The word entropy and how to compute it

SÉBASTIEN FERENCZI

*Aix Marseille Université, CNRS, Centrale Marseille, Institut de Mathématiques de Marseille, I2M - UMR 7373
163, avenue de Luminy, 13288 Marseille Cedex 9, France*

CHRISTIAN MAUDUIT

Institut Universitaire de France

*Aix Marseille Université, CNRS, Centrale Marseille, Institut de Mathématiques de Marseille, I2M - UMR 7373
163, avenue de Luminy, 13288 Marseille Cedex 9, France*

CARLOS GUSTAVO MOREIRA

Instituto de Matemática Pura e Aplicada

Estrada Dona Castorina 110

22460-320 Rio de Janeiro, RJ, Brasil

Abstract: The complexity function of an infinite word counts the number of its factors. For any positive function f , its *exponential rate of growth* $E_0(f)$ is $\liminf_{n \rightarrow \infty} \frac{1}{n} \log f(n)$. We define a new quantity, the *word entropy* $E_W(f)$, as the maximal exponential growth rate of a complexity function smaller than f . This is in general smaller than $E_0(f)$, and more difficult to compute; we give an algorithm to estimate it. We use $E_W(f)$ to compute the Hausdorff dimension of the set of real numbers whose expansions in a given base have complexity bounded by f .

1 Definitions

Let A be the finite alphabet $\{0, 1, \dots, q-1\}$. If $w \in A^{\mathbb{N}}$, and $L(w)$ the set of finite factors of w ; for any non-negative integer n , we write $L_n(w) =$

$L(w) \cap A^n$.

Definition 1.1. The *complexity function* of $w \in A^{\mathbb{N}}$ is defined for any non-negative integer n by $p_w(n) = |L_n(w)|$.

Our work concerns the study of infinite words w the complexity function of which is bounded by a given function f from $\mathbb{N}$ to $\mathbb{R}^+$. More precisely, if f is such a function, we put

$$W(f) = \{w \in A^{\mathbb{N}}, p_w(n) \leq f(n), \forall n \in \mathbb{N}\}.$$

Definition 1.2. If f is a function from $\mathbb{N}$ to $\mathbb{R}^+$, we call *exponential rate of growth* of f the quantity

$$E_0(f) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log f(n)$$

and *word entropy* of f the quantity

$$E_W(f) = \sup_{w \in W(f)} E_0(p_w).$$

Of course, if $E_0 = 0$ then E_W is zero also. Thus the study of E_W is interesting only when f has exponential growth: we are in the little-explored field of *word combinatorics in positive entropy*, or exponential complexity.

2 First properties of E_0 and E_W

If f is itself a complexity function (i.e. $f = p_w$ for some $w \in A^{\mathbb{N}}$), then $E_W(f) = E_0(f)$. But in general E_W may be much smaller than E_0 .

Indeed, for each $1 < \theta \leq q$, and $n_0 \in \mathbb{N}$ such that $\theta^{n_0+1} > n_0 + q - 1$, we define the function f by $f(1) = q$, $f(n) = n + q - 1$ for $1 \leq n \leq n_0$ and $f(n) = \theta^n$ for $n > n_0$. We have $E_0(f) = \log \theta$ and we can prove that

$$E_W(f) \leq \frac{1}{n_0} \log(n_0 + q - 1),$$

which can be made arbitrarily small, independently of θ .

We define a regularity condition for f .

Definition 2.1. We say that a function f from $\mathbb{N}$ to $\mathbb{R}^+$ satisfies the conditions $(\mathcal{C}^*)$ if

- i) for any $n \in \mathbb{N}$ we have $f(n+1) > f(n) \geq n+1$;
- ii) for any $(n, n') \in \mathbb{N}^2$ we have $f(n+n') \leq f(n)f(n')$.

But even with $(\mathcal{C}^*)$ we may have $E_W(f) < E_0(f)$. Indeed, let f be the function defined by $f(n) = \lceil 3^{n/2} \rceil$ for any $n \in \mathbb{N}$. Then it is easy to check that f satisfies conditions $(\mathcal{C}^*)$ and that $E_0(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log f(n) = \log(\sqrt{3})$. On the other hand, we have $f(1) = 2$, $f(2) = 3$; thus the language has no 00 or no 11, and this implies that $E_W(f) \leq \log(\frac{1+\sqrt{5}}{2}) < E_0(f)$.

At least, under that condition, we have the important

Theorem 2.2. *If f is a function from $\mathbb{N}$ to $\mathbb{R}^+$ satisfying the conditions $(\mathcal{C}^*)$, then $E_W(f) > \frac{1}{2}E_0(f)$.*

We can also show that the constant $\frac{1}{2}$ is optimal.

Finally, it will be useful to know that

Theorem 2.3. *For any function f from $\mathbb{N}$ to $\mathbb{R}^+$, there exists $w \in W(f)$ such that for any $n \in \mathbb{N}$ we have $p_w(n) \geq \exp(E_W(f)n)$.*

3 Algorithm

In general $E_W(f)$ is much more difficult to compute than $E_0(f)$; now we will give an algorithm which allows us to estimate with arbitrary precision

$E_W(f)$ from finitely many values of f , if we know already $E_0(f)$ and have some information on the speed with which this limit is approximated.

We assume that f satisfies conditions $\mathcal{C}^*$. We don't lose generality with this assumption, since we may always change a function f which satisfies conditions $\mathcal{C}$ by the function $\tilde{f}$ given recursively by

$$\tilde{f}(n) := \min\{f(n), \min_{1 \leq k < n} \tilde{f}(k)\tilde{f}(n-k)\},$$

which satisfies conditions $\mathcal{C}^*$, such that $\tilde{f}(n) \leq f(n), \forall n \in \mathbb{N}$ and $W(\tilde{f}) = W(f)$.

Theorem 3.1. *There is an algorithm which gives, starting from f and ε , a quantity h such that $(1 - \varepsilon)h \leq E_W(f) \leq h$. h depends explicitly on ε , $E_0(f)$, N , $f(1)$, ..., $f(N)$, for an integer N which depends explicitly on ε , $E_0(f)$, and an integer n_0 larger than an explicit function of ε and $E_0(f)$, and such that*

$$\frac{\log f(n)}{n} < (1 + \frac{E_0(f)\varepsilon}{210(4 + 2E_0(f))})E_0(f), \quad \text{for } n_0 \leq n < 2n_0.$$

We shall now give the algorithm. f is given and henceforth we omit to mention it in $E_0(f)$ and $E_W(f)$. $\varepsilon \in (0, 1)$ is also given.

Description of the algorithm

Let

$$\delta := \frac{E_0\varepsilon}{105(4 + 2E_0)} < \frac{\varepsilon}{210},$$

$$K := \lceil \delta^{-1} \rceil + 1.$$

We choose a positive integer

$$n_0 \geq K \vee \frac{4K^2}{420^3 E_0}$$

such that

$$\frac{\log f(n)}{n} < (1 + \frac{\delta}{2})E_0, \forall n \geq n_0;$$

in view of conditions $\mathcal{C}^*$, this last condition is equivalent to $\frac{\log f(n)}{n} < (1 + \frac{\delta}{2})E_0, n_0 \leq n < 2n_0$.

We choose intervals which will be so large that all the lengths of words we manipulate stay in one of them. Namely, for each $t \geq 0$, let

$$n_{t+1} := \exp(K((1 + \delta)^2 E_0 n_t + E_0)).$$

We take

$$N := n_K.$$

We choose now a set $Y \subset A^N$: for a given Y , $L_n(Y) = \cup_{\gamma \in Y} L_n(\gamma)$, we define $q_n(Y) := |L_n(Y)|$, for $1 \leq n \leq N$. We look at those Y for which $q_n(Y) \leq f(n), \forall n \leq N$, and choose one among them such that

$$\min_{1 \leq n \leq N} \frac{\log q_n(Y)}{n}$$

is maximum.

Henceforth we omit to mention Y in the $q_n(Y)$.

Proposition 3.2.

$$\min_{1 \leq n \leq N} \frac{\log q_n}{n} \geq E_W.$$

Proof. We know by Theorem ?? that there is $\hat{w} \in W(f)$ with $p_n(\hat{w}) \geq \exp(E_W n), \forall n \geq 1$. For such a word $\hat{w}$, let $X := L_N(\hat{w}) \subset A^N$. We have, for each n with $1 \leq n \leq N$, $L_n(X) = L_n(\hat{w})$ and $f(n) \geq \#L_n(\hat{w}) = p_n(\hat{w}) \geq \exp(E_W n)$. Thus X is one of the possible Y , and the result follows from the maximality of $\min_{1 \leq n \leq N} \frac{\log q_n}{n}$. $\square$

On one of the large intervals we have defined, $\frac{\log q_n}{n}$ will be almost constant:

Lemma 3.3. *There exists $r < K$, such that*

$$\frac{\log q_{n_r}}{n_r} < (1 + \delta) \frac{\log q_{n_{r+1}}}{q_{n_{r+1}}}.$$

Proof. Otherwise $\frac{\log q_{n_0}}{n_0} \geq (1 + \delta)^K \frac{\log q_{n_K}}{q_{n_K}}$: as $K > \frac{1}{\delta}$, $(1 + \delta)^K$ would be close to e for δ small enough, and is larger than $\frac{9}{4}$ as $\delta < \frac{1}{2}$; thus, as $\frac{\log q_{n_K}}{n_K} \geq E_W$ by Proposition ??, we have $\frac{\log q_{n_0}}{n_0} \geq \frac{9}{4}E_W$, but $q_{n_0} \leq f(n_0)$ hence $\frac{\log q_{n_0}}{n_0} < (1 + \frac{\delta}{2})E_0$, and this contradicts $E_0 \leq 2E_W$, which is true by Theorem ??. $\square$

Let

$$h := \frac{\log q_{n_r}}{n_r}.$$

By Proposition ?? we have

$$h \geq E_W.$$

What remains to prove is the following proposition (which, understandably, does not use the maximality of $\min_{1 \leq n \leq N} \frac{\log q_n}{n}$).

Proposition 3.4.

$$(1 - \varepsilon)h \leq E_W.$$

Proof. Our strategy is to build a word w such that, for all $n \geq 1$,

$$\exp((1 - \varepsilon)hn) \leq p_n(w) \leq f(n),$$

which gives the conclusion by definition of E_W . To build the word w , we shall define an integer m , and build successive subsets of $L_m(Y)$; for such a subset Z , we order it (lexicographically for example) and define $w(Z)$ to be the *Champernowne word* on Z : namely, if $Z = \{\beta_1, \beta_2, \dots, \beta_t\}$, we build the infinite word

$$w(Z) := \beta_1\beta_2 \dots \beta_t\beta_1\beta_1\beta_1\beta_2\beta_1\beta_3 \dots \beta_{t-1}\beta_t\beta_1\beta_1\beta_1 \dots \beta_t\beta_t\beta_t \dots$$

made by concatenation of all words in Z followed by the concatenations of all pairs of words of Z followed by the concatenations of all triples of words of Z , etc...

The word $w(Z)$ will satisfy $\exp((1 - \varepsilon)hn) \leq p_n(w(Z))$ for all n as soon as

$$|Z| \geq \exp((1 - \varepsilon)hm),$$

since, for every positive integer k , we will have at least $|Z|^k$ factors of length km in $w(Z)$.

The successive (decreasing) subsets Z of $L_m(Y)$ we build will all have cardinality at least $\exp((1 - \varepsilon)hm)$, and the words $w(Z)$ will satisfy $p_n(w(Z)) \leq f(n)$ for n in an interval which will increase at each new set Z we build, and ultimately contains all the integers.

We give only the main ideas of the remaining proof. In the first stage we define two lengths of words, $\hat{n}$ and $m > \frac{\hat{n}}{2\varepsilon}$, which will be both in the interval $[n_r, n_{r+1}]$, and a set Z_1 of words of length m of the form $\gamma\theta$, for words γ of length $\hat{n}$, such that the word $\gamma\theta\gamma$ is in $L_{m+\hat{n}}(Y)$. This is done by looking precisely at twin occurrences of words.

Let $\tilde{\varepsilon} = \frac{\varepsilon}{15} = \frac{7(4+2E_0)\delta}{E_0} > 14\delta$; then we can get such a set Z_1 with $|Z_1| \geq \exp((1 - \tilde{\varepsilon})h(m + \hat{n}))$.

In the second stage, we define a new set $Z_2 \subset Z_1$ in which all the words have the same prefix γ_1 of length $6\tilde{\varepsilon}hm$, and all the words have the same suffix γ_2 of length $6\tilde{\varepsilon}hm$, with $|Z_2| \geq |Z_1| \exp(-12\tilde{\varepsilon}hm - 2\delta h\hat{n})$, and $2\delta h\hat{n} \leq (1 - \tilde{\varepsilon})\hat{n}$, thus

$$|Z_2| \geq \exp((1 - 13\tilde{\varepsilon})hm).$$

As a consequence of the definition of Z_2 , all words of Z_2 have the same prefix of length $\hat{n}$, which is a prefix γ_0 of γ_1 ; as Z_2 is included in Z_1 , any

word of Z_2 is of the form $\gamma_0\theta$, and the word $\gamma_0\theta\gamma_0$ is in $L_{m+\hat{n}}(Y)$.

At this stage we can prove

Claim 3.5. $p_{w(Z_2)}(n) \leq f(n)$ for all $1 \leq n \leq \hat{n} + 1$.

Let us shrink again our set of words.

Lemma 3.6. *For a given subset Z of Z_2 , there exists $Z' \subset Z$, $|Z'| \geq (1 - \exp(-(j-1)\frac{E_0}{2}))^j |Z|$, such that the total number of factors of length $\hat{n} + j$ of all words $\gamma_0\theta\gamma_0$ such that $\gamma_0\theta$ is in Z' is at most $f(\hat{n} + j) - j$.*

We start from Z_2 and apply successively Lemma ?? from $j = 2$ to $j = 6\tilde{e}m$, getting $6\tilde{e}m - 1$ successive sets Z' ; at the end, we get a set Z_3 such that the total number of factors of length $\hat{n} + j$ of words $\gamma_0\theta\gamma_0$ for $\gamma_0\theta$ in Z_3 is at most $f(\hat{n} + j) - j$ for $j = 2, \dots, 6\tilde{e}m$, and $\frac{|Z_3|}{|Z_2|}$ is at least

$$\prod_{2 \leq j \leq 6\tilde{e}m - \hat{n}} (1 - \exp(-(j-1)\frac{E_0}{2}))^j \geq \prod_{j \geq 2} (1 - \exp(-(j-1)\frac{E_0}{2}))^j,$$

which implies after computations that

$$|Z_3| \geq \exp((1 - 14\tilde{e})hm).$$

We can now bound the number of short factors by using the factors we have just deleted and properties of γ_0 , γ_1 and γ_2 .

Claim 3.7. $p_{w(Z_3)}(n) \leq f(n)$ for all $1 \leq n \leq 6\tilde{e}m$.

We shrink our set again.

Let $m \geq n > 6\tilde{e}m$; in average a factor of length n of a word in Z_3 occurs in at most $\frac{m|Z_3|}{f(n)}$ elements of Z_3 . We consider the $\frac{f(n)}{mn^2}$ factors of length n which occur the least often. In total, these factors occur in at most $\frac{m|Z_3|}{f(n)} \frac{f(n)}{mn^2} = \frac{|Z_3|}{n^2}$ elements of Z_3 . We remove these words from Z_3 , for all $m \geq n > 6\tilde{e}m$, obtaining a set Z_4 with $|Z_4| \geq \exp((1 - 15\tilde{e})hm)$.

We can now control medium length factors, using again the missing factors we have just created, and γ_1 and γ_2 , but not γ_0 .

Claim 3.8. $p_{w(Z_4)}(n) \leq f(n)$ for all $1 \leq n \leq m$.

Finally we put $Z_5 = Z_4$ if $|Z_4| \leq \exp((1 - 4\tilde{\varepsilon})hm)$, otherwise we take for Z_5 any subset of Z_4 with $\lceil \exp((1 - 4\tilde{\varepsilon})hm) \rceil$ elements. In both cases we have

$$|Z_5| \geq \exp((1 - \varepsilon)hm).$$

For the long factors, we use mainly the fact that there are many missing factors of length m , but we need also some help from γ_1 and γ_2 .

Claim 3.9. $p_{w(Z_5)}(n) \leq f(n)$ for all n .

In view of the considerations at the beginning of the proof of Proposition ??, Claim ?? completes the proof of that proposition, and thus of Theorem ??. □

4 Application

We define

$$C(f) = \{x = \sum_{n \geq 0} \frac{w_n}{q^{n+1}} \in [0, 1], w(x) = w_0 w_1 \cdots w_n \cdots \in W(f)\}.$$

Then we have

Theorem 4.1.

The Hausdorff dimension of $C(f)$ is equal to $E_W(f)/\log q$.

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