DIRICHLET THEOREM FOR JACOBI-DUNKL EXPANSIONS
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The purpose of this paper is to study the pointwise convergence of the Jacobi-Dunkl series. Indeed, we recall some properties of the Jacobi-Dunkl coefficients. Then, we establish a Dirichlet type theorem for expansions in term of Jacobi-Dunkl polynomials.

1. Introduction

The Fourier series is named in honour of Jean-Baptiste Joseph Fourier (1768-1830) who introduced it for the purpose of solving the heat equation in a metal plate, publishing his initial results in [6], and pursuing his study in [7]. The question whether the Fourier series of a periodic function converges to a given function is well researched and an extensive literature exists on this subject. We mention here for example [9, 5, 12]. Indeed, mathematicians studied pointwise, absolute, uniform, quadratic convergences...

It is well known that many problems for partial differential equations are reduced to a power series expansion of the desired solution in terms of special functions or orthogonal polynomials. In particular, by using the properties of Jacobi polynomials ([15]), the Fourier-Jacobi series has been studied extensively by many authors and several results concerning the approximation of functions by partial sums of these series are proved (see e.g. [1, 11, 13, 14, 16, 17, 18]).

In this paper, we also discuss this subject. More precisely, we are interested in Jacobi-Dunkl expansions.

In [2], the author defined the Jacobi-Dunkl coefficients associated with Jacobi-Dunkl polynomials given by

\[
\psi_n^{(\alpha,\beta)}(\theta) := \begin{cases} 
R_{\lvert n \rvert}^{(\alpha,\beta)}(\cos(2\theta)) + \frac{i\lambda_n^{(\alpha,\beta)}(\alpha,\beta)}{4(\alpha + 1)} \sin(2\theta) R_{\lvert n \rvert - 1}^{(\alpha + 1,\beta + 1)}(\cos(2\theta)) & \text{if } n \in \mathbb{Z} \setminus \{0\}, \\
1 & \text{if } n = 0,
\end{cases}
\]

(1)
where \( R_{m}^{(\alpha, \beta)}(x) \), \( m \in \mathbb{N} \), is the normalized Jacobi polynomial of degree \( m \) such that \( R_{m}^{(\alpha, \beta)}(1) = 1 \), and \( \lambda_{n}^{(\alpha, \beta)} \) is given by
\[
\lambda_{n}^{(\alpha, \beta)} := 2 \text{sgn}(n) \sqrt{|n|(n + \rho)}, \quad n \in \mathbb{Z},
\]
with
\[
\alpha \geq \beta \geq -\frac{1}{2}; \quad \alpha \neq -\frac{1}{2}, \quad \text{and} \quad \rho := \alpha + \beta + 1 > 0.
\]

In the second section, we will give some preliminaries concerning these polynomials. Then, we will see more properties of the Jacobi-Dunkl coefficients in the third section. In section 4, we state a theorem about Jacobi-Dunkl convergence in quadratic mean. Finally, we will focus on pointwise convergence. We establish a Dirichlet type theorem which generalizes the classical one, see [10]. The proof is based on the asymptotic behaviour of Jacobi and Jacobi-Dunkl polynomials studied in [3] and [4].

2. Preliminaries

In this section, we will recall some properties of Jacobi and Jacobi-Dunkl polynomials. We denote by
\[
(a)_{n} := \begin{cases} 
 a(a + 1)\ldots(a + n - 1) & \text{if } n \in \mathbb{N} \setminus \{0\}, \\
 1 & \text{if } n = 0.
\end{cases}
\]

\((a)_{n}\) is called the Pochhammer symbol.

\(2F_1(a, b; c; z)\) is the Gauss hypergeometric function, given by
\[
\forall a, b \in \mathbb{C}, \forall c \in \mathbb{C} \setminus \mathbb{Z}, \forall z \in \mathbb{C}; \ |z| < 1, \quad 2F_1(a, b; c; z) := \sum_{n=0}^{+\infty} \frac{(a)_{n}(b)_{n}}{n!(c)_{n}}z^{n}.
\]

The Jacobi polynomials \( \varphi_{m}^{(\alpha, \beta)}(\theta) \), \( m \in \mathbb{N} \), \( \theta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \), are defined by
\[
\varphi_{m}^{(\alpha, \beta)}(\theta) := R_{m}^{(\alpha, \beta)}(\cos(2\theta)) = 2F_1(-m, m + \rho; \alpha + 1; (\sin \theta)^2).
\]

The Jacobi operator \( \Delta_{\alpha, \beta} \) defined on \( C^2 \left[ 0, \frac{\pi}{2} \right] \) is given by
\[
\Delta_{\alpha, \beta} := \frac{d^2}{d\theta^2} + \frac{A_{\alpha, \beta}'}{A_{\alpha, \beta}} \frac{d}{d\theta},
\]
where
\[
A_{\alpha, \beta}(\theta) := \begin{cases} 
 2^{2\rho}(\sin |\theta|)^{2\alpha+1}(\cos \theta)^{2\beta+1} & \text{if } \theta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \setminus \{0\}, \\
 0 & \text{if } \theta = 0.
\end{cases}
\]

For all \( m \in \mathbb{N} \), \( \varphi_{m}^{(\alpha, \beta)} \) is the unique even \( C^\infty \)-solution on \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) of the differential equation
\[
\begin{cases} 
 \Delta_{\alpha, \beta} u = -\lambda_{m} u, \\
 u(0) = 1, \\
 u'(0) = 0.
\end{cases}
\]
The Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$ is defined by

$$\Lambda_{\alpha,\beta}f(\theta) := \frac{d}{d\theta}f(\theta) + \frac{A'_{\alpha,\beta}(\theta)}{A_{\alpha,\beta}(\theta)}f(\theta) - f(-\theta), \quad f \in C^1 \left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right),$$

with

$$\frac{A'_{\alpha,\beta}(\theta)}{A_{\alpha,\beta}(\theta)} = (2\alpha + 1) \cot \theta - (2\beta + 1) \tan \theta, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}.$$

According to [2], the differential-difference equation

$$\Lambda_{\alpha,\beta}u(\theta) = i\lambda^{(\alpha,\beta)} u(\theta); \quad n \in \mathbb{Z}, u(0) = 1,$$

admits a unique $C^\infty$-solution on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ given by (1), which is related to the Jacobi polynomial and to its derivative by

$$\psi_n^{(\alpha,\beta)}(\theta) := \begin{cases} \varphi_n^{(\alpha,\beta)}(\theta) - \frac{i}{\lambda^{(\alpha,\beta)}} \frac{d}{d\theta} \varphi_n^{(\alpha,\beta)}(\theta) & \text{if } n \in \mathbb{Z} \setminus \{0\}, \\ 1 & \text{if } n = 0, \end{cases}$$

and satisfies

$$\forall n \in \mathbb{Z}, \forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad |\psi_n^{(\alpha,\beta)}(\theta)| \leq 1.$$ 

For all $n, p \in \mathbb{Z}$, we have the following orthogonality property

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi_n^{(\alpha,\beta)}(\theta) \psi_p^{(\alpha,\beta)}(\theta) A_{\alpha,\beta}(\theta) d\theta = (h_n^{(\alpha,\beta)})^{-1} \delta_{n,p},$$

where

$$h_n^{(\alpha,\beta)} = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\psi_n^{(\alpha,\beta)}(\theta)\right|^2 A_{\alpha,\beta}(\theta) d\theta\right)^{-1}$$

and

$$\forall n \in \mathbb{Z} \setminus \{0\}, \quad h_n^{(\alpha,\beta)} = \frac{(2|n| + \rho)\Gamma(\alpha + |n| + 1)\Gamma(\rho + |n|)}{2^{\rho+1} (\Gamma(\alpha + 1))^2 \Gamma(|n| + 1)\Gamma(\beta + |n| + 1)}.$$ 

Let $p \in [1, +\infty)$. We denote by

- $L^p_{\alpha,\beta} = L^p\left([\left[-\frac{\pi}{2}, \frac{\pi}{2}\right], A_{\alpha,\beta}(\theta) d\theta\right)$: the space of measurable functions $f$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that

$$\left\|f\right\|_{p,\alpha,\beta} = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(\theta)|^p A_{\alpha,\beta}(\theta) d\theta\right)^{\frac{1}{p}} < +\infty \quad \text{if } 1 \leq p < +\infty,$$

$$\left\|f\right\|_{\infty,\alpha,\beta} = \text{ess sup}_{\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} |f(\theta)| < +\infty \quad \text{if } p = +\infty.$$ 

- $\tilde{L}^p_{\alpha,\beta} = L^p\left([0, \frac{\pi}{2}], A_{\alpha,\beta}(\theta) d\theta\right)$: the space of measurable functions $g$ on $[0, \frac{\pi}{2}]$ such that

$$\left\|g\right\|_{p,\alpha,\beta} = \left(\int_{0}^{\frac{\pi}{2}} |g(\theta)|^p A_{\alpha,\beta}(\theta) d\theta\right)^{\frac{1}{p}} < +\infty \quad \text{if } 1 \leq p < +\infty,$$

$$\text{ess sup}_{\theta \in \left[0, \frac{\pi}{2}\right]} |g(\theta)| < +\infty \quad \text{if } p = +\infty.$$
The Jacobi coefficients (see [8]) of a function \( g \in \tilde{L}^1_{\alpha,\beta} \) are defined by

\[
\forall m \in \mathbb{N}, \quad \mathcal{F}_{\alpha,\beta}(g)(m) = \int_0^{\frac{\pi}{2}} g(\theta) \varphi_m^{(\alpha,\beta)}(\theta) A_{\alpha,\beta}(\theta) d\theta.
\]

The Jacobi-Dunkl coefficients (see [2]) of a function \( f \in L^1_{\alpha,\beta} \) are defined by

\[
\forall n \in \mathbb{Z}, \quad \mathcal{F}f(n) := \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} f(\theta) \overline{\psi_n^{(\alpha,\beta)}}(\theta) A_{\alpha,\beta}(\theta) d\theta,
\]

and satisfy

\[
\forall n \in \mathbb{Z}, \quad |\mathcal{F}f(n)| \leq \|f\|_{1,\alpha,\beta}.
\]

Now, we consider the analog of the Fourier series given by

\[
\sum_{n=-\infty}^{\infty} \mathcal{F}f(n) \psi_n^{(\alpha,\beta)}(\theta) h_n^{(\alpha,\beta)}, \quad \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].
\]

For \( n \in \mathbb{N} \), we denote its partial sum by

\[
S^f_n(\theta) := \sum_{k=-n}^{n} \mathcal{F}f(k) \psi_k^{(\alpha,\beta)}(\theta) h_k^{(\alpha,\beta)}, \quad \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].
\]

### 3. Jacobi-Dunkl coefficients

Let \( f \in L^1_{\alpha,\beta} \). We put for all \( k \in \mathbb{N} \),

\[
a_k(f) := \mathcal{F}f(k) + \mathcal{F}f(-k),
\]

and

\[
b_k(f) := \begin{cases} 
-\frac{i}{\lambda_k^{(\alpha,\beta)}} [\mathcal{F}f(k) - \mathcal{F}f(-k)] & \text{if } k \in \mathbb{N} \setminus \{0\}, \\
0 & \text{if } k = 0.
\end{cases}
\]

Hence, by (2) we can write \( S^f_n(\theta) \), for \( n \in \mathbb{N} \setminus \{0\} \) and \( \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \), as

\[
S^f_n(\theta) = \frac{a_0(f)}{2} h_0^{(\alpha,\beta)} + \sum_{k=1}^{n} \left( a_k(f) \varphi_k^{(\alpha,\beta)}(\theta) + b_k(f) \frac{d}{d\theta} \varphi_k^{(\alpha,\beta)}(\theta) \right) h_k^{(\alpha,\beta)}.
\]

**Remark 3.1.**

For all \( k \in \mathbb{N} \), we have these relations:

(1) \( \mathcal{F}f(k) = \frac{a_k(f) + i\lambda_k^{(\alpha,\beta)} b_k(f)}{2} \).

(2) \( \mathcal{F}f(-k) = \frac{a_k(f) - i\lambda_k^{(\alpha,\beta)} b_k(f)}{2} \).

**Proposition 3.2.**

For all \( k \in \mathbb{N} \), we have the following integral representations:

(1) \( a_k(f) = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \varphi_k^{(\alpha,\beta)}(\theta) A_{\alpha,\beta}(\theta) d\theta \).
(2) \( b_k(f) = \frac{2}{(\lambda_k^{(\alpha,\beta)})^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \frac{d}{d\theta} \phi_k^{(\alpha,\beta)}(\theta) A_{\alpha,\beta}(\theta) d\theta, \quad k \neq 0. \)

Proof.

(1) \( a_k(f) = \mathcal{F}f(k) + \mathcal{F}f(-k) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \left[ \psi_k^{(\alpha,\beta)}(\theta) + \psi_{-k}^{(\alpha,\beta)}(\theta) \right] A_{\alpha,\beta}(\theta) d\theta. \)

Since we know that

\[
\psi_k^{(\alpha,\beta)}(\theta) + \psi_{-k}^{(\alpha,\beta)}(\theta) = 2 \Re \left( \psi_k^{(\alpha,\beta)}(\theta) \right) = 2 \varphi_k^{(\alpha,\beta)}(\theta),
\]

then, we obtain the result.

(2) \( b_k(f) = \frac{i}{\lambda_k^{(\alpha,\beta)}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \left[ \psi_k^{(\alpha,\beta)}(\theta) - \psi_{-k}^{(\alpha,\beta)}(\theta) \right] A_{\alpha,\beta}(\theta) d\theta, \quad k \neq 0. \)

As we have

\[
\frac{\psi_k^{(\alpha,\beta)}(\theta)}{\psi_{-k}^{(\alpha,\beta)}(\theta)} = 2i \Im \left( \psi_k^{(\alpha,\beta)}(\theta) \right) = -\frac{2i}{\lambda_k^{(\alpha,\beta)}} \frac{d}{d\theta} \varphi_k^{(\alpha,\beta)}(\theta),
\]

then, we get the equality.

□

Remarks 3.3.

Let \( k \in \mathbb{N} \).

(1) If the function \( f \) is even, then

\[
b_k(f) = 0 \quad \text{and} \quad a_k(f) = 4 \int_{0}^{\frac{\pi}{2}} f(\theta) \varphi_k^{(\alpha,\beta)}(\theta) A_{\alpha,\beta}(\theta) d\theta.
\]

(2) If the function \( f \) is odd, then

\[
a_k(f) = 0 \quad \text{and} \quad b_k(f) = \frac{4}{(\lambda_k^{(\alpha,\beta)})^2} \int_{0}^{\frac{\pi}{2}} f(\theta) \frac{d}{d\theta} \varphi_k^{(\alpha,\beta)}(\theta) A_{\alpha,\beta}(\theta) d\theta, \quad k \neq 0.
\]

Proposition 3.4.

Let \( f \) be in \( L^1_{\alpha,\beta} \), a real-valued function. For all \( k \in \mathbb{N} \), we have these properties:

(1) \( \mathcal{F}f(-k) = \overline{\mathcal{F}f(k)}. \)

(2) \( a_k(f) = 2\Re (\mathcal{F}f(k)) \in \mathbb{R}. \)

(3) \( b_k(f) = \frac{2}{\lambda_k^{(\alpha,\beta)}} \Im (\mathcal{F}f(k)) \in \mathbb{R}, \quad k \neq 0. \)

Proof.

(1) \( \mathcal{F}f(-k) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \overline{\psi_{-k}^{(\alpha,\beta)}(\theta)} A_{\alpha,\beta}(\theta) d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \overline{\phi_k^{(\alpha,\beta)}(\theta)} A_{\alpha,\beta}(\theta) d\theta = \overline{\mathcal{F}f(k)}. \)
(2) \( a_k(f) = \mathcal{F}f(k) + \mathcal{F}f(-k) = \mathcal{F}f(k) + \overline{\mathcal{F}f(k)} = 2\Re(\mathcal{F}f(k)). \)

(3) For all \( k \in \mathbb{Z} \setminus \{0\} \), we have
\[
b_k(f) = \frac{i}{\lambda_k^{(\alpha,\beta)}} [\mathcal{F}f(-k) - \mathcal{F}f(k)] = \frac{i}{\lambda_k^{(\alpha,\beta)}} \left[ \overline{\mathcal{F}f(k)} - \mathcal{F}f(k) \right] = \frac{2}{\lambda_k^{(\alpha,\beta)}} \Im(\mathcal{F}f(k)).
\]

\[
\square
\]

In the following parts, we will study for a suitable given function \( f \), the convergence of the series
\[
\sum_{n=-\infty}^{+\infty} \mathcal{F}f(n)\psi_n^{(\alpha,\beta)}(\theta)h_n^{(\alpha,\beta)}.
\]

4. CONVERGENCE IN QUADRATIC MEAN

**Theorem 4.1.** For all \( f \in L_{\alpha,\beta}^2 \), we have
\[
\lim_{n \to +\infty} \|S_n^f - f\|_{2,\alpha,\beta} = 0.
\]

**Proof.** Let \( f \in L_{\alpha,\beta}^2 \) and \( n \in \mathbb{N} \).
\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| S_n^f(\theta) - f(\theta) \right|^2 A_{\alpha,\beta}(\theta) d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (S_n^f(\theta) - f(\theta)) \overline{(S_n^f(\theta) - f(\theta))} A_{\alpha,\beta}(\theta) d\theta
\]
\[
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |S_n^f(\theta)|^2 A_{\alpha,\beta}(\theta) d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S_n^f(\theta)\overline{f(\theta)} A_{\alpha,\beta}(\theta) d\theta
\]
\[
- \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta)\overline{S_n^f(\theta)} A_{\alpha,\beta}(\theta) d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(\theta)|^2 A_{\alpha,\beta}(\theta) d\theta
\]
\[
:= I_1 + I_2 + I_3 + I_4.
\]

We have by the orthogonality property (3),
\[
I_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \sum_{k=-n}^{n} \mathcal{F}f(k)\psi_k^{(\alpha,\beta)}(\theta)h_k^{(\alpha,\beta)}(\theta) \right) \left( \sum_{p=-n}^{n} \mathcal{F}f(p)\psi_p^{(\alpha,\beta)}(\theta)h_p^{(\alpha,\beta)}(\theta) \right) A_{\alpha,\beta}(\theta) d\theta
\]
\[
= \sum_{k=-n}^{n} \sum_{p=-n}^{n} \left( \mathcal{F}f(k)\overline{\mathcal{F}f(p)}h_k^{(\alpha,\beta)}h_p^{(\alpha,\beta)} \right) \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi_k^{(\alpha,\beta)}(\theta)\overline{\psi_p^{(\alpha,\beta)}(\theta)} A_{\alpha,\beta}(\theta) d\theta \right)
\]
\[
= \sum_{k=-n}^{n} \sum_{p=-n}^{n} \mathcal{F}f(k)\overline{\mathcal{F}f(k)}h_k^{(\alpha,\beta)} \left( h_k^{(\alpha,\beta)} \right)^{-1} \delta_{k,p}
\]
\[
= \sum_{k=-n}^{n} |\mathcal{F}f(k)|^2 h_k^{(\alpha,\beta)}
\]
\[
= \sum_{k=-n}^{n} |\mathcal{F}f(k)|^2 h_k^{(\alpha,\beta)}.
\]
Furthermore

\[
I_2 = - \int_{-\pi/2}^{\pi/2} \left( \sum_{k=-n}^{n} \mathcal{F} f(k) \psi_k^{(\alpha,\beta)}(\theta) h_k^{(\alpha,\beta)}(\theta) \right) \overline{f(\theta)} A_{\alpha,\beta}(\theta) d\theta
\]

\[
= - \sum_{k=-n}^{n} \mathcal{F} f(k) h_k^{(\alpha,\beta)} \left( \int_{-\pi/2}^{\pi/2} f(\theta) \overline{\psi_k^{(\alpha,\beta)}(\theta)} A_{\alpha,\beta}(\theta) d\theta \right)
\]

\[
= - \sum_{k=-n}^{n} \mathcal{F} f(k) \overline{\mathcal{F} f(k)} h_k^{(\alpha,\beta)}
\]

\[
= - \sum_{k=-n}^{n} |\mathcal{F} f(k)|^2 h_k^{(\alpha,\beta)}
\]

\[
= - I_1.
\]

We also have

\[
I_3 = I_2 = I_2 = - I_1.
\]

Then

\[
\int_{-\pi/2}^{\pi/2} |S_n^f(\theta) - f(\theta)|^2 A_{\alpha,\beta}(\theta) d\theta = \|f\|_{2,\alpha,\beta} - \sum_{k=-n}^{n} |\mathcal{F} f(k)|^2 h_k^{(\alpha,\beta)}.
\]

By the Plancherel formula [2, Theorem 3.4], we obtain

\[
\lim_{n \to +\infty} \int_{-\pi/2}^{\pi/2} |S_n^f(\theta) - f(\theta)|^2 A_{\alpha,\beta}(\theta) d\theta = 0.
\]

\[\square\]

5. Dirichlet Type Convergence

**Notation 5.1.**

For all \(n \in \mathbb{N}, \theta, \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\). We denote by

\[
D_n^{(\alpha,\beta)}(\theta, \phi) := \sum_{k=-n}^{n} \psi_k^{(\alpha,\beta)}(\theta) \overline{\psi_k^{(\alpha,\beta)}(\phi)} h_k^{(\alpha,\beta)}.
\]

\(D_n^{(\alpha,\beta)}(\theta, \phi)\) is the analog of the Dirichlet kernel associated with the Fourier series.

**Proposition 5.2.**

Let \(f \in L^1_{\alpha,\beta}, \ n \in \mathbb{N} \) and \(\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\). We have

\[
S_n^f(\theta) = \int_{-\pi/2}^{\pi/2} f(\phi) D_n^{(\alpha,\beta)}(\theta, \phi) A_{\alpha,\beta}(\phi) d\phi.
\]
Proof.
\[ S_n^{f}(\theta) = \sum_{k=-n}^{n} \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\phi) \overline{\psi_k^{(\alpha,\beta)}(\phi)} A_{\alpha,\beta}(\phi) d\phi \right) \psi_k^{(\alpha,\beta)}(\theta) h_k^{(\alpha,\beta)} \]
\[ = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\phi) \left( \sum_{k=-n}^{n} \overline{\psi_k^{(\alpha,\beta)}(\phi)} \psi_k^{(\alpha,\beta)}(\theta) h_k^{(\alpha,\beta)} \right) A_{\alpha,\beta}(\phi) d\phi \]
\[ = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\phi) D_n^{(\alpha,\beta)}(\theta, \phi) A_{\alpha,\beta}(\phi) d\phi. \]

\[ \square \]

Proposition 5.3.
Let \( n \in \mathbb{N} \) and \( \theta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \). We have
\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_n^{(\alpha,\beta)}(\theta, \phi) A_{\alpha,\beta}(\phi) d\phi = 1. \]

Proof.
\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_n^{(\alpha,\beta)}(\theta, \phi) A_{\alpha,\beta}(\phi) d\phi = \sum_{k=-n}^{n} \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overline{\psi_k^{(\alpha,\beta)}(\phi)} A_{\alpha,\beta}(\phi) d\phi \right) \psi_k^{(\alpha,\beta)}(\theta) h_k^{(\alpha,\beta)} . \]
As we know, by the orthogonality property (3), that
\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overline{\psi_k^{(\alpha,\beta)}(\phi)} A_{\alpha,\beta}(\phi) d\phi = \left( h_0^{(\alpha,\beta)} \right)^{-1} \delta_{0,k}, \]
then, we get
\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_n^{(\alpha,\beta)}(\theta, \phi) A_{\alpha,\beta}(\phi) d\phi = \psi_0^{(\alpha,\beta)}(\theta) = 1. \]
\[ \square \]

Proposition 5.4.

(1) \( \forall n \in \mathbb{N}, \forall \theta, \phi \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] ; \theta \neq \pm \phi, \) we have
\[ D_n^{(\alpha,\beta)}(\theta, \phi) = \frac{\Gamma(\alpha + n + 2) \Gamma(\rho + n + 1)}{2^{2n+n+1} \Gamma(\alpha + 1)^2 (2n + \rho + 1)n! \Gamma(\beta + n + 1)} \times \frac{1}{\cos(2\theta) - \cos(2\phi)} \times \left[ \varphi_n^{(\alpha,\beta)}(\theta) \varphi_n^{(\alpha,\beta)}(\phi) - \varphi_n^{(\alpha,\beta)}(\theta) \varphi_n^{(\alpha,\beta)}(\phi) + \frac{\lambda_n^{(\alpha,\beta)} \lambda_n^{(\alpha,\beta)} (\alpha + n + 1)}{4(n + 1)(n + \rho)} \times \left( \mathfrak{R}\psi_n^{(\alpha,\beta)}(\theta) \mathfrak{R}\psi_n^{(\alpha,\beta)}(\phi) - \mathfrak{R}\psi_n^{(\alpha,\beta)}(\theta) \mathfrak{R}\psi_n^{(\alpha,\beta)}(\phi) \right) \right] , \]
with \( \mathfrak{R}\psi_n^{(\alpha,\beta)}(\theta) = \frac{\cos(\theta) - \cos(\phi)}{\sin(\theta) \sin(\phi)}. \)

(2) \( \forall n \in \mathbb{N}, \forall \theta, \phi \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right], \) we have
\( D_n^{(\alpha,\beta)}(\theta, \phi) \in \mathbb{R}. \)
\( (a) \ D_n^{(\alpha,\beta)}(\theta, \phi) \in \mathbb{R}. \)
\( (b) \ D_n^{(\alpha,\beta)}(\theta, \theta) > 0. \)
(c) \(D_{n}^{(\alpha,\beta)}(\phi,\theta) = D_{n}^{(\alpha,\beta)}(\theta,\phi)\).

Proof.

(1) The case \(n = 0\) is obvious, and we have the result in [4, theorem 3.1], for \(n \in \mathbb{N} \setminus \{0\}\).

(2) (a) We deduce the result from (1), for \(\theta \neq \pm \phi\). We also have

\[
D_{n}^{(\alpha,\beta)}(\theta,\theta) = \sum_{k=-n}^{n} \left| \psi_{k}^{(\alpha,\beta)}(\theta) \right|^{2} h_{k}^{(\alpha,\beta)} \in \mathbb{R},
\]

and

\[
D_{n}^{(\alpha,\beta)}(\theta,-\theta) = \sum_{k=-n}^{n} \left( \psi_{k}^{(\alpha,\beta)}(\theta) \right)^{2} h_{k}^{(\alpha,\beta)}
= h_{0}^{(\alpha,\beta)} + \sum_{k=1}^{n} \left( \left( \psi_{k}^{(\alpha,\beta)}(\theta) \right)^{2} + \left( \overline{\psi}_{k}^{(\alpha,\beta)}(\theta) \right)^{2} \right) h_{k}^{(\alpha,\beta)}
= h_{0}^{(\alpha,\beta)} + 2 \Re \left( \sum_{k=1}^{n} \left( \psi_{k}^{(\alpha,\beta)}(\theta) \right)^{2} \right) \in \mathbb{R}.
\]

(b) \(D_{n}^{(\alpha,\beta)}(\theta,\theta) = h_{0}^{(\alpha,\beta)} + \sum_{k=-n,k \neq 0}^{n} \left| \psi_{k}^{(\alpha,\beta)}(\theta) \right|^{2} h_{k}^{(\alpha,\beta)} > 0\).

(c) \(D_{n}^{(\alpha,\beta)}(\phi,\theta) = \overline{D_{n}^{(\alpha,\beta)}(\theta,\phi)} = D_{n}^{(\alpha,\beta)}(\theta,\phi)\).

\[\square\]

**Theorem 5.5.**

Let \(f\) be a piecewise continuous function on \([-\pi/2,\pi/2]\) and \(\theta \in [-\pi/2,\pi/2] \setminus \{0\}\) such that

i) \(f(-\theta) = f(\theta)\),

ii) \(f\) is differentiable on \(\theta\) and \(-\theta\).

Then we have

\[
\lim_{n \to +\infty} S_{n}^{f}(\theta) = f(\theta).
\]

Proof.

Let \(n \in \mathbb{N}\) and \(\theta \in [-\pi/2,\pi/2] \setminus \{0\}\). By Proposition 5.3, we can write

\[
f(\theta) - S_{n}^{f}(\theta) = \int_{-\pi/2}^{\pi/2} [f(\theta) - f(\phi)] D_{n}^{(\alpha,\beta)}(\theta,\phi) A_{\alpha,\beta}(\phi) d\phi.
\]
From [4, Theorem 3.1], we have for all $\theta \neq \pm \phi$

$$f(\theta) - S_n^f(\theta) = \int_{-\pi/2}^{\pi/2} \frac{f(\theta) - f(\phi)}{\cos(2\theta) - \cos(2\phi)} \left[ \varphi_n^{(\alpha,\beta)}(\theta) \varphi_n^{(\alpha,\beta)}(\phi) - \varphi_n^{(\alpha,\beta)}(\theta) \varphi_n^{(\alpha,\beta)}(\phi) \right] \lambda_n^{(\alpha,\beta)} \lambda_{n+1}^{(\alpha,\beta)} A_{\alpha,\beta}(\phi) d\phi,$$

where

$$I_n^{(\alpha,\beta)} := \frac{\Gamma(\alpha + n + 2) \Gamma(\rho + n + 1)}{2^{2\rho-1} (\Gamma(\alpha + 1))^2 (2n + \rho + 1)! \Gamma(\beta + n + 1)}.$$

For all $\phi \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \setminus \{\pm \theta\}$, we put

$$g_\theta(\phi) := \frac{f(\theta) - f(\phi)}{\cos(2\theta) - \cos(2\phi)}.$$

Since we have supposed that $f$ is a piecewise continuous function on $\left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$, then $g_\theta$ is also piecewise continuous on $\left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \setminus \{\pm \theta\}$.

Furthermore, we have

$$\lim_{\phi \to \theta} g_\theta(\phi) = -\frac{1}{2} \frac{1}{\sin(2\theta)} f'(\theta).$$

And from hypothese i) of our theorem, we deduce that

$$\lim_{\phi \to -\theta} g_\theta(\phi) = \frac{1}{2} \frac{1}{\sin(2\theta)} f'(-\theta).$$

Under the assumption ii) of the theorem, these limits exist and are finite.
We still call $g_\theta$ the extension of $g_\theta$ on $\left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$. Thus, $g_\theta \in L^2_{\alpha,\beta}$.

In the following, we denote by

$$\check{g}_\theta(\phi) := g_\theta(-\phi), \quad \phi \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right],$$

$$\hat{g}_\theta^1 := (g_\theta)|_{[0, \frac{\pi}{2}]},$$

$$\hat{g}_\theta^2 := (g_\theta)|_{[-\frac{\pi}{2}, 0]},$$

$$\check{g}_\theta^2(\phi) := \check{g}_\theta(-\phi), \quad \phi \in \left[ 0, \frac{\pi}{2} \right].$$

Now, we write

$$f(\theta) - S_n^f(\theta) = I_1 + I_2 + I_3 + I_4,$$
where
\[ I_1 := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \varphi_{n+1}^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi) \, d\phi, \]
\[ I_2 := -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \varphi_{n+1}^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi) \, d\phi, \]
\[ I_3 := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \varphi_{n+1}^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi) \, d\phi, \]
\[ I_4 := -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \varphi_{n+1}^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi) \, d\phi. \]

Combining the fact that
\[ I_n^{(\alpha,\beta)} \sim \frac{1}{2\pi (\Gamma(\alpha + 1))^2 n^{2\alpha+1}}, \]
and the result (35) of [3], we get
\[ I_n^{(\alpha,\beta)} \varphi_{n+1}^{(\alpha,\beta)}(\theta) \sim \frac{\cos \left( (2n + 2 + \rho)\theta - (2\alpha + 1)\pi \right)}{\sqrt{\pi} \Gamma(\alpha + 1) A_{\alpha-\frac{1}{4}, \beta-\frac{1}{4}}(\theta)}. \]

Moreover, we have
\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \varphi_{n+1}^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi) \, d\phi = \mathcal{F}_{\alpha,\beta} \left( g_\theta^1 + g_\theta^2 \right)(n). \]

From the Parseval formula for the Jacobi coefficients (see [2]), we obtain
\[ \mathcal{F}_{\alpha,\beta} \left( g_\theta^1 + g_\theta^2 \right)(n) = o \left( n^{-(\alpha+\frac{1}{2})} \right). \]

Thus, \( \lim_{n \to +\infty} I_1 = 0. \)

We use the same proof as for \( I_1 \) to show that
\[ \lim_{n \to +\infty} I_2 = \lim_{n \to +\infty} - \left[ n^{\alpha+\frac{1}{2}} \frac{\cos \left( (2n + \rho)\theta - (2\alpha + 1)\frac{\pi}{4} \right)}{\sqrt{\pi} \Gamma(\alpha + 1) A_{\alpha-\frac{1}{4}, \beta-\frac{1}{4}}(\theta)} \mathcal{F}_{\alpha,\beta} \left( g_\theta^1 + g_\theta^2 \right)(n + 1) \right] \]
\[ = 0. \]

Otherwise, we have
\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \varphi_{n+1}^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi) \, d\phi = \frac{1}{2t} \mathcal{F} \left( g_\theta - g_\theta \right)(n). \]

By [2, corollary 3.5], we have
\[ \mathcal{F} \left( g_\theta - g_\theta \right)(n) = o \left( n^{-(\alpha+\frac{1}{2})} \right). \]
Furthermore, we get, from [3, Theorem 4.7], that
\[
\Im \psi_{n+1}^{(\alpha,\beta)}(\theta) \sim 2^{2\rho} \Gamma(\alpha + 1) \frac{n^{-\frac{(\alpha+1)}{2}}}{A_{2\alpha-1}^{-1}(\theta)} \sin \left[ (2n + 2 + \rho)|\theta| - (2\alpha + 1)\frac{\pi}{4} \right].
\]
Since we have
\[
\lim_{n \to +\infty} \frac{\lambda_n^{(\alpha,\beta)}\lambda_n^{(\alpha,\beta)}}{4(n+1)(n+\rho)} = 1,
\]
then, we get
\[
\lim_{n \to +\infty} I_3 = 0.
\]
We use the same reasons as for \(I_3\) to show that
\[
\lim_{n \to +\infty} I_4 = \lim_{n \to +\infty} -I_n^{(\alpha,\beta)} \frac{\lambda_n^{(\alpha,\beta)}\lambda_n^{(\alpha,\beta)}}{4(n+1)(n+\rho)} \Im \psi_n^{(\alpha,\beta)}(\theta) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_{\theta}(\phi) \Im \psi_{n+1}^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi) d\phi = 0.
\]
Hence, we obtain
\[
\lim_{n \to +\infty} \left[ f(\theta) - S_n^f(\theta) \right] = \lim_{n \to +\infty} I_1 + I_2 + I_3 + I_4 = 0.
\]
Which achieves the proof.

References


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