Dynamics under location uncertainty and other energy-related stochastic subgrid schemes
Valentin Ressegui, Long Li, Gabriel Jouan, Pierre Derian, Etienne Mémin, Bertrand Chapron

To cite this version:
Valentin Ressegui, Long Li, Gabriel Jouan, Pierre Derian, Etienne Mémin, et al.. Dynamics under location uncertainty and other energy-related stochastic subgrid schemes. 2019 - Workshop Conservation Principles, Data and Uncertainty in Atmosphere-Ocean Modelling, Apr 2019, Potsdam, Germany. pp.1-70. hal-02103233

HAL Id: hal-02103233
https://hal.archives-ouvertes.fr/hal-02103233
Submitted on 18 Apr 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Dynamics under location uncertainty and other energy-related stochastic subgrid schemes

V. Resseguier
L. Li, G. Jouan, P. Derian,
E. Memin, B. Chapron
Motivations

• More rigorously identified subgrid dynamics effects

• Quantification of modeling errors (UQ)

Ensemble forecasts and data assimilation
Contents

• Models under location uncertainty (LU)

• Some parameterization of the models under location uncertainty

• A new energy-budget-based stochastic scheme: WaveHyperv

• Numerical comparisons
Part I
Models under location uncertainty (LU)
LU : Adding random velocity

\[ v = w + \sigma \dot{B} \]
LU : Adding random velocity

Resolved large scales

\[ v = w + \sigma \dot{B} \]
LU : Adding random velocity

\[ \nu = w + \sigma \dot{B} \]

- Resolved large scales
- White-in-time small scales
LU: Adding random velocity

\[ \nu = \mathbf{w} + \sigma \dot{\mathbf{B}} \]

Resolved large scales

White-in-time small scales

Large scales: \( \mathbf{w} \)
Small scales: \( \sigma \dot{\mathbf{B}} \)

Variance tensor:

\[ a = a(x, x) = \mathbb{E}\left\{ \sigma dB \left( \sigma dB \right)^T \right\} \]

\[ dt \]
LU : Adding random velocity

\[ \nu = w + \sigma \dot{B} \]

Large scales:
\[ w \]
Small scales:
\[ \sigma \dot{B} \]

Variance tensor:
\[ a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} \]

Resolved large scales

White-in-time small scales

References:
- Mikulevicius & Rozovskii, 2004
- Flandoli, 2011
- Memin, 2014
- Resseguier et al. 2017 a, b, c
- Cai et al. 2017
- Chapron et al. 2018
- Yang & Memin 2019
- Resseguier et al. 2019 a,b
- Holm, 2015
- Holm and Tyranowski, 2016
- Arnaudon et al. 2017
- Cotter and al. 2017
- Crisan et al., 2017
- Gay-Balmaz & Holm 2017
- Cotter and al. 2018 a, b
- Cotter and al. 2019
LU : Adding random velocity

\[ \nu = w + \sigma \dot{B} \]

Resolved large scales

White-in-time small scales

Large scales:

\[ w \]

Small scales:

\[ \sigma \dot{B} \]

Variance tensor:

\[ a = a(x, x) = \frac{E\{\sigma dB (\sigma dB)^T\}}{dt} \]

LU

Memin, 2014
Resseguier et al. 2017 a, b, c
Cai et al. 2017
Chapron et al. 2018
Yang & Memin 2019
Resseguier et al. 2019 a,b

SALT

Holm, 2015
Holm and Tyranowski, 2016
Arnaudon et al. 2017
Cotter and al. 2017

References:

Mikulevicius & Rozovskii, 2004
Flandoli, 2011

Crisan et al., 2017
Gay-Balmaz & Holm 2017
Cotter and al. 2018 a, b
Cotter and al. 2019
Advection of tracer $\Theta$

$$\frac{D\Theta}{Dt} = 0$$

Large scales: $w$
Small scales: $\sigma \dot{B}$
Variance tensor:

$$a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}$$
Advection of tracer $\Theta$

Large scales: $w$
Small scales: $\sigma \dot{B}$
Variance tensor:
$$a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$
Advection of tracer $\Theta$

Large scales: $w$
Small scales: $\sigma \dot{B}$
Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$
Advection of tracer $\Theta$

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \lim_{\Delta t \to 0} \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{\Delta t}$$

$$\partial_t \Theta + w^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)$$
Advection of tracer $\Theta$

Large scales:
$\mathbf{w}$

Small scales:
$\sigma \dot{B}$

Variance tensor:
$a = a(x, x) = \frac{\mathbb{E}\left\{\sigma dB \left(\sigma dB\right)^T\right\}}{dt}$

$$\partial_t \Theta + \mathbf{w}^* \cdot \mathbf{\nabla} \Theta + \sigma \dot{B} \cdot \mathbf{\nabla} \Theta = \mathbf{\nabla} \cdot \left( \frac{1}{2} a \mathbf{\nabla} \Theta \right)$$
Advection
of tracer $\Theta$

Large scales: $w$
Small scales: $\sigma \dot{B}$
Variance tensor:
$a = a(x, x) = \frac{1}{dt} \mathbb{E}\{\sigma dB (\sigma dB)^T\}$

$\partial_t \Theta + \omega^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta$ = $\nabla \cdot \left( \frac{1}{2} \alpha \nabla \Theta \right)$

Drift correction
Advection
of tracer $\Theta$

Multiplicative random forcing

Large scales:
$w$
Small scales:
$\sigma \dot{B}$
Variance tensor:
$a = a(x, x) = \frac{\mathbb{E} \{ \sigma dB (\sigma dB)^T \}}{dt}$

$\partial_t \Theta + \mathbf{w}^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)$

Drift correction
Advection of tracer $\Theta$

Large scales: $\mathbf{w}$
Small scales: $\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \mathbb{E}\left\{ \sigma dB (\sigma dB)^T \right\} dt$$

Multiplicative random forcing

Drift correction

$$\partial_t \Theta + \mathbf{w}^\star \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)$$
Large scales: \( w \)
Small scales: \( \sigma \dot{B} \)

Variance tensor:

\[
a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}
dt
\]

Advection of tracer \( \Theta \)

Multiplicative random forcing

Drift correction

\[
\begin{aligned}
\partial_t \Theta + w^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta &=

\nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)
\end{aligned}
\]
Advection of tracer $\Theta$

Large scales: $w$
Small scales: $\sigma \dot{B}$
Variance tensor:

\[ a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} \]

\[ \partial_t \Theta + \nabla \cdot (w^* \Theta) + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right) \]

Multiplicative random forcing
Balanced energy exchanges
Drift correction

Large scales:

Small scales:
Variance tensor:
Part II
Some parameterization of LU models
Part II
Some parameterization
of LU models

$\sigma = ?$
Spectral model
(homogeneous and stationary $\sigma \dot{B}$)

Large scales:
- $w$

Small scales:
- $\sigma \dot{B}$

Variance tensor:
- $a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} / dt$

KE Spectrum

Reference:
Resseguier, Memin & Chapron 2017b

Code online
Spectral model
(homogeneous and stationary $\sigma \dot{B}$)

Large scales:
$w$
Small scales:
$\sigma \dot{B}$
Variance tensor:
\[
a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}\frac{dt}{dt}
\]

KE Spectrum

Fixed spectrum at small scales

Reference:
Resseguier, Memin & Chapron 2017b
Spectral model
(homogeneous and stationary $\sigma \dot{B}$)

Large scales: $w$
Small scales: $\sigma \dot{B}$
Variance tensor:

\[ a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}/dt \]

Fixed spectrum at small scales

$\sigma \dot{B} = \text{(filter)} \ast \text{(white noise)}$

Reference:
Resseguier, Memin & Chapron 2017b
Absolute Diffusivity Spectral Density
(homogeneous but non-stationary and tuning-free $\sigma \dot{B}$)

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Reference:
Resseguier, Pan & Fox-Kemper 2019a
Absolute Diffusivity
Spectral Density
(homogeneous but non-stationary and tuning-free $\sigma\dot{B}$)

Large scales: $w$
Small scales: $\sigma\dot{B}$
Variance tensor:

$$a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} dt$$

Absolute Diffusivity Spectral Density

$$A(\kappa) = E(\kappa) \tau(\kappa)$$

Reference:
Resseguier, Pan & Fox-Kemper 2019a
Absolute Diffusivity Spectral Density

(homogeneous but non-stationary and tuning-free $\sigma \dot{B}$)

$$A(\kappa) = E(\kappa) \tau(\kappa)$$

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor:
$$a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}/dt$$

Reference: Resseguier, Pan & Fox-Kemper 2019a
Absolute Diffusivity
Spectral Density
(homogeneous but non-stationary and tuning-free $\sigma \dot{B}$)

Absolute Diffusivity Spectral Density

$$A(\kappa) = E(\kappa)\tau(\kappa)$$

Large scales: $w$
Small scales: $\sigma \dot{B}$
Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Reference:
Resseguier, Pan & Fox-Kemper 2019a
Absolute Diffusivity
Spectral Density

(homogeneous but non-stationary and tuning-free $\sigma \dot{B}$)

\[
A(\kappa) = E(\kappa) \tau(\kappa)
\]

Reference:
Resseguier, Pan & Fox-Kemper 2019a
Absolute Diffusivity Spectral Density
(homogeneous but non-stationary and tuning-free $\sigma \dot{B}$)

$$A(\kappa) = E(\kappa) \tau(\kappa)$$

Reference: Resseguier, Pan & Fox-Kemper 2019a
Absolute Diffusivity Spectral Density
(homogeneous but non-stationary and tuning-free $\sigma \dot{B}$)

Large scales:
$\omega$
Small scales:
$\sigma \dot{B}$

Variance tensor:
$a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}$

$dt$

Absolute Diffusivity Spectral Density
$A(\kappa) = E(\kappa) \tau(\kappa)$

Residual non-stationary ADSD

$\sigma \dot{B} = \text{(filter)} \ast \text{(white noise)}$

Reference:
Resseguier, Pan & Fox-Kemper 2019a
Random switching of points
(heterogeneous and non-stationary $\sigma \dot{B}$)

Large scales:
$w$
Small scales:
$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Reference:
Resseguiere, Li, Jouan, Derian, Memin & Chapron 2019b
Random switching of points
(heterogeneous and non-stationary $\sigma \dot{B}$)

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E} \{ \sigma dB (\sigma dB)^T \}}{dt}$$

Reference:
Resseguiere, Li, Jouan, Derian, Memin & Chapron 2019b
Random switching of points
(heterogeneous and non-stationary $\sigma \dot{B}$)

Large scales: $w$
Small scales: $\sigma \dot{B}$
Variance tensor:

$\alpha = \alpha(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}/dt$

Reference:
Resseguier, Li, Jouan, Derian, Memin & Chapron 2019b
Random switching of points
(heterogeneous and non-stationary $\sigma \dot{B}$)

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}$$

Neighbor
Centered neighbor
Random selection

Reference:
Resseguier, Li, Jouan, Derian, Memin & Chapron 2019b
Random switching of points
(heterogeneous and non-stationary $\sigma \dot{B}$)

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E} \{ \sigma dB (\sigma dB)^T \}}{dt}$$

Reference:
Resseguier, Li, Jouan, Derian, Memin & Chapron 2019b
Random switching of points
(heterogeneous and non-stationary $\sigma \dot{B}$)

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor:
$a = a(x, x) = \mathbb{E}\left\{ \sigma dB (\sigma dB)^T \right\}$

Reference:
Resseguier, Li, Jouan, Derian, Memin & Chapron 2019b
Random switching of points
(heterogeneous and non-stationary $\sigma \dot{B}$)

Large scales:
$w$
Small scales:
$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}$$

Reference:
Resseguier, Li, Jouan, Derian, Memin & Chapron 2019b
Random switching of points
(heterogeneous and non-stationary $\sigma \dot{B}$)

Large scales:
$w$
Small scales:
$\sigma \dot{B}$

Variance tensor:
$a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}\frac{dt}{dt}$

Reference:
Resseguiery, Li, Jouan, Derian, Memin & Chapron 2019b
Random switching of points
(heterogeneous and non-stationary $\sigma \dot{B}$)

Large scales: $\mathbf{w}$
Small scales: $\sigma \dot{B}$

Variance tensor:
$a = a(x, x) = \mathbb{E} \{ \sigma dB (\sigma dB)^T \}/dt$

Reference:
Resseguijer, Li, Jouan, Derian, Memin & Chapron 2019b
Random switching of points
(heterogeneous and non-stationary $\sigma \dot{B}$)

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor:
\[ a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}/dt \]

Reference:
Resseguier, Li, Jouan, Derian, Memin & Chapron 2019b
Random switching of points
(heterogeneous and non-stationary $\sigma \dot{B}$)

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} dt$$

Reference:
Resseguier, Li, Jouan, Derian, Memin & Chapron 2019b
Random switching of points
(heterogeneous and non-stationary $\sigma \dot{B}$)

**Large scales:** $w$

**Small scales:** $\sigma \dot{B}$

Variance tensor:

\[ a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} \quad \text{dt} \]

Reference: Resseguier, Li, Jouan, Derian, Memin & Chapron 2019b
Random switching of points
(heterogeneous and non-stationary $\sigma \dot{B}$)

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor:

$$a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} dt$$

Reference:
Resseguiier, Li, Jouan, Derian, Memin & Chapron 2019b
Random switching of points
(heterogeneous and non-stationary $\sigma \dot{B}$)

Large scales:
$w$
Small scales:
$\sigma \dot{B}$

Variance tensor:
$a = a(x,x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} / dt$

Global ensemble

Local ensemble:

Neighbor
Centered neighbor
Random selection

Reference:
Resseguier, Li, Jouan, Derian, Memin & Chapron 2019b
Random switching of points
(heterogeneous and non-stationary $\sigma \dot{B}$)

Large scales:
$w$
Small scales:
$\sigma \dot{B}$

Variance tensor:
$a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}/dt$

Global ensemble

Local ensemble:

SVD

Reference:
Resseguier, Li, Jouan, Derian, Memin & Chapron 2019b
Part III
A new energy-budget-based stochastic scheme:
WaveHyperv
WaveHyperv

Transport equation

\[ \frac{Dq}{Dt} = \mathcal{L}[q] + \eta \]
WaveHyperv

\[ \frac{Dq}{Dt} = \mathcal{L}[q] + \eta \]

Usual deterministic subgrid tensor

e.g. Hyper-viscosity

\[ \mathcal{L}[q] = -\nu \Delta^4 q \]

Reference:
Resseguier, Li, Jouan, Derian, Memin & Chapron 2019b
WaveHyperv

Transport equation

\[
\frac{Dq}{Dt} = \mathcal{L}[q] + \eta
\]

Usual deterministic subgrid tensor

e.g. Hyper-viscosity

\[\mathcal{L}[q] = -\nu \Delta^4 q\]

Random forcing

built to meet the energy budget:

(Random energy intake) = \zeta \times \text{Dissipation}

Reference:
Resseguier, Li, Jouan, Derian, Memin & Chapron 2019b
# Summary of UQ methods

<table>
<thead>
<tr>
<th>Name</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>MU Spec</td>
<td>LU with homogeneous and stationary small-scale velocity</td>
</tr>
<tr>
<td>MU ADSD</td>
<td>LU with homogeneous, non-stationary and tuning-free small-scale velocity</td>
</tr>
<tr>
<td>MU SVD</td>
<td>LU with inhomogeneous and non-stationary small-scale velocity</td>
</tr>
<tr>
<td>WaveHyperv</td>
<td>Energy-budget-based stochastic scheme</td>
</tr>
<tr>
<td>PIC Spec</td>
<td>Perturbed initial conditions with homogeneous noise</td>
</tr>
<tr>
<td>PIC SVD</td>
<td>Perturbed initial conditions with inhomogeneous noise</td>
</tr>
</tbody>
</table>
Part IV

Numerical comparisons
Test case 1:

SQG

\[ \frac{D b}{D t} = -\alpha_{HV} \Delta^4 b \text{ Hyper-viscosity} \]

\[ u = \left( \text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b \]

Reference flow:

deterministic

SQG

512 x 512

\( t = 17 \text{ days} \)
Test case 1:

\[ \frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity} \]

\[ u = \left( \text{cst.} \, \nabla^\bot \Delta^{-\frac{1}{2}} \right) b \]

Reference flow:

deterministic

SQG

512 x 512
Test case 2:

\[
\frac{D b}{D t} = -\alpha_{HV} \Delta^4 b \quad \text{(Hyper-viscosity)}
\]

\[
u = \left( \text{cst.} \cdot \nabla^\perp \Delta^{-\frac{1}{2}} \right) b
\]

Reference flow:
- deterministic
- SQG
- 512 x 512

\( t = 10 \text{ days} \)
Test case 2:

**SQG**

\[
\frac{D b}{D t} = -\alpha_H V \Delta^{4} b \quad \text{Hyper-viscosity}
\]

\[
u = \left( \text{cst.} \cdot \nabla^{1} \Delta^{-\frac{1}{2}} \right) b
\]

Reference flow:

deterministic

SQG

512 x 512

\[t = 10 \text{ days}\]
## UQ metrics

<table>
<thead>
<tr>
<th>Metric</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>RMSE</strong></td>
<td><strong>Error</strong> of ensemble members</td>
</tr>
<tr>
<td><strong>Talagrand histogram (TH)</strong></td>
<td>Capacity of the ensemble to <strong>explore</strong> all reference possible values</td>
</tr>
<tr>
<td><strong>Bias^2-spread</strong></td>
<td>Capacity of the ensemble to <strong>explore</strong> all reference possible values</td>
</tr>
<tr>
<td><strong>CRPS</strong></td>
<td>Point-wise distance between the ensemble CFD and the indicator function of the event</td>
</tr>
<tr>
<td><strong>Energy Score (ES)</strong></td>
<td>Generalized CRPS for multivariate ensemble</td>
</tr>
</tbody>
</table>
Spreading VS Errors

(a) 

(b) 

(c) 

(d) 

(e) 

(f) 

\[
\frac{\text{MSE}}{B_0^2} 
\]

\[
\frac{\text{MSB} - \text{MEV}}{B_0} 
\]
Spreading VS Errors

Opposite conclusions

Spread

Error
Ensemble point-wise skills: CRPS
<table>
<thead>
<tr>
<th>MU Spec</th>
<th>MU ADSD</th>
<th>MU SVD</th>
<th>WaveHyperv</th>
<th>PIC Spec</th>
<th>PIC SVD</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="t=18%5Ctext%7Bdays%7D" alt="Image" /></td>
<td><img src="t=10%5Ctext%7Bdays%7D" alt="Image" /></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Ensemble point-wise skills: CRPS
Ensemble point-wise skills: CRPS

CRPS spatial of each model at day 19

CRPS spatial of each model at day 10
Ensemble point-wise skills:

CRPS

CRPS spatial of each model at day 19

CRPS spatial of each model at day 10

model
- mu_adsd
- mu_spec
- mu_svd
- wavhv50
- pic_spec
- pic_svd

model
- mu_adsd
- mu_spec
- mu_svd
- wavhv50
- pic_spec
- pic_svd
Conclusion
## Conclusion

<table>
<thead>
<tr>
<th>Method</th>
<th>RMSE (errors)</th>
<th>B^2-Var (spread)</th>
<th>CRPS (point-wise)</th>
<th>ES (global)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MU Spec</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>MU ADSD</td>
<td>++</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>MU SVD</td>
<td>-</td>
<td>++</td>
<td>+ +</td>
<td>++</td>
</tr>
<tr>
<td>WaveHyperv</td>
<td>-</td>
<td>+</td>
<td>++</td>
<td>+ +</td>
</tr>
<tr>
<td>PIC Spec</td>
<td>-</td>
<td>-</td>
<td>- -</td>
<td>- -</td>
</tr>
<tr>
<td>PIC SVD</td>
<td>- -</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>
## Conclusion

<table>
<thead>
<tr>
<th></th>
<th>RMSE (errors)</th>
<th>$B^2$-Var (spread)</th>
<th>CRPS (point-wise)</th>
<th>ES (global)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MU Spec</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>MU ADSD</td>
<td>++</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>MU SVD</td>
<td>-</td>
<td>++</td>
<td>++</td>
<td>++</td>
</tr>
<tr>
<td>WaveHyperv</td>
<td>-</td>
<td>+</td>
<td>++</td>
<td>++</td>
</tr>
<tr>
<td>PIC Spec</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>PIC SVD</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>
## Conclusion

### Table

<table>
<thead>
<tr>
<th>Method</th>
<th>RMSE (errors)</th>
<th>$B^2$-Var (spread)</th>
<th>CRPS (point-wise)</th>
<th>ES (global)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MU Spec</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>MU ADSD</td>
<td>++</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>MU SVD</td>
<td>-</td>
<td>++</td>
<td>++</td>
<td>++</td>
</tr>
<tr>
<td>WaveHyperv</td>
<td>-</td>
<td>+</td>
<td>++</td>
<td>++</td>
</tr>
<tr>
<td>PIC Spec</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>PIC SVD</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

### Key

- **+** indicates better performance
- **++** indicates significantly better performance
- **-** indicates worse performance
- **--** indicates significantly worse performance

### Notes

- MU Spec appears to perform well across all measures.
- MU ADSD shows strong performance in terms of spread and point-wise CRPS.
- MU SVD and WaveHyperv show mixed results, with strengths in some measures and weaknesses in others.
- PIC Spec and PIC SVD are not included in the comparison, as indicated by the red X.
## Conclusion

<table>
<thead>
<tr>
<th></th>
<th>RMSE (errors)</th>
<th>B^2-Var (spread)</th>
<th>CRPS (point-wise)</th>
<th>ES (global)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>MU Spec</strong></td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td><strong>MU ADSD</strong></td>
<td>++</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td><strong>MU SVD</strong></td>
<td>-</td>
<td>++</td>
<td>+ +</td>
<td>++</td>
</tr>
<tr>
<td><strong>WaveHyperv</strong></td>
<td>-</td>
<td>+</td>
<td>++</td>
<td>+ +</td>
</tr>
</tbody>
</table>

- **PIC Spec** are marked with a red cross indicating infeasibility.
- **PIC SVD** are also marked with a red cross indicating infeasibility.

22