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► To cite this version:

K. A Germina, Sudev Naduvath. On weakly uniform integer additive set-indexers of graphs. International Mathematical Forum, 2013, 8, pp.1827-1834. 10.12988/imf.2013.310188 . hal-02099206

HAL Id: hal-02099206

<https://hal.science/hal-02099206>

Submitted on 7 May 2019

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On Weakly Uniform Integer Additive Set-Indexers of Graphs

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Abstract

We have the notion of set-indexers, integer additive set-indexers and k -uniform integer additive set-indexers of graphs. In this paper, we initiate a study of the graphs which admit k -uniform integer additive set-indexers and introduce the notion of weakly uniform integer additive set-indexers and arbitrarily uniform integer additive set-indexers and provide some useful results on these types of set-indexers.

Mathematics Subject Classification: 05C78

Keywords: k -uniform Integer Additive Set-Indexers, Weakly k -uniform Integer Additive Set-Indexers, Arbitrarily k -uniform Integer Additive Set-Indexers

1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [7]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

The set-labeling of a graph is defined in [1] as follows. Let G be a (p, q) -graph and let X, Y, Z be non-empty sets. Then the functions $f : V(G) \rightarrow 2^X$, $f : E(G) \rightarrow 2^Y$ and $f : V(G) \cup E(G) \rightarrow 2^Z$ are called the *set-assignments* of vertices, edges and elements of G respectively. By a set-assignment of a graph, we mean any one of them. A set-assignment is called a *set-labeling* if it is injective.

Definition 1.1 [1] For a (p, q) -graph $G = (V, E)$ and a non-empty set X of cardinality n , a *set-indexer* of G is defined as an injective set-valued function $f : V(G) \rightarrow 2^X$ such that the function $f^\oplus : E(G) \rightarrow 2^X - \{\emptyset\}$ defined by $f^\oplus(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where 2^X is the set of all subsets of X and $\oplus$ is the symmetric difference of sets.

Theorem 1.2 [1] *Every graph has a set-indexer.*

Definition 1.3 Let N_0 denote the set of all non-negative integers. For all $A, B \subseteq N_0$, the sum of these sets is denoted by $A + B$ and is defined by $A + B = \{a + b : a \in A, b \in B\}$.

Definition 1.4 [6] An *integer additive set-indexer* (IASI, in short) is defined as an injective function $f : V(G) \rightarrow 2^{N_0}$ such that the induced function $g_f : E(G) \rightarrow 2^{N_0}$ defined by $g_f(uv) = f(u) + f(v)$ is also injective.

Definition 1.5 The cardinality of the labeling set of an element (vertex or edge) of a graph G is called the *set-indexing number* of that element.

Definition 1.6 [6] An IASI is said to be k -uniform if $|g_f(e)| = k$ for all $e \in E(G)$. That is, a connected graph G is said to have a k -uniform IASI if all of its edges have the same set-indexing number k . In particular, we say that a graph G has an arbitrarily k -uniform IASI if G has a k -uniform IASI for every positive integer k .

In [6], a study of a special type of 1-uniform IASI *sum square graphs* and in [2], a characterisation of 2-uniform IASI were established. The most important result given in [2] is the following.

Theorem 1.7 [2] *A graph G has a 2-uniform IASI if and only if it is a bipartite graph.*

The motivation for the study presented in this paper is from the results in [6] and [2]. As a generalisation of 1-uniform and 2-uniform IASIs, in section 2, we introduce the notion of weakly k -uniform IASI and arbitrarily k -uniform IASI and prove the conditions for a connected graph to admit a weakly or arbitrarily k -uniform IASI, where k is a positive integer. Finally, in section 3, some illustrations are also provided.

2 Uniform Integer Additive Set-Indexers

During this study our first attempt is to analyse the set-indexing number of an edge of a graph G and to study its relation with the set-indexing number of the end vertices of that edge. As a result, we establish the following results regarding the cardinality of $A + B$ defined in Definition 1.3. If either A or B is countably infinite, then clearly $A + B$ is also countably infinite and hence the study of the cardinality of $A + B$ trivial. Hence we restrict our discussion on finite sets A and B . We denote the cardinality of a set A by $|A|$.

Lemma 2.1 *Let $A, B \subseteq N_0$. Then $\max(|A|, |B|) \leq |A + B| \leq |A||B|$.*

Proof: Let $A, B \subseteq N_0$. Also let $|A| = m, |B| = n$. Assume either A or B is a singleton set. Without loss of generality, let A be a singleton set, say $A = \{q\}$ where q is a non-negative integer. That is, $m = 1$. Therefore, $A + B = \{b_1 + q, b_2 + q, \dots, b_n + q\}$. Hence $|A + B| = n$. Similarly, if $|B| = 1$, then $|A + B| = m$. Hence, $\max(|A|, |B|) \leq |A + B|$. If neither A nor B is singleton, then define $h : A \times B \rightarrow A + B$ defined by, $h(a, b) = a + b; a \in A, b \in B$. Clearly, h is injective and hence $|A + B| \leq |A||B|$. The equality holds when the function $h : A \times B \rightarrow A + B$ is a bijection. From the above two cases, we have $\max(|A|, |B|) \leq |A + B| \leq |A||B|$.

Remark 2.2 Due to Lemma 2.1, given an integer additive set-indexer f of a graph G , $\max(|f(u)|, |f(v)|) \leq |g_f(uv)| = |f(u) + f(v)| \leq |f(u)||f(v)|$, where $u, v \in V(G)$.

The characteristics of the IASIs with the property $|A + B| = \max(|f(u)|, |f(v)|)$ is of special interest. Hence we have the following definition.

Definition 2.3 An IASI f of a graph G is called a *weak IASI* if $|g_f(uv)| = \max(|f(u)|, |f(v)|)$ for all $u, v \in V(G)$.

It is observed that if $|g_f(uv)| = \max(|f(u)|, |f(v)|) \forall u, v \in V(G)$ holds for every edge uv of G , it is necessary that one of its end vertices is labeled by a singleton set. Hence we define a *weakly k -uniform IASI* as follows.

Definition 2.4 A k -uniform IASI which assigns only singleton sets and k -element sets to the vertices of a given graph G is called a *weakly k -uniform IASI*.

Theorem 2.5 *Every tree admits a weakly k -uniform IASI.*

Proof: All the sets under consideration are subsets of N_0 . Let T be the given tree and k be an arbitrary positive integer. If $k = 1$, assign distinct singleton sets to every vertex of T . Then clearly, every edge of T also has singleton sets as labeling sets. Therefore, T is 1-uniform.

Now let $k > 1$.

Step-1: Let $u_1, u_2, \dots, u_r$ be the pendant vertices of T . Assign the singleton sets $\{i\}$ to the pendant vertices $u_i, 1 \leq i \leq r$. Let $\{v_1, v_2, \dots, v_s\}$ be the set of internal vertices of T which are adjacent to some of $u_i, 1 \leq i \leq s$. Assign the k -element set $\{j, j+1, j+2, \dots, j+(k-1)\}$ to each vertex $v_j, 1 \leq j \leq s$. Then, by Theorem 2.1, each edge $u_i v_j$ has the set-indexing number k . Note that no u_i can be adjacent to two internal vertices which are adjacent to each other, since otherwise, we have a contradiction to the acyclic property of T .

Step-2: Let $\{w_1, w_2, \dots, w_l\}$ be the set of internal vertices which are adjacent to some of $v_j, 1 \leq j \leq l$, where no w_n can be adjacent to any pendant vertices. Assign singleton sets $\{r+n\}$ to each $w_n, 1 \leq n \leq l$ so that each edge $v_j w_n$ has the set-indexing number k .

The succeeding steps, the vertices that are adjacent to the just preceding labeled vertices alternately by distinct k -element sets other than the already labelled sets and distinct singleton sets, for a finite number of times, we get a weakly k -uniform IASI to the tree T .

Remark 2.6 A path P_n can be considered as a tree which has two pendant vertices. Hence by Theorem 2.5, P_n admits an arbitrary k -uniform IASI.

Theorem 2.7 *Every even cycle C_{2n} has a weakly k -uniform IASI.*

Proof: Let $C_{2n} = v_1 v_2 v_3 \dots v_{2n} v_1$ be a cycle of even length. Assign distinct singleton subsets of N_0 to the vertices $v_1, v_3, \dots, v_{2n-1}$ and assign distinct k -element subsets of N_0 to the vertices $v_2, v_4, \dots, v_{2n}$ of C_{2n} . Then by Theorem 2.1, each edge $v_i v_j$ has the set-indexing number k . Hence an even cycle has an arbitrary k -uniform IASI.

Theorem 2.8 *No odd cycle C_n has a weakly k -uniform IASI.*

Proof: By the definition of weakly k -uniform IASI, for each edge $v_i v_{i+1}$ in C_n , one of v_i (or v_{i+1}) should necessarily be labeled by a singleton set and v_{i+1} (or v_i) with a k -element set. Since n is odd, $n = 2m + 1$ for some non-negative

integer m . Let $C_n = v_1 v_2 v_3 \cdots v_{2m+1} v_1$ be a cycle of odd length. Assign distinct singleton subsets of N_0 to the vertices $v_1, v_3, \dots, v_{2m-1}$ and assign distinct k -element subsets of N_0 to the vertices $v_2, v_4, \dots, v_{2m}$ of C_n . Then by Theorem 2.1, each edge $v_i v_j$ where $i, j \leq 2n$, has the set-indexing number k . Now if we assign a singleton set to the vertex v_{2m+1} , then the edge $v_{2m+1} v_1$ has the set-indexing number 1. If we assign a k -element set to v_{2m+1} , then the edge $v_{2m} v_{2m+1}$ has the set-indexing number greater than k . Hence no odd cycle admits a weakly k -uniform IASI.

Invoking the above results, the existence of a weakly k -uniform IASI implies the bipartiteness of the given graph.

Theorem 2.9 *Let k be any positive integer. A bipartite graph G admits a weakly k -uniform IASI.*

Proof: Let $G = (X_1, X_2)$ be bipartite graph. Then assign distinct singleton sets to the vertices in X_1 and distinct k -element sets to X_2 so that every edge of G has the set-indexing number k . Therefore, G admits a weakly k -uniform IASI.

Theorem 2.10 *For any positive integer $k > 1$, a graph G admits a weakly k -uniform IASI if and only if G is bipartite.*

Proof: Proof of the necessary part follows from 2.9. Conversely, assume that G admits a weakly k -uniform IASI. That is, every edge of G has the set-indexing number k . Since G admits a weakly k -uniform IASI, one end vertex of all edges of G should necessarily be labeled by distinct singleton sets and the other end vertex by distinct k -element sets. Let $X_1 = \{u_1, u_2, u_3, \dots, u_r\}$ be the set of vertices which have been labeled by distinct singleton sets and let $X_2 = \{v_1, v_2, v_3, \dots, v_s\}$ be the set of vertices which have been labeled by k -element sets. Since $k > 1$, both X_1 and X_2 are non-empty and no two vertices of X_1 (and X_2) can be adjacent to each other. That is, (X_1, X_2) is a bipartition of the vertex set of G .

Next we establish the existence of k -uniform IASIs to bipartite graphs in the following theorem.

Theorem 2.11 *A bipartite graph G admits a k -uniform IASI.*

Proof: Let G be a bipartite graph. Let (X_1, X_2) be the bipartition of $V(G)$. Let $X = \{v_1, v_2, v_3, \dots, v_l\}$ and $Y = \{v_{l+1}, v_{l+2}, v_{l+3}, \dots, v_r\}$. Let m, n, d be positive integers with $m, n > 1$ and $d \geq 1$. Now assign the set $\{i, i + d, i + 2d, \dots, i + (m - 1)d\}$ to the vertex $v_i, 1 \leq i \leq l$ in X_1 . Let $v_{l+j}, j \geq 1$, be a vertex in X_2 which is adjacent to a vertex v_i . Now assign the set $\{l + j, l + j + d, l + j + 2d, \dots, l + j + (n - 1)d\}$ to the vertex v_{l+j} . Now the edge

$v_i v_{l+j}$ has the set-indexer $\{i+j+l, i+j+l+d, i+j+l+2d, \dots, i+j+l+(n-1)d, i+j+l+d, i+d+j+l+d, i+d+j+l+2d, \dots, i+d+j+l+(n-1)d, \dots, i+(m-1)d+j+l, i+(m-1)d+j+l+d, \dots, i+(m-1)d+j+l+(n-1)d\}$. That is, the set-indexer of $v_i v_{l+j}$ is $\{i+j+l, i+j+l+d, i+j+l+2d, \dots, i+j+l+(m-1)d, i+j+l+md, i+j+l+(m+1)d, i+j+l+(m+2)d, \dots, i+j+l+(m+n-2)d\}$. Therefore, the set-indexing number of $v_i v_{l+j}$ is $m+n-1$. Since v_i and v_{l+j} are arbitrary elements of X_1 and X_2 respectively, the above argument can be extended to all edges in G . That is, all edges of G can be assigned by a k -element set, where $k = m+n-1$. Therefore, G admits a k -uniform IASI.

Theorem 2.12 *A complete graph $K_n, n > 2$ does not admit a weakly k -uniform IASI for any positive integer $k > 1$.*

Proof: Let $k > 1$ be a positive integer. If possible, let K_n admits a weakly k -uniform IASI. Then its vertices are assigned by either singleton sets or k -element sets. As explained in the proof of Theorem 2.10, we get a bipartition (X_1, X_2) of the vertex set of K_n . This is a contradiction to the fact that no complete graph other than K_2 is bipartite. Therefore, $K_n, n > 2$ does not admit a weakly k -uniform IASI.

It is interesting to check whether implications of the existence of a weakly k -uniform IASI to the existence of an arbitrary k -uniform IASI. For any positive integer k , if there exists a weakly k -uniform IASI for a graph G , then each of its vertices is labeled by either a singleton set or by a k -element set and all edges of G have the set-indexing number k . Therefore, G has a k -uniform IASI. Since k is an arbitrary positive integer, we arrive at the following theorem.

Theorem 2.13 *A graph which has a weakly k -uniform IASI admits an arbitrarily k -uniform IASI.*

Invoking Theorem 2.13, every path, even cycle and tree have an arbitrarily k -uniform IASI. Hence we can write the following theorem.

Theorem 2.14 *Every bipartite graph admits an arbitrarily k -uniform IASI, where k is a positive integer.*

We have already established the existence of k -uniform IASIs to bipartite graphs for any positive integer k . Hence we shall look in to the existence of such uniform IASIs for other connected graphs.

It can be observed that the completely bipartite graph $K_{m,n}$ admits a k -uniform IASI for any positive integer k . For any given positive integer $k \geq 1$, the existence of k -uniform IASI is established for a bipartite graph. Now recall the Theorem 1.7 that a connected graph G admits a 2-uniform IASI if and only if it is bipartite. Hence no non-bipartite graphs can have a 2-uniform IASI. Then, what are the admissible values of $k \neq 2$, for a non-bipartite graph to have a k -uniform IASI? The following results settle this problem.

Theorem 2.15 *Let k be a positive integer. If G is a graph which admits a k -uniform IASI, then any subgraph H of G also admits the same k -uniform IASI.*

Proof: Let G be a graph which admits a k -uniform IASI and H be a subgraph of G . Let f^* be the restriction of f to $V(H)$. Then g_{f^*} is the corresponding restriction of g_f to $E(H)$. Then clearly, f^* is a set-indexer on H . This set-indexer may be called the *induced set-indexer* on H . Since $g_f(e) = k$ for all $e \in E(G)$, we have $g_{f^*}(e) = k$ for all $e \in E(H)$. Hence H admits k -uniform IASI.

Remark 2.16 As the contrapositive statement of the above theorem, we observe that if a graph G does not have a k -uniform IASI, then any supergraph of G does not admit a k -uniform IASI.

Theorem 2.17 *A complete graph K_n admits a k -uniform IASI for positive odd integer k .*

Proof: Let $\{v_1, v_2, v_3, \dots, v_n\}$ be the vertex set of the complete graph K_n . Let d and m be a positive integer. Assign the set $\{i, i + d, i + 2d, \dots, i + (m - 1)d\}$, $1 \leq i \leq n$ to each vertex v_i . Then an edge $v_i v_j$ of K_n has the indexing set $\{i + j, i + j + d, i + j + 2d, \dots, i + j + 2(m - 1)d\}$. That is, the set-indexing number of $v_i v_j$ is $2m - 1$. Since v_i and v_j are arbitrary vertices of K_n , all edges of it has the set-indexing number $2m - 1$. Then K_n admits a k -uniform IASI, where $k = 2m - 1$.

Remark 2.18 Being a subgraph of a complete graph K_n , any connected graph G admits a k -uniform IASI for all positive odd integers k .

Hence the interest of studying the graphs which admit k -uniform IASI may be related to an even integer $k > 2$. Hence what are these graphs which admit k -uniform IASI for even integer k , other than bipartite graphs? The following theorem settles this problem.

Theorem 2.19 *Let G be a connected non-bipartite graph. Then G admits a k -uniform IASI if and only if k is a positive odd integer.*

Proof: Necessary part of the result follows from Remark 2.18. Let m, n be two positive integers. Let $G = (X_1, X_2)$ be a bipartite graph. Assign m -element sets to the vertices of X_1 and n -element sets to the vertices of X_2 as in Theorem 2.11 so that every edge of G has the set-indexing number $k = m + n - 1$. Draw an edge between two vertices of X_1 (or X_2) that gives a minimal connected non-bipartite graph with given number of vertices. Without loss of generality, let $e = v_i v_j$ where $v_i, v_j \in X_1$ (or $e = v_{l+i} v_{l+j}$; $v_{l+i}, v_{l+j} \in X_2$). Let

$G_1 = G \cup e$. If possible, assume that G_1 admits a k -uniform IASI where $k > 2$ is an even integer. Then, the set-indexing number of e is $2m-1$ (or $2n-1$). Since G_1 admits a k -uniform IASI, $k > 2$, we have $m+n-1 = 2m-1 = 2n-1 = k$, which is possible only when $m = n$, a contradiction. Hence k can not be even. Hence, for a positive even integer k , by Remark 2.16, no supergraph of G_1 (including K_n) admits a k -uniform IASI for a positive integer k .

Hence, a graph G admits a k -uniform IASI, for a positive even integer $k > 2$ if and only if G is bipartite. More generally,

Theorem 2.20 *A graph G admits a k -uniform IASI, for a positive integer k , if and only if k is odd or G is bipartite.*

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Received: October 9, 2013