On the zero-dynamics of a class hybrid LTI systems: a geometric approach

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***Abstract*— The paper deals with the geometric characterization of the zero-dynamics for linear time-invariant systems with aperiodic time-driven jumps. As the intuition suggests, it is given by the restriction of the feedback dynamics to the largest subspace over which the trajectories are constrained to ensure zero output. Such a dynamics is characterized by a subset of the flowing zeros and a subset of the zeros which can be fictitiously associated to the jumping dynamics.**

***Index Terms*— Hybrid systems; Algebraic/geometric methods; Linear systems.**

I. Introduction

Nowadays, growing attention is devoted toward dynamics characterized by the interaction of both continuous and discrete-time behaviors. Those kind of systems are referred to as *hybrid systems* and typically described by set inclusions and the interconnection of suitable discrete and continuous-time models characterizing the jumping and flowing evolutions which are governing, in a combined way, the overall dynamics [1]–[6]. Among these, hybrid systems with time-driven state jumps (or impulsive systems) are of paramount importance as they allow to fully describe, for example, cyber-physical systems or dynamical analog systems interconnected to digital devices (e.g., sampled-data systems) by simultaneously catching the heterogeneous behaviors acting over the overall system. When jumps are periodic in time, several works have been developed to address important control problems such as, for example, hybrid regulation [7], [8]. In those contributions, the notion of zero-dynamics has been shown to be, as in purely continuous or discrete-time systems, a fundamental issue that cannot be discarded [9]–[11]. Still, a complete characterization of this behavior has not been provided so that its analysis is typically lead (in a conservative way) to the corresponding purely continuous or discrete-time counterpart. The work in [12] represents a first attempt toward the characterization of zeros of a hybrid system via a suitably defined hybrid transfer function in a hybrid frequency domain. However, such an approach is quite involved and suffers from generalizability to a wider context as direct integration of the trajectories is needed for the definition of the zeros and thus motivating the periodic context. Moreover, because the transfer function consists of four components that are parametrized by two complex variables (for the flow and jump behaviors), explicitly exhibiting the zeros might not be easy in general.

In this paper, we address the problem of defining the zero-dynamics for linear hybrid systems with *aperiodic* time-driven jumps in the geometric framework developed in [13], [14] for general results and recently extended to this hybrid context in, for example, [15], [16]. In doing so, no knowledge of the jumping instants is assumed. In particular, instead of focusing on the definition of the zeros, we investigate the concept of zero-dynamics subspace allowing a revealing study of the hybrid zero-dynamics as defined by a suitable combination of the zero-dynamics corresponding to the flowing and jumping behaviors. Such an approach relies upon the definition of a suitable controlled-invariant subspace (the hybrid zero-subspace) that is contained into the null-space of the output evolution and is, at the same time, invariant under the flow and jump dynamics. Connections with the hybrid zeros of the system are also established as particular subsets of couples of the flowing and *fictious* jumping zeros, associated to flowing and jumping dynamics when considered as purely continuous and discrete-time systems. In our context, the feedback laws inducing the zero-dynamics is independent upon the jumping period sequence and requires no integration of the trajectories contrarily to what proposed in [12] or in the context of invariance at large in [15].

The remaining of the paper is organized as follows: in Section II the class of systems under study is defined and the problem is settled. In Section III the hybrid zero-dynamics is characterized based on the definition of the hybrid zero-subspace which is controlled invariant under the hybrid system. Insights on the hybrid zero-dynamics are investigated in Section IV where the notion of hybrid zeros is also set. Some examples illustrate the results in Section V whereas conclusions and future perspectives are in Section VI.

II. Preliminaries and the class of systems under study

A. Notations

$\mathrm{Mat}_{\mathbb{R}}(n,m)$ defines the set of $n\times m$ matrices with real entries. Given a square matrix $A\in\mathrm{Mat}_{\mathbb{R}}(n,n)$ we denote by $|A|$ the determinant of A. The notation $\sigma(A)=\{\lambda\in\mathbb{C}\text{ s.t. }|A-\lambda I|=0\}$ defines the spectrum of A. Given a matrix $B\in\mathrm{Mat}_{\mathbb{R}}(n,m)$, we say that $V=\mathrm{span}\{s_1,\dots,s_p\}\subset\mathbb{R}^n$ is (A,B)-invariant if, for all $s_i\in V$, $As_i\in V+\mathrm{Im}B$ or, in short, $AV\subset V+\mathrm{Im}B$. Moreover, we say that F is the

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friend of a controlled invariant subspace V if $(A+BF)V \subset V$. We define the ordered set $\Pi(A,B,C) := \{V \subset \mathbb{R}^n$ s.t. $AV \subset V + \mathrm{Im}B$ and $V \subset \ker C\}$. With a slight abuse of notations, $\mathbb{R}_n^\perp$ denotes the dual space to $\mathbb{R}^n$ while ΩA denotes the subspace of $\mathbb{R}_n^\perp$ generated by the rows of ωA for $\omega \in \Omega$. Given a subspace $V \subset \mathbb{R}^n$ then $\Omega := V^\perp \subset \mathbb{R}_n^\perp$ with $\omega \in \Omega$ if, and only if, for all $s_i \in V$, $\omega s_i = 0$. I and $\mathbf{0}$ denote respectively the identity and zero matrices of suitable dimensions.

B. Hybrid systems under aperiodic jumps

Introduce the hybrid time domain $\mathscr{T} = \cup_{k=0}^{\infty}[t_k, t_{k+1}] \times \{k\}$ with $t_k - t_{k-1} := \delta_k$ for all $k \in \mathbb{N}$. Accordingly, consider the class of hybrid systems given by

$$x^+ = Fx + Gv \tag{1a}$$

$$\dot{x} = Ax + Bu \tag{1b}$$

$$y = Cx \tag{1c}$$

or, more explicitly, for $t \in [t_k, t_{k+1}[$

$$x(t_k,k) = Fx(t_k,k-1) + Gv(k), \quad \dot{x}(t,k) = Ax(t,k) + Bu(t,k)$$
$$y(t,k) = Cx(t,k)$$

with $x \in \mathbb{R}^n$, $u, v, y \in \mathbb{R}^p$ and

$$x(t_k,k-1) = e^{A\delta_k}x(t_{k-1},k-1) + \int_{t_{k-1}}^{t_k} e^{(t_k-s)A}Bu(s,k-1)\mathrm{d}s$$

for $x(t_0,-1) = x(t_0,0) = x_0$, $x(t_k,k-1) = \lim_{t\to t_k^-} x(t,k-1)$. In what follows, we denote $C^\top = \begin{pmatrix} c_1^\top & \dots & c_p^\top \end{pmatrix}$, $B = \begin{pmatrix} b_1 & \dots & b_p \end{pmatrix}$. We assume that the time domain is not known in the sense that measures (or estimates) of the jumping instants are not available. We underline that the class of system (1) under study is also referred to as impulsive systems (e.g., see [16]).

III. The hybrid zero-dynamics

The zero-dynamics of (1) is the residual dynamics the system evolves with when, for a suitable $x_0 \in \mathbb{R}^n$ such that $y_0 = 0$ and a suitable feedback law, one has $y(t,k) = 0$ for all $(t,k) \in \mathscr{T}$. From a geometric point of view [14], it is the dynamics governing the evolutions over the largest feedback-unobservable subspace. This is the point of view we shall adopt, so that the following definition of hybrid zero-dynamics will be assumed:

Definition 3.1: The zero-dynamics of (1) is the residual dynamics the system evolves with when the trajectories are constrained onto the *zero-dynamics subspace* (or, for brevity, *zero subspace*) $V_h^* \subset \mathbb{R}^n$ that is the largest subspace made unobservable under state feedback.

Accordingly, we shall characterize the zero-dynamics through the definition of the zero subspace V_h^*. As a byproduct, this will lead to a natural interpretation of the zeros of some transfer function associated to (1) as defined, for the periodical case, in [12].

To this end, when considering (1a) and (1b) as purely continuous and discrete-time dynamics with corresponding output (1c) one can define the subspaces $V_c \subset \mathbb{R}^n$ and $V_d \subset \mathbb{R}^n$ being the largest invariant subspaces that are, respectively, (A,B)-invariant and (F,G)-invariant and contained in $\ker C$; namely, V_c and V_d verify, separately,

$$AV_c \subset V_c + \mathrm{Im}B \tag{3a}$$

$$FV_d \subset V_d + \mathrm{Im}G. \tag{3b}$$

with $\dim\{V_c\} = n - r_c$ and $\dim\{V_d\} = n - r_d$ for some positive real constants r_c, r_d.

Remark 3.1: When the continuous and discrete-time dynamics associated to (1a) and (1b) admit well-defined vector relative degrees $\mathbf{r}_c = (r_c^1 \dots r_c^p)$ and $\mathbf{r}_d = (r_d^1 \dots r_d^p)$ [17, Chapter 5] with $r_c = \sum_{i=1}^p r_c^i$ and $r_d = \sum_{i=1}^p r_d^i$ [17], V_c and V_d can be further specified as

$$V_c = \cap_{i=1}^p \ker \begin{pmatrix} c_i \\ c_iA \\ \vdots \\ c_iA^{r_c^i-1} \end{pmatrix}, \quad V_d = \cap_{i=1}^p \ker \begin{pmatrix} c_i \\ c_iF \\ \vdots \\ c_iF^{r_d^i-1} \end{pmatrix}.$$

With this in mind, the zero-subspace V_h^* is thus the largest subspace that is contained in the null-space of C and, at the same time, (A,B) and (F,G)-invariant. By definition of V_c and V_d, then one has that necessarily $V_h^* \subset V_c \cap V_d \subset \ker C$ so that the following definition can be given.

Theorem 3.1: Consider the hybrid system (1) and the subspaces V_c and V_d in (3). Denote $V_{\mathrm{int}} := V_c \cap V_d$ and define V_h^* as the largest subspace contained in $V_{\mathrm{int}} \subset \ker C$ verifying

$$AV_h^* \subset V_h^* + \mathrm{Im}B \tag{4a}$$

$$FV_h^* \subset V_h^* + \mathrm{Im}G \tag{4b}$$

with $\dim\{V_h^*\} = n - r_h$ for some $r_h \in \mathbb{N}$. Then, V_h^* defines the zero-subspace for the hybrid system (1); namely, there exist K^* and H^* (the *friends of* V_h^*) verifying

$$(A+BK^*)V_h^* \subset V_h^* \quad \text{and} \quad (F+GH^*)V_h^* \subset V_h^* \tag{5}$$

so that, for all $x_0 \in V_h^*$, $y(t,k) = 0$ for all $(t,k) \in \mathscr{T}$.

The proof of Theorem 3.1 is quite straightforward. V_h^* defines the zero-subspace associated to (1) as it is the largest controlled-invariant under both (1a) and (1b) and contained in $\ker C$. In addition, when measures of the jumping times are available, double invariance (4) is no longer necessary for the definition of the zero-subspace. In that case, the necessary and sufficient condition, together with the construction algorithm, has been provided in [15] for controlled invariance of impulsive systems at large. In that case, the friends of V_h^* under (A,B) and (F,G) depend explicitly on δ_k for all $k \geq 0$.

As a consequence of Theorem 3.1, the zero-dynamics of the hybrid system (1) can be defined as its restriction onto the zero subspace V_h^*, as pointed out in Section III-A below. This results from common characteristics of the underlying geometry of the two possibly different zero-dynamics of (1a) and (1b). This aspect is revealed by the proposed state-space approach as it is not evident from the frequency domain characterization proposed in [12].

Remark 3.2: When $p = 1$, that is (1) is SISO, then r_h can be interpreted as the hybrid relative degree of (1).

Remark 3.3: When $p=1$ and $V_h^*=\{0\}$ then $r_h=n$ and there is no hybrid zero-dynamics, even if both V_c and V_d are not $\{0\}$. On the other side, whenever $r_c=r_d=1$ then $V_h^*=\ker C$ regardless A and F and, thus, the internal structure of the flow and hybrid dynamics.

In what follows, such a concept will be further clarified by linking the notion of hybrid zeros of (1) to the zeros of the single transfer functions associated to (1a) and (1b). Before doing this, an algorithm for computing V_h^* is given by extending the one in [13], [14] to this context based on the ones in [15], [18], [19].

A. On the computation of V_h^*

Let $\Pi_h(A,B,F,G,C):=\Pi(A,B,C)\cap\Pi(F,G,C)$ that is the set of subspaces being, at the same time, (A,B) and (F,G)-invariant and that are contained in $V_{\text{int}}:=V_c\cap V_d\subset\ker C$. The set $\Pi_h(A,B,F,G,C)$ is closed under subspace addition and ordered and, thus, possesses a supremal element $V_h^*:=\sup\Pi_h(A,B,F,G,C)$ [14, Chapter 4] that can be deduced starting from V_c and V_d being the maximum elements of, respectively, $\Pi(A,B,C)$ and $\Pi(F,G,C)$ and verifying (3). To this end, the following Lemma is thus useful to characterize all $V\in\Pi_h(A,B,F,G,C)$.

Lemma 3.1: Consider the matrices A,B,F,G,C defining the hybrid dynamics (1) and $\Pi_h(A,B,F,G,C):=\Pi(A,B,C)\cap\Pi(F,G,C)$. Let $V\subset\mathbb{R}^n$ and $\Omega:=V^\perp$. $V\in\Pi_h(A,B,F,G,C)$ if and only if

$$(\Omega\cap(\text{Im}B)^\perp)A+(\Omega\cap(\text{Im}G)^\perp)F\subset\Omega. \tag{6}$$

Proof: : One needs to show that (6) is equivalent to (i) $V\subset V_{\text{int}}\subset\ker C$; (ii) $AV\subset V+\text{Im}B$; (iii) $FV\subset V+\text{Im}G$. In due dual space, one gets that V verifies (i), (ii) and (iii), if and only if Ω verifies: (ib) $(\ker C)^\perp\subset(V_{\text{int}})^\perp\subset\Omega$; (iib) $(\Omega\cap(\text{Im}B)^\perp)A\subset\Omega$; $(iiib)$ $(\Omega\cap(\text{Im}G)^\perp)F\subset\Omega$. In particular, (ii) and (iii) hold true at the same time if, and only if (iib) and $(iiib)$ do and, as a consequence, (6) holds true. As a matter of fact, $\forall\omega_1\in\Omega\cap(\text{Im}B)^\perp$ and $\forall\omega_2\in\Omega\cap(\text{Im}G)^\perp$ one has $\omega_1Av=0$ and $\omega_2Fv=0$ for all $v\in V$ if, and only if, $\forall\omega_1A\in(\Omega\cap(\text{Im}B)^\perp)$ and $\forall\omega_2F\in(\Omega\cap(\text{Im}G)^\perp)$ that is $\omega_1A\in\Omega$ and $\omega_2F\in\Omega$. Setting now $\omega=\omega_1A+\omega_2F$, $\omega\in(\Omega\cap(\text{Im}B)^\perp)A+(\Omega\cap(\text{Im}G)^\perp)F$ and $\omega\in\Omega$ holding if and only if (6) does. ■

Starting from Lemma 3.1, the next result allows to construct the maximal (A,B) and (F,G) invariant subspace V_h^* that is contained in $V_{\text{int}}\subset\ker C$ and that defines the zero-subspace of the hybrid system. As typical in the geometric approach, V_h^* is deduced in the dual space by defining the minimal dimension co-subspace Ω^* verifying Lemma 3.1.

Theorem 3.2: Consider the hybrid dynamics (1) and $\Pi_h(A,B,F,G,C):=\Pi(A,B,C)\cap\Pi(F,G,C)$ with $V_{\text{int}}:=V_c\cap V_d\subset\ker C$ and V_c and V_d as in (3). Introduce the sequence $\{\Omega^\ell\}$ as $\Omega^0=V_{\text{int}}^\perp$ and

$$\Omega^\ell=V_{\text{int}}^\perp+(\Omega^{\ell-1}\cap(\text{Im}B)^\perp)A+(\Omega^{\ell-1}\cap(\text{Im}G)^\perp)F$$

with $\ell=1,2,\ldots$. Then, $\Omega^\ell\subset\Omega^{\ell-1}$ and, for some $\ell\geq\ell^*$ with $\ell^*\leq\dim(V_{\text{int}})$, $\Omega^*=\Omega^\ell=\Omega^{\ell+1}$. As a consequence, the zero-subspace V_h^* is given by

$$V_h^*:=(\Omega^*)^\perp. \tag{7}$$

Proof: By construction, one gets $\Omega^0\subset\Omega^1\subset\cdots\subset\Omega^\ell$ for $\ell=1,2,\ldots$. Thus, there exists a $\ell^*\leq\dim(V_{\text{int}})$ such that $\Omega^\ell=\Omega^{\ell+1}$ for all $\ell\geq\ell^*$. Moreover, from Lemma 3.1, $V_h^*=(\Omega^*)^\perp\in\Pi_h(A,B,F,G,C)$ and, by construction, $V_h^*=\sup\Pi_h(A,B,F,G,C)$. ■

Remark 3.4: Theorems 3.1 and 3.2 (and all the results to come) extend to the case in which the output mappings are switching between flows and jumps, that is to systems of the form (1) where (1c) is modified as follows

$$y(t,k)=\begin{cases} C_fx(t,k-1) & \text{if } t\in[t_{k-1},t_k)\\ C_jx(t_k,k-1) & \text{if } t=t_k.\end{cases}$$

In that case, one computes V_c and V_d as the subspaces associated to (A,B,C_f) and (F,G,C_j) and proceeds along the same lines.

Remark 3.5: When $G=0$, the proposed algorithm recovers the one presented in [18] for disturbance decoupling under periodic jumps. In addition, it represents an alternative to the one settled in [19] for regulation of aperiodically jumping hybrid systems.

B. An invariance-based decomposition

From Theorem 3.1, $\dim\{V_h^*\}=n-r_h$ so that, from Theorem 3.2, $\dim\{\Omega^*\}=r_h$. As a consequence, we rewrite $\Omega^*=\text{span}\{\omega_1,\ldots,\omega_{r_h}\}$ with ω_i being row vectors verifying, for all $s\in V_h^*$ that $\omega_is=0$ for $i=1,\ldots,r_h$. Introduce now the coordinate transformation

$$\begin{pmatrix} z\\ \eta\end{pmatrix}=Tx,\quad T:=\begin{pmatrix}\omega_1^\top & \ldots & \omega_{r_h}^\top & T_2^\top\end{pmatrix}^\top \tag{9}$$

with $T_2\in\text{Mat}_{\mathbb{R}}(n-r_h,n)$ being a complement so that $|T|\neq0$ and ω and $z\in\mathbb{R}^{r_h}$ and $\eta\in\mathbb{R}^{n-r_h}$. Then, one gets

$$TAT^{-1}=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix},\quad TB=\begin{pmatrix}B_1\\ B_2\end{pmatrix}$$

$$TFT^{-1}=\begin{pmatrix}F_{11} & F_{12}\\ F_{21} & F_{22}\end{pmatrix},\quad TG=\begin{pmatrix}G_1\\ G_2\end{pmatrix},\quad CT^{-1}=\begin{pmatrix}C_1 & \mathbf{0}\end{pmatrix}$$

with $\text{Im}A_{12}\subset\text{Im}B_1$ and $\text{Im}F_{12}\subset\text{Im}G_1$. Thus, in these new coordinates, the friends of V_h^* under (A,B) and (F,G) are of the form $K^*=(0\ K_r^*)$ and $H^*=(0\ H_r^*)$ and ensure

$$A_{12}+B_1K_r^*=0,\quad F_{12}+G_1H_r^*=0. \tag{10}$$

so that, when setting $K_r^*=-(B_1^\top B_1)^{-1}B_1^\top A_{12}$, $H_r^*=-(G_1^\top G_1)^{-1}G_1^\top F_{12}$ and

$$u^*=\bar{u}+K_r^*\eta,\quad v^*=\bar{v}+H_r^*\eta \tag{11}$$

the dynamics (1) get the form

$$z^+=F_{11}z+G_1\bar{v} \tag{12a}$$
$$\eta^+=F_{21}z+(F_{22}+G_2H_r^*)\eta+G_2\bar{v} \tag{12b}$$
$$\dot{z}=A_{11}z+B_1\bar{u} \tag{12c}$$
$$\dot{\eta}=A_{21}z+(A_{22}+B_2K_r^*)\eta+B_2\bar{u} \tag{12d}$$
$$y=C_1z \tag{12e}$$

From the previous representation, it is clear that when $x \in V_h^*$, then $z = 0$ so that the residual dynamics governing (1) are

$$\eta^+ = Q_d \eta, \quad \dot{\eta} = Q_c \eta \tag{13}$$

with $Q_c := F_{22} + G_2 H_r^*$ and $Q_d = A_{22} + B_2 K_r^*$. The hybrid system (13) describes the *hybrid zero-dynamics* over V_h^*.

The form (12) underlines that the feedback (11) is the one generating maximal unobservability of (1) by making the subspace V_h^* defined in Theorem 3.1 unobservable.

Remark 3.6: Contrarily to previous results for hybrid systems (e.g., [8]), thanks to the geometric characterization, the feedback laws (11) rendering the zero-dynamics invariant do not require explicit computation of the trajectories of (1) as they only depend on the matrices A, B, F, G, C and the properties they yield. In addition, the knowledge of the jumping period sequence $\{\delta_0, \delta_1, \dots\}$ is not required.

Accordingly, the following definition is straightforward.

Definition 3.2 (Minimum-phase of hybrid LTI systems): The hybrid system (1) is said to be minimum-phase when the zero-dynamics (13) are asymptotically stable.

Conditions for investigating the stability of (13) are not given as beyond the purpose of the paper. However, the reader is referred to several references on the topics for sufficient conditions and a deeper understanding on the difficulties (e.g., [20], [21] and references therein).

In what follows, further comments on the characterization of Q_c and Q_d are discussed with special emphasis on their relations with the zeros of the *transfer functions* involved.

IV. Insights to the hybrid zero dynamics

A. Jumping and flowing zeros

Unless differently specified and for the sake of simplicity, let (1) be a SISO system with $u, v, y \in \mathbb{R}$. Assume the couples (A, B) and (F, G) controllable and (A, C) and (F, C) observable. Consider now the *minimal* transfer functions associated with the individual flow and jump dynamics (1a) and (1b) when considered as purely continuous and discrete-time systems

$$\begin{aligned} P(s) &= C(s\mathrm{I} - A)^{-1}B = \frac{b_0 + \dots + b_{n-r_c} s^{n-r_c}}{a_0 + \dots + a_{n-1}s^{n-1} + s^n} \\ L(s) &= C(s\mathrm{I} - F)^{-1}G = \frac{g_0 + \dots + g_{n-r_d} s^{n-r_d}}{f_0 + \dots + f_{n-1}s^{n-1} + s^n} \end{aligned} \tag{14}$$

with $s \in \mathbb{C}$, possessing, respectively, $n - r_c$ and $n - r_d$ zeros defined by the roots of the numerators of the corresponding transfer function. The next result shows the relation among the zeros of $P(s)$ and $L(s)$ with the r_h eigenvalues of Q_c and the r_h eigenvalues of Q_d as given in (13). For the sake of compactness denote by Z_c and Z_d, respectively, the zeros of $P(s)$ and $L(s)$ that is

$$\begin{aligned} Z_c &= \{s \in \mathbb{C} \text{ s.t. } b_0 + b_1 s + \dots + b_{n-r_c} s^{n-r_c} = 0\} \\ Z_d &= \{s \in \mathbb{C} \text{ s.t. } g_0 + g_1 s + \dots + g_{n-r_d} s^{n-r_d} = 0\} \end{aligned}$$

and referred to as the sets of flowing and jumping zeros. We shall refer to $s_i \in \sigma(Q_c)$ and $z_j \in \sigma(Q_d)$ as, respectively, the hybrid-flowing and hybrid-jumping zeros.

Theorem 4.1: Consider the hybrid system (1) with $p = 1$ and zero-dynamics of dimension $n - r_h$ evolving as (13) over the zero-subspace V_h^*. Consider the transfer functions (14) and the corresponding flowing and jumping zeros in Z_c and Z_d. Then, the following inclusions hold true

$$\sigma(Q_c) \subset Z_c \tag{15a}$$

$$\sigma(Q_d) \subset Z_d. \tag{15b}$$

Proof: The proof is given only for (15b) as it follows the same lines for the flow dynamics. We first recall from [17] that, given matrices (F, G, C) then the zeros Z_d are given by the roots of the polynomial

$$\left| \begin{pmatrix} F - s\mathrm{I} & G \\ C & 0 \end{pmatrix} \right| = g_0 + g_1 s + \dots + g_{n-r_d} s^{n-r_d}. \tag{16}$$

The above polynomial is invariant under feedback and coordinate transformations so that introducing T and H^* as in (9)-(11) and $Q_d = F_{22} + G_2 H^*$ one has

$$\left| \begin{pmatrix} F - s\mathrm{I} & G \\ C & 0 \end{pmatrix} \right| = \left| \begin{pmatrix} F_{11} - s\mathrm{I} & G_1 \\ C_1 & 0 \end{pmatrix} \right| \, |Q_d - s\mathrm{I}|. \tag{17}$$

Now, by applying the Schur complement, one gets

$$\left| \begin{pmatrix} F_{11} - s\mathrm{I} & G_1 \\ C_1 & 0 \end{pmatrix} \right| = |F_{11} - s\mathrm{I}| \, |C_1 (s\mathrm{I} - F_{11})^{-1} G_1)|.$$

The polynomial $|F_{11} - s\mathrm{I}|$ defines the r_h eigenvalues of the matrix F_{11} whereas, in the SISO case $\det(C_1(s\mathrm{I} - F_{11})^{-1}G_1)$ is the transfer function associated to (F, G, C) under the feedback H^* so that

$$\left| (C_1(s\mathrm{I} - F_{11})^{-1} G_1) \right| = \frac{\bar{g}_0 + \bar{g}_1 s + \dots + \bar{g}_{\bar{m}} s^{\bar{m}}}{\det(F_{11} - s\mathrm{I})}$$

with the numerator defining the zeros with $\bar{m} < r_h$. Thus, by plugging now the above relation into (17) one gets

$$\left| \begin{pmatrix} F - s\mathrm{I} & G \\ C & 0 \end{pmatrix} \right| = (\bar{g}_0 + \bar{g}_1 s + \dots + \bar{g}_{\bar{m}} s^{\bar{m}}) |Q_d - s\mathrm{I}|. \tag{18}$$

Equating the right-hand sides of (16) and (18) one gets

$$g_0 + g_1 s + \dots + g_{n-r_d} s^{n-r_d} = |Q_d - s\mathrm{I}| (\bar{g}_0 + \bar{g}_1 s + \dots + \bar{g}_{\bar{m}} s^{\bar{m}})$$

so that necessarily, $\bar{m} = r_h - r_d$ with $|Q_d - s\mathrm{I}|$ being a factor of the polynomial identifying the zeros associated to (F, G, C). Thus, one gets that $\sigma(Q_d) \subset Z_d$. ■

It is worth to note that invariance of V_h^* (that is unobservability) under (1) is yielded under partial zero-cancelation that is by erasing the zeros of $P(s)$ and $L(s)$ in (14) making the jump and flow behaviour over V_h^* compatible. As a consequence, one does not need $Z_c \cap Z_d \neq \varnothing$ as (13) does not depend on the actual values of the zeros of (1a) and (1b) but on the common subspaces induced by the hybrid *interconnection*. Also, the definition of the hybrid zero-dynamics is independent on the poles and eigenvalues of the matrices A and F.

Remark 4.1: Those arguments extend to the MIMO case by noticing that the numerator of

$$|C_1(s\mathrm{I} - F_{11})^{-1} G_1| = \frac{\bar{g}_0 + \bar{g}_1 s + \dots + \bar{g}_{\bar{m}} s^{\bar{m}}}{\det(F_{11} - s\mathrm{I})}$$

defines the closed-loop transmission zeros with $\bar{m} < r_h$. In that case, $\sigma(Q_c)$ and $\sigma(Q_d)$ define the $n-r_h$ dimensional subset of the transmission zeros associated to, respectively, (A,B,C) and (F,G,C).

In this context, one can re-define the zero-dynamics via the definition of a suitable exosystem whose series interconnection with (1) generates an identically zero output evolution, under suitable initial condition. The following result is given by extending the usual definition of zeros (e.g., [17]).

Proposition 4.1: Let the hybrid subsystem (1) possess the zero-subspace V_h^* defined as in Theorem 3.1. Consider the exosystem $\xi^+ = Q_d\xi$, $\dot{\xi} = Q_c\xi$ with $\xi \in \mathbb{R}^{n-r_h}$ and interconnected to (1) through $u = K_r^*\xi$ and $v = H_r^*\xi$ as in (10). Then, for all $x_0 \in V_h^*$, there exists $\xi_0 \in \mathbb{R}^{n-r_h}$ such that $y(t,k) = 0$ for all $(t,k) \in \mathscr{T}$. More in details, this is given by $\xi_0 = T_2x_0$ with $T_2 \in \mathrm{Mat}_{\mathbb{R}}(n-r_h,n)$ defined as in (9).

B. On the hybrid zero-sets

The results stated so far put in light that a notion of zero-dynamics can be settled in the hybrid context: roughly speaking such a notion is related to the maximal subspace, V_h^*, which can be rendered unobservable for both the flowing and jumping dynamics under suitable state feedbacks $u = H^*x$ and $v = K^*x$. When constrained over such a subspace the hybrid dynamics evolve according to continuous and discrete-time behaviors associated to $\sigma(Q_c) \subset Z_c$ and $\sigma(Q_d) \subset Z_d$. What is peculiar of these subsets of zeros which have the same cardinality, is that they *share* the maximal unobservable subspace under feedbacks. With this in mind, the definition of *zeros-set* links the notion of hybrid zero-dynamics to the zeros of the involved transfer functions.

Definition 4.1: The *zeros-set* of the hybrid system (1) is defined as $Z_h^* = \sigma(Q_c) \times \sigma(Q_d) \subset Z_c \times Z_d$.

It must be noted that such notion of zeros-set is valid for any system resulting from the interlink of different LTI controlled continuous-time and/or discrete-time dynamics defined over the same state space $X \subset \mathbb{R}^n$, with the same output $y = Cx$.

Definition 4.1 extends to the hybrid context the notion of the set of zeros of a given transfer function. It is worth to note that the equivalent notion of *zero* of a transfer function does not have in general a hybrid counterpart since, for a fixed pair $(s_k, z_k) \in Z_h^*$, as any transfer function's zero does, the existence of a one-dimensional subspace which can be rendered unobservable under feedback is not guaranteed.

The discussion about the possibility of computing *zeros-subsets* of the zeros-set Z_h^* (not developed here for the sake of space) can be deepened starting from these simple elements and the understanding is left to two elementary examples in the sequel. Assuming that a given hybrid system has a zeros-set of cardinality at least two, the presence of zeros-subsets corresponds to the existence of feedback-unobservable subspaces shared by the flowing and jumping dynamics of dimensions less then $(n-r_h)$ (that is the maximal one); in doing this, one takes into account that unobservability can be generated only by cancelling couples of zeros in Z_h^*.

V. Some illustrating examples

A. Example 1

Consider the hybrid system (1) with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, F = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, G = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

and $C = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$. Since $r_c = r_d = 1$ then $r_h = 1$ and

$$V_h^* = V_c = V_d = \mathrm{span}\Big\{ \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^\top, \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^\top \Big\}.$$

Thus, the hybrid zero-dynamics (13) are characterized by

$$Q_c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_d = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}.$$

Then the hybrid zeros-set results to be $Z_h^* = \{(-1,1)\} \times \{1-\frac{\sqrt{2}}{2}, 1+\frac{\sqrt{2}}{2}\}$ with corresponding eigenvectors, under the coordinate transformation above, $w_1^f = (0\ 1\ -1)^\top$, $w_2^f = (0\ 1\ 1)^\top$, $w_1^j = (0\ 1\ -\sqrt{2})^\top$, $w_2^j = (0\ 1\ \sqrt{2})^\top$. From this computation, V_h^* is the unique subspace that is (A,B) and (F,G) invariant so that Z_h^* of cardinality two is the zeros-set and no zeros-subset exists.

B. Example 2

Consider the hybrid system (1) with A and F as in Example V-A and

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 0 \\ \frac{\sqrt{3}-3}{6} \end{pmatrix}, \quad C = \begin{pmatrix} \frac{\sqrt{3}-3}{6} & \frac{\sqrt{3}+3}{6} & 1 \end{pmatrix}.$$

As in the previous example, $r_h = r_c = r_d = 1$ and

$$V_h^* = V_c = V_d = \mathrm{span}\Big\{ \begin{pmatrix} 1 & 0 & \frac{3-\sqrt{3}}{6} \end{pmatrix}^\top, \begin{pmatrix} 0 & 1 & -\frac{\sqrt{3}+3}{6} \end{pmatrix}^\top \Big\}.$$

In that case, one gets the hybrid zero-dynamics (13) with

$$Q_c = \begin{pmatrix} 0 & 1 \\ \frac{3-\sqrt{3}}{6} & -\frac{3+\sqrt{3}}{6} \end{pmatrix}, \quad Q_d = \begin{pmatrix} \frac{\sqrt{3}+15}{4} & -\frac{5\sqrt{3}-3}{12} \\ \frac{3-\sqrt{3}}{6} & \frac{3-\sqrt{3}}{6} \end{pmatrix}$$

so that the zeros-set is given by $Z_h^* = \{-1, \frac{\sqrt{3}-3}{6}\} \times \{\frac{3-\sqrt{3}}{3}, \frac{3+\sqrt{3}}{4}\}$. By computing the corresponding eigenvectors, in this case the same eigenvector is associated to a pair of a continuous-time and a discrete-time eigenvalues, namely $w_2^j = (0\ \ \frac{3\sqrt{3}+4}{2}\ \ 1)^\top$, $w_1^f = w_1^j = (0\ 1\ 1)^\top$ and $w_2^f = (0\ 1\ \frac{2-\sqrt{3}}{6})^\top$. Since $s_1 = -1$ and $z_1 = \frac{3-\sqrt{3}}{3}$ share a one dimensional invariant subspace $\mathrm{span}\{w_1^f\}$ (that is an (A,B) and (F,G) invariant subspace in kerC), the system also possesses the hybrid zeros-subset $z_h = \{(-1, \frac{3-\sqrt{3}}{3})\}$ of cardinality one; it should be assumed to define a *zero-pair* as the counterpart of the zero for classical systems.

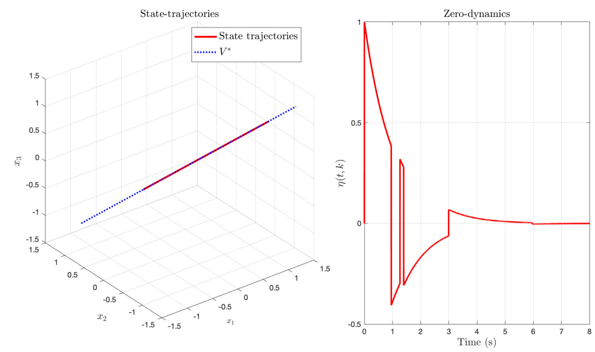

Fig. 1. $x_0 \in V_h^*$ and $y(t,k) = 0$

C. Example 3

Consider the simple example deduced by (1) when

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$$

$$F = \begin{pmatrix} -\frac{1}{9} & \frac{7}{12} & -\frac{17}{36} \\ -\frac{4}{3} & -1 & \frac{4}{3} \\ -\frac{1}{9} & \frac{1}{3} & -\frac{2}{9} \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}.$$

From Theorem 3.1, the zero-subspace is

$$V_h^* = V_d = \text{span}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \subset V_c = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}$$

that is, at the same time, (A,B) and (F,G)-invariant. Thus, $r_h = 2 = \max\{r_c, r_d\}$ with $r_c = 1$, $r_d = 2$, $Z_c = \{-1, -2\}$ and $Z_d = \{-1\}$). By specifying (9) with $T_2 = (0\ 1\ 0)$, $\omega_1 = C$ and $\omega_2 = CF$, one gets $CT^{-1} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

$$TFT^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{13}{36} & -\frac{1}{3} & -\frac{5}{3} \\ \frac{1}{5} & \frac{2}{5} & -1 \end{pmatrix}, \quad TG = \begin{pmatrix} 0 \\ -10 \\ 0 \end{pmatrix}$$

$$TAT^{-1} = \begin{pmatrix} \frac{13}{10} & \frac{3}{5} & -1 \\ -\frac{13}{20} & -\frac{3}{10} & -\frac{17}{6} \\ \frac{13}{30} & \frac{1}{5} & -1 \end{pmatrix}, \quad TB = \begin{pmatrix} 1 \\ \frac{17}{6} \\ 0 \end{pmatrix}.$$

The friends (11) of V_h^* are given by $H^* = (0\ 0\ -\frac{1}{6})$ and $K^* = (0\ 0\ -1)$ so getting that the zero-dynamics (13) evolve with $Q_c = -1$ and $Q_d = -1$. The hybrid system is minimum-phase as long as $\delta_k > 0$ for all $k \geq 0$ as the zero-dynamics is scalar. The zero-set Z_h^* is given by the only pair $\{(-1, -1)\}$ (composed by flowing and jumping zeros incidentally coincident) which is the zero-pair. For completeness, a simulation is in Figure 1 when assuming a random sequence of jumping times and $x_0 = (1\ -1\ 1)^\top \in V_h^*$.

VI. Conclusions and Perspectives

In this paper, the notion of zero-dynamics has been characterized for classes of linear time invariant hybrid systems under aperiodic time-driven jumps. Following works as [15], [22], the geometric framework contributes to a better understanding on the zero dynamics of hybrid linear systems under aperiodic jumps. Current work is toward the extension to the nonlinear context and the definition of a weaker notion of zero-dynamics revealed when constraining the output to zero only at the jumping instants as emblematic for the sampling zeros of aperiodic sampled dynamics.

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