Credit default swaps in two-dimensional models with various information flows *

Pavel V Gapeev, Monique Jeanblanc

Credit default swaps in two-dimensional models with various information flows*

Pavel V. Gapeev[†] Monique Jeanblanc[‡]

We study a credit risk model of a financial market in which the dynamics of intensity rates of two default times are described by linear combinations of three independent geometric Brownian motions. The dynamics of two default-free risky asset prices are modeled by two geometric Brownian motions which are dependent of the ones describing the default intensity rates. We obtain closed form expressions for the rational prices of both risk-free and risky credit default swaps given the reference filtration initially and progressively enlarged by the two default times. The accessible default-free reference filtration is generated by the standard Brownian motions driving the model.

1 Introduction

In the present paper, we derive closed form expressions for the rational (or no-arbitrage) prices of credit default swaps (or CDSs for short) without and with consideration of counterparty risk in a model of a financial market given the flows of information which are expressed by

*This research benefited from the support of the 'Chaire Marchés en Mutation', French Banking Federation and ILB, Labex ANR 11-LABX-0019.

[†]London School of Economics, Department of Mathematics, Houghton Street, London WC2A 2AE, United Kingdom; e-mail: p.v.gapeev@lse.ac.uk

[‡] Laboratoire de Mathématiques et Modélisation d'Évry (LaMME), UMR CNRS 8071; Univ Evry-Université Paris Saclay, 23 Boulevard de France, 91037 Évry cedex France; e-mail: monique.jeanblanc@univ-evry.fr

Mathematics Subject Classification 2010: Primary 91G40, 60G44, 60J65. Secondary 91B70, 60J60, 91G20.

Key words and phrases: Default times, credit default swaps, counterparty risk, geometric Brownian motion, initial and progressive enlargements of filtrations.

Date: May 2, 2019

the reference filtration progressively and initially enlarged by means of the default times. It is assumed that the recovery payoffs depend on the default times and the current prices of the underlying default-free risky assets taken at the times of defaults. The dynamics of market prices of the two risky assets are described by geometric Brownian motions driven by constantly correlated standard Brownian motions. The default times are given by the first times at which linear combinations of three integral processes of independent geometric Brownian motions hit certain random thresholds which are independent of each other and of the standard Brownian motions driving the model. The dependence between the default times is then expressed by means of the dynamics of their intensity rates given by linear combinations of the three independent geometric Brownian motions which are driven by standard Brownian motions constantly correlated with the ones related to the risky asset prices. The default-free reference filtration accessible from the market is generated by the standard Brownian motions driving the model. The rational prices of the resulting defaultable European style contingent claims are explicitly expressed through the transition densities of the marginal distributions of the geometric Brownian motions and their integral processes describing the model.

The credit risk models in which the default times are defined as the first times at which the associated cumulative intensity processes reach certain random thresholds were initiated by Lando [21]. The computations of conditional distributions of the default times given the observable filtrations in such a first passage intensity model with independent default intensities and correlated thresholds were presented in Schönbucher [23; Chapter X, Proposition 10.9]. Brigo and Chourdakis [7] studied the problem of pricing of CDSs in such a model with counterparty risk in which the intensities of the default times are independent of each other, but the associated random thresholds are correlated. Brigo, Capponi, and Pallavicini [6] developed the rational pricing framework for bilateral counterparty credit risk models and specified the credit and debit valuation adjustments in the cases in which the default intensity rates are expressed by means of the (strictly positive) Feller's square root diffusion processes, and the associated thresholds are correlated through a Gaussian copula. Bielecki et al. [3] provided the analytic basis for the quantitative methodology of dynamic hedging of the counterparty risk and developed the main theoretical issues of dynamic hedging of credit valuation adjustments. Assefa et al. [1] derived a model-free general counterparty risk representation formula for the credit valuation adjustment of a netted and collateralised portfolio. Some related discussions on modelling and computational aspects regarding managing of exposure to counterparty risk are provided in the recent monographs by Gregory [20], Cesari, Aquilina, and Charpillon [10], Brigo, Morini, and Pallavicini [8], and Crépey, Bielecki, and Brigo [11].

El Karoui, Jeanblanc, and Jiao [15]-[16] emphasised the roles of conditional distributions of several default times in the intensity credit risk models given the appropriate filtrations and presented general expressions for the rational prices of various defaultable European style contingent claims. In this paper, we consider a model in which the default intensity rates are explicitly given as linear combinations of three independent geometric Brownian motions which are dependent of the ones describing the dynamics of the risky asset price processes. We then use the Markov property of the resulting multi-dimensional process describing the model and apply the explicit formula from Yor [25] for the joint marginal density of a geometric Brownian motion and its integral process to derive closed form expressions for the rational prices of both risk-free and risky CDSs given the reference filtration progressively and initially enlarged by means of the default times. The model of a financial market with such dynamics of prices of dependent risky assets and default intensity rates in which the rational prices of defaultable European style contingent claims can be computed explicitly appears to be novel for the related literature, to the best of our knowledge. We also note that the model proposed in the paper keeps its Markovian feature in the filtrations which are obtained by means of the progressive and initial enlargements of the initial Brownian reference filtration. The results of this paper can naturally be extended to the case of credit risk models with more than two default times and more than two underlying risky assets of a similar dependence structure. The rational prices of CDSs and other European style defaultable contingent claims can then be expressed through the transition densities of the marginal distributions of the resulting multi-dimensional continuous Markov process describing the model.

The paper is organised as follows. In Section 2, we introduce a credit risk model of a financial market with the dependence structure of the dynamics of prices of two risky assets and two default intensity rates described above. In Section 3, we derive explicit expressions for the conditional distributions of two default times given the accessible default-free reference filtration and the observable filtrations. In Section 4, we compute closed form expressions for the rational prices of risk-free CDSs (without consideration of counterparty risk) in the models with one and two underlying risky assets given the reference filtration progressively and initially enlarged by the default times. In Section 5, we compute closed form expressions for the rational prices of risky CDSs (with consideration of counterparty risk) in the model with two underlying risky assets given the reference filtration progressively and initially enlarged by the default times. The main results of the paper are stated in Propositions 4.1-4.3, and 5.1.

2 The model

In this section, we introduce a model of a financial market with two defaultable risky assets. We also define the accessible default-free reference filtration as well as the observable filtrations and refer some known results and distribution laws.

2.1 The dynamics of default intensities and firm values

Let $(\Omega, \mathcal{G}, P)$ be a probability space supporting independent standard Brownian motions $W^j = (W_t^j)_{t \geq 0}$ and $B^j = (B_t^j)_{t \geq 0}$, $j = 0, 1, 2$, as well as the random variables U_i, $i = 1, 2$, which are uniformly distributed on $(0, 1)$. Suppose that the variables U_i, $i = 1, 2$, are independent of each other and of the processes W^j and B^j, $j = 0, 1, 2$. We define the random times τ_i, $i = 1, 2$, by

$$\tau_i = \inf \left\{ t \geq 0 \, \middle| \, \delta_i A_t^0 + \lambda_i A_t^i \geq - \ln U_i \right\} \tag{2.1}$$

where the processes $A^j = (A_t^j)_{t \geq 0}$, $j = 0, 1, 2$, are given by

$$A_t^j = \int_0^t Y_s^j \, ds \tag{2.2}$$

for all $t \geq 0$, and some $\delta_i, \lambda_i \geq 0$, $i = 1, 2$, fixed, so that the processes $(\delta_i A_t^0 + \lambda_i A_t^i)_{t \geq 0}$, $i = 1, 2$, form the *cumulative intensities*, and the processes $(\delta_i Y_t^0 + \lambda_i Y_t^i)_{t \geq 0}$, $i = 1, 2$, are the *intensity rates* of the random times τ_i, $i = 1, 2$. These notions mean that the processes $(I(\tau_i \leq t) - \delta_i A_{t \wedge \tau_i}^0 - \lambda_i A_{t \wedge \tau_i}^i)_{t \geq 0}$, $i = 1, 2$, are martingales in their natural filtrations. Assume that the processes $Y^j = (Y_t^j)_{t \geq 0}$, $j = 0, 1, 2$, admit the representations

$$Y_t^j = \exp \left(\left(\beta_j - \frac{\gamma_j^2}{2} \right) t + \gamma_j \, W_t^j \right) \tag{2.3}$$

for all $t \geq 0$, and some constants $\beta_j \in \mathbb{R}$ and $\gamma_j > 0$, $j = 0, 1, 2$. Note that the random times τ_i, $i = 1, 2$, defined in (2.1) with (2.2) and (2.3) can occur simultaneously only with probability zero, and thus, the property $P(\tau_1 = \tau_2) = 0$ holds, by construction.

Suppose that the random times τ_i, $i = 1, 2$, represent the default times of two firms (reference credits) with the value dynamics described by the processes $X^i = (X_t^i)_{t \geq 0}$, $i = 1, 2$, given by $X_t^i = (Y_t^i)^{\alpha_i} (Z_t^0)^{\zeta_i} Z_t^i$, for some α_i and $\zeta_i \in \mathbb{R}$, $i = 1, 2$, fixed. Here, the processes $Z^j = (Z_t^j)_{t \geq 0}$, $j = 0, 1, 2$, are defined by

$$Z_t^j = \exp \left(\left(\eta_j - \frac{\theta_j^2}{2} \right) t + \theta_j \, B_t^j \right) \tag{2.4}$$

for all $t \geq 0$, and some constants $\eta_j \in \mathbb{R}$ and $\theta_j > 0$, $j = 0, 1, 2$. We further assume that the discounted firm value processes $(e^{-rt} X_t^i)_{t \geq 0}$, $i = 1, 2$, are martingales with respect to the pricing measure P under which the processes Y^j and Z^j, $j = 0, 1, 2$, admit the representations in (2.3) and (2.4), where $r \geq 0$ is the interest rate of a riskless bank account. Thus, taking into account the independence of the driving processes W^j and B^j, $j = 0, 1, 2$, we may conclude that the equality

$$\beta_i \alpha_i + \frac{\gamma_i^2}{2}\, \alpha_i(\alpha_i - 1) + \eta_0 \zeta_i + \frac{\theta_0^2}{2}\, \zeta_i(\zeta_i - 1) + \eta_i = r \tag{2.5}$$

should hold, for every $i = 1, 2$.

2.2 Some filtrations and distribution laws

Let us denote by $(\mathcal{F}_t)_{t \geq 0}$ the natural filtration of the processes Y^j and Z^j, $j = 0, 1, 2$, defined by $\mathcal{F}_t = \sigma(Y_t^j, Z_t^j \,|\, 0 \leq s \leq t, j = 0, 1, 2)$, for all $t \geq 0$, which coincides with the one of the driving standard Brownian motions W^j and B^j, $j = 0, 1, 2$, given by $\sigma(W_t^j, B_t^j \,|\, 0 \leq s \leq t, j = 0, 1, 2)$, for all $t \geq 0$. We define the progressively enlarged filtrations $(\mathcal{G}_t^i)_{t \geq 0}$, $i = 1, 2$, by $\mathcal{G}_t^i = \mathcal{F}_t \vee \sigma(\tau_i \wedge t)$, and $(\mathcal{G}_t)_{t \geq 0}$ by $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau_i \wedge t) \vee \sigma(\tau_{3-i} \wedge t)$, for all $t \geq 0$. Let us also introduce the initially enlarged filtrations $(\mathcal{F}_t^i)_{t \geq 0}$, $i = 1, 2$, by $\mathcal{F}_t^i = \mathcal{F}_t \vee \sigma(\tau_i)$, for all $t \geq 0$. We actually consider the smallest right-continuous completed filtrations that contain the appropriate filtrations defined above. The default-free reference filtration $(\mathcal{F}_t)_{t \geq 0}$ reflects the information flow which is accessible for the investors trading in the market, while the filtrations $(\mathcal{G}_t^i)_{t \geq 0}$, $i = 1, 2$, and $(\mathcal{G}_t)_{t \geq 0}$ reflect the accessible information including the one about the appearance of the default times. Note that, by virtue of the independence of the random variables U_i, $i = 1, 2$, and the filtration $(\mathcal{F}_t)_{t \geq 0}$, it follows that $(\mathcal{F}_t)_{t \geq 0}$ is immersed in the filtration $(\mathcal{F}_t \vee \sigma(U_i) \vee \sigma(U_{3-i}))_{t \geq 0}$, and thus, in the smaller filtrations $(\mathcal{G}_t^i)_{t \geq 0}$, $i = 1, 2$, and $(\mathcal{G}_t)_{t \geq 0}$ (see, e.g. [5] and [17]). Moreover, by virtue of the independence of the random variable U_{3-i} and the filtration $(\mathcal{G}_t^i)_{t \geq 0}$, it follows that $(\mathcal{G}_t^i)_{t \geq 0}$ is immersed in the filtration $(\mathcal{F}_t \vee \sigma(U_i) \vee \sigma(U_{3-i}))_{t \geq 0}$, and thus, in the filtration $(\mathcal{G}_t)_{t \geq 0}$, for every $i = 1, 2$. This notion is also known as the (H)-hypothesis for the filtrations $(\mathcal{F}_t)_{t \geq 0}$ and $(\mathcal{G}_t)_{t \geq 0}$ in the literature (see, e.g. [5], [22; Chapter V, Section 4], [4; Chapter VIII, Section 3], or [2; Chapter III]). Note that the immersion of $(\mathcal{F}_t)_{t \geq 0}$ in $(\mathcal{G}_t^i)_{t \geq 0}$ is equivalent to the conditional independence of $\mathcal{G}_t^i$ and $\mathcal{F}_\infty$ with respect to $\mathcal{F}_t$, for $i = 1, 2$, while the immersion of $(\mathcal{F}_t)_{t \geq 0}$ in $(\mathcal{G}_t)_{t \geq 0}$ is equivalent to the conditional independence of $\mathcal{G}_t$ and $\mathcal{F}_\infty$ with respect to $\mathcal{F}_t$, for all $t \geq 0$ (see, e.g. [13]).

Let us now consider a filtration $(\mathcal{K}_t)_{t \geq 0}$ larger than the filtration $(\mathcal{F}_t)_{t \geq 0}$, that is, $\mathcal{F}_t \subseteq \mathcal{K}_t$, for all $t \geq 0$. Then, if $\mathcal{K}_t$ coincides with $\mathcal{F}_t$ on the event $J_t \in \mathcal{K}_t$ such that $P(J_t) > 0$, that is, if for any $K_t \in \mathcal{K}_t$, there exists an event $F_t \in \mathcal{F}_t$ such that $J_t \cap K_t = J_t \cap F_t$, then the conditional expectation $E[V \,|\, \mathcal{K}_t]$ of an integrable random variable V on the event J_t is equal to an $\mathcal{F}_t$-measurable random variable. Hence, according to the results in [12; page 122] and [4; Section 5.1], this fact leads to the equality

$$I(J_t)\, E\big[V \,\big|\, \mathcal{K}_t\big]\, P(J_t \,|\, \mathcal{F}_t) = I(J_t)\, E\big[V\, I(J_t) \,\big|\, \mathcal{F}_t\big] \tag{2.6}$$

and thus, taking into account the fact that $P(J_t \,|\, \mathcal{F}_t) > 0$ on the event J_t, we have

$$I(J_t)\, E\big[V \,\big|\, \mathcal{K}_t\big] = I(J_t)\, \frac{E[V I(J_t) \,|\, \mathcal{F}_t]}{P(J_t \,|\, \mathcal{F}_t)} \tag{2.7}$$

for any (positive) integrable random variable V and all $t \geq 0$. We further refer to the result in (2.6)-(2.7) as to the *generalised key lemma* for the filtrations $(\mathcal{K}_t)_{t \geq 0}$ and $(\mathcal{F}_t)_{t \geq 0}$. Observe that $\mathcal{G}_t^i$ coincides with $\mathcal{F}_t$ on the event $\{\tau_i > t\}$, and $\mathcal{G}_t$ coincides with $\mathcal{F}_t$ on the event $\{\tau_i \wedge \tau_{3-i} > t\}$, while $\mathcal{G}_t^i \vee \sigma(\tau_{3-i})$ coincides with $\mathcal{F}_t^{3-i} \equiv \mathcal{F}_t \vee \sigma(\tau_{3-i})$ on the event $\{\tau_i > t\}$, for all $t \geq 0$ and every $i = 1, 2$. In these cases, the expressions in (2.6)-(2.7), together with the tower property for conditional expectations, imply that, for each $\mathcal{G}_T^i$-measurable integrable random variable V_T^i, the equality

$$I(\tau_i > t)\, E\big[V_T^i \,\big|\, \mathcal{G}_t^i\big] = I(\tau_i > t)\, \frac{E[V_T^i P(\tau_i > t \,|\, \mathcal{F}_T) \,|\, \mathcal{F}_t]}{P(\tau_i > t \,|\, \mathcal{F}_t)} \tag{2.8}$$

holds, for all $t \geq 0$ and every $i = 1, 2$ (see, e.g. [2; Lemma 2.9]). Moreover, it follows that, for each $(\mathcal{F}_t)_{t \geq 0}$-progressively measurable process $V^i = (V_t^i)_{t \geq 0}$, the equality

$$E\big[V_{\tau_i}^i\, I(\tau_i > t) \,|\, \mathcal{G}_t^i\big] = I(\tau_i > t)\, E\bigg[\int_t^\infty \frac{V_u^i P(\tau_i \in du \,|\, \mathcal{F}_u)}{P(\tau_i > t \,|\, \mathcal{F}_t)} \,\bigg|\, \mathcal{F}_t\bigg] \tag{2.9}$$

holds, for all $t \geq 0$ and every $i = 1, 2$ (see, e.g. [2; Corollary 2.10]). We further refer to the results in (2.8) and (2.9) as to the *first and second part of the key lemma* for the filtrations $(\mathcal{G}_t^i)_{t \geq 0}$ and $(\mathcal{F}_t)_{t \geq 0}$, for every $i = 1, 2$.

For any (positive measurable) bounded function $\psi_i(u)$, for $u \geq 0$, let us now compute the conditional expectation

$$\Psi_t^i(\tau_{3-i}) = E\big[\psi_i(\tau_i)\, I(\tau_i > t) \,|\, \mathcal{F}_t \vee \sigma(\tau_{3-i})\big] \tag{2.10}$$

for all $t \geq 0$ and every $i = 1, 2$. For this purpose, we apply the result of [9; Proposition 2.7] to conclude that any $\mathcal{F}_t \vee \sigma(\tau_{3-i})_{t \geq 0}$-progressively measurable process can be written as $\Psi_t^i(\tau_{3-i})$

where $\Psi^i(v) = (\Psi^i_t(v))_{t\ge0}$ is $(\mathcal{F}_t)_{t\ge0}$-progressively measurable, for any $v \ge 0$ fixed, while the function $v \mapsto \Psi^i_t(v)$ is (Borel) measurable, for all $t \ge 0$ and every $i = 1, 2$. Then, we observe that, by definition of conditional expectation, for any event $F_t \in \mathcal{F}_t$, and any (positive measurable) bounded function $\varphi_{3-i}(v)$, the equality

$$
\begin{aligned}
&E\Big[\int_{v=0}^{\infty} \Psi^i_t(v)\, I(F_t)\, \varphi_{3-i}(v)\, P(\tau_{3-i} \in dv \,|\, \mathcal{F}_t)\Big] \\
&= E\Big[I(F_t) \int_{u=t}^{\infty}\int_{v=0}^{\infty} \psi_i(u)\, \varphi_{3-i}(v)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_t)\Big]
\end{aligned}
\tag{2.11}
$$

holds, for all $t \ge 0$ and every $i = 1, 2$. Hence, the application of the equality in (2.11) yields the fact that the expression

$$
\Psi^i_t(\tau_{3-i}) = \int_{u=t}^{\infty} \frac{\psi_i(u) P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_t)}{P(\tau_{3-i} \in dv \,|\, \mathcal{F}_t)}\bigg|_{v=\tau_{3-i}}
\tag{2.12}
$$

is satisfied, for all $t \ge 0$ and every $i = 1, 2$.

Let us finally refer the explicit expressions for the transition density functions of the processes (Y^j, A^j), $j = 0, 1, 2$, defined in (2.2)-(2.3) above. For this purpose, we recall from [25; page 527] that the random variable $A^{(\nu_j)}_t = \int_0^t e^{2(W^j_s + \nu_j s)} ds$ has the conditional distribution

$$
P\Big(A^{(\nu_j)}_t \in da \,\Big|\, W^j_t + \nu_j t = x\Big) = p_j(t, x, a)\, da
\tag{2.13}
$$

where the density function p_j is given by

$$
\begin{aligned}
p_j(t, x, a) = \frac{1}{\pi a^2} \exp\left(\frac{x^2 + \pi^2}{2t} + x - \frac{1 + e^{2x}}{2a}\right) \\
\times \int_0^{\infty} \exp\left(-\frac{w^2}{2t} - \frac{e^x}{a}\cosh(w)\right) \sinh(w)\, \sin\left(\frac{\pi w}{t}\right) dw
\end{aligned}
\tag{2.14}
$$

with $t, a > 0$ and $x \in \mathbb{R}$, and $\nu_j \in \mathbb{R}$ given and fixed. This fact yields that the random vector $(2(W^j_t + \nu_j t), A^{(\nu_j)}_t)$ has the distribution:

$$
P\Big(2(W^j_t + \nu_j t) \in dx, A^{(\nu_j)}_t \in da\Big) = q_j(t, x, a)\, dx da
\tag{2.15}
$$

where the density function q_j is given by

$$
\begin{aligned}
q_j(t, x, a) &= p_j\Big(t, \frac{x}{2}, a\Big) \frac{1}{2\sqrt{t}}\, \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\Big(\frac{x - 2\nu_j t}{2\sqrt{t}}\Big)^2\right) \\
&= \frac{1}{(2\pi)^{3/2} a^2 \sqrt{t}} \exp\left(\frac{\pi^2}{2t} + \Big(\frac{\nu_j + 1}{2}\Big) x - \frac{\nu_j^2}{2} t - \frac{1 + e^x}{2a}\right) \\
&\quad \times \int_0^{\infty} \exp\left(-\frac{w^2}{2t} - \frac{e^{x/2}}{a}\cosh(w)\right) \sinh(w)\, \sin\left(\frac{\pi w}{t}\right) dw
\end{aligned}
\tag{2.16}
$$

with $t, a > 0$ and $x \in \mathbb{R}$ (see also [14] and [24] for related expressions in terms of Hermite functions). Therefore, using the fact that the scaling property of W^j implies

$$P\bigg(\Big(\beta_j - \frac{\gamma_j^2}{2}\Big)\, t + \gamma_j W_t^j \le x, \int_0^t e^{(\beta_j - \gamma_j^2/2)s + \gamma_j W_s} \, ds \le a\bigg) \tag{2.17}$$
$$= P\bigg(2(W_{t'}^j + \nu_j t') \le x, \int_0^{t'} e^{2(W_s^j + \nu_j s)} \, ds \le \frac{\gamma_j^2 a}{4}\bigg)$$

with $t' = \gamma_j^2 t/4$ and $\nu_j = 2\beta_j/\gamma_j^2 - 1$, by virtue of the expressions in (2.15)-(2.16), it follows from the definition in (2.3) and the Markov property of the process (Y^j, A^j), $j = 0, 1, 2$, that the random vector $(Y_T^j/Y_t^j, (A_T^j - A_t^j)/Y_t^j)$ has the distribution

$$P\big(Y_T^j/Y_t^j \in dy, (A_T^j - A_t^j)/Y_t^j \in da\big) = P\big(Y_{T-t}^j \in dy, A_{T-t}^j \in da\big) = g_{T-t}^j(y, a)\, dyda \tag{2.18}$$

where the density function g^j is given by

$$g_{T-t}^j(y, a) = \frac{\gamma_j^2}{4y}\, q_j\left(\frac{\gamma_j^2}{4}\,(T-t), \ln(y), \frac{\gamma_j^2 a}{4}\right) \tag{2.19}$$
$$= \frac{2\sqrt{2}}{\pi^{3/2}\gamma_j^3}\, \frac{1}{a^2 y\sqrt{T-t}} \exp\left(\frac{2\pi^2}{\gamma_j^2 (T-t)} + \frac{\beta_j}{\gamma_j^2}\, \ln(y) - \left(\frac{\beta_j}{\gamma_j} - \frac{\gamma_j}{2}\right)^2 \frac{(T-t)}{2} - \frac{2(1+y)}{\gamma_j^2 a}\right)$$
$$\times \int_0^\infty \exp\left(-\frac{2w^2}{\gamma_j^2 (T-t)} - \frac{4\sqrt{y}}{\gamma_j^2 a} \cosh(w)\right) \sinh(w)\, \sin\left(\frac{4\pi w}{\gamma_j^2 (T-t)}\right) dw$$

for all $T - t, y, a > 0$, and every $j = 0, 1, 2$. Note that the formulas above were also used in [19; Section 4] for the computation of the marginal density of the posterior probability process in the one-dimensional quickest change-point detection problem.

We also recall the transition density functions of the geometric Brownian motions Z^j, $j = 0, 1, 2$, defined in (2.4) above. It follows that the random variable Z_T^j/Z_t^j has the distribution

$$P\big(Z_T^j/Z_t^j \in dz\big) = P\big(Z_{T-t}^j \in dz\big) = h_{T-t}^j(z)\, dz \tag{2.20}$$

where the density function h^j is given by

$$h_{T-t}^j(z) = \frac{1}{\theta_j z\sqrt{2\pi(T-t)}} \exp\left(-\frac{(\ln(z) - (\eta_j - \theta_j^2/2)(T-t))^2}{2\theta_j^2 (T-t)}\right) \tag{2.21}$$

for all $T - t, z > 0$, and every $j = 0, 1, 2$.

3 Conditional distributions of the default times

In this section, we derive explicit expressions for the conditional distributions of two default times given the accessible filtration generated by the market prices of the risky assets as well as given the observable filtrations.

We first compute the conditional distributions $P(\tau_i > u \,|\, \mathcal{F}_t)$ of the default times τ_i, $i = 1, 2$, given the reference filtration $(\mathcal{F}_t)_{t \ge 0}$, for all $t, u \ge 0$. In this case, we see from the independence of the random variables U_i, $i = 1, 2$, and the filtration $(\mathcal{F}_t)_{t \ge 0}$ that the equalities

$$\begin{aligned} P(\tau_i > t \,|\, \mathcal{F}_\infty) &= P\big(\delta_i A_t^0 + \lambda_i A_t^i < -\ln U_i \,\big|\, \mathcal{F}_\infty\big) \\ &= e^{-\delta_i A_t^0 - \lambda_i A_t^i} = P(\tau_i > t \,|\, \mathcal{F}_t) \quad \text{for} \quad t \ge 0 \end{aligned} \tag{3.1}$$

hold, so that the equalities

$$P(\tau_i \in dt \,|\, \mathcal{F}_\infty) = e^{-\delta_i A_t^0 - \lambda_i A_t^i} \, (\delta_i Y_t^0 + \lambda_i Y_t^i) \, dt = P(\tau_i \in dt \,|\, \mathcal{F}_t) \quad \text{for} \quad t \ge 0 \tag{3.2}$$

are satisfied, for every $i = 1, 2$. In particular, it follows from the representation in (3.1) that the equalities

$$P(\tau_i > u \,|\, \mathcal{F}_t) = P\big(\delta_i A_u^0 + \lambda_i A_u^i < -\ln U_i \,\big|\, \mathcal{F}_t\big) = e^{-\delta_i A_u^0 - \lambda_i A_u^i} \quad \text{for} \quad 0 \le u \le t \tag{3.3}$$

hold, for every $i = 1, 2$. Then, according to the tower property for conditional expectations, we obtain that the equalities

$$\begin{aligned} P(\tau_i > u \,|\, \mathcal{F}_t) &= E\big[P\big(\delta_i A_u^0 + \lambda_i A_u^i < -\ln U_i \,\big|\, \mathcal{F}_u\big) \,\big|\, \mathcal{F}_t\big] \\ &= e^{-\delta_i A_t^0 - \lambda_i A_t^i} \, E\big[e^{-\delta_i Y_t^0 (A_u^0 - A_t^0)/Y_t^0 - \lambda_i Y_t^i (A_u^i - A_t^i)/Y_t^i} \,\big|\, \mathcal{F}_t\big] \\ &= e^{-\delta_i A_t^0 - \lambda_i A_t^i} \, C_{u-t}^i(Y_t^0, Y_t^i) \quad \text{for} \quad 0 \le t < u \end{aligned} \tag{3.4}$$

are satisfied, for every $i = 1, 2$. Here, by means of the Markov property of the processes (Y^j, A^j), $j = 0, 1, 2$, and the fact that the random variables Y_u^j / Y_t^j have the same laws as Y_{u-t}^j, $j = 0, 1, 2$, for each $0 \le t < u$, taking into account the independence of the driving standard Brownian motions W^j, $j = 0, 1, 2$, we have

$$\begin{aligned} C_{u-t}^i(y_0, y_i) &= E\big[e^{-\delta_i y_0 A_{u-t}^0 - \lambda_i y_i A_{u-t}^i}\big] = E\big[e^{-\delta_i y_0 A_{u-t}^0}\big] \, E\big[e^{-\lambda_i y_i A_{u-t}^i}\big] \\ &= \int_0^\infty \int_0^\infty e^{-\delta_i y_0 a_0} \, g_{u-t}^0(y_0', a_0) \, dy_0' da_0 \int_0^\infty \int_0^\infty e^{-\lambda_i y_i a_i} \, g_{u-t}^i(y_i', a_i) \, dy_i' da_i \end{aligned} \tag{3.5}$$

for all $0 \le t < u$, and the functions g^j, $j = 0, 1, 2$, are given in (2.19) above. Moreover, taking into account the representation in (3.2), according to the tower property for conditional expectations, we obtain that the equalities

$$\begin{aligned} P(\tau_i \in du \,|\, \mathcal{F}_t) &= E\big[P(\tau_i \in du \,|\, \mathcal{F}_u) \,\big|\, \mathcal{F}_t\big] = E\big[e^{-\delta_i A_u^0 - \lambda_i A_u^i} \, (\delta_i Y_u^0 + \lambda_i Y_u^i) \,\big|\, \mathcal{F}_t\big] \, du \\ &= e^{-\delta_i A_t^0 - \lambda_i A_t^i} \, E\big[e^{-\delta_i Y_t^0 (A_u^0 - A_t^0)/Y_t^0 - \lambda_i Y_t^i (A_u^i - A_t^i)/Y_t^i} \, \big(\delta_i \, Y_t^0 (Y_u^0 / Y_t^0) + \lambda_i \, Y_t^i (Y_u^i / Y_t^i)\big) \,\big|\, \mathcal{F}_t\big] \, du \\ &= e^{-\delta_i A_t^0 - \lambda_i A_t^i} \, D_{u-t}^i(Y_t^0, Y_t^i) \, du \quad \text{for} \quad 0 \le t < u \end{aligned} \tag{3.6}$$

are satisfied, for every $i = 1, 2$. Here, by means of the Markov property of the processes (Y^j, A^j), $j = 0, 1, 2$, and the fact that the random variables Y_u^j / Y_t^j have the same laws as Y_{u-t}^j, $j = 0, 1, 2$, for each $0 \le t < u$, we have

$$
\begin{aligned}
D_{u-t}^i(y_0, y_i) &= E\big[e^{-\delta_i y_0 A_{u-t}^0 - \lambda_i y_i A_{u-t}^i} \, (\delta_i y_0 Y_{u-t}^0 + \lambda_i y_i Y_{u-t}^i)\big] \qquad (3.7) \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-\delta_i y_0 a_0 - \lambda_i y_i a_i} \, (\delta_i y_0 y_0' + \lambda_i y_i y_i') \, g_{u-t}^0(y_0', a_0) \, g_{u-t}^i(y_i', a_i) \, dy_0' da_0 dy_i' da_i
\end{aligned}
$$

for all $0 \le t < u \le T$ and every $i = 1, 2$, while the functions g^j, $j = 0, 1, 2$, are given in (2.19) above.

We proceed with computing the conditional distribution $P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_t)$ of the default times τ_i, $i = 1, 2$, given the reference filtration $(\mathcal{F}_t)_{t \ge 0}$, for all $t, u, v \ge 0$. For this purpose, we observe from the independence of the random variables U_i, $i = 1, 2$, and the filtration $(\mathcal{F}_t)_{t \ge 0}$ that the equalities

$$
\begin{aligned}
&P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_\infty) \qquad (3.8) \\
&= P\big(\delta_i A_u^0 + \lambda_i A_u^i < -\ln U_i, \delta_{3-i} A_v^0 + \lambda_{3-i} A_v^{3-i} < -\ln U_{3-i} \,\big|\, \mathcal{F}_\infty\big) \\
&= e^{-\delta_i A_u^0 - \lambda_i A_u^i - \delta_{3-i} A_v^0 - \lambda_{3-i} A_v^{3-i}} = P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_u) \quad \text{for} \quad 0 \le v \le u
\end{aligned}
$$

hold, so that the equalities

$$
\begin{aligned}
&P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_\infty) \qquad (3.9) \\
&= e^{-\delta_i A_u^0 - \lambda_i A_u^i - \delta_{3-i} A_v^0 - \lambda_{3-i} A_v^{3-i}} \, (\delta_i Y_u^0 + \lambda_i Y_u^i) \, (\delta_{3-i} Y_v^0 + \lambda_{3-i} Y_v^{3-i}) \, du dv \\
&= P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_u) \quad \text{for} \quad 0 \le v \le u
\end{aligned}
$$

are satisfied, for every $i = 1, 2$. In particular, it follows from the representation in (3.8) that the equalities

$$
\begin{aligned}
&P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_t) \qquad (3.10) \\
&= P\big(\delta_i A_u^0 + \lambda_i A_u^i < -\ln U_i, \delta_{3-i} A_v^0 + \lambda_{3-i} A_v^{3-i} < -\ln U_{3-i} \,\big|\, \mathcal{F}_t\big) \\
&= e^{-\delta_i A_u^0 - \lambda_i A_u^i - \delta_{3-i} A_v^0 - \lambda_{3-i} A_v^{3-i}} \quad \text{for} \quad 0 \le u, v \le t
\end{aligned}
$$

hold, for every $i = 1, 2$. Hence, according to the tower property for conditional expectations, we obtain

$$
\begin{aligned}
&P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_t) = E\big[P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_u) \,\big|\, \mathcal{F}_t\big] \\
&= E\big[P\big(\delta_i A_u^0 + \lambda_i A_u^i < -\ln U_i, \delta_{3-i} A_v^0 + \lambda_{3-i} A_v^{3-i} < -\ln U_{3-i} \,\big|\, \mathcal{F}_u\big) \,\big|\, \mathcal{F}_t\big] \\
&= e^{-\delta_i A_t^0 - \lambda_i A_t^{3-i} - \delta_{3-i} A_v^0 - \lambda_{3-i} A_v^i}\, E\big[e^{-\delta_i Y_t^0 (A_u^0 - A_t^0)/Y_t^0 - \lambda_i Y_t^i (A_u^i - A_t^i)/Y_t^i} \,\big|\, \mathcal{F}_t\big] \\
&= e^{-\delta_i A_t^0 - \lambda_i A_t^i - \delta_{3-i} A_v^0 - \lambda_{3-i} A_v^{3-i}}\, C_{u-t}^i(Y_t^0, Y_t^i) \quad \text{for} \quad 0 \le v \le t \le u
\end{aligned} \tag{3.11}
$$

while

$$
\begin{aligned}
&P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_t) \\
&= e^{-\delta_i A_u^0 - \lambda_i A_u^i - \delta_{3-i} A_t^0 - \lambda_{3-i} A_t^{3-i}}\, C_{v-t}^{3-i}(Y_t^0, Y_t^{3-i}) \quad \text{for} \quad 0 \le u \le t < v
\end{aligned} \tag{3.12}
$$

where $C_{u-t}^i(y_0, y_i)$ and $C_{v-t}^{3-i}(y_0, y_{3-i})$ are given as in (3.5) above, for every $i = 1, 2$. Moreover, taking into account the representation in (3.9), according to the tower property for conditional expectations, we obtain that the equalities

$$
\begin{aligned}
&P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_t) \\
&= E\big[e^{-\delta_i A_u^0 - \lambda_i A_u^i - \delta_{3-i} A_v^0 - \lambda_{3-i} A_v^{3-i}}\, (\delta_i Y_u^0 + \lambda_i Y_u^i)\, (\delta_{3-i} Y_v^0 + \lambda_{3-i} Y_v^{3-i}) \,\big|\, \mathcal{F}_t\big]\, du dv \\
&= e^{-\delta_i A_t^0 - \lambda_i A_t^i - \delta_{3-i} A_v^0 - \lambda_{3-i} A_v^{3-i}}\, (\delta_{3-i} Y_v^0 + \lambda_{3-i} Y_v^{3-i}) \\
&\quad \times E\big[e^{-\delta_i Y_t^0 (A_u^0 - A_t^0)/Y_t^0 - \lambda_i Y_t^i (A_u^i - A_t^i)/Y_t^i}\, \big(\delta_i Y_t^0 (Y_u^0 / Y_t^0) + \lambda_i Y_t^i (Y_u^i / Y_t^i)\big) \,\big|\, \mathcal{F}_t\big]\, du dv \\
&= e^{-\delta_i A_t^0 - \lambda_i A_t^i - \delta_{3-i} A_v^0 - \lambda_{3-i} A_v^{3-i}}\, (\delta_{3-i} Y_v^0 + \lambda_{3-i} Y_v^{3-i})\, D_{u-t}^i(Y_t^0, Y_t^i)\, du dv \quad \text{for} \quad 0 \le v \le t \le u
\end{aligned} \tag{3.13}
$$

are satisfied, where $D_{u-t}^i(y_0, y_i)$ is given in (3.7) above, for every $i = 1, 2$. Finally, taking into account the representation in (3.9), according to the tower property for conditional expectations, we obtain

$$
\begin{aligned}
&E\big[P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_u) \,\big|\, \mathcal{F}_v\big] \\
&= E\big[P\big(\delta_i A_u^0 + \lambda_i A_u^i < -\ln U_i, \delta_{3-i} A_v^0 + \lambda_{3-i} A_v^{3-i} < -\ln U_{3-i} \,\big|\, \mathcal{F}_u\big) \,\big|\, \mathcal{F}_v\big] \\
&= e^{-(\delta_i + \delta_{3-i}) A_v^0 - \lambda_i A_v^i - \lambda_{3-i} A_v^{3-i}}\, E\big[e^{-\delta_{3-i} Y_v^0 (A_u^0 - A_v^0)/Y_v^0 - \lambda_{3-i} Y_v^{3-i} (A_u^{3-i} - A_v^{3-i})/Y_v^{3-i}} \,\big|\, \mathcal{F}_v\big] \\
&= e^{-(\delta_i + \delta_{3-i}) A_v^0 - \lambda_i A_v^i - \lambda_{3-i} A_v^{3-i}}\, C_{u-v}^i(Y_v^0, Y_v^i) \quad \text{for} \quad 0 \le v \le u
\end{aligned} \tag{3.14}
$$

while

$$
\begin{aligned}
&E\big[P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_v) \,\big|\, \mathcal{F}_u\big] \\
&= e^{-(\delta_i + \delta_{3-i}) A_u^0 - \lambda_i A_u^i - \lambda_{3-i} A_u^{3-i}}\, C_{v-u}^{3-i}(Y_u^0, Y_u^{3-i}) \quad \text{for} \quad 0 \le u < v
\end{aligned} \tag{3.15}
$$

where $C^i_{u-v}(y_0, y_i)$ and $C^{3-i}_{v-u}(y_0, y_{3-i})$ are given as in (3.5) above, for every $i = 1, 2$. Thus, according to the tower property for conditional expectations, we obtain

$$
\begin{aligned}
&P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_t) = E\big[e^{-(\delta_i+\delta_{3-i})A^0_v - \lambda_i A^i_v - \lambda_{3-i} A^{3-i}_v}\, C^i_{u-v}(Y^0_v, Y^i_v) \,\big|\, \mathcal{F}_t\big] \qquad (3.16)\\
&= e^{-(\delta_i+\delta_{3-i})A^0_t - \lambda_i A^i_t - \lambda_{3-i} A^{3-i}_t}\, E\big[e^{-(\delta_i+\delta_{3-i})Y^0_t(A^0_v - A^0_t)/Y^0_t - \lambda_i Y^i_t(A^i_v - A^i_t)/Y^i_t - \lambda_{3-i} Y^{3-i}_t (A^{3-i}_v - A^{3-i}_t)/Y^{3-i}_t} \\
&\qquad\qquad \times C^i_{u-v}(Y^0_t(Y^0_v/Y^0_t), Y^i_t(Y^i_v/Y^i_t) \,\big|\, \mathcal{F}_t\big] \\
&= e^{-(\delta_i+\delta_{3-i})A^0_t - \lambda_i A^i_t - \lambda_{3-i} A^{3-i}_t}\, \overline{C}^i_{v-t,u-v}(Y^0_t, Y^i_t, Y^{3-i}_t) \quad \text{for} \quad 0 \le t \le v \le u
\end{aligned}
$$

while

$$
\begin{aligned}
&P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_t) = E\big[e^{-(\delta_i+\delta_{3-i})A^0_u - \lambda_i A^i_u - \lambda_{3-i} A^{3-i}_u}\, C^{3-i}_{v-u}(Y^0_u, Y^{3-i}_u) \,\big|\, \mathcal{F}_t\big] \qquad (3.17)\\
&= e^{-(\delta_i+\delta_{3-i})A^0_t - \lambda_i A^i_t - \lambda_{3-i} A^{3-i}_t}\, E\big[e^{-(\delta_i+\delta_{3-i})Y^0_t(A^0_u - A^0_t)/Y^0_t - \lambda_i Y^i_t(A^i_u - A^i_t)/Y^i_t - \lambda_{3-i} Y^{3-i}_t (A^{3-i}_u - A^{3-i}_t)/Y^{3-i}_t} \\
&\qquad\qquad \times C^{3-i}_{v-u}(Y^0_t(Y^0_u/Y^0_t), Y^{3-i}_t(Y^{3-i}_u/Y^{3-i}_t) \,\big|\, \mathcal{F}_t\big] \\
&= e^{-(\delta_i+\delta_{3-i})A^0_t - \lambda_i A^i_t - \lambda_{3-i} A^{3-i}_t}\, \overline{C}^{3-i}_{u-t,v-u}(Y^0_t, Y^{3-i}_t, Y^i_t) \quad \text{for} \quad 0 \le t < u < v
\end{aligned}
$$

for every $i = 1, 2$. Here, by virtue of the Markov property of the processes (Y^j, A^j), $j = 0, 1, 2$, and the fact that the random variables Y^j_v/Y^j_t have the same laws as Y^j_{v-t}, $j = 0, 1, 2$, for each $0 \le t < v$, we have

$$
\begin{aligned}
&\overline{C}^i_{v-t,u-v}(y_0, y_i, y_{3-i}) \qquad (3.18)\\
&= E\big[e^{-(\delta_i+\delta_{3-i})y_0 A^0_{v-t} - \lambda_i y_i A^i_{v-t} - \lambda_{3-i} y_{3-i} A^{3-i}_{v-t}}\, C^i_{u-v}(y_0 Y^0_{v-t}, y_i Y^i_{v-t})\big] \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-(\delta_i+\delta_{3-i})y_0 a_0 - \lambda_i y_i a_i - \lambda_{3-i} y_{3-i} a_{3-i}}\, C^i_{u-v}(y_0 y'_0, y_i y'_i) \\
&\quad \times\ g^0_{v-t}(y'_0, a_0)\, g^i_{v-t}(y'_i, a_i)\, g^{3-i}_{v-t}(y'_{3-i}, a_{3-i})\, dy'_0 da_0 dy'_i da_i dy'_{3-i} da_{3-i}
\end{aligned}
$$

for all $0 \le t < v < u \le T$ and every $i = 1, 2$, while the functions g^j, $j = 0, 1, 2$, are given in (2.19) above.

Let us now compute the conditional distributions $P(\tau_i > u \,|\, \mathcal{G}^i_t)$ and $P(\tau_i > u \,|\, \mathcal{G}^{3-i}_t)$ of the default times τ_i, $i = 1, 2$, given the filtrations $(\mathcal{G}^i_t)_{t \ge 0}$ and $(\mathcal{G}^{3-i}_t)_{t \ge 0}$, for all $t, u \ge 0$. In this case, we obviously have $P(\tau_i > u \,|\, \mathcal{G}^i_t) = I(\tau_i > u)$, for all $0 \le u \le t$. Then, we apply the first part of the key lemma in (2.8) for the filtrations $(\mathcal{G}^i_t)_{t \ge 0}$ and $(\mathcal{F}_t)_{t \ge 0}$, and use the expressions in (3.1) and (3.4) to get

$$
P(\tau_i > u \,|\, \mathcal{G}^i_t) = I(\tau_i > t)\, \frac{P(\tau_i > u \,|\, \mathcal{F}_t)}{P(\tau_i > t \,|\, \mathcal{F}_t)} = I(\tau_i > t)\, C^i_{u-t}(Y^0_t, Y^i_t) \quad \text{for} \quad 0 \le t < u \qquad (3.19)
$$

where $C^i_{u-t}(y_0, y_i)$ is given in (3.5) above, for every $i = 1, 2$. Moreover, taking into account the expressions in (3.1) and (3.2), we see from the independence of the random variable U_i and the filtration $(\mathcal{G}^{3-i}_t)_{t \ge 0}$ that the equalities

$$P(\tau_i > t \,|\, \mathcal{G}^{3-i}_\infty) = P\big(\delta_i A^0_t + \lambda_i A^i_t < -\ln U_i \,\big|\, \mathcal{G}^{3-i}_\infty\big) \tag{3.20}$$
$$= e^{-\delta_i A^0_t - \lambda_i A^i_t} = P(\tau_i > t \,|\, \mathcal{F}_t) \quad \text{for} \quad t \ge 0$$

hold, so that the equalities

$$P(\tau_i \in dt \,|\, \mathcal{G}^{3-i}_\infty) = e^{-\delta_i A^0_t - \lambda_i A^i_t} \, (\delta_i Y^0_t + \lambda_i Y^i_t) \, dt = P(\tau_i \in dt \,|\, \mathcal{F}_t) \quad \text{for} \quad t \ge 0 \tag{3.21}$$

are satisfied, for every $i = 1, 2$. In particular, it follows from the representation in (3.20) that the equalities

$$P(\tau_i > u \,|\, \mathcal{G}^{3-i}_t) = P\big(\delta_i A^0_u + \lambda_i A^i_u < -\ln U_i \,\big|\, \mathcal{G}^{3-i}_t\big) \tag{3.22}$$
$$= e^{-\delta_i A^0_u - \lambda_i A^i_u} = P(\tau_i > u \,|\, \mathcal{F}_t) \quad \text{for} \quad 0 \le u \le t$$

hold, for every $i = 1, 2$. Then, we apply the first part of the key lemma in (2.8) for the filtrations $(\mathcal{G}^{3-i}_t)_{t \ge 0}$ and $(\mathcal{F}_t)_{t \ge 0}$, where $\mathcal{G}^{3-i}_t$ coincides with $\mathcal{F}^{3-i}_t$ on the event $\{\tau_{3-i} \le t\}$ and with $\mathcal{F}_t$ on $\{\tau_{3-i} > t\}$, for all $t \ge 0$, and use the equality in (2.12) as well as the expressions in (3.1), (3.4), and (3.13) to get

$$P(\tau_i > u \,|\, \mathcal{G}^{3-i}_t) = I(\tau_{3-i} \le t) \, P(\tau_i > u \,|\, \mathcal{F}^{3-i}_t) + I(\tau_{3-i} > t) \, \frac{P(\tau_i > u, \tau_{3-i} > t \,|\, \mathcal{F}_t)}{P(\tau_{3-i} > t \,|\, \mathcal{F}_t)} \tag{3.23}$$
$$= e^{-\delta_i A^0_t - \lambda_i A^i_t} \, C^i_{u-t}(Y^0_t, Y^i_t) = P(\tau_i > u \,|\, \mathcal{F}_t) \quad \text{for} \quad 0 \le t < u$$

where $C^i_{u-t}(y_0, y_i)$ is given in (3.5) above, for every $i = 1, 2$.

We finally compute the conditional distributions $P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{G}_t)$ of the default times τ_i, $i = 1, 2$, given the filtration $(\mathcal{G}_t)_{t \ge 0}$, for all $t, u, v \ge 0$. In this case, we obviously have $P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{G}_t) = I(\tau_i > u, \tau_{3-i} > v)$, for all $0 \le u, v \le t$ and every $i = 1, 2$. Then, we apply the first part of the key lemma for the filtrations $(\mathcal{G}_t)_{t \ge 0}$ and $(\mathcal{F}_t)_{t \ge 0}$, where $\mathcal{G}_t$ coincides with $\mathcal{G}^i_t \vee \sigma(\tau_{3-i})$ on the event $\{v < \tau_{3-i} \le t\}$ and with $\mathcal{G}^i_t$ on $\{\tau_{3-i} > t\}$, and use the expressions in (3.14)-(3.15) and (3.19) to get

$$P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{G}_t) \tag{3.24}$$
$$= I(v < \tau_{3-i} \le t) \, P(\tau_i > u \,|\, \mathcal{G}^i_t \vee \sigma(\tau_{3-i})) + I(\tau_i > t, \tau_{3-i} > t) \, \frac{P(\tau_i > u, \tau_{3-i} > t \,|\, \mathcal{F}_t)}{P(\tau_i > t, \tau_{3-i} > t \,|\, \mathcal{F}_t)}$$
$$= I(\tau_i > t, \tau_{3-i} > v) \, C^i_{u-t}(Y^0_t, Y^i_t) \quad \text{for} \quad 0 \le v \le t \le u$$

while

$$P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{G}_t) = I(\tau_i > u, \tau_{3-i} > t)\, C^{3-i}_{v-t}(Y^0_t, Y^{3-i}_t) \quad \text{for} \quad 0 \le u \le t < v \tag{3.25}$$

where $C^i_{u-t}(y_0, y_i)$ and $C^{3-i}_{v-t}(y_0, y_{3-i})$ are given as in (3.5) above, for every $i = 1, 2$. Moreover, by means of the first part of the key lemma for the filtrations $(\mathcal{G}_t)_{t \ge 0}$ and $(\mathcal{F}_t)_{t \ge 0}$, and taking into account the expressions in (3.16) and (3.17), we have

$$\begin{aligned} P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{G}_t) &= I(\tau_i > t, \tau_{3-i} > t)\, \frac{P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_t)}{P(\tau_i > t, \tau_{3-i} > t \,|\, \mathcal{F}_t)} \\ &= I(\tau_i > t, \tau_{3-i} > t)\, \overline{C}^i_{v-t, u-v}(Y^0_t, Y^i_t, Y^{3-i}_t) \quad \text{for} \quad 0 \le t \le v \le u \end{aligned} \tag{3.26}$$

while

$$P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{G}_t) = I(\tau_i > t, \tau_{3-i} > t)\, \overline{C}^{3-i}_{u-t, v-u}(Y^0_t, Y^{3-i}_t, Y^i_t) \quad \text{for} \quad 0 \le t \le u < v \tag{3.27}$$

where $\overline{C}^i_{v-t, u-v}(y_0, y_i, y_{3-i})$ and $\overline{C}^{3-i}_{u-t, v-u}(y_0, y_{3-i}, y_i)$ are given as in (3.18) above, for every $i = 1, 2$.

4 The rational prices of risk-free CDSs (Main results)

In this section, we derive explicit expressions for the rational prices of credit default swaps without consideration of counterparty risk in the model defined above with some (continuously compounded) premia $\varkappa_i > 0$ and (non-negative measurable) deterministic recovery payoff functions $R^i_t(x_i)$, $i = 1, 2$, for all $x_i > 0$ and $0 \le t \le T$, and every $i = 1, 2$. In order to simplify the notations, without loss of generality, we further assume that the payoffs are already discounted by the dynamics of the bank account, that is equivalent to letting the interest rate r equal to zero. We compute the rational prices for the holders of risk-free CDSs in various particular cases of available information contained in the filtrations $(\mathcal{G}^i_t)_{t \ge 0}$, or $(\mathcal{G}_t)_{t \ge 0}$, or $(\mathcal{G}^i_t \vee \sigma(\tau_{3-i}))_{t \ge 0}$ defined above, for every $i = 1, 2$. In those cases, the holders of CDSs can observe only the default time τ_i, or observe the both default times τ_i, $i = 1, 2$, or observe the default time τ_i but know the default time τ_{3-i} from the beginning of observations, respectively.

4.1 The case of filtration $(\mathcal{G}^i_t)_{t \ge 0}$

Let us begin by computing the rational price $P^i = (P^i_t)_{t \ge 0}$ for the holder of a CDS in the model with the filtration $(\mathcal{G}^i_t)_{t \ge 0}$ given by

$$P^i_t = E\big[- \varkappa_i \, (\tau_i \wedge T - t)\, I(t < \tau_i) + R^i_{\tau_i}(X^i_{\tau_i})\, I(t < \tau_i \le T) \,\big|\, \mathcal{G}^i_t \big] \tag{4.1}$$

for $0 \le t \le T \wedge \tau_i$, and $P^i_t = 0$, for $t > T \wedge \tau_i$, so that the premium $\varkappa_i$ is then determined from the equation $P^i_0 = 0$, for every $i = 1,2$. In order to compute the both terms in (4.1), we apply the second part of the key lemma in (2.9) for the filtrations $(\mathcal{G}^i_t)_{t \ge 0}$ and $(\mathcal{F}_t)_{t \ge 0}$, and use Fubini's theorem for interchanging the order of conditional expectation and integration to get

$$E\big[(\tau_i \wedge T - t)\, I(t < \tau_i) \,\big|\, \mathcal{G}^i_t\big] = I(t < \tau_i)\, \frac{E[(\tau_i \wedge T - t) I(t < \tau_i) \,|\, \mathcal{F}_t]}{P(t < \tau_i \,|\, \mathcal{F}_t)} \tag{4.2}$$

$$= I(t < \tau_i)\, E\bigg[\int_t^\infty \frac{(u \wedge T - t) P(\tau_i \in du \,|\, \mathcal{F}_u)}{P(t < \tau_i \,|\, \mathcal{F}_t)} \,\bigg|\, \mathcal{F}_t\bigg] = I(t < \tau_i) \int_t^\infty \frac{(u \wedge T - t) P(\tau_i \in du \,|\, \mathcal{F}_t)}{P(t < \tau_i \,|\, \mathcal{F}_t)}$$

and

$$E\big[R^i(X^i_{\tau_i})\, I(t < \tau_i \le T) \,\big|\, \mathcal{G}^i_t\big] = I(t < \tau_i)\, \frac{E[R^i_{\tau_i}(X^i_{\tau_i}) I(t < \tau_i \le T) \,|\, \mathcal{F}_t]}{P(t < \tau_i \,|\, \mathcal{F}_t)} \tag{4.3}$$

$$= I(t < \tau_i)\, E\bigg[\int_t^T \frac{R^i_u(X^i_u) P(\tau_i \in du \,|\, \mathcal{F}_u)}{P(t < \tau_i \,|\, \mathcal{F}_t)} \,\bigg|\, \mathcal{F}_t\bigg] = I(t < \tau_i) \int_t^T \frac{E[R^i_u(X^i_u)\, P(\tau_i \in du \,|\, \mathcal{F}_u) \,|\, \mathcal{F}_t]}{P(t < \tau_i \,|\, \mathcal{F}_t)}$$

for all $0 \le t \le T$ and every $i = 1,2$. Here, we recall from the expressions in (3.1)-(3.2) and (3.6)-(3.7) that $P(t < \tau_i \,|\, \mathcal{F}_t) = e^{-\delta_i A^0_t - \lambda_i A^i_t}$ and $P(\tau_i \in dt \,|\, \mathcal{F}_t) = e^{-\delta_i A^0_t - \lambda_i A^i_t}(\delta_i Y^0_t + \lambda_i Y^i_t)$, for all $t \ge 0$, as well as $P(\tau_i \in du \,|\, \mathcal{F}_t) = e^{-\delta_i A^0_t - \lambda_i A^i_t}\, D^i_{u-t}(Y^0_t, Y^i_t)\, du$, for all $0 \le t < u$ and every $i = 1,2$. Then, taking into account the expressions in (3.2), according to the tower property for conditional expectations, for each $0 \le t < u$, we obtain that

$$\begin{aligned} &E\big[R^i_u(X^i_u)\, P(\tau_i \in du \,|\, \mathcal{F}_u) \,\big|\, \mathcal{F}_t\big] = E\big[R^i_u(X^i_u)\, e^{-\delta_i A^0_u - \lambda_i A^i_u}\, (\delta_i\, Y^0_u + \lambda_i\, Y^i_u) \,\big|\, \mathcal{F}_t\big]\, du \\ &= e^{-\delta_i A^0_t - \lambda_i A^i_t}\, E\big[R^i_u\big(X^i_t (Y^i_u/Y^i_t)^{\alpha_i} (Z^0_u/Z^0_t)^{\zeta_i} (Z^i_u/Z^i_t)\big)\, e^{-\delta_i Y^0_t (A^0_u - A^0_t)/Y^0_t - \lambda_i Y^i_t (A^i_u - A^i_t)/Y^i_t} \\ &\qquad \times\, \big(\delta_i Y^0_t (Y^0_u/Y^0_t) + \lambda_i Y^i_t (Y^i_u/Y^i_t)\big) \,\big|\, \mathcal{F}_t\big]\, du \\ &= e^{-\delta_i A^0_t - \lambda_i A^i_t}\, Q^i_{t,u-t}(X^i_t, Y^0_t, Y^i_t)\, du \end{aligned} \tag{4.4}$$

holds. Here, by means of the Markov property of the processes (Y^j, A^j) and Z^j, $j = 0,1,2$, and the fact that the random variables Y^j_u/Y^j_t and Z^j_u/Z^j_t have the same laws as Y^j_{u-t} and Z^j_{u-t}, $j = 0,1,2$, for each $0 \le t < u$, respectively, we have

$$\begin{aligned} &Q^i_{t,u-t}(x_i, y_0, y_i) \\ &= E\big[R^i_u\big(x_i (Y^i_{u-t})^{\alpha_i} (Z^0_{u-t})^{\zeta_i} Z^i_{u-t}\big)\, e^{-\delta_i y_0 A^0_{u-t} - \lambda_i y_i A^i_{u-t}}\, (\delta_i y_0 Y^0_{u-t} + \lambda_i y_i Y^i_{u-t})\big] \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty R^i_u\big(x_i (y'_i)^{\alpha_i} (z'_0)^{\zeta_i} z'_i\big)\, e^{-\delta_i y_0 a_0 - \lambda_i y_i a_i}\, (\delta_i y_0 y'_0 + \lambda_i y_i y'_i) \\ &\quad \times\, g^0_{u-t}(y'_0, a_0)\, g^i_{u-t}(y'_i, a_i)\, h^0_{u-t}(z'_0)\, h^i_{u-t}(z'_i)\, dy'_0 da_0 dy'_i da_i dz'_0 dz'_i \end{aligned} \tag{4.5}$$

for all $0 \le t < u \le T$ and every $i = 1,2$, while the functions g^j and h^j, $j = 0,1,2$, are given in (2.19) and (2.21) above.

Therefore, summarising the facts proved above, we now formulate the following assertion.

Proposition 4.1. *Suppose that $r = 0$. The rational price for the holder of a risk-free credit default swap in (4.1) is given by the sum of the expressions in (4.2) and (4.3). The latter terms are computed by means of the expressions in (3.6) and (4.4) with (3.7) and (4.5), respectively.*

4.2 The case of filtration $(\mathcal{G}_t)_{t \geq 0}$

Let us now continue by computing the rational price $\widehat{P}^i = (\widehat{P}^i_t)_{t \geq 0}$ for the holder of a CDS in the model with the filtration $(\mathcal{G}_t)_{t \geq 0}$ given by

$$\widehat{P}^i_t = E\big[- \widehat{\varkappa}_i \, (\tau_i \wedge T - t) \, I(t < \tau_i) + R^i_{\tau_i}(X^i_{\tau_i}) \, I(t < \tau_i \leq T) \, \big| \, \mathcal{G}_t \big] \tag{4.6}$$

for $0 \leq t \leq T \wedge \tau_i$, and $\widehat{P}^i_t = 0$, for $t > T \wedge \tau_i$, so that the premium $\widehat{\varkappa}_i$ is then determined from the equation $\widehat{P}^i_0 = 0$, for every $i = 1, 2$. It is seen that the equality $\widehat{\varkappa}_i = \varkappa_i$ should hold with $\varkappa_i$, $i = 1, 2$, from (4.1), since we have $\mathcal{G}_0 = \mathcal{F}_0$. In order to compute the both terms in (4.6), we apply the second part of the key lemma for the filtrations $(\mathcal{G}_t)_{t \geq 0}$ and $(\mathcal{F}_t)_{t \geq 0}$, where $\mathcal{G}_t$ coincides with $\mathcal{F}^{3-i}_t$ on the event $\{\tau_{3-i} \leq t < \tau_i\}$ and with $\mathcal{F}_t$ on $\{t < \tau_i \wedge \tau_{3-i}\}$, for all $t \geq 0$, and use the equality in (2.12) as well as Fubini's theorem for interchanging the order of conditional expectation and integration to obtain

$$\begin{aligned}
& E\big[(\tau_i \wedge T - t) \, I(t < \tau_i) \, \big| \, \mathcal{G}_t \big] \qquad\qquad (4.7) \\
&= I(\tau_{3-i} \leq t < \tau_i) \, \frac{E[(\tau_i \wedge T - t) I(t < \tau_i) \, | \, \mathcal{F}^{3-i}_t]}{P(t < \tau_i \, | \, \mathcal{F}^{3-i}_t)} \\
&\quad + I(t < \tau_i \wedge \tau_{3-i}) \, \frac{E[(\tau_i \wedge T - t) I(t < \tau_i \wedge \tau_{3-i}) \, | \, \mathcal{F}_t]}{P(t < \tau_i \wedge \tau_{3-i} \, | \, \mathcal{F}_t)} \\
&= I(\tau_{3-i} \leq t < \tau_i) \, E\bigg[\int_{u=t}^{\infty} \frac{(u \wedge T - t) P(\tau_i \in du, \tau_{3-i} \in dv \, | \, \mathcal{F}_u)}{P(t < \tau_i, \tau_{3-i} \in dv \, | \, \mathcal{F}_t)} \, \bigg| \, \mathcal{F}_t \bigg] \bigg|_{v = \tau_{3-i}} \\
&\quad + I(t < \tau_i \wedge \tau_{3-i}) \, E\bigg[\int_{u=t}^{\infty} \int_{v=t}^{\infty} \frac{(u \wedge T - t) P(\tau_i \in du, \tau_{3-i} \in dv \, | \, \mathcal{F}_{u \vee v})}{P(t < \tau_i \wedge \tau_{3-i} \, | \, \mathcal{F}_t)} \, \bigg| \, \mathcal{F}_t \bigg] \\
&= I(\tau_{3-i} \leq t < \tau_i) \int_{u=t}^{\infty} \frac{(u \wedge T - t) P(\tau_i \in du, \tau_{3-i} \in dv \, | \, \mathcal{F}_t)}{P(t < \tau_i, \tau_{3-i} \in dv \, | \, \mathcal{F}_t)} \bigg|_{v = \tau_{3-i}} \\
&\quad + I(t < \tau_i \wedge \tau_{3-i}) \int_{u=t}^{\infty} \int_{v=t}^{\infty} \frac{(u \wedge T - t) P(\tau_i \in du, \tau_{3-i} \in dv \, | \, \mathcal{F}_t)}{P(t < \tau_i \wedge \tau_{3-i} \, | \, \mathcal{F}_t)}
\end{aligned}$$

and

$$
\begin{aligned}
&E\big[R^i_{\tau_i}(X^i_{\tau_i})\,I(t<\tau_i\le T)\,\big|\,\mathcal{G}_t\big] \qquad (4.8)\\
&= I(\tau_{3-i}\le t<\tau_i)\,\frac{E[R^i_{\tau_i}(X^i_{\tau_i})\,I(t<\tau_i\le T)\,|\,\mathcal{F}^{3-i}_t]}{P(t<\tau_i\,|\,\mathcal{F}^{3-i}_t)}\\
&\quad + I(t<\tau_i\wedge\tau_{3-i})\,\frac{E[R^i_{\tau_i}(X^i_{\tau_i})\,I(t<\tau_i\wedge\tau_{3-i}\le T\wedge\tau_{3-i})\,|\,\mathcal{F}_t]}{P(t<\tau_i\wedge\tau_{3-i}\,|\,\mathcal{F}_t)}\\
&= I(\tau_{3-i}\le t<\tau_i)\,E\bigg[\int_{u=t}^{T}\frac{R^i_u(X^i_u)P(\tau_i\in du,\tau_{3-i}\in dv\,|\,\mathcal{F}_u)}{P(t<\tau_i,\tau_{3-i}\in dv\,|\,\mathcal{F}_t)}\,\bigg|\,\mathcal{F}_t\bigg]\bigg|_{v=\tau_{3-i}}\\
&\quad + I(t<\tau_i\wedge\tau_{3-i})\,E\bigg[\int_{u=t}^{T}\int_{v=t}^{\infty}\frac{R^i_u(X^i_u)\,P(\tau_i\in du,\tau_{3-i}\in dv\,|\,\mathcal{F}_{u\vee v})}{P(t<\tau_i\wedge\tau_{3-i}\,|\,\mathcal{F}_t)}\,\bigg|\,\mathcal{F}_t\bigg]\\
&= I(\tau_{3-i}\le t<\tau_i)\,\int_{u=t}^{T}\frac{E[R^i_u(X^i_u)P(\tau_i\in du,\tau_{3-i}\in dv\,|\,\mathcal{F}_u)\,|\,\mathcal{F}_t]}{P(t<\tau_i,\tau_{3-i}\in dv\,|\,\mathcal{F}_t)}\bigg|_{v=\tau_{3-i}}\\
&\quad + I(t<\tau_i\wedge\tau_{3-i})\,\int_{u=t}^{T}\int_{v=t}^{\infty}\frac{E[R^i_u(X^i_u)P(\tau_i\in du,\tau_{3-i}\in dv\,|\,\mathcal{F}_u)\,|\,\mathcal{F}_t]}{P(t<\tau_i\wedge\tau_{3-i}\,|\,\mathcal{F}_t)}
\end{aligned}
$$

for all $0\le t\le T$ and every $i=1,2$. Here, we recall from the expressions in (3.8) and (3.9) that $P(t<\tau_i\wedge\tau_{3-i}\,|\,\mathcal{F}_t)=e^{-(\delta_i+\delta_{3-i})A^0_t-\lambda_i A^i_t-\lambda_{3-i}A^{3-i}_t}$, for all $t\ge 0$, and $P(t<\tau_i,\tau_{3-i}\in dv\,|\,\mathcal{F}_t)=e^{-\delta_i A^0_t-\lambda_i A^i_t-\delta_{3-i}A^0_v-\lambda_{3-i}A^{3-i}_v}(\delta_{3-i}Y^0_v+\lambda_{3-i}Y^{3-i}_v)$, for all $0\le v\le t$ and every $i=1,2$. Moreover, it is seen from the expressions in (3.13) that $P(\tau_i\in du,\tau_{3-i}\in dv\,|\,\mathcal{F}_t)=e^{-\delta_i A^0_t-\lambda_i A^i_t-\delta_{3-i}A^0_v-\lambda_{3-i}A^{3-i}_v}\,(\delta_{3-i}Y^0_v+\lambda_{3-i}Y^{3-i}_v)\,D^i_{u-t}(Y^0_t,Y^i_t)\,du dv$, for all $0\le v\le t\le u$, where $D^i_{u-t}(y_0,y_i)$ is given in (3.7) above, for every $i=1,2$.

Observe that, taking into account the expressions in (3.9), according to the tower property for conditional expectations, for each $0\le v\le t\le u$, we obtain

$$
\begin{aligned}
&E\big[R^i_u(X^i_u)\,P(\tau_i\in du,\tau_{3-i}\in dv\,|\,\mathcal{F}_u)\,\big|\,\mathcal{F}_t\big] \qquad (4.9)\\
&= E\big[R^i_u(X^i_u)\,e^{-\delta_i A^0_u-\lambda_i A^i_u-\delta_{3-i}A^0_v-\lambda_{3-i}A^{3-i}_v}\,(\delta_i\,Y^0_u+\lambda_i\,Y^i_u)\,(\delta_{3-i}\,Y^0_v+\lambda_{3-i}\,Y^{3-i}_v)\,\big|\,\mathcal{F}_t\big]\,du dv\\
&= e^{-\delta_i A^0_t-\lambda_i A^i_t-\delta_{3-i}A^0_v-\lambda_{3-i}A^{3-i}_v}\,(\delta_{3-i}\,Y^0_v+\lambda_{3-i}\,Y^{3-i}_v)\,E\big[R^i_u\big(X^i_t(Y^i_u/Y^i_t)^{\alpha_i}(Z^0_u/Z^0_t)^{\zeta_i}(Z^i_u/Z^i_t)\big)\\
&\qquad\times e^{-\delta_i Y^0_t(A^0_u-A^0_t)/Y^0_t-\lambda_i Y^i_t(A^i_u-A^i_t)/Y^i_t}\,\big(\delta_i\,Y^0_t(Y^0_u/Y^0_t)+\lambda_i\,Y^i_t(Y^i_u/Y^i_t)\big)\,\big|\,\mathcal{F}_t\big]\,du dv\\
&= e^{-\delta_i A^0_t-\lambda_i A^i_t-\delta_{3-i}A^0_v-\lambda_{3-i}A^{3-i}_v}\,(\delta_{3-i}\,Y^0_v+\lambda_{3-i}\,Y^{3-i}_v)\,Q^i_{t,u-t}(X^i_t,Y^0_t,Y^i_t)\,du dv
\end{aligned}
$$

where $Q^i_{t,u-t}(x_i,y_0,y_i)$ is defined in (4.5) above, for every $i=1,2$. Furthermore, taking into account the expressions in (3.9), according to the tower property for conditional expectations, we obtain

$$
\begin{aligned}
& E\big[P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_u) \,\big|\, \mathcal{F}_v\big] \qquad (4.10) \\
&= E\big[e^{-\delta_i A^0_u - \lambda_i A^i_u - \delta_{3-i} A^0_v - \lambda_{3-i} A^{3-i}_v} \, (\delta_i Y^0_u + \lambda_i Y^i_u) \, (\delta_{3-i} Y^0_v + \lambda_{3-i} Y^{3-i}_v) \,\big|\, \mathcal{F}_v\big] \, dudv \\
&= e^{-(\delta_i + \delta_{3-i}) A^0_v - \lambda_i A^i_v - \lambda_{3-i} A^{3-i}_v} \, (\delta_i Y^0_v + \lambda_i Y^i_v) \, E\big[e^{-\delta_{3-i} Y^0_v (A^0_u - A^0_v)/Y^0_v - \lambda_{3-i} Y^{3-i}_v (A^{3-i}_u - A^{3-i}_v)/Y^{3-i}_v} \\
&\qquad \times \big(\delta_{3-i} Y^0_v (Y^0_u / Y^0_v) + \lambda_{3-i} Y^{3-i}_v (Y^{3-i}_u / Y^{3-i}_v)\big) \,\big|\, \mathcal{F}_v\big] \, dudv \\
&= e^{-(\delta_i + \delta_{3-i}) A^0_v - \lambda_i A^i_v - \lambda_{3-i} A^{3-i}_v} \, (\delta_{3-i} Y^0_v + \lambda_{3-i} Y^{3-i}_v) \, D^i_{u-v}(Y^0_v, Y^i_v) \, dudv \quad \text{for} \quad 0 \le v \le u
\end{aligned}
$$

while

$$
\begin{aligned}
& E\big[P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,\big|\, \mathcal{F}_u\big] \qquad (4.11) \\
&= e^{-(\delta_i + \delta_{3-i}) A^0_u - \lambda_i A^i_u - \lambda_{3-i} A^{3-i}_u} \, (\delta_i Y^0_u + \lambda_i Y^i_u) \, D^{3-i}_{v-u}(Y^0_u, Y^{3-i}_u) \, dudv \quad \text{for} \quad 0 \le u < v
\end{aligned}
$$

where $D^i_{u-v}(y_0, y_i)$ and $D^{3-i}_{v-u}(y_0, y_{3-i})$ are given as in (3.7) above, for every $i = 1, 2$. Hence, according to the tower property for conditional expectations, we obtain

$$
\begin{aligned}
& P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_t) \qquad (4.12) \\
&= E\big[e^{-(\delta_i + \delta_{3-i}) A^0_v - \lambda_i A^i_v - \lambda_{3-i} A^{3-i}_v} \, (\delta_i Y^0_v + \lambda_i Y^i_v) \, D^i_{u-v}(Y^0_v, Y^{3-i}_v) \,\big|\, \mathcal{F}_t\big] \, dudv \\
&= e^{-(\delta_i + \delta_{3-i}) A^0_t - \lambda_i A^i_t - \lambda_{3-i} A^{3-i}_t} \, E\big[e^{-(\delta_i + \delta_{3-i}) Y^0_t (A^0_v - A^0_t)/Y^0_t - \lambda_i Y^i_t (A^i_v - A^i_t)/Y^i_t - \lambda_{3-i} Y^{3-i}_t (A^{3-i}_v - A^{3-i}_t)/Y^{3-i}_t} \\
&\quad \times \big(\delta_i Y^0_t (Y^0_v / Y^0_t) + \lambda_i Y^i_t (Y^i_v / Y^i_t)\big) \, D^i_{u-v}(Y^0_t (Y^0_v / Y^0_t), Y^{3-i}_t (Y^{3-i}_v / Y^{3-i}_t)) \,\big|\, \mathcal{F}_t\big] \, dudv \\
&= e^{-(\delta_i + \delta_{3-i}) A^0_t - \lambda_i A^i_t - \lambda_{3-i} A^{3-i}_t} \, \overline{D}^i_{v-t,u-v}(Y^0_t, Y^i_t, Y^{3-i}_t) \, dudv \quad \text{for} \quad 0 \le t \le v \le u
\end{aligned}
$$

while

$$
\begin{aligned}
& P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_t) \qquad (4.13) \\
&= e^{-(\delta_i + \delta_{3-i}) A^0_t - \lambda_i A^i_t - \lambda_{3-i} A^{3-i}_t} \, \overline{D}^{3-i}_{u-t,v-u}(Y^0_t, Y^{3-i}_t, Y^i_t) \, dudv \quad \text{for} \quad 0 \le t \le u < v
\end{aligned}
$$

for every $i = 1, 2$. Here, by virtue of the Markov property of the processes (Y^j, A^j), $j = 0, 1, 2$, and the fact that the random variables Y^j_v / Y^j_t have the same laws as Y^j_{v-t}, $j = 0, 1, 2$, for each $0 \le t < v$, we have

$$
\begin{aligned}
& \overline{D}^i_{v-t,u-v}(y_0, y_i, y_{3-i}) \qquad (4.14) \\
&= E\big[e^{-(\delta_i + \delta_{3-i}) y_0 A^0_{v-t} - \lambda_i y_i A^i_{v-t} - \lambda_{3-i} y_{3-i} A^{3-i}_{v-t}} \, (\delta_i y_0 Y^0_{v-t} + \lambda_i y_i Y^i_{v-t}) \, D^i_{u-v}(y_0 Y^0_{v-t}, y_{3-i} Y^{3-i}_{v-t})\big] \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-(\delta_i + \delta_{3-i}) y_0 a_0 - \lambda_i y_i a_i - \lambda_{3-i} y_{3-i} a_{3-i}} \, (\delta_i y_0 y'_0 + \lambda_i y_i y'_i) \\
&\quad \times D^i_{u-v}(y_0 y'_0, y_{3-i} y'_{3-i}) \, g^0_{v-t}(y'_0, a_0) \, g^i_{v-t}(y'_i, a_i) \, g^{3-i}_{v-t}(y'_{3-i}, a_{3-i}) \, dy'_0 da_0 dy'_i da_i dy'_{3-i} da_{3-i}
\end{aligned}
$$

for all $0 \le t < v < u \le T$ and every $i = 1, 2$, while the functions g^j, $j = 0, 1, 2$, are given in (2.19) above.

On the other hand, taking into account the expressions in (3.9), according to the tower property for conditional expectations, for each $0 \le v \le u$, we obtain

$$
\begin{aligned}
&E\big[R_u^i(X_u^i)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_u) \,\big|\, \mathcal{F}_v\big] \qquad (4.15)\\
&= e^{-\delta_{3-i} A_v^0 - \lambda_{3-i} A_v^{3-i}}\, (\delta_{3-i}\, Y_v^0 + \lambda_{3-i}\, Y_v^{3-i})\, E\big[R_u^i(X_u^i)\, e^{-\delta_i A_u^0 - \lambda_i A_u^i}\, (\delta_i\, Y_v^0 + \lambda_i\, Y_v^i) \,\big|\, \mathcal{F}_v\big]\, dudv \\
&= e^{-(\delta_i + \delta_{3-i}) A_v^0 - \lambda_i A_v^i - \lambda_{3-i} A_v^{3-i}}\, (\delta_{3-i}\, Y_v^0 + \lambda_{3-i}\, Y_v^{3-i})\, E\big[R_u^i\big(X_v^i (Y_u^i / Y_v^i)^{\alpha_i} (Z_u^0 / Z_v^0)^{\zeta_i} (Z_u^i / Z_v^i)\big) \\
&\quad \times e^{-\delta_i Y_v^0 (A_u^0 - A_v^0)/Y_v^0 - \lambda_i Y_v^i (A_u^i - A_v^i)/Y_v^i}\, \big(\delta_i\, Y_v^0 (Y_u^0 / Y_v^0) + \lambda_i\, Y_v^i (Y_u^i / Y_v^i)\big) \,\big|\, \mathcal{F}_v\big]\, dudv \\
&= e^{-(\delta_i + \delta_{3-i}) A_v^0 - \lambda_i A_v^i - \lambda_{3-i} A_v^{3-i}}\, (\delta_{3-i}\, Y_v^0 + \lambda_{3-i}\, Y_v^{3-i})\, Q_{v,u-v}^i (X_v^i, Y_v^0, Y_v^i)\, dudv
\end{aligned}
$$

where $Q_{v,u-v}^i(x_i, y_0, y_i)$ is given as in (4.5) above, while, for each $0 \le u < v$, we get

$$
\begin{aligned}
&E\big[R_u^i(X_u^i)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,\big|\, \mathcal{F}_u\big] \qquad (4.16)\\
&= E\big[R_u^i(X_u^i)\, e^{-\delta_i A_u^0 - \lambda_i A_u^i - \delta_{3-i} A_v^0 - \lambda_{3-i} A_v^{3-i}}\, (\delta_i\, Y_u^0 + \lambda_i\, Y_u^i)\, (\delta_{3-i}\, Y_v^0 + \lambda_i\, Y_v^{3-i}) \,\big|\, \mathcal{F}_u\big]\, dudv \\
&= R_u^i(X_u^i)\, e^{-(\delta_i + \delta_{3-i}) A_u^0 - \lambda_i A_u^i - \lambda_{3-i} A_u^{3-i}}\, (\delta_i\, Y_u^0 + \lambda_i\, Y_u^i) \\
&\quad \times E\big[e^{-\delta_{3-i} Y_u^0 (A_v^0 - A_u^0)/Y_u^0 - \lambda_{3-i} Y_u^{3-i} (A_v^{3-i} - A_u^{3-i})/Y_u^{3-i}} \\
&\qquad \times \big(\delta_{3-i}\, Y_u^0 (Y_v^0 / Y_u^0) + \lambda_{3-i}\, Y_u^{3-i} (Y_v^{3-i} / Y_u^{3-i})\big) \,\big|\, \mathcal{F}_u\big]\, dudv \\
&= R_u^i(X_u^i)\, e^{-(\delta_i + \delta_{3-i}) A_u^0 - \lambda_i A_u^i - \lambda_{3-i} A_u^{3-i}}\, (\delta_i\, Y_u^0 + \lambda_i\, Y_u^i)\, D_{v-u}^{3-i}(Y_u^0, Y_u^{3-i})\, dudv
\end{aligned}
$$

where $D_{v-u}^{3-i}(y_0, y_{3-i})$ is given as in (3.7) above, for every $i = 1, 2$. Hence, according to the tower property for conditional expectations, for each $0 \le t \le u < v$, we obtain

$$
\begin{aligned}
&E\big[R_u^i(X_u^i)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_u) \,\big|\, \mathcal{F}_t\big] \qquad (4.17)\\
&= E\big[R_u^i(X_u^i)\, e^{-(\delta_i + \delta_{3-i}) A_u^0 - \lambda_i A_u^i - \lambda_{3-i} A_u^{3-i}}\, (\delta_i\, Y_u^0 + \lambda_i\, Y_u^i)\, D_{v-u}^{3-i}(Y_u^0, Y_u^{3-i}) \,\big|\, \mathcal{F}_t\big]\, dudv \\
&= e^{-(\delta_i + \delta_{3-i}) A_t^0 - \lambda_i A_t^i - \lambda_{3-i} A_t^{3-i}}\, E\big[R_u^i\big(X_t^i (Y_u^i / Y_t^i)^{\alpha_i} (Z_u^0 / Z_t^0)^{\zeta_i} (Z_u^i / Z_t^i)\big) \\
&\quad \times e^{-(\delta_i + \delta_{3-i}) Y_t^0 (A_u^0 - A_t^0)/Y_t^0 - \lambda_i Y_t^i (A_u^i - A_t^i)/Y_t^i - \lambda_{3-i} Y_t^{3-i} (A_u^{3-i} - A_t^{3-i})/Y_t^{3-i}} \\
&\quad \times \big(\delta_i Y_t^0 (Y_u^0 / Y_t^0) + \lambda_i Y_t^i (Y_u^i / Y_t^i)\big)\, D_{v-u}^{3-i}\big(Y_t^0 (Y_u^0 / Y_t^0), Y_t^{3-i} (Y_u^{3-i} / Y_t^{3-i})\big) \,\big|\, \mathcal{F}_t\big]\, dudv \\
&= e^{-(\delta_i + \delta_{3-i}) A_t^0 - \lambda_i A_t^i - \lambda_{3-i} A_t^{3-i}}\, \overline{Q}_{t,u-t,v-u}^i (X_t^i, Y_t^0, Y_t^i, Y_t^{3-i})\, dudv
\end{aligned}
$$

while, for each $0 \le t \le v < u$, we get

$$
\begin{aligned}
&E\big[R^i_u(X^i_u)\,P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_u) \,\big|\, \mathcal{F}_t\big] \qquad (4.18)\\
&= E\big[e^{-(\delta_i+\delta_{3-i})A^0_v - \lambda_i A^i_v - \lambda_{3-i}A^{3-i}_v}\,(\delta_{3-i}Y^0_v + \lambda_{3-i}Y^{3-i}_v)\,Q^i_{v,u-v}(X^i_v, Y^0_v, Y^i_v) \,\big|\, \mathcal{F}_t\big]\,du dv\\
&= e^{-(\delta_i+\delta_{3-i})A^0_t - \lambda_i A^i_t - \lambda_{3-i}A^{3-i}_t}\\
&\quad\times E\big[e^{-(\delta_i+\delta_{3-i})Y^0_t(A^0_v - A^0_t)/Y^0_t - \lambda_i Y^i_t(A^i_v - A^i_t)/Y^i_t - \lambda_{3-i}Y^{3-i}_t(A^{3-i}_v - A^{3-i}_t)/Y^{3-i}_t}\\
&\qquad\times \big(\delta_{3-i}\,Y^0_t(Y^0_v/Y^0_t) + \lambda_{3-i}\,Y^i_t(Y^i_v/Y^i_t)\big)\\
&\quad\times Q^i_{v,u-v}\big(X^i_t(Y^i_v/Y^i_t)^{\alpha_i}(Z^0_v/Z^0_t)^{\zeta_i}(Z^i_v/Z^i_t), Y^0_t(Y^0_v/Y^0_t), Y^i_t(Y^i_v/Y^i_t)\big) \,\big|\, \mathcal{F}_t\big]\,du dv\\
&= e^{-(\delta_i+\delta_{3-i})A^0_t - \lambda_i A^i_t - \lambda_{3-i}A^{3-i}_t}\,\widehat{Q}^i_{t,v-t,u-v}(X^i_t, Y^0_t, Y^i_t, Y^{3-i}_t)\,du dv
\end{aligned}
$$

for every $i = 1, 2$. Here, by virtue of the Markov property of the processes (Y^j, A^j) and Z^j, $j = 0, 1, 2$, and the fact that the random variables Y^j_u/Y^j_t and Z^j_u/Z^j_t have the same laws as Y^j_{u-t} and Z^j_{u-t}, $j = 0, 1, 2$, for each $0 \le t < u$, respectively, we have

$$
\begin{aligned}
&\overline{Q}^i_{t,u-t,v-u}(x_i, y_0, y_i, y_{3-i}) = E\big[R^i_u\big(x_i(Y^i_{u-t})^{\alpha_i}(Z^0_{u-t})^{\zeta_i}Z^i_{u-t}\big) \qquad (4.19)\\
&\times e^{-(\delta_i+\delta_{3-i})y_0A^0_{u-t} - \lambda_i y_i A^i_{u-t} - \lambda_{3-i}y_{3-i}A^{3-i}_{u-t}}\,(\delta_i y_0 Y^0_{u-t} + \lambda_i y_i Y^i_{u-t})\,D^{3-i}_{v-u}(y_0Y^0_{u-t}, y_{3-i}Y^{3-i}_{u-t})\big]\\
&= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty R^i_u\big(x_i(y'_i)^{\alpha_i}(z'_0)^{\zeta_i}z'_i\big)\,e^{-(\delta_i+\delta_{3-i})y_0a_0 - \lambda_i y_i a_i - \lambda_{3-i}y_{3-i}a_{3-i}}\\
&\quad\times (\delta_i y_0 y'_0 + \lambda_i y_i y'_i)\,D^{3-i}_{v-u}(y_0y'_0, y_{3-i}y'_{3-i})\,g^0_{u-t}(y'_0, a_0)\,g^i_{u-t}(y'_i, a_i)\,g^{3-i}_{u-t}(y'_i, a_i)\,h^0_{u-t}(z'_0)\,h^i_{u-t}(z'_i)\\
&\quad\times dy'_0da_0dy'_ida_idy'_{3-i}da_{3-i}dz'_0dz'_i
\end{aligned}
$$

for all $0 \le t < u < v \le T$, as well as

$$
\begin{aligned}
&\widehat{Q}^i_{t,v-t,u-v}(x_i, y_0, y_i, y_{3-i}) = E\big[e^{-(\delta_i+\delta_{3-i})y_0A^0_{v-t} - \lambda_i y_i A^i_{v-t} - \lambda_{3-i}y_{3-i}A^{3-i}_{v-t}} \qquad (4.20)\\
&\times (\delta_{3-i}y_0Y^0_{v-t} + \lambda_{3-i}y_{3-i}Y^{3-i}_{v-t})\,Q^i_u\big(x_i(Y^i_{v-t})^{\alpha_i}(Z^0_{v-t})^{\zeta_i}Z^i_{v-t}, y_0Y^0_{v-t}, y_{3-i}Y^{3-i}_{v-t}\big)\big]\\
&= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\delta_i+\delta_{3-i})y_0a_0 - \lambda_i y_i a_i - \lambda_{3-i}y_{3-i}a_{3-i}}\\
&\quad\times (\delta_{3-i}y_0y'_0 + \lambda_{3-i}y_{3-i}y'_{3-i})\,Q^i_{v,u-v}\big(x_i(y'_i)^{\alpha_i}(z'_0)^{\zeta_i}z'_i, y_0y'_0, y_iy'_i\big)\\
&\quad\times g^0_{v-t}(y'_0, a_0)\,g^i_{v-t}(y'_i, a_i)\,g^{3-i}_{v-t}(y'_{3-i}, a_{3-i})\,h^0_{v-t}(z'_0)\,h^i_{v-t}(z'_i)\,dy'_0da_0dy'_ida_idy'_{3-i}da_{3-i}dz'_0dz'_i
\end{aligned}
$$

for all $0 \le t < v < u \le T$ and every $i = 1, 2$.

Therefore, summarising the facts proved above, we now formulate the following assertion.

Proposition 4.2. *Suppose that $r = 0$. The rational price for the holder of a risk-free credit default swap in (4.6) is given by the sum of the expressions in (4.7) and (4.8). The latter terms are computed by means of the expressions in (3.13), (4.9), (4.12)-(4.13), and (4.17)-(4.18) with (3.7), (4.5), (4.14), and (4.19)-(4.20), respectively.*

4.3 The case of filtration $(\mathcal{G}_t^i \vee \sigma(\tau_{3-i}))_{t \geq 0}$

Let us now continue by computing the rational price $\widetilde{P}^i(\tau_{3-i}) = (\widetilde{P}_t^i(\tau_{3-i}))_{t \geq 0}$ for the holder of a CDS in the model with the filtration $(\mathcal{G}_t^i \vee \sigma(\tau_{3-i}))_{t \geq 0}$ given by

$$\widetilde{P}_t^i(\tau_{3-i}) = E\big[- \widetilde{\varkappa}_i(\tau_{3-i}) \, (\tau_i \wedge T - t) \, I(t < \tau_i) + R^i_{\tau_i}(X^i_{\tau_i}) \, I(t < \tau_i \leq T) \, \big| \, \mathcal{G}_t^i \vee \sigma(\tau_{3-i}) \big] \quad (4.21)$$

for $0 \leq t \leq T \wedge \tau_i$, and $\widetilde{P}_t^i(\tau_{3-i}) = 0$, for $t > T \wedge \tau_i$, so that the premium $\widetilde{\varkappa}_i(\tau_{3-i})$ is then determined from the equation $\widetilde{P}_0^i(\tau_{3-i}) = 0$, for every $i = 1, 2$. It is seen that $\widetilde{\varkappa}_i(\tau_{3-i})$ depends on τ_{3-i}, since we have $\mathcal{G}_0^i \vee \sigma(\tau_{3-i}) = \mathcal{F}_0 \vee \sigma(\tau_{3-i}) \equiv \sigma(\tau_{3-i})$, $i = 1, 2$. In order to compute the both terms in (4.21), we apply the second part of the key lemma for the filtrations $(\mathcal{G}_t^i \vee \sigma(\tau_{3-i}))_{t \geq 0}$ and $(\mathcal{F}_t^{3-i})_{t \geq 0}$, and use the equality in (2.12) as well as Fubini's theorem for interchanging the order of conditional expectation and integration to obtain

$$\begin{aligned} E\big[(\tau_i \wedge T - t) \, I(t < \tau_i) \, \big| \, \mathcal{G}_t^i \vee \sigma(\tau_{3-i})\big] &= I(t < \tau_i) \, \frac{E[(\tau_i \wedge T - t) \, I(t < \tau_i) \, | \, \mathcal{F}_t^{3-i}]}{P(t < \tau_i \, | \, \mathcal{F}_t^{3-i})} \quad (4.22) \\ = I(t < \tau_i) \, E\bigg[\int_{u=t}^{\infty} & \frac{(u \wedge T - t) P(\tau_i \in du, \tau_{3-i} \in dv \, | \, \mathcal{F}_{u \vee v})}{P(t < \tau_i, \tau_{3-i} \in dv \, | \, \mathcal{F}_t)} \, \bigg| \, \mathcal{F}_t \bigg] \bigg|_{v = \tau_{3-i}} \\ = I(t < \tau_i) \int_{u=t}^{\infty} & \frac{(u \wedge T - t) P(\tau_i \in du, \tau_{3-i} \in dv \, | \, \mathcal{F}_t)}{P(t < \tau_i, \tau_{3-i} \in dv \, | \, \mathcal{F}_t)} \bigg|_{v = \tau_{3-i}} \end{aligned}$$

and

$$\begin{aligned} E\big[R^i_{\tau_i}(X^i_{\tau_i}) \, I(t < \tau_i \leq T) \, \big| \, \mathcal{G}_t^i \vee \sigma(\tau_{3-i})\big] &= I(t < \tau_i) \, \frac{E[R^i_{\tau_i}(X^i_{\tau_i}) \, I(t < \tau_i \leq T) \, | \, \mathcal{F}_t^{3-i}]}{P(t < \tau_i \, | \, \mathcal{F}_t^{3-i})} \quad (4.23) \\ = I(t < \tau_i) \, E\bigg[\int_{u=t}^{T} & \frac{R^i_u(X^i_u) P(\tau_i \in du, \tau_{3-i} \in dv \, | \, \mathcal{F}_{u \vee v})}{P(t < \tau_i, \tau_{3-i} \in dv \, | \, \mathcal{F}_t)} \, \bigg| \, \mathcal{F}_t \bigg] \bigg|_{v = \tau_{3-i}} \\ = I(t < \tau_i) \int_{u=t}^{T} & \frac{E[R^i_u(X^i_u) P(\tau_i \in du, \tau_{3-i} \in dv \, | \, \mathcal{F}_u) \, | \, \mathcal{F}_t]}{P(t < \tau_i, \tau_{3-i} \in dv \, | \, \mathcal{F}_t)} \bigg|_{v = \tau_{3-i}} \end{aligned}$$

for all $0 \leq t \leq T$ and every $i = 1, 2$. Here, we recall from the expressions in (3.6)-(3.7) and (3.8)-(3.9) that $P(t < \tau_i, \tau_{3-i} \in dv \, | \, \mathcal{F}_t) = e^{-\delta_i A_t^0 - \lambda_i A_t^i - \delta_{3-i} A_v^0 - \lambda_{3-i} A_v^{3-i}} (\delta_{3-i} Y_v^0 + \lambda_{3-i} Y_v^{3-i})$, for all $0 \leq v \leq t$, and $P(t < \tau_i, \tau_{3-i} \in dv \, | \, \mathcal{F}_t) = e^{-(\delta_i + \delta_{3-i}) A_t^0 - \lambda_i A_t^i - \lambda_{3-i} A_t^{3-i}} D_{v-t}^{3-i}(Y_t^0, Y_t^{3-i}) dv$, for all $0 \leq t < v$ and every $i = 1, 2$, respectively. Note that all the terms of interest in the integrands on the right-hand sides of the expressions in (4.22) and (4.23) were computed in the previous subsection. Therefore, we may formulate the following assertion.

Proposition 4.3. *Suppose that $r = 0$. The rational price for the holder of a risk-free credit default swap in (4.21) is given by the sum of the expressions in (4.22) and (4.23). The latter terms are computed by means of the expressions in (3.13), (4.9), (4.12)-(4.13), and (4.17)-(4.18) with (3.7), (4.5), (4.14), and (4.19)-(4.20), respectively.*

5 The rational prices of risky CDSs (Conclusions)

In this section, we derive explicit expressions for the rational prices of credit default swaps in the model defined above with consideration of counterparty risk in the cases of available information contained in the filtrations $(\mathcal{G}_t)_{t \geq 0}$ or $(\mathcal{G}_t^i \vee \sigma(\tau_{3-i}))_{t \geq 0}$ defined above, for every $i = 1, 2$.

Let us consider the rational price $\widehat{\Pi}^i = (\widehat{\Pi}^i_t)_{t \geq 0}$ of a CDS in the model with consideration of counterparty risk with the filtration $(\mathcal{G}_t)_{t \geq 0}$ given by

$$\begin{aligned}\widehat{\Pi}^i_t = E\big[&- \widehat{\varkappa}_i^* \, (\tau_i \wedge \tau_{3-i} \wedge T - t) \, I(t < \tau_i \wedge \tau_{3-i}) + R^i_{\tau_i}(X^i_{\tau_i}) \, I(t < \tau_i \leq T \wedge \tau_{3-i}) \\ &+ \widehat{R}^{3-i}_{\tau_{3-i}}(X^{3-i}_{\tau_{3-i}}, Y^0_{\tau_{3-i}}, Y^{3-i}_{\tau_{3-i}}, Y^i_{\tau_{3-i}}) \, I(t < \tau_{3-i} \leq T \wedge \tau_i) \, \big| \, \mathcal{G}_t\big]\end{aligned} \tag{5.1}$$

for $0 \leq t \leq T \wedge \tau_i \wedge \tau_{3-i}$, and $\widehat{\Pi}^i_t = 0$, for $t > T \wedge \tau_i \wedge \tau_{3-i}$, so that the premium $\widehat{\varkappa}_i^*$ is then determined from the equation $\widehat{\Pi}^i_0 = 0$, for every $i = 1, 2$. Here, we set $\widehat{R}^{3-i}_t(x_{3-i}, y_0, y_{3-i}, y_i) = R^{3-i}_t(x_{3-i})(\widehat{P}^{3-i}_t)^+(x_{3-i}, y_0, y_{3-i}, y_i) - (\widehat{P}^{3-i}_t)^-(x_{3-i}, y_0, y_{3-i}, y_i)$, for all $t \geq 0$ and every $i = 1, 2$. In this case, we apply the generalised key lemma for the filtrations $(\mathcal{G}_t)_{t \geq 0}$ and $(\mathcal{F}_t)_{t \geq 0}$, where $\mathcal{G}_t$ coincides with $\mathcal{F}_t$ on the event $\{t < \tau_i \wedge \tau_{3-i}\}$, for all $t \geq 0$, and use the equality in (2.12) as well as Fubini's theorem for interchanging the order of expectation and integration to obtain

$$\begin{aligned}&E\big[(\tau_i \wedge \tau_{3-i} \wedge T - t) \, I(t < \tau_i \wedge \tau_{3-i}) \, \big| \, \mathcal{G}_t\big] \\ &= I(t < \tau_i \wedge \tau_{3-i}) \, \frac{E[(\tau_i \wedge \tau_{3-i} \wedge T - t) I(t < \tau_i \wedge \tau_{3-i}) \, | \, \mathcal{F}_t]}{P(t < \tau_i \wedge \tau_{3-i} \, | \, \mathcal{F}_t)} \\ &= I(t < \tau_i \wedge \tau_{3-i}) \, E\bigg[\int_t^\infty \int_t^\infty \frac{(u \wedge v \wedge T - t) P(\tau_i \in du, \tau_{3-i} \in dv \, | \, \mathcal{F}_{u \vee v})}{P(t < \tau_i \wedge \tau_{3-i} \, | \, \mathcal{F}_t)} \, \bigg| \, \mathcal{F}_t \bigg] \\ &= I(t < \tau_i \wedge \tau_{3-i}) \int_t^\infty \int_t^\infty \frac{(u \wedge v \wedge T - t) P(\tau_i \in du, \tau_{3-i} \in dv \, | \, \mathcal{F}_t)}{P(t < \tau_i \wedge \tau_{3-i} \, | \, \mathcal{F}_t)}\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}&E\big[R^i_{\tau_i}(X^i_{\tau_i}) \, I(t < \tau_i \leq T \wedge \tau_{3-i}) \, \big| \, \mathcal{G}_t\big] \\ &= I(t < \tau_i \wedge \tau_{3-i}) \, \frac{E[R^i_{\tau_i}(X^i_{\tau_i}) \, I(t < \tau_i \leq T \wedge \tau_{3-i}) \, | \, \mathcal{F}_t]}{P(t < \tau_i \wedge \tau_{3-i} \, | \, \mathcal{F}_t)} \\ &= I(t < \tau_i \wedge \tau_{3-i}) \, E\bigg[\int_{v=t}^\infty \int_{u=t}^{T \wedge v} \frac{R^i_u(X^i_u) P(\tau_i \in du, \tau_{3-i} \in dv \, | \, \mathcal{F}_v)}{P(t < \tau_i \wedge \tau_{3-i} \, | \, \mathcal{F}_t)} \, \bigg| \, \mathcal{F}_t \bigg] \\ &= I(t < \tau_i \wedge \tau_{3-i}) \int_{v=t}^\infty \int_{u=t}^{T \wedge v} \frac{E[R^i_u(X^i_u) P(\tau_i \in du, \tau_{3-i} \in dv \, | \, \mathcal{F}_u) \, | \, \mathcal{F}_t]}{P(t < \tau_i \wedge \tau_{3-i} \, | \, \mathcal{F}_t)}\end{aligned} \tag{5.3}$$

as well as

$$
\begin{aligned}
& E\big[\widehat{R}^{3-i}_{\tau_{3-i}}(X^{3-i}_{\tau_{3-i}}, Y^0_{\tau_{3-i}}, Y^{3-i}_{\tau_{3-i}}, Y^i_{\tau_{3-i}})\, I(t < \tau_{3-i} \le T \wedge \tau_i) \,\big|\, \mathcal{G}_t\big] \qquad (5.4) \\
& = I(t < \tau_i \wedge \tau_{3-i})\, \frac{E[\widehat{R}^{3-i}_{\tau_{3-i}}(X^{3-i}_{\tau_{3-i}}, Y^0_{\tau_{3-i}}, Y^{3-i}_{\tau_{3-i}}, Y^i_{\tau_{3-i}})\, I(t < \tau_{3-i} \le T \wedge \tau_i) \,|\, \mathcal{F}_t]}{P(t < \tau_i \wedge \tau_{3-i} \,|\, \mathcal{F}_t)} \\
& = I(t < \tau_i \wedge \tau_{3-i})\, E\bigg[\int_{u=t}^{\infty} \int_{v=t}^{T \wedge u} \frac{\widehat{R}^{3-i}_v(X^{3-i}_v, Y^0_v, Y^{3-i}_v, Y^i_v) P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_u)}{P(t < \tau_i \wedge \tau_{3-i} \,|\, \mathcal{F}_t)} \,\bigg|\, \mathcal{F}_t \bigg] \\
& = I(t < \tau_i \wedge \tau_{3-i})\, \int_{u=t}^{\infty} \int_{v=t}^{T \wedge u} \frac{E[\widehat{R}^{3-i}_v(X^{3-i}_v, Y^0_v, Y^{3-i}_v, Y^i_v) P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,|\, \mathcal{F}_t]}{P(t < \tau_i \wedge \tau_{3-i} \,|\, \mathcal{F}_t)}
\end{aligned}
$$

for all $0 \le t \le T$ and every $i = 1, 2$. Here, we recall from the expressions in (3.8) that $P(t < \tau_i \wedge \tau_{3-i} \,|\, \mathcal{F}_t) = e^{-(\delta_i + \delta_{3-i}) A^0_t - \lambda_i A^i_t - \lambda_{3-i} A^{3-i}_t}$, for all $t \ge 0$.

We also consider the rational price $\widetilde{\Pi}^i(\tau_{3-i}) = (\widetilde{\Pi}^i_t(\tau_{3-i}))_{t \ge 0}$ of a CDS in the model with consideration of counterparty risk with the filtration $(\mathcal{G}^i_t \vee \sigma(\tau_{3-i}))_{t \ge 0}$ given by

$$
\begin{aligned}
\widetilde{\Pi}^i_t(\tau_{3-i}) = E\big[& - \widetilde{\varkappa}^*_i(\tau_{3-i})\, (\tau_i \wedge \tau_{3-i} \wedge T - t)\, I(t < \tau_i \wedge \tau_{3-i}) + R^i_{\tau_i}(X^i_{\tau_i})\, I(t < \tau_i \le T \wedge \tau_{3-i}) \\
& + \widetilde{R}^{3-i}_{\tau_{3-i}}(X^{3-i}_{\tau_{3-i}}, Y^0_{\tau_{3-i}}, Y^{3-i}_{\tau_{3-i}}, Y^i_{\tau_{3-i}})\; I(t < \tau_{3-i} \le T \wedge \tau_i) \,\big|\, \mathcal{G}^i_t \vee \sigma(\tau_{3-i})\big] \qquad (5.5)
\end{aligned}
$$

for $0 \le t \le T \wedge \tau_i \wedge \tau_{3-i}$, and $\widetilde{\Pi}^i_t(\tau_{3-i}) = 0$, for $t > T \wedge \tau_i \wedge \tau_{3-i}$, so that the premium $\widetilde{\varkappa}^*_i(\tau_{3-i})$ is then determined from the equation $\widetilde{\Pi}^i_0(\tau_{3-i}) = 0$, for every $i = 1, 2$. Here, we recall that $\widetilde{R}^{3-i}_t(x_{3-i}, y_0, y_{3-i}, y_i) = R^{3-i}_t(x_{3-i})(\widetilde{P}^{3-i}_t)^+(x_{3-i}, y_0, y_{3-i}, y_i) - (\widetilde{P}^{3-i}_t)^-(x_{3-i}, y_0, y_{3-i}, y_i)$, for all $t \ge 0$ and every $i = 1, 2$. In this case, we apply the generalised key lemma for the filtrations $(\mathcal{G}^i_t \vee \sigma(\tau_{3-i}))_{t \ge 0}$ and $(\mathcal{F}^{3-i}_t)_{t \ge 0}$, where $\mathcal{G}^i_t \vee \sigma(\tau_{3-i})$ coincides with $\mathcal{F}^{3-i}_t$ on the event $\{t < \tau_i \wedge \tau_{3-i}\}$, for all $t \ge 0$, and use the equality in (2.12) as well as Fubini's theorem for interchanging the order of expectation and integration to obtain

$$
\begin{aligned}
& E\big[(\tau_i \wedge \tau_{3-i} \wedge T - t)\, I(t < \tau_i \wedge \tau_{3-i}) \,\big|\, \mathcal{G}^i_t \vee \sigma(\tau_{3-i})\big] \qquad (5.6) \\
& = I(t < \tau_i \wedge \tau_{3-i})\, \frac{E[(\tau_i \wedge \tau_{3-i} \wedge T - t) I(t < \tau_i) \,|\, \mathcal{F}^{3-i}_t]}{P(t < \tau_i \,|\, \mathcal{F}^{3-i}_t)} \\
& = I(t < \tau_i \wedge \tau_{3-i})\, E\bigg[\int_{u=t}^{\infty} \frac{(u \wedge v \wedge T - t) P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_{u \vee v})}{P(t < \tau_i, \tau_{3-i} \in dv \,|\, \mathcal{F}_t)} \,\bigg|\, \mathcal{F}_t \bigg] \bigg|_{v = \tau_{3-i}} \\
& = I(t < \tau_i \wedge \tau_{3-i})\, \int_{u=t}^{\infty} \frac{(u \wedge v \wedge T - t) P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_t)}{P(t < \tau_i, \tau_{3-i} \in dv \,|\, \mathcal{F}_t)} \bigg|_{v = \tau_{3-i}}
\end{aligned}
$$

and

$$
\begin{aligned}
&E\big[R^i_{\tau_i}(X^i_{\tau_i})\, I(t<\tau_i\le T\wedge\tau_{3-i})\,\big|\,\mathcal{G}^i_t\vee\sigma(\tau_{3-i})\big] \qquad (5.7)\\
&= I(t<\tau_i\wedge\tau_{3-i})\,\frac{E[R^i_{\tau_i}(X^i_{\tau_i})\, I(t<\tau_i\le T\wedge\tau_{3-i})\,|\,\mathcal{F}^{3-i}_t]}{P(t<\tau_i\,|\,\mathcal{F}^{3-i}_t)}\\
&= I(t<\tau_i\wedge\tau_{3-i})\, E\bigg[\int_{u=t}^{T\wedge v}\frac{R^i_u(X^i_u)P(\tau_i\in du,\tau_{3-i}\in dv\,|\,\mathcal{F}_v)}{P(t<\tau_i,\tau_{3-i}\in dv\,|\,\mathcal{F}_t)}\,\bigg|\,\mathcal{F}_t\bigg]\bigg|_{v=\tau_{3-i}}\\
&= I(t<\tau_i\wedge\tau_{3-i})\int_{u=t}^{T\wedge v}\frac{E[R^i_u(X^i_u)P(\tau_i\in du,\tau_{3-i}\in dv\,|\,\mathcal{F}_u)\,|\,\mathcal{F}_t]}{P(t<\tau_i,\tau_{3-i}\in dv\,|\,\mathcal{F}_t)}\bigg|_{v=\tau_{3-i}}
\end{aligned}
$$

as well as, on the event $\{\tau_{3-i}\le T\}$, we have

$$
\begin{aligned}
&E\big[\widetilde{R}^{3-i}_{\tau_{3-i}}(X^{3-i}_{\tau_{3-i}},Y^0_{\tau_{3-i}},Y^{3-i}_{\tau_{3-i}},Y^i_{\tau_{3-i}})\, I(t<\tau_{3-i}\le T\wedge\tau_i)\,\big|\,\mathcal{G}^i_t\vee\sigma(\tau_{3-i})\big] \qquad (5.8)\\
&= I(t<\tau_i\wedge\tau_{3-i})\,\frac{E[\widetilde{R}^{3-i}_{\tau_{3-i}}(X^{3-i}_{\tau_{3-i}},Y^0_{\tau_{3-i}},Y^{3-i}_{\tau_{3-i}},Y^i_{\tau_{3-i}})\, I(t<\tau_{3-i}\le T\wedge\tau_i)\,|\,\mathcal{F}^{3-i}_t]}{P(t<\tau_i\wedge\tau_{3-i}\,|\,\mathcal{F}^{3-i}_t)}\\
&= I(t<\tau_i\wedge\tau_{3-i})\, E\bigg[\int_{u=v}^{\infty}\frac{\widetilde{R}^{3-i}_v(X^{3-i}_v,Y^0_v,Y^{3-i}_v,Y^i_v)P(\tau_i\in du,\tau_{3-i}\in dv\,|\,\mathcal{F}_u)}{P(t<\tau_i,\tau_{3-i}\in dv\,|\,\mathcal{F}_t)}\,\bigg|\,\mathcal{F}_t\bigg]\bigg|_{v=\tau_{3-i}}\\
&= I(t<\tau_i\wedge\tau_{3-i})\int_{u=v}^{\infty}\frac{E[\widetilde{R}^{3-i}_v(X^{3-i}_v,Y^0_v,Y^{3-i}_v,Y^i_v)P(\tau_i\in du,\tau_{3-i}\in dv\,|\,\mathcal{F}_v)\,|\,\mathcal{F}_t]}{P(t<\tau_i,\tau_{3-i}\in dv\,|\,\mathcal{F}_t)}\bigg|_{v=\tau_{3-i}}
\end{aligned}
$$

for all $0\le t\le T$ and every $i=1,2$. Here, we recall from the expressions in (3.6)-(3.7) and (3.8)-(3.9) that $P(t<\tau_i,\tau_{3-i}\in dv\,|\,\mathcal{F}_t)=e^{-\delta_i A^0_t-\lambda_i A^i_t-\delta_{3-i}A^0_v-\lambda_{3-i}A^{3-i}_v}(\delta_{3-i}Y^0_v+\lambda_{3-i}Y^{3-i}_v)$, for all $0\le v\le t$, and $P(t<\tau_i,\tau_{3-i}\in dv\,|\,\mathcal{F}_t)=e^{-(\delta_i+\delta_{3-i})A^0_t-\lambda_i A^i_t-\lambda_{3-i}A^{3-i}_t}D^{3-i}_{v-t}(Y^0_t,Y^{3-i}_t)dv$, for all $0\le t<v$ and every $i=1,2$, respectively. Note that all the terms of interest in the integrands on the right-hand sides of the expressions in (5.2)-(5.4) and (5.6)-(5.8) can be computed by means of arguments similar to the ones applied in the previous subsection. Therefore, we may formulate the following assertion.

Proposition 5.1. *Suppose that $r=0$. The rational prices for the holders of a risky credit default swap in (5.1) and (5.5) are given by the sums of the expressions in (5.2)-(5.4) and (5.6)-(5.8), respectively. The latter terms are computed by means of arguments similar to the ones applied for the derivations of the expressions in (3.13), (4.9), (4.12)-(4.13), and (4.17)-(4.18) with (3.7), (4.5), (4.14), and (4.19)-(4.20), respectively.*

References

[1] Assefa, S., Bielecki, T. R., Crépey, S. and Jeanblanc, M. (2011). CVA computation for counterparty risk assessment in credit portfolios. In the Volume *Credit Risk*

Frontiers: Subprime Crisis, Pricing and Hedging, CVA, MBS, Ratings, and Liquidity. Bielecki, T. R., Brigo, D., and Patras, F. (eds.) Wiley.

[2] Aksamit, A. and Jeanblanc, M. (2017). *Enlargements of Filtrations with Finance in View.* Springer.

[3] Bielecki, T. R., Crépey, S., Jeanblanc, M. and Zargari, B. (2012). Valuation and hedging of CDS counterparty exposure in a Markov copula model. *International Journal of Theoretical and Applied Finance* **15** (1–39).

[4] Bielecki, T. R. and Rutkowski, M. (2004). *Credit Risk: Modeling, Valuation and Hedging.* (Second Edition) Springer, Berlin.

[5] Brémaud, P. and Yor, M. (1978). Changes of filtrations and of probability measures. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **45** (269-295).

[6] Brigo, D., Capponi, A. and Pallavicini, A. (2014). Arbitrage-free bilateral counterparty risk valuation under collateralization and application to credit default swaps. *Mathematical Finance* **24** (125–146).

[7] Brigo, D. and Chourdakis, K. (2009). Counterparty risk for credit default swaps: impact of spread volatility and default correlation. *International Journal of Theoretical and Applied Finance* **12** (1007–1026).

[8] Brigo, D., Morini, M. and Pallavicini, A. (2013). *Counterparty Credit Risk, Collateral and Funding: with Pricing Cases for all Asset Classes.* Wiley.

[9] Callegaro, G., Jeanblanc, M. and Zargari, B. (2013). Carthagian enlargement of filtrations. *ESAIM: Probability and Statistics* **17** (550–566).

[10] Cesari, G., Aquilina, J. and Charpillon, N. (2010). *Modelling, Pricing, and Hedging Counterparty Credit Exposure.* Springer.

[11] Crépey, S., Bielecki, T. R. and Brigo, D. (2014). *Counterparty Risk and Funding. A Tale of Two Puzzles.* Taylor and Francis.

[12] Dellacherie, C. (1972). *Capacités et Processus Stochastiques.* Springer.

[13] Dellacherie, C. and Meyer, P.-A. (1975). *Probabilités et Potentiel, Chapitres I-IV.* Hermann, Paris. (English translation, 1978: *Probabilities and Potential, Chapters I-IV.* North-Holland.)

[14] Dufresne, D. (2001). The integral of geometric Brownian motion. *Advances in Applied Probability* **33** (223–241).

[15] El Karoui, N., Jeanblanc, M. and Jiao, Y. (2010). What happens after a default: the conditional density approach. *Stochastic Processes and their Applications* **120**(7) (1011–1032).

[16] El Karoui, N., Jeanblanc, M. and Jiao, Y. (2012). Density approach in modeling successive defaults. *SIAM Journal on Financial Mathematics* **6** (1–21).

[17] Elliott, R. J., Jeanblanc, M. and Yor, M. (2000). On models of default risk. *Mathematical Finance* **10** (179-195).

[18] Gapeev, P. V. and Jeanblanc, M. (2019). Defaultable claims in switching models with partial information. *International Journal of Theoretical and Applied Finance* **??** (1001–1017).

[19] Gapeev, P. V. and Peskir, G. (2006). The Wiener disorder problem with finite horizon. *Stochastic Processes and their Applications* **116**(12) (1770–1791).

[20] Gregory, J. (2009). *Counterparty Credit Risk: The New Challenge for Global Financial Markets.* Wiley.

[21] Lando, D. (1998). On Cox processes and credit risky securities. *Review of Derivatives Research* **2** (99–120).

[22] Mansuy, R. and Yor, M. (2004). *Random Times and Enlargements of Filtrations in a Brownian Setting.* Lecture Notes in Mathematics **1873**. Springer, Berlin

[23] Schönbucher, Ph. (2003). *Credit Derivatives Pricing Models: Models, Pricing and Implementation.* Wiley.

[24] Schröder, M. (2003). On the integral of geometric Brownian motion. *Advances in Applied Probability* **35** (159–183).

[25] YOR, M. (1992). On some exponential functionals of Brownian motion. *Advances in Applied Probability* **24** (509–531).