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## **Logics of grounding**

Francesca Poggiolesi

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**Related Topics** Reductio and demonstration, ground in the work of Bolzano, the granularity of ground, the logical puzzles of ground, ground’s application to logic, ground’s application to truth-maker semantics.

## **Introduction**

The founding fathers of modern logic, illustrious scholars such as Frege, Hilbert or Russell, used this discipline to answer mathematically foundational issues; however, starting from the 1960’s logic has also been fruitfully used to deal with key philosophical notions, such as necessity, knowledge, time, probability, vagueness, and many others. But what exactly does it mean to study a philosophical notion from a logical point of view? It standardly means to use a predicate or a sentential operator to denote the philosophical concept under investigation and then to formulate and discuss axioms<sup>1</sup> that adequately characterize the predicate or the operator. For the sake of clarity let us consider the example of knowledge, which is commonly formalized via the operator  $K$  (e.g. see Hendricks and Symons (2015)). Amongst the axioms that describe the operator  $K$ , one of the most famous is the axiom  $T$ , which is

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<sup>1</sup> Although it is more common to firstly describe a novel operator or predicate via axioms, it is also possible to do that via rules.

written as  $K(A) \rightarrow A$ , where  $A$  is a formula, and can be read as “if something is known ( $K(A)$ ), then it is true ( $A$ )”. The axiom  $T$  does not tell us what someone knows, but individuates a distinctive feature of knowledge, namely that it is factive: if someone knows something, then that something is true.

Axioms characterizing a certain philosophical notion are put together to constitute *formal systems* and each formal system can be looked at from the two following perspectives:

- from a syntactic point of view, one analyzes which theorems are *provable* in a formal system  $\mathbf{S}$ . We write  $\mathbf{S} \vdash A$  to denote that the formula  $A$  is provable or derivable in, or is a theorem of, the formal system  $\mathbf{S}$ .
- from a semantic point of view, one analyzes which sentences are valid (i.e. *true* in every situation, or more technically in every *model*) in the semantics corresponding to the system  $\mathbf{S}$ . We write  $\mathbf{S} \models A$  to denote that  $A$  is valid in the semantics corresponding to the system  $\mathbf{S}$ .

The famous theorems of validity and completeness establish an equivalence between the semantic level and the syntactic level: formulas that are valid (true in every model) in the semantics corresponding to the system  $\mathbf{S}$  are all and only the theorems provable in  $\mathbf{S}$ .

Amongst the several philosophical notions that have been logically taken into account in the last fifty years, two play a prominent role: the notions of *provability* and *truth*. The peculiar logical role of these notions is clearly indicated by the fact that both have been formalized into two quite different ways. On the one hand, they have been treated just as the other philosophical notions, namely they have been formalized via predicates or sentential operators. This kind of treatment has produced impressive results: to mention a few, the formalization of truth via the predicate  $T$  has given rise to the wide literature concerning

paradoxes of truth (e.g. see Gupta (1982) and Kripke (1975)); the formalization of provability via the predicate *Bew* (from the German *Beweisbar* for *provable*) has given rise to Gödel's incompleteness theorems (1986, 1990), while the formalization of provability via a sentential operator has given rise to the modal logic of provability (Boolos (2010)).

But truth and provability have also been treated as meta-linguistic symbols: as we have said above, the metalinguistic symbol  $\vdash$  denotes provability in a formal system **S** and as such it has been widely studied by the cornerstone works of Frege, Russel, Hilbert (see van Heijenoort (1967)) and Gentzen (1969); the symbol  $\models$ , meaning truth in a certain semantic model, represents the basis of the notion of validity and has been central since the work of Tarski (1936). The importance of the treatment of truth and provability as meta-linguistic symbols can hardly be overestimated: in this respect truth and provability are the two milestone concepts on which logic itself lies and the two main tests to which any logical formal system is confronted.

Let us finally come to the notion of *grounding*, which is the central notion of this chapter and is standardly described as a relation amongst truths (or facts) that is non-causal and explanatory in nature. The concept of grounding has been long neglected or forgotten in the history of logic (exceptions can be found in this book, as chapter XXXX by Roski). This fact is all to the more astonishing once we realize that grounding seems to cover the same special logical role as the kindred notions of truth and provability: like its cousin concepts, grounding can be fruitfully formalized into two different ways, namely as (i) a predicate or sentential operator, or as (ii) a meta-linguistic relation. This double formalization, which testifies the importance of grounding as a logical notion, will structure this chapter. The next section will be dedicated to studies of grounding under the perspective of (i), which is the most developed

in the contemporary literature; while Section 3 will focus on approaches adopting perspective (ii).

In order to slightly constraint the scope of this survey, let us emphasize that here we will mean by ‘grounding’ a relation from many to one, which cannot be iterated,<sup>2</sup> and embraces a representational conception of grounding (see also chapter XXX of this book by Correia).<sup>3</sup> Most of the work developed in the formalization of grounding concerns such a notion. Moreover, we will mainly focus on the propositional logic of grounding, leaving aside the links between grounding and quantifiers: this choice is motivated by the fact that the links between quantifiers and grounding is already extensively treated by Mc Sweeney in Chapter XXX.

Finally, let us emphasize that in the last decade the literature on the logic of grounding has blossomed: there is a wide range of studies that vary in their objects, conceptions and features. In order to clarify the presentation and classification of all these studies, we first introduce some distinctions relative to the notion of grounding.

As underlined by Fine (2012b) and Correia (2014), grounding is thought to come in various types; amongst those, there are *metaphysical* grounding, *logical* grounding, *conceptual* grounding, *natural* grounding. For any type of grounding, three further distinctions apply: the

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<sup>2</sup> Litland (2018) gives a comprehensive study of the logic of iterated grounding.

<sup>3</sup> For the difference between factualist and representational conception of grounding, see Correia and Schnieder (2012). The only logic of grounding embracing a factualist conception is Correia (2010). In his work, Correia has developed his logic both at the syntactic as well as the semantic level, enriching it by a treatment of ground-theoretic equivalence. However, Krämer and Roski (2015) have shown that Correia’s logic involves theorems concerning grounding, which turn out to be opposite to our intuitions.

one between full, complete and partial grounding, the one between immediate and mediate grounding, and the one between strict and weak grounding.

According to Fine (2012b)  $A$  is a partial ground of  $C$  if  $A$  on its own or together with some other truths is a ground of  $C$ . Thus, given that  $A$  and  $B$  are the full ground of  $A \wedge B$ , each of  $A$  and  $B$  will be a partial ground of  $A \wedge B$ . The notion of full ground is never explicitly defined but it is suggested that  $A$  is a full ground of  $C$  if the truth of  $A$  is sufficient to guarantee the truth of  $C$ . Bolzano's distinction (cf. Chapter XXX by Roski) between complete and partial ground is slightly different. Following the analysis of Sebestik (1992) and Tatzel (2002), for Bolzano the multiset (a multiset is a set where the number of occurrences of the same formula counts) of all, and only, those truths each of which contributes to ground the truth  $C$  is a complete ground of  $C$ . On the other hand, each of the truths that compose the complete ground of  $C$ , as well as each strict sub-multiset of them, is said to be a partial ground of  $C$ . It seems that Bolzano and Fine are aiming at similar, if not the same, distinction, although the way they draw the line between the two concepts is different. For Fine a partial ground of a truth  $C$  can also be a full ground of  $C$ , while for Bolzano this can never be the case, since partial and complete grounds are two disjoint concepts.

Let us now move to the distinction immediate and mediate. If we could describe this distinction with proof-theoretic terminology, we would say that immediate grounding corresponds to a single (irreflexive) proof-step, while mediate grounding corresponds to a sequence of several steps of immediate grounding. In other terms, while immediate grounding is a relation that does not seem to be further reducible, mediate grounding is definable as the transitive closure of immediate grounding. As for the last distinction, strict grounding does not allow a truth to ground itself (it is irreflexive), while grounding in the weak sense allows, in fact requires, a truth to ground itself (it is reflexive).

## 2. Grounding as a predicate or as a sentential operator

In this section we introduce those logics of grounding that formalize grounding either as a predicate or as a sentential operator. The pro and cons of each approach are quite straightforward: the predicate approach provides a richer and stronger framework, but it is ontologically and formally more demanding; the operator approach is notable for its simplicity but might be more limited in its expressive power.

### 2.1. Grounding as a sentential operator

The first and most comprehensive contemporary logical study of the notion of grounding has been developed by Fine (2012a, 2012b) in two cornerstone papers where he formalizes the notion of metaphysical grounding in most of its variations, namely full and partial, and weak and strict. In what follows, we will take Fine's logic as the reference logic for grounding as a sentential operator and as the benchmark against which we present other approaches.

#### 2.1.1 Fine's Logic

*Language.* Fine adds to the first-order classical language four symbols denoting the four notions of weak full, weak partial, strict full and strict partial grounding. Accordingly, four novel types of logical formulas can be constructed in this language. Assuming that  $M, N, \dots$  denote sets of formulas and  $A, B, C, \dots$  formulas, we have:

$M \leq B$ ,  $M$  is a weak full ground for  $B$ ,

$A \ll B$ ,  $A$  is a weak partial ground for  $B$ ,

$M < B$ ,  $M$  is a strict full ground for  $B$ ,

$A < B$ ,  $A$  is a strict partial ground for  $B$ <sup>4</sup>.

*Syntax.* In this given language, Fine formulates axioms and rules that describe the four notions just introduced. These axioms and rules are divided between *pure* and *impure*:<sup>5</sup> pure rules describe the structural features of the several notions of grounding, while the impure rules describe the relationships between grounding and the other logical connectives and quantifiers. Let us start describing the former rules.

*Pure (or structural) rules.* The pure rules for grounding can be divided into two sets: those describing the relations between the four symbols introduced (they are called *Subsumption rules*), and those describing the properties that each symbol enjoys. As for the former, these outline the following relations:

- from strict grounding claims one can infer corresponding (full or partial) weak grounding claims;
- from full grounding claims one can infer corresponding (strict or weak) partial grounding claims;
- from weak full grounding claims one can infer strict full grounding claims as long as each of the grounds is a strict partial ground of the conclusion.

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<sup>4</sup> In the literature these formulas are usually called *sequents*. We use the term ‘formula’ to prevent confusion with the notion of sequent in logic, which denotes a meta-linguistic object (e.g. see Poggiolesi (2010a)).

<sup>5</sup> The distinction between pure and impure rules corresponds to the standard distinction in logic between structural and logical rules (e.g. see Poggiolesi (2010b)).

As for the structural rules introducing properties, we have that:

- weak full grounding enjoys identity and cut, namely (respectively) :

$$\begin{array}{c}
 M_1 \leq A_1, M_2 \leq A_2 \dots A_1, A_2 \dots \leq C \\
 \hline
 A \leq A \qquad \qquad \qquad M_1, M_2, \dots \leq C
 \end{array}$$

- strict partial grounding enjoys non-circularity, namely:

$$\begin{array}{c}
 A < A \\
 \hline
 \perp
 \end{array}$$

- weak partial grounding enjoys transitivity, which is also enjoyed by the following two combinations

$$\begin{array}{c}
 A \leq B, B < C \qquad \qquad A < B, B \leq C \\
 \hline
 A < C \qquad \qquad \qquad A < C
 \end{array}$$

Given the presence of the transitivity and cut rules, we assume Fine is dealing with mediate notions of grounding, although this is not specified.

*Impure (logical) axioms and rules.* The impure rules for grounding can be divided into *introduction* and *elimination* rules. The introduction rules are:<sup>6</sup>

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<sup>6</sup> The main difference between an axiom and a rule is that while an axiom does not depend on anything, in a rule the conclusion depends on its premises. Although Fine uses the word ‘rule,’ the formulas listed above are axioms rather than rules since they do not depend on

- *Conjunction*:  $A, B < A \wedge B$ .
- *Disjunction*:  $A < A \vee B$ ;  $B < A \vee B$ ;  $A, B < A \vee B$ .
- *Double negation*:  $A < \neg\neg A$ .
- *Negative conjunction*:  $\neg A < \neg(A \wedge B)$ ;  $\neg B < \neg(A \wedge B)$ ;  $\neg A, \neg B < \neg(A \wedge B)$ .
- *Negative disjunction*:  $\neg A, \neg B < \neg(A \vee B)$ .
- *Universal quantifier*:  $P(a_1), P(a_2), \dots < \forall x P x$
- *Existential quantifier*:  $P(a), E(a) < \exists x P x$ , where  $E(a)$  expresses the fact that  $a$  exists.
- *Negative universal quantifier*:  $\neg P(a_1), E(a) < \neg \forall x P x$
- *Negative existential quantifiers*:  $\neg P(a_1), \neg P(a_2), \dots, T(a_1, a_2, \dots) < \neg \exists x P x$ , where  $T(a_1, a_2, \dots)$  stands for the totality claim, i.e.  $a_1, a_2, \dots$  are all of the individuals that there are.

The elimination rules are:

- *Conjunction*: from  $M < A \wedge B$ , one can infer  $M \leq \{A, B\}$
- *Disjunction*: from  $M < A \vee B$ , one can infer  $M \leq A$ ;  $M \leq B$ ;  $M \leq \{A, B\}$ , where the semi-colon stands for a disjunctive form.
- *Double negation*: from  $M < \neg\neg A$ , one can infer  $M \leq A$ .
- *Negative conjunction*: from  $M < \neg(A \wedge B)$ , one can infer  $M \leq \neg A$ ;  $M \leq \neg B$ ;  $M \leq \{\neg A, \neg B\}$ .
- *Negative disjunction*: from  $M < \neg(A \vee B)$ , one can infer  $M \leq \{\neg A, \neg B\}$ .
- *Universal quantifier*: from  $M < \forall x P x$ , one can infer  $M \leq \{T(a_1, a_2, \dots), P(a_1), P(a_2), \dots\}$

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anything. Note that Correia (2017) has reformulated them as rules; for example, he substitutes

$A, B < A \wedge B$  with *from*  $A, B$ , *infer*  $A, B < A \wedge B$ . We prefer to be faithful to Fine's presentation.

- *Existential quantifier*: from  $M < \exists xPx$ , one can infer  $M \leq \{T(a_1, a_2, \dots), P(a_{11}), P(a_{12}), \dots\}; M \leq \{T(a_1, a_2, \dots), P(a_{21}), P(a_{22}), \dots\}; \dots$  where the  $a_{k1}, a_{k2}, \dots$  run through all of the non-empty subsets of the  $a$  for which  $P(a)$  is true.
- *Negative universal quantifier*: from  $M < \neg \forall xPx$ , one can infer  $M \leq \{T(a_1, a_2, \dots), \neg P(a_{11}), \neg P(a_{12}), \dots\}; M \leq \{T(a_1, a_2, \dots), \neg P(a_{21}), \neg P(a_{22}), \dots\}; \dots$
- *Negative existential quantifiers*: from  $M < \neg \exists xPx$ , one can infer  $M \leq \{T(a_1, a_2, \dots), \neg P(a_1), \neg P(a_2), \dots\}$ .

Let us briefly comment on the rules just introduced. As for the *Subsumption* rules, these seem natural and straightforward. As for the structural rules expressing properties enjoyed by the grounding relation, let us underline that according to the orthodox view (see Raven (2013)) (mediate) grounding is an irreflexive, asymmetric and transitive relation. Whilst transitivity and asymmetry have been widely discussed (see Chapter XXX by Thompson for further details on these discussions), irreflexivity does not seem to have been questioned; on the contrary it has led to some interesting puzzles of grounding (Fine, 2010; Krämer, 2013) which are reminiscent of puzzles of truth (cf. Chapter XXX by Krämer). Thus it is quite bizarre to find a weak notion of grounding in Fine's logic. On reflection, one realizes that Fine needs such a notion in order to formulate its elimination rules. Let us note that Fine's system does not contain the weakening rule:<sup>7</sup> we take this to be a nice feature of the system since all grounds must be relevant to the conclusion and weakening typically breaks down relevance. Finally, we remark that from the structural rules enumerated above, one can easily derive the

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<sup>7</sup> The weakening rule is the structural rule that allows one to add to the premises of a derivation a new formula; it represents the formal counterpart of the monotonicity property. Hence in a calculus where there is no weakening rule (nor explicit nor derivable nor admissible) the derivability relation is monotone or relevant.

rule of *Amalgamation* that, according to Fine (2012b), corresponds to a plausible feature of strict full grounding:

$$M_1 < C, M_2 < C, \dots M_n < C$$

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$$M_1, M_2, \dots M_n < C$$

Chapter XXX by Mc Sweeney contains an extensive discussion of the introduction rules, where the problem of the overdetermination of disjunction, as well as the several features of quantifiers, are taken into account. Let us then dwell on a point, which, although important, is often neglected in the current literature. Any discipline, from mathematics to biology, from metaphysics to moral philosophy, seem to have its own internal, explicit or implicit, criteria to test whether a result has been properly carried out. Logic is no exception in this sense and it enjoys several criteria like the subformula property, the harmony property or separability property<sup>8</sup> (e.g. see Poggiolesi (2010a) for an extensive list and discussion). These criteria are meant to ensure a formal as well as a conceptual adequacy (of the rules) of a formal system. Logics of grounding - insofar as they are logics - should be tested on the background of these criteria but unfortunately this analysis is lacking in the current literature (the only exception is Poggiolesi (2016b)). This is a regrettable gap. For example, despite its great virtues, Fine's

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<sup>8</sup> A calculus is said to satisfy the *subformula property* if, and only if, any theorem of the calculus can be proved by means of a derivation that does not contain any redundancy, i.e. no formula is introduced in the derivation if it is not essential to prove the theorem itself. Corresponding introduction and an elimination rules are said to be *harmonious* if, and only if, they satisfy a certain balance amongst them, where the idea of 'balance' has been formally captured in several different ways, see for example Francez and Dyckhoff (2012). Finally, introduction rules are said to be *separated* if, and only if, they introduce only one new symbol at a time.

introduction and elimination rules do not seem to enjoy many of the proof-theoretic properties of rules.

Finally, let us comment on the representational approach to grounding, which is present in many logics of grounding discussed in this chapter. According to this perspective, grounding should be sensitive to how facts are represented. However, not every linguistic difference should be relevant to ground-theoretic status. As an example, even for a representationalist, it should be plausible to hold that  $A \wedge B$  is ground-theoretically equivalent to  $B \wedge A$ . As we will see later, while both Correia (2017) and Poggiolesi (2018) propose a general criterion to account for that intuition, in Fine's system there is nothing of this sort: the logic does not include or rely on any proper theory of ground-theoretic equivalence.<sup>9</sup> As a result, it cannot be proved in Fine's logic that  $A \wedge B$  and  $B \wedge A$  have the same grounds, for example.

*Semantics.* Fine's semantics for its logical system of grounding is a truth-making semantics, based on the idea that a *fact*  $f$  makes true a certain sentence  $A$  (a wide discussion of truth-making semantics is developed in Chapter XXX by Fine). According to it, a true sentence  $A$  can be considered as the set of facts that make it true. In this sense, there is a strong resemblance between facts and possible worlds of Kripke semantics for modal logic (e.g. see Poggiolesi (2010b)): in Kripke's semantics a true sentence  $A$  can also be considered as the set of possible worlds where  $A$  is true. Nevertheless, there are three main differences between worlds and facts. The first is that while an accessibility relation hold among worlds of Kripke semantics, there is no accessibility relation among facts. Secondly, worlds are complete, i.e. they settle the truth-value of every proposition, while facts are partial, i.e. they can be considered as parts of possible worlds. Finally facts, differently from worlds, can be fused:

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<sup>9</sup> A recent study of this issue is Kramer (2019).

given the fact  $f$  that Mary is tall and the fact  $g$  that Mary is thin, we can have the fused fact  $f.g$  that Mary is tall and thin. The set of facts that verifies the sentence  $A$  is closed under fusion and is denoted by  $|A|$ .

We now have all the necessary notions to describe the four symbols for grounding introduced above from a semantics point of view.

- $M$  is a weak full ground for  $B$ ,  $M \leq B$  is true, if, and only if,  $f_1.f_2...f_n$  verifies  $B$  (i.e.  $f_1.f_2...f_n$  is a member of the verification set  $|B|$ ), whenever  $f_1$  verifies  $A_1$ , ..., and  $f_n$  verifies  $A_n$ , for  $M = \{A_1, \dots, A_n\}$  (i.e. whenever each of  $f_1.f_2...f_n$  is a member of the respective verification-sets of  $|A_1|$ , ...,  $|A_n|$ ).
- $A$  is a weak partial ground for  $B$ ,  $A \leqslant B$  is true, if, and only if, the set  $M$ , composed by  $A$  plus other sentences, is a weak full ground of  $B$ .
- $M$  is a strict full ground for  $B$ ,  $M < B$  is true, if, and only if,  $M \leq B$  is true, and  $B$  is not a weak partial ground for any of  $A_1, \dots, A_n$  composing  $M$
- $A$  is a strict partial ground for  $B$ ,  $A < B$  is true, if, and only if,  $A \leqslant B$  is true and  $B \leqslant A$  is not true.

Fine (2012a) shows validity and completeness between the structural rules of his system and the semantics just introduced. As Fine (2012b) says “this provides some kind of vindication both for the system and for the semantics.” However, although Fine attempts to extend this semantics to connectives and quantifiers and provides attractive definitions, semantics cannot be shown to be sound with respect to the logical part of his system (this is partly due to the fact that the semantics cannot discriminate among the semantic values of, for example,  $A$  and  $A \vee A$ ). This gap leaves open the question of the adequacy of Fine’s rules.

### 2.1.2 Other logics treating grounding as a sentential operator

**Schnieder (2011).** The main goal of Schnieder’s approach is to give a rigorous formalization of the sentential expression ‘because;’ as a result he formulates a logic which, following our distinctions, could be read as a logic capturing the notion of strict partial mediate metaphysical grounding.

Let us briefly illustrate Schnieder’s approach. In order to construct a logical system, Schnieder firstly extends the first-order language with the connective ‘because’ and secondly he adds to the classical natural deduction calculus rules describing the new connective. These rules can be divided into structural and logical rules.

*Structural rules.* Since Schnieder introduces only one new connective, he does not need to formulate subsumption rules. On the other hand, he formulates rules that describe the properties of the connective ‘because’: these are factivity, asymmetry, transitivity. As an example of these rules, here are the factivity rules:

$$\begin{array}{ccc} A \text{ because } B & & A \text{ because } B \\ \hline & & \\ A & & B \end{array}$$

*Logical rules.* As for the logical rules, Schnieder only has introduction rules; Schnieder’s introduction rules express the same relations conveyed by Fine’s introduction rules, although in a quite different form (as pointed out before, Correia (2017) correctly notes that Fine’s rules could be rewritten as Schnieder’s ones). For the sake of illustration, here are the rules concerning the grounds of a conjunction (with  $i = \{1,2\}$ ):

$$\begin{array}{c}
A_1, A_2 \\
\text{-----} \\
A_1 \wedge A_2 \text{ because } A_i
\end{array}$$

Or these are the rules concerning the grounds of a disjunction (with  $i = \{1,2\}$ ):

$$\begin{array}{c}
A_i \\
\text{-----} \\
A_1 \vee A_2 \text{ because } A_i
\end{array}$$

Schnieder’s system is proved to conservatively extend classical natural deduction. Moreover, Schnieder extends his approach to the quantifiers and underlies that other variations of his system could be obtained by substituting classical logic as the basic framework with other logics.

**deRosset (2015).** In his 2015 paper, deRosset proposes a new semantics for a subsystem of Fine’s pure logic of grounding. His motivation for constructing such a semantics, which is an alternative to Fine’s one, is philosophical. deRosset seems to be dissatisfied with Fine’s semantics for the following three reasons: (i) the semantic characterization of grounding in terms of truth-making do not seem to capture the metaphysical idea that animates grounding-enthusiasts. (ii) The rule of amalgamation, which is validated by Fine’s semantics, might be questioned. (iii) Fine’s semantics rules out a view according to which facts are “sparse,” namely there are truths that does not relate to any fact. By working with hypergraphs, assuming as primitive the relation of immediate grounding, and taking into account all subsets of the powerset of the domain of facts, deRosset avoids problems (i)-(iii).

**Correia (2017).** In his 2017 paper Correia proposes an impure logic of metaphysical grounding, where grounding is treated in its several variations (full, strict and weak,

immediate and mediate). Since he deals with the same objects as Fine's logic, let us summarize the main differences between the two. Like Fine, Correia considers both strict and weak grounding: but whilst Fine assumes weak grounding to be more fundamental, Correia does the opposite. Secondly, while Fine also covers partial grounding, Correia only analyses full grounding. In Fine's logic there is no treatment of ground-theoretic equivalence; in Correia's logic the issue is fully treated within the introduction of the symbol  $\approx$ , which is described by structural rules as well as by introduction and elimination rules that define its interaction with the other connectives. *Syntactically*, Correia's logic share the same impure grounding rules as Fine. As for the pure rules, Correia needs to add to Fine's rules those that regulate links between the notion of grounding and the notion of factual equivalence. Also there is a rule taking into account the complexity ingredient (note that an analogous rule can be found in Poggiolesi (2018)) and rules describing excluded middle and non-contradiction with grounding formulas. As for this latter rule, *Semantically*, Correia's approach relies on three main notions: the notion of structure, the notion of proposition and the notion of complexity. Each sentence of the language is interpreted into a proposition of a structure via a function. Propositions are ordered from atomic to more complex ones.

## **2.2. Grounding as a predicate**

The literature on the formalization of grounding as a predicate, although important, is not as widespread and rich as the one on grounding as a sentential operator. Indeed, with the exception of Rosen (2010) where some axioms on grounding as a predicate are suggested, the only papers where the issue is fully developed are Korbmacher (2018a, b). We dedicate the rest of the section to a brief description of his approach.

**Korbmacher (2018a, b).** The main goal of Korbmacher’s approach is to reproduce Schnieder’s logic with a predicate for grounding rather than an operator.

*Languages.* Let  $L$  be the language of Peano Arithmetic; let  $L_T$  be the language  $L$  extended with the truth predicate  $Tr$ ; finally, let  $L_{TR}^{<=}$  be the language that extends  $L_{TR}$  by the addition of the binary grounding predicate  $<=$ , which allows to construct formulas of the form  $x <= y$  to be read as  $x$  is a strict, partial, mediate grounding of  $y$ .

*Syntax.* Let **PA** be the axiomatic system of Peano arithmetic based on the language  $L$ ; **PAT** is the system that extends **PA** with the missing instances of the induction scheme over  $L_{TR}$ ; finally **PAG** is the axiomatic system that extends **PAT** with all the missing instances of the induction scheme over  $L_{TR}^{<=}$ . The predicational theory of ground **PG** consists of the axioms of **PAG** plus other axioms describing the logical as well as structural features of grounding (and truth). As for what we might call ‘pure axioms of grounding,’ they describe the following features: irreflexivity, transitivity and factivity. Just to give an idea of the form of such axioms, irreflexivity is conveyed in the following way:  $\forall x \neg(x <= x)$ . As for the impure axioms of grounding, these are divided into *upward directed axioms* and *downward directed axioms*. Each impure rule of Schnieder’s system corresponds to two axioms: one that says that if the grounds are true, then the grounds ground the conclusion, while the other saying that if the conclusion is true, then the grounds ground the true conclusion. As an example, consider again conjunction rules in Schnieder’s system:

$A_1, A_2$

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$A_1 \wedge A_2$  because  $A_i$

(with  $i = \{1,2\}$ ). These rules correspond to the following two axioms:

$\forall x \forall y (Tr(x) \wedge Tr(y) \rightarrow (x \leq x \wedge y) \wedge (y \leq x \wedge y))$  upward directed axiom

$\forall x \forall y (Tr(x \wedge y) \rightarrow (x \leq x \wedge y) \wedge (y \leq x \wedge y))$  downward directed axiom

The system **PG** is consistent but it is not a conservative extension of **PA**; it is valid, though not complete, with respect to the Schneider's logic of grounding via a certain translation.

*Semantics.* Starting from a standard model of the positive theory of truth (see Halbach (2011)), Korbmacher constructs a standard model for the predicational theory of grounding. The model is obtained by the construction of grounding trees, which were firstly introduced by Correia (2014). As Korbmacher underlies, the semantics he constructs strongly relies on the fact that the starting model is standard and would break down if applied to a non-standard model.

The predicational theory of grounding put forward by Korbmacher is proof-theoretically stronger than the operator approach (at least that of Schnieder (2011)). Moreover it provides an unique framework where grounding and truth co-exist, making easy to analyse their relations: in this framework, it is possible to formulate an answer to the famous puzzles of grounding put forward by Fine (2010) and Krämer (2013), as well as to study and analyse the ground-theoretic paradoxes of self-reference. Korbmacher suggests several ways to extend his logical system of grounding, namely via the notion of fundamentality or via a theory of definitions.

### 3. Grounding as a meta-linguistic relation

Grounding as a meta-linguistic relation can be naturally introduced via the notion of proof. At least since Aristotle (1984), two kinds of proof have been distinguished in accordance with the goals of proof (see also Chapter XXX by Malink). On the one hand, there are *proofs-that*, namely proofs by means of which we prove *that* something is true, and on the other hand, we have *proofs-why*, namely proofs by means of which we prove *why* something is true. The distinction is easy to grasp from some real-world examples. From the fact that temperatures are higher in summer than in winter we can prove that (well-functioning) thermometers are higher in summer than in winter; and conversely from the fact that thermometers are higher in summer than in winter we can prove that temperatures are higher in summer than in winter. However, we only have a proof-why in the first case: the fact about temperatures explains why – or is part of a proof why – thermometers are higher in the summer. By contrast, the fact about thermometers does not contribute to a proof why the temperatures are higher in the summer: it is not the case that thermometers being higher in summer than in winter explain why temperatures are higher in summer than in winter.

Note that the dichotomy between proofs-that and proofs-why naturally maps onto another central dichotomy for this chapter, namely that between the relation of inference and the relation of grounding. Indeed as a proof-that is about drawing inferences - in proving that something is true, a proof-that also shows from what that something can be inferred – in an analogous way, proofs-why bare open the links relating a conclusion with its grounds: in showing why something is true, a proof-why also reveals the grounds. Note that an analogous idea, although expressed in different terms, can be found in deRosset (2013).

The dichotomy proofs-that and proof-why has been regarded with a certain scepticism in contemporary logic. Whilst we have seen that the distinction seems very natural informally, certain doubts might arise about its applicability in formal contexts. In particular, while proofs-why have mainly been ignored in 20<sup>th</sup> century logic, proofs-that have been rigorously

formalized thanks to the work of Frege, Russell, Hilbert (see van Heijenoort (1967)) and, perhaps most of all, Gentzen (1969). In this literature, the meta-linguistic relation of derivability, that has been mentioned in the Introduction and is denoted by the symbol  $\vdash$ , seems to fruitfully capture the logical aspects of proofs-that. This relation is standardly defined by rules, which are divided into introduction and elimination rules. It also finds its linguistic counterpart (at least in certain logics) via a result known as the *deduction theorem*, in the (standard) material implication. This formalization, which is at the same time simple and elegant, has given rise to impressive results, such as cut-elimination, decidability, complexity, interpolation theorems, and to the whole field known today as *proof theory*.

Given the close links between proofs-that and proofs-why and the great developments of the study on proofs-that relying on this (meta-linguistic) formalization, some grounding theorists have developed an analogous formalism to capture proofs-why. Therefore, as the logical aspects of proofs-that have been formally captured by the metalinguistic symbol  $\vdash$ , new meta-linguistic symbols have been created for formally capturing the logical aspects of the notion of proof-why. Note that as the formalization of proofs-that naturally leads to a formalization of the notion of inference given their close relation, in the same way the formalization of a proof-why is also naturally a formalization of the notion of grounding.

As we have mentioned at the beginning of this chapter, logic mainly relies on the two key notions of truth (validity) and provability, formalized as meta-linguistic symbols. Formalizing (proofs-why and thus) grounding as a meta-linguistic symbol puts it on the same level as truth and provability, with consequences for the very basis of logic. As far as we understand, such was the revolutionary role attributed to grounding by the great philosopher and mathematician Bernard Bolzano, who attempted to logically capture the grounding relation (and more generally the notion of proofs-why) through the notion of exact derivability. We refer to the Chapter XXX of Roski for a detailed description of Bolzano's work.

### 3.1. Logics treating grounding as a metalinguistic relation

In the contemporary literature, there are three works where grounding is associated to a meta-linguistic relation: (i) Batchelor (2010), (ii) Correia (2014) and (iii) Poggiolesi (2016a, 2018). In (i) Batchelor introduces the idea of proofs of a special kind that provide the grounds of what is proved, but does not systematically develop a logic around this intuition. In (ii) and (iii) logics of grounding as meta-linguistic relation are analysed; however while in Poggiolesi this type of formalization is explicitly defended, Correia does not seem to explicitly defend this position<sup>10</sup>. For this reason, we start by presenting (iii) before moving on to (ii).

**Poggiolesi (2016a, 2018).** Poggiolesi's work is strongly inspired and influenced by Bolzano's insights; according to this perspective, it focuses on the notion of complete and immediate strict logical grounding, since, according to Bolzano, this is the most fundamental type of grounding and thus the appropriate starting point for analysing the notion. Just as the notion of proof-that is formalized by the metalinguistic relation of derivation and denoted by the symbol  $\vdash$ , the notion of proof-why (and thus of grounding) is formalized by the metalinguistic relation of *formal explanation* that is denoted by the symbol  $\vdash\sim$ . In accordance, (standard) inferences rules are supplemented with grounding rules, which express an authentic dependence between premises and conclusion and have the following form:

M

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<sup>10</sup> In other words, whilst one interpretation of Correia (2014) is in meta-linguistic terms, the author does not seem to rule out an alternative interpretation where grounding remains in the language.

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**C**

to be read as: the multiset  $M$  is the full and immediate strict formal explanation (or ground) of  $C$ .

*Ground-theoretic equivalence.* In Poggiolesi's account formulas that are equivalent by associativity and commutativity of conjunction and disjunction have the same grounds. This is rendered in a simple way. Bold letters, such as **A**, **B**, **C**, ... denote sets of formulas containing all formulas which are ground-theoretic equivalent. Grounding rules contain bold letters. So for example the grounding rule for conjunction has the following form

**A, B**

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**A $\wedge$ B**

which means that not only the formulas  $A$  and  $B$  formally explain (or ground) the formula  $A\wedge B$ , but they also formally explain (or ground) all formulas that are equivalent to  $A\wedge B$  by associativity and commutativity of conjunction and disjunction.

*Grounding rules for disjunction and negation.* In Poggiolesi's logic, the grounding rules for conjunction, whilst having a specific form due to the use of the meta-linguistic symbol (and the account of ground-theoretic equivalence), convey a principle analogous to all the other logics of grounding. Grounding rules for disjunction and negation in Poggiolesi's logic are more peculiar and worth a brief explanation. The rules for disjunction contain a new element called *robust condition*. Consider the example of the disjunction  $A\vee B$ , in the situation where the formula  $A$  is true. In this case,  $A$  is certainly a ground for  $A\vee B$ ; but in order for  $A$  to be the

complete ground of  $A \vee B$ <sup>11</sup>, it is necessary to specify that  $B$  is false, i.e. it is necessary to say that  $B$  does not play any role in grounding  $A \vee B$  and thus that  $A$  does the job on its own. In other words, it is the falsity of  $B$  that ensures that, or is a (robust) condition for  $A$  to be the *complete* ground of  $A \vee B$ . Thus,  $A$  is the complete and immediate formal ground of  $A \vee B$  under the robust condition that the negation of  $B$  is true. The use of robust conditions solves the problem (as Mc Sweeney puts it, see Chapter XXX) of overdetermination of disjunction and is rendered by means of square brackets, as in the following examples of grounding rules for disjunction:

$$\begin{array}{ccc}
 p, q & [\neg q] p & [\neg p] q \\
 \sim \sim & \sim \sim \sim & \sim \sim \sim \\
 p \vee q & p \vee q & p \vee q
 \end{array}$$

As concerns the negation connective, first note that Poggiolesi’s logic is the only one to have an unique rule for negation. This rule validates the grounding principles for negation analogous<sup>12</sup> to the negation axioms in Fine’s logic, but it also shows what unites them. The formulation of this rule is made possible by the use of the two meta-linguistic relations of derivability and formal explanation. The rule formalizes the following grounding analysis of the negation connective. Consider the question: what is the (complete and immediate) ground of a negative truth, such as “the ball is not cubic”? A natural answer is that the ground is “the

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<sup>11</sup> As opposed to a full ground; see Introduction.

<sup>12</sup> Note that according to Poggiolesi’s logic and in contrast with Fine’s, the grounds of formulas of the form  $\neg(A * B)$ , where  $*$  is either a conjunction or disjunction, vary depending on the form of  $A$  and  $B$ . If  $A$  and  $B$  are formulas that begin with a negation, e. g. they are of the form  $\neg p$ ,  $\neg q$ , respectively, then the grounds have the form  $p$ ,  $q$ . Otherwise, the grounds will have the form  $\neg A$ ,  $\neg B$ . This difference is required to respect a complexity measure. It is also proof-theoretically adequate, as shown in Poggiolesi (2016b).

ball is spherical”: after all, the “the ball is spherical” is true, and “the ball is spherical” and “the ball is cubic” are contradictory – one can derive a contradiction from them – so it seems to be the reason why, or the grounds of or to provide a formal explanation why “the ball is not cubic.” In formal terms, the idea here is that the ground for the truth of a formula of the form  $\neg A$  is a formula  $B$  (satisfying a certain complexity constraint) such that from  $A$  and  $B$  a contradiction is derivable.

*Justification of grounding rules.* Let us underline that almost all work on the logic of grounding does not treat the issue of *motivating* or *arguing* for certain grounding rules (or axioms) rather than others (this is also emphasized by McSweeney in Chapter XXX). Poggiolesi is an exception to this trend: all the rules of her logic of grounding are motivated by a rigorous conception of grounding developed in Poggiolesi (2016a), according to which grounding is definable in terms of (positive and negative) derivability and complexity.<sup>13</sup>

*Introduction and elimination rules for the connective ‘because.’* Just as the metalinguistic symbol of derivation is reflected in the (formal) language via material implication (the deduction theorem says that  $A \vdash C$  if, and only if,  $\vdash A \rightarrow C$ ), in an analogous way, formal explanations are reflected in the language via the ‘because’ connective, denoted by the symbol  $\triangleright$ . The new connective  $\triangleright$  is treated like the other classical connectives: by means of introduction and elimination rules that only deal with it (and do not specify the relation

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<sup>13</sup> Poggiolesi’s original definition of complete and immediate grounding is formulated for the standard connectives of classical logic. It has been extended in many ways: to the framework of relevant logic (Poggiolesi (2019)), to other less standard connectives of classical logic (Poggiolesi and Francez (2019)) and to the notion of partial (immediate and mediate) and complete and mediate grounding (Poggiolesi (2020)).

between  $\triangleright$  and the other logical connectives) and enjoy several other proof-theoretic properties. To give the reader an idea of the introduction and elimination rules for the connective  $\triangleright$ , let us describe the introduction rule. This rule mimics the rule that introduces material implication, but at the grounding level. Indeed as this latter says that if there exists a derivation from a formula  $A$  to a formula  $B$ , one can infer the formula  $A \rightarrow B$  (and ‘discharge’ or remove the  $A$ ); analogously the rule which introduces the connective  $\triangleright$  roughly says that if there exists a formal explanation from  $A$  to  $B$ , then one can infer  $A \triangleright B$  (and then discharge the  $A$ , as well as the whole formal explanation).

**Correia 2014.** In his 2014 paper, Correia treats the notion of logical grounding, strict and weak, from a syntactic as well as from a semantic perspective. Both in the syntactic as well as in the semantic approaches the central notion is that of tree: a mediate grounding relation between a set of formulas  $M$  and a formula  $A$  is a tree, where complexity is either stable or it increases. In the syntactic approach each step of the tree is determined by an introduction grounding rule, while in the semantic approach each step of the tree is determined by a transition from truth to truth. A set of formulas  $M$  strictly grounds a formula  $A$ , in symbols  $M \triangleright A$ , if, and only if, there is a tree for  $A$  from  $M$ . A set of formulas  $M$  weakly grounds a formula  $A$ , in symbols  $M \_ \triangleright A$ , if, and only if, either  $A$  belongs to  $M$  or there is a subset  $N$  of  $M$ , such that  $N \triangleright A$ .

*Syntax.* As for the structural rules, strict grounding enjoys irreflexivity and cut, whilst weak grounding enjoys reflexivity, cut and weakening. Since, as we have said in Section 2.1, grounds must be relevant to the conclusion and weakening typically leads to violations of relevance, the notion of grounding in Correia’s logic may exhibit such violation. As for the introduction rules that define a grounding tree, these convey the same principles of Fine’s logic and are expressed by means of the standard inferential line, namely they have the form:

M

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B

to be read as M grounds B. Finally, let us also note that, unlike Correia (2010, 2017), in Correia (2014) there is no treatment of grounding-theoretic equivalence.

*Semantics.* The two metalinguistic relations of strict  $\triangleright$ , as well as  $\_ \triangleright$  weak grounding, are semantically defined by trees (as in the approach proposed by Korbmacher). The links between these consequence relations and other more famous notions of consequence relations are investigated: it is shown that strict grounding is included in weak grounding, which is in its turn included in first degree entailment consequence relation, which is a sub-consequence of the classical logical consequence.

Table summing up all the logics of grounding introduced in the Chapter

	Strict/ Weak	Full (or Complete)/ Partial	Immediate/ Mediate	Metaphysical/ Logical	Syntactic/ Semantic	Operator/ Predicate/ Metalinguistic relation
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Correia 2014	S & W	F	M	L	Sy & Se	MR
Correia 2017	S & W	P	M	M	Sy & Se	O
Fine (2012a, b)	S & W	F & P	M	M	Sy & Se	O
Korbmacher (2018a, b)	S	P	M	M	Sy & Se	P
Poggiolesi (2016, 2018)	S	C	I	L	Sy	MR
Schnieder (2011)	S	P	M	M	Sy	O
deRosset(2015)	S & W	F & P	M	M	Se	O

Where in the table above the letters of every column corresponds to the initial of the words occupying the first case of the column itself.

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