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From light edges to strong edge-colouring of 1-planar graphs

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A strong edge-colouring of an undirected graph \(G\) is an edge-colouring where every two edges at distance at most 2 receive distinct colours. The strong chromatic index of \(G\) is the least number of colours in a strong edge-colouring of \(G\). A conjecture of Erdős and Nešetřil, stated back in the 80’s, asserts that every graph with maximum degree \(\Delta\) should have strong chromatic index at most roughly \(1.25\Delta^2\). Several works in the last decades have confirmed this conjecture for various graph classes. In particular, lots of attention have been dedicated to planar graphs, for which the strong chromatic index decreases to roughly \(4\Delta\), and even to smaller values under additional structural requirements.

In this work, we initiate the study of the strong chromatic index of 1-planar graphs, which are those graphs that can be drawn on the plane in such a way that every edge is crossed at most once. We provide constructions of 1-planar graphs with maximum degree \(\Delta\) and strong chromatic index roughly \(6\Delta\). As an upper bound, we prove that the strong chromatic index of a 1-planar graph with maximum degree \(\Delta\) is at most roughly \(24\Delta\) (thus linear in \(\Delta\)). The proof of this result is based on the existence of light edges in 1-planar graphs with minimum degree at least 3.

Keywords: strong edge-colouring, strong chromatic index, 1-planar graphs, light edges.

1 Introduction

Planar graphs are those graphs which can be drawn in the plane in such a way that no two edges cross. Colouring planar graphs has been one of the most active fields of graph theory, due in particular to the investigations that led to the well-known Four-Colour Theorem [1, 2]. Since then, whenever considering new graph problems, it generally makes sense wondering what happens for planar graphs. These graphs, however, are far from catching the structure of real-world graphs; for a given problem, one possible next direction can thus be to consider graph families that enclose planar ones.

One of the most natural generalizations of planar graphs is that of 1-planar graphs, which are those graphs that can be drawn on the plane in such a way that every edge is crossed at most once. These graphs were first considered by Ringel [22], as he was investigating a possible generalization of the Four-Colour Theorem. Since then, many aspects of 1-planar graphs have been considered in the literature, including structural aspects, colouring aspects, topological aspects, and so on. We refer the interested reader to the recent survey by Kobourov, Liotta and Montecchiani on this topic [18].

Our goal in this work is to initiate the study of the strong chromatic index of 1-planar graphs. For a graph \(G\), a strong edge-colouring of \(G\) is an edge-colouring where no two edges at distance at most 2 are assigned the same colour. To make it more precise, let us recall that two edges \(e, f\) are at distance 1 if they share an end, while \(e\) and \(f\) are at distance 2 if they are not at distance 1 and an end of \(e\) is adjacent to an end of \(f\). A strong edge-colouring of \(G\) can thus also be regarded as an edge-partition of \(G\) into induced matchings, or as a proper vertex-colouring of the square of the line graph of \(G\). The strong chromatic index of \(G\), denoted \(\chi'_s(G)\), is the least number of distinct colours assigned by a strong edge-colouring of \(G\).

The notion of strong edge-colouring was first introduced by Fouquet and Jolivet [12]. One of the leading conjectures in this field is that of Erdős and Nešetřil [9], stated back in the 1980’s (when no confusion is possible, we here and further denote by \(\Delta\) the maximum degree of a given graph):

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Theorem 1.2. Maximum degree $\Delta = 3$.

Conjecture 1.1 is still wide open in general. It was verified for graphs with maximum degree $\Delta$ to roughly $3\Delta$. Hence, we mean an edge one of whose ends is a $1$-vertex and the other is a $y$-vertex. Light edges in $1$-planar graphs were first studied by Fabrici and Madaras, who notably proved that $(3,\leq 29)$-edges [10]. Later on, Hudák and Šugerek [17] proved that every $1$-planar graph $G$ with $\delta(G) \geq 4$ has a $(4,\leq 13)$-, $(5,\leq 9)$-, $(6,\leq 8)$- or $(7,7)$-edge. In the latter work, the authors also provided an optimal result regarding the existence of light edges in $1$-planar graphs $G$ with $\delta(G) \geq 5$. Some other results of this sort exist, see e.g. those in [18]. However, by the time where the results of the current work were obtained, no such result on light edges in $1$-planar graphs $G$ with $\delta(G) \geq 3$ was known. To fill in this gap, we originally proved that $1$-planar graphs with minimum degree at least $3$ have $(\leq 29, \leq 29)$-edges. Independently, however, other comparable and even stronger results of this sort have appeared in the literature. We explain this situation in Section 2.

2 Light edges in $1$-planar graphs with minimum degree $3$

As an indication of what light edges one should expect to find in $1$-planar graphs with minimum degree at least $3$, let us mention the following conjecture of Hudák and Šugerek:

Conjecture 2.1 (Hudák, Šugerek [17]). Let $G$ be a $1$-planar graph of minimum degree $3$. Then $G$ contains a $(3, \leq 20)$-, $(4, \leq 13)$-, $(5, \leq 9)$-, $(6, \leq 8)$-, or $(7,7)$-edge.

Towards that conjecture, a result we proved is the following:

Theorem 2.2. Every $1$-planar graph $G$ with $\delta(G) \geq 3$ has a $(\leq 29, \leq 29)$-edge.
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Figure 1: A 1-planar graph $K^*_\Delta$ with maximum degree $\Delta \geq 5$ and strong chromatic index $6\Delta - 12$.

the authors proved that every 1-planar graph $G$ with $\delta(G) \geq 3$ has a $3, (\leq 23)$-, $(4, \leq 11)$-, $(5, \leq 9)$-, $(6, \leq 8)$-, or $(7, 7)$-edge.

Looking at the publication history of [20] and [21], it seems that our Theorem 2.2 was actually the first
result of this sort to be publicly available online, in a previous version of the current paper [4]. This is the
main reason why we keep using Theorem 2.2 in the current paper, instead of using the better results. We
however omit its proof, which the interested reader can still find in [4].

Let us finally mention that using the better results from [20] and [21] in place of Theorem 2.2 in the proof
of Theorem 3.3 (and thus of Corollary 3.4) would improve the obtained bounds, but by a constant additive
term only.

3 Application to strong edge-colouring of 1-planar graphs

Using, in particular, Theorem 2.2 we study in this section strong edge-colourings of 1-planar graphs.

3.1 Lower bounds

Let $\Delta \geq 5$, and let $K^*_\Delta$ be the graph obtained from $K_6$ by attaching $\Delta - 5$ new pendant vertices to every
vertex. It can be observed that every two edges of $K^*_\Delta$ are at distance at most 2 apart. Furthermore, $K^*_\Delta$ is
clearly 1-planar since $K_6$ is the biggest 1-planar complete graph (see e.g. [8]). Thus, for every $\Delta \geq 5$ there
are 1-planar graphs with maximum degree $\Delta$ and strong chromatic index $6\Delta - 15$.

Actually, 1-planar graphs with maximum degree $\Delta \geq 5$ and slightly larger strong chromatic index exist,
as attested by the following construction, depicted in Figure 1 Start from a $K_6$ on vertex set $\{u_1, \ldots, u_6\}$,
and replace each of the edges $u_1u_2$, $u_3u_4$ and $u_5u_6$ by a complete bipartite graph $K_{2, \Delta - 4}$. Denote the
resulting graph by $K^*_\Delta$.

By construction of $K^*_\Delta$, all $u_i$’s have degree $\Delta$ (while the other vertices have degree 2), and $K^*_\Delta$ is 1-
planar, as attested by the fact that $K_6$ is 1-planar (see Figure 1). Its strong chromatic index is deduced in
the following proposition.

Proposition 3.1. Every graph $K^*_\Delta$ has strong chromatic index $6\Delta - 12$. Consequently, for every $\Delta \geq 5$,
there exist 1-planar graphs with maximum degree $\Delta$ and strong chromatic index $6\Delta - 12$.

Proof: Every edge of $K^*_\Delta$ is incident to at least one of the $u_i$’s, while the $u_i$’s, with the exception of the
pairs $\{u_1, u_2\}$, $\{u_3, u_4\}$ and $\{u_5, u_6\}$, are all adjacent. It is easy to see that every two edges of $K^*_\Delta$ are at
The fact that Configuration A is forbidden implies that a good edge-colouring is always proper. It also implies that, for every colour $i$ assigned by $\phi$, the $i$-graph $M_i$ is well defined. We now prove that the fact that Configurations B and C are forbidden implies that each graph $M_i$ is 1-planar.
Lemma 3.2. Let $G$ be a 1-planar graph, and $\phi$ be a good edge-colouring of $G$. For every colour $i$ assigned by $\phi$, the $i$-graph $M_i$ is 1-planar.

Proof: Consider a 1-planar embedding of $G$ in the plane, and let us focus on the $i$-graph $M_i$ defined by $\phi$ for some assigned colour $i$. From the embedding of $G$, we can directly derive a corresponding embedding of $M_i$, where each vertex of $M_i$ is “shaped” just as the associated edge in $G$, and every edge of $M_i$, which results from any corresponding edge of $G$, is drawn in $M_i$ the same way as in $G$. Note that the fact that Configurations A and B are forbidden implies that, in the resulting embedding, no two vertices of $M_i$ overlap. The fact that Configuration C is forbidden implies that, in the embedding, no edge of $M_i$ goes “through” a vertex. Thus, vertices of $M_i$ are drawn in well separate locations, and the only crossing elements of $M_i$ are edges.

Now, by the embedding above, we get that the edges of $M_i$ correspond to actual edges of $G$, embedded in the similar way in the plane. From this we directly get that $M_i$ cannot have an edge crossed more than once, as otherwise $G$ would have one as well, a contradiction to the choice of its embedding.

We now prove an upper bound on the minimum number of colours in a good edge-colouring of a 1-planar graph.

Theorem 3.3. Every 1-planar graph $G$ admits a good $\mu$-edge-colouring, where $\mu = \max\{3\Delta+55, 4\Delta-1\}$.

Proof: To make our arguments work, we need to prove a stronger statement dealing with missing edges that could be involved in crossings. More precisely, we define a ghost triplet as an ordered triplet $(u,v,xy)$ where:

- $u, v, x, y$ are four pairwise distinct vertices;
- $uv \not\in E(G)$ and $xy \in E(G)$;
- $xy$ is not crossed;
- the embedding of $G$ can be extended directly to a 1-planar embedding of $G+uv$ (i.e., all vertices and edges (different from $uv$) remain drawn the same) in such a way that $uv$ and $xy$ cross.

In what follows, to prove the existence of good edge-colourings of $G$, we focus on even more restricted edge-colourings. Namely, given a set $T$ of ghost triplets of $G$ where each edge $xy$ is involved in only one triplet $(u,v,xy) \in T$, we prove that $G$ admits what we call a $\mu$-$T$-edge-colouring $\phi$, which is a good $\mu$-edge-colouring where the following bad configuration also does not appear.

4. A ghost triplet $(u, v, xy) \in T$ where an edge incident to $u$, an edge incident to $v$, and $xy$ receive the same colour by $\phi$ (Configuration D).

The proof is by induction on the number of vertices and edges of a 1-planar graph $G$. We also prove it by looking at $G$ as a graph being a subgraph of a graph with maximum degree $\Delta$. This notion of $\Delta$ is important to keep track of the number of ghost triplets $(u, v, xy)$ involving a given vertex $u$. In particular, below, the number of ghost triplets involving $u$ will never exceed $\Delta - d(u)$. The number of colours we use
is with respect to $\Delta$ (not the actual $\Delta(G)$) which is favourable, since $\Delta(G) \leq \Delta$ and there are thus more colours available (compared to what the real maximum degree of $G$ would allow).

Since the claim is obviously true when $G$ is small, we focus on the general case. Let $\Gamma$ be a fixed 1-planar embedding of $G$ in the plane, $\mathcal{T}$ be a set of ghost triplets, and consider any edge $uv$ of $G$. To use induction, we will consider the smaller graph $G' = G - uv$, with $\mathcal{T}'$ being defined from $\mathcal{T}$ as follows:

- If $uv$ is crossed by an edge $xy$, then $\mathcal{T}' = \mathcal{T} \cup \{u, v, xy\}$;
- Otherwise, i.e., $uv$ is not crossed in $G$, then $\mathcal{T}' = \mathcal{T}$.

An important point, to make the notion of ghost triplets usable, is that we consider $G'$ embedded in the plane following $\Gamma$, i.e., the 1-planar embedding of $G'$ is directly inherited from the 1-planar embedding of $G$. Note also that $\Delta(G') \leq \Delta(G) \leq \Delta$. Since $G'$ is smaller than $G$, it has a $\mu$-$\mathcal{T}'$-edge-colouring $\phi$ by the induction hypothesis, which we wish to extend to $G$ with $\mathcal{T}$, i.e., to $uv$. To that aim, we need to assign a colour $\alpha$ to $uv$ that does not create any of the Configurations A, B, C, or D. Let us describe why forbidding Configuration D is important: assume that, in $G$, edge $uv$ is crossed by an edge $xy$. If, in $G'$, one edge incident to $u$, one edge incident to $v$, and $xy$ all receive the same colour by $\phi$, then note that Configuration C would be created in $G$ no matter what colour is assigned to $uv$.

Let us now describe the constraints applying to $\alpha$.

- To avoid creating Configuration A, $\alpha$ must be different from all colours assigned by $\phi$ to the edges incident to $u$ and $v$. This is a set of $n_A \leq n_{A,u} + n_{A,v}$ forbidden colours, with $n_{A,u} = d_{G'}(u) \leq \Delta - 1$ and $n_{A,v} = d_{G'}(v) \leq \Delta - 1$.

- To avoid creating Configuration B, $\alpha$ must be different from the colour of the unique edge crossing $uv$, if it exists. This is a set of $n_B \leq 1$ forbidden colours.

- To avoid creating Configuration C, $\alpha$ must be different from:
  - the colours assigned to the $n_{C,x} = d_{G'}(x) - 1 \leq \Delta - 1$ edges incident to $x$ in $G'$, if $xy$ is the (unique) edge crossing $uv$;
  - the colours assigned to the $n_{C,u} \leq d_{G'}(u) \leq \Delta - 1$ edges crossing an edge incident to $u$;
  - the colours assigned to the $n_{C,v} \leq d_{G'}(v) \leq \Delta - 1$ edges crossing an edge incident to $v$.

This is a set of $n_C \leq n_{C,x} + n_{C,u} + n_{C,v}$ forbidden colours.

- To avoid creating Configuration D, $\alpha$ must be different from:
  - the colours assigned to the $n_{D,x} = d_{G'}(x) - 1 \leq \Delta - 1$ edges incident to $x$ in $G'$, if $uv$ is involved in a (unique) ghost triplet $(x, y, uv)$;
  - the colours assigned to the at most $n_{D,u} \leq \Delta - d_{G'}(u)$ edges $xy$ such that $(u, a, xy)$ is a ghost triplet;
  - the colours assigned to the at most $n_{D,v} \leq \Delta - d_{G'}(v)$ edges $xy$ such that $(a, v, xy)$ is a ghost triplet.

This is a set of $n_D \leq n_{D,x} + n_{D,u} + n_{D,v}$ forbidden colours.

We note that each edge distinct from $uv$ and incident to $u$ can forbid at most two colours for $uv$, namely because of Configurations A and C (the case where that edge is crossed). This is because, on the other hand, if an edge incident to $u$ is missing, we only have to deal with Configuration D, which yields only one constraint. Also, the case bringing the most constraints is when $uv$ is crossed, in which case there are at most $\Delta$ constraints because of Configurations B and C, compared, when $uv$ is not crossed, to the worst case which is when $uv$ is in a ghost triplet (in which case there are at most $\Delta - 1$ constraints because of Configuration D). From these arguments, in general the case bringing the most constraints is when $u, v$ have degree $\Delta$, and all their incident edges are crossed.

To prove our claim, we apply these arguments by considering light structures in $G$. We distinguish the following three cases.
1. Assume \( \delta(G) \geq 3 \). Since \( G \) is 1-planar, according to Theorem 3.2, it has a (\( \leq 29, \leq 29 \))-edge colouring \( uv \). We here consider \( G' = G - uv \), and \( T' \) defined as mentioned earlier. In particular, we retain the 1-planar embedding \( \Gamma \) of \( G \) for \( G' \). Since \( G' \) is smaller than \( G \), it has a \( \mu \)-\( T' \)-edge-colouring \( \phi \) by the induction hypothesis, which we wish extend to \( G \) and \( T \), i.e., to \( uv \). According to the arguments above, the worst-case scenario is when \( u \) and \( v \) have degree precisely 29 in \( G \) (or 28 in \( G' \)) and are each involved in \( \Delta - 29 \) ghost triplets, and \( uv \) is crossed by an edge \( xy \) where \( d_{G'}(x) = d_{G'}(y) = \Delta \). In that case, we have \( n_{A,u} = n_{A,v} = n_{C,u} = n_{C,v} = \Delta - 1, \) \( n_{D,u} = n_{D,v} = 0 \), and \( n_{D,u} = n_{D,v} = 0 \). There are thus at most \( 3\Delta + 54 \) colours forbidden for \( uv \), and we can thus extend \( \phi \) with an available colour.

2. Assume \( G \) has a 1-vertex \( u \) with unique neighbour \( v \). We again consider \( G' = G - uv \), and \( T' \) defined as previously. Let us consider a \( \mu \)-\( T' \)-edge-colouring \( \phi \) of \( G' \). This time, because \( d_{G}(u) = 1 \), we have \( n_{A,u} = n_{C,u} = 0 \). Then, the most constraints is when \( u \) is involved in \( \Delta - 1 \) ghost triplets, and when \( uv \) is crossed and \( v \) has degree \( \Delta \). In that very case, \( n_{A,u} = n_{C,u} = n_{D,u} = n_{D,v} = 0 \), \( n_{A,v} = n_{C,v} = n_{C,x} = n_{D,u} = \Delta - 1 \), and \( n_{B} = 1 \). Thus, there are at most \( 4\Delta - 3 \) colours forbidden for \( uv \) by \( \phi \), and we can thus extend \( \phi \) with an available colour.

3. Assume \( G \) has a 2-vertex \( u \), and let \( v \) be any neighbour of \( u \). Consider \( G' \), \( T' \) and \( \phi \) as before. Because \( d_{G}(u) = 2 \), we have \( n_{A,u} = n_{C,u} = 1 \). Then, the most constraints is when \( u \) is involved in \( \Delta - 2 \) ghost triplets, and when \( uv \) is crossed and \( v \) has degree \( \Delta \). In such a case, we have \( n_{A,u} = n_{B} = n_{C,u} = 1 \), \( n_{A,v} = n_{C,v} = n_{C,x} = \Delta - 1 \), \( n_{D,u} = n_{D,v} = 0 \), and \( n_{D,u} = n_{D,v} = 0 \). There are thus at most \( 4\Delta - 2 \) colours forbidden for \( uv \) by \( \phi \), and we can thus extend \( \phi \) with an available colour.

In all cases, we can thus extend \( \phi \) to \( uv \) because we have a pool of \( \mu \) colours while there are at most \( \mu - 1 \) constraints around. This concludes the proof of Theorem 3.3.

**Corollary 3.4.** For every 1-planar graph \( G \), \( \chi'_c(G) \leq 6 \cdot \max\{3\Delta + 55, 4\Delta - 1\} \).

**Proof:** By Theorem 3.3, \( G \) has a good \( (\max\{3\Delta + 55, 4\Delta - 1\}) \)-edge-colouring \( \phi \). Now, by Lemma 3.2, for every colour \( i \) assigned by \( \phi \), the graph \( M_i \) is 1-planar, and thus admits a proper 6-vertex-colouring \( \psi_i \). Every two adjacent edges of \( G \) are assigned different colours by \( \phi \), while, for every two edges at distance 2 being assigned colour \( i \) by \( \phi \), the two corresponding vertices in \( M_i \) receive different colours by \( \psi_i \). Thus \( \phi \) and the \( \psi_i \)'s yield a strong \( (6 \cdot \max\{3\Delta + 55, 4\Delta - 1\}) \)-edge-colouring of \( G \).

**References**


