

Non-commutative gauge theories and Zhang algebras

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NON-COMMUTATIVE GAUGE THEORIES AND ZHANG ALGEBRAS

NICOLAS GILLIERS

Abstract. We investigate lattice and continuous quantum gauge theories on the Euclidean plane with a structure group that is replaced by a Zhang algebra. Zhang algebras are non-commutative analogues of groups and contain the class of Voiculescu's dual groups. We are interested in non-commutative analogues of random gauge fields, which we describe through the random holonomy that they induce. We propose a general definition of a holonomy field with Zhang gauge symmetry, and construct a such fields starting from a quantum Lévy process on a Zhang algebra. As an application, we define higher dimensional generalizations of the so-called master field.

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1. Introduction

The present work deals with the question of defining gauge theories on non-commutative analogue of spaces of functions on a group. In 1954 Yang and Mills exposed the theory at roots of the Standard Model, which is now known as Yang–Mills theory. This theory is the culminating point of a process started decades beforehand: physicists do not longer consider symmetries as a tool to reduce computations but rather as generators of a dynamic. An output of the Yang–Mills theory is a Lagrangian, that is a dynamic, a particle, the input being, essentially, a group of symmetries and a space on which particles evolves. The $\mathbb{U}(N)$–Yang–Mills theory as now a long story and has been studied both by physicians and mathematicians. Quantizing Yang–Mills theory on a four dimensional Lorentzian space is still an open question. In two

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dimensions, the problem of defining what quantum theory is, given a surface and a compact Lie group of symmetries, has been rigorously addressed by Lévy in [1]. The problem lies in defining what a translation invariant measure on the infinite dimensional space of connections on a fiber bundle with a compact Lie group as a structure group is. A connection is most accurately described through its holonomy along any closed loops drawn on the plane. Under the non-existing Yang–Mills measure on the space of all connections on a principal fiber bundle, the process that associates to a growing family of loops the holonomy of a random connection along these loops is Brownian. In this article, we focus on the case for which the underlying surface of the theory is the plan. The master field is the limit, in a vague sense, as N tends to infinity of the $\mathbb{U}(N)$–Yang–Mills field. Studying fields that emerge the symmetry group grows is the subject of investigation of planar quantum field theory. The master field is most accurately understood and described with the help of free probability theory and is related to free Gaussian motions. Many generalizations of this field have been defined. For example, the authors in [2] considered the limit of classical holonomy fields associated with (free) infinite divisible semi-group. The present work includes these generalized master field into a larger class of fields having symmetries that show a Zhang algebra structure. As such, free generalized masters defined in [2] are considered to be one dimensional. Our main concern is to give higher dimensional counterparts of the master field by enlarging the algebras of symmetries of that field as a starting point, which is slightly different of the point of view that predominates [2].

This work is organized as follows. In Section 2.2 we define what Zhang algebras are, make a reminder on algebraic categories, let and right free comodules. In Section 8, we define the notion of monoidal structure on a category, give examples and introduce the notion of categorical independence as defined by Franz in [3] and Schürmann in [4]. The main definitions and results are located in Section 4. In particular, the definition of a categorical algebraic holonomy field can be found in Definition 19. Theorem 25 is the central result of this work. Applications of Theorem 25 can be found in Section 4.

2. Zhang algebras

In this section, we review Zhang algebras, starting with a definition. Recall that all our algebras are over the field of complex numbers, unital and associative.

The definition of Zhang algebras is very similar to that of Hopf algebras, the main difference being that the coproduct takes its values not in the tensor product, but in a free product of the algebra with itself. There is therefore a variety of notions of Zhang algebras, corresponding to various notions of free products, or more properly speaking of categorical coproducts, of algebras. Since we will use several notions of categorical coproduct, we prefer not to specify a particular one at this point, and we choose instead to adopt the more abstract point of view of *algebraic categories*. The main point of this section and the following one is to recall some basics fact on algebraic categories, monoidal categories and Zhang algebras. We use the language of category theory, functors the reader is directed to the monograph [5] for a detailed exposition. Equalities between morphisms are expressed as a commutativity property of some *diagrams*. A *diagram* is a directed graph with object labelled vertices and morphism labelled edges. We say that a diagram is commutative if the composition of morphisms along any two directed paths with same source and target yield the same result. We will frequently drop labels of edges, if it is clear from the context how edges of a diagram are morphisms tagged to increased readability.

2.1. Algebraic categories.

Recall that in a category $\mathcal{C}$, an object k is called an initial object if for every object A, there is exactly one object in $\mathrm{Hom}(k, A)$. Recall also that a coproduct of two objects A and B is the data of an object C and two morphisms $\iota_A : A \to C$ and $\iota_B : B \to C$ such that for any object D and any two morphisms $f : A \to D$ and $g : B \to D$, there exists a unique morphism $h : C \to D$ such that $f = h \circ \iota_A$ and $g = h \circ \iota_B$. If a coproduct of two objects

A and B exists, it is unique up to isomorphism and it is denoted by $A \sqcup B$. Moreover, with the current notation, the morphism h is denoted by $f \dot{\sqcup} g : A \sqcup B \to D$. The dot in the symbol $\dot{\sqcup}$ indicates that the elements of D obtained by applying f and g to the elements of A and B are multiplied in D. There is also a natural way of combining two morphisms $f : A \to D_1$ and $g : B \to D_2$ into a morphism $f \sqcup g : A \sqcup B \to D_1 \sqcup D_2$, by first forming $\iota_{D_1} \circ f : A \to D_1 \sqcup D_2$ and $\iota_{D_2} \circ g : B \to D_1 \sqcup D_2$ and then setting $f \sqcup g = (\iota_{D_1} \circ f) \dot{\sqcup} (\iota_{D_2} \circ g)$.

Definition 1 (Algebraic category)**.** An algebraic category is a category with an initial object and in which any two objects admit a coproduct.

Let, for example, Alg be the category of unital complex associative algebras. The algebra $\mathbb{C}$ is an initial object of this category. Moreover, given two algebras A and B, we can form the algebra $A \sqcup B$ freely generated by A and B, the units of A and B being identified with the unit of $A \sqcup B$. This algebra can be described as a quotient of the tensor algebra:

$$A \sqcup B = T(A \oplus B)/(a \otimes a' - aa', b \otimes b' - bb', 1_A - 1, 1_B - 1 : a, a' \in A, b, b' \in B)$$

Concretely, $A \sqcup B$ is the vector space of all formal linear combinations of alternating words in elements of A and B. Any occurrence of the units of A or B in one of these words can be ignored, and multiplication of words is given by concatenation followed, in the case where they belong to the same algebra, by the multiplication of the last letter of the first word with the first letter of the second. Then $\sqcup$ is a coproduct in the category Alg, so that Alg is an example of an algebraic category.

We will consider the following other examples of algebraic categories.

1. The category $\mathrm{Alg}^\star$ of involutive (complex unital associative) algebras. The initial object of this category is still $\mathbb{C}$. The coproduct of two objects A and B of $\mathrm{Alg}^\star$ is, as an algebra, their coproduct in Alg. Moreover, $A \sqcup B$ is endowed with the unique antimultiplicative involution which extends those of A and B. Concretely, the involution of $A \sqcup B$ reverses the order of the letters in a word, and transforms each letter according to the involutions of A and B.
2. The category $\mathrm{Alg}(R)$ of (complex unital associative) algebras endowed with a structure of bimodule over a fixed unital algebra R. The initial object in this category is R. The coproduct of two objects A and B is the coproduct of the category Alg with amalgamation over R. It can be described as

$$A \sqcup_R B = (R \oplus \bigoplus_{n \geq 1} T^n(A \oplus B))/(ar \otimes r'a' - arr'a', br \otimes r'b' - brr'b', ar \otimes b - a \otimes rb,$$
$$br \otimes a - b \otimes ra, r1_A r' - rr', r1_B r' - rr' : a, a' \in A, b, b' \in B, r, r' \in R).$$

   In English, it is the free product of A and B in which multiplication by elements of R can circulate between neighbouring factors.
3. The category of complex unital associative $\mathbb{Z}_2$-graded algebras is algebraic. A $\mathbb{Z}_2$-graded algebra is the data of an algebra and an unipotent morphism on that algebra. Morphisms are unital morphisms of algebras that preserve the grading. The free product of two graded algebras (A, D_A) and (B, D_B), is as an algebra $A \sqcup B$ and $D_{A \sqcup B} = D_A \sqcup D_B$. The initial object is $\mathbb{C}[\mathbb{Z}_2]$.
4. The category $\mathcal{G}rp$ of groups. The coproduct is the free product of groups and the initial object is the group having only one element. If G and H are groups, a word in G and H is a product of the form $s_1 s_2 \cdots s_n$, where each s_i, $i \leq n$ is either an element of the group G or an element of the group H. Such a word may be reduced using the following operations:
   1. Remove an instance of the identity element (of either G or H).
   2. Replace a pair of the form $g_1 g_2$ by its product in G, or a pair $h_1 h_2$ by its product in H, with obvious notation.

Every reduced word is an alternating product of elements of G and elements of H. The free product $G \sqcup H$ is the group whose elements are the reduced words in G an H, under the operation of concatenation followed by reduction.

5. The category biMod(R) of bimodules over a fixed unital algebra R can be endowed with two coproduct with injections. Let A and B two R-bimodules. The first product $A \sqcup_1 B$ in biMod(R) is, as a vector space, isomorphic to the sum of vector spaces $A \oplus B$. The R bimodule structure on $A \oplus B$ is the sum of the two structures:
$$r(a \oplus b)r' = rar' \oplus rbr', a \in A,\ b \in B,\ r, r' \in R.$$
The initial object is again R.
6. The category coModAlg(H) of comodule algebras over a Zhang algebra H, which we will described later (see Section 2.3).

2.2. **Zhang algebras.** We can now give the definition of a Zhang algebra of an algebraic category.

Definition 2 (Zhang algebra)**.** Let $\mathcal{C}$ be an algebraic category with initial object k and coproduct $\sqcup$. A Zhang algebra of $\mathcal{C}$ is a quadruplet (H, Δ, ϵ, S) where

1. H is an object of $\mathcal{C}$,
2. $\Delta : H \to H \sqcup H$ is a morphism of $\mathcal{C}$ such that $(\Delta \sqcup \mathrm{id}_H) \circ \Delta = (\mathrm{id}_H \sqcup \Delta) \circ \Delta$,
3. $\epsilon : H \to k$ is a morphism of $\mathcal{C}$ such that $(\epsilon \dot{\sqcup} \mathrm{id}_H) \circ \Delta = \mathrm{id}_H = (\mathrm{id}_H \dot{\sqcup} \epsilon) \circ \Delta$,
4. $S : H \to H$ is a morphism of $\mathcal{C}$ such that $(S \dot{\sqcup} \mathrm{id}_H) \circ \Delta = \eta \circ \epsilon = (\mathrm{id}_H \dot{\sqcup} S) \circ \Delta$, where η is the unique morphism from k to H.

Definition 2 is formally very similar to the definition of Hopf algebras. More succisely, replacing $\mathcal{C}$ by the category of (complex unital associative) algebras, k by $\mathbb{C}$ and $\sqcup$ by the tensor product in Definition 2 yields the exact definition of a Hopf algebra. However, the tensor product is not a coproduct in the category of algebras, and Hopf algebras are not a special case of Zhang algebras. It is instructive to understand the reason why the tensor product is not a coproduct. Consider indeed two algebras A and B and two morphisms $f : A \to D$ and $g : B \to D$. Let $\iota_A : A \to A \otimes B$ and $\iota_B : B \to A \otimes B$ be the natural morphisms. Should there exist a morphism $h : A \otimes B \to D$ such that $f = h \circ \iota_A$ and $g = h \circ \iota_B$, the relation
$$\iota_A(a)\iota_B(b) = (a \otimes 1_B)(1_A \otimes b) = a \otimes b = (1_A \otimes b)(a \otimes 1_B) = \iota_B(b)\iota_A(a)$$
would impose the equalities $h(a \otimes b) = f(a)g(b) = g(b)f(a)$, the second of which has no reason of being satisfied unless, of course, D is commutative.

The fact that the definition of Zhang algebras involves a coproduct is crucial for us, in particular because it has the consequence that we now explain. Consider a Zhang algebra H on an algebraic category $\mathcal{C}$. Then for every object A of $\mathcal{C}$, the set $\mathrm{Hom}(H, A)$ is endowed with a group structure by the formula
$$f * g = (f \dot{\sqcup} g) \circ \Delta.$$
The unit element of this group is $\eta_A \circ \epsilon_H$, where η_A is the unique morphism from k to A, and the inverse of an element f of $\mathrm{Hom}(H, A)$ is $f \circ S$.

In contrast with this situation, if H is a Hopf algebra and A is an algebra, the convolution product of two morphisms of algebras needs not be a morphism of algebras, unless A is commutative.

The reason why this fact is so important for us is that $\mathrm{Hom}(H, A)$ plays the role of a set of group-valued random variables, and it is extremely natural for us to be able to take the inverse of such a random variable, or two multiply two of them.

Let us conclude this discussion with the following theorem, which shows that Zhang algebras are in a sense exactly the right class of objects for our purposes.

Theorem 3 ([6], theorem 3.2)**.** *Let $\mathcal{C}$ be an algebraic category. Let H be an object of $\mathcal{C}$. Then $\mathsf{Hom}(H,\cdot)$ a functor from $\mathcal{C}$ to the category of groups if and only if there exists Δ,ϵ,S, such that (H,Δ,ϵ,S) is a Zhang algebra of $\mathcal{C}$.*

We end this section with examples of Zhang algebras.

1. Let V be a complex vector space. We claim that V is a Zhang algebra in the algebraic category $(\mathrm{Vect}_{\mathbb{C}},\oplus,\{0\})$. In fact, define:
$$\Delta(x)=x\oplus x\in V\oplus V,\ S(x)=-x,\ \varepsilon(x)=0.$$
It is easy to check that, with these definitions, (V,S,ε) is a Zhang algebra.
2. Let $n\geq 1$ an integer. The Dual Voiculescu group $\mathcal{O}\langle n\rangle$ is the involutive unital associative algebra generated by $2n^2$ variables; $u_{ij},\ u_{ij}^{\star}\ i,1\leq j\leq n$ subject to the relations:
$$\sum_{k=1}^{n}u_{ik}u_{jk}^{\star}=\delta_{ij},\ 1\leq i,j\leq n.$$
The dual voiculescu group is a turned into a Zhang algebra $(\mathcal{O}\langle n\rangle,\Delta,\varepsilon,S)$ if we define the structural morphisms by:
$$S(u_{ij})=u_{ji}^{\star},\ \Delta_{u_{ij}}=\sum_{k=1}^{n}u_{ik}|_1u_{kj}|_2,\ \varepsilon(u_{ij})=\delta_{ij},\ 1\leq i,j\leq n.$$
3. The rectangular unitary algebra $\mathcal{RO}\langle n\rangle$ is the involutive unital associative algebra generated by one unitary element and a set of mutually autoadjoint orthogonal projectors $\{p_i,\ i\leq n\}$. Set $\mathcal{R}=\{p_i,\ 1\leq i\leq n\}$. The rectangular unitary algebra is a bimodule algebra over $\mathcal{R}$. The algebra $\mathcal{RO}\langle n\rangle$ is a Zhang algebra in the algebraic category $\mathrm{Alg}^{\star}(\mathcal{R})$ with structural morphisms:
$$S(u)=u^{\star}=u^{-1},\ \Delta(u)=u|_1u|_2,\ \varepsilon(u)=1\in R.$$
4. Any commutative Hopf algebra is a Zhang algebra, thus if G is a group then its space of polynomial functions $\mathcal{F}(G)$ is a Zhang algebra with structure morphisms given by:
$$\Delta(f)(g,h)=f(gh),\ S(f)(g)=f(g^{-1}),\ \varepsilon(f)=f(e).$$

2.3. **Comodule algebras over Zhang algebras.** In this section, we define the category of comodule algebras over a Zhang algebra of an algebraic category $\mathcal{C}$ with initial object k and coproduct $\sqcup$. In this definition, and for every object B of $\mathcal{C}$, we will identify without further mention the objects B, $B\sqcup k$ and $k\sqcup B$. They are indeed isomorphic by the maps $\mathrm{id}_B\dot{\sqcup}\,\eta: B\sqcup k\to B$ and $\eta\,\dot{\sqcup}\,\mathrm{id}_B: k\sqcup B\to B$, where $\eta: k\to B$ is the unique element of $\mathrm{Hom}(k,B)$.

For this section, let us fix an algebraic category $\mathcal{C}$ with initial object k and coproduct $\sqcup$, and a Zhang algebra (H,Δ,ϵ,S) of this category.

Definition 4 (Comodule algebras)**.** A right H-comodule algebra of $\mathcal{C}$ is a pair (M,Ω), where M is an object of $\mathcal{C}$ and $\Omega: M\to M\sqcup H$ is a morphism such that

1. M is an object of $\mathcal{C}$,
2. $\Omega: M\to M\sqcup H$ is a morphism of $\mathcal{C}$ satisfying the following two conditions:

$$(\Omega\sqcup\mathrm{id}_H)\circ\Omega=(\mathrm{id}_M\sqcup\Delta)\circ\Omega\ \text{ and }\ (\mathrm{id}_M\sqcup\epsilon)\circ\Omega=\mathrm{id}_M. \tag{1_R}$$

The definition of a left comodule is deduced from the definition of a right comodule by replacing (1_R) by

$$(\mathrm{id}_H\sqcup\Omega)\circ\Omega=(\Delta\sqcup\mathrm{id}_M)\circ\Omega\ \text{ and }\ (\epsilon\sqcup\mathrm{id}_M)\circ\Omega=\mathrm{id}_M. \tag{1_L}$$

A morphism between two right H-comodule algebras (M, Ω_M) and (N, Ω_N) is, by definition, an element f of $\mathrm{Hom}_{\mathcal{C}}(M, N)$ which respects the structure of H-comodule in the sense that

$$\Omega_N \circ f = (f \sqcup \mathrm{id}_H) \circ \Omega_M. \tag{2}$$

We denote respectively by $\mathrm{rCoMod}_{\mathcal{C}}(H)$ and $\mathrm{lCoMod}_{\mathcal{C}}(H)$ the categories of right and left H-comodule algebras of $\mathcal{C}$. We will now state, and prove, that they are algebraic categories.

Lemma 5. *The category* $\mathrm{rCoMod}_{\mathcal{C}}(H)$ *is an algebraic category.*

Of course, an analogous statement holds for the category $\mathrm{lCoMod}_{\mathcal{C}}(H)$.

Proof. It is a simple verification that k is a right H-comodule algebra and that for every H-comodule algebra M, the unique morphism in $\mathcal{C}$ from k to $\mathcal{C}$ satisfies (2), hence is a morphism in the category of right H-comodule algebras. This shows that k is an initial element of $\mathrm{rCoMod}_{\mathcal{C}}(H)$.

We must now define a coproduct in $\mathrm{rCoMod}_{\mathcal{C}}(H)$. Let (M, Ω_M) and (N, Ω_N) be two right H-comodules. We will endow the object $M \sqcup N$ of $\mathcal{C}$ with a coaction of H. For this, we start from the map $\Omega_M \sqcup \Omega_N : M \sqcup N \to M \sqcup H \sqcup N \sqcup H$. We will compose this map with a map which, informally, forgets the origin of the factors belonging to H. Pictorially, we want a morphism $M \sqcup H_{|1} \sqcup N \sqcup H_{|2} \to M \sqcup N \sqcup H$ which sends, for instance, $nh_{|2}n'mh'_{|1}m'n''h''_{|1}h'''_{|2}$ to $nhn'mh'm'n''(h''h''')$. This map is built from the canonical maps $\iota_M : M \to M \sqcup N \sqcup H$, $\iota_N : N \to M \sqcup N \sqcup H$, $\iota_H : H \to M \sqcup N \sqcup H$ by the formula

$$\Omega_{M \sqcup N} = (\iota_M \dot{\sqcup} \iota_H \dot{\sqcup} \iota_N \dot{\sqcup} \iota_H) \circ (\Omega_M \sqcup \Omega_N).$$

We claim that $(M \sqcup N, \Omega_{M \sqcup N})$ is a coproduct of (M, Ω_M) and (N, Ω_N). The fact that $\Omega_{M \sqcup N}$ is a morphism in $\mathcal{C}$ follows from its very definition. There remains to prove that it satisfies the equalities (1_R). Let us treat the first equality in detail.

We begin by drawing (see Fig. 1) the diagram associated with the universal problem of which the pair $(M \sqcup N, \Omega_{M \sqcup N})$ is the solution.

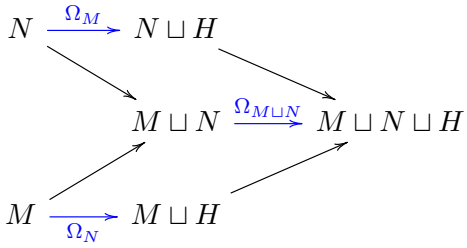


FIGURE 1. The universal problem solved by $(M \sqcup N, \Omega_{M \sqcup N})$.

Using this universal property and the fact that Ω_M and Ω_N are coactions, we draw a second diagram (see Fig. 2) in which a map $f : M \sqcup N \to M \sqcup N \sqcup H \sqcup H$ appears. We claim that the two maps of which we want to prove the equality, namely $(\Omega_{M \sqcup N} \sqcup \mathrm{id}_H) \circ \Omega_{M \sqcup N}$ and $(\mathrm{id}_{M \sqcup N} \sqcup \Delta) \circ \Omega_{M \sqcup N}$, are equal to this map.

This is done by two more diagrams. The first (Fig. 3) shows the map $(\Omega_{M \sqcup N} \sqcup \mathrm{id}_H) \circ \Omega_{M \sqcup N}$. The commutativity of this diagram and its comparison with Fig. 2 shows that this map is indeed equal to the map f.

For the second map, the diagram that we draw (Fig. 4) has four squares and the commutativity of the rightmost two needs to be checked. This is a simple verification that we leave to the reader. The equality of $(\mathrm{id}_{M \sqcup N} \sqcup \Delta) \circ \Omega_{M \sqcup N}$ with the map f, and the fact that $\Omega_{M \sqcup N}$ satisfies the first equality of (1_R), follows immediately.

The proof of the second equality of (1_R) is similar and simpler that the proof of the first, and we leave it to the reader.

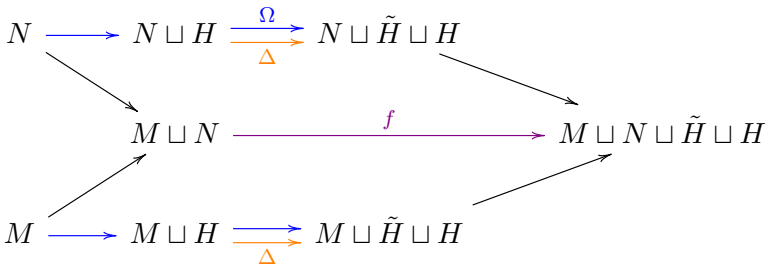


FIGURE 2. In this diagram, blue arrows indicate a coaction of H and orange arrows a coproduct of H. We use the notation $\tilde{H}$ for the sake of clarity.

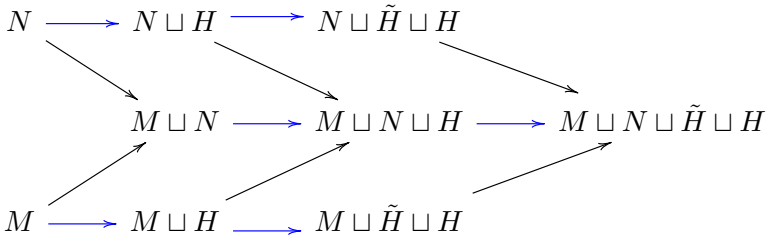


FIGURE 3. This diagram, in which blue arrows correspond to the coaction of H, shows the map $(\Omega_{M\sqcup N} \sqcup \mathrm{id}_H) \circ \Omega_{M\sqcup N}$.

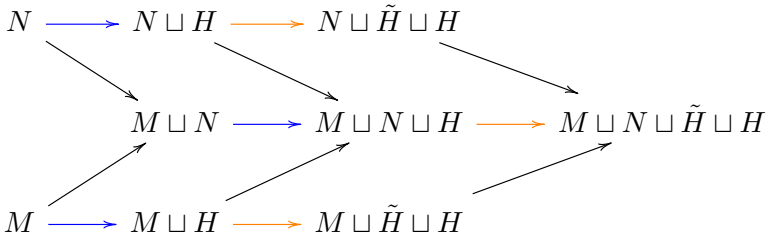


FIGURE 4. In this diagram, orange arrows correspond to the coproduct of H. We see the map $(\mathrm{id}_{M\sqcup N} \sqcup \Delta) \circ \Omega_{M\sqcup N}$ in the middle line.

We are not only stating that the free product of two comodules is a comodule, but also that the category of all comodules over the Zhang algebra H is algebraic. Three points remain to be proved. The first is the equivariance of the coproduct of two equivariant morphisms. The second is the fact that k, the initial object of $\mathcal{C}$, can be endowed with a coaction of H. The third is that, with respect to this coaction, the unique morphism from k to any comodule algebra M is equivariant.

Let us discuss the first point. Let $(M, \Omega_M), (N, \Omega_N), (C, \Omega_C)$ be three right comodules algebras. Let $f : M \to C$ and $g : N \to C$ be two morphisms of the category $\mathrm{rCoMod}_{\mathcal{C}}(H)$. We claim that $f \dot{\sqcup} g$ is equivariant with respect to coactions $\Omega_{M\sqcup N}$ and Ω_C, which means that the equality $\Omega_C \circ (f \dot{\sqcup} g) = (\mathrm{id}_H \sqcup (f \dot{\sqcup} g)) \circ \Omega_{M\sqcup N}$ holds. Fig. 5 shows a diagram in which, as before, blue arrows indicate the coaction of H. The equivariance of $f \dot{\sqcup} g$ is equivalent to the commutativity, in this diagram, of the face delimited by the gray bended arrow and the horizontal symmetry axis of the diagram. This commutativity property will be implied by commutativity of the two outer faces bounded by the violet and gray arrows. This commutativity, in turn, is implied by the associativity of the free product, drawn in Fig. 6.

The second and third points concern the initial object k. There exists an unique morphism $\eta_{k\sqcup H} : k \to k \sqcup H$ and we prove that $(k, \eta_{k\sqcup H})$ is an object of $\mathrm{rCoMod}_{\mathcal{C}}(H)$. The two morphisms $(\mathrm{id}_H \sqcup \eta_{k\sqcup H}) \circ \eta_{k\sqcup H}$ and $(\Delta \sqcup \mathrm{id}_H) \circ \eta_{k\sqcup H}$ are equal because there is a unique morphism from

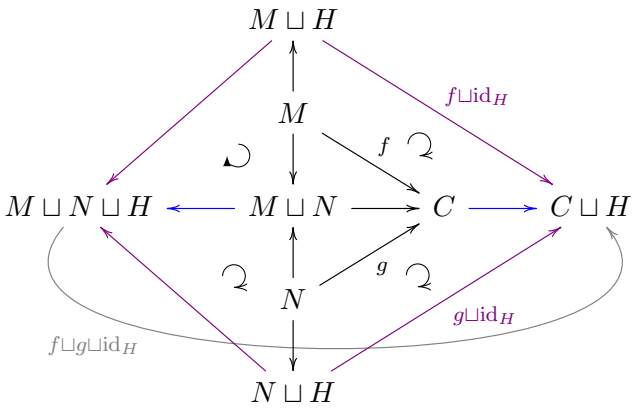


FIGURE 5. Equivariance of the map $f \dot{\sqcup} g$. The morphisms drawn in violet and gray are equal to the morphisms drawn with the same colour in Figure 6.

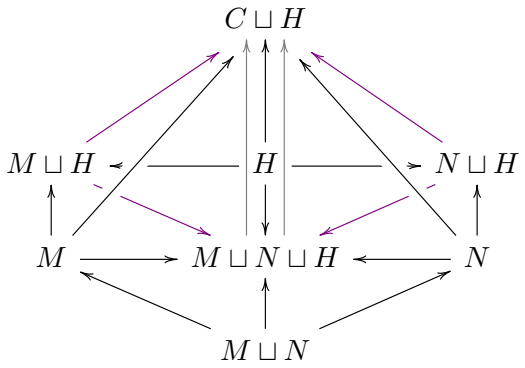


FIGURE 6. Associativity of the free product.

k to $k \sqcup H \sqcup H$. For the same reason, $(\varepsilon \sqcup \mathrm{id}_k) \circ \eta_{k \sqcup H} = \mathrm{id}_k$. The third point, that the unique map from k to any comodule algebra M is equivariant, also follows from the same argument. The proof is complete.

□

Let us denote by $\iota_1 : H \to H \sqcup H$ and $\iota_2 : H \to H \sqcup H$ the canonical maps. We define the morphism $\Omega_c : H \to H \sqcup H$ by the following formula:

$$\Omega_c = (\iota_1 \dot{\sqcup} \iota_2 \dot{\sqcup} \iota_1) \circ (\mathrm{id}_{H \sqcup H} \sqcup S) \circ (\Delta \sqcup \mathrm{id}_H) \circ \Delta.$$

For an integer $n \geq 1$, we denote by Ω_c^n the induced coaction on $H^{\sqcup n}$. It is convenient to introduce a graphical calculus to perform computations involving free products. All morphisms we handle act on free products of H with itself and are valued in the same type of objects. We do not forget that co-product on $\mathcal{C}$ comes with morphisms. Let $n \geq 1$ an integer, the morphisms from H to $H^{\sqcup n}$ are denoted $\iota_1, \ldots, \iota_n$. Let $f = (f_1, \ldots, f_n)$ a finite sequence of morphisms from H to itself. To each permutation σ of $[1, \ldots, n]$ is attached a morphism from $H^{\sqcup n}$ to $H^{\sqcup n}$, denoted f_σ and defined as the unique morphism satisfying the property: $f_\sigma \circ \iota_i = i_{\sigma(i)} \circ f_i$. Such morphisms are depicted as follows: we draw n vertical lines, labelled with the symbols $f_1, \ldots, f_n$ from left to right. We add at the beginning of the i^{th} vertical line the integer i and the integer $\sigma(i)$ at the end of the line. We draw examples in Figure 7 in case $n = 3$.

Of primary importance is the case where all the f_i 's are equal to the identity of H. In that case, we use the notation τ_σ for the morphism f_σ. In words, τ_σ relabel the letters by substituting

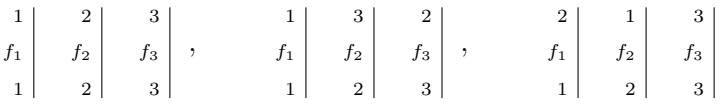


FIGURE 7. Morphisms $f_{\mathrm{id}},\ f_{(2,3)},\ f_{(1,2)}$.

the label σ_i to i. The Figure 8 shows the graphical representation of the morphism τ_{12}. To draw the graphical representations of the morphisms at stake, we make the convention that a sequence of edges starting and ending on same levels have ends labelled with increasing integers from left to right, the ends being labelled with the same integer. Using graphical calculus, the structural

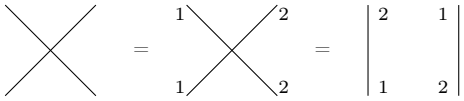


FIGURE 8. The permutation τ_{12} of labels.

morphisms S, Δ, ϵ and $\mu = id_H \dot{\sqcup} \mathrm{id}_H$ are pictured as in Fig. 9 and the relations they are subject to are drawn in Fig. 10.

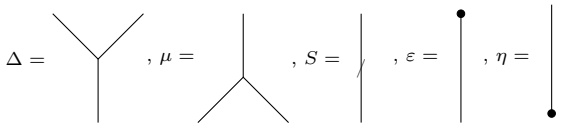


FIGURE 9. Drawings of the structural morphisms of a Zhang algebra $(H, \Delta, \varepsilon, S)$ and $\mu = \mathrm{id}_H \dot{\sqcup} \mathrm{id}_H$.

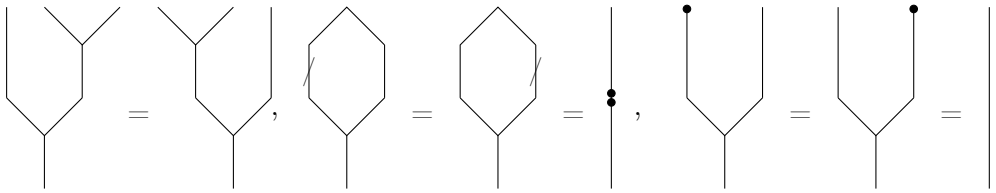


FIGURE 10. Relations amongst the structural morphisms of a Zhang algebra, from left to right: $\Delta \sqcup \mathrm{id}_H \circ \Delta = \mathrm{id}_H \sqcup \Delta \circ \Delta$, $S \sqcup \mathrm{id}_H \circ \Delta = \mathrm{id}_H \sqcup S \circ \Delta = \epsilon \circ \eta, \epsilon \sqcup \mathrm{id}_H = \mathrm{id}_H \sqcup S = \mathrm{id}_H$.

On a Zhang algebra, the antipode S is not a comorphism, however a simple relation between $S \sqcup S \circ \Delta$ and $\Delta \circ S$ can be deduced from the three structural relations drawn in Fig. 10 which is reminiscent from the fact that the antipode S of an Hopf algebra is an anti-co-morphism:

$$\tau_{(12)} \circ (S \sqcup S) \circ \Delta = \Delta \circ S$$

This last relation is pictured in Fig 11 and its proof can be found in the seminal article of Zhang [6]. Let us, however, illustrate how the graphical calculus we introduce work on this first example. The morphism $\Delta \circ S$ is the inverse of Δ in the group $\mathsf{Hom}_{\mathcal{C}}(H, H \sqcup H)$. We have to show that $((\tau_{12} \circ (S \sqcup S) \circ \Delta) \dot{\sqcup} \Delta) \circ \Delta = \eta_{H \sqcup H} \circ \epsilon$, where $\eta_{H \sqcup H}$ is the unique morphism from the initial object to $H \sqcup H$, graphical computations are performed in Fig. 12. By using the same method, it can be proved that $S^2 = \mathrm{id}_H$. The main goal of the two following lemmas (Lemma 7 and Lemma 6) is to prove that a Zhang algebra in algebraic category $\mathcal{C}$ is also a Zhang algebra in the category $rCoMod\mathcal{C}(H)$ of left comodules over H in $\mathcal{C}$.

Lemma 6. *The pair (H, Ω_c) is a right H-comodule algebra of $\mathcal{C}$.*

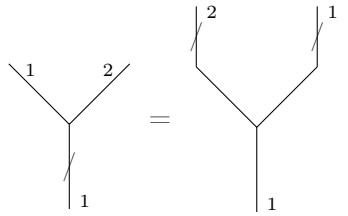


FIGURE 11. Diagram corresponding to the relation $\tau_{(12)} \circ (S \sqcup S) \circ \Delta = \Delta \circ S$.

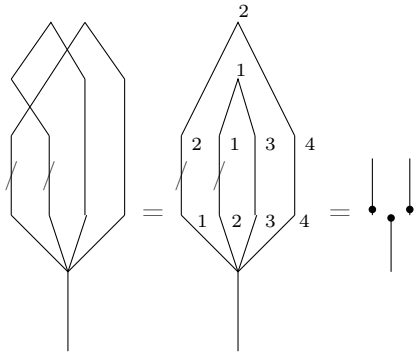


FIGURE 12. The graphical proof of the relation $\tau_{12} \circ (S \sqcup S) \circ \Delta = \Delta \circ S$.

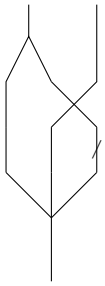


FIGURE 13. Diagram representing the coaction Ω_c.

Proof. The equality $(\mathrm{id}_H \sqcup \Omega_c) \circ \Omega_c = (\Delta \sqcup \mathrm{id}_M) \circ \Omega_c$ follows from the fact that both sides are equal to

$$(\iota_1 \dot{\sqcup} \iota_2 \dot{\sqcup} \iota_3 \dot{\sqcup} \iota_2 \dot{\sqcup} \iota_1) \circ (\mathrm{id}_{H^{\sqcup 4}} \sqcup S) \circ \Delta^4,$$

as one checks using the coassociativity of Δ and the fact that it is a morphism. Let us write more details to convince the reader with the efficiency of the graphical calculus we introduced. The co-action Ω_c has the graphical presentation showed in Fig. 13. We begin with the first relation of (1_L). In figure 14, we drew the sequence of diagram that proves the equality $(\Delta \sqcup \mathrm{id}_H) \circ \Omega_c = (\iota_1 \dot{\sqcup} \iota_2 \dot{\sqcup} \iota_3 \dot{\sqcup} \iota_2 \dot{\sqcup} \iota_1) \circ (\mathrm{id}_{H^{\sqcup 4}} \sqcup S) \circ \Delta^4$.

The verification of the equality $(\epsilon \sqcup \mathrm{id}_H) \circ \Omega_c = \mathrm{id}_H$ is simpler.

□

Furthermore, we claim that Δ, ϵ, and S, which are defined as morphisms in the category $\mathcal{C}$, are in fact morphisms in the category $\mathrm{rCoMod}\mathcal{C}(H)$. This means that they satisfy the equation (2).

Lemma 7. *$((H, \Omega_c), \Delta, \epsilon, S)$ is a Zhang algebra of the category $\mathrm{rCoMod}\mathcal{C}(H)$.*

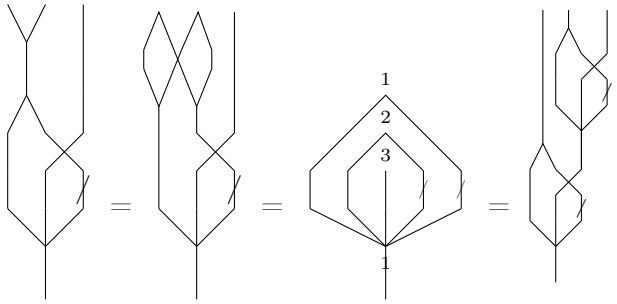


FIGURE 14. Graphical proof of the relation $(\Delta \dot{\sqcup} 1) \circ \Omega_c = (1 \dot{\sqcup} \Omega_c) \circ \Omega_c$.

Proof. We have to check that the three morphisms $\Delta,\ \epsilon,\ S$ are co-module morphisms. To that end, we use the graphical calculus introduced previously. We begin with the antipode, compatibility between S and the right co-action Ω_c means $\Omega_c \circ S = (\mathrm{id}_H \sqcup S) \circ \Omega_c$. The computations are pictured in Fig. 15, to perform them we use the relation drawn in Fig. 11.

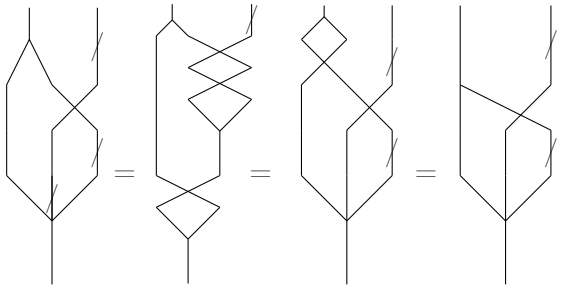


FIGURE 15. The relation $S \circ \Omega_c = (\mathrm{id}_H \sqcup S) \circ \Omega_c$.

We now turn our attention to the relation that needs to be satisfied by the counit ε. The computations are drawn in Fig. 16. We have to check that $\eta_{H \sqcup k} \circ \varepsilon = (\mathrm{id}_H \sqcup \varepsilon) \circ \Omega_c$, because we identify $H \sqcup k$ with H, the last relation is written as $\eta_H \circ \varepsilon = (\mathrm{id}_H \sqcup \varepsilon) \circ \Omega_c$. The final relation

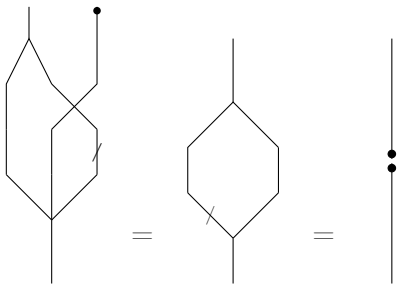


FIGURE 16. Diagrammatic proof of the relation $\eta_H \circ \varepsilon = (\mathrm{id}_H \sqcup \varepsilon) \circ \Omega_c$.

that needs to be checked implies that Δ is a comodule morphism with respect to the coactions Ω_c on H and Ω_c^2 on $H \sqcup H$:$\Omega_c^2 \circ \Delta = (\mathrm{id}_H \sqcup \Delta) \circ \Omega_c$. Once again, we perform diagrammatic computations that are pictured in Fig. 17.

□

3. Monoidal structures and inductive limits

In this section, we recall basic facts on monoidal structures and inductive (direct) limits.

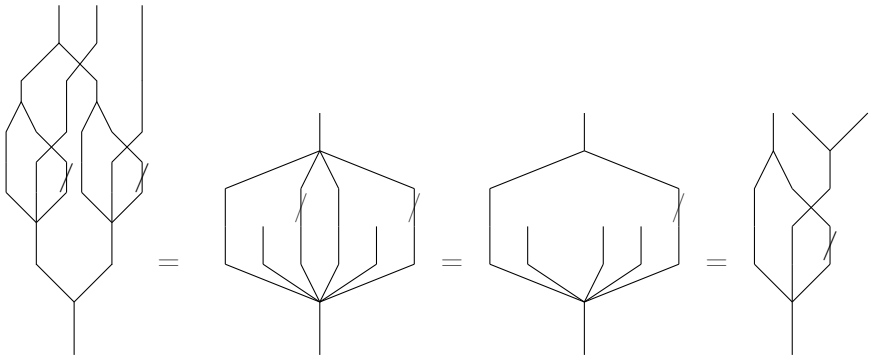


FIGURE 17. Proof of the relation $\Omega_c^2 \circ \Delta = (\mathrm{id}_H \sqcup \Delta) \circ \Omega_c$.

3.1. **Monoidal structures and independence.** In order to motivate our definitions, we start by reviewing the classical, commutative case. Let

$$X : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{S}) \text{ and } Y : (\Omega, \mathcal{F}, \mathbb{P}) \to (T, \mathcal{T}) \tag{3}$$

be two essentially bounded random variables defined on the same classical probability space. Let τ_X and τ_Y denote the linear forms defined respectively on $L^\infty(S, \mathcal{S})$ and $L^\infty(T, \mathcal{T})$ by the distributions of X and Y.

The random variables X and Y also induce homomorphisms of the algebras of measurable functions

$$j_X : L^0(S, \mathcal{S}) \to L^0(\Omega, \mathcal{F}) \text{ and } j_Y : L^0(T, \mathcal{T}) \to L^0(\Omega, \mathcal{F}),$$

defined by $j_X(f) = f \circ X$ and $j_Y(g) = g \circ Y$.

The independence of the random variables X and Y is equivalent to the existence of a morphism j making the following diagram commutative:

$$\begin{array}{ccccc} & & (L^\infty(\Omega, \mathcal{F}), \mathbb{E}) & & \\ & \overset{j_X}{\nearrow} & \uparrow j & \overset{j_Y}{\nwarrow} & \\ (L^\infty(S, \mathcal{S}), \tau_X) & \xrightarrow{\iota} & (L^\infty(S, \mathcal{S}), \tau_X) \otimes (L^\infty(T, \mathcal{T}), \tau_Y) & \xleftarrow{\iota} & (L^\infty(T, \mathcal{T}), \tau_Y) \end{array}$$

In order to generalize the notion of independence from the category of commutative probability spaces to an arbitrary category of non-commutative probability spaces, it appears that a notion of tensor product is needed. A category in which the tensor product of two objects is defined is called a monoidal category or a tensor category. A monoidal category $\mathcal{C}$ is a category $\mathcal{C}$ together with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is

(1) is associative under a natural isomorphism with compoents

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \overset{\cong}{\to} (A \otimes B) \otimes C$$

called associativity constraints,

(2) has a unit object $E \in \mathrm{Obj}(\mathcal{C})$ acting as left and right identity under natural isomorphisms with components:

$$\ell_A : E \otimes A \overset{\cong}{\to} A, \; r_A : A \otimes E \overset{\cong}{\to} A$$

called left unit constraint and right unit constraint such that the pentagon and triangle identities hold, see Fig. 18 and Fig. 19

The pentagon identities Fig.18 and triangle identities Fig.19 imply commutativity of all diagrams which contain the associativity constraint, the natural isomorphisms ℓ and r. The following definition is motivated by the fact that in a tensor category, there is in general no canonical morphism from an object to its tensor product with another object.

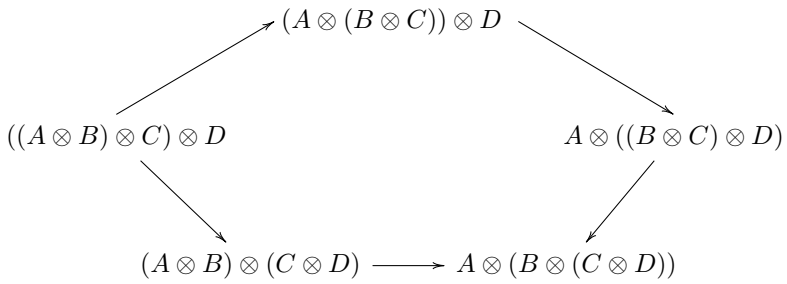


FIGURE 18. Pentagonal coherence axiom for monoidal categories

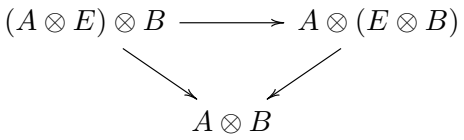


FIGURE 19. Triangle coherence axiom for monoidal categories

Definition 8 (Monoidal category with inclusions)**.** A monoidal category with inclusions $(\mathcal{C}, \otimes, \iota)$ is a tensor category $(\mathcal{C}, \otimes)$ in which, for any two objects B_1 and B_2, there exist two morphisms $\iota_{B_1} : B_1 \to B_1 \otimes B_2$ and $\iota_{B_2} : B_2 \to B_1 \otimes B_2$ such that for any two objects A_1 and A_2 and any two morphisms $f_1 : A_1 \to B_1, f_2 : A_2 \to B_2$, the following diagram commutes:

$$\begin{array}{ccccc} A_1 & \xrightarrow{\iota_{A_1}} & A_1 \otimes A_2 & \xleftarrow{\iota_{A_2}} & A_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_1 \otimes f_2} & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow{\iota_{B_1}} & B_1 \otimes B_2 & \xleftarrow{\iota_{B_2}} & B_2 \end{array}$$

Let $\mathcal{P}_i : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $i \in \{1, 2\}$ be the projections functors on the first and second component. We can reformulate Definition 8, a monoidal category with inclusions if a monoidal category with two natural transformations $\iota_1 : P_1 \Rightarrow \otimes$ and $\iota_2 : P_2 \Rightarrow \otimes$. The natural transformation ι_1 is called a *left inclusion* and ι_2 is called a *right inclusion*. We can now give a general definition of independence of two morphisms.

Definition 9. Let $(\mathcal{C}, \otimes, \iota)$ be a tensor category with injections. Two morphisms $f_1 : C_1 \to A$ and $f_2 : C_2 \to A$ are said to be *independent* if there exists a third morphism $f : C_1 \otimes C_2 \to A$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & A & & \\ & {\scriptstyle f_1} \nearrow & \uparrow{\scriptstyle f} & \nwarrow {\scriptstyle f_2} & \\ C_1 & \xrightarrow[\iota_{C_1}]{} & C_1 \otimes C_2 & \xleftarrow[\iota_{C_2}]{} & C_2 \end{array}$$

If we want to be explicit about the monoidal structure that is involved, we will say that f_1 and f_2 are $\otimes$-independent. Definition 2 defines what it means for two morphisms to be independent but not what mutual independence of a finite set of morphisms is. To define mutual independence we need further assumptions on the monoidal structural, that are detailed below. The morphism f of the last definition is called an independence morphism. In most examples this morphism, if it exists will be uniquely determined but this is not the case in general. We warn the reader: a coproduct with injections is a tensor product with injections for which any two morphisms with the same target space are independent. A tensor coproduct is not a co-product.

Let R an unital associative algebra and denote by $\mathrm{Prob}(R)$ the category of algebras endowed with a conditional expectations. We saw that the category $\mathrm{Alg}^\star$ is algebraic, the coproduct being the free product of algebras. Pick a tensor product $\otimes$ on Prob. Let A and B two probability spaces and write $A \otimes B = (C, \tau_{A\otimes B})$ with $C \in \mathrm{Alg}^\star$. From the universal property satisfied by the co-product, there exists a morphism of involutive algebras $\pi : A \sqcup B \to C$. Define the bi-functor $\otimes'$ on Prob by, for two objects A and $B \in$ Prob:

$$A \otimes' B = (A \sqcup B, \tau_{A\otimes B} \circ \pi) \tag{4}$$

and for two morphisms $f : A_1 \to A_2$, $g : B_1 \to B_2$, $\otimes'(f, g) = f \sqcup g$. We prove that $\otimes'$ is a tensor product with injections on Prob. One fact needs to be proved. Let $(A_i, \tau_{A_i}), (B_i, \tau_{B_i}) \in \{1, 2\}$ objects of Prob, $f : (A_1, \tau_{A_1}) \to (B_1, \tau_{B_1})$ and $g : (A_2, \tau_{A_2}) \to (B_2, \tau_{B_2})$ two morphisms. First, we prove that $\tau_{A_2\otimes B_2} \circ (f \sqcup g) = \tau_{A_1\otimes B_1}$. To that end, we draw the commutative diagram in Fig. 20, blue arrows are morphisms of the category Prob, while black arrows are morphisms in the category $\mathrm{Alg}^\star$. In Fig. 20, the morphisms $\iota_{A_i} : A_i \to A_i \sqcup B_i$ and $\iota_{B_i} : B_i \to A_i \sqcup B_i$ are drawn

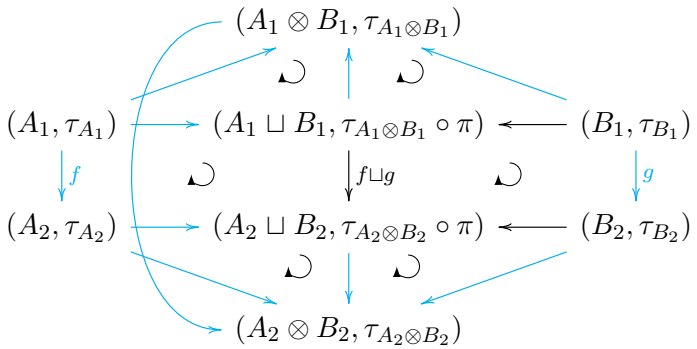


FIGURE 20.

in blue. In fact, it is easily seen that preserving the trace for these morphisms is equivalent to commutativity of the four triangles in Fig. 20. To show that the free product $f \sqcup q$ preserves the trace, it is enough to shows the commutativity of the two outer faces, the ones bordered with blue arrows. The commutativity of theses two faces is implied by the universal property satisfied by $\otimes$. In conclusion, the arrow $f \sqcup g$ is equal to a composition of blue arrow and is thus trace preserving. Assume that $A_2 = B_2$. If the two morphisms f and g are $\otimes$-independent then they are also $f \otimes' f$-independent. Hence, in the sequel, there is no loss in replacing the tensor product $\otimes$ by $\otimes'$ and thus assuming that the underlying involutive algebra of the tensor product of two probability spaces is the free product of the algebras. We provide a few examples of monoidal structures with inclusions on the categories of Probability spaces and amalgamated probability spaces, see [3] for a detailed overview.

1. The category $\mathrm{Prob}(\mathbb{C})$ of complex probability spaces can be endowed with several monoidal structures, there are three of them that will be interesting for the present work and are denoted by $\hat{\otimes}$, $*$ and $\diamond$. Let (A, E_A) and (B, E_B) two probability spaces; $E_A : A \to \mathbb{C}$ and $E_B : B \to \mathbb{C}$ are two complex positive unital linear forms.
   a. The linear for $E_A \hat{\otimes} E_B$ is a linear form on the free product of algebras $A \sqcup B$ and is defined by, for an alternating word $s \in A \sqcup B$,by:

$$\left(E_A \hat{\otimes} E_B\right)(s) = E_A(a_1 \cdots a_p) E_B(b_1 \cdots b_m). \tag{5}$$

   In the last equation, the elements $a_1, \ldots, a_p$ and $b_1, \ldots, b_m$ are indexed according to the order they appear in the word s. The bifunctor $\hat{\otimes}$ that send the pair of probability spaces $((A, E_A), (B, E_B))$ to $A \sqcup B, E_A \hat{\otimes} E_B)$ is a monoidal structure which is, in addition, symmetric. To this monoidal structure is related the notion of *universal*

tensor independence. To define $\hat{\otimes}$ we use the free product of star algebras, we could have instead used the tensor product of algebras. In fact, the functor $\otimes$ that sends (A, E_A) and (B, E_B) to $(A \otimes B, E_A \otimes E_B)$ is, first, well defined and is a monoidal structure on Prob($\mathbb{C}$). With obvious notations, two morphisms $f : (A, E_A) \to (C, E_C)$ and $g : (B, E_B) \to (C, E_C)$ are $\otimes$ independent if and only if:
1. $\forall a \in A,\ b \in B,\ [f(a), g(b)] = 0$,
2. $E_C\left(f(a_1)g(b_1)\cdots f(a_p)g(b_p)\right) = E_A\left(f(a_1\cdots a_p)\right) E_B\left(g(b_1\cdots b_p)\right)$.

The two morphims f ang g are $\hat{\otimes}$ independent if only point 1(*a*)1 holds. From the explicit expression (5), positivity of E_A and E_B implies positivity of $E_A\hat{\otimes}E_B$. Of course, the map $\tau_{A\sqcup B} : A \sqcup B \to B \sqcup A$ equal to the identity on A and B is a state preserving morphism: $\tau_{A\sqcup B} \circ E_B \otimes E_A = E_A \otimes E_B$. In that case, we say that $\hat{\otimes}$ is *symmetric.*

b. The second monoidal structure is related to the notion of *boolean independence*. The state $E_A \diamond E_B$ is defined on the free product $A \sqcup B$ and satisfies:

$$(E_A \diamond E_B)(s_1 \cdots s_p) = E_{\varepsilon_1}(s_1)\cdots E_{\varepsilon_p}(s_p).$$

with $s_1, \ldots, s_p \in \{A, B\}$, $\varepsilon_i = A$ if $s_i \in A$ and $\varepsilon_i = B$ if $s_i \in B$. Since E_A and E_B are unital map, the boolean product is well defined. The bifunctor $\diamond$ that send the pair of probability spaces (A, E_A), (B, E_B) to $(A \sqcup B, E_A \diamond E_B)$ is a symmetric monoidal structural.

c. The third and last monoidal structural we define is related to the notion of *free independence*, at the origin of free probability theory.The free product of E_A and E_B is defined by the following requirement. For all alternating word $s \in A \sqcup B$,

$$(E_A \dot{\sqcup} E_B)(s) = 0,\ s_i \in \ker(E_{\varepsilon_i}),\ i \leq p,\ j \leq m. \tag{6}$$

A formula for $E_A \dot{\sqcup} E_B(s)$ can be computed inductively. Computation of the free product of E_A and E_B are simple for words with small lengths,

$$(E_A \dot{\sqcup} E_B)(ab) = E_A(a)E_B(b),$$
$$(E_1 \dot{\sqcup} E_B)(a_1ba_2) = E_A(a_1)E_B(b)E_A(a_2) + E_A(a_1E_B(b)a_2) + E_A(a_1)E_B(b)E_A(a_2) + 2E_A(a_1)E_B(b)E_A(a_2)$$

See [7] and [8].

2. Let R be an unital associative algebra, we define two monoidal structures on the category of amalgamated probability spaces, Prob(R). In case R is commutative, we define a monoidal structure on the category comProb(R) of commutative bimodule algebras. Note that, by definition, the left and right action of R on an probability space in comProb(R) are equal. Let (A, E_A) and (B, E_B) two probability spaces in Prob(R).
   a. We begin with the amalgamated free product $E_A \dot{\sqcup}_R E_B$ of E_A and E_B. It is defined by the equation (6), for all alternating word $s \in A \sqcup_R B$ in the amalgamated free product $A \sqcup_R B$, $E_A \dot{\sqcup}_R E_B$ is defined by requiring:

$$(E_A \dot{\sqcup}_R E_B)(s) = 0,\ s_i \in \ker(E_{\varepsilon_i}),\ i \leq p,\ j \leq m.$$

   b. The amalgamated boolean product $E_A \diamond_R E_B$ of the two R bimodule maps E_A and E_B is a R bimodule map on the amalgamated free product $A \sqcup_R B$ and is defined by the equation:

$$(E_A \diamond_R E_B)(s_1 \cdots s_p) = E_{\varepsilon_1}(s_1)\cdots E_{\varepsilon_p}(s_p)$$

with $s_1, \ldots, s_p \in \{A, B\}$, $\varepsilon_i = A$ if $s_i \in A$ and $\varepsilon_i = B$ if $s_i \in B$.

3. We defined amalgamated versions of the notion of boolean and free independences, using essentially the same formulae as for the non-amalgamated case. We can not do so for tensor

independance, at least for two reasons. First, the amalgamated tensor product $A\otimes_R B$ with bimodule structure given by

$$r(a\otimes_R b)r' = ra\otimes_R br',\ a\in A,\ b\in B,\ r,r'\in R,$$

is not naturally an algebra, meaning that the canonical product $A\otimes B$ does not descend to a product on $A\otimes_R B$. Secondly, formula 5 can not be used to define an amalgamated version of tensor independance, since for all a,b $E(a_1ra_2)E(b_1)\neq E(a_1a_2)E(rb_1)$, $a_1,a_2\in A$, $b_1\in B$ and $r\in R$. Finally, in the last section of the present work we use amagamated probability spaces over commutative algebras R.

We mentioned earlier there might be an issue if we try to define independence of more than two morphisms. An additional constraint needs to be satisfied by the natural morphisms ι^1, ι^2 and the unit E. This is the content of the next definition.

Definition 10. Inclusions ι^1 and ι^2 are called compatible with unit constraints if the diagram

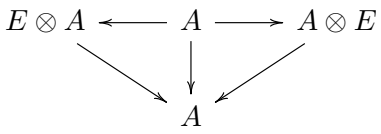


FIGURE 21. Compatibilty constraint between inclusions and unit

A more vernacular way to put constraints of Definition 10 is that inclusions ι^1 and ι^2 and left and right unit morphisms are, respectively, inverse from each other.

Theorem 11 (Theorem 3.6 in [4])**.** *Let $(\mathcal{C},\otimes,E,\alpha,\ell,r,\iota^1,\iota^2)$ a monoidal category.*

1. *If ι^1 and ι^2 are inclusions which are compatible with unit constraints, the unit object E is initial, i.e there is an unique morphism $\eta_A : E\to A$ for every object $A\in \mathrm{Obj}(\mathcal{C})$. Furthermore,*

$$(\star)\qquad \iota^1_{A,B} = (\mathrm{id}_A\otimes\eta_B)\circ r_A^{-1},\quad \iota^2_{A,B} = (\eta_A\otimes \mathrm{id}_B)\circ \ell_B^{-1}.$$

   holds for all objects $A,B,C\in\mathrm{Obj}(\mathcal{C})$.
2. *Suppose that the unit object E is an initial object. Then $\star$ read as a definition yields inclusions ι^1, ι^2 which are compatible with the unit constraints.*

As pointed out in [4], Maclane's coherence theorem can be extended to all diagrams built up from the associative constraints, left and right units, and the natural morphisms η. This extended Maclane coherence theorem implies, in particular, commutativity of the diagrams in Fig. 22 and Fig. 23 in case compatibility with the unit constraints of inclusions are satisfied. In

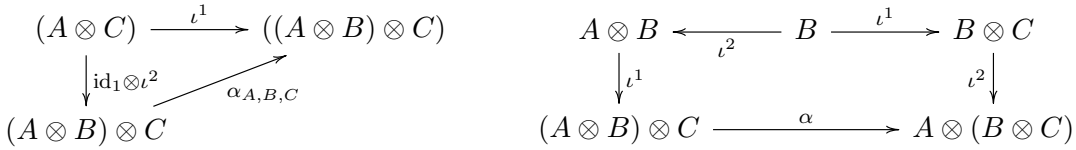


FIGURE 22.

the sequel, until the end of this article, monoidal structures with injections we consider satisfy the compatibility with unit constraints. If $1\leq i_1<\cdots<i_k\leq n$, there are unique morphisms $\iota^{i_1,\ldots,i_k;n}_{A_1,\ldots,A_n} : A_{i_1}\otimes\cdots\otimes A_{i_{i_k}}\to A_1\otimes\cdots A_n$ that constitute natural transformations $\iota^{i_1,\ldots,i_k;n}$, referred to as inclusion morphisms. In the sequel, the symbol $\mathcal{C}$ stands for a monoidal category such that the unit object is initial.

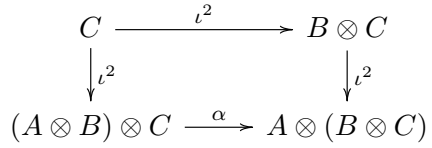


FIGURE 23.

Definition 12 (Mutual independence, Definition 3.7 in [4])**.** Let $B_1, \ldots, B_n, A$ be objects in the category $\mathcal{C}$ and $f_i : B_i \to A$ morphisms. Then $f_1, \ldots, f_n$ are called independent if there exists a morphism $h : B_1 \ldots \cdots \otimes B_n \to A$ such that the following diagram (Fig. 24):

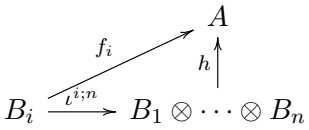


FIGURE 24. Mutual independence of a sequence of morphisms.

As a consequence of the existence of the natural injections $\iota^{i_1,\ldots,i_k}_{A_1,\ldots,A_n}$, a subsequence $(f_{i_1}, \ldots, f_{i_k})$ of a sequence of independent morphisms $(f_1, \ldots, f_n)$ is independent for any tuple $(1 \leq i_1 < \cdots < i_k \leq n)$. We warn the reader, we speak about independence of sequences of morphisms, since, with obvious notations, (f, g) may be an independent sequence without (g, f) being independent.

3.2. **Preliminaries on directed set, poset and inductive limits of $C^\star$-algebra.** In this section, we briefly recall definitions of a directed set and a poset. Let R be a associative unital algebra. The constructions of inductive limits of R-bi-module involutive algebras play prominent roles for the present work, we offer to the reader details on their construction, which otherwise can be found in [9].

An upward directed set is a set S endowed with a pre-order $\leq$, (a reflexive and transitive binary relation) with the additional property that for any pair of elements $s_1 \in S$ and $s_2 \in S$ it exists a third element $s \in S$, called an upper bound, such that $s \geq s_1$ and $s \geq s_2$. The notion of downward directed set is obtained by substituting to the existence of an upper bound with the existence a lower bound; a third element $s \in S$ such that $s \leq a$ and $s \leq b$.

An upward directed poset is an upward directed set $(S, \leq)$ with $\leq$ satisfying the extra property of being anti-symmetric $a \leq b,\ b \leq a \implies a = b$. As an example, the set of finite sequences of affine loops drawn on the plane equipped with the binary operation:

$$(\ell_1, \ldots, \ell_p) \leq \left(\ell'_1, \ldots, \ell'_p\right) \Leftrightarrow \{\ell'_1, \ldots, \ell'_p\} \subset \{\ell_1, \ldots, \ell_p\}. \tag{7}$$

is an upward directed set. Note that because we consider sequences of loops and not set of loops, $\leq$ is not anti-symmetric. The set of finite sets of loops drawn on the plane is a poset for the binary operation inducs by inclusion. For the rest of this section, we closely follow the exposition made by Ziro Takeda in its seminal article *Inductive limit and infinite direct products of $C^\star$-algebras* [9].

Definition 13. Let Γ be an increasingly directed set. A direct system in a category $\mathcal{C}$ is the data of a family of objects $\{O_\gamma, \gamma \in \Gamma\}$ in $\mathcal{C}$ and morphisms $f_{\alpha,\beta}$ for all couples (α, β) with $\alpha \leq \beta$ such that:

1. $\forall \alpha \in \Gamma,\ f_{\alpha,\alpha} = \mathrm{id}_{O_\alpha}$
2. $\forall \alpha \leq \beta \leq \gamma,\ f_{\gamma,\beta} \circ f_{\beta,\alpha} = f_{\gamma,\alpha}$

A direct system can alternatively be seen as a functor. In fact, to the upward directed set Γ is associated a category, also denoted Γ which class of objects is the set Γ. The set of

homomorphisms between two distinct elements ($\alpha \leq \beta$) contains an unique element and we denote by (α, β) this unique morphism. With the notations of the last definition, the functor O associated with the direct system is defined as:

$$O(\gamma) = O_\gamma,\ O((\alpha,\beta)) = f_{\beta,\alpha}.$$

For the rest of this section, we fix an increasingly directed set Γ.

Definition 14. Let $\mathcal{C}$ a category and let $O_\gamma,\ \gamma \in \Gamma,\ \{f_{\alpha,\beta}, \alpha \leq \beta\}$ a direct system of $\mathcal{C}$. An inductive limit is the data of an object O of $\mathcal{C}$ and morphisms $\phi_\gamma : O_\gamma \to O$ such that

1. $\phi_\beta \circ f_{\beta,\alpha} = \phi_\alpha$
2. The following universal property holds. For all objects $Y \in \mathcal{C}$ and morphisms $g_\gamma : O_\gamma \to Y$ there exists a morphism $G : O \to Y$ such that the diagram in Fig. 25 is commutative for all pairs $\alpha \leq \beta$ in Γ.

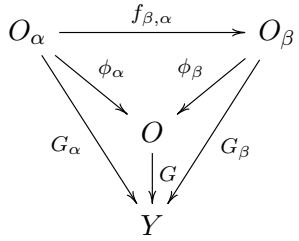


FIGURE 25. Universal property of the inductive limit.

From the universal property satisfied by inductive limits, we see that an inductive limit of a family of objects $\{A_\gamma, \gamma \in \Gamma\}$ is unique up to isomorphism. If any direct family in a category $\mathcal{C}$ admits a direct limit, we say that $\mathcal{C}$ is closed for taking inductive limits, or using the language of category theory that $\mathcal{C}$ is *inductively complete*, see [5]

Let R an unital associative algebra. The following theorem states that the category of R-bimodule involutive algebras is closed for taking inductive limits.

Theorem 15. *The categories* $\mathrm{Prob}(R)$, $\mathrm{Alg}^\star(R)$ *and* $\mathrm{biMod}(R)$ *are inductively complete.*

Proof. We prove only that $\mathrm{Prob}(R)$ is closed for taking inductive limits. Let $\mathcal{A}$ be the set of equivalence classes $\{[(\gamma, a_\gamma)],\ a_\gamma \in A_\gamma,\ \gamma \in \Gamma\}$, with

$$a_\alpha \sim a_\beta \Leftrightarrow \exists \delta \geq \alpha, \beta \text{ such that } f_{\delta,\alpha}(a_\alpha) = f_{\delta,\beta}(a_\beta).$$

Since the maps f_γ ($\gamma \in \Gamma$) are trace preserving, one has

$$\tau_\delta(f_{\delta,\alpha}(a_\alpha)) = \tau_\delta\left(f_{\delta,\beta}(a_\beta)\right) = \tau_F\left(a_F\right) \text{ for all pairs } (a_\alpha, a_\beta) \text{ with } a_\alpha \sim a_\beta$$

hence, the function $(\gamma, a_\gamma) \to \phi_\gamma(a_\gamma)$ is constant on the classes for $\sim$ and thus descends to a linear form τ on the quotient space $\bigsqcup_F \mathcal{A}_F/\sim$. The algebraic operations on $\mathcal{A}$ are defined as follows

1. Addition: $[x_\alpha] + [x_\beta] = [x_\delta + y_\delta]$, with $\delta \geq \alpha,\ \delta \geq \beta$, $x_\delta = f_{\delta,\alpha}(x_{\alpha)}$ and $y_\delta = f_{\delta,\beta}(x_\beta)$.
2. Multiplication: $[x_\alpha] \cdot [y_\beta] = [x_\delta \cdot y_\delta]$,
3. Star operation: $[x_\alpha]^\star = [x_\alpha^\star]$.
4. To define the R-bimodule structure on $\mathcal{A}$, we simply set:

$$r[x_\alpha]r' = [rx_\alpha r'],\ r, r' \in R.$$

In fact, if $[x_\alpha] = [x_\beta]$ with $\beta \geq \alpha$, then $x_\beta = f_{\beta,\alpha}(x_\alpha)$ and $rx_\beta r' = rf_{\beta,\alpha}(x_\alpha)r' = f(rx_\alpha r')$ for $r, r' \in R$

□

4. Zhang algebra holonomy fields

4.1. **Classical lattice holonomy fields.** Before giving the main definition of this paper, namely the definition of a holonomy field on a Zhang algebra (see Definition 18), we will review briefly the classical notion of a holonomy field on a lattice. Part of what we will explain in this section, in particular the content of Section 4.2, makes sense on an arbitrary surface, or even on an arbitrary graph, but for the sake of simplicity and concision, we will restrict ourselves to the framework of graphs on the Euclidean plane $\mathbb{R}^2$.

In this paper, all the paths that we will consider will be piecewise affine continuous paths on the Euclidean plane $\mathbb{R}^2$. We denote by $\mathsf{P}(\mathbb{R}^2)$ the set of all these paths. Each path c has a starting point $\underline{c}$, an end point $\overline{c}$, an orientation, but it has no preferred parametrisation. Constant paths are allowed. Given two paths c_1 and c_2 such that c_1 finishes at the starting point of c_2, the concatenation of c_1 and c_2 is defined in the most natural way and denoted by c_1c_2. Reversing the orientation of a path c results in a new path denoted by c^{-1}. A path which finishes at its starting point is called a *loop*. A loop which is as injective as possible, that is, a loop which visits twice its starting point and once every other point of its image, is called a *simple loop*. A path of the form $c\ell c^{-1}$, where c is a path and ℓ is a simple loop, is called a *lasso* (see Fig. 26).

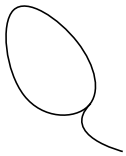


Figure 26. A lasso drawn on the plane.

Let us call *edge* a path that is either an injective path (thus with distinct endpoints) or a simple loop. By a *graph* on $\mathbb{R}^2$, we mean a finite set $\mathbb{E}$ of edges with the following properties:

1. for all edge e of $\mathbb{E}$, the edge e^{-1} also belongs to $\mathbb{E}$,
2. any two edges of $\mathbb{E}$ which are not each other's inverse meet, if at all, at some of their endpoints,
3. the union of the ranges of the elements of $\mathbb{E}$ is a connected subset of $\mathbb{R}^2$.

From the set $\mathbb{E}$, we can form the set $\mathbb{V}$ of *vertices* of the graph, which are the endpoints of the elements of $\mathbb{E}$, and the set $\mathbb{F}$ of *faces* of the graph, which are the bounded connected components of the complement of the union of the range of the edges of the graph. Although it is entirely determined by $\mathbb{E}$, it is the triple $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ that we regard as the graph.

From the graph $\mathbb{G}$, and given a vertex v, we form the set $\mathsf{L}_v(\mathbb{G})$ of all loops based at v that can be obtained by concatenating edges of $\mathbb{G}$. The operation of concatenation makes $\mathsf{L}_v(\mathbb{G})$ a monoid, with the constant loop as unit element. In order to make a group out of this monoid, one introduces the backtracking equivalence of loops and the notion of reduced loops. A loop is *reduced* if in its expression as the concatenation of a sequence of edges (which is unique) one does not find any two consecutive edges of the form ee^{-1}. We denote by $\mathsf{RL}_v(\mathbb{G})$ the subset of $\mathsf{L}_v(\mathbb{G})$ formed by reduced loops. It is however not a submonoid of $\mathsf{L}_v(\mathbb{G})$, for the concatenation of two reduced loops needs not be reduced. The appropriate operation on $\mathsf{RL}_v(\mathbb{G})$ is that of concatenation followed by reduction where, as the name indicates, one concatenates two loops, and then erases sub-loops of the form ee^{-1} until no such loops remain. It is true, although perhaps not entirely obvious, that this operation is well defined, in the sense that the order in which one erases backtracking sub-loops does not affect the final reduced loop.

From the graph $\mathbb{G}$ and the vertex v, we thus built a group $\mathsf{RL}_v(\mathbb{G})$. This group is in fact isomorphic in a very natural way with the fundamental group based at v of the subset of $\mathbb{R}^2$ formed by the union of the edges of $\mathbb{G}$. An important property of the group $\mathsf{RL}_v(\mathbb{G})$ is that it is generated by lassos (see Fig. 27)

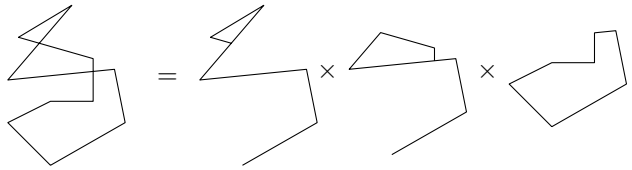


FIGURE 27. Decomposition of a loop into a product of lassos.

In fact, $\mathsf{RL}_v(\mathbb{G})$ is a free group, with rank equal to the number of faces of $\mathbb{G}$, and it admits bases formed by lassos. More precisely, we will use the following description of a basis of this group. Let us say that a lasso $c\ell c^{-1}$ surrounds a face F of the graph if the loop ℓ traces the boundary of F.

Proposition 16. *The group $\mathsf{RL}_v(\mathbb{G})$ admits a basis formed by a collection of lassos, each of which surrounds a distinct face of $\mathbb{G}$.*

In fact, the group $\mathsf{RL}_v(\mathbb{G})$ admits many such bases, but it will be enough for us to know that there exists one. A proof of this proposition, and more details about graphs in general, can be found in [10].

A classical lattice gauge field on $\mathbb{G}$ with structure group G is usually described as an element of the configuration space

$$\mathcal{C}_{\mathbb{G}} = \{g = (g_e)_{e\in\mathbb{E}} \in G^{\mathbb{E}} : \forall e \in \mathbb{E}, g_{e^{-1}} = g_e^{-1}\}.$$

This configuration space is acted on by the lattice gauge group $\mathcal{J}_{\mathbb{G}} = G^{\mathbb{V}}$, according to the formula

$$(j \cdot g)_e = j_{\overline{e}}^{-1} g_e j_{\underline{e}}.$$

Let us fix a vertex v of our graph. Any element g of the configuration space $\mathcal{C}_{\mathbb{G}}$ induces a holonomy map $\mathsf{L}_v(\mathbb{G}) \to G$, which to a loop ℓ written as a concatenation of edges $e_1 \dots e_n$ associates the element $g_{e_n} \dots g_{e_1}$ of G. This map descends to the quotient by the backtracking equivalence relation, and induces a morphism of groups $\mathsf{RL}_v(\mathbb{G}) \to G$. The action of a gauge transformation $j \in \mathcal{J}_{\mathbb{G}}$ on $\mathcal{C}_{\mathbb{G}}$ modifies this morphism by conjugating it by the element j_v. These observations can be turned in the following convenient description of the quotient $\mathcal{C}_{\mathbb{G}}/\mathcal{J}_{\mathbb{G}}$.

Proposition 17. *For all $v \in \mathbb{V}$, the holonomy map induces a bijection*

$$\mathcal{C}_{\mathbb{G}}/\mathcal{J}_{\mathbb{G}} \longrightarrow \mathrm{Hom}(\mathsf{RL}_v(\mathbb{G}), G)/G.$$

It follows from this proposition that describing a probability measure on the quotient space $\mathcal{C}_{\mathbb{G}}/\mathcal{J}_{\mathbb{G}}$ is equivalent to describing the distribution of a random group homomorphism from $\mathsf{RL}_v(\mathbb{G})$ to G, provided this random homomorphism invariant under conjugation. Combining this observation with the fact that $\mathsf{RL}_v(\mathbb{G})$ is a free group, and choosing for instance a basis $l_1, \dots, l_n$ formed by lassos surrounding the n faces of $\mathbb{G}$, we see that a probability measure on $\mathcal{C}_{\mathbb{G}}/\mathcal{J}_{\mathbb{G}}$ is the same thing as a G^n-valued random variable $(H_{l_1}, \dots, H_{l_n})$, invariant in distribution under the action of G on G^n by simultaneous conjugation on each factor.

Let us finally introduce some further notation. We denote by $\mathsf{L}_0(\mathbb{R}^2)$ the set of loops on $\mathbb{R}^2$ based at the origin and by $\mathsf{RL}_0(\mathbb{R}^2)$ the group of reduced loops. From now on, we will always assume that 0 is a vertex of all the graphs that we consider.

It is important to observe that any reduced loop on $\mathbb{R}^2$ belongs to $\mathsf{RL}_v(\mathbb{G})$ for some graph $\mathbb{G}$. Thus,

$$\mathsf{RL}_0(\mathbb{R}^2) = \bigcup_{\mathbb{G}\text{ graph}} \mathsf{RL}_0(\mathbb{G})$$

and accordingly,

$$\mathrm{Hom}(\mathsf{RL}_0(\mathbb{R}^2), G) = \varprojlim \mathrm{Hom}(\mathsf{RL}_0(\mathbb{G}), G).$$

4.2. **Holonomy field on a Zhang algebra.** In the following, we set $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let B be a unital algebra over $\mathbb{K}$. Recall from Example 2 page 3 that the category of involutive bimodule algebras over B is denoted $\mathrm{Alg}^\star(R)$.

We fix a R-probability space, that is, a pair $(\mathcal{A}, E)$ where $\mathcal{A}$ is an object of $\mathrm{Alg}^\star(R)$ and $E : \mathcal{A} \to B$ is a morphism of bimodules. We choose a monoidal structure $\otimes$ on the category Prob(R). At that point, we should make two restrictive assumptions. First, the underlying algebra $(A, \phi_A) \otimes (B, \phi_B)$ is supposed to be the free product $A \sqcup B$ with $A, B \in \mathrm{Prob}(R)$ and secondly, $\otimes$ is assumed to be symmetric with commutativity constraint maps given by $\tau_{A\otimes B} : A \otimes B \to B \otimes A$ (the canonical maps equal to identity on both A and B). As a result of these last two assumptions, any permutation of a sequence of mutually independent morphisms is a sequence of mutually independent morphisms. Also, the independence morphism associated with a sequence of independent morphisms is given by the free product of these morphisms. Finally, we choose a Zhang algebra H of $\mathrm{Alg}^\star(B)$.

Definition 18. (Probabilistic holonomy fields on a Zhang algebra) An H-algebraic holonomy field is a group homomorphism $\mathsf{H} : \mathsf{RL}_{\mathrm{Aff},0}(\mathbb{R}^2) \to \mathrm{Hom}(H, \mathcal{A})$ that satisfies the following three properties:

(1) (Gauge invariance) Let $l_1, \ldots, l_n \in \mathsf{L}_{\mathrm{Aff},0}(\mathbb{R}^2)$ a finite sequence of loops. For all morphism $\phi_H : H \to k$ of $\mathrm{Hom}(\mathcal{D})$:

$$(\phi_H \otimes \phi_{\mathcal{A}}) \circ (\mathrm{id}_B \sqcup \mathsf{H}_{\ell_1,\ldots,\ell_1})) \circ \Omega_c^n = \phi_{\mathcal{A}} \circ \mathsf{H}_{\ell_1,\ldots,\ell_n}.$$

(2) (Independence) If $(l_1, \ldots, l_n)$ and $(l'_1, \ldots, l'_m)$ are two finite sequences of loops such that $\bigcup_{i=1}^n \mathrm{Int}(l_i)$ and $\bigcup_{j=1}^m \mathrm{Int}(l'_j)$ are disjoint, then $\mathsf{H}_{l_1} \dot\sqcup \ldots \dot\sqcup \mathsf{H}_{l_n}$ and $\mathsf{H}_{l'_1} \dot\sqcup \ldots \dot\sqcup \mathsf{H}_{l'_m}$ are $\otimes$-independent.

(3) (Invariance by area-preserving homeomorphisms) For all area-preserving diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and all loops $(l_1, \ldots, l_n)$, we have the equality $E \circ (\mathsf{H}_{l_1} \dot\sqcup \ldots \dot\sqcup \mathsf{H}_{l_n}) = E \circ (\mathsf{H}_{\phi(l_1)} \dot\sqcup \ldots \dot\sqcup \mathsf{H}_{\phi(l_n)})$.

This generalisation of the usual notion of a gauge field, although already wide enough to encompass classical gauge fields with compact structure groups as well as their large N limits — the so-called master fields — is set in a context that is less general than our definition of a Zhang algebra. Indeed, in this definition, we work on the particular algebraic category $\mathrm{Alg}^\star(B)$. It is not much more difficult to extend the definition to a setting where the algebraic category in which we take our probability spaces is almost arbitrary.

Let $(\mathcal{C}, \sqcup_{\mathcal{C}}, k)$ be an algebraic category with initial object k and $(\mathcal{D}, \sqcup_{\mathcal{D}}, k)$ a second category and $F : \mathcal{C} \to \mathcal{D}$ a faithfull (injective on the homorphisms classes)and wide (surjective on the class of objects) functor of categories. To put it in words, $\mathcal{D}$ is an enlargement of the category $\mathcal{D}$: the classes of objects are the same whereas the homomorphisms classes between O and O' are larger if these objects belong to $\mathrm{Obj}(\mathcal{D})$. Morphisms of the category $\mathcal{D}$ with target spaces the initial object k are to be seen as distributions or states. In fact, define $\mathcal{B}$ as the category which objects are pairs (O, ϕ_O) with O an object of $\mathcal{C}$ and ϕ a morphism of $\mathcal{D}$ from $F(O)$ to the initial object $k \in \mathcal{D}$. A morphism between two objects (O, ϕ_O) and $(O', \phi_{O'})$ of $\mathcal{B}$ is a morphism $f : O \to O'$ of $\mathcal{C}$ such that $\phi_{O'} \circ F(f) = \phi_O$.

Let $(\mathcal{A}, \phi_{\mathcal{A}})$ an object of $\mathcal{B}$. Let $B \in \mathcal{B}$. A morphism $f : A \to B$ of the category $\mathcal{C}$ defines a morphism, denoted $\hat{f}$, of the category $\mathcal{B}$ with source $(A, \phi_B \circ F(f))$ and target space (B, ϕ_B).

We assume $\mathcal{B}$ to be endowed with a symmetric monoidal structure with injections $(\otimes, E)$ with E an initial object of $\mathcal{B}$. Denote by P_1 the obvious functor from $\mathcal{B}$ to $\mathcal{C}$. We assume this functor to conserve the monoidal structure of $\mathcal{B}$ and the product structure of $\mathcal{C}$:

$$P_1((A, \phi_A) \otimes (B, \phi_B)) = A \sqcup B, A, B \in \mathrm{Obj}(\mathcal{C}).$$

Let $\mathcal{A} = (A, \phi_{\mathcal{A}}) \in \mathcal{B}$ with $\mathcal{A}$ a H-comodule of $\mathcal{C}$, $\mathcal{A} \in \mathrm{lCoMod}\mathcal{C}(H)$ and denote by $\overline{\Omega}$ the coaction. Furthermore, let H a Zhang algebra of $\mathcal{C}$. In the following definition, we use the shorter notation $A_{\ell_1,\dots,\ell_n}$ for the free products of a sequence $(A_{\ell_1}, \dots, A_{\ell_n})$ of morphisms from H to $\mathcal{A}$.

Definition 19. (Categorical holonomy field on a Zhang algebra) An H-categorical holonomy field is a group homomorphism $\mathsf{H} : \mathsf{RL}_{\mathrm{Aff},0}(\mathbb{R}^2) \to \mathrm{Hom}_{\mathrm{lCoMod}\mathcal{C}(H)}((H, \Omega_c), \mathcal{A})$ that satisfies the following three properties:

1. (Gauge invariance) Let $\ell_1, \dots, \ell_n \in \mathsf{L}_{\mathrm{Aff},0}(\mathbb{R}^2)$ a finite sequence of loops. For all morphism $\phi_H : H \to k$ of $\mathrm{Hom}(\mathcal{D})$:
$$(\phi_H \otimes \phi_{\mathcal{A}}) \circ (\mathrm{id}_B \sqcup \mathsf{H}_{\ell_1,\dots,\ell_1})) \circ \Omega_c^n = \phi_{\mathcal{A}} \circ \mathsf{H}_{\ell_1,\dots,\ell_n}.$$
2. (Independence) If $(\ell_1, \dots, \ell_n)$ and $(\ell'_1, \dots, \ell'_m)$ are two finite sequences of loops such that $\bigcup_{i=1}^n \mathrm{Int}(l_i)$ and $\bigcup_{j=1}^m \mathrm{Int}(\ell'_j)$ are disjoint, then $\mathsf{H}_{l_1} \dot{\sqcup} \dots \dot{\sqcup} \mathsf{H}_{\ell_n}$ and $\mathsf{H}_{\ell'_1} \dot{\sqcup} \dots \dot{\sqcup} \mathsf{H}_{\ell'_m}$ are $\otimes$-independent.
3. (Invariance by area-preserving homeomorphisms) For all area-preserving diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and all loops $(\ell_1, \dots, \ell_n)$, we have the equality $\phi_{\mathcal{A}} \circ F(\mathsf{H}_{\ell_1} \dot{\sqcup} \dots \dot{\sqcup} \mathsf{H}_{\ell_n}) = \phi_{\mathcal{A}} \circ F(\mathsf{H}_{\phi(\ell_1)} \dot{\sqcup} \dots \dot{\sqcup} \mathsf{H}_{\phi(\ell_n)})$.

In the next section we construct a categorical holonomy field that satisfies a strengthened gauge invariance property; in fact the left comodule $\mathcal{A}$ is built as an inductive limit and the morphism $\phi_{\mathcal{A}}$ is gauge invariant, which means
$$(\phi_H \otimes \phi_{\mathcal{A}}) \circ \overline{\Omega}_c = \phi_{\mathcal{A}}, \text{ for all } \phi_H \in \mathrm{Hom}_D(H, k).$$
This last equation trivially implies gauge invariance property 1.

In Definition 19, we use the co-product structure $\sqcup$ of the category $\mathcal{C}$ to state what gauge invariance and invariance by area preserving homeomorphisms of the plane means. This is not compulsory as we should see now and further in the next section on the course of defining a categorical holonomy field.

Let $\ell_1, \dots, \ell_p$ a finite sequence of loops and $c_1, \dots, c_n$ a family of lassos that have disjoint bulks and such that $\ell_1, \dots, \ell_n \in \mathsf{RL}_{\mathrm{Aff},0}(c_1, \dots, c_n)$, we will see in the next section how such family is obtained. We claim that property 1 for the sequence $(\ell_1, \dots, \ell_n)$ can be obtained from the fact that **??** holds for the sequence of lassos $(c_1, \dots, c_n)$.

In fact, let w a word in M_n. We denote by m^w the unique morphism from the free product $H_w = H_{w_1} \sqcup \dots \sqcup H_{w_n}$ to $H_1 \sqcup \dots \sqcup H_n$ such that $m^w \circ (\iota^{H_w}_{H_{w_k}} = \iota^{H_1 \cdots H_n}_{H_{w_k}})$.

$$\begin{aligned}
&(\phi_H \otimes \phi_{\mathcal{A}}) \circ (\mathrm{id}_H \sqcup (\mathsf{H}_{w_1(c)} \dot{\sqcup} \mathsf{H}_{w_2(c)} \dot{\sqcup} \dots \dot{\sqcup} \mathsf{H}_{w_p(c)})) \circ \Omega \\
&\quad = (\phi_H \otimes \phi_{\mathcal{A}}) \circ (\mathrm{id}_H \sqcup (\mathsf{H}_{c_1} \dot{\sqcup} \dots \dot{\sqcup} \mathsf{H}_{c_n})) \circ (\mathrm{id}_H \sqcup m^{w_1 \cdots w_p}) \circ \Omega_c^{p_1+\dots+p_n} \circ \Delta^{p_1+\dots+p_n-1} \\
&\quad = (\phi_H \otimes \phi_{\mathcal{A}}) \circ (\mathrm{id}_H \sqcup (\mathsf{H}_{c_1} \dot{\sqcup} \dots \dot{\sqcup} \mathsf{H}_{c_n})) \circ \Omega_c^{p_1+\dots+p_n} \circ m^{w_1 \cdots w_p} \circ \Delta^{p_1+\dots+p_n-1} \\
&\quad = \phi_{\mathcal{A}} \circ \mathsf{H}_{c_1} \dot{\sqcup} \dots \dot{\sqcup} \mathsf{H}_{c_n} \circ m^{w_1 \cdots w_p} \circ \Delta^{p_1+\dots+p_n-1} = \phi_{\mathcal{A}} \circ (\mathsf{H}_{\ell_1} \dot{\sqcup} \dots \dot{\sqcup} \mathsf{H}_{\ell_p}).
\end{aligned}$$

We leave to the reader the verification that invariance by are-preserving homeomorphisms property 3 and independence property 2 hold for any sequence of affine loops if it holds for all sequence of affine lassos. In short, properties 1. – 3. of Definition 19 are equivalent to:

1′. (Independence) If $(c_1, \dots, c_n)$ is a finite sequences of lassos with two by two disjoint bulks, $\mathsf{H}_{c_1}, \dots, \mathsf{H}_{c_n}$ is a mutually independent family of morphisms.
2′. (Gauge invariance) Let $c_1, \dots, c_n \in \mathsf{L}_{\mathrm{Aff},0}(\mathbb{R}^2)$ a finite sequence of lassos with disjoint bulks. For all morphism $\phi_H : H \to k$ of $\mathrm{Hom}(\mathcal{D})$:
$$(\phi_H \otimes \phi_{\mathcal{A}}) \circ (\mathrm{id}_B \otimes \mathsf{H}_{c_1} \otimes \dots \otimes \mathsf{H}_{c_n}) \circ \Omega_c^n = \phi_{\mathcal{A}} \circ (\mathsf{H}_{c_1} \otimes \dots \otimes H_{c_n}).$$
3′. (Invariance by area-preserving homeomorphisms) For all area-preserving diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and all lassos with disjoint interiors $(c_1, \dots, c_n)$, we have the equality $\phi_{\mathcal{A}} \circ F(\mathsf{H}_{c_1} \otimes \dots \otimes \mathsf{H}_{c_n}) = \phi_{\mathcal{A}} \circ F(\mathsf{H}_{\phi(c_1)} \otimes \dots \otimes \mathsf{H}_{\phi(c_n)})$.

The distribution of a categorical holonomy field H is the collection $\{\Phi_\ell^{\mathsf{H}},\ \ell \in \mathsf{RL}_{\mathrm{Aff},0}(\mathbb{R}^2)\}$ of morphisms from H to k defined by:

$$\Phi_\ell^{\mathsf{H}} = \tau \circ \mathsf{H}_\ell,\ \ell \in \mathsf{RL}_{\mathrm{Aff},0}(\mathbb{R}^2). \tag{8}$$

We denote by $\star_\otimes$ the product on the space homomorphisms from H to k in the category $\mathcal{D}$ an defined by:

$$\alpha \star_\otimes \beta = (\alpha \otimes \beta) \circ \Delta,\ \alpha, \beta \in \mathrm{Hom}_{\mathcal{D}}(H, k).$$

Properties 1.–3. of Definition 19 for the categorical holonomy field implies the following ones for its distribution:

1. $\Phi_{\ell_1\ell_2\ell_1^{-1}} = \Phi_{\ell_2},\ \Phi_{\ell_1^{-1}} = \Phi_{\ell_1} \circ S$ for all $\ell_1, \ell_2 \in \mathsf{RL}_{\mathrm{Aff},0}(\mathbb{R}^2)$,
2. $\Phi_{\ell_1\ell_2} = \Phi_{\ell_1} \star_\otimes \Phi_{\ell_2}$, for all simple loops $\ell_1, \ell_2 \in \mathsf{RL}_{\mathrm{Aff},0}(\mathbb{R}^2)$ with disjoint interiors,
3. $\Phi_\ell = \Phi_{\psi(\ell)}$ for all area preserving homeomorphisms ψ of $\mathbb{R}^2$.

Let us now draw a comparison between the definition of a categorical Lévy process as given in [4] and Definition 19. Let γ and γ_1 be two reduced loops. We write $\gamma \prec \gamma_1$ if $\gamma_1 = \gamma\ell$ for a certain loop ℓ such that closure of interiors of the loops ℓ and γ meet only at the origin. The relation $\prec$ is a preorder:

1. It is transitive: if $\gamma \prec \gamma_1,\ \gamma_1 \prec \gamma_2$ then $\gamma_2 = \gamma_1\ell = \gamma\ell\tilde{\ell}$ and $\overline{\mathrm{Int}}(\ell) \cap \overline{\mathrm{Int}}(\gamma_1) = \emptyset$, $\overline{\mathrm{Int}}(\tilde{\ell}) \cap \overline{\mathrm{Int}}(\gamma) = \emptyset$, we obtain $\overline{\mathrm{Int}}(\gamma) \cap \overline{\mathrm{Int}}(\ell\tilde{\ell}) = \emptyset$,
2. It is reflexive.

Let H an categorical holonomy field. For all loop $\gamma \in \mathsf{RL}_{\mathrm{Aff},0}$, set $H_\gamma = (H, \phi_A \circ \mathsf{Hol}_\gamma)$. The group $\mathsf{RL}_{\mathrm{Aff},0}$ is seen as category with class of objects the set of points $\mathsf{RL}_{\mathrm{Aff},0}$ and empty morphisms sets between two different loops. The system formed by the functor $\mathsf{L} : \mathsf{RL}_{\mathrm{Aff},0} \to \mathcal{B},\ \ell \mapsto H_\ell$ and the maps:

$$\Delta : H_{\gamma\gamma^1} \to H_\gamma \otimes H_{\gamma^1},\ h \mapsto \Delta(h),\ \delta : H \to k,\ h \mapsto \varepsilon(h)$$

is a comonoidal system. In addition, if we define $\mathsf{j}^{\mathsf{H}}_{\gamma,\gamma_1} : H_{\gamma^{-1}\gamma_1} \to (\mathcal{A}, \phi_{\mathcal{A}})$ with $\gamma \prec \gamma_1$ by $\mathsf{j}^{\mathsf{Hol}}_{\gamma,\gamma_1} = \mathsf{H}_{\gamma^{-1}\gamma_1}$, then $\mathsf{j}^{\mathsf{Hol}}$ is a categorical Lévy process:

1. $j_{\gamma,\gamma} = \eta_H \circ \delta$,
2. $\left(j_{\alpha_1,\beta_1}, \ldots, j_{\alpha_p,\beta_p}\right)$, $\alpha_p \preceq \beta_p$
3. $j^{\mathsf{Hol}}_{\gamma,\gamma_1} \otimes j^{\mathsf{Hol}}_{\gamma_1,\gamma_2} \circ \Delta_{\gamma^{-1}\gamma_1,\ \gamma_1^{-1}\gamma_2} = j^{\mathsf{Hol}}_{\gamma,\gamma_2}$.

4.3. **Construction of a categorical holonomy field from a direct system.** In that section, we explain how to construct a categorical holonomy field. Of course, we need initial datas that will be of two kinds. Our exposition is divided into three steps. First, we show how to construct a morphism H from the group of reduced loops $\mathsf{RL}_{\mathrm{Aff},0}$ into the group of homomorphisms from a Zhang algebra in $\mathcal{C}$ to a certain object of the category $\mathrm{lCoMod}\mathcal{C}(H)$, starting from a direct system of objects and a projective system of morphisms. We continue our exposition by showing how to construct a categorical holonomy field that have property 2, 1 and 3 of Definition 19.

Our settings is the one of Definition 19. In addition to the hypothesis on the three categories $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ we made, we add two more of them:

1. The categories $\mathcal{C}$ and $\mathcal{D}$ are inductively complete.
2. The tensor product $\otimes$ on $\mathcal{B}$ is symmetric and we denote by τ the commutativity constraint:

$$\tau_{A,B} : (A \otimes B, \tau_{A\otimes B}) \to (B \otimes A, \tau_{B\otimes A}),\ \tau_{B,A} \circ \tau_{A,B} = \mathrm{id}_{A\otimes B},\ \tau_{A,B} \circ \tau_{B,A} = \mathrm{id}_{B\otimes A}.$$

The set of finite sequences of loops is an upward directed set as we saw in the previous section. We recall that $\mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)$ denotes the set of finite sequences of affine loops. For a finite sequence of loops $F = (l_1, \ldots, l_p)$, we use the short notation $\{F\}$ for the set $\{l_1, \ldots, l_p\}$.

We focus now on the construction of the morphism H of definition 19, regardless of the property $1-3$. We fix a direct system A of the category $\mathcal{B}$. It means that

1. $\mathsf{A}((L',L)) : \mathsf{A}(L) \to \mathsf{A}(L')$ satisfies $E_{\mathsf{A}(L')} = E_{\mathsf{A}(L)} \circ \mathsf{A}((L',L))$ for all finite sequences of loops $L' \succ L$,
2. $\mathsf{A}((L'',L')) \circ \mathsf{A}((L',L)) = A((L'',L))$ for all finite sequences of loops L, L', L'',
3. $\mathsf{A}((L,L)) = \mathrm{id}_L$ for all finite sequence $L \in \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\left(\mathbb{R}^2\right)\right)\right)$.

Below, we explain how to construct a direct system $\mathsf{A} : \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right) \to \mathcal{B}$ starting from a direct system from a subcategory $S \subset \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)$ to $\mathcal{B}$ by using the monoidal structure of $\mathcal{B}$.

We denote by $\left((\mathcal{A},E),j\right)$ the inductive limit of the direct system A in the category $\mathcal{B}$.

Let L and L' two finite sequences of affine loops of length $p \geq 1$ such that $\{L\} = \{L'\}$. The group of permutations of $[1,\ldots,p]$ is denoted $\mathcal{S}_p$. We recall that a group G defines a category whose objects are the points of G and the space of morphisms $\mathsf{Hom}(g,h)$ between two different elements g,h in G has an unique element: $L_{gh^{-1}}$ with $L_{gh^{-1}} : G \to G$ the left multiplication by gh^{-1}. In the following definition, $\mathcal{S}_p$ is seen as a category.

Definition 20. We call *commutativity constraint* of the direct system A the functors $\gamma^L_{\mathsf{A}},\ L \in \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)$ defined by, for all permutations $\sigma,\tau \in \mathcal{S}_p$ and $L \in \mathsf{L}_{\mathrm{Aff},0}$:

$$\gamma^L_{\mathsf{A}}(\sigma) = \left(\mathsf{A}(\sigma\cdot L), E_{\mathsf{A}(L)}\right), \quad \sigma\cdot\tau^{-1} \in \mathsf{Hom}(\tau,\sigma),\ \gamma^F_{\mathsf{A}}(\sigma\cdot\tau^{-1}) = \mathsf{A}((\sigma\cdot L, \tau\cdot L)). \tag{9}$$

Owing to properties 2. and 3. satisfied by the family of morphisms $\{\mathsf{A}((L,L')), L \prec L' \in \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)\}$, the equation (9) does define a co-variant functor from S_p to $\mathcal{B}$.

For a finite sequence of loops L, we denote by $\mathsf{RL}\langle L\rangle \subset \mathsf{RL}(\mathbb{R}^2)$ the subgroups of reduced loops that are concatenation (and reduction) of loops in L. For two finite sequences of loops $L \prec L'$, we define the map $\phi_{L'L}$ as:

$$\phi_{L'L} : \begin{array}{ccc} \mathsf{RL}\langle L\rangle & \to & \mathsf{RL}\langle L'\rangle \\ \ell & \mapsto & \ell \end{array}.$$

The functor $\mathsf{L} : \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}(\mathbb{R}^2)\right) \to \mathcal{G}rp$ defined by: $\mathsf{L}(L) = \mathsf{RL}\langle L\rangle$ and $\mathsf{L}((L,L')) = \phi_{L',L}$ is a direct system. The family of morphisms $\left(\phi_{L',L}\right)_{L\prec L' \in \mathcal{P}_{\mathrm{Aff},0}(\mathbb{R}^2)}$ enjoys the trivials, yet important, two following properties. Let $L,\ L'$ be two finite sequences of affine loops with $L \prec L'$, then:

$$\phi_{L,L'} = \phi_{\alpha\cdot L,\beta\cdot L'},\ \alpha \in \mathcal{S}_{\sharp L},\ \beta \in S_{\sharp L'}.$$

Also, if $L_1 \prec M_1$ and $L_2 \prec M_2$ are four finite sequences of affine loops, one has:

$$\phi_{M_1,L_1}(\ell) = \phi_{M_2,L_2}(\ell),\ \ell \in \mathsf{RL}\langle L_1\rangle \cap \mathsf{RL}\langle L_2\rangle.$$

We remind the reader with the fact that the set of affine loops drawn on the plane is the direct limit of the direct functor L:

$$\mathsf{RL}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right) = \varinjlim \mathsf{L}.$$

We explain how to construct the morphism H from definition 19 starting from a family $\{\mathsf{H}_L, L \in \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)\}$ of homomorphisms of $\mathcal{C}$ with, for each finite sequence of loops $L \in \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)$, $\mathsf{H}_L \in \mathsf{Hom}_{\mathcal{C}}\left(\mathsf{RL}\langle L\rangle,\ \mathsf{Hom}_{\mathcal{C}}(H,\mathsf{A}(L))\right)$.

Set $\overline{\mathsf{H}}_L = j_L \circ \mathsf{H}_L\ \left(L \in \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)\right)$ and assume that the following compatibility relation holds:

$$\overline{\mathsf{H}}_L = \overline{\mathsf{H}}_{L'} \circ \phi_{L',L} \tag{10}$$

Note that this last relation is implied by the following one on the family $\{\mathsf{H}_L, L \in \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)\}$:

$$\mathsf{A}((L',L)) \circ \mathsf{H}_L = \mathsf{H}_{L'} \circ \mathsf{L}((L',L)),\ L,L' \in \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}(\mathbb{R}^2)\right).$$

From the universal property of the direct limit of L and equation (10), there exists a morphism H from $\mathsf{RL}(\mathbb{R}^2)$ into $\mathrm{Hom}_{\mathcal{C}}(H,\mathcal{A})$ such that the diagram in Fig. 28 is commutative.

That it: from the two intial datas of the direct system A and projective family H we constructed a morphism $\mathsf{H} \in \mathrm{Hom}_{\mathcal{G}rp}(\mathsf{RL}(\mathbb{R}^2), \mathrm{Hom}(H,\mathcal{A}))$. Now, we focus on the question of constructing a

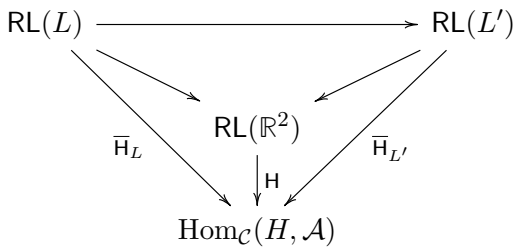


FIGURE 28. The holonomy H obtained as a solution of an universal problem.

direct system A such that property 2 of Definition 19 holds for the morphism H constructed as above.

The set, denoted lassos, of anti-clockwise oriented lassos drawn on the plane has the property that for a finite sequence F of loops drawn on the plane, there exists a set $S_F \subset \mathsf{lassos}$ such that $\mathsf{RL}(S_F) = \mathsf{RL}(F)$. In fact, two loops in F have finite self intersections, one can thus built a graph out of the set of loops F : the vertices are the intersection points and the oriented edges are the segments of the loops in F that connect two intersection points. We denote by $\mathbb{G}_F$ that graph. One can pick for S_F a set of lassos starting at the origin and surrounding, one time and in anticlockwise manner, a face of that graph and each face is the bulk of a lasso in S_F. Note that lassos in S_F have disjoint bulks.

For each anti-clockwise oriented lasso c, let E_c a morphism in the category $\mathcal{D}$ from $F(H)$ to the initial object k and set $H_c = (H, E_c) \in \mathcal{B}$. Furthermore, let $(c_1, \dots, c_p)$ a finite sequence of lassos in $\overset{\circ}{\mathcal{P}}(\mathsf{lassos})$ and define an object $\left(H_{(c_1,\dots,c_p)}, E_{(c_1,\dots,c_p)}\right)$ of the category $\mathcal{B}$ by:

$$(H_{(c_1,\dots,c_p)}, E_{(c_1,\dots,c_p)}) = (H_{c_1}, E_{c_1}) \otimes \cdots \otimes (H_{c_p}, E_{c_p}) = (H_{c_1} \sqcup \cdots \sqcup H_{c_p}, E_{c_1} \otimes \cdots \otimes E_{c_p}). \tag{11}$$

For each integer $1 \leq i \leq p$, denote by $\iota_{c_i}^{(c_1,\dots,c_p)}$ the canonical morphism from c_i to the tensor product $\otimes$ on $\mathcal{B}$ (we use the same style of notation, the greek letter ι for these morphisms because they are equal to the canonical injections associated with the co-product on $\mathcal{C}$) and define the morphism $\mathsf{H}_{(c_1,\dots,c_p)} : \mathsf{RL}\langle(c_1, \dots, c_p)\rangle \to \mathrm{Hom}_{\mathcal{C}}(H, H_{(c_1,\dots,c_p)})$ by:

$$\mathsf{H}_{(c_1,\dots,c_p)} = i_{c_1}^{(c_1,\dots,c_p)} \times \cdots \times i_{c_p}^{(c_1,\dots,c_p)}. \tag{12}$$

In the last equation, the symbol $\times$ denotes products of morphisms in the group $H_{(c_1,\dots,c_p)}$,

$$A \times B = A \dot{\sqcup} B \circ \Delta, \ A, B \in \mathrm{Hom}_{\mathcal{G}rp}(H, H_{(c_1,\dots,c_p)}).$$

In the sequel, if $(l_1, \dots, l_q)$ is a finite sequence of loops in $\mathsf{RL}\langle(c_1, \dots, c_p)\rangle$, we use the shorter notation $\mathsf{H}_{c_1,\dots,c_p}(\ell_1, \dots, \ell_p) : H_{\ell_1} \sqcup \cdots \sqcup H_{\ell_p} \to H_{(c_1,\dots,c_p)}$ with

$$\mathsf{H}_{c_1,\dots,c_p}(\ell_1, \dots, \ell_p) = \mathsf{H}_{c_1,\dots,c_p}(\ell_1) \dot{\sqcup} \cdots \dot{\sqcup} \mathsf{H}_{c_1,\cdots,c_p}(\ell_p).$$

The morphism H is the holonomy morphism of the family $\ell_1, \dots, \ell_p$ in $(c_1, \dots, c_p)$.

At that point, starting from a family of morphisms E indexed by anticlockwise lassos drawn on the plane, we constructed a family of objects in $\mathcal{B}$ indexed by finite sequences of lassos and a family H of morphisms.

We set $E_{(\ell_1,\dots,\ell_q)}^{c_1,\dots,c_p} = E_{(c_1,\dots,c_p)} \circ \mathsf{H}_{(c_1,\dots,c_p)}(\ell_1, \dots, \ell_p)$. In the classical case, the morphism $E_{\ell_1,\dots,\ell_p}^{(c_1,\dots,c_p)}$ is the distribution of the holonomies of loops $\ell_1, \dots, \ell_p$ in $(c_1, \dots, c_p)$. We emphasized the dependence of $E_{(\ell_1,\dots,\ell_q)}^{(c_1,\dots,c_p)}$ toward the set of lassos we picked initially, otherwise stated, toward the basis of the group of reduced loops $\mathsf{RL}\langle(c_1, \dots, c_p)\rangle$.

In the next part, we are concerned with making the morphisms $E^{(c_1,\dots,c_p)}_{(\ell_1,\dots,\ell_p)}$ independent from $(c_1,\dots,c_p)$, with obvious notations. To that end, we need to assume two more properties on the morphisms $E_c,\ c\in\mathsf{lassos}$.

To construct, for each finite sequence $F\in\mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)$, an object H_F of $\mathcal{B}$, we pick first an enumeration $(f_1,\dots,f_p)$ of the faces of the graph $\mathbb{G}_F$ and anticlockwise oriented lassos $(c_1,\dots,c_p)$ based at 0 that surround the faces $f_1,\cdots,f_p$. It would be natural to define:

$$(H_F,H_F)=\left(H_{\ell_1}\sqcup\cdots\sqcup H_{\ell_p},\ E_F^{c_1,\dots,c_p}\right)$$

In order for the state $E_F^{(c_1,\dots,c_p)}$ to not depend on a choice of a basis for $\mathsf{RL}(\mathbb{G}_F)$, the morphisms $(E_c)_{c\in\mathsf{lassos}(\mathbb{R}^2)}$ has to satisfy invariance properties that are stated in the next definition. We should first, motivate further their introduction. Two sequences of lassos with disjoint interiors that generate the same group of reduced loops can be characterized as follows. To a braid $\beta\in\mathcal{B}_n$, is associated a permutation σ_β, it is defined on an elementary braid $\langle i,j\rangle$ as $\sigma_{\langle i,j\rangle}=(i,j)$ for all $i<j\leq n$ and extended to $\mathcal{B}_n$ as a morphism from $\mathcal{B}_n$ to $\mathcal{S}_n$.

Proposition 21. *Let $(c_1,\dots,c_n)$ and $\left(c'_1,\dots,c'_p\right)$ be two families of lassos. For each family, we assume that the interiors of the bulks are pairwise distinct and that $\mathsf{RL}\langle c_1,\dots,c_n\rangle=\mathsf{RL}\langle c'_1,\dots,c'_p\rangle$. Then $n=p$ and it exists a braid $\beta\in\mathcal{B}_n$ such that*

$$\sigma_\beta\cdot(c'_1,\cdots,c'_n)=\beta\cdot(c_1,\cdots,c_n).$$

Proof. The proposition is a consequence of a theorem of Artin, see for example, [11]. □

This last proposition motivates the first point of Definition 22. For the second point, we consider a simple situation. Let c be a lasso that can be written as the product of two lassos $c=c_1\cdot c_p$. It is natural to ask for the morphism E_c to be equal to the holonomie's distribution of c in c_1,c_p (once again in comparison with the classical case), i.e $E_c=E^{(c)}_{(c_1,c_p)}$. This is the meaning of equation (14) in definition 22.

Definition 22. (1) The family $(H_c,E_c)_{c\in\mathsf{lassos}(\mathbb{R}^2)}$ is said to be *braid invariant* if for any sequence $(c_1,\dots,c_p)$ of lassos with disjoint interiors, one has

$$E_{\beta\cdot c}=E_{c_{\beta(1)},\dots,c_{\beta(n)}}\circ\mathsf{H}_{\sigma_\beta\cdot c}(\beta\cdot c),\qquad\text{for any braid }\beta\in\mathcal{B}_n. \tag{13}$$

(2) With the notations above, a family of morphisms $(E_c)_{c\in\mathsf{lassos}(\mathbb{R}^2)}$ on H is *infinitely divisible* if for any pair of lassos (c_1,c_2) with disjoint interiors, we have:

$$E_{c_1\cdot c_2}=E_{c_1,c_2}\circ\mathsf{H}_{(c_1,c_2)}(c_1c_2)\quad\text{ if }\quad c_1c_2\quad\text{ is a lasso.} \tag{14}$$

From now on, we assume the family $(H_c,E_c)_{c\in\mathsf{lassos}}$ to be braid invariant and infinitely divisible.

Proposition 23. *With the notations introduced so far, let $L=(\ell_1,\dots,\ell_q)$ a finite sequence of loops. Let $(c_1,\dots,c_p)$ and $(c'_1,\dots,c'_q)$ two sequences of lassos in $\overset{\circ}{\mathcal{P}}(\mathsf{lassos})$ such that $L\subset\mathsf{RL}\langle(c_1,\dots,c_p)\rangle\cap\mathsf{RL}\langle(c'_1,\dots,c'_q)$, then $E_L^{(c_1,\dots,c_p)}=E_L^{(c'_1,\dots,c'_q)}$.*

In addition, if $L'\succ L$ is a second finite sequence of loops, we have $\mathsf{H}_{\ell'_1,\dots,\ell'_q}(\ell_1,\dots,\ell_p):(H_L,E_L)\to(H_{L'},E_{L'})$.

Proof. Let L and L' be two finite sequences of loops, with $L\subset L'$, $L=(l_1,\cdots,l_p)$ and $L'=\left(\ell'_1,\ell'_2,\cdots,\ell'_{p'}\right)$. Let $(c_1,\dots,c_q)$ be a sequence of lassos in $\overset{\circ}{\mathcal{P}}(\mathsf{lassos})$ that is a basis of $\mathsf{RL}(\mathbb{G}_{L'})$. Assume that the first point of proposition 23 holds, $E_L=E_L^{(c_1,\dots,c_q)}$ and $E_{L'}=E_{L'}^{(c_1,\dots,c_q)}$. We aim at proving the equality:

$$E_{L'}^{(c_1,\dots,c_p)}\circ\mathsf{H}_{L'}(L)=E_L^{(c_1,\dots,c_p)}. \tag{$\star$}$$

Because of functorial properties of the tensor product on $\mathcal{B}$, $\mathsf{H}_{c_1,\dots,c_p}(L')\circ\mathsf{H}_{L'}(L)=\mathsf{H}_{c_1,\dots,c_p}(L)$. Hence, from the definition of the morphism $E_{L'}^{(c_1,\dots,c_p)}$, equation $(\star)$ is equivalent to:

$$E_{c_1,\dots,c_p}\circ\mathsf{H}_{(c_1,\dots,c_p)}(L)=E_L^{(c_1,\dots,c_p)}.$$

We prove now that the morphism $E_L^{c_1,\dots,c_p}$ does not depends on the sequences of lassos $(c_1,\dots,c_p)$ we pick, provided we have $L\in\mathsf{RL}\langle(c_1,\dots,c_q)\rangle$. Let $f_1,\dots,f_q$ be an enumeration of the faces of $\mathbb{G}_L$ and pick lassos $(c_1,\dots,c_p)$ that surrounds (resp.) the faces $f_1,\dots,f_p$. First, since we assumed that the tensor product is symmetric, $E_{c_1,\dots,c_p}$ does not depends on the enumeration of the faces we chose. Then, if β is a braid,

$$\begin{aligned}E_L^{\beta\cdot(c_1,\dots,c_p)}&=E_{\beta\cdot(c_1,\dots,c_p)}\circ\mathsf{H}_{\beta\cdot(c_1,\dots,c_p)}(\ell_1,\dots,\ell_q)\\&=E_{c_{\beta(1)},\dots c_{\beta(q)}}\circ\mathsf{H}_{\sigma_\beta\cdot c}(\beta\cdot c)\circ\mathsf{H}_{\beta\cdot c}(\ell_1,\dots,\ell_p)=E_{c_{\beta(1)},\dots,c_{\beta(n)}}\circ\mathsf{H}_{\sigma_\beta\cdot c}(\ell_1,\dots,\ell_q)\\&=E_L^{c_{\beta(1)},\dots,c_{\beta(n)}}=E_L^{c_{\beta(1)},\dots,c_{\beta(n)}}=E_L^{c_1,\dots,c_n}.\end{aligned}$$

In conclusion, we can pick any enumeration and basis of $\mathsf{RL}(\mathbb{G}_F)$ to compute $E_L^{(c_1,\dots,c_p)}$. Let $C'=(c'_1,\dots,c'_p)$ a finite sequence such that $L\subset\mathsf{RL}\langle(c'_1,\dots,c'_p)\rangle$.

The graph $\mathbb{G}_{C'}$ is finer than the graph $\mathbb{G}_L$ and can thus be obtained by iterative application of two transformations, starting from the graph $\mathbb{G}_L$:

1. adding a vertex on an edge,
2. connecting two vertices.

Let $\mathbb{G}_L\prec\mathbb{G}_1\prec\cdots\prec\mathbb{G}_n\prec\mathbb{G}_{C'}$ be a sequence of graphs obtained by successive applications of the transformations 1 and 2: $\mathbb{G}_{i+1}$ is obtained from $\mathbb{G}_i$ by one of the transformation 1, 2. Next, we define inductively a sequence of objects $(H^{(i)})_{0\leq i\leq n+1}$ in the category $\mathcal{B}$. We first define inductively a sequence $(c^{(1)},\dots,c^{(n+1)})$ with $c^{(i)}\in\mathring{\mathcal{P}}(\mathsf{lassos})$ such that for each $i\leq n+1$, $c^{(i)}$ is a basis of $\mathsf{RL}\langle\mathbb{G}_i\rangle$. Put $c^{(n)}=(c'_1,\dots,c'_p)$. If $\mathbb{G}_{i+1}$ is obtained by adding a vertex to $\mathbb{G}_i$, the groups of loops drawn on $\mathbb{G}^{i+1}$ is equal to the group of loops drawn on $\mathbb{G}^{(i)}$. In that case we set $c^i=c^{i+1}$. If a face f of G_i is cut into two faces f_1f_2 to obtain the graph $\mathbb{G}_{i+1}$, we obtain a basis $c^{(i)}$ of $\mathsf{RL}(\mathbb{G}_i)$ by extracting the lassos in the basis $c^{(i+1)}$ that surround nor the face f_1 nor the face f_2 and doing the product of the two lassos that surround the faces f_1, f_2 (in any order). Next we define, inductively, the sequence of objects in $\mathcal{B}$ by the equations:

$$uH^{(0)}=(H_L,E_L),\ H^{(i+1)}=\left(H_L,\ E_{c^{(i)}}\circ\mathsf{H}_{c^{(i)}}(\ell_1,\dots,\ell_p)\right). \tag{15}$$

Let $i\leq n$ an integer such that $\mathbb{G}^{(i+1)}$ is obtained by cutting a face of $G^{(i)}$ in two. We claim that the diagram in Fig. 29 is commutative diagram of morphisms in $\mathcal{C}$. In Fig. 29, the blue arrows are morphism of $\mathcal{B}$. The upper arrow in Fig. 29 is thus painted in blue owing to the infinite divisibility of the morphisms E_c, $c\in\mathsf{lassos}$.

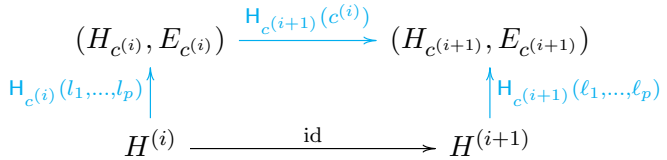


FIGURE 29.

From Fig. 29, the sequence of morphism $(E_i,i\leq n)$ is thus constant which leads readily to the conclusion, since $E_L^{(c_1,\dots,c_p)}=E_0=E_{n+1}=E_L^{(c'_1,\dots,c'_p)}$. □

Proposition 23 shows that the functor:

$$\mathsf{A}(L) = (H_L, E_L),\ \mathsf{A}(L', L) = \mathsf{H}_{L'}(L),\ L \prec L' \in \mathcal{P}\left(\mathsf{L}_{\mathrm{Aff},0}\left(\mathbb{R}^2\right)\right)$$

is well defined. We denote by $(\mathcal{A}, E)$ the inductive limit of A in $\mathcal{B}$. Properties satisfied by the tensor product $\otimes$ implies that $\left(\overline{\mathsf{H}}_L, \phi_{L,L'}\right)$ is a projective system. By construction, the property 2 holds for the morphism H constructed as above from A and $\left(\overline{\mathsf{H}}_L, \phi_{L,L'}\right)$. Assume further that each morphism $E_c,\ c \in \mathsf{lassos}$ is gauge invariant:

$$E_c \circ ((\mathsf{K} \dot{\sqcup} \mathrm{id}_H) \circ \Omega_c) = E_c,\ c \in \mathsf{lassos}.$$

for all morphism $K : H \to H_c$ such that $\hat{K}$ is independent from id_{H_c}. We prove first that the inductive limit $\mathcal{A}$ can be endowed with a co-action $\overline{\Omega}_c$ of the Zhang algebra H that makes the diagram in Fig. 30 commutative.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\overline{\Omega}_c} & H \sqcup \mathcal{A} \\ {\scriptstyle j^{\mathcal{A}}_{H_L}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}_H \sqcup j^{\mathcal{A}}_{H_L}} \\ H_L & \xrightarrow{\Omega^L_c} & H \sqcup H_L \end{array}$$

FIGURE 30.

For all pairs of loops $L \prec L'$, we saw that the holonomy $\mathsf{H}_{L'}(L)$ is gauge covariant, $\Omega^{L'}_c \circ \mathsf{H}_{L'}(L) = \mathrm{id}_H \dot{\sqcup} \mathsf{H}_{L'}(L) = \Omega^L_c$. Hence, the coaction $\overline{\Omega}_c : \mathcal{A} \to H \sqcup \mathcal{A}$ is defined by setting:

$$\overline{\Omega}_c([X_L]) = ((\mathrm{id}_H \sqcup j^{\mathcal{A}}_{H_L}) \circ \Omega_c)(X_L),\ [X_L] \in \mathcal{A}.$$

The map $\overline{\Omega}_c$ is well defined:

$$\begin{aligned} ((\mathrm{id}_H \sqcup j^{\mathcal{A}}_{H_{L'}}) \circ \Omega^{L'}_c \circ \mathsf{H}_{L'}(L))(X_L) &= (\mathrm{id}_H \sqcup (j_{H_{L'}} \circ \mathsf{H}_{L'}(L))) \circ \Omega^L_c)(E_L) \\ &= ((\mathrm{id}_H \sqcup \iota_{H_L}) \circ \Omega^L_c)(E_L). \end{aligned}$$

Property 1 is satisfied by any morphism E_L, $L \in \overset{\circ}{\mathcal{P}}(\mathsf{lassos})$, in fact, let $\phi_H : H \to k$ a morphism of $\mathcal{D}$. For all integer $1 \leq k \leq n$,

$$(\phi_H \otimes (E_{c_1} \otimes \cdots \otimes E_{c_n})) \circ (\mathrm{id}_H \sqcup i_k) \circ \Omega_c = E_{c_k} \tag{16}$$

Since the morphisms i_k are mutually independent, the morphisms $\iota_{c_i} \circ \Omega_{c_i} : H_{c_i} \to (H, \phi_H) \otimes H_{c_1} \otimes \cdots \otimes H_{c_n}$ are also mutually independent. Hence, owing to equation (16),

$$\Omega^{(n)}_c = \iota_{c_1} \circ \Omega_{c_1} \dot{\sqcup} \cdots \dot{\sqcup} \iota_{c_n} \circ \Omega_{c_n} \in \mathrm{Hom}_{\mathcal{B}}(H_{c_1} \otimes \cdots \otimes H_{c_n}, (H, \phi_H) \otimes H_{c_1} \cdots \otimes H_{c_n}). \tag{17}$$

Gauge invariance of E_L implies gauge invariance of the morphism E with respect to $\overline{\Omega}_c$ in a straightforward manner.

Invariance by are preserving homomorphisms of $\mathbb{R}^2$ is implied by invariance by area preserving homomorphisms of the family $\{E_c,\ c \in \mathsf{lassos}\}$: For each homomorphism ψ of $\mathbb{R}^2$, we have the equality $E_c = E_{\Psi(c)}$.

In the next section, we expose how to construct a family $(H, E_c),\ c \in \mathsf{lassos}$ that is braid invariant, infinitely divisible, gauge invariant and invariant by area preserving homomorphisms starting from a Lévy process on a Zhang algebra H.

4.4. **Categorical master field field from a Lévy process.** In this section, we use the same algebraic settings of the previous section. In particular, H denotes a Zhang algebra in a category $\mathcal{C}$ and $\mathcal{A} = (A, \phi_{\mathcal{A}})$ is an object of the category $\mathcal{B}$. We define the notion of Lévy process on a bialgebra B in the category $\mathcal{C}$ (that is a Zhang algebra without antipode). The tensor product $\otimes$ on $\mathcal{B}$ is supposed to satisfy the conditions stated at the beginning of the last section.

Definition 24 (Lévy process)**.** Let (B, Δ, ε) a bialgebra in $\mathcal{C}$. For all pair of times $s \leq t$, let $j_{s,t} : \mathcal{H} \to \mathcal{A}$ be a morphism of $\mathcal{C}$. We say that $j = (j_{s,t})_{s \leq t}$ is a *Lévy process* if

1. for all triple of times $u < s < t$, $æ_{u,s} \dot{\sqcup} j_{s,t} = j_{u,t}$ (Increments 1),
2. for all time $s \geq 0$ and $b \in B$, $j_{s,s}(b) = \varepsilon(b)$, (Increments 2)
3. for any tuple $(s_1 < t_1 \leq s_2 < t_2 \ldots \leq s_p < t_p)$,
$$\tau \circ j_{s_1,1} \dot{\sqcup} \cdots \dot{\sqcup} j_{s_p,t_p} = \tau \circ j_{s_1,t_1} \otimes \cdots \otimes \tau \circ j_{s_p,t_p} \text{(Independence)},$$
4. $\tau \circ j_{s,t} = \tau \circ j_{t-s}$ (Stationnarity)

In case we consider the category C of involutive algebra over the field of complex numbers and $\mathcal{B}$ the category of non-commutative probability spaces (that is the usual settings of non-commutative probability theory), since the initial object is $\mathbb{C}$, it is endowed with a natural norm and we require also the following compatiblity condition that is necessary the existence of a generators (see [12]),
$$\lim_{t \to s^+} \tau \circ j_{s,t} = \varepsilon.$$
If considering amalgamated probability spaces on an algbra R, the last condition is also required if B is endowed with a natural norm, that will be the case in the applications of the main Theorem 25 that are adressed in Section 4.5. A Lévy process on a bi-algebra is at start, a two parameters family of morphisms of the category $\mathcal{C}$ that, owing to conditions 1 and 2 are interpreted as increments. If $j = (j_t)_{t \geq 0} : H \to \mathcal{A}$ is a one parameter family of morphisms on a Zhang algebra A, by mean of the antipode, we can associate to j a set of increments
$$j_{s,t} = (j_s \circ S) \dot{\sqcup} j_t \circ \Delta.$$
The family j is said to be a Lévy process if its increments satisfy the two last conditions of Definition 24. The continuity condition 4.4 is satisfied for j if $\lim_{s \to O^+} j_s(h) = \varepsilon(h)$ for any $h \in H$. We let $j = (j_{s,t})_{s \leq t}$, $j_{s,t} : H \to (\mathcal{A}, E)$, $s \leq t$ be a $\otimes$-Lévy process on the Zhang Algebra H, valued in an object $(\mathcal{A}, E)$ of $\mathcal{B}$. Let c be a lasso drawn on the plane and denote by $|c|$ the area enclosed by the bulk of c. For each anticlockwise oriented lasso, we define the object H_c in the category $\mathcal{B}$ by $H_c = (H, E_c) = (H, E \circ j_{|c|})$.

The infinite divisible property of the family H_c, $c \in \mathsf{lassos}$ is implied by the fact that j is a Lévy process. In fact, let c_1 and c_2 two lassos such that $c_1 c_2$ is also a lasso. In that case, the area enclosed by the bulk of c is the sum of the two areas enclosed by c_1 and by c_2. Hence, $E_{c_1 c_2} = E \circ j_{|c_1|+|c_2|} = E \circ (j_{0,c_1} \times j_{|c_1|,|c_1|+|c_2|}) = (E_{c_1} \otimes E_{c_2}) \circ (\iota_{c_1} \times \iota_{c_2})$. The last equality from from $\otimes$ independence of the two increments $j_{0,|c_1|}$ and $j_{|c_1|,|c_1|+|c_2|}$. braid and gauge variance should be required for the Lévy process j in order for the objects $(H_c,\ c \in \mathsf{lasso})$ to be braid and gauge invariant. Invariance by area preserving homomorphisms is implied by the definition of the state E_c: it does only depends on the area of the bulk. The following theorem is a formal statement of the discussion concerning the construction of a categorical master field from a direct system (see Section 4.3)

Theorem 25. *Let $(\mathcal{C}, \sqcup, k)$ an algebraic category. Let $F : \mathcal{C} \to \mathcal{D}$ be a wide and faithful functor from $\mathcal{C}$ to $\mathcal{D}$. We assume that $\mathcal{C}$ and $\mathcal{D}$ are inductively complete. Define the category $\mathcal{B}$ as previously and pick a symmetric monoidal structure $\otimes$ on $\mathcal{B}$ such that P_1 is a monoidal functor. Finally, let $j = (j_{s,t})_{s \leq t}$ be braid and gauge invariant Lévy process, meaning that:*

1. *for all integer $n \geq 1$ and braid $\beta \in \mathcal{B}_n$, $\beta \cdot (j_{s_1,t_1}, \ldots, j_{s_n,t_n}) = (j_{s_1,t_1}, \ldots, j_{s_n,t_n})$ for tuples of times $s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_n < t_n$,*
2. *$(\phi_H \otimes \phi_{\mathcal{A}}) \circ (\mathrm{id}_H \sqcup j_{s_1,t_1}) \circ \Omega_c$.*

There exists a categorical master field H, in the sense of Definition 19 satisfying the following property. For all one parameter family of simple loops $\gamma = (\gamma_t)_{t \geq 0}$ based at O with

1. *for all time $t \geq 0$, $|\mathrm{Int}(\gamma_t)| = t$, and*
2. *for all times $s \leq t$, $Int(\gamma_s) \subset Int(\gamma_t)$.*

The process $(H_{\gamma_t})_{t\geq 0}$ has the same non-commutative distribution as the initial Lévy process j.

4.5. **Examples of non-commutative holonomy fields.**

4.5.1. *Classical algebraic master fields.* In this section, we specialize our construction of a probabilistic algebraic master field to classical Lévy processes (increments are tensor independent) values in a compact Lie group. Let $N \geq 1$ an integer. Let $\mathbb{K}$ be one of the three division algebras $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, the quaternions. In [12], we define a Brownian diffusion on the group $\mathbb{U}(N,\mathbb{K})$ of unitary matrices of size $N \times N$ with entries in $\mathbb{K}$, see [12]). The Brownian diffusion $\mathsf{U}_N^{\mathbb{K}}$ is the solution of a classical stochastic differential equation:

$$\begin{aligned} \mathrm{d}\mathsf{U}_N^{\mathbb{K}}(t) &= \mathrm{d}\mathsf{W}_N^{\mathbb{K}}(t)\mathsf{U}_N^{\mathbb{K}}(t) + \frac{1}{2}c_N^{\mathbb{K}}\mathsf{U}_N^{\mathbb{K}}(t) \\ \mathsf{U}_N^{\mathbb{K}}(0) &= I_N, \end{aligned}$$

with $\mathsf{W}_N^{\mathbb{K}}$ a linear Brownian motion on the Lie algebra of antihermitian matrices with entries in $\mathbb{K}$ and with respect to a conjugation invariant scalar product. The entries of $\mathsf{W}_N^{\mathbb{K}}$ are, up to symmetries, independent Brownian motions which variance scale as the inverse of the dimension N. To this diffusion are associated four Lévy processes that are of interest for the present work that are defined in [12]. Recall that we denote by $\mathcal{F}(\mathbb{U}(N,\mathbb{K}))$ the algebra of polynomial functions on $\mathbb{U}(N,\mathbb{K})$. First, we define a classical Lévy process j_N, (the increments are tensor independent), by setting for all time $s \geq 0$:

$$\begin{array}{rccc} j_N^{\mathbb{K}}(s): & \mathcal{F}(\mathbb{U}(N,\mathbb{K})) & \to & (L^\infty(\Omega,\mathcal{F},\mathbb{P}),\mathbb{E}) \\ & f & \mapsto & f(U_N^{\mathbb{K}}(s)). \end{array}$$

We recall that $\mathcal{F}(U(N,\mathbb{K})$ is an involutive Zhang algebra, being a commutative Hopf algebra with structure morphisms:

$$\Delta(f)(U,V) = f(UV),\ S(f)(U) = f(U^{-1}),\ \varepsilon(f) = f(I_N),\ \star(f) = \bar{f}.$$

The law of $j_N^{\mathbb{K}}$ is invariant by conjugation by any unitary matrices in $\mathbb{U}(N,\mathbb{K})$ since this property holds for the driving noise $\mathsf{W}_N^{\mathbb{K}}$. To prove braid invariance for $j_N^{\mathbb{K}}$, it is sufficient to prove:

$$\left(j_N^{\mathbb{K}}(t) \times j_N^{\mathbb{K}}(s,t) \times \left[j_N^{\mathbb{K}}\right]^{-1}(t),\ j_N(t)\right) \overset{\text{distrib.}}{=} \left(j_N^{\mathbb{K}}(t),\ j_N^{\mathbb{K}}(s,t)\right).$$

This last equation is readily implied by gauge invariance and independence increments throught Fubini's Theorem.

We apply Theorem 25 to obtain a Holonomy field associated with $j_N^{\mathbb{K}}$. This field is the $\mathbb{U}(N)$-*Yang–Mills field* on the plane with structure group $U(N,\mathbb{K})$.

There are two other gauge and braid invariant processes associated with the unitary diffusion $\mathsf{U}_N^{\mathbb{K}}$. The first of these quantum processes depends on two integers $n, d \geq 1$, it extracts $d \times d$ square blocks from the matrix $\mathsf{U}_N^{\mathbb{K}}$ with $N = nd$ and is defined by:

$$\begin{array}{rccc} U_{n,d}^{\mathbb{K}}: & \mathcal{O}\langle n\rangle & \to & \left(\mathcal{M}_d(L^\infty(\Omega,\mathcal{F},\mathbb{P})),\ \mathbb{E}\otimes(\frac{1}{d}\mathsf{Tr})\right) \\ & u_{ij} & \mapsto & \mathsf{U}_N^{\mathbb{K}}(i,j). \end{array} \tag{18}$$

where for $nd \times nd$ matrix A and integers $1 \leq i,j \leq n$, $A(i,j)$ is the $d \times d$ sub-matrix at position (i,j) in A.

The third quantum process we consider extracts rectangular blocks from the matrix$\mathsf{U}_N^{\mathbb{K}}$. Let $n \geq 1$ an integer and $d_N = (d_N^1, \ldots, d_N^n)$ a partition of N, which means:

$$1 \leq d_N^i,\ d_N^1 + \ldots + d_N^n = N, \quad \text{for all } 1 \leq i \leq n.$$

The Zhang algebra $\mathcal{RO}\langle n\rangle$ we call rectangular unitary algebra belongs to the category $\mathrm{Alg}^\star(R)$, we defined it in Section 2.2. Recall that R an involutive algebra generated by a complete set of

n autoadjoint projectors. The R-amalgamated quantum stochastic process $j_{d_N}^{\mathbb{K}}$ is valued in the rectangular probability space $\mathcal{M}_{d_N}(L^{\infty-}(\Omega,\mathcal{F},\mathbb{P},\mathbb{K}))$ (see [12]),

$$\begin{array}{rccc} U_{d_N}^{\mathbb{K}}: & \mathcal{RO}\langle n\rangle & \mapsto & \mathcal{M}_{d_N}(\mathbb{K}) \\ & u & \to & \mathsf{U}_N^{\mathbb{K}} \\ & p_i & \to & E_{d_n}^i. \end{array}$$

In [12], we prove for the process of square and rectangular extractions the convergence in non-commutative distributions. The limiting distributions are free (amalgamated) semi-groups, which are gauge and braid invariant. We use these semi-groups in the forthcoming section, Section 4.5.2, to construct higher dimensional counter part of the free master field.

4.5.2. *Higher dimensional probabilistic free master field.* In [12], we define a higher dimensional counterpart of a free unitary Brownian motion. More precisely, pick an integer $n \geq 1$ and a von Neumann algebra $\mathcal{A}$ endowed with a tracial state τ, $(\tau(aba^{-1}) = \tau(a),\ \tau(aa^\star) \leq 0,\ \tau(a^\star) = \overline{\tau(a)},\ a, b \in A)$. We define a free quantum Levy process $\mathsf{U}^{\langle n\rangle} = (\mathsf{U}^{\langle n\rangle}(t))_{t\geq 0}$, with $\mathsf{U}^{\langle n\rangle}(t) \in \mathcal{A}\otimes\mathcal{M}_n(\mathbb{C})$ for all time $t \geq 0$ which matricial elements are solution of a free stochastic differential system:

$$\mathrm{d}\mathsf{U}^{\langle n\rangle}(t) = \frac{\mathsf{i}}{\sqrt{n}}\mathrm{d}\mathsf{W}_t\mathsf{U}^{\langle n\rangle}(t) - \frac{1}{2}\mathsf{U}^{\langle n\rangle}(t)\mathrm{dt},\ t \geq 0.$$

From $\mathsf{U}^{\langle n\rangle}$, one can built several Lévy processes, one of them is the free unitary Brownian motion of dimension n. The dual Voiculescu group is defined in Section 2.2. Define the Lévy process $U^{\langle n\rangle} = (U^{\langle n\rangle}{}_{s,t})_{s\leq t}$ with $U^{\langle n\rangle} : \mathcal{O}\langle n\rangle \to \mathcal{A}$, by setting:

$$U^{\langle n\rangle}{}_t(u_{ij}) = \mathsf{U}^{\langle n\rangle}(t)(i,j),\ 1 \leq i, j \leq n.$$

We saw that $\mathcal{O}\langle n\rangle$ is a Zhang algebra in the category $\mathrm{Alg}^\star$ of involutive algebras. Let $\mathcal{D}$ be the category with the same objects class as the category $\mathrm{Alg}^\star$ but with morphisms class between two objects given by the set of all complex linear involutive positive maps, if $A, B \in \mathrm{Obj}(\mathcal{C})$:

$$\mathrm{Hom}_{\mathcal{D}}(A,B) = \{\tau \in \mathrm{Hom}_{\mathrm{Vect}_{\mathbb{C}}}(A,B) : \tau(aa^\star) \geq 0,\ \tau(a^\star) = \tau(a)^\star, a \in A\}.$$

The category $\mathcal{B}$ is the usual category of (complex involutive) probability spaces, Prob. Let V be a unitary matrix in $\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{A}$ and set, for all times $t \geq 0$, $\mathsf{U}^{\langle n\rangle^V}(t) = V\mathsf{U}^{\langle n\rangle}(t)V^{-1}$. If u is an element of the Voiculescu dual group $\mathcal{O}\langle n\rangle$, $\mathsf{U}^{\langle n\rangle}(t)(u)$ and $\mathsf{U}^{\langle n\rangle^V}(t)(u)$ stand for the values on u of the morphisms on $\mathcal{O}\langle n\rangle$ induced by the two matrices $\mathsf{U}^{\langle n\rangle}(t)$ and $\mathsf{U}^{\langle n\rangle^V}(t)$. We assume further that the involutive subalgebra of $\mathcal{A}$ generated by the entries of V is free from the algebra generated by the entries of $\mathsf{U}^{\langle n\rangle}(t)$ for all times $t \geq 0$.

Lemma 26. *Let $t \geq 0$ a time, $u \in \mathcal{O}\langle n\rangle$, V an unitary element of $\mathcal{M}_n(\mathbb{C})\otimes\mathcal{A}$, with the notations introduced so far,*

$$\tau \circ \mathsf{U}^{\langle n\rangle^V}(t)(u) = \tau \circ \mathsf{U}^{\langle n\rangle}(t)(u),\ t \geq, u \in \mathcal{O}\langle n\rangle. \tag{19}$$

Proof. The process U^V is solution of the following free stochastic differential system, with obvious notation,

$$\mathrm{d}\mathsf{U}^{\langle n\rangle^V}(t) = \frac{\mathsf{i}}{\sqrt{n}}\mathrm{d}\mathsf{W}_t^V\mathsf{U}^{\langle n\rangle^V}(t) - \frac{1}{2}\mathsf{U}^{\langle n\rangle^V}(t),\ t \geq 0.$$

Hence, gauge invariance of the driving noise W implies (19), property that is proved now. For an integer $m \geq 1$, we denote by NC_{2m} the set of matchings of the intervale $[\![1, 2m]\!]$. A matching $\mathsf{m} \in \mathsf{NC}_{2m}^{(2)}$ is alternatively seen as a non-crossing partition or as an involution of $[\![1, 2m]\!]$ verifying:

$$\text{for all } k < l \in [\![1, 2m]\!],\ \mathsf{m}(k) < \mathsf{m}(l).$$

We compute the cumulants of the family $\{\mathsf{W}^V(\alpha,\beta),\ 1\leq\alpha,\beta\leq n\}$ and prove that:

(20)
$$k_{2m+1}\left(\mathsf{W}^V(\alpha_1,\beta_1),\dots,\mathsf{W}^V(\alpha_{2m+1},\beta_{\beta_{2m+1}})\right)=0,\ m\geq 1,\ \text{for all } \alpha,\beta\in\{1,\dots,n\}^{2m+1},$$
$$k_{2m}\left(\mathsf{W}^V(\alpha_1,\beta_1),\dots,\mathsf{W}^V(\alpha_{2m+1},\beta_{\beta_{2m}})\right)=k_{2m}\left(\mathsf{W}(\alpha_1,\beta_1),\dots,\mathsf{W}(\alpha_{2m},\beta_{\beta_{2m}})\right)$$
$$=\sum_{\mathsf{m}\in\mathsf{NC}_{2m}}\prod_{i=1}^{2m}\delta_{\alpha_i,\beta_{\mathsf{m}(i)}}\delta_{\alpha_{\mathsf{m}(i)},\beta_i},\ \alpha,\beta\in\{1,\dots,n\}^{2m}.$$

To compute these cumulants, we use the following fondamental formula relating cumulants with products of elements of $\mathcal{A}$ as entries to the cumulants of these elements. Let $p\geq 1$ an integer and $k_p\geq 1$ an other one. We define the interval partition $\sigma=\{\{1,\dots,k_1\},\{k_1+1,\dots,k_2\},\dots,\{k_{p-1}+1,\dots,k_p\}\}$ and denote by 1_{k_p} the partition of $[\![1,k_p]\!]$ with only one block. Let $a_1,\dots,a_{k_p}\in\mathcal{A}$, then:

$$k_{k_p}(a_1\cdots a_{k_1},a_{k_1+1}\cdots a_{k_2},\dots,a_{k_{p-1}+1}\cdots a_{k_p})=\sum_{\substack{\pi\in\mathsf{NC}_{k_p}\\ \pi\vee\sigma=1_{k_p}}}k_\pi(a_1,\dots,a_{k_p}).\tag{21}$$

By using formula (21) and freeness of the matricial entries of V with W_t, it is easy to prove that the odd cumulants in (20) are equal to zero. Let $p\geq 1$ and $\alpha_1,\dots,\alpha_{2p}\in\{1,\dots,n\}^{2p}$. Consider a word $u=v_1w_1\tilde{v}_1\cdots v_1\tilde{w}\tilde{v}_1)$ with the $v^{'}s$ and the $\tilde{v}^s$ in the algebra generated by the matrix coefficients of V and the $w^{'}s$ in the algebra generated by the matrix coefficients of W_t. We say that an integer $i\leq 3p$ is white coloured if u_i is equal to v_i or v_i' and black coloured if u_i is equal to u_i.

Let σ be the interval partition $\sigma=\{\{1,2,3\},\dots,\{6p-2,6p-1,6p\}\}$. Let $\pi\in\mathsf{NC}_{6p}$ such that $\sigma\vee\pi=1_{6p}$. By using nullity of mixed cumulants having components of V and W_t as entries, we prove that $k_\pi(V(\alpha_1,k_1),\mathsf{W}(k_1,q_1),V^\star(q_1,\beta_1),\dots,V(\alpha_{2p},k_{2p}),\mathsf{W}(k_{2p},q_{2p}),V^\star(q_{2p},\beta_{2p}))$ is equal to zero if a block of π is white a black coloured. Also, the trace of π on the set of black coloured integers is a matching m, in that case,

$$k_\pi(V(\alpha_1,k_1),\mathsf{W}(k_1,q_1),V^\star(q_1,\beta_1),\dots,V(\alpha_{2p},k_{2p}),\mathsf{W}(k_{2p},q_{2p}),V^\star(q_{2p},\beta_{2p}))=$$
$$k_{\mathsf{m}}(W(k_1,q_1),\dots,W(k_{2p},q_{2p}))k_{\mathsf{K}(\mathsf{m},\pi)}(V(\alpha_1,k_1),V^\star(q_1,\beta_1),\dots,V^\star(q_{2p},\beta_{2p})).\tag{22}$$

The non crossing partition denoted $\mathsf{K}(\mathsf{m},\pi)$ is a partition of the white coloured integers of $[\![1,6p]\!]$ and equal to the complement of m in the partition π: $\mathsf{m}\cup\mathsf{K}(\mathsf{m},\pi)=\pi$. We sum (22) over non crossing partitions having the same trace $\mathsf{m}\in\mathsf{NC}_{2p}^2$ over black coloured integers and over integers $k_1,\dots,k_{2p}$,$q_1,\dots,q_{2p}$ in $\{1,\dots,n\}^2$. By using the moments-cumulants formula, we obtain:

$$\sum_{\substack{1\leq k_1,\dots,k_{2p}\leq n,\\ 1\leq q_1,\dots,q_{2p}\leq n}}\ \sum_{\substack{\pi\in\mathsf{NC}_{6p}\\ \pi_\bullet=\mathsf{m}}}k_\pi(V(\alpha_1,k_1),\mathsf{W}(k_1,q_1),V^\star(q_1,\beta_1),\dots,V(\alpha_{2p},k_{2p}),\mathsf{W}(k_{2p},q_{2p}),V^\star(q_{2p},\beta_{2p}))$$
$$=\sum_{\substack{1\leq k_1,\dots,k_{2p}\leq n,\\ 1\leq q_1,\dots,q_{2p}\leq n}}k_{\mathsf{m}}(W(k_1,q_1),\dots,W(k_{2p},q_{2p}))\tau_{\mathsf{K}(\mathsf{m},1_{6p})}(V(\alpha_1,k_1),V^\star(q_1,\beta_1),\dots,V^\star(q_{2p},\beta_{2p}))$$
$$=\sum_{\substack{1\leq k_1,\dots,k_{2p}\leq n,\\ 1\leq q_1,\dots,q_{2p}\leq n}}\prod_{i=1}^{2p}\delta_{k_i,q_{\mathsf{m}(i)}}\delta_{q_i,k_{\mathsf{m}(i)}}\tau_{\mathsf{K}(\mathsf{m},1_{6p})}(V(\alpha_1,k_1),V^\star(q_1,\beta_1),\dots,V^\star(q_{2p},\beta_{2p}))$$

To compute right hand side of the last equation, pick a block V of the partition $\mathsf{K}(\mathsf{m}, 1_{6p})$, then by using cyclicity of τ, we have:

(23)
$$\sum_{\substack{1\leq k_1,\ldots,k_{2p}\leq n,\\ 1\leq q_1,\ldots,q_{2p}\leq n}} \tau_V(V^\star(q_{i_1},\beta_{i_1})V(\alpha_{i_2},k_{i_2})\cdots V(\alpha_{i_l},k_{\mathsf{m}(i_l)})\prod_l \delta_{k_{i_l},q_{\mathsf{m}(i_l)}}\delta_{q_{i_l},k_{\mathsf{m}(i_l)}} = \prod_l \delta_{\alpha_l,\beta_{\mathsf{m}(l)}}\delta_{\beta_l,\alpha_{\mathsf{m}(l)}}.$$

The proof of the formulas (20) is now complete. □

We saw that for a classical Levy process on the Zhang algebra of function on a group, gauge invariance and independence of increments implies braid invariance. It seems difficult to prove braid invariance of $U^{\langle n\rangle}$ with the same arguments. However, we prove in [12] that $U^{\langle n\rangle}$ is a limit, in non-commutative distribution of a braid invariant process: the process $U^{\mathbb{C}}_{n,d}$ (which is not a Lévy process) that is defined in [12]. Braid invariance of the finite dimensinal approximations of $U^{\langle n\rangle}$ implies braid invariance of the latter.

We can apply Theorem 25: there exists an algebraic generalized master field associated with the higher dimensional counterpart of the free unitary Brownian motion $U^{\langle n\rangle}$.

Proposition 27. *Let $n \geq 1$ an integer. There exists a probabilistic generalized master field (see Definition 19), denoted $\Phi^{\langle n\rangle}$, such that for any one parameter family of simple loops γ with $|\gamma_t| = t$ and $\mathrm{Int}(\gamma_s) \subset \mathrm{Int}(\gamma_t)$ for all times $0 \leq s \leq t$, the process $\Phi^{\langle n\rangle}(\gamma_t)$ is a free unitary Brownian motion of dimension n.*

4.5.3. *Amalgamated probabilistic algebraic master fields.* Let $n \geq 1$ an integer. In Section 2.2, we defined the Zhang algebra $\mathcal{RO}\langle n\rangle$ as being the involutive algebra generated by one unitary element u and a complete set of mutually orthogonal projectors $\{p_i,\ 1 \leq i \leq n\}$. Set $\mathcal{R} = \langle p_i, 1 \leq i \leq n\rangle$. The algebra $\mathcal{RO}\langle n\rangle$ belongs to the category $\mathrm{lCoModAlg}(\mathcal{R})$. Let $\mathcal{D}$ be the category having the same objects class as $\mathrm{lCoModAlg}(\mathcal{R})$ and the class of morphisms between two objets being given by the positive complex $\mathcal{R}$-bimodule maps, for all $A, B \in \mathrm{Obj}(\mathcal{D})$:

$$\mathrm{Hom}_{\mathcal{D}}(A,B) = \{\phi \in \mathrm{Vect}_{\mathbb{C}}(A,B) : \phi(a^\star) = \phi(a)^\star,\ \phi(aa^\star) \geq 0,\ \phi(rar') = r\phi(a)r'\} \tag{24}$$

With the notations of Definition 19, the category $\mathcal{B}$ is the usual category of $\mathcal{R}$-amalgamated probability spaces, also known as rectangular probability spaces. Let $r = (r_1,\ldots,r_n)$ be positive real numbers such that $r_1 + \cdots + r_n = 1$. In [12], we define the semi-group E_r on the Zhang algebra $\mathcal{RO}\langle n\rangle$ and prove tha E_r is a free, with amalgamation, semi-group. We explain briefly how this semi-group is obtained.

We fix a integer $n \geq 1$. For each integer $N \geq 1$, we pick $(d_N^1,\ldots,d_N^n)$ a partition of N into n parts and we assume that the ratio $\frac{d_N^i}{N}$ converges to r_i for each integer $1 \leq i \leq n$ as $N \to +\infty$. Let $\mathbb{K}$ be one of the three division algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. In [12], we define for a partition d of N with length n the quantum process $U^{\mathbb{K}}_{\mathsf{d}}$ on the Zhang algebra $\mathcal{RO}\langle n\rangle$ that extracts rectangular blocks from the unitary Brownian motion of dimension N. This process is valued into a rectangular probability space, which we denote by $\mathcal{M}_{d_N}$. As an algebra, $\mathcal{M}_{d_N}$ is isomorphic to the space of matrices with dimension $N \times N$ and entries in the algebra of bounded variables (or random variables having moments of all order) with values in $\mathbb{K}$. The algebra $\mathcal{M}_{d_N}$ is a bimodule algebra over the commutative unital algebra generated by the projectors:

$$p_i = \sum_{j=d_1+\ldots+d_{i-1}}^{d_1+\ldots+d_i} e^j \otimes e_j,\ 1 \leq i \leq n,$$

where $(e_1,\ldots,e_n)$ denote the canonical basis of $\mathbb{K}^N$ (as a left $\mathbb{K}$-module and $(e^1,\ldots,e^n)$ is its dual basis. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the conditional expectation $\mathbb{E}_{d_N}$ on $\mathcal{M}_{d_N}$ is the mean of the trace of

the diagonal blocks:

$$\mathbb{E}_{d_N}(A) = \sum_{i=1} \frac{1}{d_N(i)} \mathbb{E}\left[\mathsf{Tr}(p_i A p_i)\right] p_i, \ \ A \in \mathcal{M}_{d_N}$$

In [12], we define the semi-group E_r as the limiting distribution of the non commutative distribution of the rectangular extraction process:

$$\mathsf{E}_{d_N} = \mathbb{E}_{d_N} \circ U^{\mathbb{K}}_{d_N}(t) \xrightarrow{N\to+\infty} \mathsf{E}_{r_1,\dots,r_n}(t), \quad \text{for all time } t \geq 0.$$

We apply our main Theorem 25 to obtain a probabilistic generalized master field $\Phi^{\langle r\rangle}$ associated with the free semi-group E_r that we name *amalgamated higher dimensional master field* with parameter r. The following proposition sums up this last discussion.

Proposition 28. *Let $r_1, \dots, r_n \geq 1$ positive real numbers. There exists a probabilistic generalized master field (see Definition 18), which we $\Phi^{\langle r_1,\dots,r_n\rangle}$, such that for all one-parameter family of simple loops γ with $|\gamma_t| = t$ and $\mathrm{Int}(\gamma_s) \subset \mathrm{Int}(\gamma_t)$ for all times $0 \leq s \leq t$, the process $\left(\Phi^{\langle r_1,\dots,r_n\rangle}(\gamma_t)\right)_{t\geq 1}$ is a free with amalgamation quantum Lévy process which non-commutative distribution at time $t \geq 0$ is $\mathsf{E}_{r_1,\dots,r_n}(t)$.*

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