# Utility and income transfer principles: Interplay and incompatibility

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**Abstract**

In this paper, it is assumed that income does not have a linear correspondence with utility. Consequently, transfers of income and transfers of utility that could improve social welfare are studied. The conditions for the fulfillment of generalized income transfer principles, relevant to any given order of stochastic dominance, are determined. The result relies on Bell polynomials and states that an income transfer principle of any order does not necessarily satisfy the utility transfer principle of the corresponding order.

**Keywords**: Bell polynomial, Inequality aversion, Transfer principle.

**JEL Classification**: D63.

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# 1 Introduction

At present, there is a consensus that individuals' well-being should be the central object of interest for public policies. However redistributive policies might be applied to transferable resources – here referred to as income in a broad sense – that do not have a linear correspondence with individuals' well-being. Therefore, the study of resource redistribution such as income transfers constitutes a non-trivial issue for analyses of well-being. Kaplow (2010) provides a framework for the interplay between the assumptions made on individual utility functions and the assumptions made on the social welfare functions (SWFs) for the purpose of income redistribution. This analysis is consistent with SWFs satisfying the Pigou–Dalton principle of income transfer, which states that a rich-to-poor income transfer improves social welfare. Kaplow shows that the hypotheses related to the utility function, being the same for all individuals in the society, play a direct role in fulfilling the Pigou-Dalton principle of income transfer. Moreover, the assumptions made on the shape of the SWF have a more subtle influence on such a fulfillment.

The Pigou–Dalton principle of utility transfer, as in Adler (2012), is among the axioms that help characterize the shape of an SWF. This principle requires that a utility transfer from a well-off agent to a less well-off one improves social welfare. Adler's (2012) ethical view can be decomposed into two aspects. On the one hand, some assumptions on the utility function have to be made, and on the other hand, the Pigou–Dalton principle of utility transfer has to be imposed to ensure that the SWF is shaped by a strictly concave mapping of utilities. The first aspect aims at defining individuals' well-being, which is formally expressed as a utility function based on idealized preferences. The question of whether utility should be determined by – empirically grounded – individuals' ordinary preferences or by – normative – idealized preferences is related to this aspect.[1] The second aspect aims at determining the shape of the SWF. The continuous, additive form of the SWF (additive separability) and the Pigou–Dalton principle of utility transfer are the key characteristics of the shape of *prioritarian* SWFs supported by Adler (2012), that outline some priority to be imposed on worse-off agents.[2] Alternatively, any (pure) utilitarian SWF is designed by a linear transformation of utility so that the Pigou–Dalton principle of utility transfer is not respected.

This approach with two aspects allows comparing policy recommendations provided by prioritarian SWFs with those provided by utilitarian SWFs. The assumptions made

[1] In Kaplow (2010), utility is assumed to be the same function for every individual. This is an empirical concept whereas the idealized preference is purely normative.

[2] See Blackorby *et al.* (2002), and *e.g.*, Bosmans *et al.* (2018) who propose prioritarian poverty measures.

on the utility function are independent of whether the social planner is utilitarian or prioritarian. In particular, as long as the utility function is assumed to be increasing and concave (*i.e.*, utility is assumed to be derived from income at a positive and decreasing rate), both utilitarian and prioritarian social planners behave in accordance with the Pigou–Dalton principle of income transfer. When income has a concave correspondence with utility, the Pigou–Dalton principle of utility transfer is not equivalent to its corresponding principle of income transfer, because the former is not necessary for the latter to hold. This relation between these two principles is relevant to discussing the implications of second-order stochastic dominance in terms of income. The dominance criterion ensures that a pair of income distributions is ranked if one distribution can be obtained from the other through a sequence of income increments and/or rich-to-poor transfers, otherwise it is inconclusive.

Higher-order dominance criteria are able to rank income distributions in cases where the second-order dominance criterion fails to provide a ranking. Gayant and Le Pape (2017) explore third-, fourth-, and higher-order dominance criteria, and especially their normative content embodied by the generalized principle of income transfer *à la* Fishburn and Willig (1984). The generalized principle of income transfer encompasses (in a recursive pattern): the second-order income transfer principle which is slightly weaker than the Pigou–Dalton condition, and the third-order income transfer principle which is similar in spirit to Kolm's (1976) diminishing transfers principle. The latter states that a given progressive transfer is increasingly valuable insofar as the recipient is poorer. It is equivalent to saying that a rich-to-poor transfer between poorer individuals coupled with a poor-to-rich transfer between richer individuals improve social welfare. One step further, the fourth-order income transfer principle states that two rich-to-poor transfers – one between much poorer individuals and the other one between much richer individuals – coupled with two poor-to-rich transfers between individuals with intermediate incomes, improve social welfare. Higher-order transfer principles of income exhibit more sensitivity to transfers occurring in the lower tail of the income distribution. Gayant and Le Pape (2017) assert that these principles display an aversion to a general degree of income inequality.

In practice, numerous redistribution policies involve complex sequences of income transfers that are relevant to higher-order transfer principles. Let us take a simple example with two generations $x$ and $y$ whose income distributions are respectively $(20, 20, 40, 60)$ and $(10, 30, 50, 50)$. Every individual of generation $x$ gives 1 unit of income to one individual of generation $y$. Before redistribution, the aggregate distribution is $(10, 20, 20, 30, 40, 50, 50, 60)$ and after redistribution it becomes $(11, 19, 19, 31, 39, 51, 51, 59)$.

The second-order dominance criterion cannot rank this pair of aggregate income distributions because the sequence of transfers needed to convert $x$ into $y$ involves two poor-to-rich transfers: 1 unit is given by the third individual to the fourth one, and 1 unit is given by the fifth individual to the sixth one. To analyze how a social planner would rank both aggregate income distributions, the fourth-order income transfer principle is necessary. Whenever the social planner respects the principle of transfer, and so refers to the fourth-order dominance criterion, the social planner judges the income redistribution to be welfare-improving.

In this paper, we compare Fishburn and Willig's (1984) income transfer principles with principles which follow the same pattern but rely on utility transfers. As with the Pigou–Dalton principle of utility transfer, the latter are axioms that help characterize the shape of an SWF. Alternatively, the income transfer principles result from the interplay between the assumptions on the utility function and the utility transfer principles. This raises two questions: (i) Which types of assumptions should we impose on the utility function so that the utility transfer principle of any given order implies the corresponding order of income transfer principle? (ii) Are the assumptions involved in an income transfer principle's being of a certain order such that they also imply the utility transfer principle of the corresponding order? Two results are obtained. (i) Assuming that the utility function is increasing concave with higher-order derivatives alternating in sign up to a given order $s \geq 2$ provides a sufficient condition for the utility transfer principle of order $s$ to imply the corresponding income transfer principle of order $s$. (ii) An alternative characterization of the conditions under which additively separable SWFs satisfy income transfer principles *à la* Fishburn and Willig (1984) is proposed. For any given order, a critical shape of SWF that corresponds to a maximum degree of convexity is determined such that the income transfer principle is satisfied, whereas SWFs beyond this shape fail to respect the principle. Such a critical shape is mathematically expressed with the aid of Bell polynomials, which are functions of the successive derivatives of the utility function. This expression epitomizes the complexity of the interplay between higher-order income and utility transfer principles. The assumptions underlying result (i) impose weaker conditions on the shape of the SWF than those ensuring the fulfillment of the utility transfer principle of order $s$. Hence, under these assumptions, it is shown that SWFs satisfying a given order of income transfer principle do not necessarily satisfy the corresponding order of utility transfer principle. The results are particularly appropriate for providing ethical assessments in a risky framework, where individual well-being should be function of some attitude to risk. The study of the interplay between utility and income transfer principles of – at least – order 3 is necessary whatever the degree of risk under consideration in an intertemporal assessment. The results shed light on the implications of assumptions such as individuals' risk aversion, prudence, temperance, and aversion to any given higher-degree risk (Ekern, 1980). These risk aversions are illustrated and discussed through the prism of income/utility transfers relevant to debates on water scarcity.

This paper is organized as follows. Section 2 presents the motivations, notation and definitions. Section 3 presents the main results and a table summarizing them. Section 4 presents an illustration of resource redistribution and water scarcity in a risky universe. Section 5 presents some conclusion.

# 2 Setup

Income is denoted by $y \in \Omega := [0, y_{\max}]$ where $y_{\max}$ is the maximum conceivable one. Let $f \in \mathscr{F}$ be a probability measure from $\Omega$ onto $[0,1]$, with $\mathscr{F}$ the set of real-valued probability measures that give nonzero mass to only a finite set of incomes, and have a total mass $\int_0^{y_{\max}} f(y)dy = 1$. An (additively separable) extended form SWF, well-known as the generalized utilitarian function, is a tool employed by a social planner in order to rank income distributions:

$$W(f) = \int_0^{y_{\max}} g \circ u(y)\ f(y)\ dy. \tag{1}$$

As determined by a social planner, $g \circ u(y)$ is the weighted utility generated by income $y$. In our approach, see also Kaplow (2010), all individuals have the same utility function, which is a particular position on how utility is inter-personally compared.[3] Under this assumption, utility is fully measurable and comparable. The function $g$ formalizes attitudes towards inequality. It is increasing in utility by assumption, *i.e.*, $g^{(1)}(u(y)) > 0$ for all $y \in \Omega$, and it belongs to the set of $s$-times differentiable functions $\mathscr{C}^s$, with $s \in \mathbb{N} := \{1, 2, \ldots\}$. Let us write $g^{(s)}(u)$ for the $s$-th derivative of $g$ with respect to $u$. From now on, utility is assumed to be non-negative, *i.e.*, $u(y) \geqslant 0$ for all $y \in \Omega$. Utility functions are assumed to be strictly increasing, $u^{(1)}(y) > 0$ for all $y \in \Omega$ , and they also belong to $\mathscr{C}^s$.[4]

[3] Actually, this is not the only informational assumption made here. Blackorby *et al.* (2002) show that an extended form SWF implies that transformed utilities $g(u)$ are unit comparable between individuals. The debate on the degree of interpersonal comparability of utility (and transformed utility) is beyond the scope of the present paper.

[4] Because of the assumption that utility is strictly increasing in income, the SWF fulfills welfarist principles. Then, the assumption made on $u$ allows the sensitivity for utility and income distributions to be studied in the same welfarist framework.

## 2.1 Motivation for the extended form SWF

Why is it important to employ an extended form SWF for analyzing redistributive policies? Because the conditions imposed on the utility function and those imposed on the $g$ function have different effects on the welfare assessment of an income and/or utility redistribution. The shape of $g$ has a straightforward effect on the assessment of utility transfers, and it has an indirect, subtle influence on how an income redistribution is judged. The function $u$ has no effect on the valuation of utility transfers but it has a direct influence on how an income redistribution is converted into a utility redistribution.

To illustrate the roles of $g$ and $u$ in assessing utility and income redistributions, consider the following isoelastic functions $u(y)$ and $w(y) := g \circ u(y)$

$$u(y) = \frac{y^{1-\alpha}}{1-\alpha} \; 0 \leq \alpha < 1 \iff u^{-1} \circ u(y) = [(1-\alpha)u(y)]^{\frac{1}{1-\alpha}}, \; 0 \leq \alpha < 1, \tag{2}$$

$$w(y) = \frac{y^{1-\phi}}{1-\phi}, \; 0 \leq \phi < 1. \tag{3}$$

Consequently, the weighted utility is expressed as:[5]

$$g(u) = \frac{[(1-\alpha)u]^{\frac{1-\phi}{1-\alpha}}}{1-\phi}. \tag{4}$$

Let us assume that the functions are parametrized such that $\phi = 0.75$, whereby the reduced form SWF is $w(y) = 4y^{0.25}$. Then, the social welfare increases by 0.76 as the result of an income transfer of amount 1 from an individual with $10,000$ to another with 1. The extended form SWF is $g \circ u(y) = 4[(1-\alpha)u(y)]^{\frac{0.25}{(1-\alpha)}}$. Several parametrizations of $u$ are consistent with this kind of transfer. Let us investigate two special cases. In case $A$, $\alpha_A = 0.5$, so that $u$ is strictly concave in income. Then, $g$ is strictly concave, $g_A(u_A(y)) = 2.83u_A^{0.5}(y)$, so that the change of social welfare is 0.76. In case $B$, $\alpha_B = 0.8$ implies that $g$ is strictly convex in utility, $g_B(u_B(y)) = 0.54u_B^{1.25}(y)$, and the change in social welfare also amounts to 0.76. Although both cases agree on the change of social welfare that would result from the income transfer, the assessment of the welfare that results from the utility transfers, similar in spirit, brings out an opposition between cases $A$ and $B$. The social welfare increases by 1.16 in case $A$ as the result of a utility transfer of amount 1 from an individual with a utility level of $10,000$ to another one with a

[5] The fact that the weighted function of utility generated by $y$ relies on the two parameters $\phi$ and $\alpha$ illustrates the non-trivial relation between $w(y)$ and $g \circ u(y)$.

utility level of 1. In case $B$, the social welfare decreases by 6.01 as the result of the same utility transfer (Table 1), whereby the social welfare increases as the result of the income transfer and decreases as the result of the utility transfer. This is because $u_B$ converts the income transfer into a utility increment (welfare gain) of 0.74 that goes to the individual with income level 1. Since $g_B$ is increasing, the social welfare increases by 0.76 as the result of this utility increment.[6] On the other hand, since $g_B$ is convex, the transfer of utility valued to be 1 is converted into a significant welfare loss (decrement) for the individual with 10,000 units of utility ($-6.75$) coupled with a slight increment of welfare for the individual with 1 unit of utility (0.74).

**Table 1. Assessments of utility and income redistributions**

| Case ↓ Transfer → | $\Delta W$ : Utility transfer | $\Delta W$ : Income transfer |
|---|---|---|
| $A$: $2.83u_A^{0.5}(y)$ | 1.16 | 0.76 |
| $B$: $0.54u_B^{1.25}(y)$ | $-6.01$ | 0.76 |

$\Delta W$: social welfare variation (ex post minus ex ante social welfare).

Cases $A$ and $B$ describe the story of two different social planners; however both behave in accordance with the income transfer principle *i.e.*, second-order stochastic dominance. The next subsection introduces generalized transfers of utility and income in order to disentangle both behaviors for extended SWFs and higher-order stochastic dominance.

## 2.2 Transfers and transfer principles

In what follows, transfers $T(\cdot)$ *à la* Fishburn and Willig (1984) are modified in order to capture a wide range of redistributive principles based either on utility or on income. The utility increment $T^1(\alpha, u(y), \delta)$ postulates that the proportion $\alpha \in (0, f(y)]$ of the population moves from utility $u(y)$ to utility $u(y) + \delta$, with $\delta > 0$. For all $f \in \mathscr{F}$ and for all $y \in \Omega$, a probability measure $h(y)$ is obtained from $f(y)$ by a utility increment whenever

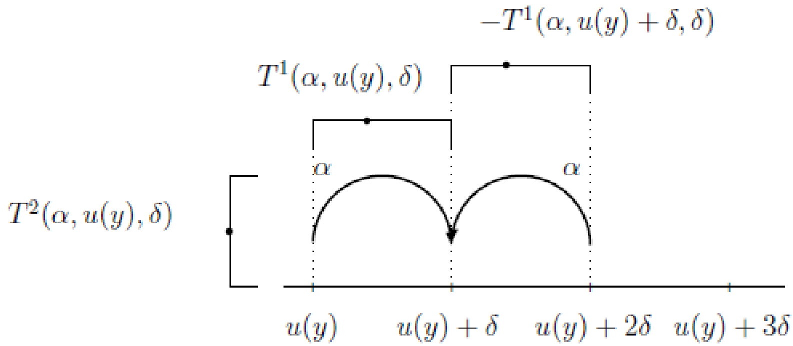
$$h(y) = \begin{cases} f(y) - \alpha & \text{at point } u(y) \ , \\ f(y) + \alpha & \text{at point } u(y) + \delta \ , \\ f(y) & \text{elsewhere.} \end{cases} \tag{5}$$

Utility increments do not preserve the mean utility generated by $f$. For higher orders, mean-preserving utility transfers are deduced recursively. As explained in Fishburn and Willig (1984), a positive transfer of order 2, $T^2(\alpha, u(y), \delta)$, is a particular case of a

[6] Rounding to two decimal places, $u_B(10,000) \approx u_B(9,999) = 31.55$; $u_B(1) = 5$ and $u_B(2) = 5.74$.

progressive transfer of utility. It is a utility transfer of amount $\delta > 0$ from a better-off $(u(y) + 2\delta)$ proportion $\alpha$ of the population to a worse-off $(u(y))$ proportion $\alpha$. Hence, a part of the population moves from the utility level $u(y)$ to $u(y) + \delta$ and another part moves from $u(y) + 2\delta$ to $u(y) + \delta$. Let us remark that $T^2$ equalizes the utility levels of the people involved in the transfer, and, as such, it is an *equalizing utility transfer*.

Figure 1: A positive utility transfer of order 2

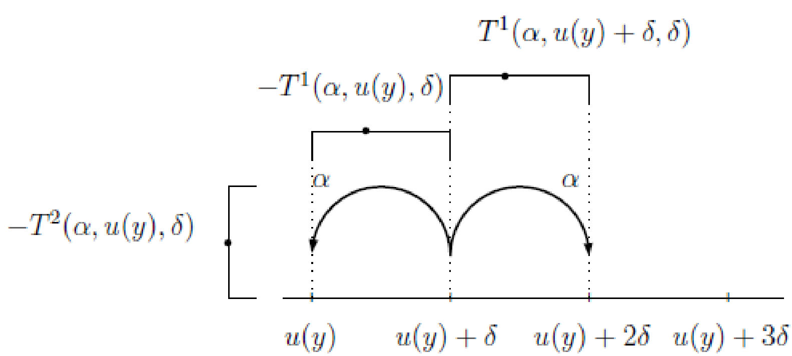



As shown in Figure 1, a utility transfer of order 2 is the sum of an increment and a decrement:

$$T^2(\alpha, u(y), \delta) = T^1(\alpha, u(y), \delta) + \left(-T^1(\alpha, u(y) + \delta, \delta)\right). \tag{6}$$

In the same fashion, $-T^2(\alpha, u(y), \delta)$ is a negative transfer of order 2 (involving utility levels at least as high as $u(y)$).[7] It is a transfer of a positive amount $\delta$ of utility that increases inequality in utility levels of two proportions $\alpha$ of the population with the same utility level $u(y) + \delta$, and, as such, it is an *inequality-increasing utility transfer*.

Figure 2: A negative utility transfer of order 2



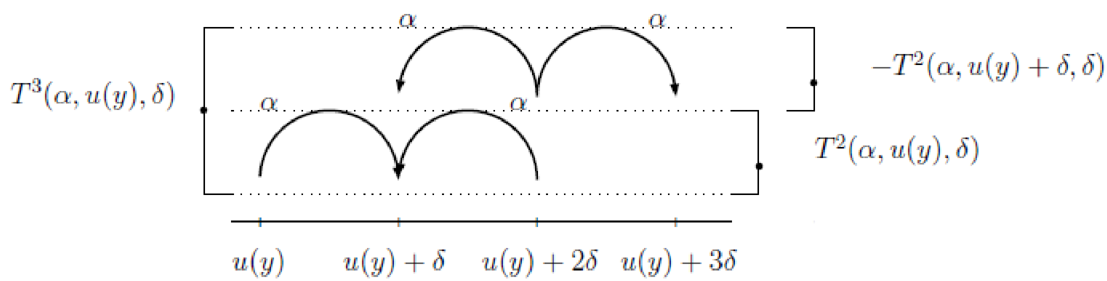

[7]Let us remark that the utility level presented as an argument of the transfer function is the lowest one involved in the transfer.

As shown in Figure 2, a negative utility transfer of order 2 can be expressed as follows:

$$-T^2(\alpha, u(y), \delta) = -T^1(\alpha, u(y), \delta) + T^1(\alpha, u(y) + \delta, \delta). \tag{7}$$

A transfer $T^3(\alpha, u(y), \delta)$ encompasses (i) an equalizing utility transfer and (ii) an inequality-increasing utility transfer involving a proportion of better-off recipients as in (i).

Figure 3: A positive utility transfer of order 3




As shown in Figure 3,

$$T^3(\alpha, u(y), \delta) = T^2(\alpha, u(y), \delta) + \left(-T^2(\alpha, u(y) + \delta, \delta)\right). \tag{8}$$

Fishburn and Willig (1984) provide a recursive and general formulation of $T^{s+1}$. We employ the same idea.

**Definition 2.1. Generalized Utility Transfer of order** $s+1$**.** *For all* $f \in \mathscr{F}$ *and for all* $y \in \Omega$*, we say that a mean-preserving utility transfer of order* $s+1$ *is given by:*

$$T^{s+1}(\alpha, u(y), \delta) := T^s(\alpha, u(y), \delta) - T^s(\alpha, u(y) + \delta, \delta), \ s \in \mathbb{N}.^8 \tag{9}$$

[8]The transfer should be such that, for $r \in \{1, \cdots, s+1\}$ even,

$$\binom{s+1}{r} \alpha \in (0, f(y)] \text{ at point } u(y) + r\delta, \text{ and } \delta > 0.$$

This condition recalls that the proportion of the population who moves from a utility level should be at most as high as the proportion of population at this utility level. Moreover it demands that no proportion moves from a utility level generated by a higher income than $y_{\max}$.

Although we presented utility transfers first, Fishburn and Willig (1984) introduced income transfers. Based on the same kind of transformation of probability measure as Eq.(5), the income increment, $T^1(\alpha, y, \delta)$ implies that a fraction $\alpha \in (0, f(y)]$ of the population moves from income $y$ to income $y + \delta$, with $\delta > 0$. Income transfers follow the same pattern as utility transfers. Although a utility transfer of order 2 equalizes the utility levels of the individuals involved, an income transfer of order 2 equalizes their income levels. We now define the concept of an income transfer of order $s + 1$.

**Definition 2.2. Generalized Income Transfer of order** $s + 1$**.** *For all* $f \in \mathscr{F}$ *and for all* $y \in \Omega$*, we say that a mean-preserving income transfer of order* $s + 1$ *is given by:*

$$T^{s+1}(\alpha, y, \delta) := T^s(\alpha, y, \delta) - T^s(\alpha, y + \delta, \delta), \ s \in \mathbb{N}.^9 \tag{10}$$

The definition of generalized transfer gives rise to the generalized transfer principle. In this context, a principle is a judgement about the welfare consequences when a transfer is carried out. If an income (resp. utility) transfer of any order from 2 to $s + 1$ does not decrease social welfare, the income (resp. utility) transfer principle of order $s + 1$ is satisfied. Consequently, this principle should incorporate all income (resp. utility) transfer principles of orders lower than $s + 1$.

**Definition 2.3. Generalized Utility and Income Transfer Principle of order** $s+1$. *For all* $f \in \mathscr{F}$*, for all* $y \in \Omega$ *and* $s \in \mathbb{N}$*, the following implications hold for utility and income transfers:*

$$(\text{UTP}^{s+1}) \qquad h = f + T^\ell(\alpha, u(y), \delta) \implies W(h) \geqslant W(f), \quad \ell = 2, \ldots, s+1;$$
$$(\text{ITP}^{s+1}) \qquad h = f + T^\ell(\alpha, y, \delta) \implies W(h) \geqslant W(f), \quad \ell = 2, \ldots, s+1.$$

The second-order income transfer principle ITP$^2$ states that a transfer does not decrease social welfare if the transfer equalizes the income levels of the individuals involved. In contrast, the classical Pigou–Dalton transfer principle requires that a rich-to-poor income transfer reduces the gap between both income levels in order to provide no social welfare reduction. Clearly, the Pigou–Dalton principle of income transfer is stronger

[9] The transfer should be such that, for $r \in \{1, \cdots, s+1\}$ even,

$$\binom{s+1}{r} \alpha \in (0, f(y + r\delta)] \text{ and } \delta > 0.$$

This condition implies that $y \in [0; y_{\max} - s\delta]$ whenever $s$ is even, and $y \in [0; y_{\max} - (s-1)\delta]$ whenever $s$ is odd. It does not appear in Fishburn and Willig (1984) because they adopt an unbounded income domain whereas here $y \in [0; y_{\max}]$.

than ITP$^2$. Moreover, ITP$^3$ is a particular case of the diminishing transfers principle.[10] These remarks remain valid when comparing UTP$^2$ with the Pigou–Dalton principle of utility transfer and UTP$^3$ with the utility diminishing transfers principle.

The following set is introduced to analyze generalized transfer principles:

$$\Gamma^{\ell+1} := \left\{ v \in \mathscr{C}^{\ell+1} \;\middle|\; (-1)^{\ell+1} v^{(\ell+1)}(x) := (-1)^{\ell+1} \frac{\partial v^{(\ell)}(x)}{\partial x} \leqslant 0, \quad \forall x \in \mathbb{R}_+ \right\},$$

which is the class of real-valued functions whose $\ell + 1$th derivative is non-positive (non-negative) if $\ell+1$ is even (odd), where $\ell$ is a non-negative integer. To be precise, Fishburn and Willig's (1984) generalized transfer principles are defined on

$$\Gamma^{\to s+1} := \left\{ v \in \mathscr{C}^{s+1} \;\middle|\; v \in \Gamma^{\ell}, \quad \forall \ell = 1, \dots, s+1 \right\},$$

which is the set of all $s+1$-times differentiable functions for which the first $s+1$ successive derivatives alternate in sign. Formally, $\Gamma^{\to s+1} = \Gamma^1 \cap \dots \cap \Gamma^{s+1}$. The following theorem determines the conditions under which income or utility transfers do not decrease social welfare.

**Theorem 2.1.** *For any given $s \in \mathbb{N}$, the two following statements are equivalent:*

(i) *$W$ satisfies $UTP^{s+1}$.*

(ii) $g \in \Gamma^{\to s+1}$.

*Moreover, the two following statements are equivalent:*

(iii) *$W$ satisfies $ITP^{s+1}$.*

(iv) $g \circ u \in \Gamma^{\to s+1}$.

*Proof.* This is a direct application of Fishburn and Willig's (1984) result (Theorem 1). □

The SWF (1) satisfies UTP$^2$ if and only if $g$ is concave. Hence, the concavity of $g$ characterizes utility inequality aversion. The SWF (1) satisfies UTP$^3$ if and only if $g \in \Gamma^{\to 3}$. Restrictions on the second-order and the third-order derivatives provide downside utility inequality aversion. This attitude amounts the exhibition of more emphasis on utility inequality for the worse-off proportions of the population. In a general way, going from condition $g \in \Gamma^{\to s}$ to $g \in \Gamma^{\to s+1}$ when $s \geqslant 3$ is equivalent to placing even more emphasis on inequality for the worst-off proportions of the population. The conditions under which SWFs (1) exhibit income inequality aversion involve the concavity of $g \circ u$. Then, the social desirability of income transfers rely both on the form of $g$ and that of $u$, *i.e.*, assumptions on the individuals' utility function.

[10] One could object that the equalizing income transfer and the inequality-increasing transfer involve different income gaps between the respective donors and recipients. But note that $W(h) > W(f)$ with $h = f + T^3(\alpha, y, \delta)$ is equivalent to $W(f + T^2(\alpha, y, \delta)) > W(f + T^2(\alpha, y + \delta, \delta))$, in which case both equalizing income transfers involve the same income gaps between the respective donors and recipients.

# 3 Results

Sufficient conditions for the fulfillment of $\text{ITP}^2$ are provided by the concavity of $g$ and $u$. The following lemma states general conditions for the fulfillment of higher-order income transfer principles.

**Lemma 3.1.** *The following statement is true for all $s \in \mathbb{N}$:*

$[\mathbf{H}^{s+1}]$ $\quad u \in \Gamma^{\to s+1}$ *and* $g \in \Gamma^{\to s+1}$ *together imply* $g \circ u \in \Gamma^{\to s+1}$.

*Proof.* See Appendix A. □

Both utility inequality aversion and decreasing marginal utility in income ensure that equalizing income transfers do not decrease social welfare. However, this statement does not deal with the possible interplay between attitudes to utility inequality and assumptions on the utility function to make a judgment on income redistribution. Indeed, assumptions of the concavity of $u$ offset (at least partially) the role of the concavity of $g$ in ensuring the concavity of $g \circ u$.

The same kind of limitation arises when a social planner considers $w = g \circ u$ as a simple function of income. The concavity of $w$ characterizes income inequality aversion but this function does not make any distinction between utility inequality aversion and assumptions on the utility function. Necessary and sufficient conditions may be derived from Lemma 3.1 and Theorem 2.1 by employing Bell polynomials. For $s, k \in \mathbb{N}$, the (exponential) Bell polynomial of order $s+1, k$, denoted by $B_{s+1,k}(\cdot)$, is given by

$$B_{s+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(s-k+2)}\right) =$$
$$\sum \frac{(s+1)!}{p_1! p_2! \cdots p_{s-k+2}!} \left(\frac{u^{(1)}}{1!}\right)^{p_1} \left(\frac{u^{(2)}}{2!}\right)^{p_2} \cdots \left(\frac{u^{(s-k+2)}}{(s-k+2)!}\right)^{p_{s-k+2}},$$

where the summation is taken over all possible sequences of non-negative integers $p_1, \ldots, p_{s-k+2}$ such that $p_1 + p_2 + \cdots = k$ and $1p_1 + 2p_2 + \cdots = s+1$. We consider utility functions with all derivatives up to $s+1$th order that alternate in sign. The following theorem states that an income transfer principle of any order does not necessarily imply the utility transfer principle of the corresponding order.

**Theorem 3.1.** *Let $u \in \Gamma^{\to s+1}$ and $g \in \{\Gamma^{\to s} \cap \mathscr{C}^{s+1}\}$ for some $s \in \mathbb{N}$. Then, the two following statements are equivalent:*

(i) *$W$ satisfies $ITP^{s+1}$.*

(ii) $(-1)^{s+1} g^{(s+1)} \circ u \leqslant (-1)^{s+1} g^{\star}(u^{(1)}, \ldots, u^{(s+1)})$ *and* $(-1)^{s+1} g^{\star}(u^{(1)}, \ldots, u^{(s+1)}) \geqslant 0$, *with*

$$g^{\star}(u^{(1)}, \ldots, u^{(s+1)}) = -\frac{\sum_{k=1}^{s} (g^{(k)} \circ u) \cdot B_{s+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(s-k+2)}\right)}{\left[u^{(1)}\right]^{s+1}}.$$

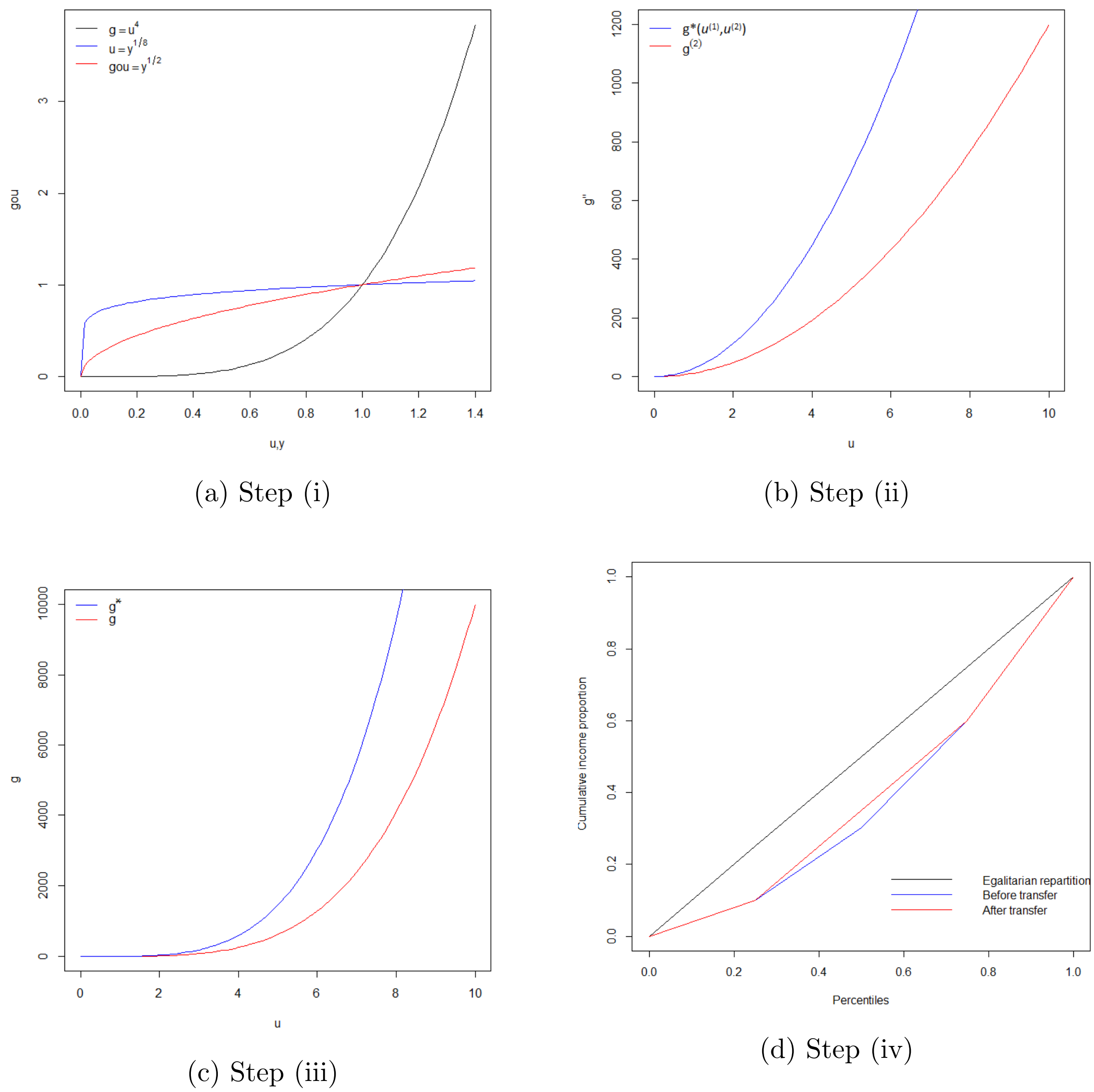
*Proof.* See Appendix A. □

The line of thought of Theorem 3.1 can be set out in four steps. (i) SWFs (1) satisfy ITP$^{s+1}$ if and only if $g \circ u \in \Gamma^{\to s+1}$. This is the result of Theorem 2.1. (ii) We provide the conditions on $g$ under which SWFs (1) satisfy an income transfer principle of some order. Formally, $(-1)^{s+1} g^{(s+1)}(u) \leqslant (-1)^{s+1} g^{\star}(u^{(1)}, \ldots, u^{(s+1)})$ if and only if SWFs (1) satisfy ITP$^{s+1}$. The function $g^{\star}$ has a critical shape such that $g \circ u \in \Gamma^{\to s+1}$. To be precise, the value of $g^{\star}(u^{(1)}, \ldots, u^{(s+1)})$ for all $y \in \Omega$ can be understood as a boundary for satisfying ITP$^{s+1}$. (iii) The assumptions on the form of $u$ ensure that the equivalence in step (ii) is interpretable. Assuming that the utility functions are members of $\Gamma^{\to s+1}$ imposes a restriction on the set of values of $(-1)^{s+1} g^{\star}(u^{(1)}, \ldots, u^{(s+1)})$ for all $y \in \Omega$. These values should be non-negative and they are comparable with the constraints imposed on $g$ under which SWFs (1) satisfy an UTP of some given order. Step (iv) is a simple corollary: The income distribution obtained by any $T^{s+1}(\alpha, y, \delta)$ stochastically dominates at the order $s+1$ the same income distribution before transfer.

Theorem 3.1 recalls that the fulfillment of ITP$^2$ by an SWF (1) does not mean that this SWF exhibits utility inequality aversion, that is, the fulfillment of UTP$^2$. From step (i), ITP$^2$ is satisfied by (1) if and only if $g \circ u \in \Gamma^{\to 2}$. From step (ii), $g^{(2)}(u) \leqslant g^{\star}(u^{(1)}, u^{(2)})$ if and only if SWFs (1) satisfy ITP$^2$. From step (iii), assuming $u \in \Gamma^{\to 2}$ implies that $g^{\star}(u^{(1)}, u^{(2)}) \geqslant 0$ for all $y \in \Omega$. In this case, $g^{\star}$ is the most convex function such that $g \circ u$ is concave. This case is depicted in Table 2, first row, first and second columns.

Let us illustrate with four figures (Figures 4a–d) the result for order 2 with $g(u) = u^4$ for all $u \geqslant 0$ and $u(y) = y^{1/8}$ for all $y \geqslant 0$.

Figure 4: Steps of the proof of Theorem 3.1




(a) Step (i)

(b) Step (ii)

(c) Step (iii)

(d) Step (iv)

The interplay between $g$ and $u$ to produce the concavity of $g \circ u$ is depicted in Figure 4a. The application of step (ii) yields: $g^{(2)}(u) = 12u^2 \leqslant g^{\star}(u^{(1)}, u^{(2)}) = 28u^2$, which is illustrated in Figure 4b. Integrating $g^{\star}(u^{(1)}, u^{(2)})$ twice yields the most convex $g$ compatible with the fulfillment of ITP$^2$. This function is $g^{\star}(u) = \frac{7}{3}u^4 + Ku + C$ with $K, C$ arbitrary constants of integration (see Figure 4c, with $K = C = 0$). As an example, consider $(\frac{1}{4}, 10; \frac{1}{4}, 25; \frac{1}{4}, 25; \frac{1}{4}, 40)$ which is obtained from an income distribution $(\frac{1}{4}, 10; \frac{1}{4}, 20; \frac{1}{4}, 30; \frac{1}{4}, 40)$ through $T^2(\frac{1}{4}, 20, 5)$ where, *e.g.*, both distributions state that one-quarter of the population has 10 (income). The fulfillment of ITP$^2$ is consistent with the fact that the Lorenz curve of the former distribution lies nowhere below the

Lorenz curve of the latter, see Figure 4d. [11]

The scope of the results of Theorem 3.1 is beyond "order 2". The fulfillment of $\text{ITP}^3$ by SWFs (1) does not mean that these SWFs exhibit downside utility inequality aversion, that is, satisfy $\text{UTP}^3$. If $u$ belongs to $\Gamma^{\to 3}$, the "critical shape function", namely $g^\star$, has a non-positive third derivative on $\Omega$ whereas downside utility inequality aversion is characterized by $g^{(3)}(u) \geqslant 0$. This case is depicted in Table 2, second row, first column.

**Table 2 Interplay between Utility and Income Transfer Principles.**

| | | | | $\text{ITP}^2$ | | |
|---|---|---|---|---|---|---|
| | | | | Respect | | Violate |
| | | | | $\text{ITP}^3$ | | |
| | | | | Respect | Violate | Violate |
| $\text{UTP}^2$ | Violate | $\text{UTP}^3$ | Violate | $g^{(2)} \geqslant 0^*$<br>but $g^{(2)} \leqslant g^{\star(2)}$<br>$g^{(3)} \leqslant 0$<br>but $g^{(3)} \geqslant g^{\star(3)}$ | $g^{(2)} \geqslant 0^*$<br>but $g^{(2)} \leqslant g^{\star(2)}$<br>$g^{(3)} \leqslant 0$<br>and $g^{(3)} \leqslant g^{\star(3)*}$ | $g^{(2)} \geqslant 0^*$<br>and $g^{(2)} \geqslant g^{\star(2)}$<br>$g^{(3)} \leqslant 0$<br>and $g^{(3)} \leqslant g^{\star(3)*}$ |
| | Respect | | Violate | $g^{(2)} \leqslant 0$<br>$g^{(3)} \leqslant 0$<br>but $g^{(3)} \geqslant g^{\star(3)}$<br>Theorem 3.1 | $g^{(2)} \leqslant 0$<br>$g^{(3)} \leqslant 0$<br>and $g^{(3)} \leqslant g^{\star(3)*}$ | Impossible<br>when $u \in \Gamma^{\to 2}$ |
| | | | Respect | $g^{(2)} \leqslant 0$<br>$g^{(3)} \geqslant 0$<br>Theorem 3.1 | Impossible<br>when $u \in \Gamma^{\to 3}$ | Impossible<br>when $u \in \Gamma^{\to 2}$ |
| Dominance | | | | TSD<br>SSD (spanning first two columns) | | |

Note: $g^{\star(2)} \equiv g^\star(u^{(1)}, u^{(2)})$ and $g^{\star(3)} \equiv g^\star(u^{(1)}, \ldots, u^{(3)})$ *for at least some defined $u(y)$

The effect of Theorem 3.1 is general: The fulfillment of $\text{ITP}^{s+1}$ by SWFs (1) does not mean that these SWFs satisfy $\text{UTP}^{s+1}$. This stems from the alternation in sign of all derivatives up to order $s+1$ of the "critical shape function" $g^\star$ when $u$ is assumed to belong to $\Gamma^{\to s+1}$. These derivatives alternate in sign, opposite to the constraints under which SWFs (1) satisfy $\text{UTP}^{s+1}$ (see Theorem 2.1).

As explained in the introduction, issues of risk-based ethical decision-making provide a relevant context for applying the results of the paper. The next section illustrates this point.

[11] Recall that the Lorenz curve plots the cumulative income share as a function of the cumulative population share. Lorenz dominance is equivalent to second-order stochastic dominance.

# 4 An illustration for risk-based ethical decision-making: The case of water scarcity

In this section, Theorem 3.1 is applied to a problem of decision-making under risk, namely, the problem of water scarcity. For the sake of simplicity, the volume of water in Alfaland is assumed to be at risk due to climate change.[12] As in the framework of Adler and Treich (2014), consumption under risk is examined when the social planner has an *ex post* view. In other words, he cares about the difference in realized utilities. For simplicity, the SWF is

$$\sum_t g(u(c_t)),$$

where $c_t$ is the water consumption in period $t$, with $t = 1, 2$. The attitude to risk of any individual is represented by a vNM utility function $u$ assumed to be cardinally equivalent to individual well-being (Bernoulli assumption).[13] The function $g$ represents the social planner's attitude to inter-temporal utility inequality.

The aim of the social planner is to determine the optimal consumption $c^*$ in the first period so that the inter-temporal social welfare is maximized. The problem is stated as follows:

$$c^* = arg \max_c g(u(c)) + Eg(u(\tilde{w} - c)),$$

where $Eg(u(\tilde{w} - c))$ stands for the expected value of $g(u(\tilde{w} - c))$ and $\tilde{w}$ is a lottery representing the risk to the volume of water in Alfaland. By monotonicity, individuals consume the overall volume of water in both periods. For technical simplicity, the rate of time preference is assumed to be null and the size of the population is constant over time.

Adler and Treich (2014) propose optimal solutions for the special case where $\tilde{w}$ embodies a second-degree risk in the terminology of Ekern (1980). As an example, $\tilde{v}$ has more second-degree risk than $\tilde{w}$ if

$$\tilde{v} = \left(2, \frac{1}{3}; 4, \frac{0}{3}; 6, \frac{2}{3}\right) \; ; \; \tilde{w} = \left(2, \frac{0}{3}; 4, \frac{2}{3}; 6, \frac{1}{3}\right)$$

where, for instance, there are two units of Alfaland's water with probability $\frac{1}{3}$ under $\tilde{v}$. According to the above notation, $\tilde{w}$ is obtained from $\tilde{v}$ through a resource transfer $T^2(\frac{1}{3}, 2, 2)$.

The precautionary savings literature states that optimal current consumption is reduced under risk if and only if, in our framework, $g \circ u \in \Gamma^{\to 3}$ (this is a direct adaptation

[12] For more information on climate change and water-resource issues, see Alavian et al. (2009).

[13] Our results hold for both views: either $u$ relies on the social planner's attitude to risk or $u$ relies empirically on individuals' attitude to risk (*e.g.*, Kaplow and Weisbach, 2011).

of Leland, 1968 and Kimball, 1990). Consequently, everything happens as if the social planner were in front of a transfer of resource of order 3, and then the social planner aims at reducing current water consumption if and only if ITP$^3$ is fulfilled. Two questions arise: (i) What hypotheses on individuals' attitudes to risk are such that the distributive judgments of the social planner imply a reduction of the current water consumption? For individuals assumed to be risk-averse and prudent ($u \in \Gamma^{\to 3}$), if the social planner respects UTP$^3$ ($g \in \Gamma^{\to 3}$), then $c^*$ under $\tilde{w}$ is at least as high as $c^*$ under $\tilde{v}$ (Lemma 3.1). (ii) For these assumptions on individuals' attitude to risk, what is the attitude to inequalities of utility such that the social planner aims at reducing current water consumption? Theorem 3.1 shows that assumptions on the degrees of risk-aversion and prudence of individuals may be sufficient for any social planner who does not adopt UTP$^3$, *i.e.*, $g^{(3)}(u(y)) < 0$ for some $u(y)$, to agree with any social planner who respects UTP$^3$: $c^*$ under $\tilde{w}$ is at least as high as $c^*$ under $\tilde{v}$.[14]

Various economic shocks, such as more or less severe droughts and floods, may affect the scarcity of water. Then, higher-degree risk effects than those explained above can reasonably be considered for the repartition of water volume in Alfaland. Let us examine how the social planner might act in these cases. For this purpose, the sign of the higher-order derivatives of $g \circ u$ becomes crucial. Eeckhoudt and Schlesinger (2008, Proposition 1) show that $c^*$ under $\tilde{w}$ is at least as high as $c^*$ under $\tilde{v}$ for every $g \circ u \in \Gamma^{\to s+1}$ if and only if $\tilde{v}$ has more $s$th degree risk than $\tilde{w}$. In our framework, the fact that $\tilde{v}$ has more $s$th degree risk than $\tilde{w}$ is equivalent to saying that $\tilde{w}$ is obtained from $\tilde{v}$ through $T^s(\alpha, y, \delta)$. However, in this two-period economy, everything happens as if the social planner were in front of a resource transfer of order $s+1$. Hence, $c^*$ under $\tilde{w}$ is at least as high as $c^*$ under $\tilde{v}$ for every social planner who respects ITP$^{s+1}$. For $u \in \Gamma^{\to s+1}$, Theorem 3.1 determines a class of functions $g$ that yields an unanimous statement in favor of the reduction of current consumption when an increase of $s$th degree risk is assessed.

Let us take an example in which $\tilde{v}$ has more third-degree risk than $\tilde{w}$. The lotteries may be expressed as follows:

$$\tilde{v} = \left(2, \frac{6}{15}; 4, \frac{1}{15}; 6, \frac{4}{15}; 8, \frac{4}{15}\right); \tilde{w} = \left(2, \frac{5}{15}; 4, \frac{4}{15}; 6, \frac{1}{15}; 8, \frac{5}{15}\right).$$

Equivalently, $\tilde{w}$ can be obtained from $\tilde{v}$ through $T^3(\frac{1}{15}, 2, 2)$. In the two-period economy, everything happens as if the social planner were in front of a resource transfer of order 4. Hence, $c^*$ under $\tilde{w}$ is at least as high as $c^*$ under $\tilde{v}$ for every social planner who

[14]Let us consider a numerical example. Let $g(u(y)) = u(y) - \frac{1}{3}u(y)^3$, so that UTP$^3$ is not fulfilled. Moreover, $u(y) = y^{\frac{1}{4}}$. The degrees of risk-aversion and prudence exhibited by $u$ are such that $c^*$ under $\tilde{w}$ is equal to 1.51 while $c^*$ under $\tilde{v}$ is equal to 1.2. The consumption difference comes from the precautionary part of the total water saving.

respects ITP[4]. Theorem 3.1 shows that if individuals display risk-aversion, prudence, and temperance ($u \in \Gamma^{\to 4}$), then any social planner with $g \in \Gamma^{\to 3}$ and with $g^{(4)}(u) \leqslant g^\star(u^{(1)}, \ldots, u^{(4)})$ argues that $c^*$ under $\tilde{w}$ is at least as high as $c^*$ under $\tilde{v}$. The critical function $g^\star(u^{(1)}, \ldots, u^{(4)})$ is

$$g^\star(u^{(1)}, \ldots, u^{(4)}) = \\ -\frac{g^{(1)}(u)B_{4,1}(u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}) + g^{(2)}(u)B_{4,2}(u^{(1)}, u^{(2)}, u^{(3)}) + g^{(3)}(u)B_{4,3}(u^{(1)}, u^{(2)})}{(u^{(1)})^4}$$

where

$$\begin{aligned} B_{4,1}(u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}) &= u^{(4)}, \\ B_{4,2}(u^{(1)}, u^{(2)}, u^{(3)}) &= 4u^{(1)}u^{(3)} + 3(u^{(2)})^2, \\ B_{4,3}(u^{(1)}, u^{(2)}) &= 6(u^{(1)})^2 u^{(2)}. \end{aligned}$$

It is easily verifiable that $g^\star(u^{(1)}, \ldots, u^{(4)})$ is non-negative as a result of the hypotheses on $u$. Moreover, Theorem 3.1 shows that assumptions on the degrees of individuals' attitudes to risk may be sufficient for any social planner who does not respect UTP[4], *i.e.*, $g^{(4)}(u(y)) > 0$ for some $u(y)$, to agree with any social planner who respects UTP[4], in words, $c^*$ under $\tilde{w}$ is at least as high as $c^*$ under $\tilde{v}$.

# 5 Conclusion

This paper has provided a characterization of the conditions under which additively separable social welfare functions satisfy generalized income transfer principles. Restrictions on the weighting function of the utility $g$ yield conditions for satisfying generalized utility transfer principles. However, conditions for satisfying generalized income transfer principles are provided by restrictions on the weighting function composed with the utility function of income. On these grounds, the main result determines the restrictions to be imposed on the weighting function for the generalized income transfer principles to be satisfied. This sheds some light on the interplay between the conditions imposed on the utility function and those imposed on the weighting function – displaying attitudes to utility inequality – as determinants of attitudes to income inequality.

When one considers utility functions with all derivatives up to $s+1$th order that alternate in sign, $s$ being a positive integer, additively separable social welfare functions that satisfy the income transfer principle of order $s+1$ do not necessarily respect the utility transfer principle of order $s+1$. This statement is relevant to applied studies in which income redistribution is justified by stochastic dominance of some order. Income redistribution consistent with stochastic dominance of order $s+1$ does not ensure a fairer utility redistribution as defined by utility transfer principles of order $s+1$. Examples of second-order stochastic dominance (SSD) and third-order stochastic dominance (TSD) are depicted in Table 1, respectively in columns 1–2, and in column 1.

Finally, it remains the open problem of analyzing the interplay between ITP and UTP when agents differ in needs. It can be shown that the interplay still holds at the order 2 for particular extended form SWFs such as Atkinson (1970) extended form SWF. However, the link between ITP and UTP is lost at the order 3 (see Appendix B).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# Appendix A. Proof of Lemma 3.1 and Theorem 3.1

For the sake of clarity, we do not make appear the argument of the $u$ function in the proofs. Obviously it does not change anything to the results.

*Proof.* Lemma 3.1:
We proceed by mathematical induction, *i.e.*, we first prove that the statement $\mathbf{H}^{s+1}$ is true for $s = 1$ and then we prove that if $\mathbf{H}^{s+1}$ is assumed to be true for any positive integer $s$, then so is $\mathbf{H}^{s+2}$. Let $w := g \circ u$. It is apparent from $w^{(2)} = (g^{(1)} \circ u) \cdot u^{(2)} + (g^{(2)} \circ u) \cdot \left[u^{(1)}\right]^2$ that $\mathbf{H}^2$ is true. Let us now assume that the statement $\mathbf{H}^{s+1}$ is true in order to show that $\mathbf{H}^{s+2}$ holds. We proceed by remarking that:

$$(g \circ u)^{(s+2)} = \left[-g^{(1)} \circ u \cdot \left(-u^{(1)}\right)\right]^{(s+1)} \quad .$$

We then have:

$$(-1)^{s+2} (g \circ u)^{(s+2)} = (-1)^{s+2} \left[-g^{(1)} \circ u \cdot \left(-u^{(1)}\right)\right]^{(s+1)} \quad .$$

Remembering Leibniz' relation for the $(s+1)$-th derivative of the product of two functions $h \cdot f$:

$$(h \cdot f)^{(s+1)} = \sum_{k=0}^{s+1} \binom{s+1}{k} \left[h^{(k)} \cdot f^{(s-k+1)}\right] \quad ,$$

then:

$$(-1)^{s+2}\,(g\circ u)^{(s+2)} = (-1)^{s+2}\sum_{k=0}^{s+1}\left(\begin{array}{c} s+1 \\ k \end{array}\right)\left[(-g^{(1)}\circ u)^{(k)}\cdot(-u^{(1)})^{(s-k+1)}\right] \ .$$

Rearranging the terms yields:

$$(-1)^{s+2}\,(g\circ u)^{(s+2)} = (-1)\sum_{k=0}^{s+1}\left(\begin{array}{c} s+1 \\ k \end{array}\right)\left[(-1)^k\,(-g^{(1)}\circ u)^{(k)}\cdot(-1)^{s-k+1}\,(-u^{(1)})^{(s-k+1)}\right] \ . \tag{A0}$$

Now we want to prove the relation $\mathbf{H}^{s+2}$. Then we assume that $-g^{(1)} \in \Gamma^{\to s+1}$ (*i.e.* $g \in \Gamma^{\to s+2}$) and $u \in \Gamma^{\to s+2}$. The induction hypothesis $\mathbf{H}^{s+1}$ is assumed to be true, then for two functions $f$ and $h$:

$$\text{if } f \in \Gamma^{\to s+1} \ \text{ and } \ h \in \Gamma^{\to s+1}\ , \text{ then } \ f\circ h \in \Gamma^{\to s+1}.$$

Let $f := -g^{(1)}$ and $h := u$ (actually if $u \in \Gamma^{\to s+2}$ then $u \in \Gamma^{\to s+1}$), then we obtain that $(-1)^k\,(-g^{(1)}\circ u)^{(k)} \leqslant 0$, for all $k = 0, \ldots, s+1$. If $u \in \Gamma^{\to s+2}$, then $(-1)^{s-k+2}u^{(s-k+2)} \leqslant 0$ for all $k = 0, \ldots, s+1$. Since Eq.(A0) can be expressed as,

$$(-1)^{s+2}\,(g\circ u)^{(s+2)} = (-1)\sum_{k=0}^{s+1}\left(\begin{array}{c} s+1 \\ k \end{array}\right)\left[(-1)^k\,(-g^{(1)}\circ u)^{(k)}\cdot(-1)^{s-k+2}\,u^{(s-k+2)}\right] \ ,$$

consequently, $(-1)^{s+2}\,(g\circ u)^{(s+2)} \leqslant 0$. As the statement $\mathbf{H}^{s+1}$ has been shown to be true for $s = 1$ and that the statement $\mathbf{H}^{s+2}$ has been proven to be true when $\mathbf{H}^{s+1}$ is invoked, then $\mathbf{H}^{s+1}$ is true. □

*Proof.* Theorem 3.1:
[(i) $\Longrightarrow$ (ii)] From Fishburn and Willig (1984), $SW(F) = \int w(y)dF(y)$ respects ITP$^{i+1}$ if and only if $(-1)^{i+1}\ w^{(i+1)}(y) \leqslant 0$ for all $i \in \{1, \ldots, s\}$. Setting $w = g\circ u$, we get from Lemma 3.1 that $(-1)^i(g\circ u)^{(i)} \leqslant 0$ for all $i \in \{1, \ldots, s\}$ since $g \in \Gamma^{\to s}$ and $u \in \Gamma^{\to s}$ (actually $u \in \Gamma^{\to s+1}$). It remains to find the conditions such that ITP$^{s+1}$ is respected, *i.e.*, $(-1)^{i+1}(g\circ u)^{(i+1)} \leqslant 0$ for $i = s$. We start the demonstration with Faà di Bruno's formula:

$$(g\circ u)^{(i+1)} = \sum_{k=1}^{i+1}(g^{(k)}\circ u)\cdot B_{i+1,k}\left(u^{(1)}, \ldots, u^{(i-k+2)}\right),$$

where,

$$B_{i+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(i-k+2)}\right) = \sum \frac{(i+1)!}{p_1!p_2!\cdots p_{i-k+2}!}\left(\frac{u^{(1)}}{1!}\right)^{p_1}\left(\frac{u^{(2)}}{2!}\right)^{p_2}\cdots\left(\frac{u^{(i-k+2)}}{(i-k+2)!}\right)^{p_{i-k+2}}, \tag{A1}$$

with $p_1, \ldots, p_{i-k+2}$ such that $p_1 + p_2 + \cdots = k$ and $1p_1 + 2p_2 + \cdots = i+1$. Since $(g \circ u)^{(i+1)} = (g^{(i+1)} \circ u) \cdot B_{i+1,i+1}(u^{(1)}) + \sum_{k=1}^{i} (g^{(k)} \circ u) \cdot B_{i+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(i-k+2)}\right)$, then ITP$^{i+1}$ is respected if and only if

$$\begin{aligned}
&(-1)^{i+1}(g \circ u)^{(i+1)} \leqslant 0 \\
\iff &(-1)^{i+1} g^{(i+1)} \circ u \leqslant (-1)^{i+1}\left[-\frac{\sum_{k=1}^{i}(g^{(k)} \circ u) \cdot B_{i+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(i-k+2)}\right)}{B_{i+1,i+1}(u^{(1)})}\right] \\
&\qquad\qquad =: (-1)^{i+1} g^{\star}(u^{(1)}, \ldots, u^{(i+1)}).
\end{aligned} \tag{A2}$$

As $B_{i+1,i+1}(u^{(1)}) = [u^{(1)}]^{i+1}$, thus:

$$g^{\star}(u^{(1)}, \ldots, u^{(i+1)}) = -\frac{\sum_{k=1}^{i}(g^{(k)} \circ u) \cdot B_{i+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(i-k+2)}\right)}{[u^{(1)}]^{i+1}}.$$

In order to derive the sign of the boundary $g^{\star}(u^{(1)}, \ldots, u^{(i+1)})$, we have to find the sign of $B_{i+1,k}(\cdot)$. Accordingly, we decompose the products of the functions $u^{(\cdot)}$ in $B_{i+1,k}(\cdot)$ in Eq.(A1) thanks to the following set:

$$\Lambda(k) := \left\{ \underbrace{u^{(1)}, \ldots, u^{(1)}}_{p_1}, \underbrace{u^{(2)}, \ldots, u^{(2)}}_{p_2}, \ldots, \underbrace{u^{(i-k+2)}, \ldots, u^{(i-k+2)}}_{p_{i-k+2}} \right\}.$$

The set $\Lambda(k)$ is decomposed into two partitions. The first one is $\Lambda^e := \{u^{(j)} \in \Lambda(k) : j \in \mathbb{E}\}$, which is the set of all derivatives $u^{(j)}$ such that the integers $j \in \mathbb{E}$, where $\mathbb{E}$ is the set of even integers without zero. The second partition is $\Lambda^o := \{u^{(j)} \in \Lambda(k) : j \in \mathbb{O}\}$ for which $j \in \mathbb{O}$, where $\mathbb{O}$ is the set of odd integers without zero. The cardinals of $\Lambda^e$ and $\Lambda^o$ are denoted by $|\Lambda^e|$ and $|\Lambda^o|$, respectively. We have by definition of Bell polynomials:

$$p_1 + p_2 + \cdots = k = |\Lambda^e| + |\Lambda^o| \tag{A3}$$

$$\sum_{\forall u^{(j)} \in \Lambda(k)} j = i+1. \tag{A4}$$

<u>Case 1: $i+1$ is even and $k$ is even.</u>

$\hookrightarrow$ Let us assume that $|\Lambda^e|$ is even. As $k$ is even this implies, from Eq.(A3), that $|\Lambda^o|$ is even too. Since $u \in \Gamma^{\rightarrow s+1}$, we have the following implication:

$$\left[\prod_{\forall u^{(j)} \in \Lambda^e}^{|\Lambda^e|} u^{(j)} \geqslant 0\,,\quad \prod_{\forall u^{(j)} \in \Lambda^o}^{|\Lambda^o|} u^{(j)} \geqslant 0\right] \implies \left[B_{i+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(i-k+2)}\right) \geqslant 0\right].$$

$\hookrightarrow$ Assume that $|\Lambda^e|$ is odd, then $|\Lambda^o|$ is odd from Eq.(A3). Then, we deduce, from Eq.(A4), that:

$$\left[\left(\sum_{\forall u^{(j)}\in\Lambda^o} j\right) \text{ is odd },\ \left(\sum_{\forall u^{(j)}\in\Lambda^e} j\right) \text{ is even}\right] \Longrightarrow \left[\left(\sum_{\forall u^{(j)}\in\{\Lambda^o\cup\Lambda^e\}} j = i+1\right) \text{ is odd}\right].$$

Since the term $i+1$ has been assumed to be even, then it yields a contradiction: $|\Lambda^e|$ and $|\Lambda^o|$ cannot be odd when $i+1$ and $k$ are even.

Case 2: $i+1$ is even and $k$ is odd.

$\hookrightarrow$ Let us assume that $|\Lambda^e|$ is odd. As $k$ is odd, we get from Eq.(A3) that $|\Lambda^o|$ is even. Since $u \in \Gamma^{\to s+1}$, we have:

$$\left[\prod_{\forall u^{(j)}\in\Lambda^e}^{|\Lambda^e|} u^{(j)} \leqslant 0\ ,\ \prod_{\forall u^{(j)}\in\Lambda^o}^{|\Lambda^o|} u^{(j)} \geqslant 0\right] \Longrightarrow \left[B_{i+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(i-k+2)}\right) \leqslant 0\right].$$

$\hookrightarrow$ If $|\Lambda^e|$ is even, then $|\Lambda^o|$ is odd by Eq.(A3). This case is impossible since $i+1$ is even whereas we obtain a contradiction:

$$\left[\left(\sum_{\forall u^{(j)}\in\Lambda^o} j\right) \text{ is odd },\ \left(\sum_{\forall u^{(j)}\in\Lambda^e} j\right) \text{ is even}\right] \Longrightarrow \left[\left(\sum_{\forall u^{(j)}\in\{\Lambda^o\cup\Lambda^e\}} j = i+1\right) \text{ is odd}\right].$$

Case 3: $i+1$ is odd and $k$ is even.

$\hookrightarrow$ Let us assume that $|\Lambda^e|$ is odd, then $|\Lambda^o|$ is odd by Eq.(A3). Since $u \in \Gamma^{\to s+1}$, we have:

$$\left[\prod_{\forall u^{(j)}\in\Lambda^e}^{|\Lambda^e|} u^{(j)} \leqslant 0\ ,\ \prod_{\forall u^{(j)}\in\Lambda^o}^{|\Lambda^o|} u^{(j)} \geqslant 0\right] \Longrightarrow \left[B_{i+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(i-k+2)}\right) \leqslant 0\right].$$

$\hookrightarrow$ If $|\Lambda^e|$ is even, then $|\Lambda^o|$ is even by Eq.(A3). We have a contradiction since $i+1$ is assumed to be odd whereas:

$$\left[\left(\sum_{\forall u^{(j)}\in\Lambda^o} j\right) \text{ is even },\ \left(\sum_{\forall u^{(j)}\in\Lambda^e} j\right) \text{ is even}\right] \Longrightarrow \left[\left(\sum_{\forall u^{(j)}\in\{\Lambda^o\cup\Lambda^e\}} j = i+1\right) \text{ is even}\right].$$

Case 4: $i+1$ is odd and $k$ is odd.

$\hookrightarrow$ Let us assume that $|\Lambda^e|$ is even, thus $|\Lambda^o|$ is odd by Eq.(A3). Since $u \in \Gamma^{\to s+1}$, we have:

$$\left[\prod_{\forall u^{(j)}\in\Lambda^e}^{|\Lambda^e|} u^{(j)} \geqslant 0\ ,\ \prod_{\forall u^{(j)}\in\Lambda^o}^{|\Lambda^o|} u^{(j)} \geqslant 0\right] \Longrightarrow \left[B_{i+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(i-k+2)}\right) \geqslant 0\right].$$

$\hookrightarrow$ If $|\Lambda^e|$ is odd, then $|\Lambda^o|$ is even by Eq.(A3). This yields a contradiction since $i+1$ is assumed to be odd whereas:

$$\left[\left(\sum_{\forall u^{(j)}\in\Lambda^o} j\right) \text{ is even },\ \left(\sum_{\forall u^{(j)}\in\Lambda^e} j\right) \text{ is even}\right] \implies \left[\left(\sum_{\forall u^{(j)}\in\{\Lambda^o\cup\Lambda^e\}} j=i+1\right) \text{ is even}\right].$$

Final Remark: By definition, we have $B_{i+1,1}(\cdot) = [u^{(i+1)}]^1$ and $B_{i+1,i+1}(\cdot) = [u^{(1)}]^{i+1}$. Since $g\in\Gamma^{\to s}$, we have therefore $(-1)^k g^{(k)}\circ u \leqslant 0$, for all $k=1,\ldots,i$. Then, from Cases 1 to 4 that:

$$(-1)^{i+1}(g^{(k)}\circ u)\cdot B_{i+1,k}\left(u^{(1)},u^{(2)},\ldots,u^{(i-k+2)}\right) \leqslant 0,\ \forall k=1,\ldots,i.$$

Hence, from Eq.(A2), $(-1)^{s+1}g^{\star}(u^{(1)},\ldots,u^{(s+1)}) \geqslant 0$.

[(ii) $\implies$ (i)] Let us assume that:

$$(-1)^{s+1}g^{(s+1)}\circ u \leqslant (-1)^{s+1}\left[-\frac{\sum_{k=1}^{s}(g^{(k)}\circ u)\cdot B_{s+1,k}\left(u^{(1)},u^{(2)},\ldots,u^{(s-k+2)}\right)}{\left[u^{(1)}\right]^{s+1}}\right].$$

We have:

$$(-1)^{s+1}\left[g^{(s+1)}\circ u\left[u^{(1)}\right]^{s+1} + \sum_{k=1}^{s}(g^{(k)}\circ u)\cdot B_{s+1,k}\left(u^{(1)},u^{(2)},\ldots,u^{(s-k+2)}\right)\right] \leqslant 0. \quad \text{(A5)}$$

From Faà di Bruno's formula, Eq.(A5) becomes:

$$(-1)^{s+1}(g\circ u)^{(s+1)} \leqslant 0. \quad \text{(A6)}$$

From Fishburn et Willig's (1984) Theorem 1 and Lemma 3.1, if $g\in\Gamma^{\to s}$, $u\in\Gamma^{\to s}$ and Eq.(A6), then $W$ satisfies every ITP up to the order $s+1$, which ends the proof. □

# Appendix B. Interplay between UTP and ITP with heterogeneous agents

It is shown below that when households differ in needs, there exists an interpretable relation between UTP and ITP at the order 2, not at the order 3. Households are described by income $y\in\Omega$ and type $h\in\mathscr{H}=\{1,2,\ldots,H\}$. We assume that there is a finite number of types $H\geq 2$. Let $F$ be the joint cumulative distribution function $F(y,h)$ that is the proportion of households whose income is at most $y$ and whose type does not exceed $h$. Formally,

$$F(y,h)=\sum_{r=1}^{h}\int_0^y f(\varepsilon,r)d\varepsilon,\ \forall h\in\mathscr{H},\ \forall y\in\Omega,$$

where $f(y,h)$ is the joint probability measure from $\Omega\times\mathscr{H}$ onto $[0,1]$. Any given bivariate $f$ that gives nonzero mass to only a finite number of incomes, and has a total mass $\sum_{r\in\mathscr{H}}\int_{y\in\Omega} f(y,r)dy = 1$ belongs to the set $\mathscr{F}^*$. For the sake of clarity, we restrict the SWFs to the class $\mathscr{W}_A$ of Atkinson (1970) SWFs:

$$W_A(f) = \begin{cases} \sum_{h=1}^{H}\int_0^{y_{\max}} \frac{[u_h(y)]^{1-\rho}}{1-\rho}\ f(y,h)\ dy \text{ if } \rho\in\mathbb{R} \text{ and } \rho\neq 1; \\ \\ \sum_{h=1}^{H}\int_0^{y_{\max}} \ln\left(u_h(y)\right)\ f(y,h)\ dy \text{ if } \rho = 1. \end{cases} \tag{B1}$$

The parameter $\rho$ exhibits the attitude to inequality of utility. Households' types are defined following the definition of needs provided by Moyes (2012).

**Definition 5.1. Households' types (needs):** *For any given $\ell\in\{1,2\}$ and $y\in\Omega$:*

$$(-1)^{\ell}u_1^{(\ell)}(y) \leqslant (-1)^{\ell}u_2^{(\ell)}(y) \leqslant \cdots \leqslant (-1)^{\ell}u_H^{(\ell)}(y) < 0\ ; \tag{B2a}$$

$$0 < u_1(y) \leqslant u_2(y)\ \leqslant \cdots \leqslant u_H(y). \tag{B2b}$$

Utility transfers of order 2 and 3, inspired from Definition 2.1, are analyzed in the cases where donors and recipients have different types. Let $m$ be obtained from $f\in\mathscr{F}^*$ through an income increment $T^1(\alpha,y,\delta,k)$:

$$m(y,k) = \begin{cases} f(y,k)-\alpha & \text{at level } y\ , \\ f(y,k)+\alpha & \text{at level } y+\delta\ , \\ f(y,k) & \text{anywhere else.} \end{cases}$$

**Definition 5.2. Utility and Income transfer of order s between heterogeneous agents.** *For all $f\in\mathscr{F}^*$, for all $y\in\Omega$, and for given $k$ and $h\in\mathscr{H}$ with $k<h$, utility and income transfers of order $s$ between heterogeneous agents are given by, respectively:*

$$T^s(\alpha,u_{\{k,h\}}(y),\delta) := T^{s-1}(\alpha,u_k(y),\delta) - T^{s-1}(\alpha,u_h(y)+\delta,\delta),\ \ s\in\{2,3\}$$
$$T^s(\alpha,y,\delta,k,h) := T^{s-1}(\alpha,y,\delta,k) - T^{s-1}(\alpha,y+\delta,\delta,h),\ \ s\in\{2,3\}.$$

Utility and income transfer principles of order 2 and 3 are defined as follows.

**Definition 5.3. Utility and Income Transfer Principles**. *For all $f,m\in\mathscr{F}^*$, for all $y\in\Omega$, for given $k,h\in\mathscr{H}$ and $s\in\{2,3\}$, the following implications hold for utility and income transfers, respectively:*

$$(\mathrm{UTP}^s) \qquad m = f + T^{\ell}(\alpha,u(y),\delta) \implies W(m)\geqslant W(f), \quad \ell = 2,s;$$
$$(\mathrm{ITP}^s) \qquad m = f + T^{\ell}(\alpha,y,\delta) \implies W(m)\geqslant W(f), \quad \ell = 2,s.$$

Let us define the following set of SWFs:

$$\mathscr{W}_A^P = \{W \in \mathscr{W}_A : \rho > 0\}.$$

**Proposition 5.1.** *If Eq.*(B2a) *and Eq.*(B2b) *hold, then the following assertions are equivalent:*
*(i)* $W_A$ *respects* $UTP^2$ *and* $UTP^3$*.*
*(ii)* $W_A \in \mathscr{W}_A^P$*.*

*Proof.* See Dubois (2016, pp. 79-83). □

Let us now introduce the following set:

$$\mathscr{W}_A^{P*} = \left\{ W \in \mathscr{W}_A : \rho > \rho^* = \sup_{y,\delta,k<h} \frac{\ln\left(\frac{u_k^{(1)}(y)}{u_h^{(1)}(y+\delta)}\right)}{\ln\left(\frac{u_k(y)}{u_h(y+\delta)}\right)},\ \rho^* \leqslant 0 \right\}.$$

**Proposition 5.2.** *If Eq.*(B2a) *and Eq.*(B2b) *hold, then the following assertions are equivalent:*
*(i)* $W_A$ *respects* $ITP^2$*.*
*(ii)* $W_A \in \mathscr{W}_A^{P*}$*.*

*Proof.* See Dubois (2016, pp. 87-89). □

Proposition 5.3 shows that the interplay between UTP$^2$ and ITP$^2$ still holds in a framework with agents who differ in needs. Clearly, $\mathscr{W}_A^P \subseteq \mathscr{W}_A^{P*}$, therefore if a social planner who behaves in accordance with Eq.(B1) respects UTP$^2$, then the social planner respects ITP$^2$, the converse being not true. However, a step further, the interplay between ITP$^3$ and UTP$^3$ is lost.

**Proposition 5.3.** *Consider that Eq.*(B2a) *and Eq.*(B2b) *hold. If* $W_A$ *respects* $ITP^3$*, then:*

$$\frac{-\rho}{u_k(y)^{1+\rho}}[u_k^{(1)}(y)]^2 + \frac{u_k^{(2)}(y)}{u_k(y)^{\rho}} < \frac{-\rho}{u_h(y+\delta)^{1+\rho}}[u_h^{(1)}(y+\delta)]^2 + \frac{u_h^{(2)}(y+\delta)}{u_h(y+\delta)^{\rho}}.$$

*Proof.* See Dubois (2016, p. 95). □

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