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Arithmetic integer additive set-indexers of graph operations

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Abstract. Let $\mathbb{N}_0$ be the set of non-negative integers. An integer additive set-indexer (IASI) of a graph G is an injective function $f : V(G) \to 2^{\mathbb{N}_0}$ such that the induced function $f^+ : E(G) \to 2^{\mathbb{N}_0}$ defined by $f^+(uv) = f(u) + f(v)$; $uv \in E(G)$, is also injective, where $f(u) + f(v)$ is the sum set of the sets $f(u)$ and $f(v)$. A graph G which admits an IASI is called an IASI graph. An arithmetic IASI is an IASI f, under which the elements of the set-labels of all vertices and edges of a given graph G are in arithmetic progressions. In this paper, we discuss about admissibility of arithmetic IASIs by certain operations and products of graphs.

Keywords: Integer additive set-indexers; Arithmetic IASIs; Isoarithmetic IASIs; Biarithmetic IASIs.

Mathematics Subject Classification 2010: 05C78.

1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [11] and for more about graph labeling, we refer to [7]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

The *sum set* of two non-empty sets A and B is the set denoted by $A + B$ and is defined as $A + B = \{a + b : a \in A, b \in B\}$. Note that if either A or B is countably infinite, then $A + B$ will also be a countably infinite set. Hence, we consider only finite sets for our study. Using the concepts of the sum sets of two sets, the notion of integer additive set-indexers was introduced as follows.

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Definition 1.1. [8] Let $\mathbb{N}_0$ be the set of non-negative integers. An *integer additive set-indexer* (IASI, in short) of a graph G is an injective function $f : V(G) \to 2^{\mathbb{N}_0}$ such that the induced function $f^+ : E(G) \to 2^{\mathbb{N}_0}$ defined by $f^+(uv) = f(u) + f(v)$ is also injective, where $f(u) + f(v)$ is the sum set of the sets $f(u)$ and $f(v)$. A graph G which admits an IASI is called an *IASI graph*.

The existence of IASIs for given graphs has been discussed in [18] and proposed the following theorem.

Theorem 1.2. Every graph G admits an integer additive set-indexer.

Note that all sets we mention here are finite sets consisting of non-negative integers. We denote the cardinality of a set A by $|A|$. Then, we have the following definition.

Definition 1.3. The cardinality of the labeling set of an element (vertex or edge) of a graph G is called the *set-indexing number* of that element.

It has been proved that the existence of an IASI is hereditary in nature. Therefore, we can find an IASI for every subgraph of G by restricting the domain of the IASI of G to the vertex set of that subgraph. Similarly, in many occasions, we can extend or modify an IASI of a graph G to its super graphs or certain associated graphs of it, in accordance with certain conditions. Such IASIs are called *induced IASIs* of the related graphs concerned.

In [9], the vertex set V of a graph G is defined to be *l-uniformly set-indexed*, if all the vertices of G have the set-indexing number l.

By the term, an arithmetically progressive set, (AP-set, in short), we mean a set whose elements are in an arithmetic progression. We call the common difference of the set-label of an element of the given graph, the *deterministic index* of that element. The *deterministic ratio* of an edge of a graph G is the ratio $r \geq 1$, between the deterministic indices of its end vertices.

Definition 1.4. A graph G is said to be a *vertex arithmetic IASI graph* if the set-labels of all whose vertices are AP-sets and a graph G is said to be an *edge arithmetic IASI graph* if the set-labels of all of whose edges are AP-sets.

It is proved in [20] that if G is an edge arithmetic IASI graph implies it is a vertex arithmetic IASI graph.

Proposition 1.5. If f is a vertex-arithmetic IASI defined on G such that all vertices of G have the same deterministic index d, then f is an edge-arithmetic IASI of G such that all the edges of G also have the same deterministic index d, with respect to the associated function f^+ of f.

Definition 1.6. [20] An *arithmetic integer additive set-indexer* of a graph G is an integer additive set-indexer f, with respect to which, the set-labels of all elements of a given graph G are AP-sets. A graph that admits an arithmetic IASI is called an *arithmetic IASI graph*.

With respect to a function $f : V(G) \to 2^{\mathbb{N}_0}$, if all vertices of G are labeled by distinct AP-sets, but the set-labels of the edges of G are not AP-sets, then f is called a *semi-arithmetic IASI*.

Theorem 1.7. [20] A graph G admits an arithmetic IASI f if and only if f is a vertex arithmetic IASI and the deterministic ratio of any edge of G is a positive integer, which is less than or equal to the set-indexing number of its end vertex having smaller deterministic index.

In other words, if v_i and v_j are two adjacent vertices of G, with deterministic indices d_i and d_j respectively with respect to an IASI f of G, where $d_i \leq d_j$, then f is an arithmetic IASI if and only if $d_j = k\, d_i$, where k is a positive integer such that $1 \leq k \leq |f(v_i)|$. In view of this fact, we have the following notions.

Definition 1.8. [21] If all the set-labels of all elements of a graph G consist of arithmetic progressions with the same common difference d, then the corresponding IASI is called *isoarithmetic IASI*.

That is, if all the elements of an arithmetic IASI graph G have the same deterministic index under an IASI f, then G is called an isoarithmetic IASI graph.

Definition 1.9. An arithmetic IASI f of a graph G, under which the deterministic ratio of each edge e of G is a positive integer greater than 1 and less than or equal to the set-indexing number of the end vertex of e having smaller deterministic index is called a *biarithmetic IASI*.

In other words, a biarithmetic IASI of a graph G is an arithmetic IASI f of G, for which the deterministic indices of any two adjacent vertices v_i and v_j in G, denoted by d_i and d_j respectively, holds the condition $d_j = kd_i$ where k is an integer such that $1 < k \leq |f(v_i)|$ (or $d_i = k\, d_j$, where $1 < k \leq |f(v_j)|$).

In general, all edges of G may not have the same deterministic ratio. Hence, we introduce the following notion.

Definition 1.10. [20] Let f be a biarithmetic IASI defined on a graph G. If the deterministic ratio of every edge of G is the same, say k, then f is called an *identical biarithmetic IASI* of G and G is called an *identical biarithmetic IASI graph*.

The following theorem establishes the hereditary nature of the existence of different types of arithmetic IASIs.

Theorem 1.11. [20, 21] A subgraph of an arithmetic (or isoarithmetic or biarithmetic) IASI graph is also an arithmetic (or isoarithmetic or biarithmetic) IASI graph.

A characterisation of an identical biarithmetic IASI graph, done in [21], is as follows.

Theorem 1.12. [21] A graph G admits an identical biarithmetic IASI if and only if it is bipartite.

Certain studies in this area have been made in [20] and [21]. Further studies regarding the properties and characteristics of graphs whose elements have AP-sets as their set-labels are still of special interest and appear to be much promising. In this paper, we investigate the admissibility of different types of (induced) arithmetic integer additive set-indexers by certain operations and products of arithmetic IASI graphs.

2 Arithmetic IASIs of Graph Operations

In this section, we discuss the admissibility of induced arithmetic IASI by the union and join of two arithmetic IASI graphs. First, recall the definition of the union of two graphs.

Definition 2.1. [3] The union of two given graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, denoted by $G_1 \cup G_2$, is the graph whose vertex set is $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$.

The following result establishes the existence of an (induced) arithmetic IASI for the union of two arithmetic IASI graphs.

Proposition 2.2. The union of two arithmetic IASI graphs admits an induced arithmetic IASI graph.

Proof. let f_1 and f_2 be the arithmetic IASIs defined on G_1 and G_2 respectively. Define a function f on $G = G_1 \cup G_2$ by

$$f(v) = \begin{cases} f_1(v) & \text{if } \ v \in G_1 \\ f_2(v) & \text{if } \ v \in G_2. \end{cases}$$

Since both f_1 and f_2 are arithmetic IASIs, then f is also an arithmetic IASI on $G_1 \cup G_2$. □

It is to be noted that if G_1 and G_2 have a common vertex, say v, then we have $f_1(v) = f_2(v)$.

An immediate question that arises in this context is whether the union of two isoarithmetic IASI graphs admits an isoarithmetic IASI. The theorem given below answers this question.

Theorem 2.3. The union of two isoarithmetic IASI graphs admits an induced isoarithmetic IASI graph if and only if all the vertices in both G_1 and G_2 have the same deterministic index.

Proof. Let f_1 and f_2 be the isoarithmetic IASIs defined on G_1 and G_2 respectively. Define a function $f : V(G_1 \cup G_2) \to 2^{\mathbb{N}_0}$ as

$$f(v) = \begin{cases} f_1(v) & \text{if } \ v \in G_1 \\ f_2(v) & \text{if } \ v \in G_2. \end{cases}$$

Assume that the vertices of both G_1 and G_2 have the same deterministic index. Then, all the vertices of $G_1 \cup G_2$ have the same deterministic index. Therefore, f is arithmetic IASI on $G_1 \cup G_2$.

Conversely, assume that $G_1 \cup G_2$ admits an isoarithmetic IASI, say f. Therefore, by Theorem 1.11, its subgraphs G_1 and G_2 also admit induced isoarithmetic IASIs which are the restrictions f_1 and f_2 of f to G_1 and G_2 respectively. □

The admissibility of biarithmetic IASI by the union of two biarithmetic IASI graphs follows immediately from Proposition 2.2. Hence, we now proceed to verify the existence of an (induced) arithmetic IASI for the join of two arithmetic IASI graphs. Let us recall the definition of the join of two graphs.

Definition 2.4. [11] The *join* of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, denoted by $G_1 + G_2$, is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup E_{ij}$, where $E_{ij} = \{u_i v_j : u_i \in G_1, v_j \in G_2\}$.

Note that, in $G_1 + G_2$, every vertex of G_1 is adjacent to all vertices of G_2 (and vice versa). Hence, all arithmetic IASIs of G_1 and G_2 need not constitute arithmetic IASIs for their join $G_1 + G_2$. Hence, we look for a necessary and sufficient condition required for $G_1 + G_2$ to admit an arithmetic IASI in the following theorem.

Theorem 2.5. The join of two arithmetic IASI graphs admits an arithmetic IASI if and only if the deterministic index of every vertex of one graph is an integral multiple or divisor of the deterministic index of every vertex of the other graph, where this integer is less than or equal to the set-indexing number of the vertex having smaller deterministic index.

Proof. Let G_1 and G_2 be the given arithmetic IASI graphs. Let $E_{ij} = \{u_i v_j : u_i \in G_1, v_j \in G_2\}$ so that $G_1 + G_2 = G_1 \cup G_2 \cup \langle E_{ij} \rangle$.

Assume that $G_1 + G_2$ admits an arithmetic IASI, say f. Therefore, by Theorem 1.7, for all edges in $\langle E_{ij} \rangle$ also, the deterministic index of one end vertex is a positive integral multiple of the deterministic index of the other end vertex, where this integer is less than or equal to the set-indexing number of the end vertex having smaller deterministic index. Since every vertex, say u_i, in G_1 is adjacent to every vertex, say v_j of G_2 (and vice versa), deterministic index of u_i is either a positive integral multiple or a divisor of the deterministic index of v_j, where this integer is less than or equal to the set-indexing number of the vertex having smaller deterministic index.

Conversely, assume, without loss of generality, that the deterministic index of every vertex of G_1 is a positive integral multiple or a divisor of the deterministic index of every vertex of G_2 such that this integer is less than or equal to the set-indexing number of the vertex having smaller deterministic index. Hence, by Theorem 1.7, $\langle E_{ij} \rangle$ admits an arithmetic IASI, say f'. Define a function $f : V(G_1 + G_2) \to 2^{\mathbb{N}_0}$ such that

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in G_1 \\ f_2(v) & \text{if } v \in G_2 \\ f'(v) & \text{if } v \in \langle E_{ij} \rangle \end{cases}$$

Hence, for every edge in G_1+G_2, the deterministic index of one end vertex is an integral multiple of the deterministic index of the other end vertex, since both G_1 and G_2 are arithmetic IASI graphs. Then, by Theorem 1.7, G_1+G_2 admits an arithmetic IASI. $\square$

In view of the above theorem, we note that the join of two arithmetic IASI graphs is an arithmetic IASI if and only if the deterministic ratio of every edge in the induced subgraph $\langle E_{ij}\rangle$ of G_1+G_2 is a positive integer which lies between 1 and the set-indexing number of its end vertex having minimum deterministic index. Hence, we can restate the above theorem as

Theorem 2.6. The join G of two arithmetic IASI graphs G_1 and G_2 admits an arithmetic IASI if and only if the induced subgraph $\langle E_{ij}\rangle$ of G admits an (induced) arithmetic IASI, where $E_{ij}=\{u_iv_j : u_i\in G_1, v_j\in G_2\}$.

The following result establishes the admissibility of arithmetic IASIs by the join of two isoarithmetic IASI graphs.

Theorem 2.7. The join of two isoarithmetic IASI graphs is an arithmetic IASI graph if and only if the deterministic index of the elements of one graph is a positive integral multiple of the deterministic index of the elements of the other, where this integer is less than or equal to the minimum of the set-indexing numbers of the elements of the graph whose vertices have the smaller deterministic index.

Proof. Let G_1 and G_2 admit isoarithmetic IASIs f_1 and f_2 respectively and let $E_{ij}=\{v_iu_j : v_i\in G_1, u_j\in G_2\}$ be such that $G_1+G_2=G_1\cup G_2\cup\langle E_{ij}\rangle$. Note that all the elements of G_1 have the same deterministic index, say d_1 and all the elements of G_2 also have the same deterministic index, say d_2. Let that $d_1\neq d_2$.

Let us now assume that G_1+G_2 is an arithmetic IASI graph. Therefore, by Theorem 1.11, its subgraph $\langle E_{ij}\rangle$ admits an induced arithmetic IASI. Then, by Theorem 1.7, for any edge in E_{ij}, the deterministic index of one end vertex of it is a positive integral multiple of the deterministic index of the other, where this integer is less than or equal to the set-indexing number of the end vertex having smaller deterministic index. That is, $d'_j=k\ d_i;\ k\leq|f_1(v_i)|$, where d_i is the deterministic index of the vertex v_i in G_1 and d'_j is the deterministic index of the vertex u_j in G_2, with respect to the arithmetic IASI concerned.

Conversely, assume, without loss of generality, that the deterministic index of the elements of G_2 is an integral multiple of the deterministic index of the elements of G_1, where this integer is less than or equal to the minimum among the cardinalities of all set-labels of G_1. Therefore, by Theorem 1.7, $\langle E_{ij}\rangle$ admits an arithmetic IASI, say f'. Now define a function $f:V(G_1+G_2)\to 2^{\mathbb{N}_0}$ such that

$$f(v)=\begin{cases} f_1(v) & \text{if } \ v\in G_1\\ f_2(v) & \text{if } \ v\in G_2\\ f'(v) & \text{if } \ v\in\langle E_{ij}\rangle\end{cases}$$

Clearly, f is an arithmetic IASI of G_1+G_2. $\square$

Let us now explore the conditions required for the join of two isoarithmetic IASI graphs to have an induced isoarithmetic IASI.

Proposition 2.8. The join of two isoarithmetic IASI graphs admits an isoarithmetic IASI if and only if all the vertices in both G_1 and G_2 have the same deterministic index.

Proof. First assume that $G_1 + G_2$ admits an isoarithmetic IASI, say f. Then, the deterministic index of all vertices of $G_1 + G_2$ is the same,say d. Since, G_1 and G_2 are subgraphs of $G_1 + G_2$, by Theorem 1.11, G_1 and G_2 admit the induced isoarithmetic IASI of f. All the vertices in both G_1 and G_2 have the same deterministic index.

Conversely, the vertices in both G_1 and G_2 have the same deterministic index. Let $E_{ij} = \{u_i v_j : u_i \in G_1, v_j \in G_2\}$. Then, every edge in $\langle E_{ij} \rangle$ must have the same deterministic index of its end vertices as that of both G_1 and G_2. Hence, by Theorem 1.7, $G_1 + G_2$ admits an isoarithmetic IASI. □

On examining Theorem 2.7 and Proposition 2.8, we arrive at a conclusion for the question regarding the existence of an arithmetic IASI, which is not an isoarithmetic IASI, for the join of two isoarithmetic IASI graphs as provided in the following theorem.

Theorem 2.9. The join G of two isoarithmetic IASI graphs admits an arithmetic IASI, that is not an isoarithmetic IASI, if and only if the induced IASI of the induced subgraph $\langle E_{ij} \rangle$ of G is an identical biarithmetic IASI of $\langle E_{ij} \rangle$.

The admissibility of an induced biarithmetic IASI by the join of two biarithmetic IASI graphs is a particular case of Theorem 2.5. More over, we can notice that the join of two isoarithmetic IASI graphs will never admits an induced biarithmetic IASI. Now, the only case that remains to be considered in this context is the existence of identical biarithmetic IASI for the join of two identical biarithmetic IASI graphs. Hence, we have

Proposition 2.10. Let G_1 and G_2 be two graphs which admit identical biarithmetic IASI. Then, $G_1 + G_2$ does not admit an identical biarithmetic IASI.

Proof. If possible, let $G_1 + G_2$ admits an identical biarithmetic IASI. Since G_1 and G_2 are identical biarithmetic IASI graphs, for every pair of adjacent vertices in them, the deterministic index of one is a positive integral multiple of the deterministic index of the other and this positive integer is unique for all such pair of vertices in G_1 and G_2. Let v_i be a vertex of G_1 with deterministic index d_i and let u_j and u_l be two adjacent vertices in G_2 with deterministic indices d'_j and d'_l respectively. By Theorem 1.7, $d'_l = k.d'_j$ for some positive integer k.

Now, v_i is adjacent to both and u_j and u_l in G_1+G_2. Then, by Theorem 1.7, we have $d_i = k.d'_j$, $d'_l = k.d'_j$ and $d_i = k.d'_j$, all of which do not hold simultaneously. Therefore, G_1+G_2 does not admit an identical biarithmetic IASI. Hence, G_1+G_2 is not an identical biarithmetic IASI graph. □

3 Arithmetic IASIs of Graph Products

In this section, we discuss the admissibility of arithmetic IASI by certain graph products. First, recall the definition of the Cartesian product of two graphs.

Definition 3.1. [10] The *Cartesian product* of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, denoted by $G_1 \Box G_2$, is the graph with vertex set $V_1 \times V_2$ defined as follows. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two points in $V_1 \times V_2$. Then, u and v are adjacent in $G_1 \Box G_2$ whenever $[u_1 = v_1$ and u_2 is adjacent to $v_2]$ or $[u_2 = v_2$ and u_1 is adjacent to $v_1]$.

The Cartesian product $G_1 \Box G_2$ of two graphs G_1 and G_2 is a graph obtained by taking $|V(G_2)|$ copies of G_1, one copy for each vertex v_i of G_2 and joining the corresponding vertices of two copies G_1 if the corresponding vertices in G_2, are adjacent in G_2. We say that two copies of a graph are *adjacent* if the corresponding vertices of these copies are adjacent.

The following theorem establishes the admissibility of arithmetic IASI by the Cartesian product of two arithmetic IASI graphs.

Theorem 3.2. The Cartesian product of two arithmetic IASI graphs G_1 and G_2 admits an arithmetic IASI if and only if the deterministic ratio of every edge between corresponding vertices of any two adjacent copies of G_1 (or G_2) in $G_1 \Box G_2$, is a positive integer which lies between 1 and l, where l is the set-indexing number of its end vertex having smaller deterministic index.

Proof. Let $U = \{u_1, u_2, \ldots, u_n\}$ be the vertex set of G_1 and $V = \{v_1, v_2, \ldots, v_m\}$ be the vertex set of G_2. For $1 \leq j \leq |V(G_2)|$, let G_{1j}, be the j-th copy of G_1 in $G_1 \Box G_2$. Therefore, $G_{1j} = \langle U_j \rangle$ where $U_j = \{u_{ij} : 1 \leq i \leq |V(G_1)|, 1 \leq j \leq |V(G_2)|\}$. Now, for all values of j, the graphs induced by the set of vertices $\{u_{ij} : 1 \leq i \leq |V(G_1)|\}$ are graphs isomorphic to G_1 and similarly for all values of i, the graphs induced by the set of vertices $\{u_{ij} : 1 \leq j \leq |V(G_2)|\}$ are the graphs isomorphic to G_2. Without loss of generality, let $\langle U_1 \rangle = G_1$ and $\langle V_1 \rangle = G_2$, where $V_1 = \{u_{i1} : 1 \leq i \leq |V(G_2)|\}$. Also, the corresponding vertices of G_{1r} and G_{1s} are adjacent in $G_1 \Box G_2$ if the vertices v_r and v_s are adjacent in G_2.

Now, let f_1 and f_2 be the arithmetic IASIs of G_1 and G_2 respectively. Since G_1 is an arithmetic IASI graph, for two adjacent vertices u_r and u_s in G_1, we have $d_s = k_l.d_r$, where $k_l \leq |f_1(u_r)|$, is a positive integer with $1 \leq l \leq |E(G_1)|$. Similarly, since G_2 is an arithmetic IASI graph, for two adjacent vertices v_r and v_s in G_2, we have $d'_s = k'_l.d'_r$, where $k'_l \leq |f_2(v_r)|$, is a positive integer with $1 \leq l \leq |E(G_2)|$, where d_i is the deterministic index of the vertex u_i in G_1 and d'_j is the deterministic index of the vertex v_j in G_2. Label the vertices of $\langle U_1 \rangle$ by the same set-labels of G_1 itself and label the vertices of the copies $\langle U_r \rangle$ and $\langle U_s \rangle$ in such a way that the deterministic indices of all vertices in U_s are integral multiple of the deterministic indices of the corresponding vertices of U_r by a unique positive integer $k'_l = \frac{d'_s}{d'_r}$, where d'_r and d'_s are the deterministic indices of the corresponding vertices v_r and v_s in G_2 of the copies $\langle U_r \rangle$ and $\angle U_s \rangle$ of G_1 respectively. Then, for every pair of adjacent vertices in $G_1 \Box G_2$, the deterministic index of one vertex is an integral multiple of the deterministic index of the other. Hence, by Theorem 1.7, $G_1 \Box G_2$ is an arithmetic IASI graph.

Conversely, assume that $G_1 \Box G_2$ admits an arithmetic IASI. Now, we can take the same set-labels of the copy U_1 as the set-labels of the vertices of G_1 and the same set-labels of the graph $\langle V_1 \rangle$, defined above, as the set-labels of the graph G_2. Clearly, these set-labelings of G_1 and G_2 are arithmetic IASIs. Therefore G_1 and G_2 are arithmetic IASI graphs. □

Invoking Proposition 3.2, we now establish the following results.

Corollary 3.3. The Cartesian product of two isoarithmetic IASI graphs admits an isoarithmetic IASI if and only if all vertices in both G_1 and G_2 have the same deterministic index.

Proof. The proof is immediate form Theorem 3.2 by taking $k_l = 1$ and $k'_l = 1$. □

Let us now investigate the admissibility of an arithmetic IASI by the Cartesian product of two identical biarithmetic IASI graphs.

The admissibility of biarithmetic IASI by the cartesian product of two biarithmetic IASI graphs is a particular case of Theorem 3.2. On this occasion, the only thing remaining is to verify whether the Cartesian product of two identical biarithmetic IASI graphs admits an induced identical biarithmetic IASI. Hence, we establish the following result.

Proposition 3.4. The Cartesian product of two identical biarithmetic IASI graphs admits an identical biarithmetic IASI.

Proof. Since G_1 and G_2 are identical biarithmetic IASI graphs, by Theorem 1.12, both G_1 and G_2 are bipartite. Therefore, since the Cartesian product of two bipartite graphs is also a bipartite graph, $G_1 \times G_2$ is bipartite. Hence, $G_1 \times G_2$ admits an identical biarithmetic IASI. The labeling of the vertices in $G_1 \times G_2$ follows from Theorem 3.2 by taking $k_l = k'_l = k$, a unique positive integer. □

Next, we proceed to verify the admissibility of arithmetic IASI by the corona of two graphs. Now, recall the definition of corona of two graphs.

Definition 3.5. [11] The *corona* of two graphs G_1 and G_2, denoted by $G_1 \circ G_2$, is the graph obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 and then joining the i-th point of G_1 to every point in the i-th copy of G_2.

The following theorem discusses the existence of an induced arithmetic IASI for the corona of two arithmetic IASI graphs.

Theorem 3.6. Let G_1 and G_2 be two arithmetic IASI graphs. Then, the corona $G_1 \circ G_2$ admits an arithmetic IASI if and only if the deterministic index of every vertex of one graph is an integral multiple or a divisor of the deterministic index of every vertex of the other, where this integer is less than or equal to the set-indexing number of the vertex having smaller deterministic index.

Proof. Let $U = \{u_1, u_2, \ldots, u_n\}$ be the vertex set of G_1 and $V = \{v_1, v_2, \ldots, v_m\}$ be the vertex set of G_2. Let G_{2i}; $1 \leq i \leq n$, be the n copies of G_2 in $G_1 \circ G_2$. Therefore, $G_{2i} = \langle V_i \rangle$ where $V_i = \{v_{ij} : 1 \leq i \leq |V(G_1)| \text{ and } 1 \leq j \leq |V(G_2)|\}$. Now, for any value of $i = r$, the graphs induced by the set of vertices $\{v_{rj} : 1 \leq j \leq |V(G_2)|\}$ is a graph isomorphic to G_2. Without loss of generality, let $\langle V_1 \rangle = G_2$. Also, all the vertices of $\langle V_i \rangle$ are adjacent to the vertex u_i of G_1 in $G_1 \circ G_2$.

Now, assume that G_1 and G_2 admit arithmetic IASIs f_1 and f_2 respectively. Since G_1 is an arithmetic IASI graph, for two adjacent vertices u_r and u_s in G_1, we have $d_s = k_l.d_r$, where $k_l \leq |f_1(u_r)|$ is a positive integer with $1 \leq l \leq |E(G_1)|$. Similarly, since G_2 is an arithmetic IASI graph, for two adjacent vertices v_r and v_s in G_2, we have $d'_s = k'_l.d'_r$, where $k'_l \leq |f_2(v_r)|$ is a positive integer with $1 \leq l \leq |E(G_2)|$, where d_i is the deterministic index of the vertex u_i in G_1 and d'_j is the deterministic index of the vertex v_j in G_2. Label the vertices of $\langle V_1 \rangle$ by the same set-labels of G_2 itself and label the vertices of the copies $\langle V_r \rangle$; $1 < r \leq n$, by distinct sets in such a way that the deterministic indices of the corresponding vertices in V_1 and V_r have the same deterministic indices. Then, for every pair of adjacent vertices in $G_1 \circ G_2$, the deterministic index of one vertex is an integral multiple of the deterministic index of the other. Hence, by Theorem 1.7, $G_1 \circ G_2$ is an arithmetic IASI graph.

Conversely, assume that $G_1 \circ G_2$ admits an arithmetic IASI. Since G_1 is a subgraph of $G_1 \circ G_2$, by Theorem 1.11, G_1 admits an arithmetic IASI. Also, we can take the same set-labels of the copy $\langle V_1 \rangle$ as the set-labels of the vertices of G_2 and the same set-labels of the component graph $\langle V_1 \rangle$ of $G_1 \circ G_2$, defined above, as the set-labels of the graph G_2. Clearly, these set-labelings of G_1 and G_2 are arithmetic IASIs. Therefore G_1 and G_2 are arithmetic IASI graphs. □

Invoking Proposition 3.2, we now establish the following results.

Proposition 3.7. The corona of two isoarithmetic IASI graphs admits an isoarithmetic IASI if and only if all vertices in both G_1 and G_2 have the same deterministic index.

Proof. The proof is immediate form Theorem 3.6. □

Proposition 3.8. The corona of two identical biarithmetic IASI graphs does not admit an identical biarithmetic IASI.

Proof. Since the corona of two bipartite graphs is not a bipartite graph, by theorem 1.12, $G_1 \circ G_2$ does not admit an identical biarithmetic IASI. □

4 Arithmetic IASIs of Graph Complements

In this section, we enquire in to the admissibility of arithmetic IASI by complements of given arithmetic IASI graphs. Note that the vertices of a graph G and its complements have the same set-labels and hence the same deterministic indices.

Theorem 4.1. The complement of an arithmetic IASI graph G admits an arithmetic IASI f if and only if the deterministic index of any vertex of G is an integral multiple or divisor of the deterministic index of every other vertex of G, where this integer lies between 1 and $\min|f(v_i)|$ for all $v_i \in V(G)$.

Proof. First assume that the deterministic index of any vertex of G is an integral multiple or divisor of the deterministic index of every other vertex of G, where this integer lies between 1 and $\min|f(v_i)|$. Hence, for any pair of adjacent vertices in $\overline{G}$ also, the deterministic index of one vertex is an integral multiple of the deterministic index of the other. Hence by Theorem 1.7, $\overline{G}$ admits an arithmetic IASI.

Conversely, assume that $\overline{G}$ admits an arithmetic IASI. Since every pair of vertices in $V(G)$ are either adjacent in G or in its complement $\overline{G}$ and both G and $\overline{G}$ are arithmetic IASI graphs, for every pair of vertices, the deterministic index of one vertex must be a positive integral multiple of the deterministic index of the other, where this integer is less than or equal to the set-indexing number of the end vertex having smaller deterministic index. This completes the proof. □

Let us now verify the existence of an isoarithmetic IASI for the complement of an isoarithmetic IASI graph.

Proposition 4.2. The complement of an isoarithmetic IASI graph admits an (induced) isoarithmetic IASI.

Proof. Let G admits an isoarithmetic IASI graph, say f. Then, the deterministic index of all vertices of G (and $\overline{G}$) with respect to f are the same. Therefore, f is an induced isoarithmetic IASI for $\overline{G}$ also. □

Proposition 4.3. The complement of an identical biarithmetic IASI never admits an identical biarithmetic IASI.

Proof. The complement of a bipartite graph G is not a bipartite graph. Hence, by Theorem 1.12, $\overline{G}$ does not admit an identical biarithmetic IASI. □

5 Conclusion

In this paper, we have discussed some characteristics of certain graph operations and products which admit arithmetic IASIs. We have not addressed certain problems in this area which are still open. The following are some of the open problems we have identified in this area.

Problem 5.1. Examine the necessary and sufficient conditions for the existence of non-identical biarithmetic IASIs for given graphs.

Problem 5.2. Discuss the necessary and sufficient conditions for the existence of arithmetic IASIs for corona of two isoarithmetic IASI graphs and for the corona of two identical biarithmetic graphs.

Problem 5.3. Discuss the necessary and sufficient conditions for the existence of arithmetic IASIs for some other products, such as lexicographic product, tensor product, strong product, rooted product etc., of arithmetic IASI graphs.

Problem 5.4. Discuss the necessary and sufficient conditions for the existence of arithmetic IASIs of all types for certain powers of arithmetic IASI graphs.

Problem 5.5. Discuss the necessary and sufficient conditions for the existence of arithmetic IASIs for different graph classes.

Problem 5.6. Characterise certain graphs and graph classes in accordance with their admissibility of identical and non-identical biarithmetic IASIs.

Certain IASIs under which the vertices of a given graph are labeled by different standard sequences of non negative integers, are also worth studying. The problems of establishing the necessary and sufficient conditions for various graphs and graph classes to have certain IASIs still remain unsettled.

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