High-order composite finite element exact sequences based on tetrahedral-hexahedral-prismatic-pyramidal partitions

Philippe R. B. Devloo^{a,*}, Omar Durán^a, Sônia M. Gomes^b, Mark Ainsworth^c

^a*FEC-Universidade Estadual de Campinas, Campinas-SP, Brazil*

^b*IMECC-Universidade Estadual de Campinas, Campinas-SP, Brazil*

^c*Division of Applied Mathematics, Brown University, USA*

Abstract

The combination of tetrahedral and hexahedral elements in a single conformal mesh requires pyramids or prisms to make the transition between triangular and quadrilateral faces. This paper presents high order exact sequences of finite element approximations in $H^1(\Omega)$, $\mathbf{H}(\text{curl}, \Omega)$, $\mathbf{H}(\text{div}, \Omega)$, and $L^2(\Omega)$ based on such kind of three dimensional mesh configurations. The approach is to consider composite polynomial approximations based on local partitions of the pyramids into two or four tetrahedra. The traces associated with triangular faces of these tetrahedral elements are constrained to match the quadrilateral shape functions on the quadrilateral face of the pyramid, in order to maintain conformity with shared neighboring hexahedron, or prism. Two classes of composite exact sequences are constructed, one using classic Nédélec spaces of first kind, and a second one formed by enriching these spaces with properly chosen higher order functions with vanishing traces. Projection-based interpolants satisfying the commuting diagram property are presented in a general form for each type of element. The interpolants are expressed as the sum of linearly independent contributions associated with vertices, edges, faces, and volume, according to the kind of traces appropriate to the space under consideration. Furthermore, we study applications to the mixed formulation of Darcy's problems based on compatible pairs of approximations in $\{\mathbf{H}(\text{div}, \Omega), L^2(\Omega)\}$ for such tetrahedral-hexahedral-prismatic-pyramidal meshes. An error analysis is outlined, showing same (optimal) orders of approximation in terms of the mesh size as one would obtain using purely hexahedral or purely tetrahedral partitions. Enhanced accuracy for potential and flux divergence variables are obtained when enriched space configurations are applied. The predicted convergence orders are verified for some test problems.

Keywords: Finite element exact sequences, hierarchical shape functions, projection-based interpolants, tetrahedral-hexahedral-prismatic-pyramidal meshes, mixed finite elements, enhanced accuracy.

1. Introduction

Generally, three dimensional finite element approximations are based on meshes consisting of only hexahedral or tetrahedral elements. While hexahedral elements have advantages thanks to their tensor-product nature and the fact that they often require fewer degrees-of-freedom to reach a desired accuracy, approximations using tetrahedral elements are mandated when the problem involves more complicated domains. As a result, the use of hexahedra-dominant meshes suggests itself, with the aim of combining advantages from

*Corresponding author - Phone +55 19 997005477

Email addresses: phil@fec.unicamp.br (Philippe R. B. Devloo), oduran@unicamp.br (Omar Durán), soniag@ime.unicamp.br (Sônia M. Gomes), mark_ainsworth@brown.edu (Mark Ainsworth)

regular hexahedral elements as far as possible, whilst tetrahedral elements are employed where meshing using hexahedral elements is infeasible. This kind of hybrid mesh is also needed when dealing with directional or *hp*-adaptive meshes. However, the transition between triangular and quadrilateral faces in a conformal hybrid mesh requires pyramids or prisms [7]. Ideally, scientific finite element software should support the use of such tetrahedral-hexahedral-prismatic-pyramidal meshes.

A peculiarity of a pyramidal element, not shared by any other element, is their four faces abut at its apex. This fact contributes to making the construction of scalar continuous vertex shape functions for pyramidal elements a tricky point. As described in [20, 8] and references therein, one approach consists of using rational functions, but this is unattractive and entails having to deal with rational functions and their derivatives. An alternative approach, which we pursue in the current article, is to split the pyramid into tetrahedra, and use usual piecewise polynomials basis functions on the sub-tetrahedra (see [2] and other references therein for a discussion in the setting of H^1 , and [3] for a lowest order de Rham sequence).

The development of $\mathbf{H}(\text{curl})$ -conforming or $\mathbf{H}(\text{div})$ -conforming approximations based on pyramidal elements has its challenges. Rational functions have been used in [9, 10, 28] for the construction of high order pyramid vector shape functions. However, within an exact sequence, there is the additional difficulty of ensuring the conformity of the spaces. For instance, in mixed methods for Darcy's flows satisfying divergence conformity of flux approximations in $\mathbf{H}(\text{div}, \Omega)$ with the scalar functions used for approximations of the potential variable, this is not a trivial task in the case of rational polynomials.

We shall construct an arbitrary order exact sequence of finite element approximation spaces $V_h \subset H^1(\Omega)$, $\mathbf{X}_h \subset \mathbf{H}(\text{curl}, \Omega)$, $\mathbf{V}_h \subset \mathbf{H}(\text{div}, \Omega)$, and $W_h \subset L^2(\Omega)$, based on tetrahedral-hexahedral-prismatic-pyramidal meshes $\mathcal{T}_h$ for a region Ω , along with projection-based interpolants satisfying the commuting diagram property. Enriched exact sequences U_h^+ , $\mathbf{X}_h^+$, $\mathbf{V}_h^+$, and W_h^+ are also proposed, in which local spaces are constructed by enriching the original ones with properly chosen higher order shape functions with vanishing traces.

The commutative property of the de Rham diagram is crucial in proving stability of mixed finite element methods [27]. The same structure is also useful for robust large scale discretizations of complex coupled multiphysics systems, enabling each component to be discretized with appropriate physical fidelity, as discussed in [11]. The de Rham diagram also plays a role in the analysis of multigrid and overlapping Schwarz methods for the solution of discrete systems resulting from approximating problems in $\mathbf{H}(\text{curl})$ and $\mathbf{H}(\text{div})$, as discussed in [31]. There are several works in the literature dealing with exact finite element sequences for meshes consisting of elements of a single geometry. A comprehensive presentation of the subject can be found in [19], for 1D, 2D, and 3D elements.

The paper is organized as follows. The main results regarding high order exact sequences of finite element approximation spaces for hybrid meshes containing pyramids are described in Section 5. The necessary tools, and exact sequences valid for tetrahedral, hexahedral, and prismatic elements are discussed in Section 2 and Section 3. The composite H^1 , $\mathbf{H}(\text{curl})$ and $\mathbf{H}(\text{div})$ -conforming polynomial functions based on tetrahedra sub-partitions needed for pyramids are detailed in Section 4. A key idea is that, by choosing a sufficiently high polynomial degree for the tetrahedra inside a pyramid, the corresponding traces associated with their triangular faces can be constrained to coincide with standard face functions on any quadrilateral face of the pyramid that might be

shared by a neighboring hexahedron or prism. The interpolants satisfying the commuting diagram property are defined in Section 6, where use is made of the general guidelines expounded in [18, 19], valid for all element geometry and the different space configurations. Section 7 describes applications using hybrid meshes to the mixed formulation for Darcy’s problem. An error analysis based on approximation properties of the interpolants is performed, showing the same (optimal) orders of approximation in terms of the mesh size h that one would expect when using purely hexahedral or purely tetrahedral partitions. These rates are verified for some model problems in Section 8. Finally, Appendix A describes the hierarchy of scalar and vector shape functions used in the algorithms needed for the global assembly of local approximation spaces, satisfying inter-element conformity conditions, as explained in Appendix B.

All of the numerical examples were implemented using the NeoPZ computational platform¹, which is an open-source object-oriented project providing a comprehensive set of high performance tools required for finite element simulations, including hierarchical high order vector and scalar shape functions, data structure allowing the identification of face and internal shape functions of different degrees, and procedures for shape function constraints in two or three dimensions.

The NeoPZ framework makes a strong distinction between geometric modeling, generation of approximation spaces, and definition of the variational statement. The classes that define approximation spaces (computational mesh) always refers to a geometric mesh. It is composed of a list of computational elements and nodes. Computational elements can form either continuous, discontinuous, or $\mathbf{H}(\text{div})$ computational meshes. Each computational element/side can have an independent polynomial order. As such, p -adaption can be constructed for all of the above spaces, and others. For h -refined meshes, the library provides an interface to compute shape function constraints to handle hanging nodes/sides [15, 22]. Thus, code capabilities are available for a user to implement hp -adaptive strategies, without limitation on hanging sides and/or distribution of approximation orders, and to solve multiphysics coupled systems of partial differential equations by combining different kinds of finite element approximation spaces chosen for each field [26]. The newly developed $\mathbf{H}(\text{div})$ -conforming spaces for tetrahedral-hexahedral-prismatic-pyramidal meshes have been incorporated in this platform.

2. Notation and preliminaries

Consider a partition $\mathcal{T}_h = \{K\}$ for a region Ω formed of tetrahedra, hexahedra, prisms or pyramids (denoted by T-H-Pr-P partition). The elements $K \in \mathcal{T}_h$ are assumed to be the image of a single master element $\hat{K}$ in the sense that there is a diffeomorphism $F_K : \hat{K} \rightarrow K$ mapping the points $\hat{\mathbf{x}} = (\hat{x}, \hat{y}, \hat{z}) \in \hat{K}$ to $\mathbf{x} = (x, y, z) = F_K(\hat{\mathbf{x}}) \in K$. For simplicity, only conforming affine (or asymptotically affine) meshes, without hanging faces, are considered. The sets $\mathcal{V}(F)$ and $\mathcal{E}(F)$ contain vertices and edges of a face F of K , $\mathbf{n}^F$ denotes the outer normal vector of K on the face F , and $\boldsymbol{\tau}^e$ is the tangential vector of an edge e ; its orientation is assumed to be fixed.

This section describes some required tools that are usually applied for purely tetrahedral, hexahedral and prismatic meshes, whose master elements are taken to be:

- Tetrahedron: $\hat{T} = \{(\hat{x}, \hat{y}, \hat{z}); \hat{x} \geq 0, \hat{y} \geq 0, \hat{z} \geq 0, \hat{x} + \hat{y} + \hat{z} \leq 1\}$.

¹<http://github.com/labmec/neopz>

- Hexahedron: $\hat{H} = [-1, 1] \times [-1, 1] \times [-1, 1]$.
- Prism: $\hat{P}r = \{(\hat{x}, \hat{y}, \hat{z}); \hat{x} \geq 0, \hat{y} \geq 0, \hat{x} + \hat{y} \leq 1, 0 \leq \hat{z} \leq 1\}$.

We postpone the discussion of pyramids until later sections.

Standard constructions of finite element subspaces $U(K) \subset H^1(K)$, $\mathbf{M}^{\text{curl}}(K) \subset \mathbf{H}(\text{curl}, K)$, $\mathbf{M}^{\text{div}}(K) \subset \mathbf{H}(\text{div}, K)$ and $W(K) \subset L^2(K)$ are defined by mapping the corresponding polynomial spaces defined in $\hat{K}$ as follows:

- Scalar functions $U(K) \subset H^1(K)$ and $W(K) \subset L^2(K)$ are given by

$$\begin{aligned} U(K) &= \mathbb{F}_K U(\hat{K}) = \left\{ \varphi \mid \varphi \circ F_K \in U(\hat{K}) \right\}, \\ W(K) &= \mathbb{F}_K W(\hat{K}) = \left\{ \varphi \mid \varphi \circ F_K \in W(\hat{K}) \right\}. \end{aligned}$$

- Vector spaces $\mathbf{M}^{\text{curl}}(K) \subset \mathbf{H}(\text{curl}, K)$ and $\mathbf{M}^{\text{div}}(K) \subset \mathbf{H}(\text{div}, K)$ are set as

$$\begin{aligned} \mathbf{M}^{\text{curl}}(K) &= \mathbb{F}_K^{\text{curl}} \mathbf{M}^{\text{curl}}(\hat{K}) = \left\{ \mathbf{q} \mid \mathbf{q}(\mathbf{x}) = [DF_K^{-T} \hat{\mathbf{q}}](\hat{\mathbf{x}}), \hat{\mathbf{q}} \in \mathbf{M}^{\text{curl}}(\hat{K}) \right\}, \\ \mathbf{M}^{\text{div}}(K) &= \mathbb{F}_K^{\text{div}} \mathbf{M}^{\text{div}}(\hat{K}) = \left\{ \mathbf{q} \mid \mathbf{q}(\mathbf{x}) = [J_K^{-1} DF_K \hat{\mathbf{q}}](\hat{\mathbf{x}}), \hat{\mathbf{q}} \in \mathbf{M}^{\text{div}}(\hat{K}) \right\}, \end{aligned}$$

where $\mathbf{x} = F_K(\hat{\mathbf{x}})$, DF_K is the Jacobian matrix of F_K , and $J_K = \det(DF_K)$.

The global space is defined in terms of the local spaces whilst enforcing the appropriate conformity condition on the element boundary ∂K on the trace for each kind of subspace: $U_h \subset H^1(\Omega)$, $\mathbf{X}_h \subset \mathbf{H}(\text{curl}, \Omega)$, $\mathbf{V}_h \subset \mathbf{H}(\text{div}, \Omega)$, and $W_h \subset L^2(\Omega)$. Specifically, for H^1 , the trace is the value of the function itself; $\mathbf{H}(\text{curl})$ trace is the tangential components of the vector valued function across the boundary (which takes values on edges, and two components across faces, but is not defined at vertices); $\mathbf{H}(\text{div})$ trace is the normal component of the vector valued function across the boundary faces (but is not defined at vertices or edges).

We shall need various types of scalar polynomial spaces $P_k(\hat{K})$, depending on the element geometry: $P_k(\hat{T}) = \mathbb{P}_k$; $P_k(\hat{H}) = \mathbb{Q}_{k,k,k}$; $P_k(\hat{P}r) = \mathbb{W}_{k,k}$. The associated vector polynomial spaces are denoted by $\mathbf{P}_k(\hat{K}) = P_k(\hat{K})^3$. In the above definitions, $\mathbb{P}_k$ are the polynomials of total degree not greater than k , $\mathbb{Q}_{k,k,k}$ are polynomials of maximum degree k in each coordinate, and $\mathbb{W}_{p,q} = \mathbb{P}_p(\hat{x}, \hat{y}) \otimes \mathbb{P}_q(\hat{z})$. Homogeneous polynomials of total degree k shall also be used, which are denoted by $\mathbb{P}_k$.

Several kinds of hierarchical bases have been constructed in the literature for $P_k(\hat{K})$ and $\mathbf{P}_k(\hat{K})$ [1, 4, 33, 20, 28, 16]. Their constructions generally follow a topological classification induced by conformity requirements at each space level. Roughly speaking, the hierarchical bases can be split into two parts, according to the corresponding traces of the shape functions over $\partial \hat{K}$.

$$\begin{aligned} B_k^{H^1}(\hat{K}) &= B_k^{\partial, H^1}(\hat{K}) \cup \mathring{B}_k^{H^1}(\hat{K}) \\ \mathbf{B}_k^{\text{curl}}(\hat{K}) &= \mathbf{B}_k^{\partial, \text{curl}}(\hat{K}) \cup \mathring{\mathbf{B}}_k^{\text{curl}}(\hat{K}), \\ \mathbf{B}_k^{\text{div}}(\hat{K}) &= \mathbf{B}_k^{\partial, \text{div}}(\hat{K}) \cup \mathring{\mathbf{B}}_k^{\text{div}}(\hat{K}), \end{aligned}$$

where $\mathring{B}_k^{H^1}(\hat{K})$, $\mathring{\mathbf{B}}_k^{\text{curl}}(\hat{K})$ and $\mathring{\mathbf{B}}_k^{\text{div}}(\hat{K})$ are formed by interior shape functions (i.e., functions vanishing the corresponding traces over $\partial \hat{K}$). Conversely, the shape functions in $B_k^{\partial, H^1}(\hat{K})$, $\mathbf{B}_k^{\partial, \text{curl}}(\hat{K})$ and in $\mathbf{B}_k^{\partial, \text{div}}(\hat{K})$ have non-zero traces. Appendix A describes some example of these shape functions.

3. Exact sequences of space configurations for $\hat{T}$, $\hat{H}$, and $\hat{P}r$

The following exact sequences valid for tetrahedral, hexahedral, and prismatic elements shall be considered.

3.1. Nédélec's Second Kind sequence for $\hat{T}$

For tetrahedral elements, one possible exact sequence, known as tetrahedral Nédélec's Second Kind elements [30] (and here denoted by sequence of Class 2 for short). The subspaces $U_k(\hat{T}) \subset H^1(\hat{T})$, $\mathbf{M}_k^{\text{curl}}(\hat{T}) \subset \mathbf{H}(\text{curl}, \hat{T})$, $\mathbf{M}_k^{\text{div}}(\hat{T}) \subset \mathbf{H}(\text{div}, \hat{T})$, $W_k(\hat{T}) \subset L^2(\hat{T})$, and the corresponding traces on the faces and edges of $\hat{T}$ are summarized in Table 1. Note that, for convenience, all the subspaces in this sequence are indexed by k , despite the fact that this does not necessarily indicate the degree of their functions, which decreases from $k + 2$ at the H^1 -space level to $k - 1$ at the L^2 -space level.

Space	$\hat{T}$	$\hat{F}_\Delta$	$\hat{e}$
$H^1(\hat{T})$	$\supset \mathbb{P}_{k+2}$	$\xrightarrow{\text{tr}} \mathbb{P}_{k+2}$	$\xrightarrow{\text{tr}} \mathbb{P}_{k+2}$
$\downarrow \nabla$	$\downarrow \nabla$		
$\mathbf{H}(\text{curl}, \hat{T})$	$\supset [\mathbb{P}_{k+1}]^3$	$\xrightarrow{\text{tr}} [\mathbb{P}_{k+1}]^2$	$\xrightarrow{\text{tr}} \mathbb{P}_{k+1}$
$\downarrow \nabla \times$	$\downarrow \nabla \times$		
$\mathbf{H}(\text{div}, \hat{T})$	$\supset [\mathbb{P}_k]^3$	$\xrightarrow{\text{tr}} \mathbb{P}_k$	
$\downarrow \nabla \cdot$	$\downarrow \nabla \cdot$		
$L^2(\hat{T})$	$\supset \mathbb{P}_{k-1}$		

Table 1: Exact polynomial space sequence of Class 2 for $\hat{T}$, and the corresponding traces on faces and edges

3.2. Nédélec's First Kind sequences for $\hat{T}$, $\hat{H}$, and $\hat{P}r$

Let us also consider the Nédélec's First Kind elements, referring to the paper [29]. Herein they are denoted by sequences of Class 1, to be considered for $\hat{T}$, $\hat{H}$, and $\hat{P}r$. The definitions of the subspaces $U_k(\hat{K}) \subset H^1(\hat{K})$, $\mathbf{M}_k^{\text{curl}}(\hat{K}) \subset \mathbf{H}(\text{curl}, \hat{K})$, $\mathbf{M}_k^{\text{div}}(\hat{K}) \subset \mathbf{H}(\text{div}, \hat{K})$, and $W_k(\hat{K}) \subset L^2(\hat{K})$ are summarized in Table 2, together with their traces on faces and edges. Roughly speaking, the vector polynomial spaces $\mathbf{M}_k^{\text{curl}}(\hat{K})$ and $\mathbf{M}_k^{\text{div}}(\hat{K})$ are constructed by augmenting $P_k(\hat{K})^3$ with shape functions of higher degree. Thus, the index k in $\mathbf{M}_k^{\text{curl}}(\hat{K})$ can be seen as the degree of edge traces, and in $\mathbf{M}_k^{\text{div}}(\hat{K})$ it corresponds to the degree of the normal traces. More details can be found in [33, 28].

4. Composite finite element exact sequences for the pyramid

As mentioned earlier, our basic approach to the construction of finite element spaces on pyramids consists in subdividing the master element

$$\hat{P} = \{(\hat{x}, \hat{y}, \hat{z}); \hat{z} - 1 \leq \hat{x}, \hat{y} \leq 1 - \hat{z}, 0 \leq \hat{z} \leq 1\}$$

into sub-tetrahedra. Thus, we consider a conformal partition $\mathcal{T}^{\hat{P}}$ consisting of tetrahedra T_ℓ . The simplest case is the division into two tetrahedra, as shown in Figure 1. A pyramid is illustrated with its vertices a_i , and the quadrilateral basis $\hat{F}_\square = [a_0, a_1, a_2, a_3]$. The two

Space		$\hat{T}$		$\hat{F}_\Delta$		$\hat{e}$
$H^1(\hat{T})$	$\supset$	$\mathbb{P}_{k+1}$	$\xrightarrow{\text{tr}}$	$\mathbb{P}_{k+1}$	$\xrightarrow{\text{tr}}$	$\mathbb{P}_{k+1}$
$\downarrow \nabla$		$\downarrow \nabla$				
$\mathbf{H}(\text{curl}, \hat{T})$	$\supset$	$\mathbb{P}_k^3 \oplus \hat{\mathbf{x}} \times \tilde{\mathbb{P}}_k^3$	$\xrightarrow{\text{tr}}$	$\mathbb{P}_k^2 \oplus \hat{\mathbf{x}} \times \tilde{\mathbb{P}}_k^2$	$\xrightarrow{\text{tr}}$	$\mathbb{P}_k$
$\downarrow \nabla \times$		$\downarrow \nabla \times$				
$\mathbf{H}(\text{div}, \hat{T})$	$\supset$	$\mathbb{P}_k^3 \oplus \hat{\mathbf{x}} \tilde{\mathbb{P}}_k$	$\xrightarrow{\text{tr}}$	$\mathbb{P}_k$		
$\downarrow \nabla \cdot$		$\downarrow \nabla \cdot$				
$L^2(\hat{T})$	$\supset$	$\mathbb{P}_k$				
Space		$\hat{H}$		$\hat{F}_\square$		$\hat{e}$
$H^1(\hat{H})$	$\supset$	$\mathbb{Q}_{k+1,k+1,k+1}$	$\xrightarrow{\text{tr}}$	$\mathbb{Q}_{k+1,k+1}$	$\xrightarrow{\text{tr}}$	$\mathbb{P}_{k+1}$
$\downarrow \nabla$		$\downarrow \nabla$				
$\mathbf{H}(\text{curl}, \hat{H})$	$\supset$	$\mathbb{Q}_{k,k+1,k+1} \times \mathbb{Q}_{k+1,k,k+1} \times \mathbb{Q}_{k+1,k+1,k}$	$\xrightarrow{\text{tr}}$	$\mathbb{Q}_{k,k+1} \times \mathbb{Q}_{k+1,k}$	$\xrightarrow{\text{tr}}$	$\mathbb{P}_k$
$\downarrow \nabla \times$		$\downarrow \nabla \times$				
$\mathbf{H}(\text{div}, \hat{H})$	$\supset$	$\mathbb{Q}_{k+1,k,k} \times \mathbb{Q}_{k,k+1,k} \times \mathbb{Q}_{k,k,k+1}$	$\xrightarrow{\text{tr}}$	$\mathbb{Q}_{k,k}$		
$\downarrow \nabla \cdot$		$\downarrow \nabla \cdot$				
$L^2(\hat{H})$	$\supset$	$\mathbb{Q}_{k,k}$				
Space		$\hat{P}r$		$\hat{F}$		$\hat{e}$
$H^1(\hat{P}r)$	$\supset$	$\mathbb{W}_{k+1,k+1}$	$\xrightarrow{\text{tr}}$	$\mathbb{Q}_{k+1,k+1}(\hat{F}_\square)$ $\mathbb{P}_{k+1}(\hat{F}_\Delta)$	$\xrightarrow{\text{tr}}$ $\xrightarrow{\text{tr}}$	$\mathbb{P}_{k+1}$ $\mathbb{P}_{k+1}$
$\downarrow \nabla$		$\downarrow \nabla$				
$\mathbf{H}(\text{curl}, \hat{P}r)$	$\supset$	$\left[\left(\mathbb{P}_k^2 \oplus \hat{\mathbf{x}} \times \tilde{\mathbb{P}}_k^2 \right) \otimes \mathbb{P}_k \right] \times \mathbb{W}_{k,k-1}$	$\xrightarrow{\text{tr}}$	$\mathbb{Q}_{k,k+1} \times \mathbb{Q}_{k+1,k}(\hat{F}_\square)$ $\mathbb{P}_k^2 \oplus \hat{\mathbf{x}} \times \tilde{\mathbb{P}}_k^2(\hat{F}_\Delta)$	$\xrightarrow{\text{tr}}$ $\xrightarrow{\text{tr}}$	$\mathbb{P}_k$ $\mathbb{P}_k$
$\downarrow \nabla \times$		$\downarrow \nabla \times$				
$\mathbf{H}(\text{div}, \hat{P}r)$	$\supset$	$\left[\left(\mathbb{P}_k^2 \oplus \hat{\mathbf{x}} \tilde{\mathbb{P}}_k \right) \otimes \mathbb{P}_{k-1} \right] \times \mathbb{W}_{k-1,k}$	$\xrightarrow{\text{tr}}$	$\mathbb{Q}_{k,k}(\hat{F}_\square)$ $\mathbb{P}_k(\hat{F}_\Delta)$		
$\downarrow \nabla \cdot$		$\downarrow \nabla \cdot$				
$L^2(\hat{P}r)$	$\supset$	$\mathbb{W}_{k,k}$				

Table 2: Exact polynomial space sequences of Class 1 for $\hat{T}$, $\hat{H}$ and $\hat{P}r$, and their traces on faces and edges

tetrahedra T_1 and T_2 , with virtual interface $\Gamma = T_1 \cap T_2$ form the partition $\mathcal{T}^{\hat{P}}$. This subdivision induces a partition $\mathcal{T}^{\hat{F}_\square} = \{F_{\Delta_i}\}$ into two triangles F_{Δ_1} and F_{Δ_2} . One could also consider sub-divisions of four (or more), which may be preferable on the grounds of simplifying the coding.

Figure 1: Pyramid divided into two tetrahedra.

By analogy with the exact sequences for $\hat{T}$ in the previous section, we consider two types of composite finite element space configurations for $\hat{P}$.

Composite sequence of Class 1 for $\hat{P}$

$$U_p^1(\hat{P}) = \mathbb{P}_{p+1}^{H^1}(\mathcal{T}^{\hat{P}}) = \{v \in H^1(\hat{P}); v|_{T_\ell} \in \mathbb{P}_{p+1}(T_\ell), \forall T_\ell \in \mathcal{T}^{\hat{P}}\}, \quad (1)$$

$$\mathbf{M}_p^{1,\text{curl}}(\hat{P}) = \mathbf{M}_p^{\text{curl}}(\mathcal{T}^{\hat{P}}) = \{\mathbf{q} \in \mathbf{H}(\text{curl}, \hat{P}); \mathbf{q}|_{T_\ell} \in \mathbf{M}_p^{\text{curl}}(\hat{T}_\ell), \forall T_\ell \in \mathcal{T}^{\hat{P}}\}, \quad (2)$$

$$\mathbf{M}_p^{1,\text{div}}(\hat{P}) = \mathbf{M}_p^{\text{div}}(\mathcal{T}^{\hat{P}}) = \{\mathbf{q} \in \mathbf{H}(\text{div}, \hat{P}); \mathbf{q}|_{T_\ell} \in \mathbf{M}_p^{\text{div}}(\hat{T}_\ell), \forall T_\ell \in \mathcal{T}^{\hat{P}}\}, \quad (3)$$

$$W_p^1(\hat{P}) = \mathbb{P}_p^{L^2}(\mathcal{T}^{\hat{P}}) = \{v \in L^2(\hat{P}); v|_{T_\ell} \in \mathbb{P}_p(T_\ell), \forall T_\ell \in \mathcal{T}^{\hat{P}}\}. \quad (4)$$

Composite sequence of Class 2 for $\hat{P}$

$$U_p^2(\hat{P}) = \mathbb{P}_{p+2}^{H^1}(\mathcal{T}^{\hat{P}}) = \{v \in H^1(\hat{P}); v|_{T_\ell} \in \mathbb{P}_{p+2}(T_\ell), \forall T_\ell \in \mathcal{T}^{\hat{P}}\}, \quad (5)$$

$$\mathbf{M}_p^{2,\text{curl}}(\hat{P}) = \mathbb{P}_{p+1}^{\text{curl}}(\mathcal{T}^{\hat{P}}) = \{\mathbf{q} \in \mathbf{H}(\text{curl}, \hat{P}); \mathbf{q}|_{T_\ell} \in \mathbb{P}_{p+1}(T_\ell)^3, \forall T_\ell \in \mathcal{T}^{\hat{P}}\}, \quad (6)$$

$$\mathbf{M}_p^{2,\text{div}}(\hat{P}) = \mathbb{P}_p^{\text{div}}(\mathcal{T}^{\hat{P}}) = \{\mathbf{q} \in \mathbf{H}(\text{div}, \hat{P}); \mathbf{q}|_{T_\ell} \in \mathbb{P}_p(T_\ell)^3, \forall T_\ell \in \mathcal{T}^{\hat{P}}\}, \quad (7)$$

$$W_p^2(\hat{P}) = \mathbb{P}_{p-1}^{L^2}(\mathcal{T}^{\hat{P}}) = \{v \in L^2(\hat{P}); v|_{T_\ell} \in \mathbb{P}_{p-1}(T_\ell), \forall T_\ell \in \mathcal{T}^{\hat{P}}\}. \quad (8)$$

Clearly, composite pyramidal elements form exact sequences just as the spaces defined in the foregoing section form exact sequences on tetrahedral elements. Table 3 is a summary of these pyramidal sequences, together with the trace spaces on each sub-element.

5. Finite element exact sequences for T-H-Pr-P partitions

Let $\mathcal{T}_h = \{K\}$ be a T-H-Pr-P partition of Ω and consider approximation spaces of the form

$$U_h = \{\varphi \in H^1(\Omega) : \varphi|_K \in U(K)\}, \quad (9)$$

$$\mathbf{X}_h = \{\boldsymbol{\sigma} \in \mathbf{H}(\text{curl}, \Omega); \boldsymbol{\sigma}|_K \in \mathbf{M}^{\text{curl}}(K)\}, \quad (10)$$

$$\mathbf{V}_h = \{\boldsymbol{\sigma} \in \mathbf{H}(\text{div}, \Omega); \boldsymbol{\sigma}|_K \in \mathbf{M}^{\text{div}}(K)\}, \quad (11)$$

$$W_h = \{\varphi \in L^2(\Omega) : \varphi|_K \in W(K)\}, \quad (12)$$

Composite space configuration of Class 1

Space	$\hat{P}$	$\hat{F}$	$\hat{e}$
$H^1(\hat{P})$	$\supset \mathbb{P}_{p+1}^{H^1}(\mathcal{T}^{\hat{P}})$	$\xrightarrow{\text{tr}} \mathbb{P}_{p+1}(\mathcal{T}^{\hat{F}\square})$ $\mathbb{P}_{p+1}(\hat{F}_\Delta)$	$\xrightarrow{\text{tr}} \mathbb{P}_{p+1}$ $\xrightarrow{\text{tr}} \mathbb{P}_{p+1}$
$\downarrow \nabla$	$\downarrow \nabla$		
$\mathbf{H}(\text{curl}, \hat{P})$	$\supset \mathbf{M}_p^{\text{curl}}(\mathcal{T}^{\hat{P}})$	$\xrightarrow{\text{tr}} \mathbb{P}_p^2 \oplus \hat{\mathbf{x}} \times \tilde{\mathbb{P}}_p^2(\mathcal{T}^{\hat{F}\square})$ $[\mathbb{P}_p]^2 \oplus \hat{\mathbf{x}} \times \tilde{\mathbb{P}}_p^2(\hat{F}_\Delta)$	$\xrightarrow{\text{tr}} \mathbb{P}_p$ $\xrightarrow{\text{tr}} \mathbb{P}_p$
$\downarrow \nabla \times$	$\downarrow \nabla \times$		
$\mathbf{H}(\text{div}, \hat{P})$	$\supset \mathbf{M}_p^{\text{div}}(\mathcal{T}^{\hat{P}})$	$\xrightarrow{\text{tr}} \mathbb{P}_p(\mathcal{T}^{\hat{F}\square})$ $\mathbb{P}_p(\hat{F}_\Delta)$	
$\downarrow \nabla \cdot$	$\downarrow \nabla \cdot$		
$L^2(\hat{P})$	$\supset \mathbb{P}_p^{L^2}(\mathcal{T}^{\hat{P}})$		

Composite space configuration of Class 2

Space	$\hat{P}$	$\hat{F}$	$\hat{e}$
$H^1(\hat{P})$	$\supset \mathbb{P}_{p+2}^{H^1}(\mathcal{T}^{\hat{P}})$	$\xrightarrow{\text{tr}} \mathbb{P}_{p+2}(\mathcal{T}^{\hat{F}\square})$ $\mathbb{P}_{p+2}(\hat{F}_\Delta)$	$\xrightarrow{\text{tr}} \mathbb{P}_{p+2}$ $\xrightarrow{\text{tr}} \mathbb{P}_{p+2}$
$\downarrow \nabla$	$\downarrow \nabla$		
$\mathbf{H}(\text{curl}, \hat{P})$	$\supset \mathbb{P}_{p+1}^{\text{curl}}(\mathcal{T}^{\hat{P}})$	$\xrightarrow{\text{tr}} \mathbb{P}_{p+1}(\mathcal{T}^{\hat{F}\square})^2$ $\mathbb{P}_{p+1}(\hat{F}_\Delta)^2$	$\xrightarrow{\text{tr}} \mathbb{P}_{p+1}$ $\xrightarrow{\text{tr}} \mathbb{P}_{p+1}$
$\downarrow \nabla \times$	$\downarrow \nabla \times$		
$\mathbf{H}(\text{div}, \hat{P})$	$\supset \mathbb{P}_p^{\text{div}}(\mathcal{T}^{\hat{P}})$	$\xrightarrow{\text{tr}} \mathbb{P}_p(\mathcal{T}^{\hat{F}\square})$ $\mathbb{P}_p(\hat{F}_\Delta)$	
$\downarrow \nabla \cdot$	$\downarrow \nabla \cdot$		
$L^2(\hat{P})$	$\supset \mathbb{P}_{p-1}^{L^2}(\mathcal{T}^{\hat{P}})$		

 Table 3: Exact composite space sequences of Class 1 and of Class 2 for the pyramid $\hat{P}$, and their traces on faces and edges

where the local spaces when $K \in \{T, H, Pr\}$ are chosen to be one of the variants described in Section 3 for a given index k . Some care must be exercised in the choice of local spaces on the pyramids, however, owing to the need to enforce inter-element conformity. For instance, suppose on the reference elements we chose the space configuration of Class 1 for T , H and Pr . Then, the appropriate choice of composite spaces on the pyramids is given by $U_p^2(\hat{P})$, $\mathbf{M}_p^{2,\text{curl}}(\hat{P})$, $\mathbf{M}_p^{2,\text{div}}(\hat{P})$ and $W_p^2(\hat{P})$, where the index p differs, depending on the type of neighbors: $p = 2k$ for a pyramid sharing its quadrilateral face with a hexahedron or a prism, with $p = k + 1$ otherwise. This choice of p is necessary since the quadrilateral faces are associated with traces in spaces containing tensor product type polynomials of degree k and the sub-tetrahedral are associated with polynomials of total

degree p . More precisely if we consider the constrained composite pyramidal spaces to be:

$$U(\hat{P}) = \{\varphi \in U_p^2(\hat{P}); \varphi|_{\hat{F}} \in P^{H^1}(\hat{F}), \varphi|_{\hat{e}} \in P^{H^1}(\hat{e})\}, \quad (13)$$

$$\mathbf{M}^{\text{curl}}(\hat{P}) = \{\mathbf{v} \in \mathbf{M}_p^{2,\text{curl}}(\hat{P}); \mathbf{v} \cdot \boldsymbol{\tau}^{\hat{e}}|_{\hat{e}} \in P^{\text{curl}}(\hat{e}); \mathbf{v} \wedge \mathbf{n}^{\hat{F}}|_{\hat{F}} \in \mathbf{P}^{\text{curl}}(\hat{F})\}, \quad (14)$$

$$\mathbf{M}^{\text{div}}(\hat{P}) = \{\mathbf{v} \in \mathbf{M}_p^{2,\text{div}}(\hat{P}); \mathbf{v} \cdot \mathbf{n}^{\hat{F}}|_{\hat{F}} \in P^{\text{div}}(\hat{F})\}, \quad (15)$$

where $\hat{F} \subset \partial\hat{P}$ and $\hat{e} \subset \partial\hat{P}$, and set $W(\hat{P}) = W_p^2(\hat{P})$, then the face and edge trace spaces are determined as follows:

Trace spaces for $U(\hat{P})$

- $P^{H^1}(\hat{F}_{\Delta}) = \mathbb{P}_{k+1}(\hat{F}_{\Delta})$ for triangular faces.
- $P^{H^1}(\hat{F}_{\square}) = \mathbb{Q}_{k+1,k+1}(\hat{F}_{\square})$ for quadrilateral faces shared with a hexahedron or a prism
- $P^{H^1}(\hat{F}_{\square}) = \mathbb{P}_{k+1}(\mathcal{T}^{\hat{F}_{\square}})$ for quadrilateral faces shared by two pyramids.
- $P^{H^1}(\hat{e}) = \mathbb{P}_{k+1}(\hat{e})$ for all edges.

Trace spaces for $\mathbf{M}^{\text{curl}}(\hat{P})$

- $\mathbf{P}^{\text{curl}}(\hat{F}_{\Delta}) = \mathbb{P}_k^2 \oplus \hat{\mathbf{x}} \times \tilde{\mathbb{P}}_k^2(\hat{F}_{\Delta})$ for triangular faces.
- $\mathbf{P}^{\text{curl}}(\hat{F}_{\square}) = \mathbb{Q}_{k,k+1} \times \mathbb{Q}_{k+1,k}(\hat{F}_{\square})$ for quadrilateral faces shared with a hexahedron or a prism.
- $\mathbf{P}^{\text{curl}}(\hat{F}_{\square}) = [\mathbb{P}_k]^2 \oplus \mathbf{x} \times [\tilde{\mathbb{P}}_k]^2(\mathcal{T}^{\hat{F}_{\square}})$ for quadrilateral faces shared by two pyramids.
- $\mathbf{P}^{\text{curl}}(\hat{e}) = \mathbb{P}_k(\hat{e})$ for all edges.

Trace spaces for $\mathbf{M}^{\text{div}}(\hat{P})$

- $P^{\text{div}}(\hat{F}_{\Delta}) = \mathbb{P}_k(\hat{F}_{\Delta})$ for triangular faces.
- $P^{\text{div}}(\hat{F}_{\square}) = \mathbb{Q}_{k,k}$ for a quadrilateral face shared with a hexahedron or a prism.
- $P^{\text{div}}(\hat{F}_{\square}) = \mathbb{P}_k(\mathcal{T}^{\hat{F}_{\square}})$ for quadrilateral faces shared by two pyramids.

By choosing the composite pyramidal spaces as above, we have the following result.

Theorem 5.1. *For the pyramid $\hat{P}$, the constrained composite spaces $U(\hat{P})$, $\mathbf{M}^{\text{curl}}(\hat{P})$, $\mathbf{M}^{\text{div}}(\hat{P})$ inherit the compatibility of their unconstrained counterparts, and combined with $W(\hat{P})$, they form an exact sequence.*

Remarks

1. Composite sequence of Class 1 for pyramids: With some minor modifications, composite sequence of Class 1 for pyramids can also be chosen.

2. Enriched space configurations for $\hat{T}$, $\hat{H}$, and $\hat{P}r$: Consider any of the exact sequences of spaces $U_k(\hat{K})$, $\mathbf{M}_k^{\text{curl}}(\hat{K})$, $\mathbf{M}_k^{\text{div}}(\hat{K})$ and $W_k(\hat{K})$ presented in Section 3 (of Class 1 for $\hat{T}$, $\hat{H}$, and $\hat{P}r$, or of Class 2 for $\hat{T}$), and factorize them as follows:

$$U_k(\hat{K}) = U_k^\partial(\hat{K}) \oplus \mathring{U}_k(\hat{K}) \quad (16)$$

$$\mathbf{M}_k^{\text{curl}}(\hat{K}) = \mathbf{M}_k^{\partial, \text{curl}}(\hat{K}) \oplus \mathring{\mathbf{M}}_k^{\text{curl}}(\hat{K}), \quad (17)$$

$$\mathbf{M}_k^{\text{div}}(\hat{K}) = \mathbf{M}_k^{\partial, \text{div}}(\hat{K}) \oplus \mathring{\mathbf{M}}_k^{\text{div}}(\hat{K}). \quad (18)$$

where $\mathring{U}_k(\hat{K})$, $\mathring{\mathbf{M}}_k^{\text{div}}(\hat{K})$, and $\mathring{\mathbf{M}}_k^{\text{div}}(\hat{K})$ are the corresponding internal functions, and $U_k^\partial(\hat{K})$, $\mathbf{M}_k^{\partial, \text{curl}}(\hat{K})$ and $\mathbf{M}_k^{\partial, \text{div}}(\hat{K})$ are the complementary trace components. Enriched versions may be defined as follows:

$$U_k^+(\hat{K}) = U_k^\partial(\hat{K}) \oplus \mathring{U}_{k+1}(\hat{K}), \quad (19)$$

$$\mathbf{M}_k^{\text{curl},+}(\hat{K}) = \mathbf{M}_k^{\partial, \text{curl}}(\hat{K}) \oplus \mathring{\mathbf{M}}_{k+1}^{\text{curl}}(\hat{K}), \quad (20)$$

$$\mathbf{M}_k^{\text{div},+}(\hat{K}) = \mathbf{M}_k^{\partial, \text{div}}(\hat{K}) \oplus \mathring{\mathbf{M}}_{k+1}^{\text{div}}(\hat{K}), \quad (21)$$

$$W_k^+(\hat{K}) = W_{k+1}(\hat{K}). \quad (22)$$

In other words, they are obtained by pruning from $U_{k+1}(\hat{K})$, $\mathbf{M}_{k+1}^{\text{curl}}(\hat{K})$ and $\mathbf{M}_{k+1}^{\text{div}}(\hat{K})$ the trace functions of the highest degree, keeping just those ones with traces of degree not greater than k .

Theorem 5.2. *The enriched spaces $U_k^+(\hat{K})$, $\mathbf{M}_k^{\text{curl},+}(\hat{K})$, $\mathbf{M}_k^{\text{div},+}(\hat{K})$, and $W_k^+(\hat{K})$, defined by the equations (19)-(22), obtained from exact sequences $U_k(\hat{K})$, $\mathbf{M}_k^{\text{curl}}(\hat{K})$, $\mathbf{M}_k^{\text{div}}(\hat{K})$, and $W_k(\hat{K})$, factorized as in (16)-(18), form exact sequences as well. Furthermore, both sequences share the same trace spaces. Consequently, the enriched spaces can be used in the construction of finite element exact sequences for T-H-Pr-P partitions in combination with composite pyramidal elements, just as their original space versions do.*

Proof: By construction, it is clear that the original sequence and its enriched version share the same trace spaces. Since the vector functions in $\nabla \mathring{U}_{k+1}(\hat{K}) \subset \mathbf{M}_{k+1}^{\text{curl}}(\hat{K})$ are normal to $\partial \hat{K}$, we conclude that $\nabla \mathring{U}_{k+1}(\hat{K}) \subset \mathring{\mathbf{M}}_{k+1}^{\text{curl}}(\hat{K})$, implying the gradient compatibility of the pair $\{U_k^+(\hat{K}), \mathbf{M}_k^{\text{curl},+}(\hat{K})\}$. Similarly, since $\nabla \times \mathring{\mathbf{M}}_{k+1}^{\text{curl}}(\hat{K}) \subset \mathbf{M}_{k+1}^{\text{div}}(\hat{K})$ is divergence free, Green's formula imply that for $\mathbf{q} = \nabla \times \phi \in \nabla \times \mathring{\mathbf{M}}_{k+1}^{\text{curl}}(\hat{K})$

$$0 = (\nabla \times \phi, \nabla \varphi)_{\hat{K}} = \langle \varphi, \mathbf{q} \cdot \hat{\mathbf{n}} \rangle_{\partial \hat{K}}, \forall \varphi \in H^1(\hat{K}),$$

meaning that $\mathbf{q} \cdot \hat{\mathbf{n}} = 0$, i.e., $\mathbf{q} \in \mathring{\mathbf{M}}_{k+1}^{\text{div}}(\hat{K})$. Therefore, space enrichment does not affect the rotational compatibility. The divergence compatibility of general enriched pairs $\{\mathbf{M}_k^{\text{div},+}(\hat{K}), W_k^+(\hat{K})\}$ was studied in [16, 21]. A counting argument shows that equality of the image of a differential operator (∇ , $\nabla \times$ or $\nabla \cdot$) with the kernel of the next one holds instead of only inclusion. $\square$

6. Interpolants satisfying the commuting property

The purpose of this section is to define interpolants satisfying the commuting de Rham diagram property (23) for the sequence of spaces based on T-H-Pr-P meshes proposed in

the previous section

$$\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\nabla} & \mathbf{H}(\text{curl}, \Omega) & \xrightarrow{\nabla \times} & \mathbf{H}(\text{div}, \Omega) & \xrightarrow{\nabla \cdot} & L^2(\Omega) \\
\cup & & \cup & & \cup & & \\
H^2(\Omega) & & \mathbf{H}^1(\text{curl}, \Omega) & & \mathbf{H}^1(\text{div}, \Omega) & & \\
\downarrow \Pi_h^{\text{grad}} & & \downarrow \Pi_h^{\text{curl}} & & \downarrow \Pi_h^{\text{div}} & & \downarrow \Lambda_h \\
U_h & \xrightarrow{\nabla} & \mathbf{X}_h & \xrightarrow{\nabla \times} & \mathbf{V}_h & \xrightarrow{\nabla \cdot} & W_h
\end{array} \tag{23}$$

where $\mathbf{H}^1(\text{curl}, \Omega) = \{\mathbf{q} \in \mathbf{H}^1(\Omega); \nabla \times \mathbf{q} \in H^1(\Omega)\}$, and $\mathbf{H}^1(\text{div}, \Omega)$ is defined similarly. The interpolant Λ_h is the usual L^2 -projection over W_h .

Our basic approach to defining the remaining operators Π_h^{grad} , Π_h^{curl} and Π_h^{div} makes use of the projection-based interpolation operators proposed by L. Demkowicz and several coworkers, as summarized in [19]. They admit a general form, without requiring any specific geometric aspect, and are valid for general exact sequences of approximation spaces, including some classes of constrained spaces occurring in hp contexts (which include the enriched space configurations for purely tetrahedral, hexahedral and prismatic meshes as particular cases). Thus, they are appropriate for the current context of spaces for T-H-Pr-P meshes, and shall be described in a general form for the particular case of interest of composite constrained spaces for pyramids $K = P$.

The interpolants are expressed by the combination of local versions of them, $\pi^{K,\text{grad}}$, $\pi^{P,\text{curl}}$, and $\pi^{P,\text{div}}$, over the constrained local spaces $U(P)$, $\mathbf{M}^{\text{curl}}(P)$, and $\mathbf{M}^{\text{div}}(P)$, which are defined for sufficiently smooth functions. In each case, the interpolants take the form of a sum of linearly independent contributions associated with vertices, edges, faces, and volume, according to the kind of traces appropriate to the space under consideration.

6.1. Interpolant $\pi^{K,\text{grad}}$ over $U(P)$

For simplicity, we assume $u \in H^2(\Omega)$. The interpolant $\pi^{P,\text{grad}}u \in U(P)$ can be expressed as the sum of four contributions, which are applied recursively

$$\begin{aligned}
\pi^{P,\text{grad}}u &= \pi_V^{P,\text{grad}}u + \pi_E^{P,\text{grad}}(u - \pi_V^{P,\text{grad}}u) \\
&+ \pi_F^{P,\text{grad}}(u - \pi_V^{P,\text{grad}}u - \pi_E^{P,\text{grad}}u) \\
&+ \pi_I^{P,\text{grad}}(u - \pi_V^{P,\text{grad}}u - \pi_E^{P,\text{grad}}u - \pi_F^{P,\text{grad}}u),
\end{aligned}$$

where the projections $\pi_V^{P,\text{grad}}$, $\pi_E^{P,\text{grad}}$, $\pi_F^{P,\text{grad}}$ and $\pi_I^{P,\text{grad}}$ are defined as follows:

- $\pi_V^{P,\text{grad}}u$ is the linear interpolant of u at the vertices a of P :

$$\pi_V^{P,\text{grad}}u(a) = u(a). \tag{24}$$

- $\pi_E^{P,\text{grad}}u|_e \in P^{H^1}(e) \cap H_0^1(e)$:

$$\int_e \frac{\partial}{\partial s} (u - \pi_V^{P,\text{grad}}u - \pi_E^{P,\text{grad}}u) \frac{\partial \varphi}{\partial s} ds = 0, \quad \forall \varphi \in P^{H^1}(e) \cap H_0^1(e). \tag{25}$$

- $\pi_F^{P,\text{grad}} u|_F \in P^{H^1}(F) \cap H_0^1(F)$:

$$\int_F \nabla_F(u - \pi_V^{K,\text{grad}} u - \pi_E^{P,\text{grad}} u - \pi_F^{P,\text{grad}} u) \cdot \nabla_F \varphi \, dF = 0, \quad \forall \varphi \in P^{H^1}(F) \cap H_0^1(F). \quad (26)$$

- $\pi_I^{P,\text{grad}} u \in U(P) \cap H_0^1(P)$:

$$\int_P \nabla(u - \pi_V^{P,\text{grad}} u - \pi_E^{P,\text{grad}} u - \pi_F^{P,\text{grad}} u - \pi_I^{P,\text{grad}} u) \cdot \nabla \varphi \, dP = 0, \quad \forall \varphi \in U(P) \cap H_0^1(P). \quad (27)$$

6.2. Interpolant $\pi^{P,\text{curl}}$ over $\mathbf{M}^{\text{curl}}(P)$

Let $\mathbf{q} \in \mathbf{H}^1(\text{curl}, \Omega)$. Then the interpolant $\pi^{P,\text{curl}}$ is given by

$$\pi^{P,\text{curl}} \mathbf{q} = \pi_E^{P,\text{curl}} \mathbf{q} + \pi_F^{P,\text{curl}} (\mathbf{q} - \pi_E^{P,\text{curl}} \mathbf{q}) + \pi_I^{P,\text{curl}} (\mathbf{q} - \pi_E^{P,\text{curl}} \mathbf{q} - \pi_F^{P,\text{curl}} \mathbf{q}),$$

where the projections $\pi_E^{P,\text{curl}}$, $\pi_F^{P,\text{curl}}$, and $\pi_I^{P,\text{curl}}$ are defined as follows:

- $\pi_E^{P,\text{curl}} \mathbf{q}|_e \in \mathbf{P}^{\text{curl}}(e)$:

$$\int_e (\mathbf{q} - \pi_E^{K,\text{curl}} \mathbf{q}) \cdot \boldsymbol{\tau}_e \varphi \, de = 0, \quad \forall \varphi \in P^{\text{curl}}(e). \quad (28)$$

- $\pi_F^{P,\text{curl}} \mathbf{q}|_F \in \mathbf{P}^{\text{curl}}(F) \cap \mathbf{H}_0^1(\text{curl}, F)$: $\forall \boldsymbol{\phi} \in \mathbf{P}^{\text{curl}}(F) \cap \mathbf{H}_0^1(\text{curl}, F)$

$$\int_F \nabla \times (\mathbf{q} - \pi_E^{K,\text{curl}} \mathbf{q} - \pi_F^{K,\text{curl}} \mathbf{q}) \wedge \mathbf{n}^F \cdot \nabla \times \boldsymbol{\phi} \, dF = 0, \quad (29)$$

$$\int_F (\mathbf{q} - \pi_E^K \mathbf{q} - \pi_F^{K,\text{curl}} \mathbf{q}) \wedge \mathbf{n}^F \cdot \boldsymbol{\phi} \, dF = 0, \quad \text{when } \nabla \times \boldsymbol{\phi} = 0. \quad (30)$$

- $\pi_I^{P,\text{curl}} \mathbf{q} \in \mathbf{M}^{\text{curl}}(P) \cap \mathbf{H}_0^1(\text{curl}, P)$: for all $\boldsymbol{\sigma} \in \mathbf{M}^{\text{curl}}(P) \cap \mathbf{H}_0^1(\text{curl}, P)$:

$$\int_P \nabla \times (\mathbf{q} - \pi_E^{P,\text{curl}} \mathbf{q} - \pi_F^{P,\text{curl}} \mathbf{q} - \pi_I^{P,\text{curl}} \mathbf{q}) \cdot \nabla \times \boldsymbol{\sigma} \, dP = 0, \quad (31)$$

$$\int_P (\mathbf{q} - \pi_E^{P,\text{curl}} \mathbf{q} - \pi_F^{P,\text{curl}} \mathbf{q} - \pi_I^{P,\text{curl}} \mathbf{q}) \cdot \boldsymbol{\sigma} \, dP = 0, \quad \text{when } \nabla \times \boldsymbol{\sigma} = 0. \quad (32)$$

6.3. Interpolant $\pi^{P,\text{div}}$ over $\mathbf{M}^{\text{div}}(P)$

For $\mathbf{q} \in \mathbf{H}^1(\text{div}, \Omega)$, the interpolant $\pi^{P,\text{div}}$ is given by

$$\pi^{P,\text{div}} \mathbf{q} = \pi_F^{P,\text{div}} \mathbf{q} + \pi_I^{P,\text{div}} (\mathbf{q} - \pi_F^{P,\text{div}} \mathbf{q}),$$

where the projections $\pi_F^{P,\text{div}}$, and $\pi_I^{P,\text{div}}$ are defined as follows:

- $\pi_F^{P,\text{div}} \mathbf{q}|_F \in P^{\text{div}}(F)$:

$$\int_F (\mathbf{q} - \pi_F^{P,\text{div}} \mathbf{q}) \cdot \mathbf{n} \, \phi \, dF = 0, \quad \forall \phi \in P^{\text{div}}(F). \quad (33)$$

- $\pi_I^{P,\text{div}} \in \mathbf{M}^{\text{div}}(P) \cap \mathbf{H}_0^1(\text{div}, P)$: for all $\sigma \in \mathbf{M}^{\text{div}}(P) \cap \mathbf{H}_0^1(\text{div}, P)$:

$$\int_P \nabla \cdot (\mathbf{q} - \pi_F^{P,\text{div}} \mathbf{q} - \pi_I^{P,\text{div}} \mathbf{q}) \nabla \cdot \sigma \, dP = 0, \quad (34)$$

$$\int_P (\mathbf{q} - \pi_F^{P,\text{div}} \mathbf{q} - \pi_I^{P,\text{div}} \mathbf{q}) \cdot \sigma \, dP = 0, \text{ when } \nabla \cdot \sigma = 0. \quad (35)$$

The commutation properties can be verified as in [19]. The only new point is the issue regarding the pyramidal elements. However, thanks to our use of composite spaces in pyramids, the commuting property is a consequence of the corresponding results hold for the tetrahedra in the sub-partition. These facts are collected in the following theorem.

Theorem 6.1. *For pyramids $P \in \mathcal{T}_h$, the local projection-based interpolants $\pi^{P,\text{grad}}$, $\pi^{P,\text{curl}}$, and $\pi^{P,\text{div}}$ over the local spaces $U(P)$, $\mathbf{M}^{\text{curl}}(P)$, and $\mathbf{M}^{\text{div}}(P)$, respectively, are well defined and bounded. Furthermore, these interpolants are compatible with those defined in tetrahedra, hexahedra, and prisms. Consequently, global interpolation operators can be defined so that their restriction to an element is $\pi^{K,\text{grad}}$, $\pi^{P,\text{curl}}$, or $\pi^{P,\text{div}}$. These global interpolants satisfy the commuting diagram property.*

7. Application to the mixed formulation for Darcy's problems

Consider a Darcy's flow defined in a polygonal region $\Omega \subset \mathbb{R}^3$:

$$\begin{aligned} \nabla \cdot \sigma &= f \text{ in } \Omega, \\ \sigma &= -\mathcal{K} \nabla u \text{ in } \Omega, \\ u &= u_D \text{ on } \partial\Omega_D, \\ \sigma \cdot \mathbf{n} &= \sigma_N \text{ on } \partial\Omega_N, \end{aligned}$$

where $\mathcal{K}$ is a symmetric positive definite tensor, $f \in L^2(\Omega)$, $u_D \in C(\partial\Omega_D)$, and $\sigma_N \in L^2(\partial\Omega_N)$ are prescribed Dirichlet and Neumann boundary data. For the present study, we seek an approximation in spaces W_h and $\mathbf{V}_h$, based on T-H-Pr-P meshes $\mathcal{T}_h$, described in the previous sections. As studied in [13], the discrete variational mixed formulation consists of: find $u_h \in W_h$ and $\sigma_h \in \mathbf{V}_h$, $\sigma_h \cdot \mathbf{n} = \sigma_N$ on $\partial\Omega_N$, such that for all $\mathbf{q} \in \mathbf{V}_h$, with $\mathbf{q} \cdot \mathbf{n} = 0$ on $\partial\Omega_N$, and all $v \in W_h$ the following system of equations holds

$$\int_{\Omega} \mathcal{K}^{-1} \sigma_h \cdot \mathbf{q} \, d\Omega - \int_{\Omega} u_h \nabla \cdot \mathbf{q} \, d\Omega = - \int_{\partial\Omega_D} u_D \mathbf{q} \cdot \mathbf{n} \, ds, \quad (36)$$

$$\int_{\Omega} \nabla \cdot \sigma \, \text{deR} h_h v \, d\Omega = \int_{\Omega} f v \, d\Omega. \quad (37)$$

Stability and error analysis

The stability analysis follows standard lines based on Brezzi's Theorem. Ellipticity on the kernel follows the usual argument. Using Fortin's criterion [27], the discrete inf-sup condition is assured by the existence of bounded interpolants satisfying the commuting diagram property $\Lambda_h \nabla \cdot \mathbf{q} = \nabla \cdot \Pi_h^{\text{div}} \mathbf{q}$, thanks to Section 6. Furthermore, error analysis of approximate solutions $(\sigma_h, u_h) \in \mathbf{V}_h \times W_h$ can be performed by separate expressions of flux, flux divergence and potential errors in terms of projection errors of the corresponding exact solutions, as stated in [6], Theorem 6.1. A general overview of these aspects can be found in [13, 24, 5].

Classic arguments are usually applied to derive projection error estimations. For that, the mesh family $\mathcal{T}_h$ is required to have some shape regularity. According to [25], for affine meshes, it is sufficient to assume that there exists a constant γ , independent of h and K , such that the shape constants $\gamma(K) := h(K)/\rho(K) \leq \gamma$, where $h(K)$ denotes the diameter of K , and $\rho(K)$ is the diameter of the largest ball contained in K such that K is star-shaped with respect to it.

As for the proofs of Theorem 4.1 and Theorem 4.2 in [6], the rates of convergence depend on the total degrees of the vector and scalar polynomials that can be represented in the local spaces $\mathbf{M}^{\text{div}}(K)$ and $W(K)$. For all element geometry, $\mathbb{P}_k^3 \subset \mathbf{M}^{\text{div}}(K)$, the same property holding for the enriched spaces $\mathbf{M}^{\text{div},+}(K)$. For the scalar spaces, $\mathbb{P}_k \subset W_k(K)$, and $\mathbb{P}_{k+1} \subset W_k^+(K)$. Consequently, the following projection error estimates hold for such space configurations.

Theorem 7.1. *Consider the projections Π_h^{div} and Λ_h for the approximation space configurations $\{\mathbf{V}_h, W_h\}$, based on T-H-Pr-P shape-regular affine meshes $\mathcal{T}_h$. Suppose that the local spaces $\mathbf{M}^{\text{div}}(K)$ and $W(K)$ of Class 1, or $\mathbf{M}^{\text{div},+}(K)$ and $W_k^+(K)$ of Class 1^+ are used for hexahedra, tetrahedra, and prismatic elements, combining them with constrained composite local spaces $\mathbf{M}^{\text{div}}(P)$ and $W(P)$ for pyramids. Then, the following error estimates hold for sufficient regular $\mathbf{q}$ and u*

$$\|\mathbf{q} - \Pi_h^{\text{div}} \mathbf{q}\|_{\mathbf{L}^2(\Omega)} \lesssim h^{k+1} \|\mathbf{q}\|_{\mathbf{H}^{k+1}(\Omega)}, \quad (38)$$

$$\|\nabla \cdot (\mathbf{q} - \Pi_h^{\text{div}} \mathbf{q})\|_{L^2(\Omega)} \lesssim h^{t+1} \|\nabla \cdot \mathbf{q}\|_{H^{t+1}(\Omega)}, \quad (39)$$

$$\|u - \Lambda_h u\|_{L^2(\Omega)} \lesssim h^{t+1} \|u\|_{H^{t+1}}, \quad (40)$$

where $t = k$ for the Class 1 case, and $t = k + 1$ for the enriched version of Class 1^+ .

Here and in what follows we write $a \lesssim b$ whenever $a \leq Cb$ for a constant C not depending on essential quantities. The leading constants in (38) and (39) depend only on the shape regularity constant of $\mathcal{T}_h$. Applying these projection errors, the error analysis we are looking for the numerical solutions of the mixed formulation is summarized in the next theorem.

Theorem 7.2. *Consider approximation space configurations $\{\mathbf{V}_h, W_h\}$ based on T-H-Pr-P shape-regular affine meshes $\mathcal{T}_h$ of a convex domain Ω , for which projections Π_h and Λ_h are defined, as stated in Theorem 7.1. Let $\boldsymbol{\sigma}_h$ and $u_h \in U_h$, satisfying (37)-(37), be approximate solutions for the exact flux $\boldsymbol{\sigma}$ and potential u , which are supposed to be regular enough. The following estimates hold:*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{L}^2(\Omega)} \lesssim h^{k+1} \|\boldsymbol{\sigma}\|_{\mathbf{H}^{s+1}(\Omega)}, \quad (41)$$

$$\|\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L^2(\Omega)} \lesssim h^{t+1} \|\nabla \cdot \boldsymbol{\sigma}\|_{H^{r+1}}, \quad (42)$$

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h^{q+1} \|u\|_{H^{q+1}}, \quad (43)$$

where $t = k$ when spaces of Class 1 are used, $t = k + 1$ for the Class 1^+ case, and $q = \min\{k + 1, t\}$.

Observe that the estimates (41)-(43) show the same orders of approximation as one would obtain using purely hexahedral, purely tetrahedral or purely prismatic partitions, as derived in [16, 21].

8. Numerical Examples

The foregoing developments will now be illustrated for some simple examples. For the implementations, the hierarchical bases $\mathbf{B}^{\text{div}}(\hat{K})$ for $\mathbf{P}_k(\hat{K})$ are the ones constructed in [16]. The adopted local spaces $\mathbf{M}^{\text{div}}(K)$ are the $BDFM_{k+1}$ spaces for tetrahedra, corresponding to the enriched version of the BDM_k space of Class 2, and RT_k spaces for hexahedra. In both cases, the flux normal traces and potential spaces have degree k . Their corresponding enriched versions $BDFM_{k+1}^+$ and RT_k^+ are also used for the simulations. For pyramids, the constrained composite space configuration of Class 2 is adopted.

In order to apply static condensation, the degrees of freedom of the flux $\boldsymbol{\sigma}$ are organized in two parts: $\boldsymbol{\sigma}|_K \in \mathring{\mathbf{M}}^{\text{div}}(K)$, and $\boldsymbol{\sigma}^\partial|_K \in \mathbf{M}^{\text{div},\partial}(K)$, referring to internal and face components of $\boldsymbol{\sigma}$, respectively. For the variable u , take u_0 to be formed by constant values on each element K (the choice of any other degree of freedom for u corresponding to a shape function in each element with nonzero average also works), and let u_i denote the remaining degrees of freedom except u_0 . Then, static condensation is applied by eliminating the degrees of freedom $\boldsymbol{\sigma}$, and u_i , to get a condensed system in terms of $\boldsymbol{\sigma}^\partial$, and u_0 , of dimension proportional of the trace space dimension over the mesh skeleton. The H^1 -conforming simulations adopt spaces $U(K)$ spanned by the hierarchical shape functions described in [20]. The MKL/Pardiso solver has been used for the resolution of the global condensed linear systems.

8.1. Test problem 1

Consider a Poisson test problem defined on the region $\Omega = [0, 1]^3$, which exact solution is equal to $u_{ex}(x, y, z) = \sin(\pi x) \sin(\pi y) \sin(\pi z)$. The domain Ω is partitioned by an affine hexahedral-pyramidal mesh (H-P) constructed from a uniform partition of $2^n \times 2^n \times 2^n$ cubic elements. Following an alternating pattern, the cubes are divided into six pyramids, such that all internal quadrilateral interfaces are shared by a pyramid. Figure 2 illustrates this H-P mesh construction for $n = 1$ and 2. The convergence rates of the approximate solutions obtained with different $\mathbf{H}(\text{div})$ -conforming finite element approximations described in the previous sections are tested, and the evolution of the L^2 -errors as a function of the mesh size $h = 2^{-n}$ are presented for approximations with $k = 1, 2$ and 3, and $n = 1, 2$ and 3.

Figure 2: Problem 1: H-P meshes formed by $2^n \times 2^n \times 2^n$ uniform cubic partitions, which are every other one divided into six pyramids, for $n = 1$ (left side), and $n = 2$ (right side).

Pyramids sub-divided into two or four tetrahedra

As illustrated in Figure 3, the composition of the pyramids by two or by four tetrahedra has practically no effect on the solution accuracy. The composition of pyramids by four tetrahedra does not increase the number of equations in the condensed systems, but it requires more internal degrees-of-freedom, increasing the effort of matrix assembly. However, this strategy may be preferable for its more simple code implementation.

Figure 3: Problem 1: L^2 -errors in u (left side) and σ (right side) using space configurations of Class 1 for H-P meshes, with $k = 1, 2$ and 3; the pyramids subdivided by two ($T = 2$, continuous lines) or by four ($T = 4$, dashed lines) internal tetrahedra.

Approximations with enriched space configurations

In the plots of Figure 4, we compare the errors of the potential and flux variables when space configurations of Class 1 and Class 1⁺ are applied to the H-P mesh. The pyramids are subdivided by two tetrahedra. As predicted by the error analysis, one unit higher order $k + 2$ of convergence is obtained for potential and divergence variables when using enriched spaces, and that the errors for the flux are almost the same in both classes of space configurations, especially for high order cases $k = 2$ and $k = 3$.

Figure 4: Problem 1: L^2 -errors in u (left side), σ (middle), and divergence (right side) using space configurations of Class 1 (continuous lines) and of Class 1⁺ (dashed lines) based on H-P meshes; the pyramids are subdivided by two tetrahedra.

Hexahedral-pyramidal meshes versus purely tetrahedral or purely hexahedral meshes

Error curves versus the number of equations in the condensed systems are presented in Figure 5 for the potential and flux variables. Three scenarios are analyzed for the mixed method based on space configuration of Class 1, using hexahedral-pyramidal meshes (with pyramids by four tetrahedra), purely tetrahedral, and purely hexahedral meshes, for $k = 1$ and $k = 2$. For comparison, results for the mixed formulation with space configuration of Class 1⁺, and for the H^1 -conforming method are also included, both based on the hexahedral-pyramidal meshes.

It can be observed that the error magnitudes for flux and potential variables obtained with the mixed method based on space configuration of Class 1 for hexahedral-pyramidal meshes are in between the errors for purely hexahedral meshes and purely tetrahedral meshes. The results are plausible as the meshes including pyramids are composed of hexahedral and pyramid elements. The reduced efficiency of tetrahedral meshes is compatible with previous publications [23]. The results confirm the improved convergence of the potential for space configuration of Class 1⁺, as a function of degrees of freedom, for the H-P meshes. This type of enhanced accuracy was first documented in [22] for meshes of single element geometry. Observe that the errors of these enriched formulations are comparable to the errors of H^1 approximations with polynomial order $k + 1$, as illustrated by the plots for $k = 1$.

8.2. Test problem 2: Thiem-Dupuit radial flow

Consider the application of the mixed formulation for a Darcy's problem with Thiem-Dupuit radial flow as an exact solution. It is defined form on a circular reservoir which is completely perforated by a vertical well. The pressure distribution depends only on the radius variable r , by the expression

$$u_{exact} = p_0 - \frac{Q\eta}{2\pi\kappa H_{np}} \ln \left(\frac{r}{r_0} \right).$$

where H_{np} is the reservoir of thickness (net-pay), κ is the constant permeability, r_0 is the reservoir radius, where the pressure remains unchanged at a value p_0 , η is the constant fluid viscosity, and Q is the constant fluid rate. Due to its logarithmic variation, most of the pressure drop occurs in the wellbore region. For this problem, consider the computational domain approximated by an octahedral region, and the input data displayed in Table 4.

Property	Value
Reservoir dimensions	$H_{np} = 10 \text{ m}, r_w = 0.2 \text{ m}, r_o = 50 \text{ m}$
Permeability	$\kappa = 10^{-13} \text{ m}^2$
Viscosity	$\eta = 0.001 \text{ Pa s}$
External pressure	$p_o = 25 \times 10^6 \text{ Pa}$
Flow rate	$Q = 0.01 \text{ m}^3 \text{ s}^{-1}$

Table 4: Input data for the Thiem-Dupuit radial flow.

The computational cylindrical domain is replaced by a region where the outer and inner curved boundaries are approximated by planar faces, and which are partitioned by tetrahedral-hexahedral-pyramidal meshes (T-H-P) generated by Gmsh ². At the coarsest

²<http://gmsh.info/>

Figure 5: Problem 1: L^2 -errors for u (left side) and σ (right side) versus the number of equations in the condensed systems, for $k = 1$ (top side) and $k = 2$ (bottom side), obtained using mixed formulation with space configurations of Class 1 (continuous lines), based on H-P meshes (MF (H-P)), purely tetrahedral meshes (MF (T)), and purely hexahedral meshes (MF (H)); with space configurations of Class 1^+ (dashed lines) based on hexahedral-pyramidal meshes (MF (H-P)); and H^1 -conforming formulation (dotted lines) based on hexahedral-pyramidal meshes (H^1 (H-P)).

level, there is an unstructured tetrahedral partition of a small cube around the wellbore (length with $= 5 \times r_w$); the most part of the region is filled by structured hexahedral elements, directionally refined towards the central well; the transition between these two set of elements is obtained by one layer of pyramids, as illustrated in Figure 6. Following, the coarsest mesh is uniformly refined.

In the plots in Figure 7, the errors of the potential and flux variables of Problem 2 are compared when space configurations of Class 1 and Class 1^+ based on such T-H-P meshes are applied. The pyramids are subdivided by four tetrahedra, and $k = 1$ or $k = 2$. The predicted convergence rates are verified.

Error curves versus the number of equations in the condensed systems are presented in Figure 8 for the potential and flux variables. Three scenarios are compared for space configurations of Class 1: using the hexahedral-pyramidal-tetrahedral meshes (with pyramids subdivided by four tetrahedra), unstructured purely tetrahedral, and structured purely

Figure 6: Problem 2: an overview of the T-H-P mesh (top), the core unstructured tetrahedral part (bottom left side), and the trapezoidal ring (bottom right side) used to connect the bulk structured hexahedra and the tetrahedral elements.

hexahedral meshes.

Figure 7: Problem 2: L^2 -errors in u (left side) and σ (right side) using space configurations of Class 1 (continuous lines) and of Class 1^+ (+, dashed lines), for $k = 1$ and $k = 2$, based on the T-H-P meshes; pyramids are subdivided by 4 tetrahedra.

Figure 8: Problem 2: L^2 -errors for u (left side) and σ (right side) versus the number of equations in the condensed systems, obtained using space configurations of Class 1, for $k = 1$ (continuous lines), and $k = 2$ (dashed lines), based on the H-T-P meshes, purely tetrahedral meshes (T), and purely hexahedral meshes (H); pyramids are subdivided by 4 tetrahedra.

8.3. Problem 3: flow past a spherical object

Let us consider the simulation of the steady flow problem

$$\Delta u = 0$$

of an incompressible, inviscid fluid past a spherical obstacle of radius $a = 0.1$ and centered at the origin, having constant velocity magnitude $V = 1$. The exact solution is given by the formula

$$u_{ex}(r, \theta) = -a \left[\frac{r}{a} + \frac{1}{2} \left(\frac{a}{r} \right)^2 \right] \cos \theta.$$

The computation region Ω is a cube with the sphere placed at its center. Dirichlet boundary condition is set over $\partial\Omega$. To construct the meshes, at the coarsest level there is a bulk part partitioned by structured hexahedral elements, directionally refined towards a small cube of side length 0.75 around the sphere, where an unstructured triangular partition is created. The transition between these two set of elements is filled with pyramids, as illustrated in Figure 9. Following, the coarsest mesh is uniformly refined.

Figure 9: Problem 3: T-H-P mesh (left side), and zoom in (right side).

The space configuration of Class 1 is applied, where the pyramids are subdivided by four tetrahedra, for $k = 1$ and $k = 2$. The results are plotted in Figure 10, showing the expected convergence errors of order $k + 1$ for potential and flux variables. Error curves versus the number of equations in the condensed systems are presented in Figure 11 for the potential and flux variables obtained with T-H-P meshes, unstructured purely tetrahedral, and structured purely hexahedral meshes. The expressively lower error of the hexahedral meshes can be understood by the fact that these meshes are almost aligned with the flux directions. In a flux aligned mesh, the flux over the faces perpendicular to the flow is zero and the flux over the orthogonal faces is almost constant. Tetrahedral meshes will never attain such configuration.

Figure 10: Problem 3: L^2 -errors in u (left side) and σ (right side) using space configurations of Class 1 for $k = 1$ (continuous lines) and $k = 2$ (dashed lines), based on the T-H-P meshes; pyramids are subdivided by 4 tetrahedra.

Figure 11: Problem 3: L^2 -errors for u (left side) and σ (right side) versus the number of equations in the condensed systems, obtained using space configurations of Class 1, for $k = 1$ (continuous lines), and $k = 2$ (dashed lines), based on the T-H-P meshes, purely tetrahedral meshes (T), and purely hexahedral meshes (H); pyramids are subdivided by 4 tetrahedra.

9. Conclusions

Two classes of composite hierarchical high order exact sequences of finite element approximations applicable to meshes combining hexahedra, tetrahedra, prisms, and pyramids are presented: one for Nédélec spaces of first kind, and a second one for enhanced versions of these spaces with properly chosen higher order internal bubble functions. Inside a pyramid, composite polynomial approximations are based on local tetrahedral partitions where traces over their triangular faces are constrained to shape functions on the quadrilateral face shared by a neighboring hexahedron or prism. The construction of projection-based interpolants satisfying the commuting de Rham diagram property and space assembly use hierarchic shape functions. Numerical results implementing Darcy's mixed formulations confirm the theoretical convergence rates. Improved convergence orders for the pressure variable were verified for space configurations enriched with bubble functions.

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Appendix A. Hierarchy of shape functions

Shape functions for the scalar spaces $P_k(\hat{K})$

The bases $B_k^{H^1}(\hat{K})$ for the spaces $P_k(\hat{K})$ are associated with each basic components of $\hat{K}$, namely, vertex, edge, face or the volume $\hat{K}$ as follows. For instance, following the construction and notation in [20], the shape functions with non-zero trace contribution can be grouped in the form

$$B_k^{\partial, H^1}(\hat{K}) = \left\{ \varphi^{\hat{a}}, \varphi_n^{\hat{e}}, \varphi_{n_1, n_2}^{\hat{F}} \right\}.$$

Here, the vertex functions $\varphi^{\hat{a}}$ are Lagrangian first order basis functions such that $\varphi^{\hat{a}}(\hat{a}) = 1$, and which vanish on all remaining vertices, and on edges and faces not containing $\hat{a}$. The edge functions $\varphi_n^{\hat{e}}$ vanish on all edges of $\hat{K}$ different from $\hat{e}$, and on all faces not sharing $\hat{e}$. The face functions $\varphi_{n_1, n_2}^{\hat{F}}$ vanish on all edges, vertices, and faces of $\hat{K}$, and on faces other than $\hat{F}$. The internal shape functions vanish on all faces, edges and vertices of $\hat{K}$, and are expressed in the form

$$\mathring{B}_k^{H^1}(\hat{K}) = \left\{ \varphi_{n_1, n_2, n_3}^{\hat{K}} \right\}.$$

The parameters n, n_1, n_2 and n_3 used in the above definitions determine the polynomial orders of the corresponding shape function, depending of the element geometry.

Bases $\mathbf{B}_k^{\text{curl}}(\hat{K})$ for $\mathbf{P}_k(\hat{K})$

As discussed in [1, 33, 28], shape functions in $\mathbf{B}_k^{\hat{K}, \text{curl}}$ are connected to edges and faces of $\hat{K}$, and to $\hat{K}$ itself. For instance, according to the classification adopted in [1] for the tetrahedron, the shape functions having some kind of non zero (tangential) trace may be associated to edges or to faces of $\hat{K}$. Namely,

$$\mathbf{B}_k^{\partial, \text{curl}}(\hat{K}) = \left\{ \phi_{\ell}^{\hat{e}}, \phi_{\ell}^{\hat{F}, \hat{e}}, \phi_{(i), \ell, m}^{\hat{F}} \right\},$$

where $\phi_{\ell}^{\hat{e}}$ are called edge functions, their tangential traces $\phi_{\ell}^{\hat{e}} \cdot \boldsymbol{\tau}_{\hat{e}'}$ vanishing for all edges $\hat{e}' \neq \hat{e}$, $\phi_{\ell}^{\hat{F}, \hat{e}}$ are referred as edge-based face functions, since they vanish over all faces not sharing $\hat{e}$ with $\hat{F}$. However, the tangential trace components of $\phi_{\ell}^{\hat{F}, \hat{e}}$ are non-zero only over $\hat{F}$. The face bubble functions $\phi_{(i), \ell, m}^{\hat{F}}$ have vanishing tangential traces over all faces other than $\hat{F}$. The interior shape functions, with vanishing tangential traces, fall into two categories

$$\mathring{\mathbf{B}}_k^{\text{curl}}(\hat{K}) = \left\{ \phi_{\ell, m}^{\hat{K}, \hat{F}}, \phi_{(d), \ell, m, n}^{\hat{K}} \right\}.$$

The face-based interior functions $\phi_{\ell, m}^{\hat{K}, \hat{F}}$ have non-zero normal traces on certain faces, and the interior bubble functions $\phi_{(d), \ell, m, n}^{\hat{K}}$ vanish on all faces of $\hat{K}$. The parameters ℓ, m, n determine the shape function degrees, their ranges depending of the element geometry, $i = 1, 2$ and $d \in \{1, 2, 3\}$ indexing appropriate face and internal vector fields used in their constructions.

Bases $\mathbf{B}_k^{\hat{K}, \text{div}}$ for $\mathbf{P}_k(\hat{K})$

The hierarchical bases $\mathbf{B}_k^{\hat{K}, \text{div}}$ are chosen to be the ones proposed in [16]. The face shape functions can be expressed as

$$\mathbf{B}_k^{\partial, \text{div}}(\hat{K}) = \left\{ \Phi^{\hat{F}, \hat{a}}, \Phi_n^{\hat{F}, \hat{e}}, \Phi_{n_1, n_2}^{\hat{F}} \right\},$$

where $\hat{F}$ ranges over the faces of $\hat{K}$, $\hat{a}$ and $\hat{e}$ indicating the vertices and edges of $\hat{F}$. The normal component of a face shape function vanishes on every face other than the one with which it is associated. The internal shape functions, vanishing the normal components over all faces of $\hat{K}$, are classified as

$$\mathring{\mathbf{B}}_k^{\text{div}}(\hat{K}) = \left\{ \Phi_n^{\hat{K}, \hat{e}}, \Phi_{(i), n_1, n_2}^{\hat{K}, \hat{F}}, \Phi_{(d), n_1, n_2, n_3}^{\hat{K}} \right\},$$

where $\hat{e}$ and $\hat{F}$ vary over all edge and faces of $\hat{K}$. The parameters n, n_1, n_2 and n_3 determine the shape function degree, their ranges depending of the element geometry, $i = 1, 2$ and $d = 1, 2, 3$ are indexes for internal vector fields associated with faces and volume, respectively.

Appendix B. Finite element assembly algorithms for the spaces based on T-H-Pr-P partitions

The finite element sub-assembly of the matrices associated with spaces $U_h \subset H^1(\Omega)$, $\mathbf{X}_h \subset \mathbf{H}(\text{curl}, \Omega)$, $\mathbf{V}_h \subset \mathbf{H}(\text{div}, \Omega)$ is dictated by compatibility between the traces on the inter-element boundaries. The treatment of interfaces between conformal hexahedra, tetrahedra and prisms is well documented. It only remains to consider the case when pyramidal elements are involved.

In contrast to the exact sequences considered in [31] for macro-elements partitioned by tetrahedra, where hanging edges or faces are disallowed over macro-element interfaces, for sequences based on pyramids we are faced with handling the case where a pair of curvilinear triangular faces (arising from the base of the pyramid) abut a quadrilateral face of neighboring hexahedral or prismatic element. Furthermore, an additional difficulty arises from the fact that the pyramidal order on the curvilinear triangular faces must be double the order to accommodate the tensor product polynomial spaces on the hexahedral and prismatic elements. A detailed discussion of these issues in the case of H^1 -spaces is given in [2].

Appendix B.1. Triangle-triangle interface

Suppose a pyramid P shares a triangular face F_Δ with a neighboring element T , and that F_Δ is a face of the internal tetrahedron $T_j \in \mathcal{T}^P$. For conciseness, we assume T is a tetrahedron but the arguments apply equally well to the case where the neighboring element is a prism or another pyramid.

By definition of $U(P)$, the edge and face shape functions $\varphi_{T_j}^{e,n}$ and $\varphi_{n_1, n_2, T_j}^{F_\Delta}$ of $U(T_j)$ are of degree no larger than $k+1$, having traces in $P^{H^1}(e) = \mathbb{P}_{k+1}(e)$, and $P^{H^1}(F_\Delta) = \mathbb{P}_{k+1}(F_\Delta)$, respectively. By construction, the space $U(T)$ has the same trace spaces on the face F_Δ . Consequently, the constraints which must be applied on F_Δ are identical to those required in the case of a tetrahedra-tetrahedra interface [1, 15].

Let ϕ_{ℓ, T_j}^e , $\phi_{\ell, T_j}^{F_\Delta, e}$, and $\phi_{(i), \ell, m, T_j}^{F_\Delta}$ be the edge, edge-based and bubble face shape functions of $\mathbf{M}^{\text{curl}}(T_j)$ connected to F_Δ . Recall that, by definition of $\mathbf{M}^{\text{curl}}(P)$, these shape functions are supposed to be restricted to those ones having tangential traces in $P^{\text{curl}}(e) = \mathbb{P}_k(e)$ and $P^{\text{curl}}(F_\Delta) = [\mathbb{P}_k]^2 \oplus \mathbf{x} \times [\tilde{\mathbb{P}}_k]^2(F_\Delta)$. Similarly, from the T side, consider the corresponding shape functions $\phi_{\ell, T}^e$, $\phi_{\ell, T}^{F_\Delta, e}$, and $\phi_{(i), \ell, m, T}^{F_\Delta}$ of $\mathbf{M}^{\text{curl}}(T) = \mathbf{M}^{\text{curl}}(T)$ having the same tangential trace spaces over F_Δ . Therefore, the trace compatibility of $\mathbf{M}^{\text{curl}}(P)$ and $\mathbf{M}^{\text{curl}}(T)$ is obtained by the usual arguments hold by constrained k -order conformal tetrahedra spaces [1].

An analogous analysis holds for the assembly of $\mathbf{M}^{\text{div}}(P)$ and $\mathbf{M}^{\text{div}}(T)$, for which the normal traces over F_Δ are supposed to be included in $P^{\text{div}}(F_\Delta) = \mathbb{P}_k(F_\Delta)$. This means that the face shape functions $\Phi_{T_j}^{F_\Delta, a}$, $\Phi_{n, T_j}^{F_\Delta, e}$ and $\Phi_{n_1, n_2, T_j}^{F_\Delta}$ from the pyramid side are supposed to be restricted to the ones of degree no greater than k , sharing the same traces with the corresponding ones $\Phi_T^{F_\Delta, a}$, $\Phi_{n, T}^{F_\Delta, e}$ and $\Phi_{n_1, n_2, T}^{F_\Delta}$ from the tetrahedron side.

Following, the assembly arguments described in [1] for constrained k -order conformal tetrahedra spaces apply for the constrained spaces $\mathbf{M}^{\text{div}}(P)$ and $\mathbf{M}^{\text{div}}(T)$.

Appendix B.2. Quadrilateral pyramid face

Let $F_{\square}$ be the quadrilateral face of a pyramid P . Thus, $F_{\square}$ is subdivided by a triangular partition $\mathcal{T}^{F_{\square}}$ formed by disjoint triangular parts $F_{\Delta,j} = F_{\square} \cap T_j$, $T_j \in \mathcal{T}^P$. There are two cases to be considered: (1) $F_{\square}$ is shared by another pyramid P' or (2) by a hexahedon H , or prism Pr .

Case 1: The foregoing assumptions mean that the sub-tetrahedra are neighboring pyramidal elements are compatible in the sense that the triangular faces coincide on the shared interface. This means that Case 1 reduces to triangle-triangle interface case discussed in Section Appendix B.1

Case 2: The most delicate case arises when the quadrilateral face of a pyramid abuts a neighboring hexahedron or prismatic element. In this case, the trace spaces are given by $P^{H^1}(F_{\square}) = \mathbb{Q}_{k+1,k+1}(F_{\square})$, $\mathbf{P}^{\text{curl}}(F_{\square}) = \mathbb{Q}_{k,k+1} \times \mathbb{Q}_{k+1,k}(F_{\square})$, and $P^{\text{div}}(F_{\square}) = \mathbb{Q}_{k,k}(F_{\square})$. We consider the hexahedron neighbor case, for prisms the procedures are similar. On the pyramid side the piecewise spaces on the triangles must be constrained to the trace spaces on the hexahedron. For instance, at $H^1(\Omega)$ space level the procedure is explained in [2].

Assembly of $U(H)$ and $U(P)$

Let $u_P \in U(P)$ and consider the expansion of $u_P|_{F_{\Delta,j}}$ in terms of shape functions associated with T_j

$$\begin{aligned} u_P|_{F_{\Delta,j}} &= \sum_{a \in \mathcal{V}(F_{\Delta,j})} \alpha_{a,T_j} \varphi_{T_j}^a|_{F_{\Delta,j}} + \sum_{e \in \mathcal{E}(F_{\Delta,j})} \sum_n \beta_{e,n,T_j} \varphi_{n,T_j}^e|_{F_{\Delta,j}} \\ &+ \sum_{n_1,n_2} \gamma_{n_1,n_2,T_j} \varphi_{n_1,n_2,T_j}^{F_{\Delta,j}}|_{F_{\Delta,j}}. \end{aligned} \quad (\text{B.1})$$

Recall that, by definition of the constrained space $U(P)$, the edge and face traces of u_P over $F_{\Delta,j}$ should live in $\mathbb{P}_{k+1}(e)$ and in $\mathbb{Q}_{k+1,k+1,k+1}$ restricted to $F_{\Delta,j}$. Thus, multiplying coefficients for those shape functions φ_{n,T_j}^e and $\varphi_{n_1,n_2,T_j}^{F_{\Delta,j}}$ failing to verify these properties were set to be zero, so these functions do not appear in (B.1).

Given a function $u_H \in U(H)$, the goal is to determine the multiplying coefficients α_{a,T_j} , β_{e,n,T_j} and γ_{n_1,n_2,T_j} in expansion (B.1) that guarantee the desired trace continuity of u_H and u_P over the triangular part $F_{\Delta,j}$. This is done in a sequence of steps, by imposing trace continuity over the basic elements of $F_{\Delta,j}$, starting with vertices, then throughout edges and the face $F_{\Delta,j}$ itself.

1. Let a be a vertex of $F_{\Delta,j}$. One knows that $\varphi_{n,T_j}^e(a) = 0$, $\varphi_{n_1,n_2,T_j}^{F_{\Delta,j}}(a) = 0$, and $\varphi_{T_j}^{a*}(a) = \delta_{a,a*}$. Imposing $u_H(a) = u_P(a)$, one obtains the formula $\alpha_{a,T_j} = u_H(a)$, and set

$$u_P^V|_{F_{\Delta,j}} = \sum_{a \in \mathcal{V}(F_{\Delta,j})} \alpha_{a,T_j} \varphi_{T_j}^a|_{F_{\Delta,j}}.$$

2. By incorporating the vertex term to the left hand side of the expression (B.1) we obtain

$$(u_P - u_P^V)|_{F_{\Delta,j}} = \sum_{e \in \mathcal{E}(F_{\Delta,j})} \sum_n \beta_{e,n,T_j} \varphi_{n,T_j}^e|_{F_{\Delta,j}} + \sum_{n_1,n_2} \gamma_{n_1,n_2,T_j} \varphi_{n_1,n_2,T_j}^{F_{\Delta,j}}|_{F_{\Delta,j}}.$$

For a given edge $e \in \mathcal{E}(F_{\Delta,j})$, observe that $\varphi_{n,T_j}^{e*}|_e = 0$, for $e^* \neq e$, and $\varphi_{n_1,n_2,T_j}^{F_{\Delta,j}}|_e = 0$. Consequently, $(u_P - u_P^V)|_e = \beta_{e,n,T_j} \varphi_{n,T_j}^e|_e \in \mathring{P}^{H^1}(e)$. Thus, in order to enforce continuous trace $u_H|_e = u_P|_e$ across e , let the coefficients β_{e,n,T_j} be determined by forcing $(u_P - u_P^V)|_e \in \mathring{P}^{H^1}(e)$ to be the $L^2(e)$ -projection of $u_H|_e$ over $\mathring{P}^{H^1}(e)$

$$\int_e (u_H - u_P - u_P^V) \varphi \, de = 0, \quad \forall \varphi \in \mathring{P}^{H^1}(e),$$

and set $u_P^E|_{F_{\Delta,j}} = \sum_{e \in \mathcal{E}(F_{\Delta,j})} \sum_n \beta_{e,n,T_j} \varphi_{n,T_j}^e|_{F_{\Delta,j}}$.

3. By incorporating the vertex and edge terms to the left hand side of the expression (B.1), using the values of α_{a,T_j} and β_{e,n,T_j} determined in the previous steps, one obtains the function

$$u_P^F|_{F_{\Delta,j}} = (u_P - u_P^V - u_P^E)|_{F_{\Delta,j}} = \sum_{n_1,n_2} \gamma_{n_1,n_2,T_j} \varphi_{n_1,n_2,T_j}^{F_{\Delta,j}}|_{F_{\Delta,j}} \in \mathring{P}^{H^1}(F_{\Delta,j}).$$

To conclude the continuous trace enforcement $u_H|_{F_{\Delta,j}} = u_P|_{F_{\Delta,j}}$, let the coefficients γ_{n_1,n_2,T_j} be obtained by forcing $u_P^F|_{F_{\Delta,j}}$ to be the $L^2(F_{\Delta,j})$ -fitting of $u_H|_{F_{\Delta,j}}$ over $\mathring{P}^{H^1}(F_{\Delta,j})$, such that

$$\int_{F_{\Delta,j}} (u_H - u_P^F) \varphi \, dF_{\Delta,j} = 0, \quad \forall \varphi \in \mathring{P}^{H^1}(F_{\Delta,j}).$$

Assembly of $\mathbf{M}^{\text{curl}}(H)$ and $\mathbf{M}^{\text{curl}}(P)$

For a function $\mathbf{q}_P \in \mathbf{M}^{\text{curl}}(P)$, the expansion of $\mathbf{q}_P|_{F_{\Delta,j}}$ in terms of shape functions associated with T_j has the form:

$$\begin{aligned} \mathbf{q}_P|_{F_{\Delta,j}} &= \sum_{e \in \mathcal{E}(F_{\Delta,j})} \sum_{\ell} \left[\alpha_{e,\ell,T_j} \boldsymbol{\phi}_{\ell,T_j}^e|_{F_{\Delta,j}} + \beta_{e,\ell,T_j} \boldsymbol{\phi}_{\ell,T_j}^{F_{\Delta,j},e}|_{F_{\Delta,j}} \right] \\ &+ \sum_{(i)} \sum_{\ell} \sum_m \gamma_{(i),\ell,m,T_j} \boldsymbol{\phi}_{(i),\ell,m,T_j}^{F_{\Delta,j}}|_{F_{\Delta,j}}. \end{aligned} \quad (\text{B.2})$$

By definition of the constrained space $\mathbf{M}^{\text{curl}}(P)$, the edge and face tangential traces of $\mathbf{q}_P$ over $F_{\Delta,j}$ should live in $\mathbb{P}_k(e)$ and in $P^{\text{curl}}(F_{\square})|_{F_{\Delta,j}} = \mathbb{Q}_{k,k+1} \times \mathbb{Q}_{k+1,k}|_{F_{\Delta,j}}$. Thus, the shape functions $\boldsymbol{\phi}_{\ell,T_j}^e$, $\boldsymbol{\Phi}_{n,T_j}^{F_{\Delta,j}}|_{F_{\Delta,\ell}}$ and $\boldsymbol{\Phi}_{n_1,n_2,T_j}^{F_{\Delta,j}}|_{F_{\Delta,\ell}}$ failing to be verify these properties do not appear in (B.2). The goal is to determine the constraints on the multiplying coefficients α_{e,ℓ,T_j} , β_{e,ℓ,T_j} and $\gamma_{(i),\ell,m,T_j}$ appearing in expansion (B.2) that guarantee the desired tangential trace compatibility over $F_{\Delta,j}$ of $\mathbf{q}_P$ with a given function $\mathbf{q}_H \in \mathbf{M}^{\text{curl}}(H)$.

1. Observe that over a given edge $e \in \mathcal{E}(F_{\Delta,j})$, $\boldsymbol{\phi}_{\ell,T_j}^{e*} \cdot \boldsymbol{\tau}^e|_e = 0$ if $e^* \neq e$, $\boldsymbol{\phi}_{\ell,T_j}^{F_{\Delta,j},e*} \cdot \boldsymbol{\tau}^e|_e = 0$, for all edges e^* , and $\boldsymbol{\phi}_{(i),\ell,m,T_j}^{F_{\Delta,j}} \cdot \boldsymbol{\tau}^e|_e = 0$, implying that $\mathbf{q}_P \cdot \boldsymbol{\tau}^e|_e = \sum_{\ell} \alpha_{e,\ell,T_j} \boldsymbol{\phi}_{\ell,T_j}^e \cdot \boldsymbol{\tau}^e|_e$. Consequently, the coefficients α_{e,ℓ,T_j} are obtained by forcing $\mathbf{q}_P \cdot \boldsymbol{\tau}^e|_e \in P^{\text{curl}}(e)$ to be the $L^2(e)$ -projection of $\mathbf{q}_H \cdot \boldsymbol{\tau}_e$ over $P^{\text{curl}}(e)$. This means that

$$\int_e (\mathbf{q}_H - \mathbf{q}_P) \cdot \boldsymbol{\tau}^e \varphi \, de = 0, \quad \forall \varphi \in P^{\text{curl}}(e),$$

which determines the edge term

$$\mathbf{q}_P^E|_{F_{\Delta,j}} = \sum_{e \in \mathcal{E}(F_{\Delta,j})} \sum_{\ell} \alpha_{e,\ell,T_j} \boldsymbol{\phi}_{\ell,T_j}^e|_{F_{\Delta,j}}.$$

2. By incorporating the edge term in the left hand side of expression (B.2), one obtains the function

$$\begin{aligned}
(\mathbf{q}_P - \mathbf{q}_P^E)|_{F_{\Delta,j}} &= \sum_{e \in \mathcal{E}(F_{\Delta,j})} \sum_{\ell} \beta_{e,\ell,T_j} \phi_{\ell,T_j}^{F_{\Delta,j},e} \Big|_{F_{\Delta,j}} \\
&+ \sum_{(i)} \sum_{\ell} \sum_m \gamma_{(i),\ell,m,T_j} \phi_{(i),\ell,m,T_j}^{F_{\Delta,j}} \Big|_{F_{\Delta,j}}.
\end{aligned}$$

Following, the coefficients β_{e,ℓ,T_j} and $\gamma_{(i),\ell,m,T_j}$ are determined by setting $(\mathbf{q}_P - \mathbf{q}_P^E) \wedge \mathbf{n}^{F_{\Delta,j}}|_{F_{\Delta,j}} \in \mathring{\mathbf{P}}^{\text{curl}}(F_{\Delta,j})$ as the $L^2(F_{\Delta,j})$ -projection of $\mathbf{q}_P \wedge \mathbf{n}^{F_{\Delta,j}}|_{F_{\Delta,j}}$ over $\mathring{\mathbf{P}}^{\text{curl}}(F_{\Delta,j})$, such that

$$\int_{F_{\Delta,j}} (\mathbf{q}_H - \mathbf{q}_P - \mathbf{q}_P^E) \wedge \mathbf{n}^{F_{\Delta,j}} \phi \, dF_{\Delta,j} = 0, \quad \forall \phi \in \mathring{\mathbf{P}}^{\text{curl}}(F_{\Delta,j}).$$

Assembly of $\mathbf{M}^{\text{div}}(H)$ and $\mathbf{M}^{\text{div}}(P)$

Let $\mathbf{q}_P \in \mathbf{M}^{\text{div}}(P)$. By definition of the constrained space $\mathbf{M}^{\text{div}}(P)$, the normal component $\mathbf{q}_P \cdot \mathbf{n}^{F_{\square}}|_{F_{\Delta,j}}$ should live in $P^{\text{div}}(F_{\square})|_{F_{\Delta,j}} = \mathbb{Q}_{k,k}|_{F_{\Delta,j}}$. Consider the expansion of $\mathbf{q}_P|_{F_{\Delta,j}}$ in terms of shape functions associated with the tetrahedron T_j

$$\begin{aligned}
\mathbf{q}_P|_{F_{\Delta,j}} &= \sum_{a \in \mathcal{V}(F_{\Delta,j})} \alpha_{a,T_j} \Phi_{T_j}^{F_{\Delta,j},a} \Big|_{F_{\Delta,j}} + \sum_{e \in \mathcal{E}(F_{\Delta,j})} \sum_n \beta_{e,n,T_j} \Phi_{n,T_j}^{F_{\Delta,j},e} \Big|_{F_{\Delta,j}} \\
&+ \sum_{n_1,n_2} \gamma_{n_1,n_2,T_j} \Phi_{n_1,n_2,T_j}^{F_{\Delta,j}} \Big|_{F_{\Delta,j}}.
\end{aligned} \tag{B.3}$$

Thus, those shape functions $\Phi_{T_j}^{F_{\Delta,j},a}$, $\Phi_{n,T_j}^{F_{\Delta,j},e}$ or $\Phi_{n_1,n_2,T_j}^{F_{\Delta,j}}$ whose normal components restricted to $F_{\Delta,j}$ fail to be a polynomial of maximum degree k in each variable do not appear in expansion (B.3).

Giving a function $\mathbf{q}_H \in \mathbf{M}^{\text{div}}(H)$, the goal is to determine the coefficients α_{a,T_j} , β_{e,n,T_j} and γ_{n_1,n_2,T_j} of expansion (B.3) so that the normal trace jump $\mathbf{q}_P \cdot \mathbf{n}^{F_{\square}} + \mathbf{q}_P \cdot \mathbf{n}^{F_{\square}}|_{F_{\Delta,j}}$ vanishes. This is performed by the $L^2(F_{\Delta,j})$ -fitting

$$\int_{F_{\Delta,j}} \varphi(q_P + \mathbf{q}_P) \cdot \mathbf{n}^{F_{\square}} \, dF_{\Delta,j} = 0, \quad \forall \varphi \in P^{\text{div}}(F_{\square}) \Big|_{F_{\Delta,j}}.$$

Remark All L^2 -fittings used in the above assembly schemes are computed by exact quadrature formulae for the polynomials involved.