Projections in enlargements of filtrations under Jacod's hypothesis and examples

Pavel V Gapeev, Monique Jeanblanc, Dongli Wu

Projections in enlargements of filtrations under Jacod's hypothesis and examples*

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In this paper, we consider two kinds of enlargements of a Brownian filtration $\mathbb{F}$: the initial enlargement with a random time τ, denoted by $\mathbb{F}^{(\tau)}$, and the progressive enlargement with τ, denoted by $\mathbb{G}$. We assume Jacod's equivalence hypothesis, that is, the existence of a positive $\mathbb{F}$-conditional density for τ. Then, starting with the predictable representation of an $\mathbb{F}^{(\tau)}$-martingale $Y(\tau)$ in terms of a standard $\mathbb{F}^{(\tau)}$-Brownian motion, we consider its projection on $\mathbb{G}$, denoted by $Y^{\mathbb{G}}$, and on $\mathbb{F}$, denoted by y. We show how to obtain the coefficients which appear in the predictable representation property for $Y^{\mathbb{G}}$ (and y) in terms of $Y(\tau)$ and its predictable representation. In the last part, we give examples of conditional densities.

1 Introduction

In this paper, we consider two kinds of enlargement of a Brownian filtration $\mathbb{F}$ generated by a Brownian motion W: the initial enlargement with a random time (a positive random variable) τ, denoted by $\mathbb{F}^{(\tau)}$, and the progressive enlargement with τ, denoted by $\mathbb{G}$. We assume Jacod's

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equivalence hypothesis, that is, the existence of a positive conditional density for τ (see (2.1) below in Section 2 for a precise definition). Processes considered in the filtration $\mathbb{F}^{(\tau)}$ will be denoted as $Y(\tau)$ since, under Jacod's condition, any $\mathcal{F}_t^{(\tau)}$-measurable random variable is of the form $Y_t(\omega, \tau(\omega))$ for some $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$ measurable function $(\omega, u) \to Y_t(\omega, u)$ (see [1; Proposition 4.22]), processes considered in $\mathbb{G}$ will be indicated with the superscript $\mathbb{G}$, as $Y^{\mathbb{G}}$. Processes denoted without these symbols are $\mathbb{F}$-adapted as Y or $y(u)$.

In Section 2, we recall basic facts on enlargements of filtrations. In Section 3, we start with an $\mathbb{F}^{(\tau)}$-martingale $Y(\tau)$, which admits a decomposition

$$Y_t(\tau) = Y_0(\tau) + \int_0^t y_s(\tau)\, dW_s(\tau) \tag{1.1}$$

for all $t \geq 0$, where $W(\tau)$ is an $\mathbb{F}^{(\tau)}$-Brownian motion and, for any $u \geq 0$ the process $y(u)$ is $\mathbb{F}$-predictable and for any t the map $u \to y_t(u)$ is Borelian, and we study its $\mathbb{G}$-optional projection $Y^{\mathbb{G}}$ and its $\mathbb{F}$-optional projection y. From the predictable representation properties [5], we obtain the existence of two $\mathbb{G}$-predictable processes $\beta^{\mathbb{G}}$ and $\gamma^{\mathbb{G}}$ such that

$$Y_t^{\mathbb{G}} = \mathbb{E}\big[Y_t(\tau) \,\big|\, \mathcal{G}_t\big] = Y_0^{\mathbb{G}} + \int_0^t \beta_s^{\mathbb{G}}\, dW_s^{\mathbb{G}} + \int_0^t \gamma_s^{\mathbb{G}}\, dM_s^{\mathbb{G}} \tag{1.2}$$

for all $t \geq 0$, where $M^{\mathbb{G}}$ is the compensated $\mathbb{G}$-martingale of the default process $\mathbb{1}_{\{\tau \leq t\}}$, and the existence of an $\mathbb{F}$-predictable process σ such that

$$Y_t = \mathbb{E}[Y_t(\tau)|\mathcal{F}_t] = y_0 + \int_0^t \sigma_s dW_s\,.$$

We show how to compute $\beta^{\mathbb{G}}, \gamma^{\mathbb{G}}$ and σ in terms of $Y(\tau)$ and $y(\tau)$. We apply these computations to compare equivalent martingale measures for a financial market with price process following an $\mathbb{F}$-geometric Brownian motion considered in the two enlarged filtrations in Section 4. In the last section, we give a family of example of conditional densities.

In the whole paper, "positive" means "strictly positive".

2 Enlargement of filtrations

We recall that, if $\mathbb{H} \subset \mathbb{K}$, the $\mathbb{H}$-optional projection of a $\mathbb{K}$-martingale μ is the $\mathbb{H}$-optional process ν such that $\mathbb{E}[\mu_\vartheta \mathbb{1}_{\{\vartheta<\infty\}}|\mathcal{H}_\vartheta) = \nu_\vartheta \mathbb{1}_{\{\vartheta<\infty\}}$ for any $\mathbb{H}$-stopping time ϑ. This optional projection satisfies $\mathbb{E}(\mu_t|\mathcal{H}_t) = \nu_t$. By abuse of language, we shall call $\mathbb{E}(\mu_t|\mathcal{H}_t)$ the $\mathbb{H}$-optional projection of μ.

We consider a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with a standard Brownian motion $W = (W_t)_{t\geq 0}$ and a positive random variable τ with the positive density g on $\mathbb{R}^+$ under $\mathbb{P}$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the natural right-continuous and completed filtration of W and $\mathbb{G}$ is the progressive enlargement of $\mathbb{F}$ with τ. We assume Jacod's equivalence hypothesis (see [1; Chapter IV; Section 5]), that is, the existence of a positive function $(\omega, t, u) \to p_t(\omega, u)$ such that

(i): the map $u \mapsto p_t(u; \omega)$ is Borelian, $(dt \times d\mathbb{P})$ a.s.

(ii): for each $u \geq 0$, the process $p(u) = (p_t(u))_{t\geq 0}$ is a continuous $\mathbb{F}$-martingale,

(iii): for each $t \geq 0$, for every bounded Borel function f, we have

$$\mathbb{E}\big[f(\tau) \,\big|\, \mathcal{F}_t\big] = \int_0^\infty f(u)\, p_t(u)\, g(u)\, du\,. \tag{2.1}$$

Note that $\int_0^\infty p_t(u)\, g(u)\, du = 1$, for all $t \geq 0$, and $p_0(u) = 1$, for each $u \geq 0$. Moreover, it follows from the predictable representation theorem in a Brownian filtration that, for each $u \geq 0$, there exists an $\mathbb{F}$-predictable process $\varphi(u) = (\varphi_t(u))_{t\geq 0}$ such that the positive martingale $p(u)$ admits the representation

$$p_t(u) = p_0(u) + \int_0^t p_s(u)\, \varphi_s(u)\, dW_s \tag{2.2}$$

so that it takes the form of the Doléans-Dade stochastic exponential

$$p_t(u) = p_0(u) \exp\bigg(\int_0^t \varphi_s(u)\, dW_s - \frac{1}{2} \int_0^t \varphi_s^2(u)\, ds \bigg) \tag{2.3}$$

for all $t \geq 0$. It can be shown that $u \mapsto \varphi_t(u)$ is a Borel function on $[0, \infty)$, for all $t \geq 0$.

We introduce the supermartingale G defined as $G_t := \mathbb{P}(\tau > t \,|\, \mathcal{F}_t)$ which can be written in terms of the conditional density as

$$G_t = \int_t^\infty p_t(u)\, g(u)\, du\ , t \geq 0\,. \tag{2.4}$$

Note that G is a positive continuous $\mathbb{F}$-supermartingale called the Azéma supermartingale of τ which admits a Doob-Meyer decomposition $G_t = N_t - A_t$, where $N = (N_t)_{t\geq 0}$ is an $\mathbb{F}$-martingale under $\mathbb{P}$ and $A = (A_t)_{t\geq 0}$ is a nondecreasing $\mathbb{F}$-predictable process. Then, by means of Jacod's equivalence hypothesis, we have

$$A_t = \int_0^t p_u(u)\, g(u)\, du \quad \text{and} \quad N_t = 1 - \int_0^t \big(p_t(u) - p_u(u)\big)\, g(u)\, du \tag{2.5}$$

for all $t \geq 0$ (see [7; Subsection 4.2.1]). Moreover, by means of Itô-Wentzell's lemma (see, e.g. Kunita [19], Wentzell [22], or [14; Theorem 1.5.3.2]), we have

$$dN_t = -\int_0^t d_t p_t(u)\, g(u)\, du = -\Big(\int_0^t \varphi_t(u)\, p_t(u)\, g(u)\, du \Big)\, dW_t \tag{2.6}$$

where d_t means that one makes use of the stochastic differential, and the associated predictable covariation processes have the form

$$d\langle W, N \rangle_t = -\Big(\int_0^t \varphi_t(u)\, p_t(u)\, g(u)\, du \Big)\, dt \quad \text{and} \quad d\langle W, p(u) \rangle_t = p_t(u)\, \varphi_t(u)\, dt \tag{2.7}$$

for all $t, u \geq 0$.

It follows from Jacod's theorem for initial enlargements of filtrations ([13; Corollaire 1.11] or [1; Chapter IV, Proposition 4.40]) that the process $W(\tau) = (W_t(\tau))_{t \geq 0}$ defined by

$$W_t(\tau) := W_t - \int_0^t \frac{d\langle p(u), W \rangle_s|_{u=\tau}}{p_s(\tau)} = W_t - \int_0^t \varphi_s(\tau)\, ds \tag{2.8}$$

for all $t \geq 0$, is a $\mathbb{P}$-Brownian motion with respect to the initially enlarged filtration $\mathbb{F}^{(\tau)}$ defined by $\mathbb{F}^{(\tau)} = (\mathcal{F}_t \vee \sigma(\tau))_{t \geq 0}$. Note that, according to the results of Fontana [9], the filtration $\mathbb{F}^{(\tau)}$ is right-continuous.

It follows from the results of Grorud and Pontier [12] and also Amendinger [2], that the process $(1/p_t(\tau))_{t \geq 0}$ forms an $\mathbb{F}^{(\tau)}$-martingale under $\mathbb{P}$, and thus, since $p_0(\tau) = 1$ holds, we have $\mathbb{E}[1/p_t(\tau)] = 1$, for all $t \geq 0$. Therefore, we can define a probability measure $\mathbb{P}^*$ on $\mathcal{F}_t^{(\tau)}$ by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}\bigg|_{\mathcal{F}_t^{(\tau)}} = \frac{1}{p_t(\tau)} = \exp\bigg(-\int_0^t \varphi_s(\tau)\, dW_s(\tau) - \frac{1}{2} \int_0^t \varphi_s^2(\tau)\, ds \bigg) \tag{2.9}$$

for all $t \geq 0$. It is proved in [12] (see also [1; Chapter IV, Proposition 4.37]) that the probability measure $\mathbb{P}^*$ defined in (2.9) coincides with $\mathbb{P}$ on $\mathbb{F}$ and on $\sigma(\tau)$, and the random time τ is independent of $\mathbb{F}$ under $\mathbb{P}^*$. This fact particularly implies that $\mathbb{P}^*(\tau > u \,|\, \mathcal{F}_t) = \mathbb{P}^*(\tau > u) = \mathbb{P}(\tau > u)$ holds, for all $t, u \geq 0$, as well as that the process W is a $(\mathbb{P}, \mathbb{F})$ standard Brownian motion with respect to $\mathbb{F}$ under $\mathbb{P}^*$. Note that, by Girsanov's theorem, following the arguments of Callegaro et al. [5], we can recover that the process $W(\tau)$ from (2.8) is a $(\mathbb{P}, \mathbb{F}^{(\tau)})$-Brownian motion.

Let us now introduce the progressively enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ which is the smallest right-continuous filtration containing $\mathbb{F}$ and making τ a stopping time.

Note that, according to the hypothesis that the positive random variable τ has a positive density with support $\mathbb{R}^+$, the σ-algebra $\mathcal{G}_0$ is trivial. It is known (see Jeulin [17; Chapter III] or Callegaro et al. [5; Chapter I, Section 2]) that any $\mathbb{G}$-predictable process $Z^{\mathbb{G}}$ can be written as

$$Z_t^{\mathbb{G}} = Z_t^0\, \mathbb{1}_{\{t\le\tau\}} + Z_t^1(\tau)\, \mathbb{1}_{\{\tau<t\}} \tag{2.10}$$

for all $t \ge 0$, where the process Z^0 is $\mathbb{F}$-predictable and, for any $u \ge 0$, the process $Z^1(u)$ is $\mathbb{F}$-predictable. In that case, using the key lemma for the computation of Z^0 (see, e.g. [1; Chapter II, Lemma 2.9]), we get

$$Z_t^1(u) = Z_t(u), \text{ for } t \ge u, \quad \text{and} \quad Z_t^0 = \frac{1}{G_t}\int_t^\infty Z_t(u)\, p_t(u)\, g(u)\, du\,. \tag{2.11}$$

We shall say that Z^0 is the $\mathbb{F}$-predictable reduction of $Z^{\mathbb{G}}$.

The decomposition (2.10), proved for any progressive enlargement in Jeulin does not extend in general to optional processes (see Barlow [3] for a counter example). However, under Jacod's equivalence hypothesis, any $\mathbb{G}$-optional process $U^{\mathbb{G}}$ can be written as

$$U_t^{\mathbb{G}} = U_t^0\, \mathbb{1}_{\{t<\tau\}} + U_t^1(\tau)\, \mathbb{1}_{\{\tau\le t\}} \tag{2.12}$$

for all $t \ge 0$, where U^0 is $\mathbb{F}$-optional and, for each $u \ge 0$, $U^1(u)$ is $\mathbb{F}$-optional (see Song [21]). In particular, we shall use that result for the $\mathbb{G}$-optional projection of an integrable $\mathbb{F}^{(\tau)}$-optional process $U(\tau)$, i.e., for

$$U_t^{\mathbb{G}} = \mathbb{E}\big[U_t(\tau)\,\big|\,\mathcal{G}_t\big] \tag{2.13}$$

In that case, similarly to (2.11), we have

$$U_t^1(u) = U_t(u), \text{ for } t \ge u, \quad \text{and} \quad U_t^0 = \frac{1}{G_t}\int_t^\infty U_t(u)\, p_t(u)\, g(u)\, du \tag{2.14}$$

for all $t \ge 0$. The process U^0 is called the optional reduction of U. Note that, working in a Brownian filtration under Jacod's equivalence hypothesis, U^0 and $U^1(u)$ being continuous are also $\mathbb{F}$-predictable.

Then, it follows from the results of Jeanblanc and Le Cam [16] that the process $W^{\mathbb{G}} = (W_t^{\mathbb{G}})_{t\ge0}$ defined by

$$\begin{aligned} W_t^{\mathbb{G}} &:= W_t - \int_0^{t\wedge\tau} \frac{d\langle W, N\rangle_s}{G_s} - \int_{t\wedge\tau}^t \frac{d\langle W, p(u)\rangle_s|_{u=\tau}}{p_s(\tau)} \\ &= W_t + \int_0^{t\wedge\tau} \frac{1}{G_s}\bigg(\int_0^s \varphi_s(u)\, p_s(u)\, g(u)\, du\bigg)\, ds - \int_{t\wedge\tau}^t \varphi_s(\tau)\, ds \qquad (2.15) \\ &= W_t + \int_0^t \alpha_s^{\mathbb{G}}\, ds \qquad (2.16) \end{aligned}$$

is a $(\mathbb{P},\mathbb{G})$-standard Brownian motion, where $\alpha^{\mathbb{G}}$ is the $\mathbb{G}$-predictable process with the decomposition

$$\alpha_t^{\mathbb{G}} = \mathbb{1}_{\{t\le\tau\}}\frac{1}{G_t}\int_0^t \varphi_t(u)\,p_t(u)\,g(u)\,du\mathbb{1}_{\{\tau<t\}}\varphi_t(u) := \mathbb{1}_{\{t\le\tau\}}\,\alpha_t^0 + \mathbb{1}_{\{\tau<t\}}\,\alpha_t^1(\tau) \tag{2.17}$$

for all $t\ge 0$. Observe from (2.8) and (2.15) that

$$W_t(\tau) = W_t^{\mathbb{G}} - \int_0^{t\wedge\tau}\big(\alpha_s^0+\varphi_s(\tau)\big)\,ds \tag{2.18}$$

for all $t\ge 0$.

We define the $\mathbb{G}$-martingale

$$M_t^{\mathbb{G}} := \mathbb{1}_{\{\tau\le t\}} - \int_0^t \lambda_s^{\mathbb{G}}\,ds = \mathbb{1}_{\{\tau\le t\}} - \int_0^{t\wedge\tau}\lambda_s^0\,ds \tag{2.19}$$

for all $t\ge 0$, where $\lambda_s^{\mathbb{G}} = \mathbb{1}_{\{s\le\tau\}}\lambda_s^0$ and $\lambda_s^0 = p_s(s)g(s)/G_s$ (see, e.g. [5; Subsection 1.2] or [1; Chapter II]).

We recall that the process $W(\tau)$ enjoys the $\mathbb{F}^\tau$-predictable representation property and that the pair $(W^{\mathbb{G}}, M^{\mathbb{G}})$ enjoys the $\mathbb{G}$-predictable representation property (see, e.g. [5; Proposition 4.3]):
Any $\mathbb{F}^{(\tau)}$-martingale $Y(\tau)$ has the form

$$Y_t(\tau) = Y_0(\tau) + \int_0^t y_s(\tau)\,dW_s(\tau) \tag{2.20}$$

for all $t\ge 0$, where Y_0 is a (deterministic) function and $y(u)$, $u\ge 0$, is a family of $\mathbb{F}$-predictable processes. In particular, any $(\mathbb{P},\mathbb{F}^{(\tau)})$-martingale is continuous, and if $Y(\tau)$ is square integrable on $[0,T]$, then $\mathbb{E}[\int_0^T(y_s(\tau))^2ds]<\infty$.
Any $\mathbb{G}$-martingale admits the representation

$$Y_t^{\mathbb{G}} = Y_0^{\mathbb{G}} + \int_0^t \beta_s^{\mathbb{G}}\,dW_s^{\mathbb{G}} + \int_0^t \gamma_s^{\mathbb{G}}\,dM_s^{\mathbb{G}} \tag{2.21}$$

where $\beta^{\mathbb{G}}$ and $\gamma^{\mathbb{G}}$ are $\mathbb{G}$-predictable processes.

3 Martingales and projections

3.1 Projections of $\mathbb{F}^{(\tau)}$ martingales on $\mathbb{G}$

Proposition 3.1 *Let $Y(\tau)$ be an $\mathbb{F}^{(\tau)}$-martingale with representation* (2.20). *Its $\mathbb{G}$-optional projection $Y^{\mathbb{G}}$ admits the following representation*

$$Y_t^{\mathbb{G}} = \mathbb{E}\big[Y_t(\tau)\,\big|\,\mathcal{G}_t\big] = Y_0^{\mathbb{G}} + \int_0^t \beta_s^{\mathbb{G}}\,dW_s^{\mathbb{G}} + \int_0^t \gamma_s^{\mathbb{G}}\,dM_s^{\mathbb{G}} \tag{3.1}$$

where $Y^{\mathbb{G}}_0 = \mathbb{E}[Y_0(\tau)]$*, and the* $\mathbb{G}$*-predictable processes* $\beta^{\mathbb{G}}$ *and* $\gamma^{\mathbb{G}}$ *admit, from* (2.10)*, the decomposition*

$$\beta^{\mathbb{G}}_t = \beta^0_t \, 1\!\!1_{\{t\le\tau\}} + \beta^1_t(\tau)\, 1\!\!1_{\{\tau<t\}} \tag{3.2}$$

$$\gamma^{\mathbb{G}}_t = \gamma^0_t \, 1\!\!1_{\{t\le\tau\}} + \gamma^1_t(\tau)\, 1\!\!1_{\{\tau<t\}}\,. \tag{3.3}$$

Then,

$$\beta^0_t = \frac{1}{G_t}\int_t^\infty \big(y_t(u) + (\alpha^0_t + \varphi_t(u))\,Y_t(u)\big)\,p_t(u)\,g(u)\,du, \quad \beta^1_t(u) = y_t(u)\,, \quad for\, t \ge u \tag{3.4}$$

$$\gamma^0_t = Y_t(t) - Y^0_t\,, \tag{3.5}$$

for all $t \ge 0$*, and* Y^0 *is given by (2.11). Note that for any choice of* γ^1*, the property* $\int_0^t \gamma^{\mathbb{G}}_s dM^{\mathbb{G}}_s = \int_0^t \gamma^0_s dM^{\mathbb{G}}_s$ *holds.*

Proof: In a first step, we assume that $Y(\tau)$ (hence $Y^{\mathbb{G}}$), are square integrable on $[0,T]$ We start to determine $\beta^{\mathbb{G}}$ (which is square integrable). For this purpose, we observe that for any bounded $\mathbb{G}$-predictable process $n^{\mathbb{G}}$, using the tower property and the fact that $W^{\mathbb{G}}$ is a $\mathbb{G}$-martingale orthogonal to $M^{\mathbb{G}}$, we have

$$\mathbb{E}\Big[Y_t(\tau)\int_0^t n^{\mathbb{G}}_s\, dW^{\mathbb{G}}_s\Big] = \mathbb{E}\Big[Y^{\mathbb{G}}_t\int_0^t n^{\mathbb{G}}_s dW^{\mathbb{G}}_s\Big] \tag{3.6}$$
$$= \mathbb{E}\Big[\int_0^t \beta^{\mathbb{G}}_s\, dW^{\mathbb{G}}_s \int_0^t n^{\mathbb{G}}_s\, dW^{\mathbb{G}}_s\Big] = \mathbb{E}\Big[\int_0^t \beta^{\mathbb{G}}_s\, n^{\mathbb{G}}_s\, ds\Big]$$

for all $t \ge 0$. Recall that, from (2.18), we have

$$W^{\mathbb{G}}_t = W_t(\tau) + \int_0^{t\wedge\tau} \big(\alpha^0_s + \varphi_s(\tau)\big)\,ds \tag{3.7}$$

for all $t \ge 0$. We introduce the continuous $\mathbb{G}$-martingale

$$V^{\mathbb{G}}_t = \int_0^t n^{\mathbb{G}}_s\, dW^{\mathbb{G}}_s = \int_0^t n^{\mathbb{G}}_s\, dW_s(\tau) + \int_0^t \big(\alpha^0_s + \varphi_s(\tau)\big)\, 1\!\!1_{\{s\le\tau\}}\, n^{\mathbb{G}}_s\, ds \tag{3.8}$$

for all $t \ge 0$. By means of the integration by parts, we have

$$\mathbb{E}\big[Y_t(\tau)\, V^{\mathbb{G}}_t\big] = \mathbb{E}\Big[+\int_0^t Y_s(\tau)\, dV^{\mathbb{G}}_s + \langle Y(\tau), V^{\mathbb{G}}\rangle^{\mathbb{F}(\tau)}_t \int_0^t V^{\mathbb{G}}_s\, dY_s(\tau)\Big] \tag{3.9}$$

for all $t \ge 0$.

A martingale X is square integrable if $\sup_{t\le T} \mathbb{E}(X^2_t) < \infty$.

We recall that a local martingale M satisfying $\mathbb{E}[(\langle M\rangle_T)^{1/2}] < \infty$ is a martingale. Let us prove that the $\mathbb{F}^{(\tau)}$-local martingale $M_t = \int_0^t V_s^{\mathbb{G}}\, dY_s(\tau)$ is a martingale. One has

$$\begin{aligned}\mathbb{E}[(\langle M\rangle_T)^{1/2}) &= \mathbb{E}\left[\Big(\int_0^T (V_s^{\mathbb{G}})^2 (y_s(\tau))^2 ds\Big)^{1/2} ds\right] \leq \mathbb{E}\Big[\sup_{s\leq T} |V_s^{\mathbb{G}}| \Big(\int_0^T (y_s(\tau))^2 ds\Big)^{1/2}\Big] \\ &\leq \mathbb{E}[\sup_{s\leq T} |V_s^{\mathbb{G}}|^2] + \mathbb{E}\Big[\int_0^T y_s(\tau)^2 ds\Big]\,.\end{aligned}$$

From Burkholder's inequality {footnote Burkholder's inequality states that for any $p \geq 1$, $\mathbb{E}[\sup_{t\leq T} |M_t|^p)] \leq C_p \mathbb{E}[\langle M\rangle_T)^{p/2}]$ where C_p is a constant depending only on p. the quantity $\mathbb{E}[\sup_{s\leq T} |V_s^{\mathbb{G}}|^2]$ is smaller than $C_2\mathbb{E}[\int_0^T (n_s^{\mathbb{G}})^2 ds]$ which is bounded. Furthermorcdote, $\mathbb{E}[\int_0^T y_s(\tau)^2 ds]$ is bounded. Hence, $\int_0^\cdot V_s^{\mathbb{G}}\, dY_s(\tau)$ is a martingale and its expectation is null. The boundeness of $n^{\mathbb{G}}$ and the square integrability of $Y(\tau)$ imply that $\int_0^t Y_s(\tau) n_s^{\mathbb{G}} dW_s(\tau)$ is a martingale, so that

$$\mathbb{E}\left[\int_0^t V_s^{\mathbb{G}}\, dY_s(\tau) + \int_0^t Y_s(\tau)\, dV_s^{\mathbb{G}}\right] = \mathbb{E}\left[\int_0^t Y_s(\tau)\left(\alpha_s^0 + \varphi_s(\tau)\right) 1\!\!1_{\{s\leq\tau\}}\, n_s^{\mathbb{G}}\, ds\right]. \tag{3.10}$$

Recall that $\langle Y(\tau), V^{\mathbb{G}}\rangle_t^{\mathbb{F}^{(\tau)}} = \int_0^t y_s(\tau) n_s^{\mathbb{G}} ds$, for all $t \geq 0$. Then, we have

$$\mathbb{E}\big[Y_t(\tau)\, V_t^{\mathbb{G}}\big] = \mathbb{E}\left[\int_0^t \Big(Y_s(\tau)\left(\alpha_s^0 + \varphi_s(\tau)\right) 1\!\!1_{\{s\leq\tau\}} + y_s(\tau)\Big)\, n_s^{\mathbb{G}}\, ds\right] \tag{3.11}$$

hence, using the tower property

$$\mathbb{E}\Big[\int_0^t \beta_s^{\mathbb{G}}\, n_s^{\mathbb{G}}\, ds\Big] = \mathbb{E}\Big[\int_0^t \mathbb{E}\big[y_s(\tau) + Y_s(\tau)\big(\alpha_s^0 + \varphi_s(\tau)\big) 1\!\!1_{\{s\leq\tau\}} \,\big|\, \mathcal{G}_s\big]\, n_s^{\mathbb{G}}\, ds\Big] \tag{3.12}$$

for all $t \geq 0$ and for any bounded $n^{\mathbb{G}}$ It follows that

$$\beta_t^{\mathbb{G}} = \mathbb{E}\big[y_t(\tau) + Y_t(\tau)\left(\alpha_t^0 + \varphi_t(\tau)\right) 1\!\!1_{\{t\leq\tau\}} \,\big|\, \mathcal{G}_t\big]\,, \tag{3.13}$$

and the $\mathbb{F}$-predictable reduction of $\beta^{\mathbb{G}}$ is

$$\beta_t^0 = \frac{1}{G_t}\int_t^\infty \Big(y_t(u) + \big(\alpha_t^0 + \varphi_t(u)\big)\, Y_t(u)\Big)\, p_t(u)\, g(u)\, du \tag{3.14}$$

and on $\{t > \tau\}$, one has, as expected, that $\beta_t^{\mathbb{G}} = \mathbb{E}[y_t(\tau)\,|\,\mathcal{G}_t] = y_t(\tau)$ so that $\beta_t^1(u) = y_t(u)$.

In the second step, we determine $\gamma^{\mathbb{G}}$. For this purpose, on the one hand, for any bounded $\mathbb{G}$-predictable process $n^{\mathbb{G}}$, using the fact that $M^{\mathbb{G}}$ is a $\mathbb{G}$-martingale orthogonal to $W^{\mathbb{G}}$ and that $M^{\mathbb{G}}$ is flat after τ ($M^{\mathbb{G}}_t = M_{t\wedge\tau}$), and that the equality $d\langle M^{\mathbb{G}}\rangle_t = \lambda^0_t \mathbb{1}_{\{t\leq\tau\}} dt$ holds, we have

$$\begin{aligned}
\mathbb{E}\left[Y_t(\tau)\int_0^t n^{\mathbb{G}}_s\, dM^{\mathbb{G}}_s\right] &= \mathbb{E}\left[Y_t(\tau)\int_0^t n^0_s\, dM^{\mathbb{G}}_s\right] = \mathbb{E}\left[Y^{\mathbb{G}}_t\int_0^t n^0_s\, dM^{\mathbb{G}}_s\right] \qquad (3.15)\\
= \mathbb{E}\left[\int_0^t \gamma^0_s\, dM^{\mathbb{G}}_s \int_0^t n^0_s\, dM^{\mathbb{G}}_s\right] &= \mathbb{E}\left[\int_0^t \gamma^0_s\, \lambda^0_s\, \mathbb{1}_{\{s<\tau\}}\, n^0_s\, ds\right] = \mathbb{E}\left[\int_0^t \gamma^0_s\, \lambda^0_s\, G_s\, n^0_s\, ds\right]
\end{aligned}$$

for all $t \geq 0$, where n^0 is the $\mathbb{F}$-predictable reduction of $n^{\mathbb{G}}$ and where the last equality comes from the tower property.

Recall that $M^{\mathbb{G}}$ is a predictable bounded variation process in the initially enlarged filtration. Let $U^{\mathbb{G}}_t = \int_0^t n^0_s\, dM^{\mathbb{G}}_s$. By integration by parts, using the fact that $Y(\tau)$ is continuous, its bracket (in $\mathbb{F}^{\mathbb{G}}$ with the $\mathbb{F}^{(\tau)}$-predictable bounded variation process $U^{\mathbb{G}}$ is null

$$\mathbb{E}[Y_t(\tau)U^{\mathbb{G}}_t] = \mathbb{E}[\int_0^t Y_s(\tau) n^0_s dM^{\mathbb{G}}_s + \int_0^t U^{\mathbb{G}}_s dY_s(\tau)]\,.$$

The same methodology than in the first part for $\int V^{\mathbb{G}} dY_s(\tau)$ proves that the process $\int_0^{\cdot} U^{\mathbb{G}}_s dY_s(\tau)$ is a martingale. Furthermore, setting $H_t = \mathbb{1}_{\{t\leq\tau\}}$, we have

$$\begin{aligned}
\mathbb{E}[\int_0^t Y_s(\tau) n^0_s dM^{\mathbb{G}}_s] &= \mathbb{E}[\int_0^t Y_s(\tau) n^0_s dH_s - \int_0^t Y_s(\tau) n^0_s \lambda^0_s \mathbb{1}_{\{s<\tau\}} ds\\
&= \mathbb{E}[\mathbb{1}_{\{\tau\leq t\}} Y_\tau(\tau) n^0_\tau - \int_0^t \mathbb{E}[Y_s(\tau)\mathcal{G}_s] n^0_s \lambda^0_s \mathbb{1}_{\{s<\tau\}} ds]\\
&= \mathbb{E}[\int_0^t Y_s() n^0_u p_t(s) g(s) ds - \int_0^t Y^0_s n^0_s \lambda^0_s \mathbb{1}_{\{s<\tau\}} ds\\
&= \mathbb{E}[\int_0^t [Y_s(s) p_s(s) - Y^0_u \lambda^0_u G_u] n^0_u du]
\end{aligned}$$

where in the last equality, we used the fact that $p(u)$ is an $\mathbb{F}$-martingale. To conclude, we have

$$\mathbb{E}\left[\int_0^t \gamma^0_s\, \lambda^0_s\, G_s\, n^0_s\, ds\right] = \mathbb{E}\left[\int_0^t (Y_s(s)\ p_s(s)\, g(s) - \lambda^0_s\, G_s\, Y^0_s)\, n^0_s\, ds\right] \qquad (3.16)$$

and using the fact that $\lambda^0_s G_s = p_s(s) g(s)$, one has, for any $\mathbb{F}$ adapted bounded process n^0

$$\mathbb{E}\left[\int_0^t (\gamma^0_s - Y_s(s) + Y^0_s)\, \lambda^0_s\, G_s\, n^0_s\, ds\right] = 0 \qquad (3.17)$$

for all $t \geq 0$, so that the expression in (3.5) holds. The result obtained for square integrable martingales $Y(\tau)$ extends to all martingales. □

3.2 Projection of $\mathbb{F}^{(\tau)}$-martingales on $\mathbb{F}$

Proposition 3.2 *Let $Y(\tau)$ be an $\mathbb{F}^{(\tau)}$-martingale of the form $dY_t(\tau) = y_t(\tau)dW_t(\tau)$. Its $\mathbb{F}$-optional projection is $Y_t = \mathbb{E}[Y_t(\tau) \,|\, \mathcal{F}_t] = Y_0 + \int_0^t \sigma_s dW_s$, where the $\mathbb{F}$-predictable process σ is given by*

$$\sigma_t = \int_0^\infty \big(y_t(u) + Y_t(u)\,\varphi_t(u)\big)\, p_t(u)\, g(u)\, du \tag{3.18}$$

for all $t \geq 0$.

Proof: It follows from the predictable representation property in the filtration $\mathbb{F}$ that there exists an $\mathbb{F}$-predictable process σ such that

$$\mathbb{E}\big[Y_t(\tau) \,\big|\, \mathcal{F}_t\big] = Y_0 + \int_0^t \sigma_s\, dW_s \tag{3.19}$$

holds for all $t \geq 0$. On the one hand, for any bounded $\mathbb{F}$-adapted process n, we have

$$\mathbb{E}\bigg[Y_t(\tau) \int_0^t n_s\, dW_s\bigg] = \mathbb{E}\bigg[Y_t \int_0^t n_s\, dW_s\bigg] = \mathbb{E}\bigg[\int_0^t \sigma_s\, n_s\, ds\bigg] \tag{3.20}$$

for all $t \geq 0$. On the other hand, using (2.8) and an integration by parts on the left-hand side lead to, assuming that $Y(\tau)$ is square integrable

$$\begin{aligned}
\mathbb{E}\bigg[Y_t(\tau) \int_0^t n_s\, dW_s\bigg] &= \mathbb{E}\bigg[Y_t(\tau) \bigg(\int_0^t n_s\, dW_s(\tau) + \int_0^t n_s\, \varphi_s(\tau)\, ds\bigg)\bigg] \qquad (3.21) \\
= \mathbb{E}\bigg[\int_0^t \big(y_s(\tau) + Y_s(\tau)\,\varphi_s(\tau)\big)\, n_s\, ds\bigg] &= \mathbb{E}\bigg[\int_0^t \mathbb{E}\big[y_s(\tau) + Y_s(\tau)\,\varphi_s(\tau) \,|\, \mathcal{F}_s\big]\, n_s\, ds\bigg]
\end{aligned}$$

for all $t \geq 0$. Hence, we have

$$\sigma_t = \mathbb{E}\big[y_t(\tau) + Y_t(\tau)\,\varphi_t(\tau) \,|\, \mathcal{F}_t\big] = \int_0^\infty \big(y_t(u) + Y_t(u)\,\varphi_t(u)\big)\, p_t(u)\, g(u)\, du \tag{3.22}$$

for all $t \geq 0$. □

3.3 Projection of $\mathbb{G}$-martingales on $\mathbb{F}$

Proposition 3.3 *Consider the $\mathbb{G}$-martingale $Y^{\mathbb{G}} = (Y_t^{\mathbb{G}})_{t\geq 0}$ defined by*

$$Y_t^{\mathbb{G}} = Y_0^{\mathbb{G}} + \int_0^t \beta_s^{\mathbb{G}}\, dW_s^{\mathbb{G}} + \int_0^t \gamma_s^{\mathbb{G}}\, dM_s^{\mathbb{G}} = Y_t^0\, \mathbb{1}_{\{t<\tau\}} + Y_t^1(\tau)\, \mathbb{1}_{\{\tau \leq t\}}\,. \tag{3.23}$$

Then, its $\mathbb{F}$-optional projection is $Y_t = \mathbb{E}[Y^{\mathbb{G}}_t \,|\, \mathcal{F}_t] = Y_0 + \int_0^t \eta_s dW_s$, where the $\mathbb{F}$-predictable process η satisfies

$$\eta_t = \mathbb{E}\big[\beta^{\mathbb{G}}_t - Y^{\mathbb{G}}_t\, \alpha^{\mathbb{G}}_t \,|\, \mathcal{F}_t\big] = (\beta^0_t - Y^0_t\, \alpha^0_t)\, G_t + \int_0^t \big(\beta^1_t(u) - Y^1_t(u)\, \alpha^1_t(u)\big)\, p_t(u)\, g(u)\, du \quad (3.24)$$

for all $t \geq 0$.

Proof: We proceed as before and consider, for a bounded $\mathbb{F}$-adapted process n, the quantity $\mathbb{E}\left[Y^{\mathbb{G}}_t \int_0^t n_s dW_s\right]$, for all $t \geq 0$. On the one hand, using the same procedure as in the analysis of (3.12), we get

$$\mathbb{E}\left[Y^{\mathbb{G}}_t \int_0^t n_s\, dW_s\right] = \mathbb{E}\left[\int_0^t \eta_s\, n_s\, ds\right] \quad (3.25)$$

for all $t \geq 0$. On the other hand, setting $U_t = \int_0^t n_s dW_s$, by means of integration by parts and (2.15), we obtain, using the same methodology as before

$$\mathbb{E}\left[Y^{\mathbb{G}}_t \int_0^t n_s\, dW_s\right] = \mathbb{E}\big[Y^{\mathbb{G}}_t\, U_t\big] = \mathbb{E}\left[\int_0^t U_s\, dY^{\mathbb{G}}_s + \int_0^t Y^{\mathbb{G}}_s\, n_s\, dW_s + \int_0^t \beta^{\mathbb{G}}_s\, n_s\, ds\right] \quad (3.26)$$

$$= \mathbb{E}\left[\int_0^t \big(\beta^{\mathbb{G}}_s - Y^{\mathbb{G}}_s\, \alpha^{\mathbb{G}}_s\big)\, n_s\, ds\right] = \mathbb{E}\left[\int_0^t \mathbb{E}\big[\beta^{\mathbb{G}}_s - Y^{\mathbb{G}}_s\, \alpha^{\mathbb{G}}_s \,|\, \mathcal{F}_s\big]\, n_s\, ds\right]$$

for all $t \geq 0$. □

Remark 3.4 *Since $\mathbb{E}[Y_t(\tau)|\mathcal{F}_t] = \mathbb{E}[\mathbb{E}[Y_t(\tau)|\mathcal{G}_t]|\mathcal{F}_t]$ we have $\sigma = \eta$ where these quantities are defined in (3.18) and (3.24). Indeed, from (3.13),*

$$\begin{aligned}\mathbb{E}[\beta^{\mathbb{G}}_t - \alpha^{\mathbb{G}}_t Y^{\mathbb{G}}_t|\mathcal{F}_t] &= \mathbb{E}[y_t(\tau) + Y_t(\tau)(\alpha^0_t + \varphi_t(\tau))1\!\!1_{\{t\leq\tau\}} - \alpha^0_t Y^0_t 1\!\!1_{\{t\leq\tau\}} - \alpha^1(\tau) Y^1_t(\tau) 1\!\!1_{\{\tau<t\}}|\mathcal{F}_t] \\ &= \mathbb{E}[y_t(\tau) + Y_t(\tau)\varphi_t(\tau)|\mathcal{F}_t]\,. \end{aligned} \quad (3.27)$$

3.4 Change of probability measures

3.4.1 $\mathbb{F}^{(\tau)}$ versus $\mathbb{G}$

Note that, from PRP, and positive $\mathbb{F}^{(\tau)}$-martingale has the form

$$L_t(\tau) = L_0(\tau)\mathcal{E}(\zeta(\tau)\cdot W(\tau))_t, = L_0(\tau)\exp\left(\int_0^t \zeta_s(\tau) dW_s(\tau) - \frac{1}{2}\int_0^t (\zeta_s(\tau))^2 ds\right),\ t \geq 0 \quad (3.28)$$

where L_0 is a positive function and $\zeta(\tau)$ is $\mathbb{F}^{(\tau)}$-predictable, or in a closed form

$$L_t(\tau) = L_0(\tau)\exp\Big(\int_0^t \zeta_s(\tau) dW_s(\tau) - \frac{1}{2}\int_0^t (\zeta_s(\tau))^2 ds\Big)$$

and any positive $\mathbb{G}$-martingale has the form

$$C\mathcal{E}(\mu^{\mathbb{G}} \cdot W^{\mathbb{G}})_t \, \mathcal{E}(\psi^{\mathbb{G}} \cdot M^{\mathbb{G}})_t \,, \tag{3.29}$$

where C is a positive constant and $\mu^{\mathbb{G}}$ and $\psi^{\mathbb{G}}$ are $\mathbb{G}$ predictable and $\psi^{\mathbb{G}} > -1$. We recall that

$$\mathcal{E}(\psi^{\mathbb{G}} \cdot M^{\mathbb{G}})_t = \exp\left(\int_0^{t\wedge\tau} \psi^{\mathbb{G}}_s \lambda^0_s ds\right) (1 + \psi^{\mathbb{G}}_\tau)^{H_t} \,.$$

Proposition 3.5 *Let $L(\tau) = (L_t(\tau))_{t\geq 0}$ be of the form* (3.28)*. Then, its $\mathbb{G}$-optional projection $L^{\mathbb{G}}$ satisfies*

$$L^{\mathbb{G}}_t = \mathbb{E}\big[L_t(\tau) \,|\, \mathcal{G}_t\big] = \mathbb{E}\big[L_0(\tau)\big] + \int_0^t L^{\mathbb{G}}_{s-} \, \mu^{\mathbb{G}}_s \, dW^{\mathbb{G}}_s + \int_0^t L^{\mathbb{G}}_{s-} \, \psi^{\mathbb{G}}_s \, dM^{\mathbb{G}}_s \tag{3.30}$$

where the $\mathbb{G}$-predictable processes $\mu^{\mathbb{G}}$ and $\psi^{\mathbb{G}}$ are given by

$$\mu^{\mathbb{G}}_t = 1\!\!1_{\{t\leq\tau\}} \frac{1}{L^0_t G_t} \int_t^\infty L_t(u)\big(\zeta_t(u) + \varphi_t(u) + \alpha^0_t\big) \, p_t(u) \, g(u) \, du + 1\!\!1_{\{\tau<t\}} \, \zeta_t(\tau) \tag{3.31}$$

$$\psi^{\mathbb{G}}_t = \frac{L_t(t)}{L^0_t} - 1 \tag{3.32}$$

where L^0 is the optional reduction of $L^{\mathbb{G}}$.

Proof: This is an application of Proposition 3.2. □

Note that $\psi^{\mathbb{G}}_t > -1$, as it must be. If $\mathbb{E}[L_0(\tau)] = 1$, and $L(\tau)$ is a positive $(\mathbb{P}, \mathbb{F}^{(\tau)})$-martingale with initial value $\ell(\tau)$, one can associate to it the change of probability measure defined by $d\widetilde{\mathbb{P}} = L_t(\tau) d\mathbb{P}$ on $\mathcal{F}^{(\tau)}_t$, for any $t \geq 0$. The choice $L_0 \equiv 1$ is equivalent to $\widetilde{\mathbb{P}}(\tau > u) = \mathbb{P}(\tau > u)$, for each $u \geq 0$. Indeed, since τ is $\mathcal{F}_0 \vee \sigma(\tau)$-measurable, we have, using tower property $\widetilde{\mathbb{P}}(\tau > u) = \mathbb{E}\left[\ell(\tau) 1\!\!1_{\{\tau>u\}}\right]$ and the equality $\mathbb{E}\left[L_0(\tau) 1\!\!1_{\{\tau>u\}}\right] = \mathbb{E}\left[1\!\!1_{\{\tau>u\}}\right]$ holds, for each $u \geq 0$, which implies that $L_0 \equiv 1$.

In particular, in the case $L_t(\tau) = p_0(tau)/p_t(\tau)$, which satisfies $dL_t(\tau) = -L_t(\tau)\varphi_t(\tau) dW_t(\tau)$, one obtains (recall that P^* is defined in (2.9)) $d\mathbb{P}^*|_{\mathcal{G}_t} = L^{\mathbb{G},*}_t d\mathbb{P}|_{\mathcal{G}_t}$ with

$$L^{\mathbb{G},*}_t = \mathbb{E}(\frac{p_0(\tau)}{p_t(\tau)} | \mathcal{G}_t) = \mathcal{E}(\mu^{\mathbb{G},*} \cdot W^{\mathbb{G}})_t \, \mathcal{E}(\psi^{\mathbb{G},*} \cdot M^{\mathbb{G}})_t \tag{3.33}$$

with

$$\mu^{\mathbb{G},*}_t = 1\!\!1_{\{t\leq\tau\}} \, \alpha^0_t - 1\!\!1_{\{\tau<t\}} \, \varphi_t(\tau) \quad \text{and} \quad \psi^{\mathbb{G},*}_t = 1\!\!1_{\{t\leq\tau\}} \left(\frac{G_t}{p_t(t)(1 - F(t))} - 1\right) \tag{3.34}$$

where $F(t) = \int_0^t g(u)du$.

From the definition of $L^{\mathbb{G},*}$ one has

$$L_t^{\mathbb{G},*} = 1\!\!1_{\{t \le \tau\}} \frac{1 - F(t)}{G_t} + 1\!\!1_{\{\tau < t\}} L_t(\tau) \tag{3.35}$$

for all $t \ge 0$.

3.4.2 $\mathbb{G}$ versus $\mathbb{F}$

From PRP, any positive $(\mathbb{P}, \mathbb{G})$-martingale $L^{\mathbb{G}} = (L_t^{\mathbb{G}})_{t \ge 0}$ can be written as

$$L_t^{\mathbb{G}} = L_0^{\mathbb{G}} + \int_0^t L_s^{\mathbb{G}} \mu_s^{\mathbb{G}} \, dW_s^{\mathbb{G}} + \int_0^t L_{s-}^{\mathbb{G}} \psi_s^{\mathbb{G}} \, dM_s^{\mathbb{G}} = L_t^0 \, 1\!\!1_{\{t < \tau\}} + L_t^1(\tau) \, 1\!\!1_{\{\tau \le t\}} \, . \tag{3.36}$$

Let $L = (L_t)_{t \ge 0}$ be its $\mathbb{F}$-optional projection $L_t = \mathbb{E}\left[L_t^{\mathbb{G}} \,|\, \mathcal{F}_t\right]$, for all $t \ge 0$. Then, we have $L_t = L_0 + \int_0^t L_s \eta_s dW_s$, where $L_0 = \mathbb{E}[L_0(\tau)]$ and η is the $\mathbb{F}$-predictable process

$$\eta_t = \frac{L_t^0}{L_t} \left(\mu_t^0 - \alpha_t^0\right) G_t + \int_0^t L_t^1(u) \left(\mu_t^1(u) - \alpha_t^1(u)\right) p_t(u) \, g(u) \, du \tag{3.37}$$

for all $t \ge 0$.

3.4.3 $\mathbb{F}^{(\tau)}$ versus $\mathbb{F}$

Let $L(\tau)$ be a positive $\mathbb{F}^{(\tau)}$-martingale of the form (3.28), then its $\mathbb{F}$-optional projection is $L_t = \mathbb{E}[L_t(\tau) \,|\, \mathcal{F}_t] = L_0 + \int_0^t L_s \sigma_s dW_s$ where $L_0 = \mathbb{E}[L_0(\tau)]$ and σ is the $\mathbb{F}$-predictable process

$$\sigma_t = \frac{1}{\ell_t} \int_0^\infty L_t(u) \left(\zeta_t(u) + \varphi_t(u)\right) p_t(u) \, g(u) \, du \tag{3.38}$$

for all $t \ge 0$.

3.5 Stability of the Brownian property

As we mentioned before, due to the fact that the processes $W^{\mathbb{G}}$ and $M^{\mathbb{G}}$ enjoy the predictable representation property in $\mathbb{G}$, any locally equivalent probability measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}$ on $\mathcal{G}_t$ is given by

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{G}_t} = \mathcal{E}(\mu^{\mathbb{G}} \cdot W^{\mathbb{G}})_t \, \mathcal{E}(\psi^{\mathbb{G}} \cdot M^{\mathbb{G}})_t \tag{3.39}$$

for all $t \ge 0$, where $\mu^{\mathbb{G}}$ and $\psi^{\mathbb{G}}$ are $\mathbb{G}$ predictable and $\psi^{\mathbb{G}} > -1$. Note that the change of probability measure does not affect the intensity if and only if $\psi^{\mathbb{G}} = 0$.

For a probability measure $\mathbb{Q}$ which is locally equivalent to $\mathbb{P}$ on $\mathbb{G}$, let us denote by $\mathbb{Q}^{\mathbb{F}}$ its restriction to $\mathbb{F}$, which is locally equivalent to $\mathbb{P}$ on $\mathbb{F}$, with its density being of the form $d\mathbb{Q}^{\mathbb{F}}/d\mathbb{P} = \mathcal{E}(\eta\cdot W)_t$, for all $t \geq 0$. Under $\mathbb{Q}^{\mathbb{F}}$, the process $W^{\mathbb{Q},\mathbb{F}}$ defined by $W_t^{\mathbb{Q},\mathbb{F}} = W_t - \int_0^t \eta_s ds$ is a $(\mathbb{Q}^{\mathbb{F}},\mathbb{F})$-standard Brownian motion. Let us provide a set of probability measures $\mathbb{Q}$ which are equivalent to $\mathbb{P}$ on $\mathbb{G}$ such that the $(\mathbb{Q}^{\mathbb{F}},\mathbb{F})$-standard Brownian motion $W^{\mathbb{Q},\mathbb{F}}$ is a $(\mathbb{Q},\mathbb{G})$-standard Brownian motion.

Proposition 3.6 *Let $L^{\mathbb{G}}$ be a positive $(\mathbb{P},\mathbb{G})$-martingale of the form $\mathcal{E}(\mu^{\mathbb{G}}\cdot W^{\mathbb{G}})$, and define the probability measure $\mathbb{Q}$ by*

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{G}_t} = L_t^{\mathbb{G}} \tag{3.40}$$

for all $t \geq 0$. The process $W^{\mathbb{Q},\mathbb{F}}$ is a $(\mathbb{Q},\mathbb{G})$-standard Brownian motion if and only if $\mu^{\mathbb{G}} - \alpha^{\mathbb{G}}$ is $\mathbb{F}$-adapted. Then, we have $\eta_t = \mu_t^{\mathbb{G}} - \alpha_t^{\mathbb{G}}$, for all $t \geq 0$.

Proof: On the one hand, we have seen in (2.15) that $W_t^{\mathbb{G}} = W_t + \int_0^t \alpha_s^{\mathbb{G}} ds$, for all $t \geq 0$. Hence, we have $W_t^{\mathbb{G}} = W_t^{\mathbb{Q},\mathbb{F}} + \int_0^t \sigma_s ds + \int_0^t \alpha_s^{\mathbb{G}} ds$, for all $t \geq 0$. On the other hand, we see that $W_t^{\mathbb{G}} - \int_0^t \mu_s^{\mathbb{G}} ds$ is a $(\mathbb{Q},\mathbb{G})$-standard Brownian motion. It follows that $W_t^{\mathbb{Q},\mathbb{F}} + \int_0^t \sigma_s ds + \int_0^t \alpha_s^{\mathbb{G}} ds - \int_0^t \mu_s^{\mathbb{G}} ds$ is a $\mathbb{G}$-standard Brownian motion. Therefore, $W^{\mathbb{Q},\mathbb{F}}$ is a $(\mathbb{Q},\mathbb{G})$-standard Brownian motion if and only if $\sigma_t + \alpha_t^{\mathbb{G}} - \mu_t^{\mathbb{G}} = 0$, for all $t \geq 0$. Note that indeed, if $\alpha_t^{\mathbb{G}} - \mu_t^{\mathbb{G}}$ is $\mathbb{F}$ adapted, then, from (3.24) $L_t\sigma_t = \mathbb{E}(L_t^{\mathbb{G}}(\alpha_t^{\mathbb{G}} - \mu_t^{\mathbb{G}})|\mathcal{F}_t) = L_t(\alpha_t^{\mathbb{G}} - \mu_t^{\mathbb{G}})$ where $L_t = \mathbb{E}(L_t^{\mathbb{G}}|\mathcal{F}_t)$. □

Examples 3.7 (i) If the process $\mu^{\mathbb{G}}$ is such that

$$\mu_t^0 = 0 \tag{3.41}$$

then $\mathbb{Q}$ is equal to $\mathbb{P}$ on $\mathcal{G}_\tau$ and the immersion holds for $\mathbb{F}$ and $\mathbb{G}$ under $\mathbb{Q}$. The choice $\mu_t^0 = 0$ leads to $\eta = \alpha^0$, hence $\mu_t^1(u) = \alpha_t^1(u) - \alpha_t^0 = -\varphi_t(u) - \alpha_t^0$, for all $t \geq u$. Therefore, we have

$$\begin{aligned}
&\mathcal{E}(\mu^1\cdot W^{\mathbb{G}})_t && (3.42)\\
&= \mathbb{1}_{\{t<\tau\}} + \mathbb{1}_{\{\tau\leq t\}} \exp\left(\int_\tau^t \alpha_s^1(\tau)\, dW_s^{\mathbb{G}} - \frac{1}{2}\int_\tau^t (\alpha_s^1)^2\, ds\right) \exp\left(-\int_\tau^t \alpha_s^0\, dW_s^{\mathbb{G}} - \frac{1}{2}\int_\tau^t (\alpha_s^0)^2\, ds\right)\\
&= \mathbb{1}_{\{t<\tau\}} + \mathbb{1}_{\{\tau\leq t\}} \frac{p_\tau(\tau)}{p_t(\tau)}\frac{Z_t}{Z_\tau}
\end{aligned}$$

where we have used the fact that, on $\{\tau < t\}$, one has $p_\tau(\tau)/p_t(\tau) = \exp\Big(\int_\tau^t \alpha_s^1 dW_s + \frac{1}{2}\int_\tau^t (\alpha_s^1)^2 ds\Big)$ and $Z_t = \mathcal{E}(-\alpha^0 \cdot W)_t = G_t e^{\Lambda_t}$, for all $t \geq 0$. More precisely, we have

$$L_t^{\mathbb{G}} = \mathbb{1}_{\{t<\tau\}} + \mathbb{1}_{\{\tau \leq t\}} \frac{p_\tau(\tau)\, e^{\Lambda_t} G_t}{p_t(\tau) e^{\Lambda_\tau} G_\tau} \tag{3.43}$$

and

$$\eta_t = \frac{G_t}{1 - F(t)} \tag{3.44}$$

This example was presented in [1; Chapter V].

(ii) the case $\mu^0 = \alpha^0$ leads to $\eta = 0$ and $\mu^1 = \alpha^1$, that is, this is the case in which $\mathbb{Q} = \mathbb{P}^*$.

4 Equivalent Martingale Measures

consider the case where ν and σ are processes Let us consider a model of a financial market in which the stock price process $S = (S_t)_{t \geq 0}$ started at $S_0 = 1$ satisfies the stochastic differential equations (written in various filtrations)

$$dS_t = S_t\big(\nu\, dt + \sigma\, dW_t\big) = S_t\big(\nu_t^{\mathbb{G}}\, dt + \sigma\, dW_t^{\mathbb{G}}\big) = S_t\Big(\big(\nu + \sigma\varphi_t(\tau)\big)\, dt + \sigma\, dW_t(\tau)\Big) \tag{4.1}$$

for some $\sigma > 0$ fixed, where, from (2.15), we have

$$\nu_t^{\mathbb{G}} = \nu + \sigma\alpha_t^{\mathbb{G}} = \nu + \sigma\, \alpha_t^0\, \mathbb{1}_{\{t \leq \tau\}} - \varphi_t(\tau)\, \mathbb{1}_{\{\tau < t\}}\,. \tag{4.2}$$

We assume that a riskless asset is traded with a null interest rate. It is straightforward to show that, for positive function ℓ satisfying $\mathbb{E}[\ell(\tau)] = 1$, the positive martingale

$$Y_t(\tau) = \ell(\tau) \exp\bigg(\int_0^t y_s(\tau)\, dW_s(\tau) - \frac{1}{2}\int_0^t y_s^2(\tau)\, ds\bigg) \tag{4.3}$$

is a Radon-Nikodým density of an equivalent martingale measure on $\mathbb{F}^{(\tau)}$ (i.e. such that $SY(\tau)$ is an $\mathbb{F}^{(\tau)}$-martingale) if and only if

$$y_t(\tau) = -\varphi_t(\tau) - \frac{\nu}{\sigma} \tag{4.4}$$

for all $t \geq 0$. We denote by $L(\tau) = (L_t(\tau))_{t \geq 0}$ such a positive martingale defined by

$$L_t(\tau) = L_0(\tau)\, \exp\bigg(-\int_0^t \Big(\varphi_s(\tau) + \frac{\nu}{\sigma}\Big)\, dW_s(\tau) - \frac{1}{2}\int_0^t \Big(\varphi_s(\tau) + \frac{\nu}{\sigma}\Big)^2 ds\bigg) \tag{4.5}$$

for all $t \geq 0$. Then, we define a new probability measure $\widetilde{\mathbb{P}}$ which is locally equivalent to $\mathbb{P}$ on $\mathbb{F}^{(\tau)}$ by

$$\left.\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t^{(\tau)}} = L_t(\tau) \tag{4.6}$$

for all $t \geq 0$. Actually, there exists infinitely many such probabilities, which differ from each other by the choice of the initial value L_0, that is, by the choice of the law of τ (under $\widetilde{\mathbb{P}}$): $\widetilde{\mathbb{P}}(\tau > u) = \mathbb{E}\left[L_0(\tau)\mathbb{1}_{\{\tau>u\}}\right] = \int_u^\infty L_0(v)g(v)dv$, for all $u \geq 0$. The case $L_0 \equiv 1$ corresponds to the situation in which $\widetilde{\mathbb{P}}(\tau > u) = \mathbb{P}(\tau > u)$, for any $u \geq 0$. Note that, by virtue of Girsanov's theorem, the process $\widetilde{W}(\tau)$ defined as

$$\widetilde{W}_t(\tau) = W_t(\tau) + \int_0^t \left(\varphi_s(\tau) + \frac{\nu}{\sigma}\right) ds \tag{4.7}$$

is a $(\widetilde{\mathbb{P}}, \mathbb{F}^{(\tau)})$-standard Brownian motion.

Working in the filtration $\mathbb{G}$, it easily seen that the set of $\mathbb{G}$-equivalent martingale measures (i.e. the set of positive $\mathbb{G}$-martingales $Y^{\mathbb{G}}$ such that $SY^{\mathbb{G}}$ is a $\mathbb{G}$-martingale) is

$$dY_t^{\mathbb{G}} = Y_{t-}^{\mathbb{G}}\left(-\frac{\nu_t^{\mathbb{G}}}{\sigma}\,dW_t^{\mathbb{G}} + \gamma_t^{\mathbb{G}}\,dM_t^{\mathbb{G}}\right) \tag{4.8}$$

so that

$$Y_t^{\mathbb{G}} = \mathcal{E}(-(\nu^{\mathbb{G}}/\sigma)\cdot W^{\mathbb{G}})_t\,\mathcal{E}(\gamma^{\mathbb{G}}\cdot M^{\mathbb{G}})_t \tag{4.9}$$

where γ is any $\mathbb{G}$-predictable process, with $\gamma > -1$.

Let us now consider $L^{\mathbb{G}}$, the $\mathbb{G}$-optional projection of $L_t(\tau)$. The process $L^{\mathbb{G}}$ defines an equivalent martingale measure for S and can be written as in Section 3.4

$$L_t^{\mathbb{G}} = 1 + \int_0^t L_{s-}^{\mathbb{G}}\,\mu_s^{\mathbb{G}}\,dW_s^{\mathbb{G}} + \int_0^t L_{s-}^{\mathbb{G}}\,\psi_s^{\mathbb{G}}\,dM_s^{\mathbb{G}} = \mathcal{E}(\mu^{\mathbb{G}}\cdot W^{\mathbb{G}})_t\,\mathcal{E}(\psi^{\mathbb{G}}\cdot M^{\mathbb{G}})_t \tag{4.10}$$

where the processes $\mu^{\mathbb{G}}$ and $\psi^{\mathbb{G}}$ are given in (3.31). Using the fact that $L_t^0 = \frac{1}{G_t}\int_0^t L_t(u)p_t(u)g(u)du$, one has

$$\begin{aligned}
\mu_t^{\mathbb{G}} &= \mathbb{1}_{\{t\leq\tau\}}\frac{1}{G_t L_t^0}\left(\int_0^t \left(-\frac{\nu}{\sigma} - \alpha_t^0\right) L_t(u)\,p_t(u)\,g(u)\,du + \mathbb{1}_{\{\tau<t\}}\,\varphi_t(\tau)\right. &(4.11)\\
&= -\mathbb{1}_{\{t\leq\tau\}}\left(\frac{\nu^{\mathbb{G}}}{\sigma} + \alpha_t^0\right) + \mathbb{1}_{\{\tau<t\}}\,\varphi_t(\tau) = -\mathbb{1}_{\{t\leq\tau\}}\,\frac{\nu_t^{\mathbb{G}}}{\sigma} + \mathbb{1}_{\{\tau<t\}}\,\varphi_t(\tau)\\
\psi_t^{\mathbb{G}} &= \mathbb{1}_{\{t\leq\tau\}}\left(\frac{L_t(t)}{L_t^0} - 1\right). &(4.12)
\end{aligned}$$

We obtain here an equivalent martingale measures.

However, the set of all equivalent martingale measures is larger, since it is the family $\mathcal{E}(\mu^{\mathbb{G}} \cdot W^{\mathbb{G}})\mathcal{E}(\gamma^{\mathbb{G}} \cdot M^{\mathbb{G}})$, for any $\mathbb{G}$-predictable process $\gamma^{\mathbb{G}} = (\gamma_t^{\mathbb{G}})_{t\geq 0}$ such that $\gamma_t^{\mathbb{G}} > -1$, for all $t \geq 0$. The compensated $(\mathbb{P}, \mathbb{G})$-martingale of $H = (H_t)_{t\geq 0}$ with $H_t = \mathbb{1}_{\{\tau \leq t\}}$, for all $t \geq 0$ is the (uniformly integrable) $(\mathbb{P}, \mathbb{G})$-martingale

$$M_t^{\mathbb{G}} = H_t - \int_0^{\tau\wedge t} \frac{p_s(s)g(s)}{G_s}\, ds = H_t - \int_0^{t\wedge\tau} \lambda_s^0\, ds \tag{4.13}$$

(see, e.g., [5; Subsection 1.2] or [1]). We introduce $L^{\mathbb{G}}$ as

$$L_t^{\mathbb{G}} = \mathcal{E}(\mu^{\mathbb{G}} \cdot W^{\mathbb{G}})\mathcal{E}(\gamma^{\mathbb{G}} \cdot M^{\mathbb{G}})$$

and the measure $\widehat{\mathbb{P}}$ on $\mathbb{G}$ with

$$\left.\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{G}_t} = L_t^{\mathbb{G}} \tag{4.14}$$

By virtue of Girsanov's theorem, we have that the process $\widehat{W}$ defines as

$$\widehat{W}_t^{\mathbb{G}} = W_t^{\mathbb{G}} - \int_0^t \mu_s^{\mathbb{G}}\, ds \tag{4.15}$$

is a standard Brownian motion under $\widehat{\mathbb{P}}$ with respect to $\mathbb{G}$. Moreover, the process

$$\widehat{M}_t^{\mathbb{G}} = M_t^{\mathbb{G}} - \int_0^{t\wedge\tau} \gamma_s^0\, \lambda_s^0\, ds = H_t - \int_0^{t\wedge\tau} (1 + \gamma_s^0)\, \lambda_s^0\, ds \tag{4.16}$$

is a (uniformly integrable) $(\widehat{\mathbb{P}}, \mathbb{G})$-martingale. The above change of probability changes the Brownian motion and the intensity of the default. The specific choice $\gamma = 0$ leads to a change of probability measure which does not affect the intensity.

Remark 4.1 Assume for simplicity that S is a martingale under $\mathbb{P}$. By definition S is also a $\mathbb{Q}^{\mathbb{F}}$ martingale, hence $W = W^{\mathbb{G}}$. Then, for any $\zeta_T \in \mathcal{F}_T$ and any equivalent martingale measure $\mathbb{Q}^{\mathbb{G}}$ on $\mathbb{G}$, =one has $\mathbb{E}_{\mathbb{Q}^{\mathbb{G}}}[\zeta_T \,|\, \mathcal{G}_t] = \mathbb{E}_{\mathbb{P}}[\zeta_T \,|\, \mathcal{F}_t]$, for $0 \leq t \leq T$. The first reason is that ζ_T is hedgeable in $\mathbb{F}$, and thus in $\mathbb{G}$. The second reason is that from $\zeta_T = x + \int_0^T x_s dW_s x + \int_0^T x_s dW_s^{\mathbb{G}}$ and using the orthogonality of $W^{\mathbb{G}}$ and $M^{\mathbb{G}}$, setting $\zeta_t = x + \int_0^t x_s dW^{\mathbb{G}}$, the product $\mathcal{E}(\gamma \cdot M^{\mathbb{G}})_t \zeta_t$ is a martingale, hence, by Bayes' formula

$$\mathbb{E}_{\mathbb{Q}^{\mathbb{G}}}\big[\zeta_T \,\big|\, \mathcal{G}_t\big] = \frac{1}{\mathcal{E}(\gamma \cdot M^{\mathbb{G}})_t}\, \mathbb{E}_{\mathbb{P}}\big[\mathcal{E}(\gamma \cdot M^{\mathbb{G}})_T \zeta_T \,\big|\, \mathcal{G}_t\big] = \zeta_t \tag{4.17}$$

for $0 \leq t \leq T$.

Remark 4.2 We can extend easily the study to the case where the interest rate is $\mathbb{G}$-adapted with $r_t(\tau) = r_t^0 1\!\!1_{\{t<\tau\}} + r_t^1(\tau) 1\!\!1_{\{\tau \le t\}}$, where the processes $r^0 = (r_t^0)_{t\ge 0}$ and $r^1(u) = (r_t^1(u))_{t \ge 0}$ are $\mathbb{F}$-adapted, for each $u \ge 0$ fixed, and we define the discounted process $\widetilde{S}(\tau) = (\widetilde{S}_t(\tau))_{t\ge 0}$ by

$$\widetilde{S}_t(\tau) = \exp\bigg(- \int_0^t r_s(\tau)\, ds \bigg)\, S_t \tag{4.18}$$

for all $t \ge 0$.

5 Examples of conditional density

In the literature, there are some explicit examples of random times satisfying Jacod's Hypothesis, among them the Gaussian ones given in Crépey et al. [4] and the one in [10]. Here, we shall work in the same framework as in [10], with the difference that we do not make use of a change of probability measure, and we are working under the given probability $\mathbb{P}$.

Let $W = (W_t)_{t\ge 0}$ be a standard Brownian motion with its natural filtration $(\mathcal{F}_t)_{t\ge 0}$. Let F be the cumulative distribution function defined by $F(t) = \int_0^t g(s) ds$, with the given probability density function g. For some constants μ and $\sigma > 0$, we define $G = (G_t)_{t \ge 0}$ as the unique strong solution of the stochastic differential equation

$$dG_t = -G_t \, \frac{g(t)}{1-F(t)}\, dt - \frac{\mu}{\sigma}\, G_t (1 - G_t)\, dW_t \quad (G_0 = 1)\,. \tag{5.1}$$

for all $t \ge 0$. It is easily seen from the expressions in (5.1) that G is a supermartingale valued in $[0,1]$. We also define the process $X = (X_t)_{t\ge 0}$ by

$$X_t = \int_0^t \mu\, (1 - G_s)\, ds + \int_0^t \sigma\, dW_s \tag{5.2}$$

so that, we have $\mathcal{F}_t^X \subseteq \mathcal{F}_t$ with $\mathcal{F}_t^X = \sigma(X_s \,|\, 0 \le s \le t)$, for all $t \ge 0$.

Let us now provide the multiplicative decomposition for the supermartingale G. For this purpose, we define the process $Z = (Z_t)_{t\ge 0}$ by:

$$Z_t = \exp\bigg(\frac{\mu}{\sigma^2}\, X_t - \frac{\mu^2}{2\sigma^2}\, t \bigg) \tag{5.3}$$

and the process $Y = (Y_t)_{t \ge 0}$ by

$$Y_t = \int_0^t \frac{g(s)}{Z_s}\, ds + \frac{1 - F(t)}{Z_t} \tag{5.4}$$

for all $t \geq 0$. Then, it is shown by means of standard arguments based on Itô's formula that the process Z solves the stochastic differential equation

$$dZ_t = Z_t \, \frac{\mu}{\sigma^2} \, dX_t = \Big(\frac{\mu}{\sigma}\Big)^2 (1 - G_t) \, Z_t \, dt + \frac{\mu}{\sigma} \, Z_t \, dW_t \quad (Z_0 = 1) \tag{5.5}$$

so that the process $1/Z$ takes the expression

$$d\bigg(\frac{1}{Z_t}\bigg) = -\frac{1}{Z_t} \, \frac{\mu}{\sigma^2} \, dX_t + \frac{1}{Z_t} \Big(\frac{\mu}{\sigma}\Big)^2 dt \quad \bigg(\frac{1}{Z_0} = 1\bigg) \tag{5.6}$$

and thus, the process Y admits the representation

$$dY_t = \Big(\frac{\mu}{\sigma}\Big)^2 \, \frac{(1 - F(t))G_t}{Z_t} \, dt - \frac{\mu}{\sigma} \, \frac{(1 - F(t))}{Z_t} \, dW_t \quad (Y_0 = 1). \tag{5.7}$$

Observe from the expression in (5.1) that $G_t = N_t - A_t$ with $dN_t = -(\mu/\sigma)G_t(1 - G_t)dW_t$ and that $p_t(t) = G_t/(1 - F(t))$, so that the intensity of τ is a deterministic function $\lambda(t) = p_t(t)g(t)/G_t = g(t)/(1 - F(t))$, for all $t \geq 0$.

Let us now consider the stochastic differential

$$d(G_t Z_t Y_t) = Y_t \, d(G_t Z_t) + (G_t Z_t) \, dY_t + d\langle GZ, \, Y \rangle_t \tag{5.8}$$

for which we first compute, setting $\lambda(t) = g(t)/(1 - F(t))$,

$$\begin{aligned} d(G_t Z_t) &= Z_t \, dG_t + G_t \, dZ_t + d\langle G, Z \rangle_t = Z_t \bigg(- \lambda(t) \, G_t \, dt - \frac{\mu}{\sigma} \, G_t(1 - G_t) \, dW_t \bigg) \\ &\quad + G_t \bigg(\Big(\frac{\mu}{\sigma}\Big)^2 (1 - G_t) \, Z_t \, dt + \frac{\mu}{\sigma} \, Z_t \, dW_t \bigg) - \Big(\frac{\mu}{\sigma}\Big)^2 G_t(1 - G_t) \, Z_t \, dt \\ &= -\lambda(t) \, G_t \, Z_t \, dt + \frac{\mu}{\sigma} \, G_t \, Z_t \, [1 - 1 + G_t] \, dW_t \\ &= -\lambda(t) \, G_t \, Z_t \, dt + \frac{\mu}{\sigma} \, G_t^2 \, Z_t \, dW_t \, . \end{aligned} \tag{5.9}$$

Then, we can proceed with computing the expression (5.8) and obtain

$$\begin{aligned} d(G_t Z_t Y_t) &= Y_t \bigg(- \lambda(t) \, G_t \, Z_t \, dt + \frac{\mu}{\sigma} \, G_t^2 \, Z_t \, dW_t \bigg) \\ &\quad + G_t Z_t \bigg(\Big(\frac{\mu}{\sigma}\Big)^2 \frac{(1 - F(t))G_t}{Z_t} \, dt - \frac{\mu}{\sigma} \, \frac{(1 - F(t))}{Z_t} \, dW_t \bigg) - \Big(\frac{\mu}{\sigma}\Big)^2 (1 - F(t)) \, G_t^2 \, dt \\ &= -\lambda(t) \, G_t \, Z_t \, Y_t \, dt + \frac{\mu}{\sigma} \, G_t \, [G_t Z_t Y_t - (1 - F(t))] \, dW_t \end{aligned} \tag{5.10}$$

so that, by using the fact that $d(1 - F(t)) = -g(t)dt$, we get

$$\begin{aligned} d[G_t Z_t Y_t - (1 - F(t))] &= \Big(- \lambda(t) \, G_t Y_t \, Z_t + g(t) \Big) \, dt + \frac{\mu}{\sigma} \, G_t \, [G_t Z_t Y_t - (1 - F(t))] \, dW_t \\ &= -\frac{g(t)}{1 - F(t)} \, [G_t Z Y_t - (1 - F(t))] \, dt + \frac{\mu}{\sigma} \, G_t \, [G_t Z_t Y_t - (1 - F(t))] \, dW_t \end{aligned} \tag{5.11}$$

from where it is seen that the process $V_t := G_t Z_t Y_t - (1 - F(t))$ is a solution of the stochastic differential equation $dV_t = V_t(-\lambda(t)\, dt + (\mu/\sigma)\, G_t\, dW_t)$, started at $G_0 Z_0 Y_0 - (1 - F(0)) = 0$, and thus, we have $G_t Z_t Y_t - (1 - F(t)) = 0$, for all $t \geq 0$.

Furthermore, by applying Itô's formula we have

$$dY_t = -\frac{\mu}{\sigma^2}\, \frac{1 - F(t)}{Z_t}\, dX_t + \Big(\frac{\mu}{\sigma}\Big)^2\, \frac{1 - F(t)}{Z_t}\, dt \tag{5.12}$$

and thus

$$d\bigg(\frac{1}{Y_t}\bigg) = \frac{\mu}{\sigma}\, \frac{1}{Y_t^2}\, \frac{1 - F(t)}{Z_t}\, dW_t = \frac{\mu}{\sigma}\, \frac{G_t}{Y_t}\, dW_t \tag{5.13}$$

and

$$d\bigg(\frac{1}{Z_t Y_t}\bigg) = \frac{1}{Z_t}\, d\bigg(\frac{1}{Y_t}\bigg) + \frac{1}{Y_t}\, d\bigg(\frac{1}{Z_t}\bigg) + d\bigg\langle \frac{1}{Z}, \frac{1}{Y} \bigg\rangle_t = -\frac{\mu}{\sigma}\, \frac{1 - G_t}{Z_t Y_t}\, dW_t \tag{5.14}$$

and hence, the processes $1/Y$ and $1/(ZY)$ are $(\mathcal{F}_t)_{t \geq 0}$-martingales. We recall that $G_t = \frac{1-F(t)}{L_t Y_t}$. Then, we can construct a family of positive $\mathbb{F}$-martingales $M(u) = (M_t(u))_{t \geq 0}$, for $u \geq 0$, by

$$M_t(u) = \frac{1}{Y_t}\, \bigg(\int_u^{u \vee t} \frac{g(s)}{Z_s}\, ds + \frac{1}{Z_t} \int_{u \vee t}^{\infty} g(s)\, ds \bigg). \tag{5.15}$$

Indeed, from

$$M_t(u) = \mathbb{1}_{\{t<u\}}\, \frac{1}{Y_t Z_t} \int_u^{\infty} g(s)\, ds + \mathbb{1}_{\{u<t\}}\, \frac{1}{Y_t}\, \bigg(\int_u^t \frac{g(s)}{Z_s}\, ds + \frac{1}{Z_t} \int_t^{\infty} g(s)\, ds \bigg) \tag{5.16}$$

one has:

on $\{t \leq u\}$, $M_t(u) = (1 - F(u))/(Y_t Z_t)$ is an $\mathbb{F}$-martingale

on $\{u < t\}$, $M_t(u) = (1/Y_t)\Big(\int_u^t (g(s)/Z_s)\, ds + (1/Z_t) \int_t^{\infty} g(s)\, ds \Big)$ which leads to

$$d_t M_t(u) = \bigg(\int_0^u \frac{g(s)}{Z_s}\, ds \bigg)\, d\bigg(\frac{1}{Y_t}\bigg) + \big(1 - F(t)\big)\, d\bigg(\frac{1}{Y_t Z_t}\bigg) \tag{5.17}$$

so that $M(u)$ forms a martingale, because $1/Y$ and $1/(ZY)$ are martingales. Note that the process $M(u)$ is valued in $(0,1)$, and $M_t(u)$ is decreasing with respect to u, for any $t \geq 0$ fixed, with $M_t(0) = 1$. Furthermore, we have $M_t(t) = G_t$, $M_0(u) = \int_u^{\infty} g(s) ds$, and $M_t(u) = \int_u^{\infty} p_t(s) g(s) ds$ where $p_t(u) = g(u)/(Z_{u \wedge t} Y_t)$.

It is therefore possible to construct, on an extended probability space, a random time τ and a probability measure $\mathbb{Q}$ such that $\mathbb{Q}$ and $\mathbb{P}$ coincide on $\mathbb{F}$ and $\mathbb{Q}(\tau > u \,|\, \mathcal{F}_t) = M_t(u)$, for $t, u \geq 0$ (see [15]). It is straightforward to check the important property $\int_0^{\infty} p_t(u) g(u) du = 1$, for any $t \geq 0$.

Let us now thus compute the dynamics of $p(u)$ by means of the integration by parts and thus

$$d_t p_t(u) = d_t\left(\frac{1}{Z_{u\wedge t} Y_t}\right) = \frac{\mu}{\sigma}\left(1\!\!1_{\{u\le t\}}\frac{G_t}{Z_u Y_t} - 1\!\!1_{\{u>t\}}\frac{1-G_t}{Z_t Y_t}\right) dW_t \tag{5.18}$$

so that

$$d_t p_t(u) = p_t(u)\,\frac{\mu}{\sigma}\left(1\!\!1_{\{u\le t\}}\, G_t - 1\!\!1_{\{u>t\}}\,(1-G_t)\right) dW_t \tag{5.19}$$

and

$$\varphi_t(u) = \frac{\mu}{\sigma}\left(1\!\!1_{\{u\le t\}}\, G_t - 1\!\!1_{\{u>t\}}\,(1-G_t)\right) = \frac{\mu}{\sigma}\left(1\!\!1_{\{u\le t\}} - (1-G_t)\right). \tag{5.20}$$

In the progressively enlarged filtration, $dW_t^{\mathbb{G}} = dW_t - \alpha_t^{\mathbb{G}} dt$, where

$$\begin{aligned}\alpha_t^0 &= \frac{\mu}{\sigma}\frac{1}{G_t}\int_0^t G_t\,\frac{g(u)}{Z_u Y_t}\,du = \frac{\mu}{\sigma}\frac{1}{Y_t}\int_0^t \frac{g(u)}{Z_u}\,du \qquad (5.21)\\ &= \frac{\mu}{\sigma}\frac{1}{Y_t}\left(Y_t - \frac{1-F(t)}{Z_t}\right) = \frac{\mu}{\sigma}\left(1 - \frac{1-F(t)}{Y_t Z_t}\right) = \frac{\mu}{\sigma}\,(1-G_t)\\ \alpha_t^1(\tau) &= -\frac{\mu}{\sigma}\,G_t\,, \qquad (5.22)\end{aligned}$$

and

$$dW_t^{\mathbb{G}} = dW_t - \frac{\mu}{\sigma}\left((1-G_t)\,1\!\!1_{\{t\le\tau\}} - G_t\,1\!\!1_{\{\tau<t\}}\right)dt\,. \tag{5.23}$$

We can check that the property $\int_0^\infty \mathbb{E}(\alpha_t^{\mathbb{G}}|\mathcal{F}_t)dt = 1$ holds. Moreover, from

$$dW_t^{\mathbb{G}} = dW_t - \frac{\mu}{\sigma}\left(1\!\!1_{\{\tau\le t\}} - (1-G_t)\right) dt \tag{5.24}$$

we deduce that

$$dX_t = \mu\,1\!\!1_{\{\tau\le t\}}\,dt + \sigma\,dW_t^{\mathbb{G}}\,. \tag{5.25}$$

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