Modelling a faillibilistic and perspectivistic reasoning

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arnaud.kohler@pacariane.com

Abstract

It is commonly accepted that propositional logic is insufficient to capture human reasoning. To extend its capabilities, we propose to automatically integrate silent propositions into the set of atomic propositions. We call them thoughts. They are used to define a semantic interpretation function on models. Our contribution to the family of non-monotonic formalisms is to formalise a faillibilistic and perspectivistic reasoning. We illustrate the interest of these properties by developing an example of application.

1 Introduction

Formalisms based on propositional logic L_p classically focus on modelling knowledge that is deemed to be true or false. It is insufficient to model the full diversity and complexity of human reasoning, which obviously also exploits inconsistent or uncertain information. Another difficulty is to move from a monotonic semantic interpretation (what is supposed to be true given a state of knowledge remains true even if a new piece of knowledge appears) to a non-monotonic semantic interpretation (what is supposed to be true given a state of knowledge can become false if a new piece of knowledge appears).

Many propositions have been presented to address these needs: epistemic modal logics, paraconsistent logics, default logic, intuitionistic logic, adaptive logics, or multivalued logics for example. They each manage to capture different properties. But they have not succeeded in modelling the many modes of reasoning empirically observed in humans (D. Andler [2]).

These formalisms most often approach human reasoning through the principle of proof inherited from mathematics. In this article, we propose the contextual logic L_c, which uses the principle of non-refutability. To achieve this, L_c models a faillibilistic reasoning:

> *Each piece of knowledge is uncertain, and our belief is constructed by identifying those that are justifiable.*

We will find that the exercise leads, mechanically, to a perspectivist reasoning:

> *Belief is constructed by aggregating some different justifiable points of view, accepting the possibility that they are incomplete, incorrect, or inconsistent with each other.*

L_c remains within the monotonic syntax of propositional logic L_p and enrich the set of atomic propositions by *silent* propositions. We call them thoughts. Integrated and consumed automatically, they identify the formulae belonging to the set of knowledge. We use them to define a non-monotonic semantic interpretation function based on the analysis of their behaviour in models of the theory.

We first present the propositional logic to share the vocabulary we use and the associated definitions. The supposed limits of the syntax of L_p are recalled. The principles of L_c and its main properties are described. The assumed limitations of the propositional logic are then revisited. To illustrate our point, we conclude our presentation by developing an example of application of L_c. It calls upon a sufficiently broad knowledge base to demonstrate, through a practical case, the non-monotonic expressiveness of the language and to give meaning to the various technical examples used throughout the text.

2 The propositional logic

In this article, we refer to several formal languages. We do not detail them in general so as not to make the presentation unnecessarily heavy, inviting the reader to refer to the many documents available on these formalisms. However, we think it is useful to pause for a moment on the propositional logic L_p.

This paragraph does not contain anything new. There are many equivalent ways of modeling a formal language. Here we share the definitions of the vocabulary and symbols that we use out of habit.

<u>The syntax of L_p</u>

The language of the propositional logic L_p is composed of the set P_{Lp} of atomic propositions, the negation connector $\neg$, the implication connector $\rightarrow$, and the parenthesis symbols, which are used according to classical mathematical rules. The rules for forming a **well-formed formula** are:

- any atomic proposition is a well-formed formula,
- if f and g are well-formed formulae, then the expressions (f), $\neg f$ and $f \rightarrow g$ are well-formed formulae,
- a well-formed formula is obtained only by applying the two precedent rules a finite number of times.

Let f, g and h be some well-formed formulae. The following formulae are some **axioms**:

- $f \rightarrow (g \rightarrow f)$
- $(f \rightarrow (g \rightarrow h)) \rightarrow ((f \rightarrow g) \rightarrow (f \rightarrow h))$
- $(\neg f \rightarrow \neg g) \rightarrow ((\neg f \rightarrow g) \rightarrow f)$

This axiom scheme is sufficient to cover all the axioms of L_p. For example, $a \rightarrow (a \rightarrow a)$ is an axiom obtained by applying the first formula for f is a and g is a. $f \rightarrow f$ is another axiom, which can be demonstrated with this scheme and the **theorem formation rules**:

- any axiom is a theorem,
- let f and g be two well-formed formulae. If f and $f \rightarrow g$ are theorems, then g is a theorem (this rule is called the *modus ponens*),
- a theorem can only be obtained by applying the two previous rules a finite number of times.

The statement f is a theorem is denoted $\vdash_{Lp} f$. A **theory** E_{Lp} is a set of well-formed formulae. The formulae $f \in E_{Lp}$ represent the **hypotheses** of E_{Lp}. A formula f is said **to be provable** in E_{Lp} if, and only if, it can be produced from E_{Lp} by applying the theorem formation rules, for all hypotheses of E_{Lp} behaving as theorems. In this case, f is said a theorem of E_{Lp}, and this is denoted $E_{Lp} \vdash_{Lp} f$.

A theory is said to be **inconsistent** if it produces the negation of a theorem. Otherwise, it is said to be **consistent**.

The semantic of L_p

Classically, the logician's attitude is to consider in formal language only mathematical symbols:

> "*A formal language is, by definition, a language with only syntax and no semantics*" – (translation) J. Hebenstreit, Enclycopedia Universalis

What is relevant is the study of the mechanisms and laws of reasoning, modelled by syntactic rules. Any reference to semantic content is discarded. However, it is possible to attribute a meaning to connectors if it is strictly symbolic and univocal.

Let E_{Lp} be a theory of L_p, and f and g two well-formed formulae. The **syntactic interpretation function** of L_p is defined by a function I_{Lp} such that:

- $I_{Lp}(E_{Lp}, f) = true$ or $I_{Lp}(E_{Lp}, f) = false$,
- If f is a hypothesis of E_{Lp}, then $I_{Lp}(E_{Lp}, f) = true$.

The meaning of the connectors is then defined by:

- $I_{Lp}(E_{Lp}, \neg f) = true$ if, and only if, $I_{Lp}(E_{Lp}, f) = false$,
- $I_{Lp}(E_{Lp}, f \rightarrow g) = true$ if, and only if, $I_{Lp}(E_{Lp}, f) = false$ or $I_{Lp}(E_{Lp}, g) = true$.

The symbol $\vDash_{Lp}$ is defined by:

$$E_{Lp} \vDash_{Lp} f \text{ if, and only if, } I_{Lp}(E_{Lp}, f) = true$$

The syntactic interpretation of an axiom is always *true*, and L_p is correct and complete: everything that is produced (using $\vdash_{Lp}$) is *true* (according to $\vDash_{Lp}$), and everything that is *true* is produced: $E_{Lp} \vDash_{Lp} f$ if, and only if, $E_{Lp} \vdash_{Lp} f$.

If the constraints of a theory do not allow the *true* or *false* truth value of a formula to be calculated, it is said to have an *unknown* value for this theory – for example, the value of $I_{Lp}(\{a \rightarrow b\}, a)$: a can be *true* (in this case, b is *true*) or *false* (in this case, b can be *true* or *false*).

This remark introduces the definition of **model**. A model of a theory E_{Lp} is obtained by associating to each atomic proposition only one truth value (*true* or exclusively *false*) such that the result verifies I_{Lp}. E_{Lp} is consistent if it has at least one model. It is inconsistent otherwise.

For example, the theory $\{a \rightarrow b, c\}$ is verified by three models:

$$\{(a, true), (b, true), (c, true)\}$$
$$\{(a, false), (b, true), (c, true)\}$$
$$\{(a, false), (b, false), (c, true)\}$$

So, it is consistent. As a counter example, $\{a, \neg a\}$ does not accept a model: if a is assumed to be *true*, $\neg a$ is not verified - and *vice versa*. a is true and false. It is inconsistent.

We end this paragraph by presenting three often used connectors. To simplify the expression of formulae, the language is extended to disjunction (denoted $\vee$), conjunction (denoted $\wedge$) and equivalence (denoted $\leftrightarrow$) connectors. For f and g two well-formed formulae, they are defined by:

- $f \vee g$ is equivalent to $(\neg f \rightarrow g)$,
- $f \wedge g$ is equivalent to $\neg (f \rightarrow \neg g)$,
- $f \leftrightarrow g$ is equivalent to $(f \rightarrow g) \wedge (g \rightarrow f)$.

A **literal** is an atomic proposition or the negation of an atomic proposition. A **clause** is a disjunction of literals. A formula is said to be in **conjunctive normal form** if it is a conjunction of clauses. Any well-formed formula admits a logically equivalent rewriting in conjunctive normal form (A. Thayse, [17]). For example, a conjunctive normal form of the formula $((\neg f \rightarrow g) \rightarrow h)$ is $((\neg f \vee h) \wedge (\neg g \vee h))$. P. Siegel [16] proposes a linear complexity process that rewrites any well-formed formula into a conjunctive normal form.

3 The limits of L_p

Modelling human reasoning with L_p faces several difficulties. This article focuses on three of them.

1) A difficulty is to move from a monotonic semantic interpretation to a **non-monotonic** semantic interpretation. This topic is covered in paragraph 5.

2) Human reasoning sometimes seems incoherent. But whatever f and g two well-formed formulae of L_p, $\{f, \neg f\} \vdash_{Lp} g$. This is the **explosion principle**. It forbids the appearance of a syntactic inconsistency in a theory. This topic is covered in paragraph 6.

3) Human reasoning uses **semantic links between propositions**. But the symmetrical behaviour of connectors prohibits this type of modelling in L_p. Put more explicitly with an example, $f \rightarrow (g \rightarrow h)$ is syntactically equivalent to $g \rightarrow (f \rightarrow h)$: f and g have the same behaviour in the formula, and it is not possible to model a privileged relationship between one of them and h. This topic is covered in paragraph 7.

Two other difficulties are the modelling of induction and abduction, and the computational complexity of the algorithms. We do not address them in this article.

The successive failures of logic researchers to solve these problems have led many to conclude that the modelling of human reasoning probably escapes L_p, and logic formalisms more generally (D. Andler [2]).

4 The contextual logic

Let us consider a thought. We perceive it in the sense defined by R. Descartes [4]:

> "*By the name of thought, I understand all that is so much in us that we are immediately aware of it*" (translation)

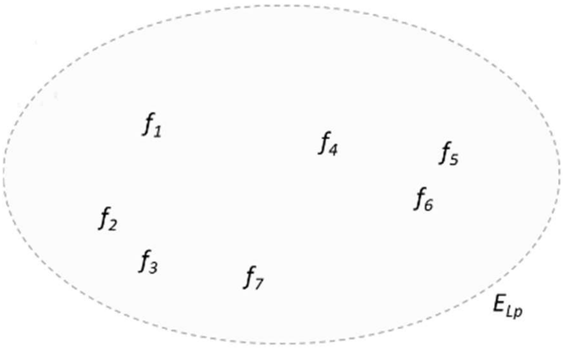
and we describe it with a set of sentences. For example, we can have the thought of a painted picture. Consider the Mona Lisa, by Leonardo da Vinci. We can describe it with some sentences. But even if this description would be ideally complete and perfect, we are immediately aware that it is not the thought that we have of the Mona Lisa.

We model this observation by distinguishing two notions in the syntax of the language: a unit sign c, which symbolises a thought, and a combination of signs f, which reproduces the sentences that describe it. This leads to the need to define a relationship between c and f. To this end, we consider the following postulate [8]:

Contextual postulate *Let L be a formal language with the functions of syntactic production* $\vdash_L$ *and of syntactic interpretation* $\vDash_L$. *A well-formed formula f of L is a set of signs that has no meaning. Its meaning is carried by a thought, which is an atomic proposition of L "which is not pronounced". For c symbolising this thought, the relation between c and f is* $c \vDash_L f$.

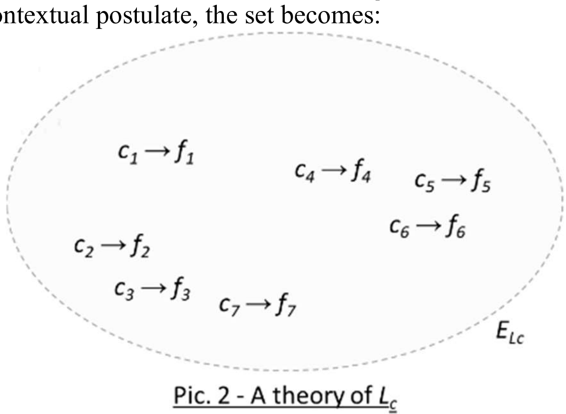
The expression $c \vDash_L f$ asserts neither the thought c nor the sentence f. It models that the sentence f expresses the thought c. c is an atomic proposition which respects the syntactic properties of L.

For example, consider seven sentences $f_1, f_2, \ldots, f_7$ of L_p such that:



Pic. 1 - A theory of L_p

E_{Lp} admits a syntactic behaviour, but it has no semantic meaning according to contextual postulate. To overcome this, we need to consider the thoughts $c_1, c_2, \ldots, c_7$. $c \vDash_L f$ is equivalent to $\vDash_L c \rightarrow f$ if L is L_p. So, after applying contextual postulate, the set becomes:



Pic. 2 - A theory of L_c

E_{Lc} is a theory of L_p. It models: "*each thought c_i is expressed by a sentence f_i*". We thus agree with L. Wittgenstein when he states [18]:

> "*We should not say: The complex sign aRb says that a is in the relation R with b, but: That a is in a certain relation R with b says that aRb*" (translation)

The application of contextual postulate to a formalism L produces the contextualised logic L. By language convention, we call contextual logic, denoted L_c, the contextualised propositional logic.

Each contextual formula takes a form $c \rightarrow f$, for c a thought and f a well-formed formula in the sense of L_p. Expressions in L_c accept a pre-order:

- an atomic proposition that is not a thought is of rank 0,
- a thought is of rank 1 or higher. We will see later that we propose to automatically handle the assignment of a rank to a thought,
- the rank of a well-formed formula in the sense of L_p is equal to the maximum rank of the atomic propositions (including thoughts) that compose it.

We define a well-formed formula in the sense of L_c to be a formula $c \rightarrow f$, for c a thought of rank i and f a well-formed formula in the sense of L_p of rank j, such that $i > j$.

This defines a syntactical restriction. Without these notions of rank and well-formed formula in the sense of L_c, thoughts would have a reflexive property that is not easy to understand: it could be expressed by a sentence containing it: c could participate in f in $c \rightarrow f$.

Knowledge reflexivity is a concept defended by J. Pitrat [12], but we don't master it. The restricted language we propose remains sufficient to cover the level of expressiveness that we want to achieve in this article. However, we do not know if this restriction is necessary.

Given the syntax $c \rightarrow f$ of the contextual formulae, the set $\{(c_i, false), c_i \text{ are the thoughts}\}$ characterises some models that verify any contextual theory. For example, $\{(c_1, false), (c_2, false), \ldots, (c_7, false)\}$ characterises some models that verify $\{c_1 \rightarrow f_1, c_2 \rightarrow f_2, \ldots, c_7 \rightarrow f_7\}$. The first consequence is that each contextual theory admits at least one model. So, it is always consistent.

The second consequence is that any thought is possibly false. In L_c, uncertainty is intrinsically embedded in the syntax. To remedy this problem, we adopt the following principles:

- because any formula can be false, we cannot interrogate a contextual theory with a question such as "Is f true (or false)?". But we can say: "What can I conclude if I suppose that f is true (or false)?".

  Notation *f is an event (a question, a fact, a new thought, etc.) which generates a need for semantic interpretation. We call this a* ***stimulus***.
- because every thought is possibly false, we propose to relativize the semantic interpretation to the subsets of thoughts identified as *the most relevant*. We cannot conclude that f is true or false, but we can say "f is true (or false) with respect to the most relevant sets of thoughts".

For example, let a and b be two atomic propositions of L_p, and c_1, c_2 and c_3 be three thoughts. Consider the following set:

$$E_{Lc} = \{c_1 \rightarrow a, c_2 \rightarrow \neg a, c_3 \rightarrow b\}$$

We cannot prove that a or b is true or false. But we can say that a is true considering $\{c_1\}$, or that a is false and b is true considering $\{c_2, c_3\}$, etc. There are many possible combinations, so we should define a method for selecting "the most relevant sets of thoughts". For this purpose, we need some definitions.

Definitions *Let E_{Lc} be a theory of L_c and i and j be 2 integers such that $0 < i \leq j$.*

- *A set of thoughts is called a* ***context****.*
- *A context is said to be* ***of rank i to j*** *if all the thoughts in it are of rank i to j. A context of rank i to i is said of rank i.*
- *A context that is verified by at least one model of E_{Lc} is called a* ***possible*** *(or a consistent)* ***context****.*
- *A context that does not check any model of E_{Lc} is called an* ***impossible*** *(or an inconsistent)* ***context****.*
- *An impossible context is called a* ***minimal impossible context*** *if each of its strict subsets is possible.*
- *A possible context that has no strict extension that checks E_{Lc} is called a* ***maximal context****.*
- *A possible context is called the* ***credible context*** *if it has no intersection with a minimal impossible context and if all its strict extensions have an intersection with a minimal impossible context.*

In the following, and in accordance with common practice, we invariably use the notions of conjunction of formulae (for example: $c_1 \wedge c_2$) or of set of formulae (for example: $\{c_1, c_2\}$) to designate the same object. A conjunction of thoughts also means a context.

Example *Let a, b and c be three atomic propositions of L_p, and c_1, c_2, c_3, c_4 and c_5 be five thoughts. Consider the following set:*

$$E_{Lc} = \{c_1 \rightarrow a, c_2 \rightarrow \neg a, c_3 \rightarrow b, c_4 \rightarrow \neg b, c_5 \rightarrow c\}$$

$c_1 \wedge c_2$ and $c_3 \wedge c_4$ are the only two minimal impossible contexts. So, c_5 is the credible context, and there are four maximal contexts: $\{c_1, c_3, c_5\}$, $\{c_1, c_4, c_5\}$, $\{c_2, c_3, c_5\}$ and $\{c_2, c_4, c_5\}$.

We see that, for a given theory, there are possibly several maximal contexts (potentially empty) and a single credible context (potentially empty). They are obtained by calculating the minimal impossible contexts in a first step. The different possible combinations of thoughts then produce them.

We use these definitions to define the function that identifies the contexts considered most relevant for semantic interpretation.

Definition *Let E_{Lc} be a theory of L_c, S be a stimulus, i and j be two integers such that $0 < i < j$, and k be the maximal rank of the thoughts of E_{Lc}. The relevant contexts are defined as follows:*

- *calculation of the* ***maximal epistemic contexts****: if $k < j$ then there is an empty maximal epistemic context, else calculation on $\{E_{Lc}, S\}$ of the maximal contexts of rank j to k,*
- *then enrichment of each maximal epistemic context C, by the credible context of rank i to j-1 on $\{E_{Lc}, S, C\}$.*

This defines the set of ***epistemic contexts****. It is denoted $C_{ELc, S, i, j}$.*

This definition presents the notion of epistemic contexts. They are *the most relevant sets of thoughts*, which meets the need we identified earlier. Other definitions are possible, for example by using the ranks of thoughts more finely. Epistemic contexts are sufficient for the modelling needs presented in this article.

Example *Let a and b be two atomic propositions of L_p, c_1 and c_2 be two thoughts of rank 1, and c_3 and c_4 be two thoughts of rank 2. Consider the following set:*

$$E_{Lc} = \{c_1 \rightarrow a, c_2 \rightarrow \neg b, c_3 \rightarrow c_1, c_4 \rightarrow \neg c_1\}$$

Let i=1 and j=2, and we consider the stimulus is empty. $\{c_3, c_4\}$ is incoherent, so $\{c_3\}$ and $\{c_4\}$ are the two maximal contexts of rank 2. Let's add to each the credible context of rank 1 associated with it to calculate the two epistemic contexts. We obtain:

- *$\{c_1, c_2\}$ is the credible context of rank 1 considering $\{E_{Lc}, S, c_3\}$, so we have $\{c_3, c_1, c_2\}$,*
- *$\{c_2\}$ is the credible context of rank 1 considering $\{E_{Lc}, S, c_4\}$, so we have $\{c_4, c_2\}$.*

We are now able to define the semantic interpretation function of L_c.

Definition *Let E_{Lc} be a theory, S be a stimulus and i and j be two integers such that $0 < i < j$. Considering E_{Lc}, S, i and j, a sentence f, called a piece of* ***belief****, is said:*

- ***conceivable*** *if, and only if, there is at least one epistemic context C_1 such that $\{E_{Lc}, S, C_1\} \vDash_{Lp} f$, and there is at least one epistemic context C_2 such that $\{E_{Lc}, S, C_2\} \vDash_{Lp} \neg f$,*
- ***credible*** *if, and only if, there is at least one epistemic context C_1 such that $\{E_{Lc}, S, C_1\} \vDash_{Lp} f$, and there is no epistemic context C_2 such that $\{E_{Lc}, S, C_2\} \vDash_{Lp} \neg f$,*
- ***improbable*** *if, and only if, there is at least one epistemic context C_1 such that $\{E_{Lc}, S, C_1\} \vDash_{Lp} \neg f$, and there is no epistemic context C_2 such that $\{E_{Lc}, S, C_2\} \vDash_{Lp} f$,*
- ***not interpretable*** *in other cases.*

$\{E_{Lc}, S, C \in C_{ELc, S, i, j}\}$ is called a ***semantic perspective****.*

This definition presents the basic semantic interpretation function of L_c. It can be enriched, for example by distinguishing true formulae in all semantic perspectives. This version is sufficient for the modelling needs presented in this article.

Example *Let us return to the epistemic contexts of the previous example. We obtain:*

- *$\{c_3, c_1, c_2\}$: the associated semantic perspective says that a is true and b is false,*
- *$\{c_4, c_2\}$: the associated semantic perspective says that b is false.*

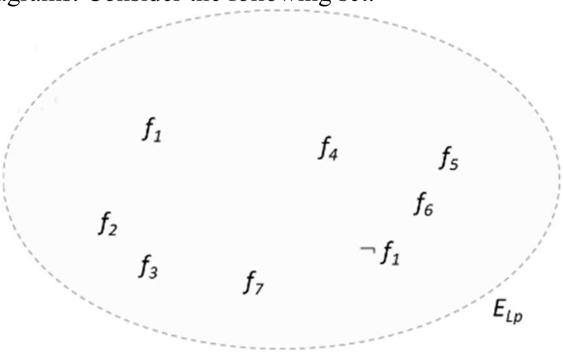
So, according to the semantic vocabulary of L_c, a is credible and b is improbable.

We have just presented the mathematical definition of the semantic interpretation function of L_c. This is not compatible with our natural language habits. Therefore, we will allow ourselves some linguistic shortcuts, for example: a context is said to be a belief or a piece of knowledge, and *vice versa*, a semantic perspective is said to be a perspective, a conceivable expression is said to be true and false, or possible, a credible expression is said to be true, conceivable, or possible, and an improbable expression is said to be false, conceivable, incredible, or impossible. We will use them in a way that does not create confusion.

5 The properties of L_c

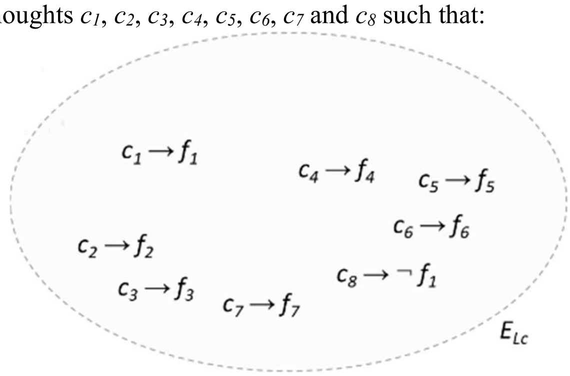
We now present the properties of L_c. We remain on a technical observation and not discuss their relevance. Indeed, each property echoes philosophical concepts and deserves a dedicated article. The debates are rich, and there are as many defenders as detractors. We do not bring new philosophical elements to enrich these exchanges – only a few practical findings that we will share in paragraphs 6 and 7.

To facilitate the understanding of what is to come, we propose to illustrate the principles of L_c with some small diagrams. Consider the following set:



Pic. 3 - A syntactically inconsistent theory of L_p

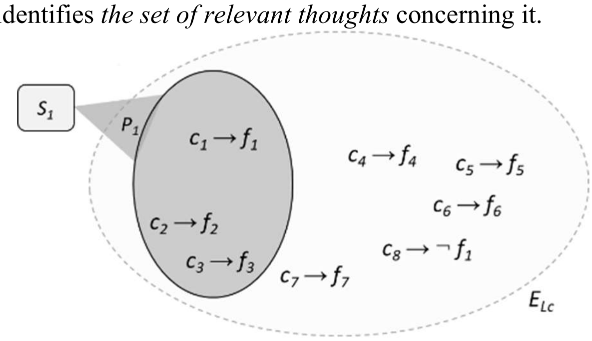
It is syntactically inconsistent, and it has no semantic meaning according to contextual postulate. Let us apply it by enriching the set of atomic propositions with the thoughts c_1, c_2, c_3, c_4, c_5, c_6, c_7 and c_8 such that:



Pic. 4 - A semantically inconsistent theory of L_c

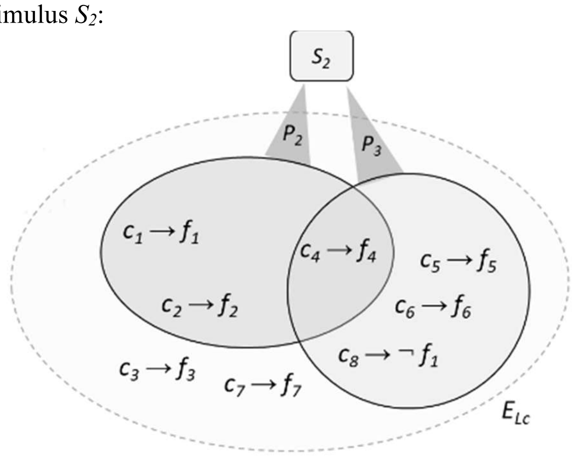
The resulting set E_{Lc} is a kind of dictionary of thoughts: each thought c_i is expressed by a formula f_i. It is syntactically consistent even though the thoughts may be semantically incoherent with each other. This result is obtained at the cost of an absolute uncertainty: all thoughts are potentially false.

To analyse a situation (formalised by a stimulus), L_c identifies *the set of relevant thoughts* concerning it.



Pic. 5 – The semantic interpretation of the stimulus S_1

In this example (presented for illustrative purposes only), the stimulus S_1 generates the semantic perspective P_1. Its epistemic context is $\{c_1, c_2, c_3\}$. c_1, c_2 and c_3 are the most relevant thoughts considering S_1. Assume another stimulus S_2:



Pic. 6 – The semantic interpretation of the stimulus S_2

S_2 is seen through two mutually incoherent semantic perspectives P_2 and P_3.

The syntax production function is monotonic

L_c respects the syntax of the propositional logic and is therefore syntactically monotonic: whatever f and g are contextually well-formed formulae, if a theory E_{Lc} produces f then $\{E_{Lc}, g\}$ produces f. Note that the syntactic interpretation function is mechanically also monotonic.

The semantic interpretation function is non-monotonic

L_c decorrelates the syntactic interpretation function from the semantic interpretation function. Syntax produces a set of formulae according to the rules of the contextualised formalism. Semantics then provides an interpretation by analysing the behaviour of the thoughts in the models of the theory. They are considered as they are produced by the syntactic rules of L_p.

The models of a theory can change if a new piece of knowledge is introduced. So, $C_{ELc, i, j}$ must be recalculated in this case, and L_c has a non-monotonic semantic:

considering a stimulus S, a formula f can be credible considering $\{E_{Lc}, S\}$ and incredible considering $\{\{E_{Lc}, g\}, S\}$, for g a contextual well-formed formula. We present some examples of use in the following paragraphs.

The semantic is faillibilistic

Non-monotonicity is a matter of completeness of knowledge: a belief that is true in one state may be false in an enriched state. Faillibilism (K. Popper [13]) is a more radical philosophical principle. It assumes that knowledge is impossible: all belief can, at any time, be questioned – and possibly contradicted.

A consequence of contextual postulate is that every proposition (which is not an axiom) is possibly false. To avoid this, the solution is to consider that what is not explicitly false is credible and will remain so until it is explicitly contradicted or challenged. We illustrate this with some examples which we develop in the following paragraphs.

The semantic is perspectivistic

Perspectivism (F. Kaulbach [7]) refers to philosophical doctrines that defend the idea that our perception of reality is composed of the sum of the perspectives we have on it.

In L_c, belief is not the consequence of a global point of view built on the whole of thoughts, but the juxtaposition of several points of view built from distinct subsets of thoughts each considered to be relevant.

Atomic propositions of L_p are attributes and not assertions

In L_c, a mechanical consequence of the application of contextual postulate is that it is not possible to deduce that a proposition is true or false according to L_p. A proposition can only be interpreted in relation to a set of thoughts, called a context. It characterises it. Consider, for example, the sentence:

"*If Tweety is a bird, then it flies*".

Its modelling in predicate logic can be:

$$Bird(Tweety) \rightarrow Fly(Tweety)$$

In L_c, this assertion is modelled by:

$$\{c_1 \rightarrow (Tweety \rightarrow Bird), c_2 \rightarrow (Bird \rightarrow Fly)\}$$

which allows for several readings - for example: *Bird* and *Fly* are attributes of the stimulus *Tweety* if we consider the context $\{c_1, c_2\}$.

Reasoning is introspective

By distinguishing expression and thought, and by modelling a relationship between them, contextual postulate brings a capacity for introspective reasoning to the formal language: the thoughts can reason about themselves using the constraints carried by the sentences that express them.

We have finished with the presentation of L_c. The following sections show how to use its properties to provide answers to the two last difficulties identified in paragraph 3 (the first being covered as we have just observed).

Considering the definition of epistemic contexts, i and j can theoretically take any value. According to the work of J. Pitrat [12], there are probably cognitive thresholds limiting human reasoning abilities. In the rest of this document, we use the thresholds *2* and *3*, which are sufficient to cover the expected level of expressiveness expected in this article.

We will see in paragraph 7 for what we reserve rank 1. And by writing convention, we now note $c_{i,j}$ the thoughts. i singularizes the atomic proposition and j indicates its rank.

6 Modelling an inconsistent information

For example, consider a set of L_p's propositions $\{a, b, c\}$ and let be the following set:

$$E_{Lp} = \{a \rightarrow b, a \rightarrow \neg b, c, a\}$$

It is inconsistent because $E_{Lp} \vdash_{Lp} b \wedge \neg b$. But E_{Lp} has no meaning according to contextual postulate. Let us now place ourselves in the contextual logic framework. Considering the set of thoughts $\{c_{10,2}, c_{20,2}, c_{30,2}, c_{40,2}\}$, we assume the following theory:

$$\begin{aligned} E_{Lc} = \{&c_{10,2} \rightarrow (a \rightarrow b), \\ &c_{20,2} \rightarrow (a \rightarrow \neg b), \\ &c_{30,2} \rightarrow c, \\ &c_{40,2} \rightarrow a\} \end{aligned}$$

Note that thoughts name formulas, which will allow us to carry out introspective reasoning. E_{Lc} is consistent, and there is an incoherence between the three thoughts $c_{10,2}$, $c_{20,2}$, and $c_{40,2}$ because:

$$\{E_{Lc}, c_{10,2}, c_{20,2}, c_{40,2}\} \vdash_{Lp} b \wedge \neg b$$

$\{c_{10,2}, c_{20,2}, c_{40,2}\}$ is a minimal impossible context. So, $\{c_{30,2}\}$ is the only epistemic context. If the stimulus is empty, we obtain one perspective which says $\{c\}$, and $\{a, c\}$ if the stimulus is $\{a\}$.

This is a first result showing the possibility of exploiting inconsistent beliefs in L_c. The solution is to get around the problem by considering that the thoughts $c_{10,2}$, $c_{20,2}$ and $c_{40,2}$ are not credible because they produce an inconsistency.

We now want to address this inconsistency, by modelling that $a \rightarrow b$ (i.e., the thought $c_{10,2}$) is not always true - or, put differently, is sometimes true and sometimes false. Let's use two new thoughts, $c_{11,3}$ and $c_{12,3}$:

$$\begin{aligned} E_{Lc} = \{&c_{10,2} \rightarrow (a \rightarrow b), \\ &\mathbf{c_{11,3} \rightarrow c_{10,2}}, \\ &\mathbf{c_{12,3} \rightarrow \neg c_{10,2}}, \\ &c_{20,2} \rightarrow (a \rightarrow \neg b), \\ &c_{30,2} \rightarrow c, \\ &c_{40,2} \rightarrow a\} \end{aligned}$$

Considering a is the stimulus, let's calculate the epistemic contexts. There are 2 maximum contexts of rank 3: $\{c_{11,3}\}$ and $\{c_{12,3}\}$. Let us extend each of them to their associated credible contexts of rank 2:

- considering $\{E_{Lc}, a, c_{11,3}\}$, $\{c_{20,2}, c_{40,2}\}$ is the only minimal impossible context, so $\{c_{10,2}, c_{30,2}\}$ is the credible context of the rank 2 in this case,
- considering $\{E_{Lc}, a, c_{12,3}\}$, $\{c_{10,2}\}$ is the only minimal impossible context, and $\{c_{20,2}, c_{30,2}, c_{40,2}\}$ is the credible context in this.

In fine, considering the stimulus $\{a\}$, we obtain two epistemic contexts:

- $\{c_{11,3}, c_{10,2}, c_{30,2}\}$ which says $\{a, b, c\}$ is true,
- $\{c_{12,3}, c_{20,2}, c_{30,2}, c_{40,2}\}$ which says $\{a, \neg b, c\}$ is true.

Taking a as the stimulus, we conclude that c is credible (or true), and that b is conceivable (or true and false). The formalism does this by modelling an epistemic information: the belief $a \rightarrow b$ (i.e., $c_{10,2}$) is true (what $c_{11,3}$ formalises) and false (what $c_{12,3}$ formalises).

We have used a single contradiction $\{c_{11,3}, c_{12,3}\}$ to illustrate our point. If multiple contradictions (two contradictions $\{c_{x1,3}, c_{x2,3}\}$ and $\{c_{y1,3}, c_{y2,3}\}$ for example), the different cases are managed on the maximal epistemic contexts ($\{c_{x1,3}, c_{y1,3}\}$, $\{c_{x1,3}, c_{y2,3}\}$, $\{c_{x2,3}, c_{y1,3}\}$, and $\{c_{x2,3}, c_{y2,3}\}$ with the example).

We obtain by combination the set of relevant perspectives. This is illustrated in the example that we develop at the end of this article.

Comparison with other formalisms

In this section, we point out the major gaps in the treatment of inconsistent or incomplete information between L_c and other non-classical formalisms.

The first and, from our point of view, the main difference is that contextual logic does not quit the syntax of propositional logic. Contrary to what is commonly shared, it is not necessary to add new connectors or to modify the syntactic rules of L_p to model a notion of inconsistency. It's not the only gap.

Paraconsistent and multivalued logics aim to tolerate inconsistencies by escaping the principle of explosion. The approach, theorised by J. Lukaszewicz [10], is either to weaken Aristotle's principles to limit the inferential capacities of language or to add a third truth value to indicate that the piece of knowledge concerned is both true and false.

L_c addresses the issue of uncertainty and inconsistency through its perspectivistic property: it models that something is simultaneously true according to some thoughts and false according to others. L_c is therefore not a paraconsistent or a multivalued formalism: it preserves the syntactic rules of L_p. Therefore, it does not escape the principle of explosion. If one retains a reference context that syntactically produces $f \wedge \neg f$, then it produces any belief g whatsoever.

Inconsistency is accepted in the semantic interpretation of L_c. It remains non-tolerable in its syntax.

Another difference between L_c and other non-classical formalisms is its faillibilistic property: noting that nothing is true, it takes as credible what is possible. This property gives L_c a particular behaviour, which does not allow it to fully capture modal logics or default logics, for example.

Default logic is proposed by R. Reiter [15]. To reason with uncertain information, he extends production rules by expressions of the form ($a : b / c$) which read:

If a is true and if b is possible then c is produced

In L_c, the thought that b is possible "generates the thought b". However, related to R. Reiter's syntax, L_c's expressiveness is limited to normal default rules, of the form ($a : b / b$) [8].

Modal epistemic logics extend the expressiveness of languages by adding a new connector for reasoning about the quality of the interpretative value. The most widely used epistemic modal connector is the alethic connector $\Box$. $\Box f$ usually expresses that f is necessary, and its dual $\neg\Box\neg f$, denoted $\Diamond f$, that f is possible. The language relies on the semantics of possible worlds of S.A. Kripke [9] to benefit from a syntactic interpretation function of $\Box$.

We have proposed a relationship between modal epistemic logics and L_c [8]. This requires an evolution of the definition of epistemic context, using ranks to capture the imbrications of the monadic connector (rank i for $\Box$, rank $i+1$ for $\Box\Box$, etc.). It models the sets $\{\Box f\}$ and $\{\Diamond f, \Diamond\neg f\}$, but the set restricted to $\{\Diamond f\}$ is interpreted as $\{\Box f\}$: f is considered necessary if the possibility of its opposite is not explicitly expressed.

The behaviour of L_c is equivalent to adding a default rule to the K system: $\{\Diamond f : \Box f / \Box f\}$. In the framework of Kripke's semantics, this expression can be understood as:

If I know an accessible world in which f is true, and if I do not know an accessible world in which ¬ f is true, then I consider that f is true in all accessible worlds.

Paraconsistent, default and modal epistemic logics deal with the issue of incoherence by evolving the syntactic capabilities of the language. D. Batens proposes another approach [3]. He considers that there are several reasoning strategies, and that the solution consists in choosing the one that is best adapted to the situation. These are adaptive logics. For example, let be the following set of formulae:

$$E_{La} = \{\neg p,\ \neg q,\ p \vee q,\ p \vee r,\ q \vee r\}$$

It is incoherent, and therefore explosive in the context of propositional logic because $\{\neg p, \neg q\}$ contradicts $p \vee q$. If one adopts a strategy favouring reliable reasoning, it is not possible to deduce r: it would be unwise to conclude anything using the first three formulae. However, if we choose a strategy that minimises abnormalities, and assume that at least two of the first three formulae are true, then r is produced.

In contextual logic, the set becomes:

$$E_{Lc} = \{c_{1,2} \rightarrow \neg p,\ c_{2,2} \rightarrow \neg q,\ c_{3,2} \rightarrow p \vee q,\ c_{4,2} \rightarrow p \vee r,\ c_{5,2} \rightarrow q \vee r\}$$

Assume that the stimulus is empty. $\{c_{4,2}, c_{5,2}\}$ is the reference context because $\{c_{1,2}, c_{2,2}, c_{3,2}\}$ is a minimal impossible context. As far as we know, r is not interpretable. Using epistemic contexts that retain the maximal credible contexts at rank 2 is therefore a prudent strategy.

So, L_c is not an adaptive logic. Both formalisms have the capacity to adapt their semantic interpretation to local characteristics: L_c chooses to use or not a thought depending on the stimulus. But its principle is not to adapt its reasoning according to the typology of the situation. It uses a unique analysis strategy, based on the definition of epistemic contexts.

We end this comparative section with the circumscription logic of J. McCarthy [11]. It consists in extending the set of atomic propositions by some atomic propositions that indicate the epistemic character of a formula. For example, "$a \rightarrow b$ is true except in atypical cases" is modelled by $((a \rightarrow b) \vee abnormal)$.

The atomic proposition *abnormal* carries the exceptional behaviours when needed. The models of the theory are then analysed to select those that minimise the abnormalities.

This approach, which consists in seeking a solution by enriching the set of atomic propositions and then analysing the models of the theory, is most certainly the closest to ours. We have shown that contextual logic can capture its expressive capacity by adapting the definition of epistemic contexts to meet the minimality criterion [8].

However, beyond this result, the choice to minimise abnormalities seems reasonable but can easily be questioned with use cases. This difficulty is shared with adaptive logics, or more generally with the concept of epistemic rooting proposed by P. Gardenfors and D. Makinson [6].

Indeed, these methods suppose the existence of an order relation (on pieces of knowledge or on reasoning strategies) which would oversee selecting the information in case of incoherence. In L_c, syntactic consistency is guaranteed. It is therefore not necessary to manage this in the formalism.

7 Modelling a predicate information

L_p sees a proposition as a whole, which is given a universal value. It is necessary to decompose this whole when we wish to use a singular value. To this end, predicate logic meets this need by allowing the desired relationship to be modelled directly in the elementary proposition.

It then becomes possible to model that *Socrates is a man*, and to deduce that *Socrates is mortal* because *a man is mortal*:

$$Man(Socrates)$$
$$Man(Socrates) \rightarrow Mortal(Socrates)$$

This syllogism uses the link between *Man* and *Socrates* to deduce the association with *Mortal*. In this context, G. Frege [5] theorised the notion of universal quantifier.

As a classical example of use:

$$\forall\, x\; Man(x) \rightarrow Mortal(x)$$

which reads: whatever x is, if x is a man then x is mortal. We note two enrichments with respect to the native modelling capabilities of the propositional logic:

– the notion of the universal quantifier $\forall$. We will come back to the subject of universal connectors or universal quantifier at the paragraph 9,
– the possibility of breaking down an atomic proposition of L_p into several singular instances. In our example, the proposition *he is a man* is modelled by using two distinct units, *Man* and x. The expression *Man(x)* creates a syntactic relationship, which formalises a semantic link, between these two-unit elements. x then allows to create a semantic link between the two propositions *Man(x)* and *Mortal(x)*.

We have seen in paragraph 3 that it is not possible to model a relation between two atomic propositions in L_p because of the symmetric behaviour of connectors. We now present how this point can be solved in L_c.

Suppose in L_p the set of propositions $\{a, b, c, d, e, f\}$ and the following theory:

$$E_{Lp} = \{a \rightarrow b,\; c \rightarrow d,\; a \rightarrow \neg c,\; c \rightarrow (a \rightarrow e),\; c \rightarrow (a \rightarrow f)\}$$

We want to model that $a \rightarrow e$ is a predicate of c – i.e., c is the subject of $a \rightarrow e$. The problem is that the formula $c \rightarrow (a \rightarrow e)$ is syntactically equivalent to $a \rightarrow (c \rightarrow e)$. Now, consider the following set in L_c:

$$E_{Lc} = \{c_{10,2} \rightarrow (a \rightarrow b),\; c_{20,2} \rightarrow (c \rightarrow d),\; c_{30,2} \rightarrow (a \rightarrow \neg c),\; \mathbf{c_{40,2} \rightarrow (c \rightarrow c_{41,1})},\; \mathbf{c_{41,1} \rightarrow (a \rightarrow e)},\; c_{50,2} \rightarrow (c \rightarrow (a \rightarrow f))\}$$

$c_{40,2}$ and $c_{41,1}$ model the predicative piece of knowledge. They break the syntactic symmetry and introduce two new pieces of information: the predicate is syntactically distinguished by the thought $c_{41,1}$, and $c_{40,2}$ says that $c_{41,1}$ is true if the subject c is true. It is important to note that $c_{41,1}$ is of rank 1, so, by definition, it does not belong to an epistemic context. It only appears later, in the contexts in which $c_{40,2}$ is true if c is true.

Let $c \wedge a$ be the stimulus. $\{c_{10,2}, c_{20,2}, c_{40,2}, c_{50,2}\}$ is its epistemic context. The perspective associated says $\{c, a, b, d, e, f, c_{41,1}\}$. The presence of $c_{41,1}$ indicates that $a \rightarrow e$ is a predicate, but the relationship with its subject c is not apparent. It can be found by applying the following method:

– calculate the perspectives of the stimulus c. It says $\{c, \neg a, d, c_{41,1}\}$. The predicate $c_{41,1}$ is obtained by $c_{40,2}$,
– calculate the perspectives of the stimulus a. It says $\{a, \neg c, b\}$. The thought $c_{41,1}$ is not syntactically produced,
– apply the predicates associated with each perspective to the other perspectives. This produces $\{e\}$ by applying $c_{41,1}$ to the perspective of a. e is associated with the stimulus c associated with the applied predicate,
– and finally, calculate the perspectives of the stimulus $c \wedge a$. This produces $\{f\}$.

If the thought of a predicate appears in a perspective, then it expresses that its subject is its stimulus. Unlike in predicate logic, the notion of predicate is not carried by the syntax of L_c. It is interpreted from the semantics of perspectives. This method extends the consumption of epistemic contexts by a recursive function:

The semantics of a composite universe is obtained by crossing the semantics of the objects that compose it.

The semantics of a universe composed of three objects A, B and C can only be partially obtained by analysing the perspectives of $A \wedge B \wedge C$. To obtain a complete perception, we must analyse $\{A\}$, $\{B\}$, $\{C\}$, $\{A, B\}$, $\{A, C\}$, $\{B, C\}$ and $\{A, B, C\}$ separately, and cross-reference the properties associated with these seven different objects.

We call this method the **generalised contextual semantics**. It allows exhaustively capturing the characteristics of each object, of each combination of objects, and to calculate cross-predictive inferences. It seems to produce redundancies. We have not studied whether technical optimizations are possible.

8 An example of application

After presenting the theoretical principles of contextual logic, we propose to develop an example of application to clarify our purpose, and to illustrate the knowledge modelling capabilities of L_c. To do this, we use the example of the bird Tweety, a classical case study in the literature on non-monotonicity and belief revision.

Example *Consider the following knowledge, which we call the E_{NL} (for Natural Language) set:*

Birds and felines are animals [01 and 02]. Birds are not felines [03]. Animals are diurnal [04]. Diurnal animals are not nocturnal [05]. Birds fly [06]. They are insectivorous [07] and gregarious [08]. Felines are carnivorous [09] and solitary [10]. Solitaires are not gregarious [11]. Insectivores are not carnivorous [12]. Swallows, sparrows, ostriches, and owls are birds [13, 14, 15 and 16]. Swallows are not sparrows [17], ostriches [18], or owls [19]. Sparrows are not ostriches [20] or owls [21]. Ostriches are not owls [22]. Ostriches do not fly [23]. Owls are solitary [24], nocturnal [25], carnivorous [26], and insectivorous [27]. Cats and lions are felines [28 and 29]. Cats are not lions [30]. Cats are nocturnal [31]. Lions are gregarious [32]. Carnivores are hunters [33]. Herbivores are prey for hunters [34]. Hunters attack prey [35]. If the prey is larger than the hunter, the latter does not attack [36]. Ostriches are larger than cats [37] and owls [38].

E_{NL} contains a lot of inconsistent, epistemic, and predicative information - for example: birds fly and do not fly, birds are insectivores and carnivorous, and hunters attack prey (and not the reverse).

We have chosen to take a relatively large knowledge base to model several cross-cases, and we will develop below the example in detail. What follows is therefore quite tedious to read. We apologize to the readers for this. The objective is to give them enough element to reproduce the exercise if they wish.

This knowledge is deemed to escape the syntax of propositional logic. We are however going to show that it is sufficient to model and exploit them.

In a first time, we propose to translate it into the syntax of L_p by the following formulae.

01 *Bird → Animal*
02 *Feline → Animal*
03 *Bird → ¬ Feline*
04 *Animal → Diurnal)*
05 *Diurnal → ¬ Nocturnal*
06 *Bird → Fly*
07 *Bird → Insectivore*
08 *Bird → Gregarious*
09 *Feline → Carnivore*
10 *Feline → Solitary*
11 *Gregarious → ¬ Solitary*
12 *Insectivore → ¬ Carnivore*
13 *Swallow → Bird*
14 *Sparrow → Bird*
15 *Ostrich → Bird*
16 *Owl → Bird*
17 *Swallow → ¬ Sparrow*
18 *Swallow → ¬ Ostrich*
19 *Swallow → ¬ Owl*
20 *Sparrow → ¬ Ostrich*
21 *Sparrow → ¬ Owl*
22 *Ostrich → ¬ Owl*
23 *Ostrich → ¬ Fly*
24 *Owl → Solitary*
25 *Owl → Nocturnal*
26 *Owl → Carnivore*
27 *Owl → Insectivore*
28 *Cat → Feline*
29 *Lion → Feline*
30 *Cat → ¬ Lion*
31 *Cat → Nocturnal*
32 *Lion → Gregarious*
33 *Carnivore → Hunter*
34 *Herbivore →* (*Hunter → Prey*)
35 *Hunter →* (*Prey → Attack*)
36 *Hunter →* (*Prey →* (*Larger → ¬ Attack*))
37 *Ostrich →* (*Cat → Larger*)
38 *Ostrich →* (*Owl → Larger*)

Let us consider that the knowledge is entered in the order in which it appears, according to the following algorithm:

For each formula f_i
    If f_i has a subject, then PK
    Else
        Creating the thought $c_{i0,2}$
        Creating the formula $c_{i0,2} \rightarrow f_i$
        If there is a contradiction, then EM

We assume there is a monitor accompanying the learning. For example, we don't have a grammatical module allowing us to identify the subjects of each sentence. So, the system queries its instructor. Up to formula 34, the instructor's response is that there is none. Indeed, the identification of the subject is not always necessary. Doing it systematically would only make the presentation more cumbersome.

The sentences are processed one after the other (the description of the modules EM and PK will come later).

01 $c_{010.2}$ *→ (Bird → Animal)*
02 $c_{020.2}$ *→ (Feline → Animal)*
..
24 $c_{240.2}$ *→ (Owl → Solitary)*

The formula is integrated, and the system tested its semantic consistency. Until formula 23, there is no question. The formula 24 is then integrated, and a potential inconsistency is detected: owls are birds [16], birds are gregarious [08], owls are solitary [24] and gregarious is not solitary [11]. So, owls do not exist, or they are solitary and not solitary.

We consider that there is no automatic solution to resolve an inconsistency. The properties of L_c make it possible to record it as it arrives, and additional information can possibly bring precision. For this, the system interrogates its instructor. He has three possible answers:

1) he does not know, or he considers it is normal: the semantic inconsistency is accepted, and the system moves on to the next step,
2) he indicates that one of the pieces of knowledge involved in the inconsistency is false. In this case, the system applies the EM module to the indicated formula,
3) he indicates that one of the pieces of knowledge involved in the inconsistency is true and false: it is an alethic information. In this case, birds are gregarious [08] is sometimes true and sometimes false. Then the system applies the EM module to the indicated formula.

The EM module is:

if f_i is false *// Case 2*
  Creating the thoughts $c_{i2,3}$
  Creating the formula $c_{i2,3} \rightarrow \neg\, c_{i0,2}$
  // The combinatorics on the maximal contexts
  // of T_m means that $c_{i0,2}$ will no longer be
  // retained in the epistemic contexts
else if f_i is true and false *// Case 3*
  Creating the thoughts $c_{i1,3}$ et $c_{i2,3}$
  Creating the formula $c_{i1,3} \rightarrow c_{i0,2}$
  Creating the formula $c_{i2,3} \rightarrow \neg\, c_{i0,2}$

We identify below the formulae that are affected by this module. We will only use the third last case which is the most interesting.

04 $Animal \rightarrow Diurnal$
06 $Bird \rightarrow Fly$
07 $Bird \rightarrow Insectivore$
08 $Bird \rightarrow Gregarious$
10 $Feline \rightarrow Solitary$
12 $Insectivore \rightarrow \neg\, Carnivore$
35 $Hunter \rightarrow (Prey \rightarrow Attack)$

Let's continue the treatment. We obtain:

01 $c_{010.2} \rightarrow (Bird \rightarrow Animal)$
02 $c_{020.2} \rightarrow (Feline \rightarrow Animal)$
03 $c_{030.2} \rightarrow (Bird \rightarrow \neg\, Feline)$
04 $c_{040.2} \rightarrow (Animal \rightarrow Diurnal)$
$c_{041.3} \rightarrow c_{040.2}$
$c_{042.3} \rightarrow \neg\, c_{040.2}$
05 $c_{050.2} \rightarrow (Diurnal \rightarrow \neg\, Nocturnal)$
..
32 $c_{320.2} \rightarrow (Lion \rightarrow Gregarious)$
33 $c_{330.2} \rightarrow (Carnivore \rightarrow Hunter)$

We come to the formula number 34: $Herbivore \rightarrow (Hunter \rightarrow Prey)$. Before integration, the system asks if it contains a subject.

Until now, we had not used this possibility so as not to make the presentation unnecessarily heavy. But, within the framework of this formula, the identification of the subject is necessary. So, the answer is yes, for *Herbivore*.

The PK module is then applied:

// f_i is a clause, so it is of type $g \rightarrow h$
// with g is a conjunction of literals
// and h is a disjunction of literals
// g is indicated by the instructor
Creating the thoughts $c_{i0,2}$ et $c_{i3,1}$
Creating the formula $c_{i0,2} \rightarrow g \rightarrow c_{i3,1}$
Creating the formula $c_{i3,1} \rightarrow h$

We identify below the formulae that are affected by this module.

34 $Herbivore \rightarrow (Hunter \rightarrow Prey)$
35 $Hunter \rightarrow (Prey \rightarrow Attack)$
36 $Hunter \rightarrow (Prey \rightarrow (Larger \rightarrow \neg\, Attack))$
37 $Ostrich \rightarrow (Cat \rightarrow Larger)$
38 $Ostrich \rightarrow (Owl \rightarrow Larger)$

After application to the whole of E_{NL}, we obtain:

01 $c_{010.2} \rightarrow (Bird \rightarrow Animal)$
02 $c_{020.2} \rightarrow (Feline \rightarrow Animal)$
03 $c_{030.2} \rightarrow (Bird \rightarrow \neg\, Feline)$
04 $c_{040.2} \rightarrow (Animal \rightarrow Diurnal)$
$c_{041.3} \rightarrow c_{040.2}$
$c_{042.3} \rightarrow \neg\, c_{040.2}$
05 $c_{050.2} \rightarrow (Diurnal \rightarrow \neg\, Nocturnal)$
06 $c_{060.2} \rightarrow (Bird \rightarrow Fly)$
$c_{061.3} \rightarrow c_{060.2}$
$c_{062.3} \rightarrow c_{060.2}$
07 $c_{070.2} \rightarrow (Bird \rightarrow Insectivore)$
$c_{071.3} \rightarrow c_{070.2}$
$c_{072.3} \rightarrow \neg\, c_{070.2}$
08 $c_{080.2} \rightarrow (Bird \rightarrow Gregarious)$
$c_{081.3} \rightarrow c_{080.2}$
$c_{082.3} \rightarrow \neg\, c_{080.2}$
09 $c_{090.2} \rightarrow (Feline \rightarrow Carnivore)$
10 $c_{100.2} \rightarrow (Feline \rightarrow Solitary)$
$c_{101.3} \rightarrow c_{100.2}$
$c_{102.3} \rightarrow \neg\, c_{100.2}$
11 $c_{110.2} \rightarrow (Gregarious \rightarrow \neg\, Solitary)$
12 $c_{120.2} \rightarrow (Insectivore \rightarrow \neg\, Carnivore)$
$c_{121.3} \rightarrow c_{120.2}$
$c_{122.3} \rightarrow \neg\, c_{120.2}$
13 $c_{130.2} \rightarrow (Swallow \rightarrow Bird)$
14 $c_{140.2} \rightarrow (Sparrow \rightarrow Bird)$
15 $c_{150.2} \rightarrow (Ostrich \rightarrow Bird)$
16 $c_{160.2} \rightarrow (Owl \rightarrow Bird)$
17 $c_{170.2} \rightarrow (Swallow \rightarrow \neg\, Sparrow)$
18 $c_{180.2} \rightarrow (Swallow \rightarrow \neg\, Ostrich)$
19 $c_{190.2} \rightarrow (Swallow \rightarrow \neg\, Owl)$
20 $c_{200.2} \rightarrow (Sparrow \rightarrow \neg\, Ostrich)$
21 $c_{210.2} \rightarrow (Sparrow \rightarrow \neg\, Owl)$
22 $c_{220.2} \rightarrow (Ostrich \rightarrow \neg\, Owl)$
23 $c_{230.2} \rightarrow (Ostrich \rightarrow \neg\, Fly)$
24 $c_{240.2} \rightarrow (Owl \rightarrow Solitary)$
25 $c_{250.2} \rightarrow (Owl \rightarrow Nocturnal)$
26 $c_{260.2} \rightarrow (Owl \rightarrow Carnivorous)$
27 $c_{270.2} \rightarrow (Owl \rightarrow Insectivore)$
28 $c_{280.2} \rightarrow (Cat \rightarrow Feline)$
29 $c_{290.2} \rightarrow (Lion \rightarrow Feline)$
30 $c_{300.2} \rightarrow (Cat \rightarrow \neg\, Lion)$
31 $c_{310.2} \rightarrow (Cat \rightarrow Nocturnal)$
32 $c_{320.2} \rightarrow (Lion \rightarrow Gregarious)$
33 $c_{330.2} \rightarrow (Carnivore \rightarrow Hunter)$
34 $c_{340.2} \rightarrow (Herbivore \rightarrow c_{343.1})$
$c_{343.1} \rightarrow (Hunter \rightarrow Prey)$
35 $c_{350.2} \rightarrow (Hunter \rightarrow c_{353.1})$
$c_{351.3} \rightarrow c_{350.2}$

$c_{352.3} \rightarrow \neg\, c_{350.2}$
$c_{353.1} \rightarrow (Prey \rightarrow Attack)$
36 $c_{360.2} \rightarrow (Hunter \rightarrow c_{363.1})$
$c_{363.1} \rightarrow (Prey \rightarrow (Larger \rightarrow \neg\, Attack))$
37 $c_{370.2} \rightarrow (Ostrich \rightarrow c_{373.1})$
$c_{373.1} \rightarrow (Cat \rightarrow Larger)$
38 $c_{380.2} \rightarrow (Ostrich \rightarrow c_{383.1})$
$c_{383.1} \rightarrow (Owl \rightarrow Larger)$

We have converted E_{NL} into a set E_{Lc} that is a theory of L_p. It is consistent. L_c uses the syntactic rules of propositional logic. Here are some examples obtained by applying the generalised contextual semantic:

- if the stimulus is *Bird*: birds are animals, diurnal, gregarious, insectivore, and fly,
- if the stimulus is *Swallow*: swallows are birds, animals, diurnal, gregarious, insectivore, and fly,
- if the stimulus is *Ostrich*: ostriches are birds, animal, diurnal, gregarious, insectivore, and do not fly,
- if the stimulus is *Owl*: owls are birds, animals, nocturnal, solitary, carnivorous, and fly,
- if the stimulus is *Cat*: cats are feline, animals, diurnal, carnivorous and solitary,
- if the stimulus is *Lion*: lions are feline, animals, diurnal, carnivorous and gregarious,
- if the stimulus is {*Cat*, *Sparrow*}: the sparrow is attacked,
- if the stimulus is {*Cat*, *Owl*, *Sparrow*, *Ostrich*}: the sparrow is in a bad way, but the ostrich can go about its business,
- and if the stimulus is {*Lion*, *Ostrich*}: the ostrich would have some reason to be worried.

Informatic tool

The different steps are described in detail, which probably makes reading tedious at times. The goal is to allow the readers to control and to reproduce the process if they wish.

We use a classical propositional logic solver to calculate the minimum inconsistent contexts, and a classical combinatorial algorithm using the minimum inconsistent contexts to calculate the epistemic contexts.

These tools are common and easily accessible. We hold those which we developed at the disposal of the readers who would like it.

Use case

We use the example of Tweety, which may seem simplistic or even naive. Its first interest is to be understandable by all, while illustrating all the theoretical problems of non-monotonicity and of belief revision.

Any other subject could have done the trick. The syntactic and semantic rules used were defined by the theory, and we did not integrate any behavioural rules specific to the subject.

Modelled information

While strictly preserving the syntactic rules of propositional logic, L_c allows us to define a semantic function on the models of the theories of L_p. It exploits:

- semantically incoherent pieces of knowledge: *Birds fly. Ostriches are birds. Ostriches do not fly* (thoughts $c_{060,2}$, $c_{150,2}$ and $c_{230,2}$),
- predicative pieces of knowledge: *Herbivores are prey for hunters* (thoughts $c_{340,2}$ and $c_{343,1}$),
- and alethic modal pieces of knowledge: *Birds are generally insectivorous* (thoughts $c_{070,2}$, $c_{071,3}$ and $c_{072,3}$).

The example contains several different cases to illustrate their use thanks to the combinatorics calculated for the identification of reference contexts. And we have adjoined a case which simultaneously uses a predicate and an alethic modality: $Hunter \rightarrow (Prey \rightarrow Attack)$ (thought $c_{350,2}$). It is false if the prey is large (thought $c_{360,2}$).

9 Some points of clarification and opening

In the previous paragraphs, we have certainly not answered all the questions that our presentation raises. But there are also some topics that we have deliberately skimmed over so as not to make our remarks totally indigestible. We are not going to develop them, for the same reason. But, for readers who wish to explore these concepts in more depth, we offer to list them below, in a thought-provoking format.

The notion of subject

We do not use the notion of subject on all formulas. In absolute terms, this should have been done in the first sentence. But the result would have been unreadable, without any contribution to what we wished to demonstrate.

In fact, this remark is not insignificant. It raises some questions as: are we saying the same thing with $a \rightarrow b$ and $\neg\, b \rightarrow \neg\, a$? The question is about the meaning of connectors. It has been extensively studied, notably by J. Lukaszewicz [10].

Without elaborating on the subject, let us point out to readers interested in this topic that L_c proposes an answer. $c \rightarrow f$ says that if c is true then f is true. But if c is false, then f can be true or false. The fact that a thought is false does not imply that the expression that describes it is false in the sense of syntactic interpretation.

Thoughts

They are indeed *silent* atomic propositions. They appear completely automatically in the syntax. In the algorithms of paragraph 10, the communication interface with the instructor does not see them, and only acts through the sentences of L_p.

Conjunctive normal form

To expand the example, we have chosen to associate a new thought with each clause of the conjunctive normal form of the theory. This has an impact on the semantic interpretation: for the same stimulus, the interpretation of the set $\{c_1 \rightarrow f, c_2 \rightarrow g\}$ may be different from that which would be obtained on the set $\{c \rightarrow f \wedge g\}$.

Our choice has no theoretical basis. It is pragmatic: the conversion of a set of formulae into its conjunctive normal form is achieved by a linear algorithm, and each clause identifies a unique formula and therefore a unique thought. We have chosen the simplest technical solution.

Epistemic context

Concerning the semantic interpretation function, we propose the definition of epistemic contexts to identify the relevant sets of thoughts. Other definitions are possible.

Another method, perhaps more purist, would have consisted in giving the different possible definitions and comparing their mathematical properties. It requires a lot of methodical work that we have not done.

Universal connector

A difference between predicate logic and contextual logic is that the latter has no universal quantifier.

In fact, L_c natively models a form of quantifier. For example, assume a knowledge base consisting of a single piece of information: $\{c \rightarrow (Bird \rightarrow Fly)\}$. This says that *Birds fly*. Given this piece of knowledge, it would be the same to say *All bird fly*.

L_c is faillibilistic. In this context, asserting that a state is universal does not make sense: a state is universal as long as it is not contradicted. In the same vein, epistemic modalities are not expressed by a dedicated connector, but are deduced from a semantic interpretation.

Stimulus

Contextual logic introduces the notion of stimulus. Its concept is necessary to generate a possible reaction. We use it in the different examples as an imposed external event *f*. Relations with facts are carried by thoughts, according to an anchoring principle that remains to be defined.

10 Conclusion

Contextual logic is obtained by applying the contextual postulate on propositional logic. It proposes to formalise a relation between thoughts and languages. This brings the possibility of modelling an introspective reasoning.

The principle of proof is the foundation of mathematical philosophy. But introspective reasoning automatically generates the impossibility to demonstrate that something is true. Faced with the need for decidability, L_c uses the principle of non-refutability, which is backed by faillibilism and perspectivism. These are some old and still open subjects (H. Albert [1] and W. Quine [14] for example). Our work proposes to reconcile the different theories, by an answer based on a “semantic of uncertainty” modelled by a “certain syntax”.

The historical ambition of logic is to model the process of human reasoning. This article, by presenting an application of the contextual postulate, performs a first clearing. It addresses, or rather it flies over, some basic concepts of AI, mixing mathematics, computer science, philosophy, and cognitive science. We are especially aware that this presentation leaves many open questions.

The syntactic dimension of formal languages is a vast field of work. For many years, it has been the subject of substantial studies, and many discoveries probably remain to be made in this area. Our contribution is to show that the syntax of the simplest known formal language, namely propositional logic, is sufficient to model complex reasoning which until now was deemed to escape it.

Contextual logic has a non-monotonic semantic and exploits some inconsistent information and some predicative forms. We obtain this by remaining in the syntax of L_p. By presenting this result, we invite to study the possibilities of enriching the semantic interpretation capacities of formal language theories by revisiting the definitions and meanings associated with atomic propositions.

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