditions are satisfied:*

$$\sum_{k\geq 1} 2^k \gamma_k^{1/2} \leq 1, \quad \limsup_{k\to\infty} \frac{\tau^{(k)}}{\delta^{(k)}} < 1 \quad \textit{and} \quad 0 < \limsup_{k\to\infty} \frac{\delta^{(k)}}{\Delta^{(k)}} \leq \frac{1}{5}\cdot$$

Then T is frequently hypercyclic but $c(T) < 1$, so that T is not ergodic.

PROOF. The frequent hypercyclicity of T follows from Theorem 7.2. Let us show that $c(T) < 1$. Recall that $c(T)$ is characterized by the following property (see [**30**, Rem. 4.6]): for quasi-all hypercyclic vectors x for T in the Baire category sense, we have

$$c(T) = \overline{\mathrm{dens}}\, \mathcal{N}_T\big(x, B(0,\varepsilon)\big) \quad \text{for every } \varepsilon > 0. \tag{34}$$

Let us fix a hypercyclic vector x for T satisfying (34). By Lemma 6.11 combined with the end of the proof of Theorem 7.2, we know that there exists an $\varepsilon > 0$ such that

$$\underline{\mathrm{dens}}\, \mathcal{N}_T\big(x, B(0,\varepsilon)^c)\big) \geq \liminf_{l\to\infty} \inf_{J\geq N_l} \frac{1}{J+1}\, \#\big\{0 \leq j \leq J\,;\, \|P_l T^j P_l\, x\| \geq X_l/2\big\}$$

(recall that we proved above that under the assumption (3) of Theorem 7.2, the assumptions of Theorem 6.10 are satisfied with $C = 1/4$, so that $2CX_l = X_l/2$). Using an inequality obtained at the end of the proof of Theorem 7.2, we get

$$\underline{\mathrm{dens}}\, \mathcal{N}_T\big(x, B(0,\varepsilon)^c\big) \geq \liminf_{k\to\infty} \left(1 - 2\delta^{(k)}\Big(\frac{1}{\Delta^{(k)} - \delta^{(k)} + 1} + \frac{1}{\Delta^{(k)}}\Big)\right)$$

so that

$$\begin{aligned}\overline{\mathrm{dens}}\, \mathcal{N}_T\big(x, B(0,\varepsilon)\big) &\leq \limsup_{k\to\infty} \left(\frac{2\delta^{(k)}}{\Delta^{(k)} - \delta^{(k)} + 1} + \frac{2\delta^{(k)}}{\Delta^{(k)}}\right) \\ &= \limsup_{k\to\infty} \left(\frac{2\delta^{(k)}}{\Delta^{(k)}} \cdot \frac{1}{1 - \dfrac{\delta^{(k)}}{\Delta^{(k)}} + \dfrac{1}{\Delta^{(k)}}} + \frac{2\delta^{(k)}}{\Delta^{(k)}}\right).\end{aligned}$$

It follows that

$$c(T) \leq \limsup_{k\to\infty} \left(\frac{2\delta^{(k)}}{\Delta^{(k)}} \cdot \frac{1}{1 - \dfrac{\delta^{(k)}}{\Delta^{(k)}} + \dfrac{1}{\Delta^{(k)}}} + \frac{2\delta^{(k)}}{\Delta^{(k)}}\right). \tag{35}$$

Our assumption thus implies that

$$c(T) \leq \frac{2/5}{1 - 1/5} + \frac{2}{5} = \frac{1}{2} + \frac{2}{5} < 1,$$

which proves that T is not ergodic. □

EXAMPLE 7.7. Let C be a positive integer, and consider the operator of $\mathrm{C}_{+,1}$-type T on $\ell_p(\mathbb{N})$ associated to the parameters

$$\tau^{(k)} = 2^{Ck}, \quad \delta^{(k)} = 2 \cdot 2^{Ck} \quad \text{and} \quad \Delta^{(k)} = 10 \cdot 2^{Ck}, \quad k \geq 1.$$

If C is sufficiently large, T is frequently hypercyclic but not ergodic. Besides, it is also not topologically mixing.

Proof. With this choice of parameters, and assuming that $C \geq 2$, we have

$$\gamma_k = 2^{\,(2\cdot 2^{C(k-1)}-2^{Ck})} \cdot 10^{1-\frac{1}{p}} \cdot 2^{Ck(1-\frac{1}{p})} \leq 10 \cdot 2^{-\frac{1}{2}\,2^{Ck}} \cdot 2^{Ck}$$

for every $k \geq 1$. So it is not hard to check that if C is sufficiently large, the sequence $(\gamma_k)_{k\geq 1}$ is non-increasing and satisfies $\sum_{k\geq 1} 2^k \gamma_k^{1/2} \leq 1$. The other assumptions of Theorem 7.6 are clearly satisfied, and hence T is frequently hypercyclic but not ergodic.

In order to show that T is also not topologically mixing if C is large enough, we use Proposition 6.16. If n is any integer of the form $n = 2^{k_0} - 1$, where $k_0 \geq 1$ is any integer, then

$$\begin{aligned}\sum_{l>n} C_l\,(b_{l+1}-b_l)^{1-\frac{1}{p}} \Big(\sup_{b_{l+1}-2(b_{n+1}-b_n)\leq k<b_{l+1}} \prod_{s=k+1}^{b_{l+1}-1} |w_s| \Big) &= \sum_{k>k_0} 2^{k-1} 2^{\delta^{(k-1)}-\tau^{(k)}} (\Delta^{(k)})^{1-\frac{1}{p}} \\ &= \sum_{k>k_0} 2^{k-1}\gamma_k,\end{aligned}$$

If C is sufficiently large, then $\sum_{k=1}^{\infty} 2^{k-1}\gamma_k < 1$; so we deduce from Proposition 6.16 (by considering $K := 1$ and $n := 2^{k_0} - 1$ for every $k_0 \geq 1$) that T is not topologically mixing. Notice that T also satisfies the assumptions of Corollary 7.4. □

Since the conditions of Theorem 7.2 make no difference between $\mathcal{U}$-frequent and frequent hypercyclicity, we will need to introduce another family of operators of C_+-type in order to construct examples of $\mathcal{U}$-frequent hypercyclic operators on $\ell_p(\mathbb{N})$ which are not frequently hypercyclic. This we do in the next section.

3. Operators of $\mathrm{C}_{+,2}$-type: UFHC does not imply FHC

In this section, we impose the following restrictions on the parameters v, w, and b of a C_+-type operator $T_{v,\,w,\,\varphi,\,b}$: for every $k \geq 1$,

$$v^{(k)} = 2^{-\tau^{(k)}} \quad \text{and} \quad w_i^{(k)} = \begin{cases} 2 & \text{if } \ 1 \leq i \leq \delta^{(k)} \\ 1 & \text{if } \ \delta^{(k)} < i < \Delta^{(k)} - 3\delta^{(k)} \\ 1/2 & \text{if } \ \Delta^{(k)} - 3\delta^{(k)} \leq i < \Delta^{(k)} - 2\delta^{(k)} \\ 2 & \text{if } \ \Delta^{(k)} - 2\delta^{(k)} \leq i < \Delta^{(k)} - \delta^{(k)} \\ 1 & \text{if } \ \Delta^{(k)} - \delta^{(k)} \leq i < \Delta^{(k)} \end{cases}$$

where $(\tau^{(k)})_{k\geq 1}$ and $(\delta^{(k)})_{k\geq 1}$ are two strictly increasing sequences of integers, satisfying $4\delta^k < \Delta^{(k)}$ for every $k \geq 1$. We call operators satisfying these conditions *operators of $C_{+,2}$-type*. Observe that we still have with this definition

$$\prod_{i=1}^{\Delta^{(k)}-1} w_i^{(k)} = 2^{\,\delta^{(k)}} \quad \text{for every } k \geq 1.$$

This choice of weights is motivated by the differences between the assumptions of Theorem 5.14 and those of Theorem 5.35. If we assume for simplicity that $\delta^{(k)} = 2\tau^{(k)}$ for every $k \geq 1$ and if we consider the vector $x := 2^{-\frac{\delta^{(k)}}{2}} e_{b_{2^{k-1}}+\Delta^{(k)}-2\delta^{(k)}}$, we observe that $T^{2\delta^{(k)}}x$ is close to e_0 and that the orbit of x follows the orbit of e_0 during an interval of time proportional to $\delta^{(k)}$. In view of Theorem 5.14, it thus seems likely that T will be $\mathcal{U}$-frequently hypercyclic. However, if $\lim_{k\to\infty} \frac{\delta^{(k)}}{\Delta^{(k)}} = 0$, it also seems plausible that T will not satisfy the assumptions of Theorem 5.35, and hence will possibly not be frequently hypercyclic.

3.1. How to be FHC or UFHC. In this section, we present some sufficient conditions for an operator of $C_{+,2}$-type to be $\mathcal{U}$-frequently hypercyclic.

THEOREM 7.8. *Let $T = T_{v,\,w,\,\varphi,\,b}$ be an operator of $C_{+,2}$-type on $\ell_p(\mathbb{N})$.*

(1) *If* $\limsup_{k\to\infty} \left(\delta^{(k)} - \tau^{(k)}\right) = \infty$, *then T is chaotic.*

(2) *If* $\limsup_{k\to\infty} \dfrac{\tau^{(k)}}{\delta^{(k)}} < 1$, *then T is chaotic and $\mathcal{U}$-frequently hypercyclic.*

(3) *If* $\limsup_{k\to\infty} \dfrac{\delta^{(k)} - \tau^{(k)}}{\Delta^{(k)}} > 0$, *then T is chaotic and frequently hypercyclic.*

PROOF. The proofs of assertions (1) and (3) are completely similar to the ones given in the proof of Theorem 7.1. As for (2), it is a consequence of Theorem 6.8. Let us fix

$$0 < \alpha < \frac{1}{3} - \frac{1}{3} \limsup_{k\to\infty} \frac{\tau^{(k)}}{\delta^{(k)}}.$$

Fix also $C \geq 1$, and an integer $k_0 \geq 1$. There exists an integer $k \geq k_0$ such that

$$\frac{\tau^{(k)}}{\delta^{(k)}} < 1 - 3\alpha \quad \text{and} \quad \alpha\,\delta^{(k)} > \log_2 C.$$

We then have

$$|v^{(k)}| \cdot \prod_{i=\Delta^{(k)}-2\delta^{(k)}}^{\Delta^{(k)}-1} |w_i^{(k)}| = 2^{\,\delta^{(k)}-\tau^{(k)}} > 2^{\,3\alpha\delta^{(k)}} > C$$

and, for every $0 \leq m' \leq 2\alpha\delta^{(k)}$:

$$|v^{(k)}| \cdot \prod_{i=m'+1}^{\Delta^{(k)}-1} |w_i^{(k)}| \geq 2^{\,(1-2\alpha)\delta^{(k)}-\tau^{(k)}} > 2^{\,\alpha\delta^{(k)}} > C$$

Applying Theorem 6.8 with $m = 2\,\delta^{(k)}$, it follows that T is $\mathcal{U}$-frequently hypercyclic. □

We see from this proof that the new structure of the weights $(w_j^{(k)})_{1\leq j<\Delta^{(k)}}$, $k \geq 1$, compared with the case of operators of $C_{+,1}$-type, allows us to provide different conditions for $\mathcal{U}$-frequent hypercyclicity and frequent hypercyclicity of operators of $C_{+,2}$-type. Our next result gives, under some additional assumptions, a *necessary and sufficient* condition for frequent hypercyclicity of operators of $C_{+,2}$-type which, combined with (2) of Theorem 7.8, will ultimately allow us to construct $\mathcal{U}$-frequently hypercyclic operators of $C_{+,2}$-type which are not frequently hypercyclic.

THEOREM 7.9. *Let $T = T_{v,\,w,\,\varphi,\,b}$ be an operator of $C_{+,2}$-type on $\ell_p(\mathbb{N})$, and for every $k \geq 1$, set $\gamma_k := 2^{\,\delta^{(k-1)}-\tau^{(k)}} \left(\Delta^{(k)}\right)^{1-\frac{1}{p}}$. Suppose that the sequence $(\gamma_k)_{k\geq 1}$ is non-increasing and that it satisfies the following three conditions:*

$$\sum_{k\geq 1} 2^k \gamma_k^{1/2} \leq 1, \qquad \limsup_{k\to\infty} \frac{\tau^{(k)}}{\delta^{(k)}} < 1, \quad \textit{and} \quad \lim_{k\to\infty} \frac{\delta^{(k)}}{\delta^{(k+1)}} = 0.$$

Then T is frequently hypercyclic if and only if $\limsup_{k\to\infty} \delta^{(k)}/\Delta^{(k)} > 0$.

PROOF. Since $\limsup_{k\to\infty} \frac{\tau^{(k)}}{\delta^{(k)}} < 1$, the condition $\limsup_{k\to\infty} \frac{\delta^{(k)}}{\Delta^{(k)}} > 0$ is equivalent to

$$\limsup_{k\to\infty} \frac{\delta^{(k)} - \tau^{(k)}}{\Delta^{(k)}} > 0;$$

and by assertion (3) of Theorem 7.8, this condition implies that T is frequently hypercyclic.

The proof of the converse assertion proceeds exactly as in the proof of the implication (1) $\Longrightarrow$ (3) in Theorem 7.2. Suppose that $\limsup_{k\to\infty} \frac{\delta^{(k)}}{\Delta^{(k)}} = 0$, and let $x \in \ell_p(\mathbb{N})$ be a hypercyclic vector

for T. Our aim is to prove that x cannot be a frequently hypercyclic vector for T, using Theorem 6.10. We set

$$\beta_l := 4\,\gamma_k \quad \text{for every } l \in [2^{k-1}, 2^k),\ k \geq 1.$$

This sequence $(\beta_l)_{l\geq 1}$ is non-increasing and satisfies $\sum_{l\geq 1}\sqrt{\beta_l} \leq 1$. Setting

$$X_l := \Big\| \sum_{i=b_l}^{b_{l+1}-1} \Big(\prod_{s=i+1}^{b_{l+1}-1} w_s \Big) x_i e_i \Big\| \quad \text{for} \quad l \in [2^{k-1}, 2^k),\ k \geq 1,$$

we have $\|P_l\, x\| \leq X_l$.

Also, by Proposition 6.12, we have for every $l \geq 0$, every $0 \leq n < l$, and every $1 \leq N \leq \Delta^{(k)}$, where $l \in [2^{k-1}, 2^k)$, that

$$\sup_{j\geq 0} \|P_n T^j P_l\, x\| \leq \frac{1}{4}\,\beta_l X_l$$

and

$$\sup_{0\leq j\leq N} \|P_n T^j P_l\, x\| \leq \frac{1}{4}\,\beta_l \cdot \Big(\prod_{i=\Delta^{(k)}-N+1}^{\Delta^{(k)}-1} |w_i^{(k)}| \Big) \cdot \|P_l\, x\|.$$

At this point, we diverge from the proof of Theorem 7.2, and set $N_l := \delta^{(k)}$ for every $l \in [2^{k-1}, 2^k)$ and every $k \geq 1$. We then have $\sup\limits_{0\leq j\leq N_l} \|P_n T^j P_l\, x\| \leq \dfrac{1}{4}\,\beta_l\, \|P_l\, x\|$ for every $l \geq 0$ and every $0 \leq n < l$. In order to check assumption (C') of Theorem 6.10, we use again Proposition 6.13. We have $w_i^{(k)} = 1$ for every $i \in (\delta^{(k)}, \Delta^{(k)} - 3\delta^{(k)})$ and

$$\prod_{i=\delta^{(k)}+1}^{\Delta^{(k)}-1} w_i^{(k)} = 1.$$

It follows from Proposition 6.13 that

$$\frac{1}{J+1}\ \#\big\{0 \leq j \leq J\,;\, \|P_l T^j T_l\, x\| \geq X_l/2\big\} \geq 1 - 8\delta^{(k)}\Big(\frac{1}{J+1} + \frac{1}{\Delta^{(k)}}\Big)$$

for every $J \geq 0$, every $k \geq 1$, and every $l \in [2^{k-1}, 2^k)$. Now, we have $\min(\varphi^{-1}(l)) = 2^k$ for every $l \in [2^{k-1}, 2^k)$, so that $N_{\min(\varphi^{-1}(l))} = \delta^{(k+1)}$.

For every $k \geq 1$, we have

$$\inf_{J\geq \delta^{(k+1)}} \frac{1}{J+1}\ \#\Big\{0 \leq j \leq J\,;\, \|P_l T^j T_l\, x\| \geq X_l/2\Big\} \geq 1 - \frac{8\delta^{(k)}}{\delta^{(k+1)}} - \frac{8\delta^{(k)}}{\Delta^{(k)}}.$$

Hence

$$\liminf_{l\to\infty} \inf_{J\geq \delta^{(k+1)}} \frac{1}{J+1}\ \#\Big\{0 \leq j \leq J\,; \|P_l T^j T_l\, x\| \geq X_l/2\Big\}$$
$$\geq \liminf_{k\to\infty} \Big(1 - \frac{8\delta^{(k)}}{\delta^{(k+1)}} - \frac{8\delta^{(k)}}{\Delta^{(k)}}\Big) = 1,$$

since $\lim_{k\to\infty} \frac{\delta^{(k)}}{\delta^{(k+1)}} = 0$ and $\limsup_{k\to\infty} \frac{\delta^{(k)}}{\Delta^{(k)}} = 0$. Theorem 6.10 thus implies that T is not frequently hypercyclic $\square$

3.2. UFHC but not FHC. As a direct consequence of Theorems 7.8 and 7.9, we now obtain

THEOREM 7.10. *Let $T = T_{v,\,w,\,\varphi,\,b}$ be an operator of $C_{+,2}$-type on $\ell_p(\mathbb{N})$. For every $k \geq 1$, set $\gamma_k := 2^{\delta^{(k-1)}-\tau^{(k)}}\big(\Delta^{(k)}\big)^{1-\frac{1}{p}}$. Suppose that the sequence $(\gamma_k)_{k\geq 1}$ is non-increasing, and that the following conditions are satisfied:*

$$\sum_{k\geq 1} 2^k \gamma_k^{1/2} \leq 1, \quad \limsup_{k\to\infty} \frac{\tau^{(k)}}{\delta^{(k)}} < 1, \quad \lim_{k\to\infty} \frac{\delta^{(k)}}{\delta^{(k+1)}} = 0 \quad \textit{and} \quad \lim_{k\to\infty} \frac{\delta^{(k)}}{\Delta^{(k)}} = 0.$$

Then T is $\mathcal{U}$-frequently hypercyclic but not frequently hypercyclic.

Here is a concrete example, the proof of which is left to the reader.

EXAMPLE 7.11. If we consider the operator of $C_{+,2}$-type associated to the parameters

$$\tau^{(k)} = 2^{Ck^2}, \quad \delta^{(k)} = 2^{Ck^2+1} \quad \text{and} \quad \Delta^{(k)} = 2^{2Ck^2+4}, \quad k \geq 1,$$

where C is a sufficiently large integer, then T is $\mathcal{U}$-frequently hypercyclic but not frequently hypercyclic.

4. Operators of C_2-type: chaos plus mixing do not imply UFHC

In this section, we introduce yet another class of C-type operators $T = T_{v,\,w,\,\varphi,\,b}$ on $\ell_p(\mathbb{N})$.

We consider increasing sequences $(a_k)_{k\geq 1}$, $(f_k)_{k\geq 1}$, $(\delta^{(k)})_{k\geq 1}$, $(\tau^{(k)})_{k\geq 1}$ and $(\Delta^{(k)})_{k\geq 1}$ of integers such that $0 \leq a_k < f_k < \Delta^{(k)} - 4\delta^{(k)}$ for every $k \geq 1$, and we denote by $(J_k)_{k\geq 0}$ the partition of $\mathbb{N}$ into consecutive finite intervals defined as follows: $J_0 = \{0\}$ and $\#J_k = (f_k - a_k)(\sum_{i=0}^{k-1} \#J_i)$ for every $k \geq 1$.

We then require that:

- $\varphi(n) = \left\lfloor \frac{n - \min J_k}{f_k - a_k} \right\rfloor$ for every $n \in J_k$;
- the blocks $[b_n, b_{n+1})$, $n \in J_k$, all have the same size $\Delta^{(k)}$;
- the sequence v is given by

$$v_n = 2^{-\tau^{(k)}} \qquad \text{for every } n \in J_k;$$

- If j belongs to the interval $[b_n, b_{n+1})$ with $n \in J_k$ and $(n - \min J_k) \mod (f_k - a_k) = l$, then the weight w_j is given by

$$w_j = \begin{cases} 2 & \text{if } \ b_n < j \leq b_n + \delta^{(k)} \\ 1 & \text{if } \ b_n + \delta^{(k)} < j < b_{n+1} - a_k - l - 2\delta^{(k)} \\ 1/2 & \text{if } \ b_{n+1} - a_k - l - 2\delta^{(k)} \leq j < b_{n+1} - a_k - l - \delta^{(k)} \\ 2 & \text{if } \ b_{n+1} - a_k - l - \delta^{(k)} \leq j < b_{n+1} - a_k - l \\ 1 & \text{if } \ b_{n+1} - a_k - l \leq j < b_{n+1}. \end{cases}$$

If these conditions are met, we call $T = T_{v,\,w,\,\varphi,\,b}$ an *operator of C_2-type.*

Note that the family of C_2-type operators contain that of $C_{+,2}$-type operators. Indeed, if we consider the sequences $(a_k)_{k\geq 1}$ and $(f_k)_{k\geq 1}$ given by $a_k = \delta^{(k)}$ and $f_k = a_k + 1$ for every $k \geq 1$, we get back the definition of $C_{+,2}$-type operators. The specificity of these new operators lies in the fact that for every $k \geq 1$ and every $n \in \bigcup_{k'<k} J_{k'}$, the set J_k contains $f_k - a_k$ integers m for which $\varphi(m) = n$, and, for each of these integers, the central block of weights $(1/2, \ldots, 1/2, 2, \cdots, 2)$ is translated in a similar way.

4.1. How to be topologically mixing. By using Proposition 6.14, we can show that under suitable assumptions on the sequences $(a_k)_{k\geq 1}$ and $(f_k)_{k\geq 1}$, an operator of C_2-type can be topologically mixing.

PROPOSITION 7.12. *Let T be an operator of C_2-type on $\ell_p(\mathbb{N})$. Suppose that*

$$\lim_{k\to\infty}(\delta^{(k)}-\tau^{(k)})=\infty,$$

and that the set $\bigcup_{k\geq 1}[a_k+\delta^{(k)}, f_k+\delta^{(k)})$ is cofinite. Then T is topologically mixing.

PROOF. With the notations of Proposition 6.14, we have to show that the set $S_{\eta,n}$ is cofinite. First of all, observe that the set $\mathcal{N}_{\eta,n}$ contains, for every $k\geq 1$ such that $2^{\delta^{(k)}-\tau^{(k)}}>1/\eta$, the set $\varphi^{-1}(n)\cap J_k$. Since $\delta^{(k)}-\tau^{(k)}$ tends to infinity as k tends to infinity, there exists an integer $k_0\geq 1$ such that $\mathcal{N}_{\eta,n}$ contains the set

$$\bigcup_{k\geq k_0}(\varphi^{-1}(n)\cap J_k).$$

Recalling that

$$S_{\eta,n}=\bigcup_{m\in\mathcal{N}_{\eta,n}}\Big\{s\in[1,b_{m+1}-b_m);\ |v_m|\prod_{i=b_{m+1}-s}^{b_{m+1}-1}|w_i|>\frac{1}{\eta}\Big\},$$

we deduce that $S_{\eta,n}$ contains the set

$$\bigcup_{k\geq k_0}\bigcup_{m\in\varphi^{-1}(n)\cap J_k}\Big\{s\in[1,b_{m+1}-b_m);\ |v_m|\prod_{i=b_{m+1}-s}^{b_{m+1}-1}|w_i|\geq 2^{\delta^{(k)}-\tau^{(k)}}\Big\},$$

which in its turn contains the set

$$\bigcup_{k\geq k_0}\bigcup_{m\in\varphi^{-1}(n)\cap J_k}\Big\{s\in[1,b_{m+1}-b_m);\ \prod_{i=b_{m+1}-s}^{b_{m+1}-1}|w_i|=2^{\delta^{(k)}}\Big\}$$

(recall that $v_m=2^{-\tau^{(k)}}$ for every integer m belonging to J_k). Fix an integer $k\geq k_0$, and let m be an integer of the form $m=\min J_k+n(f_k-a_k)+l$, where $0\leq l<f_k-a_k$. Then m belongs to J_k, $\varphi(m)=n$ and $\prod_{i=b_{m+1}-a_k-l-\delta^{(k)}}^{b_{m+1}-1}|w_i|=2^{\delta^{(k)}}$. It follows that $S_{\eta,n}$ contains every integer of the form $a_k+l+\delta^{(k)}$, where $0\leq l<f_k-a_k$, *i.e.* that

$$\bigcup_{k\geq k_0}[a_k+\delta^{(k)}, f_k+\delta^{(k)})\subseteq S_{\eta,n}.$$

Since the set on the left hand side is cofinite by assumption, the desired result follows from Proposition 6.14. □

4.2. How not to be UFHC. We now give some conditions ensuring that a C_2-type operator fails to be $\mathcal{U}$-frequently hypercyclic. This will rely on the following more general version of Proposition 6.13.

PROPOSITION 7.13. *Let T be an operator of C-type on $\ell_p(\mathbb{N})$, and let $x\in\ell_p(\mathbb{N})$. Fix $l\geq 0$, and define*

$$X_l:=\Big\|\sum_{k=b_l}^{b_{l+1}-1}\Big(\prod_{s=k+1}^{b_{l+1}-1}w_s\Big)x_ke_k\Big\|.$$

Suppose that there exist three integers $0\leq k_0<k_1<k_2\leq b_{l+1}-b_l$ such that

$$|w_{b_l+k}|=1\quad\textit{for every } k\in(k_0,k_1)\cup(k_2,b_{l+1}-b_l)\quad\textit{and}\quad\prod_{s=b_l+k_0+1}^{b_{l+1}-1}|w_s|=1.$$

Then we have for every $J \geq 0$

$$\frac{1}{J+1}\,\#\Big\{0 \leq j \leq J\,;\, \|P_l T^j P_l\, x\| \geq X_l/2\Big\} \geq 1 - 4\big(k_2 - k_1 + k_0\big)\,\cdot\,\Big(\frac{1}{J+1} + \frac{1}{b_{l+1} - b_l}\Big).$$

Proof. The proof is similar to the proof of Proposition 6.13 except that the sets I_j, $j \geq 0$, have to be defined as follows:

$$I_j := \Big\{k \in [b_l, b_{l+1})\,;\, j + k - b_l \mod (b_{l+1} - b_l) \text{ does not belong to } [k_0, k_1) \cup [k_2, b_{l+1} - b_l)\Big\}.$$

We can then deduce that there exists an integer j_0 such that for every $J \geq 0$:

$$\frac{1}{J+1}\,\#\Big\{0 \leq j \leq J\,;\, \|P_l T^j P_l\, x\| \geq X_l/2\Big\} \geq 1 - \frac{1}{J+1}\,\#\Big\{0 \leq j \leq J\,;\, I_j \cap I_{j_0} \neq \emptyset\Big\}. \tag{36}$$

Now, we remark that if we set $i_j := j \mod (b_{l+1} - b_j)$ for every $j \geq 0$, we have

$$I_j = \begin{cases} [b_l, b_l + k_0 - i_j) \cup [b_l + k_1 - i_j, b_l + k_2 - i_j) \cup [b_{l+1} - i_j, b_{l+1}) & \text{if } 0 \leq i_j < k_0 \\ [b_l + k_1 - i_j, b_l + k_2 - i_j) \cup [b_{l+1} - i_j, b_{l+1} + k_0 - i_j) & \text{if } k_0 \leq i_j \leq k_1 \\ [b_l, b_l + k_2 - i_j) \cup [b_{l+1} - i_j, b_{l+1} + k_0 - i_j) \cup [b_{l+1} + k_1 - i_j, b_{l+1}) & \text{if } k_1 < i_j < k_2 \\ [b_{l+1} - i_j, b_{l+1} + k_0 - i_j) \cup [b_{l+1} + k_1 - i_j, b_{l+1} + k_2 - i_j) & \text{if } k_2 < i_j < b_{l+1} - b_l. \end{cases}$$

This particular structure of the sets I_j implies that

$$\#\,\Big\{0 \leq j \leq J\,;\, I_j \cap I_{j_0} \neq \emptyset\Big\} \leq 4(k_2 - k_1 + k_0)\,\cdot\,\Big(\Big\lfloor \frac{J}{b_{l+1} - b_l}\Big\rfloor + 1\Big),$$

and this yields the desired result. □

Proposition 7.14. *Let T be an operator of C_2-type on $\ell_p(\mathbb{N})$, and define as usual a sequence $(\gamma_k)_{k\geq 1}$ by setting $\gamma_k = 2^{\delta^{(k-1)} - \tau^{(k)}}\big(\Delta^{(k)}\big)^{1-\frac{1}{p}}$ for every $k \geq 1$. Suppose that the sequence $(\gamma_k)_{k\geq 1}$ is non-increasing, and that the following two conditions are satisfied:*

$$2\sum_{k\geq 1} \#J_k\, \gamma_k^{1/2} \leq 1 \quad \textit{and} \quad \lim_{k\to\infty} \frac{\delta^{(k)}}{a_k} = 0.$$

Then T is not $\mathcal{U}$-frequently hypercyclic.

Proof. Let $x \in \ell_p(\mathbb{N})$ be a hypercyclic vector for T. Our aim is to prove that x cannot be a $\mathcal{U}$-frequently hypercyclic vector for T, using Theorem 6.10. We set

$$\beta_l := 4\,\gamma_k \quad \text{for every } l \in J_k \text{ and every } k \geq 1.$$

This sequence $(\beta_l)_{l\geq 1}$ is non-increasing and satisfies $\sum_{l\geq 1}\sqrt{\beta_l} \leq 1$. Setting

$$X_l := \Big\|\sum_{k=b_l}^{b_{l+1}-1}\Big(\prod_{j=k+1}^{b_{l+1}-1} w_j\Big)\, x_k e_k\Big\| \quad \text{for every } l \in I_k \text{ and every } k \geq 1,$$

we have $\|P_l\, x\| \leq X_l$ and, by Proposition 6.12,

$$\sup_{j\geq 0}\, \|P_n T^j P_l\, x\| \leq \frac{1}{4}\,\beta_l X_l$$

for every $0 \leq n < l$, and

$$\sup_{0\leq j\leq N}\, \|P_n T^j P_l\, x\| \leq \frac{1}{4}\,\beta_l\,\Big(\sup_{b_{l+1}-N\leq k<b_{l+1}} \prod_{s=k+1}^{b_{l+1}-1} |w_s|\Big)\, \|P_l\, x\|$$

for every $1 \le N \le \Delta^{(k)}$ and $0 \le n < l$ with l belonging to J_k. Setting $N_l := a_k$ for every $l \in J_k$ and every $k \ge 1$, we deduce that

$$\sup_{0 \le j \le N_l} \|P_n T^j P_l\, x\| \le \frac{1}{4}\, \beta_l\, \|P_l\, x\|$$

for every $l \ge 0$ and every $0 \le n < l$. In order to check assumption (C) of Theorem 6.10, we use Proposition 7.13. Let n belong to J_k, and define $r = (n - \min J_k) \mod (f_k - a_k)$. We have $w_{b_n+j} = 1$ for every $j \in (\delta^{(k)}, \Delta^{(k)} - a_k - r - 2\delta^{(k)}) \cup (\Delta^{(k)} - a_k - r, \Delta^{(k)})$ and

$$\prod_{j=b_n+\delta^{(k)}+1}^{b_{n+1}-1} w_j = 1.$$

It follows from Proposition 7.13 that

$$\frac{1}{J+1}\, \#\big\{0 \le j \le J\,;\, \|P_l T^j T_l\, x\| \ge X_l/2\big\} \ge 1 - 12\delta^{(k)}\Big(\frac{1}{J+1} + \frac{1}{\Delta^{(k)}}\Big)$$

for every $J \ge 0$, every $k \ge 1$, and every $l \in J_k$. For every $k \ge 1$, we thus have

$$\inf_{J \ge a_k} \frac{1}{J+1}\, \#\Big\{0 \le j \le J\,;\, \|P_l T^j T_l\, x\| \ge X_l/2\Big\} \ge 1 - \frac{12\delta^{(k)}}{a_k} - \frac{12\delta^{(k)}}{\Delta^{(k)}}.$$

Since $\lim_{k\to\infty} \frac{\delta^{(k)}}{\Delta^{(k)}} \le \lim_{k\to\infty} \frac{\delta^{(k)}}{a_k} = 0$, it follows that

$$\begin{aligned}\liminf_{l\to\infty} \inf_{J \ge N_l} \frac{1}{J+1}\, \#\Big\{0 \le j \le J\,; \|P_l T^j T_l\, x\| \ge X_l/2\Big\} \\ \ge \liminf_{k\to\infty} \Big(1 - \frac{12\delta^{(k)}}{a_k} - \frac{12\delta^{(k)}}{\Delta^{(k)}}\Big) = 1.\end{aligned}$$

By Theorem 6.10, T is not $\mathcal{U}$-frequently hypercyclic. □

4.3. Chaotic and mixing operators which are not UFHC. From Propositions 7.12 and 7.14, we immediately deduce

THEOREM 7.15. *Let T be an operator of C_2-type on $\ell_p(\mathbb{N})$, and for every $k \ge 1$, set $\gamma_k = 2^{\delta^{(k-1)} - \tau^{(k)}} \big(\Delta^{(k)}\big)^{1-\frac{1}{p}}$. Suppose that the sequence $(\gamma_k)_{k\ge1}$ is non-increasing and that the following three conditions are met:*

$$2 \sum_{k\ge1} \#J_k\, \gamma_k^{1/2} \le 1, \quad \lim_{k\to\infty} \frac{\delta^{(k)}}{a_k} = 0 \quad \text{and} \quad \lim_{k\to\infty} (\delta^{(k)} - \tau^{(k)}) = \infty.$$

If the set

$$\bigcup_{k\ge1} [a_k + \delta^{(k)}, f_k + \delta^{(k)})$$

is cofinite, then T is chaotic and topologically mixing but not $\mathcal{U}$-frequently hypercyclic.

Here is a concrete example of such a chaotic and topologically mixing operator which is not $\mathcal{U}$-frequently hypercyclic.

EXAMPLE 7.16. Let C be a positive integer. Consider the operator of C_2-type T on $\ell_p(\mathbb{N})$ associated to the parameters

$$\Delta^{(k)} = 2^{2Ck^2+5}, \quad \delta^{(k)} = 2^{Ck^2+1}, \quad \tau^{(k)} = 2^{Ck^2}, \quad a_k = k\delta^{(k)} \quad \text{and} \quad f_k = \frac{1}{2}\Delta^{(k)}, \quad k \ge 1.$$

If C is sufficiently large, then T is chaotic and topologically mixing but not $\mathcal{U}$-frequently hypercyclic.

PROOF. The operator T is well-defined since

$$\Delta^{(k)} - 4\delta^{(k)} = 2^{2Ck^2+5} - 2^{Ck^2+3} > 2^{2Ck^2+4} = f_k \quad \text{and} \quad f_k = 2^{2Ck^2+4} \geq k2^{Ck^2+1} = a_k$$

for every $k \geq 1$. Moreover, the set

$$\bigcup_{k\geq 1} [a_k + \delta^{(k)}, f_k + \delta^{(k)})$$

is cofinite since $f_k + \delta^{(k)} \geq a_{k+1} + \delta^{(k+1)}$ if k is large enough. Indeed,

$$f_k + \delta_k - a_{k+1} - \delta^{(k+1)} \geq f_k - 2a_{k+1} = 2^{2Ck^2+4} - k2^{C(k+1)^2+2},$$

and the quantity on the right hand side tends to infinity as k tends to infinity. Finally, since $\#J_k \leq \Delta^{(k)} k \,.\, \#J_{k-1}$ and $\#J_0 = 1$, we have $\#J_k \leq \prod_{j=1}^{k} j\Delta^{(j)} \leq 2^{8Ck^3}$ for every $k \geq 1$. It follows that the remaining assumptions of Theorem 7.15 are also satisfied if C is sufficiently large. □

REMARK 7.17. The operators constructed in [**40**] (which are chaotic and not $\mathcal{U}$-frequently hypercyclic) are never topologically mixing. Indeed, the parameters in this construction satisfy the following three conditions:

- the quantity $\delta_n/(b_{n+1} - b_n)$ tends to 0 as n tends to infinity;
- for every $n \geq 1$, $b_{n+1} - b_n$ is a multiple of $2(b_n - b_{n-1})$;
- for every $n \geq 1$, $2^{\delta_{n-1}-\tau_n}(b_{n+1} - b_n) \leq 2^{-2(n+1)}$.

Therefore, for any such operator T, there exists an integer n_0 such that

$$\delta_n < \frac{1}{2}(b_{n+1} - b_n) \quad \text{for every } n \geq n_0.$$

It follows that for every integer $n \geq n_0$ such that 1 does not belong to the set $\bigcup_{j\geq 0} \varphi^j(n)$, $b_{n+1} - b_n$ does not belong to $\mathcal{N}_T(B(0,1), B(3e_{b_1}, 1))$. Indeed, we have for any $k \geq 0$

- $T^{b_{n+1}-b_n} e_k = e_k$ if $k < b_n$;
- $P_1 T^{b_{n+1}-b_n} e_k = 0$ if k belongs to $[b_n, b_{n+1}[$ since 1 does not belong to $\bigcup_{j\geq 0} \varphi^j(n)$;
- $\|P_1 T^{b_{n+1}-b_n} P_l x\| \leq 2^{-2(l+1)} \|P_l x\|$ if $l > n$, by Proposition 6.12.

Hence, if x is any vector of the unit ball $B(0,1)$, the vector $y := T^{b_{n+1}-b_n} x$ satisfies

$$|y_{b_1}| \leq |x_{b_1}| + \sum_{l>n} \|P_1 T^{b_{n+1}-b_n} P_l x\| < 1 + \sum_{l>n} \frac{1}{2^{2(l+1)}} \leq 2.$$

Thus y does not belong to the ball $B(3e_{b_1}, 1)$. Since there are infinitely many integers $n \geq n_0$ such that $\bigcup_{j\geq 0} \varphi^j(n)$ does not contain 1, this shows that T is indeed not topologically mixing.

5. Chaos plus mixing plus FHC do not imply ergodicity

We have seen in Section 2 that there exist operators which are chaotic and frequently hypercyclic but not ergodic. In this section, we show that one can even find topologically mixing operators with this property. The examples we give are C_2-type operators.

THEOREM 7.18. *Let T be an operator of C_2-type on $\ell_p(\mathbb{N})$, and for every $k \geq 1$, set $\gamma_k = 2^{\delta^{(k-1)}-\tau^{(k)}} \big(\Delta^{(k)}\big)^{1-\frac{1}{p}}$. Assume that the sequence $(\gamma_k)_{k\geq 1}$ is non-increasing and that the following conditions are satisfied:*

$$2\sum_{k\geq 1} \#J_k \, \gamma_k^{1/2} \leq 1, \quad \limsup_{k\to\infty} \frac{\delta^{(k)}}{\Delta^{(k)}} > 0, \quad \limsup_{k\to\infty} \frac{\delta^{(k)}}{a_k} < \frac{1}{24} \quad \textit{and} \quad \lim_{k\to\infty} \big(\frac{1}{2}\delta^{(k)} - \tau^{(k)}\big) = \infty.$$

If the set

$$\bigcup_{k\geq 1} [a_k + \delta^{(k)}, f_k + \delta^{(k)})$$

is cofinite, then T is topologically mixing, chaotic, frequently hypercyclic and not ergodic.

PROOF. Since this proof is quite similar to others already given in detail, we only sketch it. By Proposition 7.12, we already know that since $\lim_{k\to\infty}(\delta^{(k)}-\tau^{(k)})=\infty$ and the set $\bigcup_{k\geq 1}[a_k+\delta^{(k)}, f_k+\delta^{(k)})$ is cofinite, the operator T is topologically mixing. On the other hand, for any $k\geq 1$ any $n\in J_k$, we have

$$|v_n|\prod_{j=b_n+1}^{b_{n+1}-1}|w_j| = 2^{\delta^{(k)}-\tau^{(k)}} \quad \text{and}$$

$$|v_n|\prod_{j=b_n+m'+1}^{b_{n+1}-1}|w_j| \geq 2^{\frac{\delta^{(k)}}{2}-\tau^{(k)}} \qquad \text{for every } 0\leq m'<\tfrac{1}{2}\delta^{(k)}-1.$$

Since $\lim_{k\to\infty}(\frac{1}{2}\delta^{(k)}-\tau^{(k)})=\infty$ and $\limsup_{k\to\infty}\frac{\delta^{(k)}}{\Delta^{(k)}}>0$, we can then deduce the frequent hypercyclicity of T as in Theorem 6.9. Finally, T is not ergodic because $c(T)<1$. Indeed, there exists $x\in \mathrm{HC}(T)$ such that $c(T)=\overline{\mathrm{dens}}\ N(x,B(0,\varepsilon))$ for every $\varepsilon>0$; and since $2\sum_{k\geq 1}\#J_k\,\gamma_k^{1/2}\leq 1$, it follows from Lemma 6.11 combined with the end of the proof of Proposition 7.14 that there exists $\varepsilon>0$ such that

$$\begin{aligned}\overline{\mathrm{dens}}\ N(x,B(0,\varepsilon)) &\leq 1-\liminf_{l\to\infty}\inf_{J\geq N_l}\frac{1}{J+1}\,\#\big\{0\leq j\leq J\,;\,\|P_lT^jP_l\,x\|\geq X_l/2\big\}\\ &\leq \limsup_{k\to\infty}\frac{12\delta^{(k)}}{a_k}+\frac{12\delta^{(k)}}{\Delta^{(k)}}\leq \limsup_{k\to\infty}\frac{24\delta^{(k)}}{a_k}<1.\end{aligned}$$

□

EXAMPLE 7.19. If we take

$$\Delta^{(k)}=2^{C(k+4)},\quad \delta^{(k)}=2^{Ck},\quad \tau^{(k)}=\frac{\delta^{(k)}}{4},\quad a_k=2^{C(k+1)}\quad\text{and}\quad f_k=2^{C(k+3)}$$

for a sufficiently large constant C, then T is topologically mixing, chaotic, frequently hypercyclic and not ergodic.

REMARK 7.20. In the above example, as well as in Example 7.7, we prove the non-ergodicity of the operator T by checking that $c(T)<1$. It would be interesting to know if there exist frequently hypercyclic non-ergodic operators satisfying $c(T)=1$.

6. C-type operators with few eigenvalues

In this section, we exhibit a class of C-type operators having only countably many unimodular eigenvalues. This provides further examples of hypercyclic operators on $\ell_p(\mathbb{N})$ with only countably many unimodular eigenvalues, and such that the associated unimodular eigenvectors span the space (as mentioned previously, the question of the existence of such operators was raised by Flytzanis in [**23**]).

The general idea of the forthcoming construction is the following: if $T=T_{v,w,\varphi,b}$ is an operator of C-type on $\ell_p(\mathbb{N})$ and if the sequence $v=(v_n)_{n\geq 1}$ decreases extremely fast, then the unimodular eigenvalues of T must be roots of unity. This is not such a surprising statement if one considers what happens in the "degenerate" case where the sequence v is identically equal to 0: indeed, in this case the operator T has the form $T=\bigoplus_{n\geq 0}C_n$ where the operators C_n, $n\geq 0$, are finite dimensional cyclic operators satisfying $C_n^{2(b_{n+1}-b_n)}=I$. Thus any eigenvalue λ of T must satisfy $\lambda^{2(b_{n+1}-b_n)}=1$ for some integer $n\geq 0$.

Before starting our construction, we determine the spectrum of the operators of C-type on $\ell_p(\mathbb{N})$ which we consider here:

LEMMA 7.21. *Let T be a hypercyclic C-type operator on $\ell_p(\mathbb{N})$ such that*

$$\lim_{n\to\infty} \prod_{j=b_n+1}^{b_{n+1}-1} |w_j| = \infty.$$

Then the spectrum of T is the closed disk $\overline{D}(0,R)$, where

$$R := \limsup_{N\to\infty} \left(\sup_{\substack{n\geq 0 \\ b_{n+1}-b_n>N}} \sup_{b_n\leq k<b_{n+1}-N} (w_{k+1}w_{k+2}\cdots w_{k+N}) \right)^{1/N}.$$

If T is either a $C_{+,1}$-type or a $C_{+,2}$-type operator on $\ell_p(\mathbb{N})$, the spectrum of T is thus the closed disk $\overline{D}(0,2)$.

PROOF. Using the notation employed in the proof of Lemma 6.2, we observe that T is a compact perturbation of the direct sum operator $C := \bigoplus_{n\geq 0} C_{w,\,b,\,n}$ on $\ell_p(\mathbb{N})$. Also, the assumption of Lemma 7.21 implies that C is itself a compact perturbation of the forward weighted shift S on $\ell_p(\mathbb{N})$ defined by

$$Se_k = \begin{cases} w_{k+1}\, e_{k+1} & \text{if } k \in [b_n, b_{n+1}-1),\ n\geq 0 \\ 0 & \text{if } k = b_n - 1,\ n \geq 1. \end{cases}$$

So T^* is a compact perturbation of S^*, and it then follows from the Fredholm alternative that if $\lambda \in \mathbb{C}$ is any element of $\sigma(T^*)\setminus\sigma(S^*)$, then λ is an eigenvalue of T^*. But as T is hypercyclic, its adjoint has no eigenvalue, and it follows that $\sigma(T^*)$ is contained in $\sigma(S^*)$. Hence, $\sigma(T)$ is contained in $\sigma(S)$. Conversely, the same argument shows that any $\lambda \in \sigma(S)\setminus\sigma(T)$ is an eigenvalue of S. But the only eigenvalue of S is 0, so that $\sigma(S)$ is contained in $\sigma(T)\cup\{0\}$. Now, it is well-known that the $\sigma(S)$ is the closed disk $\overline{D}(0,R)$ (see for instance [**47**]), so that $\sigma(T)\subseteq\overline{D}(0,R)\subseteq\sigma(T)\cup\{0\}$. Since $\sigma(T)$ is closed, it follows that $\sigma(T)=\overline{D}(0,R)$, and this concludes the proof of Lemma 7.21. □

We now come back to our construction of C-type operators with few unimodular eigenvalues. In order to simplify the expressions involved in the results which we are about to state, we adopt the following notation: if $T = T_{v,\,w,\,\varphi,\,b}$ is an operator of C-type, we set

$$\Delta b_n := b_{n+1} - b_n \qquad \text{for every } n \geq 0.$$

Also, we will say that an increasing sequence of positive integers $(n(m))_{m\geq 0}$ is a *φ-sequence* if

$$n(m) = \varphi(n(m+1)) \quad \text{for all } m \geq 0.$$

THEOREM 7.22. *Let $T = T_{v,w,\varphi,b}$ be an operator of C-type on $\ell_p(\mathbb{N})$. Assume that for every φ-sequence $(n(m))_{m\geq 0}$, we have*

$$\limsup_{m\to\infty} |v_{n(m)}| \cdot 2^{n(m)} (\Delta b_{n(m)})^m \left(\prod_{j=1}^{m-1} \frac{|v_{n(j)}|}{\Delta b_{n(j)}} \prod_{\nu=b_{n(j)}+1}^{b_{n(j)+1}-1} |w_\nu| \right) < \infty. \tag{37}$$

Then each unimodular eigenvalue of T is a root of unity. More precisely, any such eigenvalue λ must satisfy $\lambda^{\Delta b_n} = 1$ for some integer $n \geq 0$.

This statement may not be particularly enlightening. Corollary 7.24 below should look more intuitive.

For the proof of Theorem 7.22, we will need the following lemma.

LEMMA 7.23. *Fix $\lambda \in \mathbb{T}$, and let $(p_m)_{m\geq 1}$ be a sequence of positive integers tending to infinity such that p_{m+1} is a multiple of p_m for every $m \geq 1$. Assume that $\lambda^{p_m} \neq \pm 1$ for every $m \geq 1$. Then there exist infinitely many integers $m \geq 1$ such that*

$$\left|\lambda^{p_j} \pm 1\right| \geq \frac{p_j}{p_{m+1}} \qquad \textit{for every } 1 \leq j \leq m.$$

Proof. Writing λ as $\lambda = e^{i\theta}$ with $\theta \in \mathbb{R} \setminus \pi\mathbb{Z}$, we have for every $j \geq 1$

$$\left|\lambda^{p_j} \pm 1\right| \geq |\sin(p_j\theta)| = \sin\left(\operatorname{dist}(p_j\theta, \pi\mathbb{Z})\right) \geq \frac{2}{\pi}\operatorname{dist}(p_j\theta, \pi\mathbb{Z}) = \frac{2p_j}{\pi}\operatorname{dist}\left(\theta, \frac{\pi}{p_j}\mathbb{Z}\right).$$

Since p_m is a multiple of p_j for every $j \leq m$ and every $m \geq 1$, it follows that

$$\left|\lambda^{p_j} \pm 1\right| \geq \frac{2p_j}{\pi}\operatorname{dist}\left(\theta, \frac{\pi}{p_m}\mathbb{Z}\right) \quad \text{for every } m \geq 1 \text{ and every } 1 \leq j \leq m.$$

So it suffices to show that there exist infinitely many integers $m \geq 1$ such that

$$\operatorname{dist}\left(\theta, \frac{\pi}{p_m}\mathbb{Z}\right) \geq \frac{\pi}{2p_{m+1}}.$$

Towards a contradiction assume that there exists an integer m_0 and, for every $m \geq m_0$, an integer k_m such that

$$\left|\theta - \frac{k_m\pi}{p_m}\right| < \frac{\pi}{2p_{m+1}} \quad \text{for every } m \geq m_0.$$

We then have

$$\left|\frac{k_m\pi}{p_m} - \frac{k_{m+1}\pi}{p_{m+1}}\right| < \frac{\pi}{2p_{m+1}} + \frac{\pi}{2p_{m+2}} \leq \frac{\pi}{p_{m+1}},$$

and since p_{m+1} is a multiple of p_m, it follows that

$$\frac{k_m\pi}{p_m} = \frac{k_{m+1}\pi}{p_{m+1}} \quad \text{for every } m \geq m_0.$$

Since p_m tends to infinity as m tends to infinity, we conclude that $\theta = \frac{k_{m_0}\pi}{p_{m_0}}$, so that $\lambda^{p_{m_0}} = \pm 1$, which stands in contradiction with our initial assumption. □

We can now give the proof of Theorem 7.22.

Proof. Let $\lambda \in \mathbb{T}$ be a unimodular eigenvalue of T, and towards a contradiction, assume that $\lambda^{\Delta b_n} \neq 1$ for every $n \geq 0$. Note that since Δb_{n+1} is an even multiple of Δb_n, we also have $\lambda^{\Delta b_n} \neq -1$ for every $n \geq 0$.

Let $x \in \ell_p(\mathbb{N}) \setminus \{0\}$ be an eigenvector of T associated to the eigenvalue λ. The equation $Tx = \lambda x$ yields that for every $n \geq 0$ and every $k \in [b_n, b_{n+1})$, we have

$$x_k = \frac{\prod_{\nu=b_n+1}^{k} w_\nu}{\lambda^{k-b_n}} x_{b_n} \tag{38}$$

and

$$\begin{aligned} \lambda x_{b_n} &= -\frac{1}{\prod_{\nu=b_n+1}^{b_{n+1}-1} w_\nu} x_{b_{n+1}-1} + \sum_{l\in\varphi^{-1}(n)} v_l x_{b_{l+1}-1} \\ &= -\frac{1}{\lambda^{\Delta b_n - 1}} x_{b_n} + \sum_{l\in\varphi^{-1}(n)} v_l \frac{\prod_{\nu=b_l+1}^{b_{l+1}-1} w_\nu}{\lambda^{\Delta b_l - 1}} x_{b_l}. \end{aligned} \tag{39}$$

Since $\sum_{l>n} 2^{n-l} = 1$, we deduce from the last identity that for every $n \geq 0$, there exists $l > n$ with $\varphi(l) = n$ such that

$$|v_l| \prod_{\nu=b_l+1}^{b_{l+1}-1} |w_\nu||x_{b_l}| \geq 2^{n-l}\left|\lambda + \frac{1}{\lambda^{\Delta b_n - 1}}\right| |x_{b_n}| = 2^{n-l}|\lambda^{\Delta b_n} + 1|\,|x_{b_n}|.$$

Thus, if we set

$$\varepsilon_n = |1 + \lambda^{\Delta b_n}|,$$

the following property holds true: for every $n \geq 0$, there exists $l > n$ with $\varphi(l) = n$ such that

$$|x_{b_l}| \geq \frac{2^{n-l}\varepsilon_n}{|v_l| \prod_{\nu=b_l+1}^{b_{l+1}-1} |w_\nu|} |x_{b_n}|.$$

Let us choose an integer $n(0)$ such that $x_{b_{n(0)}} \neq 0$ (since x is non-zero, such an integer does exist by (38)). The argument given just above allows us to construct a φ-sequence $(n(m))_{m\geq 0}$ such that

$$|x_{b_{n(m+1)}}| \geq \frac{2^{n(m)-n(m+1)}\varepsilon_{n(m)}}{|v_{n(m+1)}| \prod_{\nu=b_{n(m+1)}+1}^{b_{n(m+1)+1}-1} |w_\nu|} |x_{b_{n(m)}}| \quad \text{for every } m \geq 0.$$

We then have

$$|x_{b_{n(m)}}| \geq \frac{2^{n(0)-n(m)} \left(\prod_{j=0}^{m-1} \varepsilon_{n(j)}\right)}{\left(\prod_{j=1}^{m} |v_{n(j)}|\right)\left(\prod_{j=1}^{m} \prod_{\nu=b_{n(j)}+1}^{b_{n(j)+1}-1} |w_\nu|\right)} |x_{b_{n(0)}}| \quad \text{for every } m \geq 1,$$

and hence (by (38) again)

$$|x_{b_{n(m)+1}-1}| = \frac{2^{n(0)-n(m)} \left(\prod_{j=0}^{m-1} \varepsilon_{n(j)}\right)}{\left(\prod_{j=1}^{m} |v_{n(j)}|\right)\left(\prod_{j=1}^{m-1} \prod_{\nu=b_{n(j)}+1}^{b_{n(j)+1}-1} |w_\nu|\right)} |x_{b_{n(0)}}|.$$

If we now set

$$\alpha_m := \frac{\left(\prod_{j=1}^{m} |v_{n(j)}|\right)\left(\prod_{j=1}^{m-1} \prod_{\nu=b_{n(j)}+1}^{b_{n(j)+1}-1} |w_\nu|\right)}{2^{n(0)-n(m)} \left(\prod_{j=0}^{m-1} \varepsilon_{n(j)}\right)},$$

it suffices to prove that α_m does not tend to infinity as m tends to infinity in order to obtain a contradiction, since this would imply that x does not belong to $c_0(\mathbb{N})$. In view of our assumption (37), this will be verified if we are able to show that

$$\prod_{j=0}^{m-1} \varepsilon_{n(j)} \geq (\Delta b_{n(m)})^{-m} \prod_{j=0}^{m-1} \Delta b_{n(j)} \quad \text{for infinitely many } m \geq 1. \tag{40}$$

At this point, we apply Lemma 7.23, considering the sequence $(p_m)_{m\geq 1}$ defined by setting $p_m := \Delta b_{n(m-1)}$ for every $m \geq 1$. The conclusion is that there exist infinitely many integers $m \geq 1$ such that

$$\varepsilon_j = \left|1 + \lambda^{\Delta b_{n(j)}}\right| \geq \frac{\Delta b_{n(j)}}{\Delta b_{n(m)}} \quad \text{for every } 0 \leq j \leq m-1,$$

which gives (40). Theorem 7.22 is proved. □

Even though it does show up naturally in the above proof, the condition appearing in Theorem 7.22 may look a bit strange since it involves *all φ-sequences* $(n(m))_{m\geq 0}$. However, its general meaning is clear: v_n should go to 0 quite fast as n goes to infinity. Here is a consequence of Theorem 7.22 that makes it rather transparent. Recall that if T is a $\mathrm{C}_{+,1}$-type or a $\mathrm{C}_{+,2}$-type operator on $\ell_p(\mathbb{N})$, then $v_n = 2^{-\tau^{(k)}}$ for every $n \in [2^{k-1}, 2^k)$ and every $k \geq 1$.

Corollary 7.24. *Let T be a $C_{+,1}$-type or a $C_{+,2}$-type operator on $\ell_p(\mathbb{N})$. Assume that $\Delta^{(k)}/\Delta^{(k-1)}$ tends to infinity and $k \log \Delta^{(k)} = O\big(\Delta^{(k-1)}\big)$ as k tends to infinity. If*

$$\lim_{k\to\infty} 2^{-\tau^{(k)}} M^{\Delta^{(k-1)}} = 0 \quad \textit{for every } M > 0, \tag{41}$$

then all the unimodular eigenvalues of T must be roots of unity.

Proof. Let $(n(m))_{m\geq 0}$ be a φ-sequence, and for each $m \geq 0$, let us denote by α_m the quantity appearing in (37), namely

$$\alpha_m = v_{n(m)} \cdot 2^{n(m)} (\Delta b_{n(m)})^m \left(\prod_{j=1}^{m-1} \frac{v_{n(j)}}{\Delta b_{n(j)}} \prod_{\nu=b_{n(j)}+1}^{b_{n(j)+1}-1} w_\nu\right).$$

For any $m \geq 0$, let us denote by k_m the unique positive integer such that $n(m)$ belongs to the interval $[2^{k_m-1}, 2^{k_m})$. Then $m \leq k_m$ because $(n(m))$ is a φ-sequence.

Observe first that the partial products

$$\prod_{j=0}^{m-1} \frac{v_{n(j)}}{\Delta b_{n(j)}}, \quad m \geq 1$$

remain bounded (in fact, they tend quickly to 0). Also, since the sequence w is bounded, there exists a constant $A > 1$ such that

$$\prod_{j=1}^{m-1} \prod_{\nu=b_{n(j)}+1}^{b_{n(j)+1}-1} |w_\nu| \leq A^{\sum_{j=1}^{m-1} \Delta^{(k_j)}} \leq A^{\sum_{i=0}^{k_m-1} \Delta^{(i)}} \qquad \text{for every } m \geq 1.$$

Since $\Delta^{(k)}/\Delta^{(k-1)}$ tends to infinity as k tends to infinity, it follows that there exists a positive constant B such that

$$\prod_{j=1}^{m-1} \prod_{\nu=b_{n(j)}+1}^{b_{n(j)+1}-1} |w_\nu| \leq B^{\Delta^{(k_m-1)}} \quad \text{for every } m \geq 1.$$

Moreover, there also exists a positive constant C such that $2^{n(m)} \leq 2^{2^{k_m}} \leq C^{\Delta^{(k_m-1)}}$ for every $m \geq 1$ (again because $\Delta^{(k)}/\Delta^{(k-1)}$ tends to infinity). Lastly, we have

$$\big(\Delta b_{n(m)}\big)^m = \big(\Delta^{(k_m)}\big)^m \leq \big(\Delta^{(k_m)}\big)^{k_m} \quad \text{for every } m \geq 1,$$

and since $k \log \Delta^{(k)} = O\big(\Delta^{(k-1)}\big)$ the quantity $\big(\Delta^{(k_m)}\big)^{k_m}$ is dominated by $D^{\Delta^{(k_m-1)}}$ for some positive constant D. Putting things together, and remembering that $v_{n(m)} = 2^{-\tau_{k_m}}$ for every $m \geq 1$, we obtain that there exists a positive constant M such that

$$\alpha_m \leq 2^{-\tau^{(k_m)}} M^{\Delta^{(k_m-1)}} \quad \text{for every } m \geq 1.$$

By (41), this concludes the proof of Corollary 7.24. □

Remark 7.25. Condition (41) is compatible with those appearing for example in Theorems 7.2, 7.6 or 7.10.

Remark 7.26. Assume that $b_1 = 1$. If $\mathbf{n} = (n(m))_{m\geq 0}$ is a φ-sequence such that $n(0) = 0$, the proof of Theorem 7.22 shows that if $\lambda \in \mathbb{T}$ and if we set

$$x_{\mathbf{n},\lambda} := e_0 + \sum_{m=1}^{\infty} \frac{\prod_{l=1}^{m}(1+\lambda^{\Delta b_{n(l-1)}})}{\Big(\prod_{l=1}^{m} v_{n(l)}\Big)\Big(\prod_{l=1}^{m-1}\prod_{\nu=b_{n(l)}+1}^{b_{n(l)+1}-1} w_\nu\Big)} \sum_{j=b_{n(m)}}^{b_{n(m)+1}-1} \frac{\lambda^{b_{n(k)+1}-1-j}}{\prod_{\nu=j+1}^{b_{n(k)+1}-1} w_\nu} e_j,$$

then $x_{\mathbf{n},\lambda}$ is an eigenvector of $T = T_{v,\,w,\,\varphi,\,b}$ associated to the eigenvalue λ, provided that the series defining $x_{\mathbf{n},\lambda}$ is convergent. This has the following two interesting consequences.

(1) An operator of C-type T may quite well have a unimodular eigenvalue λ such that the associated eigenspace $\ker(T-\lambda)$ is not one-dimensional. This happens in particular if $\lambda^{\Delta b_n} = -1$ for some integer n and if there exists $m \neq n$ such that $\Delta b_m = \Delta b_n$.

(2) If we assume that $v_{n(l)} = \Big(\prod_{\nu=b_{n(l-1)}+1}^{b_{n(l-1)+1}-1} w_\nu\Big)^{-1}$, we obtain the following expression for $x_{\mathbf{n},\lambda}$:

$$x_{\mathbf{n},\lambda} = e_0 + \sum_{m=1}^{\infty} \frac{\prod_{l=1}^{m}(1+\lambda^{\Delta b_{n(l-1)}})}{v_{n(1)}} \sum_{j=b_{n(m)}}^{b_{n(m)+1}-1} \frac{\lambda^{b_{n(m)+1}-1-j}}{\prod_{\nu=j+1}^{b_{n(m)+1}-1} w_\nu} e_j.$$

This vector is well-defined for some unimodular numbers λ which are not roots of unity, provided that the sequence $(\Delta b_{n(l-1)})_{l\geq 1}$ is suitably chosen. Therefore, operators of C-type can have unimodular eigenvalues which are not roots of unity.

7. $C_{+,1}$-type with many eigenvalues

We have just seen that a large class of C-type operators have few eigenvalues. In this section, we show that, on the other hand, C-type operators (in fact, $C_{+,1}$-type operators) can also have lots of eigenvalues, to the point of being mixing in the Gaussian sense.

In what follows, we consider C_+-type operators $T = T_{v,\,w,\,\varphi,\,b}$. As already observed, the map φ is then given by

$$\varphi(n) = n - 2^{k-1} \qquad \text{if} \quad 2^{k-1} \le n < 2^k, \quad k \ge 1.$$

From this, it follows that a sequence $\mathbf{n} = (n(m))_{m\ge 0}$ with $n(0) = 0$ is a φ-sequence if and only if there exists an increasing sequence of positive integers $(k_m)_{m\ge 1}$ such that

$$n(m) = 2^{k_1-1} + \cdots + 2^{k_m-1} \qquad \text{for every } m \ge 1.$$

More precisely, k_m is for each $m \ge 1$ the unique positive integer such that $2^{k_m-1} \le n(m) < 2^{k_m}$. In this case, we say that $\mathbf{n}$ is *the φ-sequence associated to the sequence* $(k_m)_{m\ge 1}$.

With these notations, the formula of Remark 7.26 defining $x_{\mathbf{n},\lambda}$ for a φ-sequence $\mathbf{n}$ and an element λ of $\mathbb{T}$ can be rewritten as follows:

$$\begin{aligned} x_{\mathbf{n},\lambda} &= e_0 + \sum_{m=1}^{\infty} \frac{\prod_{l=1}^{m}(1+\lambda^{\Delta^{(k_{l-1})}})}{2^{-\sum_{l=1}^{m}\tau^{(k_l)} + \sum_{l=1}^{m-1}\delta^{(k_l)}}} \sum_{j=b_{n(m)}}^{b_{n(m)+1}-1} \frac{\lambda^{b_{n(m)+1}-1-j}}{\max\bigl(1, 2^{\delta^{(k_m)}-(j-b_{n(m)})}\bigr)}\, e_j \\ &= e_0 + \sum_{m=1}^{\infty} \frac{\prod_{l=1}^{m}(1+\lambda^{\Delta^{(k_{l-1})}})}{2^{\tau^{(k_1)} + \sum_{l=2}^{m}\left(\delta^{(k_{l-1})}-\tau^{(k_l)}\right)}} \sum_{j=b_{n(m)}}^{b_{n(m)+1}-1} \frac{\lambda^{b_{n(m)+1}-1-j}}{\max\bigl(1, 2^{\delta^{(k_m)}-(j-b_{n(m)})}\bigr)}\, e_j. \end{aligned}$$

We now introduce the notion of a *good φ-sequence.*

Definition 7.27. We will say that a φ-sequence $\mathbf{n}$ with $n(0) = 0$ is a good φ-sequence if $x_{\mathbf{n},\lambda}$ is well-defined for every $\lambda \in \mathbb{T}$ and if the map $\lambda \mapsto x_{\mathbf{n},\lambda}$ is continuous from $\mathbb{T}$ into $\ell_2(\mathbb{N})$.

The study of these sequences will allow us to show that under some conditions, a $C_{+,1}$-type operator on $\ell_2(\mathbb{N})$ can be mixing in the Gaussian sense.

Lemma 7.28. *Let (v, w, φ, b) define a $C_{+,1}$-type operator. Assume that $b_1 = 1$, and that*

$$\sum_{m=1}^{\infty} 4^{m-\sum_{k=1}^{m}(\delta^{(k-1)}-\tau^{(k)})}\, \Delta^{(m)} < \infty. \tag{42}$$

Then, for any integer $n \ge 1$, there exists a good φ-sequence $\mathbf{n}$ passing through n, i.e. such that $n(r) = n$ for some integer $r \ge 1$.

Proof. Write n as $n = 2^{k_1-1} + \cdots + 2^{k_r-1}$ with $1 \le k_1 < \cdots < k_r$, and set $k_{r+i} := k_r + i$ for every $i \ge 1$. Then the φ-sequence $\mathbf{n}$ associated to $(k_m)_{m\ge 1}$ passes through n. Moreover, since $k_{r+i} := k_r + i$ for every $i \ge 1$, it follows from (42) the series

$$\sum_{m\ge 1} 4^{m-\sum_{l=1}^{m}(\delta^{(k_{l-1})}-\tau^{(k_l)})}\, \Delta^{(k_m)}$$

is convergent. Since $\left|\prod_{l=1}^{m}(1+\lambda^{\Delta^{(k_l)}})\right| \le 2^m$ for every $m \ge 1$ and every $\lambda \in \mathbb{T}$, this implies that $\mathbf{n}$ is a good φ-sequence. $\square$

Lemma 7.29. *Let $\mathbf{n}$ be a good φ-sequence. If $y = \sum_{j\ge 0} y_j e_j$ is a vector of $\ell_2(\mathbb{N})$ which is orthogonal to $x_{\mathbf{n},\lambda}$ for every $\lambda \in \mathbb{T}$, then*

$$y_0 = 0 \quad \text{and} \quad y_j = 0 \quad \text{for every } j \in \bigcup_{m\ge 1} [b_{n(m)}, b_{n(m)+1}).$$

PROOF. Let $(k_m)_{m\geq 1}$ be the increasing sequence of integers associated to $\mathbf{n}$, and set $k_0 := 0$. By assumption and in view of the definition of $x_{\mathbf{n},\lambda}$, we have

$$y_0 + \sum_{m=1}^{\infty} \prod_{l=1}^{m} \big(1+\lambda^{\Delta^{(k_{l-1})}}\big) \sum_{b_{n(m)}\leq j<b_{n(m)+1}} c_{m,j}\lambda^{b_{n(m)+1}-1-j} y_j = 0 \tag{43}$$

for every $\lambda \in \mathbb{T}$, where the $c_{m,j}$ are non-zero scalars which do not depend on λ.

Equality (43) applied to $\lambda := -1$ yields that $y_0 = 0$. Indeed, since $\Delta^{(k_0)} = b_1 - b_0 = 1$, we have $(1+\lambda^{\Delta^{(k_0)}}) = 0$, and the factor $(1+\lambda^{\Delta^{(k_0)}})$ appears in each of the terms $\prod_{l=1}^{m}\big(1+\lambda^{\Delta^{(k_{l-1})}}\big)$. Now, take λ such that $\lambda^{\Delta^{(k_1)}} = -1$. Then $\prod_{l=1}^{m}(1+\lambda^{\Delta^{(k_{l-1})}}) = 0$ for every $m \geq 2$. Moreover, $(1+\lambda^{\Delta^{(k_0)}})$ is non-zero because $\Delta^{(k_1)}$ is an even multiple of $\Delta^{(k_0)}$. So we get

$$\sum_{b_{n(1)}\leq j<b_{n(1)+1}} c_{1,j}\lambda^{b_{n(1)+1}-1-j} y_j = 0 \tag{44}$$

for every λ such that $\lambda^{\Delta^{(k_1)}} = -1$. Denoting by Λ_1 the set of all such elements λ, and recalling that $n(1) = 2^{k_1-1}$, we can rewrite (44) as

$$\sum_{s=0}^{\Delta^{(k_1)}-1} z_s\lambda^s = 0 \quad \text{for every } \lambda \in \Lambda_1,$$

where $z_s = c_{1,b_{n(1)+1}-1-s} y_{b_{n(1)+1}-1-s}$ for every $0 \leq s < \Delta^{(k_1)}$. Since Λ_1 has cardinality $\Delta^{(k_1)}$, it follows that $z_s = 0$ for every $0 \leq s < \Delta^{(k_1)}$, and hence that $y_j = 0$ for every $b_{n(1)} \leq j < b_{n(1)+1}$.

Continuing in this way, we obtain that for every $m \geq 1$, $y_j = 0$ for every j belonging to the interval $[b_{n(m)}, b_{n(m)+1})$, which proves our claim. □

We can now state

THEOREM 7.30. *Let $T = T_{v,\,w,\,\varphi,\,b}$ be a $C_{+,1}$-type operator on $\ell_2(\mathbb{N})$. Suppose that $b_1 = 1$ and that*

$$\sum_{m=1}^{\infty} 4^{m-\sum_{k=1}^{m}(\delta^{(k-1)}-\tau^{(k)})}\,\Delta^{(m)} < \infty.$$

Then T is mixing in the Gaussian sense.

PROOF. Let us denote by $\mathbf{G}$ the set of all good φ-sequences, and set $E_{\mathbf{n}}(\lambda) := x_{\mathbf{n},\lambda}$ for every $\mathbf{n} \in \mathbf{G}$ and every $\lambda \in \mathbb{T}$. Then the maps $E_{\mathbf{n}} : \mathbb{T} \to \ell_2(\mathbb{N})$, $\mathbf{n} \in \mathbf{G}$, are continuous eigenvector fields for T defined on $\mathbb{T}$. Lemmas 7.28 and 7.29 then imply that $\overline{\text{span}}\,\big\{E_{\mathbf{n}}(\lambda);\ \mathbf{n} \in \mathbf{G}\,,\ \lambda \in \mathbb{T}\big\} = \ell_2(\mathbb{N})$. By [**9**], it follows that T is strongly mixing in the Gaussian sense. □

EXAMPLE 7.31. If we consider the $C_{+,1}$-type operator T on $\ell_2(\mathbb{N})$ associated to the data $\delta^{(k)} = 2k$, $\tau^{(k)} = \frac{1}{2}\delta^{(k)}$ and $\Delta^{(k)} = 2^{k+1}$, $k \geq 1$, then T is mixing in the Gaussian sense.

REMARK 7.32. Condition (42) is incompatible with the "additional assumptions" of Theorem 7.2. Indeed, the latter imply that γ_k tends to zero as k tends to infinity, and hence that $\tau^{(k)} - \delta^{(k-1)}$ tends to infinity.

8. Infinite direct sums of frequently hypercyclic operators

A well-known open question, dating back to [**6**], asks whether the direct sum $T \oplus T$ of a frequently hypercyclic operator T with itself has to be frequently hypercyclic. (This is the analogue of Herreros "$T\oplus T$ problem" for frequent hypercyclicity.) By [**33**], $T\oplus T$ is hypercyclic as soon as T is $\mathcal{U}$-frequently hypercyclic; that is, $\mathcal{U}$-frequently hypercyclic operators are weakly mixing. Actually, the direct sum of any two $\mathcal{U}$-frequently hypercyclic operators is hypercyclic. Indeed, let T_1 and T_2 be two bounded operators acting respectively on the Banach spaces X_1 and X_2, and let U_1, V_1 and U_2, V_2 be non-empty open subsets of X_1 and X_2 respectively. By [**33**], the set $\mathcal{N}_{T_1}(U_1, V_1)$ has

bounded gaps. Also, since T_2 is weakly mixing, the set $\mathcal{N}_{T_2}(U_2, V_2)$ is *thick*, *i.e.* it contains arbitrarily long intervals (this is a classical result, see e.g. [**24**]). Hence $\mathcal{N}_{T_1}(U_1, V_1) \cap \mathcal{N}_{T_2}(U_2, V_2) \neq \emptyset$ for any U_1, V_1, U_2, V_2, *i.e.* $T_1 \oplus T_2$ is hypercyclic.

Apart from this consequence of [**33**], nothing seems to be known concerning this question from [**6**]. More generally, it seems to be unknown whether the direct sum $T_1 \oplus T_2$ of two frequently hypercyclic operators is necessarily frequently hypercyclic, or even $\mathcal{U}$-frequently hypercyclic. To the best of our knowledge, this question is open even for infinite direct sums of frequently hypercyclic operators (that is, we couldn't locate any counterexample). Our aim in this section is to use operators of C_{+1}-type on $\ell_p(\mathbb{N})$ to prove the following result.

THEOREM 7.33. *Let $p > 1$. There exists a sequence $(T_n)_{n\geq 1}$ of frequently hypercyclic operators on $\ell_p(\mathbb{N})$ such that the ℓ_p-sum operator $T = \bigoplus_{n\geq 1} T_n$ acting on $X = \bigoplus_{n\geq 1} \ell_p(\mathbb{N})$ is not $\mathcal{U}$-frequently hypercyclic.*

The proof of Theorem 7.33 relies on Theorem 7.6, combined with an elementary lemma providing an upper bound for the parameter $c(T)$ of a direct sum operator:

LEMMA 7.34. *Let T_1 and T_2 be two bounded operators acting respectively on the Banach spaces X_1 and X_2. Fix $p > 1$, and let $T = T_1 \oplus_{\ell_p} T_2$ be the ℓ_p-sum operator of T_1 and T_2, acting on $X = X_1 \oplus_{\ell_p} X_2$. Moreover, assume that T is hypercyclic. Then $c(T) \leq \min(c_1(T), c_2(T))$.*

PROOF. Let $c \in [0, 1]$ be such that $\overline{\text{dens}}\,\mathcal{N}_T(x, B_X(0, 1)) \geq c$ for a comeager set of vectors x of X. For any such vector $x = x_1 \oplus x_2$, with $x_1 \in X_1$ and $x_2 \in X_2$, there exists a subset D_x of $\mathbb{N}$ with $\overline{\text{dens}}\, D_x \geq c$ such that $\left(\|T_1^n x_1\|^p + \|T_2^n x_2\|^p\right)^{1/p} < 1$ for every $n \in D_x$. Thus $\overline{\text{dens}}\,\mathcal{N}_{T_1}(x_1, B_{X_1}(0, 1)) \geq c$ for a comeager subset of vectors $x_1 \in X_1$, so that $c(T_1) \geq c$; and likewise, $c(T_2) \geq c$. Hence $\min(c_1(T), c_2(T)) \geq c$ for any c as above, which proves the lemma. □

As an easy consequence of Lemma 7.34, we obtain

LEMMA 7.35. *For each $n \geq 1$, let T_n be a bounded operator on a Banach space X_n. Fix $p > 1$, and let $T = \bigoplus_{\ell_p} T_n$ be the ℓ_p-sum of the operators T_n, $n \geq 1$, acting on the space $X = \bigoplus_{\ell_p} X_n$. Moreover, assume that T is hypercyclic. Then*

$$c(T) = \inf_{n\geq 1} c(T_1 \oplus \cdots \oplus T_n) \leq \inf_{n\geq 1} c(T_n).$$

PROOF. It follows directly from Lemma 7.34 that

$$c(T) \leq \inf_{n\geq 1} c(T_1 \oplus \cdots \oplus T_n) \leq \inf_{n\geq 1} c(T_n),$$

so we only have to prove that

$$c(T) \geq \inf_{n\geq 1} c(T_1 \oplus \cdots \oplus T_n).$$

If the infimum $\inf_{n\geq 1} c(T_1 \oplus \cdots \oplus T_n)$ is equal to 0, there is nothing to prove. Suppose that $\inf_{n\geq 1} c(T_1 \oplus \cdots \oplus T_n) > 0$ and consider a number c such that $0 \leq c < \inf_{n\geq 1} c(T_1 \oplus \cdots \oplus T_n)$. Let U be a non-empty open subset of X. There exist an integer $n \geq 1$ and, for every $1 \leq i \leq n$, a non-empty open subset U_i of X_i, such that U contains the set $U_1 \oplus \cdots \oplus U_n \oplus 0 \oplus \cdots$. Since $c < \inf_{n\geq 1} c(T_1 \oplus \cdots \oplus T_n)$, there exist a vector $x = x_1 \oplus \cdots \oplus x_n \in U_1 \oplus \cdots \oplus U_n$ and a subset D of $\mathbb{N}$ with $\overline{\text{dens}}\, D \geq c$ such that $\left(\sum_{i=1}^n \|T_i^k x_i\|^p\right)^{1/p} < 1$ for every $k \in D$. Setting $x = x_1 \oplus \cdots \oplus x_n \oplus 0 \oplus \cdots$, we deduce that x is a vector of U which satisfies $\overline{\text{dens}}\,\mathcal{N}_T(x, B_X(0, 1)) \geq c$. The set of vectors $x \in X$ such that $\overline{\text{dens}}\,\mathcal{N}_T(x, B_X(0, 1)) \geq c$ is thus dense in X, and the usual argument shows that this set is in fact comeager in X. It follows that $c(T) \geq c$, from which we deduce that $c(T) \geq \inf_{n\geq 1} c(T_1 \oplus \cdots \oplus T_n)$. This concludes the proof of Lemma 7.34. □

We are now ready for the proof of Theorem 7.33.

Proof of Theorem 7.33. Let $(T_n)_{n\geq 1}$ be a sequence of operators satisfying the assumptions of Theorem 7.6. Denoting for each $n \geq 1$ by $(\tau_n^{(k)})_{k\geq 1}$, $(\delta_n^{(k)})_{k\geq 1}$, and $(\Delta_n^{(k)})_{k\geq 1}$ the sequences of parameters associated to the operator T_n, we suppose that

$$0 < \limsup_{k\to\infty} \frac{\delta_n^{(k)}}{\Delta_n^{(k)}} \leq 2^{-n}$$

for every $n \geq 1$. By Theorem 7.6, all the operators T_n are frequently hypercyclic. On the other hand, it follows from (35) that

$$\begin{aligned} c(T_n) &\leq \limsup_{k\to\infty} \left(2\frac{\delta_n^{(k)}}{\Delta_n^{(k)}} \cdot \frac{1}{1 - \dfrac{\delta_n^{(k)}}{\Delta_n^{(k)}} + \dfrac{1}{\Delta_n^{(k)}}} + 2\frac{\delta_n^{(k)}}{\Delta_n^{(k)}} \right) \\ &\leq 2 \cdot 2^{-n} \cdot \frac{1}{1 - 2^{-n}} + 2.2^{-n} \leq 6 \cdot 2^{-n} \quad \text{for every } n \geq 1, \end{aligned}$$

so that $c(T_n)$ tends to 0 as n tends to infinity. By Lemma 7.35, the operator $T = \bigoplus_{\ell_p} T_n$ (which is hypercyclic) satisfies $c(T) = 0$. Hence T is not $\mathcal{U}$-frequently hypercyclic. □

CHAPTER 8

A few questions

We conclude the monograph with a short list of questions. Most of them have already been stated, at least implicitly. Some of them will perhaps be found interesting and not hopelessly intractable.

QUESTION 8.1. For which Banach spaces X with separable dual do the "typicality" results proved here for Hilbert spaces hold true?

QUESTION 8.2. Do there exist ergodic operators on $\mathcal{H}$ without eigenvalues?

QUESTION 8.3. Do there exist at least Hilbert space operators admitting non-trivial invariant measures but no eigenvalues?

QUESTION 8.4. Let $M > 1$. What is the descriptive complexity of $\mathrm{ERG}_M(\mathcal{H})$ and $\mathrm{INV}_M(\mathcal{H})$, with respect to $\mathtt{SOT}$ and/or $\mathtt{SOT}^*$? In particular, are these sets Borel in $\mathfrak{B}_M(\mathcal{H})$?

QUESTION 8.5. Let $M > 1$. Is $\mathrm{UFHC}_M(\mathcal{H})$ a true $\mathbf{\Pi}^0_4$ set in $(\mathfrak{B}_M(\mathcal{H}), \mathtt{SOT}^*)$? Is $\mathrm{UFHC}_M(\mathcal{H}) \cap \mathrm{CH}_M(\mathcal{H})$ a true difference of $\mathbf{\Sigma}^0_3$ sets?

QUESTION 8.6. Let $M > 1$. Is $\mathrm{FHC}_M(\mathcal{H})$ Borel in $\mathfrak{B}_M(\mathcal{H})$?

QUESTION 8.7. Let $T \in \mathfrak{B}(\mathcal{H})$. Assume that T is hypercyclic, and that there exists a sequence $(u_i)_{i\geq 1}$ of unimodular eigenvectors with rationally independent eigenvalues spanning a dense linear subspace of $\mathcal{H}$. Is T ergodic, or at least frequently hypercyclic? $\mathcal{U}$-frequently hypercyclic?

QUESTION 8.8. Let $\boldsymbol{\lambda}$ be a sequence of distinct unimodular complex numbers tending to 1, and let $\boldsymbol{\omega}$ be a bounded sequence of positive numbers. If the operator $T_{\boldsymbol{\lambda},\boldsymbol{\omega}} = D_{\boldsymbol{\lambda}} + B_{\boldsymbol{\omega}}$ acting on $\ell_2(\mathbb{N})$ is hypercyclic, is it necessarily ergodic, or at least frequently hypercyclic? $\mathcal{U}$-frequently hypercyclic?

QUESTION 8.9. When exactly is an operator of the form $T_{\boldsymbol{\lambda},\boldsymbol{\omega}} = D_{\boldsymbol{\lambda}} + B_{\boldsymbol{\omega}}$ hypercyclic?

QUESTION 8.10. Let $M > 1$. Is the class of chaotic operators belonging to $\mathfrak{T}_M(\mathcal{H})$ comeager in $\mathfrak{T}_M(\mathcal{H})$?

QUESTION 8.11. Let T be a Banach space operator. Assume that T has a dense set of uniformly recurrent points. Does it follow that T has a non-zero periodic point?

QUESTION 8.12. Let T be a hypercyclic operator. Assume that $c(T) > 0$ and that T admits an invariant measure with full support. Does it follow that T is $\mathcal{U}$-frequently hypercyclic?

QUESTION 8.13. Let X be a Banach space, and let $T \in \mathfrak{B}(X)$ have the OSP. Are the ergodic measures for T dense in the space of all T-invariant measures?

QUESTION 8.14. Let $T \in \mathfrak{B}(\mathcal{H})$. Assume that T is frequently hypercyclic and satisfies $c(T) = 1$. Does it follow that T is ergodic?

QUESTION 8.15. On which Banach spaces is it possible to construct frequently hypercyclic operators which are not ergodic and/or $\mathcal{U}$-frequently hypercyclic operators which are not frequently hypercyclic?

Short list of abbreviations

For the reader's convenience, we provide here, in more or less alphabetical order, the list of the abbreviations we use for the various classes of operators on $\mathcal{H}$ considered in the monograph.

$\mathrm{c}^1(\mathcal{H})$	operators T such that $c(T) = 1$
$\mathrm{CEV}(\mathcal{H})$	operators with only countably many eigenvalues
$\mathrm{CH}(\mathcal{H})$	chaotic operators
$\mathrm{DCH}(\mathcal{H})$	distributionally chaotic operators
$\mathrm{DDCH}(\mathcal{H})$	densely distributionally chaotic operators
$\mathrm{ERG}(\mathcal{H})$	ergodic operators
$\mathrm{FHC}(\mathcal{H})$	frequently hypercyclic operators
$\text{G-ERG}(\mathcal{H})$	operators which are ergodic in the Gaussian sense
$\text{G-INV}_f(\mathcal{H})$	operators admitting a nontrivial Gaussian invariant measure
$\text{G-MIX}(\mathcal{H})$	operators which are mixing in the Gaussian sense
$\mathrm{INV}(\mathcal{H})$	operators admitting a nontrivial invariant measure
$\mathrm{INV}_f(\mathcal{H})$	operators admitting an invariant measure with full support
$\mathrm{MIX}(\mathcal{H})$	mixing operators
$\mathrm{NEV}(\mathcal{H})$	operators without eigenvalues
$\mathrm{PSPAN}(\mathcal{H})$	operators with perfectly spanning unimodular eigenvectors
$\mathrm{SPAN}(\mathcal{H})$	operators with spanning unimodular eigenvectors
$\mathfrak{T}_0(\mathcal{H})$	upper triangular operators whose diagonal coefficients have modulus 1
$\mathfrak{T}(\mathcal{H})$	operators in $\mathfrak{T}_0(\mathcal{H})$ with distinct diagonal coefficients
$\mathfrak{T}_{\mathrm{ind}}(\mathcal{H})$	operators in $\mathfrak{T}_0(\mathcal{H})$ with rationally independent diagonal coefficients
$\mathrm{TMIX}(\mathcal{H})$	topologically mixing operators
$\mathrm{UFHC}(\mathcal{H})$	upper frequently hypercyclic operators

Bibliography

[1] S. Bartoll, F. Martínez-Giménez, A. Peris, *The specification property for backward shifts*. J. Difference Equ. Appl. **18** (2012), 599–605.

[2] S. Bartoll, F. Martínez-Giménez, A. Peris, *Operators with the specification property*. J. Math. Anal. Appl. **436** (2016), 478–488.

[3] F. Bayart and T. Bermúdez, *Semigroups of chaotic operators*. Bull. Lond. Math. Soc. **41** (2009), 823–830.

[4] F. Bayart and S. Grivaux, *Hypercyclicity and unimodular point spectrum*, J. Funct. Anal. **226** (2005), 281–300.

[5] F. Bayart and S. Grivaux, *Frequently hypercyclic operators*. Trans. Amer. Math. Soc. **358** (2006), 5083–5117.

[6] F. Bayart and S. Grivaux, *Invariant Gaussian measures for operators on Banach spaces and linear dynamics*. Proc. Lond. Math. Soc. **94** (2007), 181–210.

[7] F. Bayart and É. Matheron, *Hypercyclic operators failing the Hypercyclicity Criterion on classical Banach spaces*. J. Funct. Anal **250** (2007), 426–441.

[8] F. Bayart and É. Matheron, *Dynamics of linear operators*. Cambridge Tracts in Mathematics **179**, Cambridge University Press (2009).

[9] F. Bayart and É. Matheron, *Mixing operators and small subsets of the circle*. J. Reine Angew. Math. **715** (2016), 75–123.

[10] F. Bayart and I. Z. Rusza, *Difference sets and frequently hypercyclic weighted shifts*. Erg. Th. Dynam. Systems **35** (2015), 691–709.

[11] N. Bernardes, A. Bonilla, V. Müller and A. Peris, *Distributional chaos for linear operators*. J. Funct. Anal. **265** (2013), 2143–2163.

[12] J. Bès and A. Peris, *Hereditarily hypercyclic operators*. J. Funct. Anal. **167** (1999), 94–112.

[13] J. Bès, Q. Menet, A. Peris and Y. Puig, *Recurrence properties of hypercyclic operators*. Math. Ann. **366** (2016), 545–572.

[14] A. Bonilla and K.-G. Grosse-Erdmann, *Frequently hypercyclic operators and vectors*. Ergodic Theory Dynam. Systems **27** (2007), 383–404.

[15] A. Bonilla, K.-G. Grosse-Erdmann, *Upper frequent hypercyclicity and related notions*. Preprint (2016), available at http://arxiv.org/abs/1601.07276.

[16] R. Bowen, *Topological entropy and axiom* A. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, Amer. Math. Soc. (1970) 23–41.

[17] K. Chan, *The density of hypercyclic operators on a Hilbert space*. J. Operator Theory **47** (2002), 131–143.

[18] I. Dayan and E. Glasner, *Hypercyclic operators, Gauss measures and Polish dynamical systems*. Israel J. Math. **208** (2015), 79 – 99.

[19] M. De La Rosa and C. Read, *A hypercyclic operator whose direct sum is not hypercyclic*. J. Operator Theory **61** (2009), 369–380.

[20] T. Eisner, *A "typical" contraction is unitary*. Enseign. Math. **56** (2010), 403–410.

[21] T. Eisner and A. Serény, *Category theorems for stable operators on Hilbert spaces*. Acta Sci. Math. (Szeged) **74** (2008), 259–270.

[22] T. Eisner and T. Mátrai, *On typical properties of Hilbert space operators*. Israel J. Math. **195** (2013), 247–281.

[23] E. Flytzanis, *Unimodular eigenvalues and linear chaos in Hilbert spaces*. Geom. Funct. Anal. **5** (1995), 1–13.

[24] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*. Princeton University Press (1981).

[25] E. Glasner, *Ergodic theory via joinings*. Mathematical Surveys and Monographs **101**, Amer. Math. Soc. (2003).

[26] E. Glasner and B. Weiss, *A universal hypercyclic representation*. J. Funct. Anal. **268** (2015), 3478–3491.

[27] S. Grivaux, *Construction of operators with prescribed behaviour*. Arch. Math. **81** (2003), 291–299.

[28] S. Grivaux, *A new class of frequently hypercyclic operators*. Indiana Univ. Math. J. **60** (2011), 1177–1201.

[29] S. Grivaux, *Some new examples of universal hypercyclic operators in the sense of Glasner and Weiss*. Trans. Amer. Math. Soc. **369** (2017), 7589–7629.

[30] S. Grivaux and É. Matheron, *Invariant measures for frequently hypercyclic operators*. Adv. Math. **265** (2014), 371–427.

[31] S. Grivaux and S. Shkarin, *Non-mixing hypercyclic operators*. Unpublished work (2007).

[32] K. -G. Grosse-Erdmann and A. Peris, *Linear Chaos*. Universitext, Springer (2011).

[33] K. -G. Grosse-Erdmann and A. Peris, *Frequently dense orbits*. C. R. Math. Acad. Sci. **341** (2005), 123–128.

[34] P. R. Halmos, *Lectures on ergodic theory.* Chelsea Publishing Co. (1956).
[35] D. Herrero, *Limits of hypercyclic and supercyclic operators.* J. Funct. Anal. **99** (1991), 179–190.
[36] D. Herrero, *Hypercyclic operators and chaos.* J. Operator Th. **29** (1992), 93–103.
[37] G. Kalisch, *On operators on separable Banach spaces with arbitrary prescribed point spectrum.* Proc. Amer. Math. Soc **34** (1972), 207–208.
[38] A. Kechris, *Classical descriptive set theory*. Graduate Texts in Mathematics, **156**, Springer (1995).
[39] A. S. Kechris and A. Louveau, *Descriptive set theory and the structure of sets of uniqueness.* London Math. Soc. Lecture Notes Series **128**, Cambridge University Press (1987).
[40] Q. Menet, *Linear chaos and frequent hypercyclicity*. Trans. Amer. Math. Soc. **369** (2017), 4977–4994.
[41] T. K. S. Moothathu, *Two remarks on frequent hypercyclicity.* J. Math. Anal. Appl. **408** (2013), 843–845.
[42] M. Murillo-Arcilla and A. Peris, *Strong mixing measures for linear operators and frequent hypercyclicity*. J. Math. Anal. Appl. **398** (2013), 462–465.
[43] K. R. Parthasarathy, *On the category of ergodic measures.* Illinois J. Math. **5** (1961), 648–656.
[44] G. Pedersen, *Analysis now.* Springer (1989).
[45] H. Salas, *A hypercyclic operator whose adjoint is also hypercyclic.* Proc. Amer. Math. Soc. **112** (1991), 765–770.
[46] O. Sarig, *Lecture notes on ergodic theory.* Lecture notes (2009), available online.
[47] A. Shields, *Weighted shift operators and analytic function theory.* Topics in operator theory, Math. Surveys **13**, 49 – 128, Amer. Math. Soc (1974).
[48] S. Shkarin, *Universal elements for non-linear operators and their applications.* J. Math. Anal. Appl. **348** (2008), 193–210.
[49] S. Shkarin, *On the spectrum of frequently hypercyclic operators.* Proc. Amer. Math. Soc. **137** (2009), 123–134.
[50] B. Schweizer and J. Smítal, *Measure of chaos and a spectral decomposition of dynamical systems on the interval.* Trans. Amer. Math. Soc. **344** (1994), 737–754.
[51] K. Sigmund, *On dynamical systems with the Specification Property.* Trans. Amer. Math. Soc. **190** (1974), 285–299.
[52] S. Todorcevic, *Topics in topology.* Lecture Notes in Mathematics **1652**, Springer (1997).
[53] R. E. Waterman, *Invariant subspaces, similarity and isometric equivalence of certain commuting operators on* L_p. Pacific J. Math. **48** (1973), 593–613.
[54] N. Weaver, *Set theory and cyclic vectors.* J. Operator Theory **52** (2004), 133–138.