A Note on α -Gini Measures

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Abstract

This note provides a characterization of α-Gini inequality measures. These measures generalize the standard Gini index by including one sensitivity parameter α, which captures different value judgments. The α-Gini measures are shown to be weakly decomposable and unit-consistent. Weak decomposition provides within-group and between-group inequalities. Unit consistency keeps unchanged the ranking of two income distributions when the income units vary. It is shown that the α-Gini measures are relevant with either "leftist" or"rightist" views.

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1 Introduction

The Gini index is one of the most popular indices employed in economics. It is expressed in various ways and its applications are concerned with concentration, income inequality, poverty, segregation, diversity, health, mobility and many other fields. In those recent decades, the Gini index has been generalized in order to bring out wider families of income inequality, for example, the $S-$Gini family due to Donaldson and Weymark (1980), the extended Gini index proposed by Yitzhaki (1983), and among others, the $\mathcal{P}-$Gini family introduced by Gajdos (2002). The measures of those families may depend on a parameter of sensitivity towards inequality. The parametrization and the structure of those measures vary from one family to another, however each member of these families inherits from the basic properties satisfied by the Gini index, highlighting either its connection with the Lorenz curve or with Yaari's (1987) dual social welfare function.

The aim of this note is to propose a characterization of the α-Gini measures, introduced by Chameni (2006) and Ebert (2010). A subgroup decomposition property (WDEC) is proposed and appears to be a generalization of Ebert's (2010) weak decomposition properties (DEC) and ($\widehat{DEC}(\varepsilon)$). Such a property allows for computing within- and between-group inequalities. As mentioned in Ebert (2010), contrary to the traditional additive decomposition in which the between-group inequality term consists in comparing the mean income of the subgroups only, the weak decomposition outlines the comparison of each income pairwise between every pairs of subgroups. On this basis, rich-to-poor transfers may occur between subgroups. This constitutes an important step for the implementation of redistributive actions targeting specific parts of the population in which inequality is concentrated.

Income redistribution is embodied by value judgments included in any measure of inequality (see *e.g.*, Kolm, 1976a, 1976b), such as blue"leftist" (absolute) or "rightist" (relative) views. Zheng (2007) proposes the *unit consistency* property (UC) in order to capture and to enlarge such value judgments. It is an ordinal condition postulating that the ranking between two income distributions remains unchanged when income units differ. Thanks to basic axioms, combining (UC) with (WDEC), we provide a characterization of the α-Gini measures, which are consistent with either *leftist* or *rightist* value judgments.

The outline of the paper is as follows. In Section 2, the notations are introduced and also the standard axioms usually employed in the literature on inequality measurement (2.1). The different formulations of the decomposition axioms are motivated (2.2). The main results are presented in Section 3: the interest of combining weak decomposition and unit-consistency properties in order to characterize the α-Gini measures. Section 4 closes the paper.

2 Notations and Axioms

We first expose the notations and the usual axiomatic properties. Then, we present the various formulations underlying the concept of weak decomposition recently introduced in the literature.

2.1 Notations and Standards axioms

Let us consider a population of n individuals, $i = 1, \ldots, n$, with $n \in \mathbb{N}$, $\mathbb{N}$ being the set of positive integers. The income distribution is $\mathbf{x} := (x_1, \ldots, x_n) \in \mathbb{R}^n_+$, where $\mathbb{R}^n_+$ represents the $n-$dimensional non-negative Euclidean space ($\mathbb{R}^n_{++}$ being its positive part). The set of all admissible income distributions (with variable size n) is denoted $\mathscr{X}_+ := \bigcup_{n \geqslant 1} \mathbb{R}^n_+$ with $\mathscr{X}_{++} := \mathscr{X}_+ \setminus \bigcup_{n \geqslant 1} \mathbf{0}^n$ ($\mathbf{0}^n$ being a vector of zeros of size n). The population may be partitioned into G exhaustive and exclusive subgroups of size n_g, $g = 1, \ldots, G$ such that $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^g, \ldots, \mathbf{x}^G)$ and $\mathbf{x}^g \in \mathscr{X}_+$. The vector of subgroup population sizes is $\boldsymbol{n} := (n_1, \ldots, n_G)$ such that $n = n_1 + \cdots + n_G$. The vector of subgroup arithmetic means is $\boldsymbol{\mu} := (\mu(\mathbf{x}^1), \ldots, \mu(\mathbf{x}^g), \ldots, \mu(\mathbf{x}^G))$, such that $\mu(\mathbf{x}^g) > 0$ for all $g = 1, \ldots, G$, and $\mu(\mathbf{x})$ is the arithmetic mean of the population. The arithmetic mean between individual's i income and individual's j income is denoted $\mu(x_i, x_j)$, such that $\mu(x_i, x_j) > 0$. The vector of ones of size $n(\mathbf{x}) \equiv n$ is denoted $\mathbf{1}_{n(\mathbf{x})}$. A replication of an income distribution $\mathbf{x}$ by order k, for $k \geqslant 2$, is $\mathbf{x}^{[k]} = (\underbrace{x_1, \ldots, x_1}_{k \text{ times}}, \ldots, \underbrace{x_n, \ldots, x_n}_{k \text{ times}})$.

The inequality measure is a function $I : \mathscr{X}_+ \longrightarrow \mathbb{R}_+$ defined as a sequence of indices $I : \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+$ such that there exists an inequality index for every n.

As a first requirement, the inequality measure is supposed to be continuous.

Axiom 2.1. – Continuity – (CN). $I(\mathbf{x})$ *satisfies continuity if, for all* $\mathbf{x} \in \mathscr{X}_+$, $I(\mathbf{x})$ *is a continuous function.*

The inequality measure remains unchanged if individuals are permuted within the income distribution. The inequality measure is not affected by a rearrangement of the components of vector $\mathbf{x}$.

Axiom 2.2. – Anonymity – (AN). $I(\mathbf{x})$ *satisfies anonymity if, for all* $\mathbf{x} \in \mathscr{X}_+$ *and all permutation matrices* Π *of size* $n \times n$, $I(\mathbf{x}) = I(\Pi\mathbf{x})$.

The inequality measure is normalized, that is, the measure is null when incomes are identical.

Axiom 2.3. – Normalization – (NM). $I(\mathbf{x})$ *satisfies normalization if, for all* $\mathbf{x} \in \mathscr{X}_+$ *such that* $\mathbf{x} = \varepsilon \mathbf{1}_{n(\mathbf{x})}$ *with* $\varepsilon > 0$, $I(\mathbf{x}) = 0$.

The inequality measure is invariant by replication. Such a property introduced by Dalton (1920), the Population Principle, allows comparisons of inequality measures for different population sizes.

Axiom 2.4. – Replication Invariance – (PP). *$I(\mathbf{x})$ satisfies replication invariance if, for all $\mathbf{x} \in \mathscr{X}_+$ and $k \geqslant 2$, $I(\mathbf{x}^{[k]}) = I(\mathbf{x})$.*

By definition, inequality measures may be considered in absolute or relative terms. Absolute and relative inequality measures are characterized by invariance properties recalled below.

Axiom 2.5. – Invariance by translation – (INV). *$J(\mathbf{x})$ satisfies invariance by translation if, for all $\mathbf{x} \in \mathscr{X}_+$ and $\theta > 0$, $J(\mathbf{x} + \varepsilon \mathbf{1}_{n(\mathbf{x})}) = J(\mathbf{x})$.*

Axiom 2.6. – Scale Invariance – (SI). *$I(\mathbf{x})$ satisfies scale invariance if, for all $\mathbf{x} \in \mathscr{X}_+$ and $\theta > 0$, $I(\theta \mathbf{x}) = I(\mathbf{x})$.*

Note that the scale invariance property corresponds to a homogeneity of degree zero requirement. Note that rescaling an absolute inequality measure by the mean, one obtains a relative inequality measure. For the sake of generality, it is assumed that the absolute measures may be rescaled by any given real-valued function of the mean income.

Definition 2.1. *Let $J : \mathscr{X}_+ \longrightarrow \mathbb{R}_+$ denotes an absolute inequality measure and $I : \mathscr{X}_+ \longrightarrow \mathbb{R}_+$ a relative one. For any function $f : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$, the relation between absolute inequality measures $J(\mathbf{x})$ and relative inequality measures $I(\mathbf{x})$ is the following:*

$$I(\mathbf{x}) = \begin{cases} \frac{J(\mathbf{x})}{f(\mu(\mathbf{x}))}, & \forall \mathbf{x} \in \mathscr{X}_{++} \\ 0, & \forall \mathbf{x} \in \bigcup_{n \geqslant 1} \mathbf{0}^n. \end{cases}$$

2.2 Axioms of weak decomposition

Subgroup decomposition properties are of interest to deal with a population composed of heterogeneous agents. In 2010 a weaker scheme of decomposition than the usual one, defined by Bourguignon (1979) or Shorrocks (1980), is axiomatized by Ebert (2010). This new property enables Pigou-Dalton transfers to be performed between precise individuals of distinct subgroups rather than the mean of the subgroups (see Ebert, 2010). The between-group inequality component is based on the comparison between each and every pairs of incomes rather than the use of the mean incomes of the subgroups. The weakly decomposable measures are well suited for the study of the inequality between and within different subgroups of the population. In particular, they outline two components of inequality, which are relevant with Kolm's (1999) aggregation principles for pairwise inequality measures, such as the Gini indices, which capture envy between each and every pairs of individuals. Ebert (2010) investigates two types of decomposition schemes. The first one exhibits weights being functions of the subgroup sizes.

Axiom 2.7. – Decomposition – (DEC). *$I(\mathbf{x})$ satisfies weak decomposability if, for all income distributions $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2) \in \mathscr{X}_+$ subdivided into 2 exhaustive and exclusive subgroups,*

$$I(\mathbf{x}) = \alpha^1(\boldsymbol{n})I(\mathbf{x}^1) + \alpha^2(\boldsymbol{n})I(\mathbf{x}^2) + \beta(\boldsymbol{n}) \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} I(x_i^1, x_j^2),$$

where x_i^1 (x_j^2) stands for the income of the $i-th$ ($j-th$) individual in subgroup 1 (2) and $\alpha^1(\boldsymbol{n})$, $\alpha^2(\boldsymbol{n})$ and $\beta(\boldsymbol{n})$ strictly positive weighting functions.

Ebert(2010) also suggests an alternative version of the weak decomposition property (DEC) with weights embodied by functions of subgroup sizes and income shares.

Axiom 2.8. – Decomposition – ($\widehat{\mathbf{DEC}}(\varepsilon)$). *$I(\mathbf{x})$ satisfies weak decomposability if, for all income distributions $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2) \in \mathscr{X}_+$ subdivided into 2 exhaustive and exclusive subgroups,*

$$I(\mathbf{x}) = \alpha^1(\boldsymbol{n}) \cdot \frac{\mu(\mathbf{x}^1)^\varepsilon}{\mu(\mathbf{x})^\varepsilon} \cdot I(\mathbf{x}^1) + \alpha^2(\boldsymbol{n}) \cdot \frac{\mu(\mathbf{x}^2)^\varepsilon}{\mu(\mathbf{x})^\varepsilon} \cdot I(\mathbf{x}^2) \\ + \ \beta(\boldsymbol{n}) \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \frac{\mu(x_i^1, x_j^2)^\varepsilon}{\mu(\mathbf{x})^\varepsilon} \cdot I(x_i^1, x_j^2),$$

where $\varepsilon > 0$, and where x_i^1 (x_j^2) stands for the income of the $i-th$ ($j-th$) individual in subgroup 1 (2) and $\alpha^1(\boldsymbol{n})$, $\alpha^2(\boldsymbol{n})$ and $\beta(\boldsymbol{n})$ strictly positive weighting functions.

In order to deal with a weaker axiom of decomposition, no functional form is imposed on the income shares. We assume that there exist strictly positive weighting functions denoted $\alpha(n_1, n)$, $\alpha(n_2, n)$, $\beta(2, n)$, and $\xi(\mu(\mathbf{x}^g), \mu(\mathbf{x}))$ such that the functions $\alpha(\cdot, \cdot)$ have the same structure for all subgroups.

Axiom 2.9. – Weak Decomposition – (WDEC). *$I(\mathbf{x})$ satisfies weak decomposability if, for all income distributions $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2) \in \mathscr{X}_+$ subdivided into 2 exhaustive and exclusive subgroups,*

$$I(\mathbf{x}) = \alpha(n_1, n) \cdot \xi(\mu(\mathbf{x}^1), \mu(\mathbf{x})) \cdot I(\mathbf{x}^1) + \alpha(n_2, n) \cdot \xi(\mu(\mathbf{x}^2), \mu(\mathbf{x})) \cdot I(\mathbf{x}^2) \\ + \ \beta(2, n) \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \xi(\mu(x_i^1, x_j^2), \mu(\mathbf{x})) I(x_i^1, x_j^2),$$

where $n = n_1 + n_2$ such that $n_1, n_2 \geqslant 1$, with $\alpha(\cdot, n)$, $\beta(2, n)$, and $\xi(\cdot, \mu(\mathbf{x}))$ strictly positive weighting functions.

In the sequel, (WDEC) is used to characterize the family of weakly decomposable inequality measures being unit-consistent.[1]

[1]See also Mornet (2016) for a characterization of weakly decomposable inequality measures with a particular emphasis on the characterization of the weight functions $\alpha(\cdot)$ and $\beta(\cdot)$. In Mornet (2016) the link between unit consistency and weak decomposition is not investigated and the weight functions are more general.

3 The Family of α-Gini Measures

The unit-consistency requirement is first introduced and combined with the weak decomposition property in order to expose our main results.

Unit consistency is an ordinal and general property. According to Zheng (2007), changing the units of the income distributions must preserve their ranking with respect to the inequality measure $I(\cdot)$.

Axiom 3.1. – Unit consistency – (UC). $I(\mathbf{x})$ *satisfies unit consistency if, for all* $\mathbf{x}, \mathbf{y} \in \mathscr{X}_+$,

$$[I(\mathbf{x}) < I(\mathbf{y})] \implies [I(\theta\mathbf{x}) < I(\theta\mathbf{y})],\ \forall\theta \in \mathbb{R}_{++}.$$

An immediate consequence of (UC) if the following.

Proposition 3.1. (Zheng, 2007) *An inequality index* $I(\mathbf{x})$ *is unit-consistent if, and only if, for all* $\mathbf{x} \in \mathscr{X}_{++}$ *and* $\theta \in \mathbb{R}_{++}$, *there exists a continuous function* $f(\cdot,\cdot)$ *which is also increasing in the second argument such that:*

$$I(\theta\mathbf{x}) = f[\theta, I(\mathbf{x})].$$

On the one hand, it is noteworthy that Proposition 3.1 is valid for all $x \in \mathscr{X}_+$ as well. On the other hand, unit consistency (UC) can be interpreted as a *weak currency independence* property (see Zoli, 2012). Besides, it enables a wide spectrum of value judgments to be captured, from leftist points of view to rightist ones. The use of (UC) is of interest to derive a new class of inequality measures which are also weakly decomposable. Following Zheng (2007), when (UC) is combined with a proper subgroup decomposition property it provides a homogeneity condition.

Lemma 3.1. *If an inequality measure* $I(\mathbf{x})$ *satisfies* (CN), (NM), (WDEC), (PP) *and* (UC) *then,*

$$I(\theta\mathbf{x}) = \theta^\alpha I(\mathbf{x}),\ \forall\theta \in \mathbb{R}_{++},\ \forall\alpha \in \mathbb{R}.$$

Proof. See the Appendix. □

Corollary 3.1. *The previous result holds true for* $(\widehat{\text{DEC}}(\varepsilon))$ *or* (DEC).

Proof. Using the same approach as in the proof of Lemma 3.1 yields the desired result. □

Measures which are both unit-consistent and weakly decomposable are also homogeneous of degree α, for all $\alpha \in \mathbb{R}$.[2] A similar result has been obtained by Zheng (2007) with the additive decomposition.

Now, adding anonymity (AN) yields the family of weakly decomposable and unit-consistent inequality measures.

[2]When the degree of homogeneity is null ($\alpha = 0$), the expression corresponds to scale invariance (SI).

Theorem 3.1. *An inequality measure* $I(\mathbf{x})$ *satisfies* (CN), (NM), (AN), (WDEC), (UC) *and* (PP) *if, and only if,*

$$I(\mathbf{x}) = \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} h\left(\frac{\mu(x_i, x_j)}{\mu(\mathbf{x})}\right) I(x_i, x_j),$$

where $I(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is homogeneous of degree* $\alpha \in \mathbb{R}$*, symmetric and continuous such that* $I(z, z) = 0$*, and where* $h : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ *is continuous.*

Proof. See the Appendix. □

From Theorem 3.1 several subclasses of inequality measures embodying specific points of view may be deduced. For instance, the class of absolute weakly decomposable inequality measures relevant with the unit consistency property is characterized to capture the *leftist* point of view.

Proposition 3.2. *An absolute inequality measure* $J(\mathbf{x})$ *satisfies* (CN), (NM), (AN), (WDEC), (UC), *and* (PP) *if, and only if,*

$$J(\mathbf{x}) = c \cdot \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j|^{\alpha}, \ c > 0, \ \alpha \neq 0.$$

Proof. See the Appendix. The logical independence of the characterizing axioms is also exposed in Appendix. □

This last expression corresponds to an extension of the Gini mean difference, that is, the absolute α-Gini measures. Ebert (2010) axiomatically derives those measures from (DEC), whereas we derive them by (WDEC) weaker than (DEC). The α parametrization is convenient in particular for the implementation of redistributive actions (rich-to-poor transfers), see for instance Chameni (2006, 2013). The larger is α, the more the measures are sensitive to transfers occurring at the tails of the income distribution, see Mornet, Zoli *et al.* (2013).

Besides, since there exists a link between absolute and relative inequality measures, the latter may be directly deduced from the former in order to capture the *rightist* point of view.[3]

Corollary 3.2. *A relative inequality measure* $I(\mathbf{x})$ *that satisfies* (CN), (NM), (AN), (WDEC), (UC), *and* (PP) *is given by,*

$$I(\mathbf{x}) = c \cdot \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{|x_i - x_j|^{\alpha}}{\mu^{\alpha}(\mathbf{x})}, \ c > 0, \ \alpha \neq 0.$$

Proof. See the Appendix. □

Those measures are the relative α-Gini measures that respect ($\widehat{\text{DEC}}(\varepsilon)$).

[3]By Lemma 3.1, a unit consistent index is homogeneous of degree α, then (SI) inequality measures are included in the family of (UC) inequality measures. Therefore, invoking the rightist point of view (SI) will restrict the unit consistent inequality measures to homogeneous functions of degree 0.

4 Concluding Remarks

This note introduces a characterization of the α-Gini measures being either absolute or relative. Instead of dealing with separate axioms of weak decomposition, as in Ebert (2010), a general axiom of weak decomposition has been used to derive absolute and relative measures. Those "leftist" and "rightist" views are consistent with some well-known principles of transfers such as Pigou-Dalton ($\alpha \geqslant 1$), see Chameni (2006), and the principle of concentration ($\alpha > 0$), see Ebert (2010). The α-Gini measures are also relevant with the strong principle of diminishing transfers ($\alpha > 2$), either at the top of the income distribution or at the bottom, see Mornet, Zoli *et al.* (2013).

Appendix

Lemma 3.1 *If an inequality measure* $I(\mathbf{x})$ *satisfies* (CN)*,* (NM)*,* (PP)*,* (WDEC) *and* (UC)*, then:*

$$I(\theta\mathbf{x}) = \theta^{\alpha} I(\mathbf{x}),\ \forall\theta \in \mathbb{R}_{++},\ \forall\alpha \in \mathbb{R}.$$

Proof. <u>STEP 1 :</u> $f(\theta, \cdot)$ is linear.
Consider an income distribution $\mathbf{x} := (\mathbf{x}^1, \mathbf{x}^2) \in \mathscr{X}_+$ of size n subdivided into 2 subgroups, such that $n := (n_1 + n_2)$ and $n_g \geqslant 1$ for $g = 1, 2$. Let $I(\mathbf{x})$ be a continuous (CN) and weakly decomposable (WDEC) inequality measure, thus:

$$\begin{aligned} I(\mathbf{x}) &= \alpha(n_1, n)\xi(\mu(\mathbf{x}^1), \mu(\mathbf{x})) \cdot I(\mathbf{x}^1) + \alpha(n_2, n)\xi(\mu(\mathbf{x}^2), \mu(\mathbf{x})) \cdot I(\mathbf{x}^2) \\ &+\ \beta(2, n) \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \xi(\mu(x_i^1, x_j^2), \mu(\mathbf{x})) \cdot I(x_i^1, x_j^2). \end{aligned}$$

Now assume $\mathbf{x}^1 = (x_1^1, x_1^1, \ldots, x_1^1)$ and $\mathbf{x}^2 = (x_2^2, x_2^2, \ldots, x_2^2)$. From (NM) it follows that $I(\mathbf{x}^1) = I(\mathbf{x}^2) = 0$, then the previous equation becomes:

$$\begin{aligned} I(\mathbf{x}) &= \beta(2, n) \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \xi(\mu(x_i^1, x_j^2), \mu(\mathbf{x})) \cdot I(x_i^1, x_j^2) \\ &= n_1 n_2 \beta(2, n) \xi(\mu(x_1^1, x_2^2), \mu(\mathbf{x})) \cdot I(x_1^1, x_2^2). \end{aligned} \tag{A1}$$

Assuming that the income distribution $\mathbf{x}$ is multiplied by $\theta \in \mathbb{R}_{++}$, since $\xi(\mu(\theta x_1^1, \theta x_2^2), \mu(\theta\mathbf{x})) = \xi(\theta\mu(x_1^1, x_2^2), \theta\mu(\mathbf{x}))$, we have:

$$I(\theta\mathbf{x}) = n_1 n_2 \beta(2, n) \xi(\theta\mu(x_1^1, x_2^2), \theta\mu(\mathbf{x})) \cdot I(\theta x_1^1, \theta x_2^2).$$

Let $n_1 = n_2$, then by (PP):

$$I(\theta\mathbf{x}^1, \theta\mathbf{x}^2) \overset{(\mathrm{PP})}{=} I(\theta x_1^1, \theta x_2^2) = n_1 n_2 \beta(2, n) \xi(\theta\mu(x_1^1, x_2^2), \theta\mu(\mathbf{x})) \cdot I(\theta x_1^1, \theta x_2^2).$$

Thus,

$$\xi(\theta\mu(x_1^1, x_2^2), \theta\mu(\mathbf{x})) = \frac{1}{n_1 n_2 \beta(2, n)},$$

i.e., $\xi(\cdot,\cdot)$ is independent of θ.

Now, let us invoke unit consistency (UC). Whenever $I(\mathbf{x})$ is unit consistent, so does the inequality index $I(\mathbf{x}^g)$ of any given subgroup g, that is, $I(\mathbf{x}^g) < I(\mathbf{y}^g) \Longrightarrow I(\theta\mathbf{x}^g) < I(\theta\mathbf{y}^g)$, for all $\mathbf{x}^g, \mathbf{y}^g \in \mathscr{X}_+$. Suppose, without loss of generality, that $n(\mathbf{x}^g) = n(\mathbf{y}^g) = n_g$ and $\mu(\mathbf{x}^g) = \mu(\mathbf{y}^g) = \mu_g$, then:

$$
\begin{aligned}
&[\alpha(n_g,n)\xi(\mu_g,\mu(\mathbf{x}))I(\mathbf{x}^g) < \alpha(n_g,n)\xi(\mu_g,\mu(\mathbf{x}))I(\mathbf{y}^g)]\\
\Longrightarrow\ &[\alpha(n_g,n)\xi(\theta\mu_g,\theta\mu(\mathbf{x}))I(\theta\mathbf{x}^g) < \alpha(n_g,n)\xi(\theta\mu_g,\theta\mu(\mathbf{x}))I(\theta\mathbf{y}^g)].
\end{aligned}
$$

By Proposition 3.1 (Zheng, 2007), for all $\mathbf{x}^g \in \mathscr{X}_+$ and $\theta \in \mathbb{R}_{++}$, there exists a continuous and increasing function $f(\theta,\cdot)$ in both arguments such that:

$$\alpha(n_g,n)\xi(\theta\mu_g,\theta\mu(\mathbf{x}))I(\theta\mathbf{x}^g) = f[\theta, \alpha(n_g,n)\xi(\mu_g,\mu(\mathbf{x}))I(\mathbf{x}^g)].$$

Because $\xi(\cdot,\cdot)$ is independent of θ,

$$\alpha(n_g,n)\xi(\mu_g,\mu(\mathbf{x}))I(\theta\mathbf{x}^g) = f[\theta, \alpha(n_g,n)\xi(\mu_g,\mu(\mathbf{x}))I(\mathbf{x}^g)],$$

that is, by (UC),

$$\alpha(n_g,n)\xi(\mu_g,\mu(\mathbf{x}))f[\theta, I(\mathbf{x}^g)] = f[\theta, \alpha(n_g,n)\xi(\mu_g,\mu(\mathbf{x}))I(\mathbf{x}^g)].$$

Setting $\tilde{n} := n$, $\tilde{n}_g := n_g$, $\tilde{\mu}_g := \mu_g$ and $\tilde{\mu} := \mu(\mathbf{x})$ and recalling that $f[\theta,\cdot] = f_\theta[\cdot]$ yields:

$$af_\theta(t) = f_\theta(at),$$

for some positive constant $a := \alpha(\tilde{n}_g,\tilde{n})\xi(\tilde{\mu}_g,\tilde{\mu})$. Therefore, f_θ is linear:

$$f_\theta(t) = at,\ \forall a \in \mathbb{R}_{++}.$$

Since $f_\theta(\cdot) = f(\theta,\cdot)$, then for any given $\mathbf{x} \in \mathscr{X}_+$ and $\theta \in \mathbb{R}_{++}$:

$$f(\theta, I(\mathbf{x})) = a(\theta)I(\mathbf{x}), \tag{A4}$$

for some continuous and real function $a(\cdot)$.

<u>STEP 2 :</u> Homogeneity of $I(\mathbf{x})$.
Now consider an income distribution $\mathbf{x} = (\mathbf{x}^1,\mathbf{x}^2) \in \mathscr{X}_+$ such that $\mathbf{x}^1 = (x_1^1, x_2^1, \ldots, x_{n_1}^1)$ and $\mathbf{x}^2 = (x_1^2, x_2^2, \ldots, x_{n_2}^2)$. From (A4), (WDEC) and using the fact that $\xi(\cdot,\cdot)$ is independent of $\theta \in \mathbb{R}_{++}$, we get:

$$
\begin{aligned}
I(\theta\mathbf{x}) &= \alpha(n_1,n)\xi(\mu(\mathbf{x}^1),\mu(\mathbf{x}))\cdot f_\theta[I(\mathbf{x}^1)] +\ \alpha(n_2,n)\xi(\mu(\mathbf{x}^2),\mu(\mathbf{x}))\cdot f_\theta[I(\mathbf{x}^2)]\\
&\quad +\ \beta(2,n)\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\xi(\mu(x_i^1,x_j^2),\mu(\mathbf{x}))\cdot f_\theta[I(x_i^1,x_j^2)]\\
&= a(\theta)\Big[\alpha(n_1,n)\xi(\mu(\mathbf{x}^1),\mu(\mathbf{x}))\cdot I(\mathbf{x}^1) + \alpha(n_2,n)\xi(\mu(\mathbf{x}^2),\mu(\mathbf{x}))\cdot I(\mathbf{x}^2)\\
&\quad +\ \beta(2,n)\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\xi(\mu(x_i^1,x_j^2),\mu(\mathbf{x}))\cdot I(x_i^1,x_j^2)\Big]\\
&= a(\theta)I(\mathbf{x}).
\end{aligned} \tag{A5}
$$

Then, for $\theta, \gamma \in \mathbb{R}_{++}$, it follows as in Zheng (2007) that:

$$f[\theta\gamma, I(\mathbf{x})] = a(\theta\gamma)I(\mathbf{x}),$$

or

$$f[\theta, I(\gamma\mathbf{x})] = a(\theta)f[\gamma, I(\mathbf{x})] = a(\theta)a(\gamma)I(\mathbf{x}).$$

Equating the two above equations yields:

$$a(\theta\gamma) = a(\theta)a(\gamma).$$

The solution is $a(\theta) = \theta^\alpha$, for all $\alpha \in \mathbb{R}$. Then:

$$I(\theta\mathbf{x}) = \theta^\alpha I(\mathbf{x}),$$

for all $\mathbf{x} \in \mathscr{X}_+$ and α a real constant. Therefore $I(\mathbf{x})$ is homogeneous of degree α. □

Theorem 3.1 *An inequality measure $I(\mathbf{x})$ satisfies* (CN)*,* (NM)*,* (AN)*,* (WDEC)*,* (UC)*,* (PP) *if, and only if,*

$$I(\mathbf{x}) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} h\left(\frac{\mu(x_i, x_j)}{\mu(\mathbf{x})}\right) I(x_i, x_j),$$

where $I(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is homogeneous of degree $\alpha \in \mathbb{R}$, symmetric and continuous such that $I(z, z) = 0$, and where $h : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ is continuous.

Proof. ($\Longrightarrow$) Let us prove the necessity part of the theorem by invoking (CN), (NM), (AN), (WDEC), (UC), and (PP).

The expression of (WDEC) may be generalized to G subgroups.[4]

Weak decomposition (WDEC). *For all $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^G) \in \mathscr{X}_+$ of size $n = (n_1 + \cdots + n_G)$ where $n_g \geqslant 1$ and $g \in \{1, \ldots, G\}$ such that $G \geqslant 2$, an inequality measure is weakly decomposable, if there exist positive weighting functions $\alpha(n_g, n)$, $\beta(2, n)$ and $\xi(\mu(\mathbf{x}^g), \mu(\mathbf{x}))$ such that:*

$$I(\mathbf{x}) = \sum_{g=1}^{G}\alpha(n_g, n)I(\mathbf{x}^g) + \sum_{k=2}^{G}\sum_{\ell=1}^{k-1}\sum_{i=1}^{n_k}\sum_{j=1}^{n_\ell}\beta(2, n)\xi(\mu(x_i^k, x_j^\ell), \mu(\mathbf{x}))I(x_i^k, x_j^\ell).$$

STEP 1 : We show that the inequality measure $I(\mathbf{x})$ can be rewritten with the weights $\beta(2, n)$ and $\xi(\mu(x_i, x_j), \mu(\mathbf{x}))$ only. We consider that each individual constitutes one

[4] The proof of such a result is based on a reasoning by induction which can be provided upon request.

subgroup in the overall distribution $\mathbf{x}$, that is, $n = G = \sum_g n_g$ with $n_g = 1$ for all $g \in \{1, 2, \ldots, G\}$. Then according to (WDEC):

$$
\begin{aligned}
I(\mathbf{x}) =& \alpha(1,n)I(x_1) + \cdots + \alpha(1,n)I(x_n) + \\
& \sum_{k=2}^{G} \beta(2,n)\xi(\mu(x_k,x_1),\mu(\mathbf{x}))I(x_k,x_1) + \\
& \sum_{k=3}^{G} \beta(2,n)\xi(\mu(x_k,x_2),\mu(\mathbf{x}))I(x_k,x_2) + \\
& + \cdots + \\
& + \beta(2,n)\xi(\mu(x_n,x_{n-1}),\mu(\mathbf{x}))I(x_n,x_{n-1}).
\end{aligned} \tag{A6}
$$

From (NM), $I(x_1) = I(x_2) = \ldots = I(x_n) = 0$. According to (AN), $I(x_i,x_j) = 1/2 \cdot I(x_i,x_j) + 1/2 \cdot I(x_j,x_i)$, then (A6) becomes:

$$
I(\mathbf{x}) = \frac{\beta(2,n)}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \xi(\mu(x_i,x_j),\mu(\mathbf{x}))I(x_i,x_j). \tag{A7}
$$

<u>STEP 2 :</u> We show that the inequality measure depends on the scalar n^2. Let us assume that the income distribution $\mathbf{x} \in \mathscr{X}_+$ is replicated k times, such that $\mathbf{x}^{[k]} = ((x_1, \ldots, x_n), \ldots, (x_1, \ldots, x_n,)) \in \mathscr{X}_+$. Expression (A7) yields:

$$
\begin{aligned}
I(\mathbf{x}^{[k]}) &= \frac{\beta(2,kn)}{2} \sum_{\ell=1}^{k\cdot n} \sum_{m=1}^{k\cdot n} \xi(\mu(x_\ell,x_m),\mu(\mathbf{x}))I(x_\ell,x_m) \\
&= \frac{\beta(2,kn)}{2} k^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \xi(\mu(x_i,x_j),\mu(\mathbf{x}))I(x_i,x_j).
\end{aligned}
$$

Invoking the principle of replication invariance (PP), $I(\mathbf{x}^{[k]}) = I(\mathbf{x})$, yields:

$$
k^2 \frac{\beta(2,kn)}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \xi(\mu(x_i,x_j),\mu(\mathbf{x}))I(x_i,x_j) = \frac{\beta(2,n)}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \xi(\mu(x_i,x_j),\mu(\mathbf{x}))I(x_i,x_j),
$$

thus,

$$
\frac{1}{2}\left[k^2\beta(2,kn) - \beta(2,n)\right] \sum_{i=1}^{n} \sum_{j=1}^{n} \xi(\mu(x_i,x_j),\mu(\mathbf{x}))I(x_i,x_j) = 0.
$$

It follows:

$$
\beta(2,kn) = \frac{1}{k^2}\beta(2,n) \quad \text{or alternatively} \quad \beta(2,kn) = \frac{1}{n^2}\beta(2,k).
$$

Setting $c = \beta(2,1)$, we get for $n \geqslant 2$:

$$\beta(2,n) = \frac{c}{n^2}, \quad \text{with } c \text{ a real constant.}$$

In the particular case where $n = 2$ such that $\mathbf{x} = (x_i, x_j)$ with $x_i \neq x_j$, we get from (A7):

$$I(\mathbf{x}) = \frac{\beta(2,2)}{2}[I(x_i,x_j) + I(x_j,x_i)].$$

Recalling that from (AN) $I(x_i,x_j) = 1/2 \cdot I(x_i,x_j) + 1/2 \cdot I(x_j,x_i)$, we deduce that $\beta(2,2) = 1$. Since we have shown that $\beta(2,n) = \frac{c}{n^2}$, thus for $n = 2$, we have:

$$\beta(2,2) = \frac{c}{2^2} = 1 \implies c = 4.$$

Hence:

$$\beta(2,n) = \frac{4}{n^2}.$$

We finally have:

$$I(\mathbf{x}) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi(\mu(x_i,x_j),\mu(\mathbf{x}))I(x_i,x_j),$$

such that $I(\cdot,\cdot) : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ with $I(z,z) = 0$ by (NM), and where $I(\cdot,\cdot)$ is continuous and symmetric by (CN) and (AN).

<u>STEP 3 :</u> Homogeneity. From Lemma 3.1 we have that $\xi(\mu(\mathbf{x}^g),\mu(\mathbf{x}))$ is homogeneous of degree 0 since $I(\cdot,\cdot)$ is homogeneous of degree α. Then, from Aczél (1966), p. 229:

$$\xi(\mu(\mathbf{x}^g),\mu(\mathbf{x})) = \xi\left(\frac{\mu(\mathbf{x}^g)}{\mu(\mathbf{x})},1\right) =: h\left(\frac{\mu(\mathbf{x}^g)}{\mu(\mathbf{x})}\right),$$

where $h : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ since by definition $\mu(\mathbf{x}^g) > 0$ for all $g = 1,\dots,G$ such that h is continuous by (CN).

($\Longleftarrow$) The sufficiency part of the theorem is left to the reader. □

Proposition 3.2 *An absolute inequality measure* $J(\mathbf{x})$ *satisfies* (CN), (NM), (AN), (WDEC), (UC), *and* (PP) *if, and only if,*

$$J(\mathbf{x}) = c \cdot \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|^\alpha, \; c > 0, \; \alpha \neq 0.$$

Proof. ($\Longrightarrow$) Assume that $J(\mathbf{x})$ satisfies (CN), (NM), (AN), (WDEC), (UC), (PP) and (INV) since it is absolute. From Theorem 3.1:

$$J(\mathbf{x}) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} h\left(\frac{\mu(x_i,x_j)}{\mu(\mathbf{x})}\right) J(x_i,x_j). \tag{A8}$$

Let $\varepsilon \in \mathbb{R}_+$ such that $\varepsilon \neq \mu(\mathbf{x})$, then by (INV):

$$\begin{aligned} J(\mathbf{x}) = & \; J(\mathbf{x} - \varepsilon \mathbf{1}_{n(\mathbf{x})}) \\ = & \; \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} h\left(\frac{\mu(x_i, x_j) - \varepsilon}{\mu(\mathbf{x}) - \varepsilon} \right) J(x_i - \varepsilon, x_j - \varepsilon). \end{aligned} \tag{A9}$$

By Lemma 3.1, $J(x_i, x_j)$ is homogeneous of degree $\alpha \in \mathbb{R}$. Note that $J(x_i, x_j)$ inherits from translation invariance (INV). Setting $\varepsilon = -x_j$ (or $\varepsilon = -x_i$), it follows from (INV) and (AN) that:

$$J(x_i, x_j) = J(x_i - x_j, 0) = J(0, x_j - x_i) = |x_i - x_j|^{\alpha} J(1, 0).$$

Setting $a = J(1, 0) > 0$, with $\alpha \neq 0$ to get a well defined inequality measure, then equating equations (A8) and (A9):

$$a \, \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} J(x_i, x_j) \left[h\left(\frac{\mu(x_i, x_j)}{\mu(\mathbf{x})} \right) - h\left(\frac{\mu(x_i, x_j) - \varepsilon}{\mu(\mathbf{x}) - \varepsilon} \right) \right] = 0.$$

From Theorem 3.1 $h(u/v) = F(u, v)$ such that F is homogeneous of degree zero. Then, setting $n = 2$, it follows from the previous expression that:

$$F(\mu(x_i, x_j), \mu(\mathbf{x})) = F(\mu(x_i, x_j) - \varepsilon, \mu(\mathbf{x}) - \varepsilon).$$

The function $F : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is homogeneous of degree zero (SI) and translatable (INV) if, and only if, $F(u, v) = b$, with b any given real. Thus, setting $b > 0$ in order to get a well defined inequality measure and $c = a \cdot b$, we get:

$$J(\mathbf{x}) = c \cdot \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j|^{\alpha}, \; \alpha \neq 0, \; c > 0.$$

($\Longleftarrow$) The sufficiency part is left to the reader.
(Independence) Let us check the independence of the axioms. We must find indices

that respect all axioms but one (continuity is left to the reader):

$$J_1(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{|x_i - x_j|^\alpha + c}{n^2},\ \forall\alpha \neq 0,\ \forall c > 0 \qquad \text{(Not NM)}$$

$$J_2(\mathbf{x}) = \sum_{i=1}^{n} \frac{(x_i^2 - \mu^2(\mathbf{x}))}{\alpha(1-\alpha)n},\ \forall\alpha \neq 0 \qquad \text{(Not WDEC)}$$

$$J_3(\mathbf{x}) = \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} |x_i - x_j|^\alpha,\ \forall\alpha \neq 0 \qquad \text{(Not PP)}$$

$$J_4(\mathbf{x}) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} (x_i - x_j)^\alpha,\ \alpha \in \{1,3,5,\ldots\} \qquad \text{(Not AN)}$$

$$J_5(\mathbf{x}) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} |x_i^\alpha - x_j^\alpha|,\ \forall\alpha \neq 0 \qquad \text{(Not INV)}$$

$$J_6(\mathbf{x}) = \frac{2}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n} \left[e^{|x_j - x_i|} - 1\right] \qquad \text{(Not UC)}$$

Note that $J_2(\mathbf{x})$ is a particular case of Zheng's (2007) entropy index being additively decomposable and unit consistent. □

Corollary 3.2 *A relative inequality measure $I(\mathbf{x})$ that satisfies* (CN)*,* (NM)*,* (AN)*,* (WDEC)*,* (UC)*, and* (PP) *is given by*

$$I(\mathbf{x}) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{c\ |x_i - x_j|^\alpha}{\mu^\alpha(\mathbf{x})},\ c > 0,\ \alpha \neq 0.$$

Proof. From Theorem 3.1:

$$I(\mathbf{x}) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} h\left(\frac{\mu(x_i, x_j)}{\mu(\mathbf{x})}\right) I(x_i, x_j).$$

Since the measure is relative and because it is issued from $J(\mathbf{x})$, then:

$$I(\mathbf{x}) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} h\left(\frac{\mu(x_i, x_j)}{\mu(\mathbf{x})}\right) \frac{J(x_i, x_j)}{f(\mu(x_i, x_j))}.$$

From Proposition 3.2, $J(x_i, x_j) = \frac{c \cdot 2}{4}|x_i - x_j|^\alpha$, it follows that $f(\mu(x_i, x_j))$ must be homogeneous of degree α up to a constant, *i.e.*, $f(\mu(x_i, x_j)) = d \cdot \mu^\alpha(x_i, x_j)$ such that $d > 0$. Thus,

$$I(\mathbf{x}) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} h\left(\frac{\mu(x_i, x_j)}{\mu(\mathbf{x})}\right) \frac{\frac{c \cdot 2}{4}|x_i - x_j|^\alpha}{d \cdot \mu^\alpha(x_i, x_j)},\ \forall\alpha \neq 0.$$

Taking a distribution $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2) = (x_1^1, \ldots, x_1^1, x_2^2, \ldots, x_2^2)$ with n_1 terms x_1^1 and n_2 terms x_2^2 yields by (NM) and (WDEC):

$$I(\mathbf{x}) = \frac{c}{n^2}\; n_1 n_2\; h\left(\frac{\mu(x_1, x_2)}{\mu(\mathbf{x})}\right) \frac{|x_1 - x_2|^\alpha}{d \cdot \mu^\alpha(x_1, x_2)}. \tag{A10}$$

Since $I(\mathbf{x}) = J(\mathbf{x})/f(\mu(\mathbf{x}))$, we get from Proposition 3.2:

$$I(\mathbf{x}) = c\; \frac{2}{n^2}\; n_1 n_2\; \frac{|x_1 - x_2|^\alpha}{d \cdot \mu^\alpha(\mathbf{x})}. \tag{A11}$$

Equating (A10) and (A11) entails:

$$h\left(\frac{\mu(x_1, x_2)}{\mu(\mathbf{x})}\right) = 2\; \frac{\mu^\alpha(x_1, x_2)}{\mu^\alpha(\mathbf{x})}.$$

Hence, in general,

$$h\left(\frac{\mu(x_i, x_j)}{\mu(\mathbf{x})}\right) = 2\; \frac{\mu^\alpha(x_i, x_j)}{\mu^\alpha(\mathbf{x})},\ \forall \alpha \neq 0,$$

and this ends the proof. □

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