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Abstract. Of concern is the study of the long-time existence of solutions to the Euler–Arnold equation of the right-invariant $H^{\frac{3}{2}}$-metric on the diffeomorphism group of the circle. In previous work by Escher and Kolev it has been shown that this equation admits long-time solutions if the order s of the metric is greater than $\frac{3}{2}$, the behaviour for the critical Sobolev index $s = \frac{3}{2}$ has been left open. In this article we fill this gap by proving the analogous result also for the boundary case. The behaviour of the $H^{3/2}$-metric is, however, still different from its higher order counter parts, as it does not induce a complete Riemannian metric on any group of Sobolev diffeomorphisms.

1. Introduction

In this article, we prove longtime existence for solutions of the geodesic initial value problem of the right invariant $H^{3/2}$-metric on the group of smooth diffeomorphisms on the circle. The interest in (fractional) order metrics on diffeomorphism groups is fuelled by their relations to various prominent PDEs of mathematical physics: In the seminal article [1], Arnold showed in 1965 that Euler's equations for the motion of an incompressible, ideal fluid have a geometric interpretation as the geodesic equations on the group of volume preserving diffeomorphisms. Since then, an analogous result has been found for a whole variety of PDEs, including the Burgers equation, the Hunter–Saxton equation, the Camassea–Holm equation [6, 18] or the modified Constantin–Lax–Majda (mCLM) equation [10, 14]. Building up on the pioneering work of Ebin and Marsden [11], these geometric interpretations have been used to obtain rigourous well-posedness and stability results for the corresponding PDEs [9, 24, 23, 21, 3, 15, 19].

Motivated by the analysis on the mCLM equation, Escher and Kolev recently studied fractional order Sobolev metrics on the diffeomorphism group of the circle [13, 12]. In their investigations, they showed that the geodesic equation of the class of Sobolev metrics of order s is locally well-posed if $s \geq \frac{1}{2}$ and globally well-posed if $s > \frac{3}{2}$. The question of global existence of solutions of the geodesic equation for the critical index $s = \frac{3}{2}$ was left unanswered. Towards this direction, Preston and Washabaugh proved in [22] that the Weil–Petersson metric on the universal Teichmüller space, which is of critical order $\frac{3}{2}$, possesses smooth global solutions. In this article, we extend their analysis to obtain a global existence result for general Sobolev metrics of order 3/2 on the diffeomorphism group of the circle and thus give a positive answer for the critical index.

Metric completeness. In [2], it was shown that the Sobolev metric of order $s > \frac{3}{2}$ extends smoothly to a *strong* Riemannian metric on the group of Sobolev diffeomorphisms $\mathcal{D}^s(S^1)$. This allowed the authors to use results on strong, right invariant metrics to show that the metric is not only geodesically complete, but (as a metric on $\mathcal{D}^s(S^1)$) also metrically complete, see also [5]. This result is not true anymore for the critical index $s = \frac{3}{2}$, as $\mathcal{D}^s(S^1)$ is only a topological group for $s > \frac{3}{2}$; the metric extends only to a smooth, *weak* Riemannian metric on the Sobolev completion $\mathcal{D}^q(S^1)$, for high enough $q > \frac{3}{2}$.

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Metrics of lower order. For certain examples of metrics of order $s < \frac{3}{2}$, it has been shown that solutions of the Euler–Arnold equation can blowup in finite time. This includes the L^2-metric (Burgers equation), the $H^{1/2}$-metric (mCLM equation) [22, 4] and the H^1-metric (Camassa–Holm equation) [6, 7]. We conjecture that blowup of solutions occurs for every metric of order $s < \frac{3}{2}$. This result would provide a complete characterization for the solution behaviour of the geodesic equation of fractional order metrics on the group of diffeomorphisms. As we are not able to show this result at the present time, we leave this question open for future research.

2. Right invariant Sobolev metrics on $\mathrm{Diff}(S^1)$

Let $\mathrm{Diff}(S^1)$ denote the group of smooth and orientation preserving diffeomorphisms on the circle. The space $\mathrm{Diff}(S^1)$ is an open subset of the Fréchet manifold of all smooth functions $C^\infty(S^1, S^1)$ and thus, itself a Fréchet manifold. Furthermore, composition and inversion are smooth maps and $\mathrm{Diff}(S^1)$ is a Fréchet-Lie group, where the Lie algebra is the space of vector fields on the circle, equipped with the negative of the usual Lie-bracket on vector fields, *i.e.*:

$$[u, v] = u_x w - u w_x \ .$$

See [16] for more details on diffeomorphism groups as infinite dimensional Lie groups. Given an inner product on the space of vector fields we can extend this using right translations to obtain a right invariant metric on the diffeomorphism group. Thus, to define a *right-invariant* metric, it remains only to specify the inner product on the space of vector fields. The most natural choice is given by the standard L^2-inner product:

$$\langle u, v\rangle_{L^2} = \int_{S^1} uv \, dx \ .$$

This metric (which corresponds to the Burgers equation) has been studied in great detail and it has been shown, in particular, that:

(1) the induced geodesic distance of the metric is vanishing [20];
(2) the exponential map is not a C^1 diffeomorphism [8].

More generally, we consider an inner product on $C^\infty(S^1)$ which writes

$$\langle u, v\rangle_A = \int_{S^1} (Au)v \, dx \ .$$

where $A : C^\infty(S^1) \to C^\infty(S^1)$, the so-called *inertia operator* is self-adjoint, with respect to the L^2-inner product *i.e.*

$$\int_{S^1} (Au)v \, dx = \int_{S^1} u(Av) \, dx, \qquad \forall u, v \in C^\infty(S^1)\,,$$

and positive definite *i.e.*

$$\int_{S^1} (Au)u \, dx > 0, \qquad \forall u \in C^\infty(S^1)\,.$$

The corresponding right invariant metric on $\mathrm{Diff}(S^1)$ reads as

$$G^A_\varphi(h, k) = \langle h\circ\varphi^{-1}, k\circ\varphi^{-1}\rangle_A = \int_{S^1} (A(h\circ\varphi^{-1}))k\circ\varphi^{-1} \, dx,$$

where $h, k \in T_\varphi\mathrm{Diff}(S^1)$. If we assume, furthermore, that A is invertible and commutes with differentiation, then, the Euler–Arnold equation of the metric G^A is given by:

$$m_t + u m_x + 2 m u_x = 0, \qquad m = Au, \qquad u(0) = u_0 \in C^\infty(S^1). \tag{1}$$

In this equation, m is the so-called momentum associated to the velocity u. If the inertia operator A is a *nice operator* of order $s > \frac{3}{2}$, the global existence of smooth solutions to this equation has been shown by Escher and Kolev in [13]. In this article, the question for the behaviour of the solutions for the boundary index $s = \frac{3}{2}$ was raised. The aim of the present work is to give a positive answer to this former open problem.

3. The right-invariant $H^{3/2}$-metric on $\mathrm{Diff}(S^1)$

We will now formally introduce the class of *nice operators* we are interested in. We will assume that A is a continuous linear operator on $C^\infty(S^1)$ that commutes with differentiation. In that case, A is a *Fourier-multiplier*, *i.e.*

$$(Au)(x) = \sum_{k\in\mathbb{Z}} a(k)\hat{u}(k)\exp(2i\pi kx),$$

where $\hat{u}(k)$ is the k-th Fourier coefficients of the vector field u, see [13, App. A]. The sequence $a : \mathbb{Z} \to \mathbb{C}$ is called the *symbol* of A and we will use the notation $A = \mathrm{op}(a(k))$ or equivalently $A = a(D)$. For a more detailed introduction to the theory of Fourier multipliers in the context of $\mathrm{Diff}(S^1)$ we refer to the article [13]. The most important example in our context is the inertia operator for the fractional order Sobolev metric H^s, which reads:

$$A = \mathrm{op}\left((1+k^2)^{s/2}\right). \tag{2}$$

In this article we will be interested in the specific case $s = 3$.

Definition 1. Given $r \in \mathbb{R}$, a Fourier multiplier $a(D)$ is of class $\mathcal{S}^r$ iff a extends to a smooth function $\mathbb{R}^d \to \mathbb{C}^d$ and satisfies moreover the following condition:

$$a^{(l)}(\xi) = O(|\xi|^{r-l}), \qquad \forall l \in \mathbb{N}.$$

Remark 1. Note that a Fourier multiplier $a(D)$ of class $\mathcal{S}^r$ extends to a *bounded linear operator*

$$H^q(S^1) \to H^{q-r}(S^1)$$

for any $q \geq r$.

Definition 2. If a Fourier multiplier $a(D)$ extends to a bounded linear operator

$$H^q(S^1) \to H^{q-r}(S^1)$$

for q big enough and $r \in \mathbb{R}$, we will say that $a(D)$ is *of order less than* r.

We will impose a slightly more restrictive condition on the symbol class, by requiring that a has a series representation of the form

$$a(\xi) = \sum_{k=0}^{\infty} \tilde{a}_{3-k}\,|\xi|^{3-k}\ . \tag{3}$$

In terms of the operator $A = a(D)$ this translates to

$$A = a_3(HD)^3 + R_2 = a_3(HD)^3 + a_2(HD)^2 + R_1,$$

where

$$R_k := \sum_{j=-\infty}^{k} a_j(HD)^j$$

and $H := \mathrm{op}(-i\,\mathrm{sign}(k))$ denotes the Hilbert transform. Using the property $H^2 = -\,\mathrm{Id}$, we can then rewrite A to obtain

$$A = -a_3HD^3 + R_2 = -a_3HD^3 - a_2D^2 + R_1. \tag{4}$$

Note, that the remainder terms R_k are Fourier multipliers of order k.

Finally, in order to ensure local well-posedness of the Euler–Arnold equation (1) (see [13]), we will require, furthermore, an ellipticity condition on A.

Definition 3. A Fourier multiplier $A = a(D)$ in the class $\mathcal{S}^r$ is called *elliptic* if

$$\left(1 + |\xi|^2\right)^{r/2} \lesssim |a(\xi)|, \qquad \forall \xi \in \mathbb{R}^d.$$

In the following definition, we summarize the assumptions on the class of operators we will consider.

Definition 4. An operator $A \in L(C^\infty(S^1))$ is in the class $\mathcal{E}^3_{\text{cl}}$ iff the following conditions are satisfied:

(1) A is a Fourier multiplier of class $\mathcal{S}^3$;
(2) A is elliptic;
(3) $a(\xi)$ is real for all $\xi \in \mathbb{R}$;
(4) $a(\xi)$ is positive for all $\xi \in \mathbb{R}$;
(5) $a(\xi)$ has a series expansion of the form (3).

Remark 2. Note that assumption (1) guarantees that the operator is of order three and commutes with differentiation; The ellipticity condition (2) is required to show local well-posedness of the geodesic equation. Condition (3) guarantees that A is L^2-self-adjoint and (4) that it is a positive definite operator. Assumption (5) is a technical condition, that is essential for our long-time existence proof.

Example 1. A trivial example for an operator within the class $\mathcal{S}^3_{\text{cl}}$ is the operator $A = 1 - H\partial_\theta^3$. Another example consist of the Sobolev metric of fractional order $\frac{3}{2}$ as defined in (2). The ellipticity of this inertia operator has been shown in [13]. To see that this operator satisfies assumption (5), one only needs to make a series expansion of the Fourier multiplier $(1 + k^2)^{\frac{3}{2}}$.

4. Global Well-Posedness of the EPDiff equation.

In this section, we will prove the global existence of solutions to the EDDiff equation of metrics of order 3/2:

Theorem 1. *Let G be the right invariant metric on $\operatorname{Diff}(S^1)$ with inertia operator A in the class $\mathcal{E}^3_{\text{cl}}$ and let u_0 be an H^s velocity field on S^1, for some $s > \frac{3}{2}$. Then the solution $u(t)$ of the Euler–Arnold-equation (1) of the metric G with $u(0) = u_0$ remains in H^s for all time. In particular if u_0 is C^∞ then so is $u(t)$ for all $t > 0$. Thus the space $\left(\operatorname{Diff}(S^1), G\right)$ is geodesically complete.*

Note, that this result implies in particular the completeness of the $H^{3/2}$-metric, as the inertia operator $A = \operatorname{op}\left((1 + k^2)^s\right)$ is of class $\mathcal{E}^3_{\text{cl}}$, c.f. Example 1.

Proof. According to [12, Theorem 5.6], we only need to show that for any solution $u(t)$ of the Euler–Arnold equation (1), the norm $\|u_x(t)\|_{L^\infty}$ is *bounded on every bounded time interval.*

Let $\varphi(t)$ be the flow of the time dependant vector field $u(t)$, we have

$$\partial_t(u_x \circ \varphi) = (u_{tx} + uu_{xx}) \circ \varphi.$$

Now, from (1), we get

$$\begin{aligned} u_{tx} &= -A^{-1}D(uAu_x + 2u_xAu) \\ &= -A^{-1}\left(D^2(uAu) + D(u_xAu)\right). \end{aligned}$$

Thus, observing that $\|w\|_{L^\infty} = \|w \circ \eta\|_{L^\infty}$, for every $w \in \mathrm{C}^\infty(S^1)$ and $\eta \in \operatorname{Diff}(S^1)$, we have

$$\|u_x(t)\|_{L^\infty} \leq \|u_x(0)\|_{L^\infty} + \int_0^t \|Q(u(s))\|_{L^\infty}\, ds$$

where

$$Q(u) := uu_{xx} - A^{-1}\left(D^2(uAu) + D(u_xAu)\right).$$

Therefore, thanks to Grönwall inequality, it is sufficient to show that

$$\|Q(u)\|_{L^\infty} \leq \alpha \|u\|_{H^{3/2}}^2 + \beta \|u\|_{H^{3/2}} \|u_x\|_{L^\infty},$$

for some positive constants α, β, because the norm $\|u\|_{H^{3/2}}$ is equivalent to the norm $\|u\|_A$, given by

$$\|u\|_A^2 := \int_{S^1} uAu\,dx,$$

which is an integral constant. The remaining estimate for $Q(u)$ will be achieved in Lemma 1 below. □

To prove the bound for $Q(u)$ we will use the decomposition (4) of A to further expand $Q(u)$ in a sum of terms that can be bounded separately. We have:

$$\begin{aligned}
D^2(uAu) + D(u_xAu) &= -a_3D^2(uHu_{xxx}) + D^2(uR_2u) - a_3D(u_xHu_{xxx}) + D(u_xR_2u)\\
&= -a_3D^3(uHu_{xx}) + a_3D^2(u_xHu_{xx}) + D^2(uR_2u)\\
&\quad - a_3D^2(u_xHu_{xx}) + a_3D(u_{xx}Hu_{xx}) + D(u_xR_2u)\\
&= -a_3D^3(uHu_{xx}) + D^2(uR_2u) + a_3D(u_{xx}Hu_{xx}) + D(u_xR_2u)\\
&= a_3HD^3H(uHu_{xx}) - R_2H(uHu_{xx}) + R_2H(uHu_{xx})\\
&\quad + D^2(uR_2u) + a_3D(u_{xx}Hu_{xx}) + D(u_xR_2u)\\
&= -AH(uHu_{xx}) + a_3D(u_{xx}Hu_{xx}) + R_2H(uHu_{xx}) + D^2(uR_2u) + D(u_xR_2u).
\end{aligned}$$

Further expanding with $R_2 = -a_2D^2 + R_1$, we get:

$$\begin{aligned}
R_2H(uHu_{xx}) &= -a_2D^2H(uHu_{xx}) + R_1H(uHu_{xx})\\
&= -a_2D^2H(uHu_{xx}) + R_1HD(uHu_x) - R_1H(u_xHu_x),\\
D^2(uR_2u) &= -a_2D^2(uu_{xx}) + D^2(uR_1u),\\
D(u_xR_2u) &= -a_2D(u_xu_{xx}) + D(u_xR_1u)\\
&= -\frac{a_2}{2}D^2(u_x^2) + D(u_xR_1u).
\end{aligned}$$

Summing up and rearranging all the terms, we get finally:

$$Q(u) = \sum_{i=1}^{8} Q_i(u),$$

where

$$\begin{aligned}
Q_1(u) &= H(uHu_{xx}) + uu_{xx}, & Q_2(u) &= -a_3A^{-1}D(u_{xx}Hu_{xx}),\\
Q_3(u) &= a_2A^{-1}D^2\big(H(uHu_{xx}) + uu_{xx}\big), & Q_4(u) &= \frac{a_2}{2}A^{-1}D^2(u_x^2),\\
Q_5(u) &= -A^{-1}D^2(uR_1u), & Q_6(u) &= -A^{-1}R_1HD(uHu_x),\\
Q_7(u) &= A^{-1}R_1H(u_xHu_x), & Q_8(u) &= -A^{-1}D(u_xR_1u).
\end{aligned}$$

To achieve the proof of Theorem 1, it only remains to show that all the quadratic terms $Q_i(u)$ are bounded either by $\|u\|_{H^{3/2}}^2$ or by $\|u\|_{H^{3/2}} \|u_x\|_{L^\infty}$:

Lemma 1. *For $i = 1, \dots, 8$, there exists $\kappa_i > 0$ such that:*

$$\|Q_i(u)\|_{L^\infty} \leq \kappa_i \|u\|_{H^{3/2}}^2, \qquad \textit{for} \quad i = 1,2,3,5,6,7,$$

and

$$\|Q_i(u)\|_{L^\infty} \leq \kappa_i \|u_x\|_{L^\infty} \|u\|_{H^{3/2}}, \qquad \textit{for} \quad i = 4,8.$$

Proof. Using Theorem 2, we can bound the supremum norm of Q_1 via

$$\|Q_1(u)\|_{L^\infty} \leq \kappa_1 \|u\|^2_{H^{3/2}} .$$

Using Theorem 3, with $B = A^{-1}D$, we obtain:

$$\|Q_2(u)\|_{L^\infty} \leq \kappa_2 \|u\|^2_{H^{3/2}} \qquad \text{and} \qquad \|Q_7(u)\|_{L^\infty} \lesssim \|u\|^2_{H^{1/2}} \leq \kappa_7 \|u\|^2_{H^{3/2}} .$$

Since $A^{-1}D^2$ is of order ≤ -1, we can use Lemma 4 to bound Q_3 via:

$$\|Q_3(u)\|_{L^\infty} \lesssim \left\|\left(H(uHu_{xx}) + uu_{xx}\right)\right\|_{L^2} \leq \left\|\left(H(uHu_{xx}) + uu_{xx}\right)\right\|_{L^\infty} \leq \kappa_3 \|u\|^2_{H^{3/2}} ,$$

the last inequality following from Theorem 2. For Q_5, we apply Lemma 5 with $B_1 = R_1$ of order ≤ 1 and $B_2 = A^{-1}D^2$ of order ≤ -1, which yields $\|Q_5(u)\|_{L^\infty} \leq \kappa_3 \|u\|^2_{H^{3/2}}$. Finally, using Lemma 4, we get

$$\|Q_4(u)\|_{L^\infty} \lesssim \left\|u_x^2\right\|_{L^2} \leq \|u_x\|_{L^\infty} \|u_x\|_{L^2} \lesssim \|u_x\|_{L^\infty} \|u\|_{H^{3/2}} ,$$

then

$$\|Q_6(u)\|_{L^\infty} \lesssim \|uHu_x\|_{L^2} \lesssim \|u\|_{L^\infty} \|Hu_x\|_{L^2} \lesssim \|u\|^2_{H^{3/2}} ,$$

and

$$\|Q_8(u)\|_{L^\infty} \lesssim \|u_x R_1 u\|_{L^2} \lesssim \|u_x\|_{L^\infty} \|R_1 u\|_{L^2} \lesssim \|u_x\|_{L^\infty} \|u\|_{H^{3/2}} .$$

□

Appendix A. Norm-bounds

In this appendix, we will collect some estimates that were needed in the proof of the main result of this article. The principal estimate is the following which relies on a computation performed in [4].

Lemma 2. *Let $u\colon S^1 \to \mathbb{R}$ be a smooth function and $F(u) := uu_{xx} + H(uHu_{xx})$. Then*

$$\|F(u)\|_{L^\infty} \leq 4 \|u\|^2_{H^{3/2}} .$$

Proof. Since $F(u + c) = F(u)$ for every $c \in \mathbb{R}$, it is enough to show this estimate when u has vanishing mean value. Express u in a Fourier basis $u(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx}$. It was then shown in [4, Theorem 17] that

$$F(u)(x) = 2\sum_{n=1}^{\infty}(2n-1)\left|\sum_{k=n}^{\infty} u_k e^{ikx}\right|^2 .$$

But

$$\begin{aligned}\left|\sum_{k=n}^{\infty} u_k e^{ikx}\right|^2 &= \left|\sum_{k=n}^{\infty}\left(\sqrt{k(k+1)}u_k e^{ikx}\right)\left(\frac{1}{\sqrt{k(k+1)}}\right)\right|^2 \\ &\leq \left(\sum_{k=n}^{\infty} k(k+1)\left|u_k\right|^2\right)\left(\sum_{k=n}^{\infty}\frac{1}{k(k+1)}\right) \\ &= \frac{1}{n}\sum_{k=n}^{\infty} k(k+1)\left|u_k\right|^2 .\end{aligned}$$

Thus we get

$$\begin{aligned}\|F(u)\|_{L^\infty} &\leq 2\sum_{n=1}^{\infty}\frac{(2n-1)}{n}\left(\sum_{k=n}^{\infty} k(k+1)\left|u_k\right|^2\right) \\ &\leq 4\sum_{k=n}^{\infty} k(k+1)\left|u_k\right|^2 \leq 4\|u\|^2_{H^{3/2}} .\end{aligned}$$

□

Next, we will formulate the following estimate, which is a slight modification of [22, Theorem 8].

Lemma 3. *Let $u\colon S^1 \to \mathbb{R}$ be a smooth function and set $G(u) := B(u_x H u_x)$, where B is a Fourier multiplier of order less than -2. Then*

$$\|G(u)\|_{L^\infty} \leq C \|u\|^2_{H^{1/2}},$$

for some constant $C > 0$.

Proof. The proof is similar as one of [22, Theorem 8]. The first part of the proof, can be copied word by word from [22]. Express u in a Fourier basis $u(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx}$, and let $h = u_x H u_x$. Then we have

$$\begin{aligned}(u_x H u_x)(x) &= i \sum_{m,n\in\mathbb{Z}} mn\, u_m u_n (\operatorname{sign} n) e^{i(m+n)x} \\ &= i \sum_{k\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} |n|\,(k-n) u_{k-n} u_n \right) e^{ikx} \\ &= i \sum_{k\in\mathbb{Z}} h_k e^{ikx},\end{aligned}$$

where $h_k = \sum_{n\in\mathbb{Z}} |n|\,(k-n) u_{k-n} u_n$. Now let us simplify h_k. For $k > 0$, we have

$$\begin{aligned}h_k &= \sum_{n=1}^{\infty} n(k-n) u_n u_{k-n} + \sum_{n=1}^{\infty} n(k+n)\overline{u_n} u_{k+n} \\ &= \sum_{n=1}^{k-1} n(k-n) u_n u_{k-n} + \sum_{m=1}^{\infty} (k+m)(-m) u_{k+m}\overline{u_m} + \sum_{n=1}^{\infty} n(k+n)\overline{u_n} u_{k+n},\end{aligned}$$

where we used the substitution $m = n - k$. Clearly the middle term cancels the last term, so

$$h_k = \sum_{n=1}^{k-1} n(k-n) u_n u_{k-n}.$$

It is easy to see that $h_0 = 0$ due to cancellations, while if $k < 0$, we get

$$h_k = -\sum_{n=1}^{|k|-1} n(|k|-n)\overline{u_n u_{|k|-n}} = -\overline{h_{|k|}}.$$

Note in particular that $h_1 = h_{-1} = 0$. We thus obtain

$$(u_x H u_x)(x) = \sum_{k=2}^{\infty} \left(i h_k e^{ikx} - i\overline{h_k} e^{-ikx} \right).$$

From here we slightly differ from the proof of [22], although the idea remains the same. Applying B to the function h yields

$$G(u)(x) = B(h)(x) = \sum_{k=2}^{\infty} \left(i b(k) h_k e^{ikx} - i b(-k) \overline{h_k} e^{-ikx} \right).$$

We estimate only the first part of the sum, the second is similar:

$$\begin{aligned}\|G(u)\|_{L^\infty} &\leq \sum_{k=2}^{\infty}\sum_{n=1}^{k-1} b(k)n(k-n)\,|u_n|\,|u_{k-n}| \\ &= \sum_{n=1}^{\infty}\sum_{k=n+1}^{\infty} b(k)n(k-n)\,|u_n|\,|u_{k-n}| \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b(n+m)nm\,|u_n|\,|u_m|\,.\end{aligned}$$

Using the assumption on the symbol of B, we then have

$$\begin{aligned}\|G(u)\|_{L^\infty} &\leq C\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{nm}{1+(m+n)^2}\,|u_n|\,|u_m| \\ &\leq 2C\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\sqrt{nm}\,|u_n|\,|u_m|}{n+m} \\ &\leq 2C\pi\left(\sum_{n=1}^{\infty} n\,|u_n|^2\right) \leq C\pi\,\|u\|_{H^{1/2}}^2\,,\end{aligned}$$

where the inequality in the last line is precisely the well-known Hilbert double series theorem. □

Finally, we will provide the following two estimates.

Lemma 4. *Let $u\colon S^1 \to \mathbb{R}$ be a smooth function and B be a Fourier multiplier of order $s < -1/2$. Then*

$$\|Bu\|_{L^\infty} \leq C\,\|u\|_{L^2}\,,$$

for some constant $C > 0$.

Proof. Since B is of order $s < -1/2$, there exists a constant $\tilde{C}$ such that

$$b(m) \leq \tilde{C}(1+m^2)^{s/2}$$

Express u in a Fourier basis $u(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx}$. Then, we have

$$(Bu)(x) = \sum_{m\in\mathbb{Z}} b(m)u_m e^{imx},$$

and thus

$$\begin{aligned}\|Bu\|_{L^\infty} &\leq \sum_{m\in\mathbb{Z}} |b(m)u_m| \\ &\leq \tilde{C}\sum_{m\in\mathbb{Z}}(1+m^2)^{s/2}\,|u_m| \\ &\leq \tilde{C}\left(\sum_{m\in\mathbb{Z}}(1+m^2)^{s}\right)^{1/2}\|u\|_{L^2} \\ &\leq C\,\|u\|_{L^2}\,.\end{aligned}$$

□

Lemma 5. *Let $u, v\colon S^1 \to \mathbb{R}$ be smooth functions, B_1 be Fourier multiplier operator of order $k_1 \leq 3/2$ and B_2 be Fourier multiplier operator of order $k_2 < -1/2$. Then, we have*

$$\|B_2(uB_1v)\|_{L^\infty} \leq C\,\|u\|_{H^{3/2}}\,\|v\|_{H^{3/2}}\,,$$

for some constant $C > 0$.

Proof. Since B_2 is of order $k_2 < -1/2$, by virtue of Lemma 4, we get

$$\|B_2(uB_1v)\|_{L^\infty} \leq C_1 \|uB_1v\|_{L^2}$$

Now, we will recall the following inequality (see [17, Lemma 2.3]) on pointwise multiplication in Sobolev spaces, valid for $q > 1/2$ and $0 \leq \rho \leq q$:

$$\|uw\|_{H^\rho} \lesssim \|u\|_{H^q} \|w\|_{H^\rho} .$$

We deduce from it, using $q = 3/2$ and $\rho = 0$, that

$$\|uB_1v\|_{L^2} \lesssim \|u\|_{H^{3/2}} \|B_1v\|_{L^2} \lesssim \|u\|_{H^{3/2}} \|v\|_{H^{k_1}} \lesssim \|u\|_{H^{3/2}} \|v\|_{H^{3/2}} ,$$

which achieves the proof. □

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