Fractional chromatic number, maximum degree and girth

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FRACTIONAL CHROMATIC NUMBER, MAXIMUM DEGREE AND GIRTH

FRANÇOIS PIROT AND JEAN-SÉBASTIEN SERENI

Abstract. We prove new lower bounds on the independence ratio of graphs of maximum degree $\Delta \in \{3,4,5\}$ and girth $g \in \{6,\ldots,12\}$, establishing notably that $i(4,10) \geq 1/3$ and $i(5,8) \geq 2/7$. We also demonstrate that every graph G of girth at least 7 and maximum degree Δ has fractional chromatic number at most $\min_{k\in\mathbb{N}} \frac{2\Delta+2^{k-3}+k}{k}$. In particular, the fractional chromatic number of a graph of girth 7 and maximum degree Δ is at most $\frac{2\Delta+9}{5}$ when $\Delta \in [3,8]$, at most $\frac{\Delta+7}{3}$ when $\Delta \in [8,20]$, at most $\frac{2\Delta+23}{7}$ when $\Delta \in [20,48]$, and at most $\frac{\Delta}{4}+5$ when $\Delta \in [48,112]$.

1. Introduction

Independent sets in graphs are fundamental objects, at the heart of several problems and notions such as graph colouring. Of particular interest is the order $\alpha(G)$ of a largest independent set in a graph G, which often is divided by the number of vertices of G: this is the *independence ratio* of G,

$$\mathrm{ir}(G) \coloneqq \frac{\alpha(G)}{|V(G)|}.$$

Since a k-colouring of a graph is a partition of the vertex set into k independent sets, it follows that the independence ratio of a graph is a lower bound on its chromatic number. For instance, the 4-colour theorem thus implies that every planar graph has independent ratio at least $\frac{1}{4}$. Interestingly enough, no one seems to know how to prove this last statement, sometimes called the "Erdős-Vizing conjecture", without using the 4-colour theorem — or a proof of a similar nature and length.

The independence ratio of a graph has often been studied in relation with the *girth*, which is the length of a smallest cycle in the graph. A first result in this direction is the celebrated introduction of the so-called "deletion method" in graph theory by Erdős, who used it to demonstrate the existence of graphs with arbitrarily large girth and chromatic number. The latter is actually established by proving that the independence ratio of the graph is arbitrarily large. As a large girth is not strong enough a requirement to imply a constant upper bound on the chromatic number, a way to pursue this line of research is to express the upper bound in terms of the maximum degree $\Delta(G)$ of the graph G considered. This also applies to the independence ratio. Letting $g(G)$ stand for the girth of the graph G, that is, the length of a shortest cycle in G if G is not a forest and $+\infty$ otherwise, we define $i(\Delta, g)$ to be the infimum of the independence ratios among all graphs of maximum degree Δ and girth at least g.

$$i(\Delta, g) \coloneqq \inf \left\{ \frac{\alpha(G)}{|V(G)|} \;\middle|\; G \text{ graph with } \Delta(G) \leq \Delta \text{ and } g(G) \geq g \right\}.$$

We moreover define $\mathrm{i}_\infty(\Delta)$ to be the limit of the values taken by $i(\Delta, g)$ as g tends to infinity (which exists as $(i(\Delta, g))_{g\in\mathbb{N}}$ is a non-increasing sequence of positive rational numbers).

In 1979, Staton [17] established that $i(\Delta, 4) \geq \frac{5}{5\Delta-1}$, in particular implying that $i(3,4) \geq \frac{5}{14}$. The two graphs depicted in Figure 1, called the graphs of Fajtlowicz and of Locke, have fourteen vertices each, girth 5, and no independent set of order 6. It follows that $i(3,4) = \frac{5}{14} = i(3,5)$. It is known that the graphs of Fajtlowicz and of Locke are the only two cubic triangle-free and connected graphs with independence ratio $\frac{5}{14}$. This follows from a result of Fraughnaugh and Locke [9] for

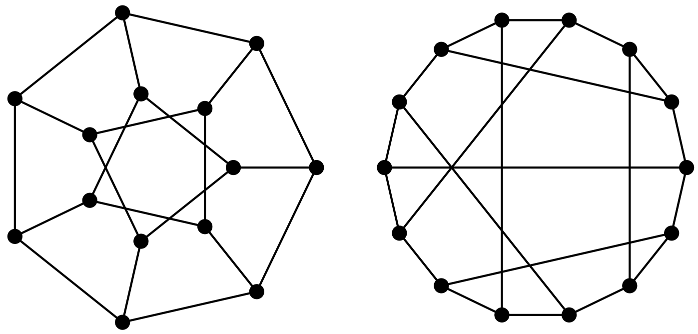

FIGURE 1. The two cubic triangle-free connected graphs with independence ratio $\frac{5}{14}$.

graphs with more than 14 vertices completed by an exhaustive computer check on graphs with at most 14 vertices performed by Bajnok and Brinkmann [1].

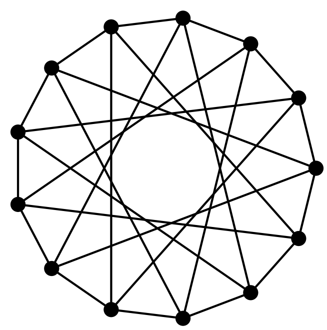

FIGURE 2. The only known 4-regular triangle-free connected graph of independence ratio $\frac{4}{13}$.

In 1983, Jones [10] reached the next step by establishing that $i(4,4) = \frac{4}{13}$. Only one connected graph is known to attain this value: it has 13 vertices and is represented in Figure 2. The value of $i(\Delta, 4)$ when $\Delta \geq 5$ is still unknown; the best general lower bound is due to Shearer [16]. He also provides a lower bound for $i(\Delta, 6)$ as a consequence of a stronger result on graphs with no cycle of length 3 or 5.

Theorem 1 (Shearer [16]). *For every non-negative integer d, set*

$$f(d) := \begin{cases} 1 & \text{if } d = 0, \\ \frac{1+(d^2-d)f(d-1)}{d^2+1} & \text{if } d \geq 1. \end{cases}$$

If G is a triangle-free graph on n vertices with degree sequence $d_1, \ldots, d_n$, then

$$\alpha(G) \geq \sum_{i=1}^{n} f(d_i).$$

Δ	upper bound of $i(\Delta,4)^{-1}$	upper bound on $i(\Delta,6)^{-1}$
2	$5/2 = 2.5$	$7/3 \approx 2.33333$
3	$50/17 \approx 2.94118$	$14/5 = 2.8$
4	$425/127 \approx 3.34646$	$119/37 \approx 3.21622$
5	$2210/593 \approx 3.72681$	$3094/859 \approx 3.60186$
6	$8177/2000 \approx 4.0885$	$57239/14432 \approx 3.96612$
7	$408850/92177 \approx 4.43549$	$408850/94769 \approx 4.31417$
8	$13287625/2785381 \approx 4.77049$	$13287625/2857957 \approx 4.64934$
9	$1089585250/213835057 \approx 5.09545$	$1089585250/219060529 \approx 4.9739$
10	$11004811025/2033474038 \approx 5.41183$	$11004811025/2080503286 \approx 5.28949$

TABLE 1. Upper bounds on $i(\Delta,4)$ and on $i(\Delta,6)$ for $\Delta \in \{2,\dots,10\}$ deduced from Theorems 1 and 2.

Theorem 2 (Shearer [16]). *For every non-negative integer d, set*

$$f(d) := \begin{cases} 0 & \text{if } d = 0,\\ \frac{4}{7} & \text{if } d = 1,\\ \frac{1+(d^2-d)f(d-1)}{d^2+1} & \text{if } d \geq 2. \end{cases}$$

If G is a graph on n vertices with degree sequence $d_1,\dots,d_n$ and with no 3-cycle and no 5-cycle, then

$$\alpha(G) \geq \sum_{i=1}^{n} f(d_i) - \frac{n_{11}}{7},$$

where n_{11} is the number of pairs of adjacent vertices of degree 1 in G.

From now on, we rather consider $i(\Delta,g)^{-1} = 1/i(\Delta,g)$, because this quantity is a lower bound on the fractional chromatic number (defined later on) of the graphs considered. Theorems 1 and 2 allow us to compute upper bounds on $i(\Delta,4)^{-1}$ and on $i(\Delta,6)^{-1}$ for small values of Δ, as indicated in Table 1. When $\Delta \geq 5$, these bounds are the best known ones.

We are not aware of any non trivial lower bounds on $i(5,4)^{-1}$ and $i(6,4)^{-1}$. Figures 3 and Figure 4 show graphs illustrating that $i(5,4)^{-1} \geq \frac{10}{3} \approx 3.33333$ and $i(6,4)^{-1} \geq \frac{29}{8} = 3.625$. These two graphs are circulant graphs, which are Cayley graphs over $\mathbb{Z}_n$.

The value of $i(3,g)^{-1}$ has also been studied when g goes to infinity. Kardoš, Král' and Volec [11] proved the existence of an integer g_0 such that $i(3,g_0)^{-1} \leq 2.2978$. More strongly, their upper bound holds for the fractional chromatic number of every (sub)cubic graph of girth at least g_0. In the other direction, Bollobás [4] proved a general upper bound on $i(\Delta,g)^{-1}$.

Theorem 3 (Bollobás, 1981). *Let $\Delta \geq 3$. Let α be a real number in $(0,1)$ such that*

$$\alpha(\Delta \ln 2 - \ln(\alpha)) + (2-\alpha)(\Delta-1)\ln(2-\alpha) + (\alpha-1)\Delta\ln(1-\alpha) < 2(\Delta-1)\ln 2.$$

For every positive integer g, there exists a Δ-regular graph with girth at least g and independence ratio less than $\alpha/2$.

Theorem 3 allows us to compute lower bounds on $\mathrm{i}_\infty(\Delta)^{-1}$ for small values of Δ, and also provides a general lower bound [4, Corollary 3], which are all presented in Table 2.

The fractional chromatic number $\chi_f(G)$ of a graph G is a refinement of the chromatic number. It is the fractional solution to a linear program the integer solution of which is the chromatic number. Let G be a given graph; we define $\mathcal{S}_{\max}(G)$ to be the set of all maximal independent sets of G

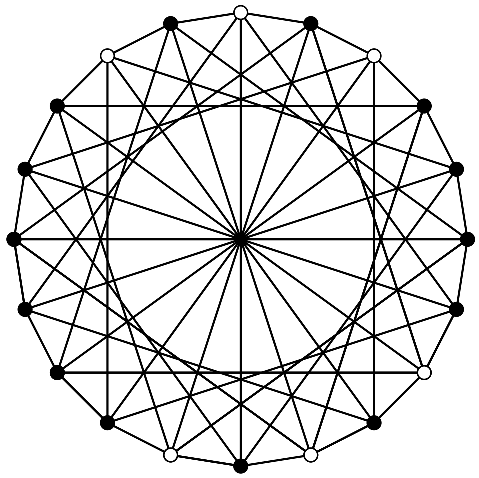

FIGURE 3. A 5-regular triangle-free (vertex-transitive) graph with independence ratio $\frac{3}{10}$. It is the Cayley graph over $\mathbb{Z}_{20}$ with generating set $\{\pm 1, \pm 6, \pm 10\}$. There is no independent set of order 7, and the white vertices form an independent set of order 6.

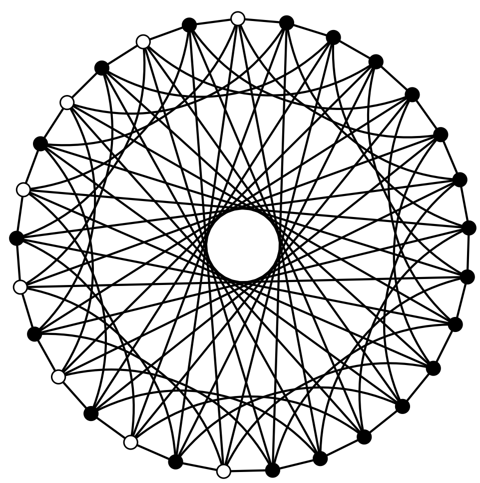

FIGURE 4. A 6-regular triangle-free (vertex-transitive) graph with independence ratio $\frac{8}{29}$. It is the Cayley graph over $\mathbb{Z}_{29}$ with generating set $\{\pm 1, \pm 5, \pm 13\}$. There is no independent set of order 9, and the white vertices form an independent set of order 8.

and $\mathcal{S}_\alpha(G)$ to be the set of all maximum independent sets of G. Then $\chi_f(G)$ is the solution of the

Δ	lower bound on $\mathrm{i}_\infty(\Delta)^{-1}$
2	2
3	2.17835
4	2.3775
5	2.57278
6	2.76222
7	2.94606
8	3.1249
9	3.29931
10	3.46981
11	3.63684
12	3.80074
Δ	$\dfrac{\Delta}{2\ln\Delta}$

TABLE 2. Lower bounds on $\mathrm{i}_\infty(\Delta)^{-1}$ implied by Theorem 3.

following linear program.

$$\min \sum_{S\in\mathcal{S}_{\max}(G)} w_S$$

$$\text{such that} \begin{cases} w_S \in [0,1] & \text{for each } S \in \mathcal{S}_{\max} \\ \displaystyle\sum_{\substack{S\in\mathcal{S}_{\max}\\ v\in S}} w_S \geq 1 & \text{for each } v \in V(G). \end{cases}$$

A *fractional colouring of weight w of G* is any instance within the domain of the above linear program such that $\sum w_S = w$. You can note that a k-colouring of G is a special case of a fractional colouring of weight k of G, where $w_S = 1$ if S is a monochromatic class of the k-colouring, and $w_S = 0$ otherwise. Note also that if G is a clique, then any fractional colouring of G is of weight at least $|V(G)|$. This allows us to write the following inequalities

$$\omega(G) \leq \chi_f(G) \leq \chi(G) \leq \Delta(G) + 1,$$

where $\omega(G)$ is the maximum order of a clique in G, and $\Delta(G)$ is the maximum degree of G. Equality holds between $\omega(G)$ and $\chi(G)$, and so in particular between $\omega(G)$ and $\chi_f(G)$, when G is a perfect graph. Those are the graphs that contain no odd hole nor odd antihole, as was conjectured by Berge [2] in 1961, and proved by Chudnovsky *et al.* [6] in 2006. On the other side, the characterisation of the graphs G for which equality holds between $\chi(G)$ and $\Delta(G)+1$ was established by Brooks [5] in 1941, and those graphs are cliques and odd cycles. Since $\chi_f(C_{2k+1}) = \frac{k}{2k+1}$, the only graphs G such that $\chi_f(G) = \Delta(G) + 1$ are cliques. Moreover, a relation between the independence ratio of G and its fractional chromatic number is obtained by observing that

$$\mathrm{ir}(G)^{-1} = \frac{|V(G)|}{\alpha(G)} \leq \chi_f(G),$$

where equality holds in particular when G is vertex-transitive.

Very recently, Molloy [14] proved the best known extremal upper bounds for the chromatic number of graphs of given clique number and maximum degree.

Theorem 4 (Molloy, 2019)**.**

- *If G is a triangle-free graph, then*

$$\forall\varepsilon, \exists\Delta_\varepsilon, \quad \Delta(G) \geq \Delta_\varepsilon \Rightarrow \chi(G) \leq (1+\varepsilon)\frac{\Delta(G)}{\ln\Delta(G)}.$$

- *If G is a graph with $\omega(G) > 2$, then*

$$\chi(G) \leq 200\omega(G)\frac{\Delta(G)\ln\ln\Delta(G)}{\ln\Delta(G)}.$$

The first bound is sharp up to a multiplicative factor in a strong sense, since as shown by Bollobás [4, Corollaries 3 and 4] for all integers g and $\Delta \geq 3$ there exists a graph with maximum degree Δ, girth at least g and chromatic number at least $\frac{\Delta}{2\ln\Delta}$.

There remains however a substantial range of degrees not concerned by the bound for triangle-free graphs given by Theorem 4, namely when $\Delta(G)$ is smaller than Δ_ε, which is greater than $20^{2/\varepsilon}$. To this date, the best known general upper bound in terms of clique number and maximum degree for the *fractional* chromatic number[1] is due to Molloy and Reed [15, Theorem 21.7, p. 244].

Theorem 5 (Molloy and Reed, 2002)**.** *For every graph G,*

$$\chi_f(G) \leq \frac{\omega(G)+\Delta(G)+1}{2}.$$

If one considers a convex combination of the clique number and the maximum degree plus one for an upper bound on the (fractional) chromatic number of a graph, then because the chromatic number of a graph never exceeds its maximum degree plus one, the aim is to maximise the coefficient in front of the clique number. The convex combination provided by Theorem 5 (which is conjectured to hold, after taking the ceiling, also for the chromatic number), is best possible. Indeed, for every positive integer k the graph $G_k \coloneqq C_5 \boxtimes K_k$ is such that $\omega(G_k) = 2k, \Delta(G_k) = 3k-1, \chi_f(G_k) = \frac{5k}{2} = \frac{\omega(G_k)+\Delta(G_k)+1}{2}$.

A local form of Theorem 5 exists: it was first devised by McDiarmid (unpublished) and appearing as an exercise in Molloy and Reed's book [15]. A published version is found in the thesis of Andrew King [12, Theorem 2.10, p. 12].

Theorem 6 (McDiarmid, unpublished)**.** *Let G be a graph, and set $f_G(v) \coloneqq \frac{\omega_G(v)+\deg_G(v)+1}{2}$ for every $v \in V(G)$, where $\omega_G(v)$ is the order of a largest clique in G containing v. Then*

$$\chi_f(G) \leq \max\{f_G(v) \,:\, v \in V(G)\}.$$

In Subsection 3.1, we slightly strengthen the local property of Theorem 6 as a way to illustrate the arguments used later on.

Our first contribution is to establish an upper bound on the fractional chromatic number of graphs of girth at least 7.

Theorem 7. *Let $f(x) \coloneqq \min\limits_{k\in\mathbb{N}} \frac{2x+2^{k-3}+k}{k}$. If G is a graph of girth at least 7, then G admits a fractional colouring such that for every induced subgraph H of G, the restriction of c to H has weight at most $f(\max\{\deg_G(v) \,:\, v \in V(H)\})$. In particular,*

$$\chi_f(G) \leq f(\Delta(G)).$$

[1] For the chromatic number, the reader is referred to a nice theorem of Kostochka [13], which for instance implies that every graph with maximum degree at most 5 and girth at least 35 has chromatic number at most 4 (Corollary 2 in *loc. cit.*). The general upper bound on the chromatic number guaranteed by Kostochka's theorem is never less than the floor of half the maximum degree plus two.

Remark 1. In Theorem 7, if $x \geq 3$ then the minimum of the function $k \to \frac{2x+2^{k-3}+k}{k}$ (over $\mathbb{N}$) is attained for $k = [4 + \log_2 x - \log_2 \log_2 x]$. So if $x \geq 3$, then $f(x) = (2\ln 2 + o(1))x/\ln x$, which is off by a multiplicative factor $2\ln 2$ from the best known extremal value for triangle-free graphs. However, up to x of the order of 10^7, this is smaller than the best known explicit upper bound for fractional colouring [7], namely

$$\min_{\lambda>0} \quad \frac{\lambda+1}{\lambda} e^{W(x\ln(1+\lambda))},$$

where W is the Lambert function, defined as the reciprocal of $z \mapsto ze^z$. We also note that for every non-negative integer x, the minimum of the function $k \to \frac{2x+2^{k-3}+k}{k}$ (over $\mathbb{N}$) is attained in an integer greater than 3.

We also provide improved upper bounds on the inverse independence ratio of graphs of maximum degree in $\{3,4,5\}$ and girth in $\{6,\dots,12\}$. In particular, these are upper bounds on the fractional chromatic number of vertex-transitive graphs in these classes. These upper bounds are obtained *via* a systematic computer-assisted method.

Theorem 8. *The values presented in Table 3 are upper bounds on $i(\Delta,g)^{-1}$ for $\Delta \in \{3,4,5\}$ and $g \in \{6,\dots,12\}$.*

Δ \ g	6	7	8	9	10	11	12
3	$30/11 \approx 2.727272$	$30/11$	2.625224	2.604167	2.557176	2.539132	2.510378
4	$41/13 \approx 3.153846$	$41/13$	3.038497	3.017382	3		
5	$69/19 \approx 3.631579$	3.6	3.5				

TABLE 3. Upper bounds on $i(\Delta,g)^{-1}$ for $\Delta \in \{3,4,5\}$ and $g \in \{6,\dots,12\}$.

The bounds provided by Theorem 8 when $\Delta \in \{3,4\}$ and $g = 7$ are the same as those for $g = 6$. It seems that this could be a general phenomenon. A computation is currently running to determine an upper bound on $i(3,13)^{-1}$, which we expect to be 2.5. We therefore offer the following conjecture.

Conjecture 1. *The values presented in Table 4 are upper bounds on $i(\Delta,g)^{-1}$ for $\Delta \in \{3,4,5\}$ and $g \in \{6,8,10,12\}$.*

Δ \ g	6	8	10	12
3		2.604167	2.539132	2.5
4		3.017382	3	
5	3.6	3.5		

TABLE 4. Conjectured upper bounds on $i(\Delta,g)^{-1}$ for $\Delta \in \{3,4,5\}$ and $g \in \{6,\dots,12\}$.

Notation. We introduce some notation before establishing a few technical lemmas, from which we will prove Theorems 7 and 8. If v is a vertex of a graph G and r a non-negative integer, then $N_G^r(v)$ is the set of all vertices of G at distance exactly r from v in G, while $N_G^r[v] := \bigcup_{j=0}^r N_G^j(v)$. Recall that $\mathcal{S}_{\max}(G)$ is the set of all maximal independent sets of G and $\mathcal{S}_\alpha(G)$ the set of all maximum independent sets of G. If w is a mapping from $\mathcal{S}_{\max}(G)$ to $\mathbb{R}$ then for every vertex $v \in V(G)$ we set

$$w[v] \coloneqq \sum_{\substack{S\in\mathcal{S}_{\max}(G)\\ v\in S}} w(S).$$

Further, if X is a collection of maximal independent sets of G, then $w(X) \coloneqq \sum_{S\in X} w(S)$. If S in an independent set of a graph G, a vertex v is *covered* by S if v belongs to S or has a neighbour in S. A vertex that is not covered by S is *uncovered* (by S). If G is a graph rooted at a vertex v, then for every positive integer d, the set of all vertices at distance d from v in G is a *layer* of G.

2. Technical lemmas

In this section we present the tools needed for the proofs of the main theorems.

2.1. Greedy fractional colouring algorithm.

Our results on fractional colouring are obtained using a greedy algorithm analysed in a recent work involving the first author [7]. This algorithm is a generalisation of an algorithm first described in the book of Molloy and Reed [15, p. 245] for the uniform distribution over maximum independent sets. The setting here is, for each induced subgraph H of the graph we wish to fractionally colour, a probability distribution over the independent sets of H. We shall use only distributions over maximal independent sets.

Lemma 1 (de Joannis de Verclos *et al.*, 2018)**.** *Fix a positive integer r. Let G be a graph and suppose that every vertex $v \in V(G)$ is assigned a list $(\alpha_j(v))_{j=0}^r$ of $r+1$ real numbers. Suppose that for each induced subgraphs H of G, there is a probability distribution on $\mathcal{S}_{\max}(H)$ such that, writing $\mathbf{S}_H$ for the random independent set from this distribution,*

$$\forall v \in V(H), \qquad \sum_{j=0}^{r} \alpha_j(v)\mathbb{E}\left[\left|N_H^j(v) \cap \mathbf{S}_H\right|\right] \geq 1.$$

The greedy fractional algorithm *defined by Algorithm 1 produces a fractional colouring w of G such that the restriction of w to any subgraph H of G is a fractional colouring of H of weight at most $\max\{\gamma_\alpha(v) : v \in V(H)\}$, where $\gamma_\alpha(v) \coloneqq \sum_{j=0}^r \alpha_j(v)\left|N_G^j(v)\right|$. In particular, $\chi_f(G) \leq \max\{\gamma_\alpha(v) : v \in V(G)\}$.*

Algorithm 1 The greedy fractional algorithm

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for $I \in \mathcal{S}_{\max}(G)$ **do**
  $w(I) \leftarrow 0$
end for
$H \leftarrow G$
while $|V(H)| > 0$ **do**
  $\iota \leftarrow \min\left\{\min_{v\in V(H)} \frac{1-w[v]}{\mathbb{P}[v\in \mathbf{S}_H]}, \min_{v\in V(H)} \gamma_\alpha(v) - w(\mathcal{S}_{\max}(G))\right\}$
  for $S \in \mathcal{S}_{\max}(H)$ **do**
    $w(S) \leftarrow w(S) + \mathbb{P}[\mathbf{S}_H = S]\,\iota$
  end for
end while

2.2. **Independence ratio.** We state two lemmas which can be proved in similar ways. We only present the proof of the second one, the argument for the first lemma being very close but a little simpler.

Lemma 2. *Let r be a positive integer and G be a d-regular graph on n vertices. Assume that there exists a probability distribution p on $\mathcal{S}_{\max}(G)$ such that*

$$\forall v \in V(G), \quad \sum_{i=0}^{r} \alpha_i \mathbb{E}\left[\mathbf{X}_i(v)\right] \geq 1, \tag{1}$$

where $\mathbf{X}_i(v)$ is the random variable counting the number of paths of length i between v and a vertex belonging to a random independent set $\mathbf{S}$ chosen following p. Then

$$\frac{n}{\alpha(G)} \leq \alpha_0 + \sum_{i=1}^{r} \alpha_i d(d-1)^{i-1}. \tag{2}$$

Lemma 3. *Let r be a positive integer and G be a d-regular graph on n vertices. Assume that there exists a probability distribution p on $\mathcal{S}_{\max}(G)$ such that*

$$\forall e \in E(G), \quad \sum_{i=0}^{r} \alpha_i \mathbb{E}\left[\mathbf{X}_i(e)\right] \geq 1, \tag{3}$$

where $\mathbf{X}_i(e)$ is the random variable counting the number of paths of length $i+1$ starting with e and ending at a vertex belonging to a random independent set $\mathbf{S}$ chosen following p. Then

$$\frac{n}{\alpha(G)} \leq \sum_{i=0}^{r} 2\alpha_i (d-1)^i. \tag{4}$$

Proof. Given an edge e of G, the contribution of an arbitrary vertex $v \in \mathbf{S}$ to $\mathbf{X}_i(e)$ is the number of paths of length $i+1$ starting at v and ending with e. It follows that the total contribution of any vertex $v \in \mathbf{S}$ to $\sum_{e \in E(G)} \mathbf{X}_i(e)$ is the number of paths of G with length $i+1$ that start at v, which is $d(d-1)^i$ since G is a d-regular graph. Consequently,

$$\sum_{e \in E(G)} \mathbf{X}_i(e) = \sum_{v \in V(G)} \mathbb{P}\left[v \in \mathbf{S}\right] d(d-1)^i.$$

We now sum (3) over all edges of G.

$$\sum_{e \in E(G)} \sum_{i=0}^{r} \alpha_i \mathbb{E}\left[\mathbf{X}_i(e)\right] \geq |E(G)| = \frac{nd}{2}$$

$$\sum_{i=0}^{r} \alpha_i \sum_{e \in E(G)} \mathbb{E}\left[\mathbf{X}_i(e)\right] \geq \frac{nd}{2}$$

$$\sum_{i=0}^{r} \alpha_i \sum_{v \in V(G)} \mathbb{P}\left[v \in \mathbf{S}\right] d(d-1)^i \geq \frac{nd}{2}$$

$$\sum_{i=0}^{r} 2\alpha_i \mathbb{E}\left[|\mathbf{S}|\right] (d-1)^i \geq n$$

$$\sum_{i=0}^{r} 2\alpha_i (d-1)^i \geq \frac{n}{\alpha(G)}$$

□

The next lemma allows us to generalise Lemmas 2 and 3 to non-regular graphs. To this end, we use a standard argument coupled with the existence of specific vertex-transitive type-1 regular graphs with any given degree and girth. These are provided by a construction of Exoo and Jajcay [8] in the proof of their Theorem 19, which is a direct generalisation of a construction for cubic graphs designed by Biggs [3, Theorem 6.2]. We can formulate their theorem as follows, the mentioned edge-colouring following simply from the fact that the graph constructed is a Cayley graph obtained from a generating set consisting only of involutions.

Theorem 9 (Exoo & Jajcay, 2013)**.** *For every integers d and g both at least* 3*, there exists a vertex transitive d-regular graph with girth at least g and chromatic index d.*

Lemma 4. *From any graph G of maximum degree d and girth g, we can construct a d-regular graph $\varphi(G)$ of girth g whose vertex set can be partitioned into induced copies of G, and such that any vertex $v \in G$ can be sent to any of its copies through an automorphism.*

Proof. Set $k \coloneqq \sum_{v \in G}(d - \deg(v))$. Let G' be the supergraph of G obtained by adding k vertices $v'_1, \ldots, v'_k$ each of degree 1, such that all other vertices have degree d. We let e'_i be the edge of G' incident to v'_i, for each $i \in \{1, \ldots, k\}$. By Theorem 9, there exists a vertex-transitive k-regular graph H of girth at least g together with a proper edge-colouring c using k colours. Let $n(H)$ be the number of vertices of H and write $V(H) = \{1, \ldots, n(H)\}$.

We construct $\varphi(G)$ by starting from the disjoint union of $n(H)$ copies $G_1, \ldots, G_{n(H)}$ of G. For each edge $e = \{i, j\} \in E(H)$, letting u_e be the vertex of G incident to the edge $e'_{c(e)}$ in G', we add an edge between the copy of u_e in G_i and that in G_j.

Any cycle in $\varphi(G)$ either is a cycle in G, and hence has length at least g, or contains all the edges of a cycle in H, and hence has length at least g. It follows that $\varphi(G)$ has girth g.

The last statement follows readily from the fact that H is vertex transitive. ☐

Corollary 1. *Let d and g be integers greater than two. If there exists a constant $B = B(d, g)$ such that every d-regular graph H with girth g has independence ratio at least B, then every graph G with maximum degree d and girth g also has independence ratio at least B. In particular, if Lemma 2 or Lemma 3 can be applied to the class of d-regular graphs of girth g, then the conclusion also holds for the class of graphs with maximum degree d and girth g, that is, for $i(d, g)$.*

Proof. Let G be a graph with maximum degree d and girth g on n vertices. Let $\varphi(G)$ be the graph provided by Lemma 4. In particular, $|V(\varphi(G))| = kn$ where k is the number of induced copies of G partitioning $V(\varphi(G))$. By assumptions, $\varphi(G)$ contains an independent set I of order at least $B \cdot kn$. Letting I_i be the set of vertices of the i-th copy of G contained in I, by the pigeon-hole principle there exists $i \in \{1, \ldots, k\}$ such that $|I_i| \geq B \cdot n$, and hence G has independence ratio at least B. ☐

3. Local fractional colourings

3.1. **A local Reed's bound.** For the sake of illustration, we begin by showing how Lemma 1 can be used to prove Theorem 5. We actually establish a slight strengthening of Theorem 6, the local form of Theorem 5. The argument relies on the relation (5) below [12, Lemma 2.11], which is a local version of the relation (21.10) appearing in Molloy and Reed's book [15]. The short argument, however, stays the same and we provide it here only for explanatory purposes, since it is the inspiration for the argument used in the proof of Theorem 7.

Proposition 1. *Let G be a graph, and set $f_G(v) \coloneqq \frac{\omega_G(v)+\deg_G(v)+1}{2}$ for every $v \in V(G)$, where $\omega_G(v)$ is the order of a largest clique in G containing v. Then G admits a fractional colouring c such that the restriction of c to any induced subgraph H of G has weight at most $\max_{v \in V(H)} f_G(v)$. In particular,*

$$\chi_f(G) \leq \max \{ f_G(v) \,:\, v \in V(G) \} .$$

Proof. We demonstrate the statement by applying Lemma 1. To this end, we use the uniform distribution on maximum independent sets. Specifically, for every induced subgraph H of G we let $\mathbf{S}_H$ be a maximum independent set of H, drawn uniformly at random. Let $v \in V(H)$ be any vertex. We shall prove that

$$\frac{\omega(v)+1}{2}\mathbb{P}\left[v \in \mathbf{S}_H\right] + \frac{1}{2}\mathbb{E}\left[|N(v) \cap \mathbf{S}_H|\right] \geq 1. \tag{5}$$

The conclusion then follows by applying Lemma 1, with $r = 1$, $\alpha_0(v) = \frac{1}{2} \cdot (\omega(v)+1)$ and $\alpha_1(v) = \frac{1}{2}$ for every vertex $v \in V(G)$.

It remains to establish (5). We let $\mathbf{R} := \mathbf{S}_H \backslash N[v]$, and we condition on the following random events.

(i) Let X_k be the random event that $\mathbf{W} := N[v] \backslash N(\mathbf{R})$ is a clique of size $k \in \{1, \dots, \omega(v)\}$. It follows that exactly one vertex from $\mathbf{W}$ belongs to $\mathbf{S}_H$, and every vertex in $\mathbf{W}$ has equal probability $1/k$ to be in $\mathbf{S}_H$. It follows that

$$\frac{\omega(v)+1}{2}\mathbb{P}\left[v \in \mathbf{S}_H \mid X_k\right] + \frac{1}{2}\mathbb{E}\left[|N(v) \cap \mathbf{S}_H| \mid X_k\right] = \frac{\omega(v)+1}{2k} + \frac{k-1}{2k} \geq 1.$$

(ii) Let Y be the random event that $\mathbf{W}$ is not a clique. Note that Y is the complementary event to the union of the events X_k. In this case, $|\mathbf{W}\backslash\{v\} \cap \mathbf{S}_H| \geq 2$, and $v \notin \mathbf{S}_H$, since $\mathbf{S}_H$ is a maximum independent set. It follows that

$$\frac{\omega(v)+1}{2}\mathbb{P}\left[v \in \mathbf{S}_H \mid Y\right] + \frac{1}{2}\mathbb{E}\left[|N(v) \cap \mathbf{S}_H| \mid Y\right] \geq \frac{1}{2} \times 2 = 1$$

The validity of (5) follows by summing over all possible sets R for which there exists a maximum independent set S of H such that $R = S \setminus N[v]$. □

We finish by noting that the bound provided by Theorem 6 is best possible over the class of unicyclic triangle-free graphs if one uses the fractional greedy colouring of Lemma 1 together with any probability distribution on the *maximum* independent sets of the graph.

Lemma 5. *If the probability distribution used in Lemma 1 gives positive probability only to maximum independent sets, then the greedy fractional colouring algorithm can return a fractional colouring of weight up to $\frac{d+3}{2}$ in general for graphs of degree d, should they be acyclic when d is odd, or have a unique cycle (of length 5) when d is even.*

Proof. We prove the statement by induction on the positive integer d.

- If $d = 1$, then let G_1 consist only of an edge. The algorithm returns a fractional colouring of G_1 of weight 2.
- If $d = 2$, then let G_2 be the cycle of length 5. The algorithm returns a fractional colouring of G_2 of weight $\frac{5}{2}$.
- If $d > 2$, then let G_d be obtained from G_{d-2} by adding two neighbours of degree 1 to every vertex. This creates no new cycles, so G_d is acyclic when d is odd, and contains a unique cycle, which is of length 5, when d is even.

  For every $d \geq 3$, the graph G_d contains a unique maximum independent set, namely $S_0 := V(G_d)\backslash V(G_{d-2})$. After the first step of the algorithm applied to G_d, all the vertices in S_0 have weight 1, and we are left with the graph G_{d-2} where every vertex has weight 0. By the induction hypothesis, the total weight of the fractional colouring returned by the algorithm is therefore $1 + \frac{(d-2)+3}{2} = \frac{d+3}{2}$.

□

3.2. **A stronger bound for graphs of girth 7.** Lemma 5 implies that if we are to prove a better bound than that given by Theorem 6, we need to use a probability distribution that gives a non-zero probability to non-maximum independent sets. Moreover, we need to be able to make a local analysis of the possible outcomes for the random independent set, independently from its exterior shape. Only few probability distributions have this property. One of them is the hard-core distribution, which we use together with Lemma 1 in order to prove Theorem 7.

For any induced graph H of a graph G, we let $\mathbf{S}_H$ be a random independent set of H, drawn from $\mathcal{S}_{\max}(H)$ according to the hard-core distribution with fugacity $\lambda > 0$. This means that

$$\forall S_0 \in \mathcal{S}_{\max}(H), \quad \mathbb{P}\left[\mathbf{S}_H = S_0\right] = \frac{\lambda^{|S_0|}}{\sum\limits_{S \in \mathcal{S}_{\max}(H)} \lambda^{|S|}}.$$

From now on, let G be a graph of girth (at least) 7 and H an induced subgraph of G. If $v \in V(H)$ then

$$\mathcal{R}_{H,v} \coloneqq \left\{ I \setminus N_H^2[v] \;:\; I \in \mathcal{S}_{\max}(H) \right\}.$$

Set $\mathbf{R}_v \coloneqq \mathbf{S}_H \backslash N_H^2[v]$, and $\mathbf{X}_i(v) \coloneqq \mathbf{S}_H \cap N_H^i(v)$. We establish the following assertion.

(A). *Using the hard-core distribution on $\mathcal{S}_{\max}(H)$ with fugacity $\lambda = 4$, it holds that for every vertex $v \in V(H)$, every set $R_0 \in \mathcal{R}_{H,v}$ and every integer $k \geq 4$,*

$$\frac{2^{k-3}+k}{k}\mathbb{E}\left[\mathbf{X}_0(v) \mid \mathbf{R}_v = R_0\right] + \frac{2}{k}\mathbb{E}\left[\mathbf{X}_1(v) \mid \mathbf{R}_v = R_0\right] \geq 1.$$

Proof. The subset $\mathbf{S}_H \setminus \mathbf{R}_v$ consists of an independent set of G contained in $\mathbf{W}_0 \coloneqq N_H^2[v] \backslash N(\mathbf{R}_v)$. It could hold that some vertices in $\mathbf{W}_0$ are forced to belong to this independent set, namely when one of their neighbours in $V(H) \setminus \mathbf{W}_0$ is not covered by $\mathbf{R}_v$. Let $\mathbf{W}_f$ be the set of those vertices, and $\mathbf{W}$ be obtained by removing those vertices and their neighbours:

$$\begin{aligned} \mathbf{W}_f &\coloneqq \left\{ v \in \mathbf{W}_0 \;:\; (N(v) \backslash \mathbf{W}_0) \not\subseteq N(\mathbf{R}_v) \right\}, \\ \mathbf{W} &\coloneqq \mathbf{W}_0 \backslash N[\mathbf{W}_f]. \end{aligned}$$

Note that the subgraph of H induced by $\mathbf{W}$ is a forest of maximum degree d, and the tree containing v has depth at most 2. It is enough to establish (A) when this subgraph is a tree.

Let $R_0 \in \mathcal{R}_{H,v}$ be any fixed realisation of $\mathbf{R}_v$, and let us condition on the random event that $\mathbf{R}_v = R_0$. Let W, W_f and W_0 be the respective (deterministic) values of $\mathbf{W}$, $\mathbf{W}_f$ and $\mathbf{W}_0$ in this setting. It turns out that $\mathbf{S}_H \cap W$ is an independent set drawn according to the hard-core distribution with fugacity λ from $\mathcal{S}_{\max}(H[W])$.

To see this, let $S \in \mathcal{S}_{\max}(H)$ be any realisation of $\mathbf{S}_H$ such that $S \backslash N_H^2[v] = R_0$. Let $S_v \coloneqq S \cap W$; we show that $S_v \in \mathcal{S}_{\max}(H[W])$. First, we show that $W_f \subseteq S$. Indeed, if $u \in W_f$, then u has at least one neighbour $u' \in V(H) \backslash W_0$ that is uncovered by R_0. Because H is of girth 7, the vertex u is the only neighbour of u' in W_0. The maximality of S implies that u' must be covered by W_0, and hence $u \in S$. Second, if there is a vertex $u \in W$ that is uncovered by S_v, then the maximality of S implies that u must be covered by $S \backslash W$, and hence either by R_0 or by W_f. None is possible since $N(R_0)$ and $N(W_f)$ are both disjoint from W by construction, so we have a contradiction.

On the other hand, given any set $S_v \in \mathcal{S}_{\max}(H[W])$, the set $R_0 \cup W_f \cup S_v$ is a valid realisation of $\mathbf{S}_H$. Indeed, any vertex in W is covered by S_v, and any vertex in $V(H) \setminus W$ is covered either by R_0 or by W_f, so $R_0 \cup W_f \cup S_v$ is a maximal stable set of H.

In conclusion, the set of realisations of $\mathbf{S}_H \cap W$ is exactly $\mathcal{S}_{\max}(H[W])$, and each such realisation S_v has a probability proportional to $\lambda^{|S_v|+|W_f|+|R_0|}$, and hence proportional to $\lambda^{|S_v|}$ since R_0 and W_f are fixed. This finishes to establish that $\mathbf{S}_H \cap W$ follows that hard-core distribution with fugacity λ on $\mathcal{S}_{\max}(H[W])$.

We let W_i be the set of vertices of W at distance i from v in W, for $i \in \{0,1,2\}$, and $W_{1,j}$ be the subset of vertices of W_1 with j neighbours in W_2. We set $x_j \coloneqq |W_{1,j}|$. Thus $|W_1| = \sum_{j=0}^{d-1} x_j$ and $|W_2| = \sum_{j=1}^{d-1} j x_j$.

Note that $W_1 \in \mathcal{S}_{\max}(H[W])$ and that $\mathbb{P}[\mathbf{S}_H \cap W = W_1]$ is proportional to $\lambda^{\sum_{j=0}^{d-1} x_j}$. In order to ease the following computations and verifications, we compute a weight $w(S)$ for each independent set $S \in \mathcal{S}_{\max}(H[W])$ that is proportional to $\mathbb{P}[\mathbf{S}_H \cap W = S]$, such that $w(W_1) = 1$.

There is exactly one maximal independent set S_0 that contains v, namely $S_0 \coloneqq \{v\} \cup W_2$, of normalised weight $w_0 \coloneqq \lambda^{1+\sum_{j\geq 0}(j-1)x_j}$. Every other maximal independent set $S \in \mathcal{S}_{\max}(H[W]) \setminus \{S_0, W_1\}$ contains $W_{1,0}$. In addition, for every vertex $u \in W_1 \setminus W_{1,0}$, the set S either contains u or it contains all the neighbours of u in W_2. Therefore, it follows that if $x_0 > 0$, then the sum of the weights of these other independent sets is

$$T \coloneqq \sum_{i_1 \leq x_1, \ldots, i_{d-1} \leq x_{d-1}} \prod_{j=1}^{d-1} \binom{x_j}{i_j} \left(\lambda^{j-1}\right)^{i_j} = \prod_{j=1}^{d-1} \left(1 + \lambda^{j-1}\right)^{x_j}.$$

If $x_0 = 0$, then the sum of their weights is $T - \frac{w_0}{\lambda}$, since there is no independent set containing W_2 in whole and not v in this case.

We let $D \coloneqq w_0 + T$ if $x_0 > 0$, and $D \coloneqq T + w_0\left(1 - \frac{1}{\lambda}\right)$ otherwise. It follows that

$$\mathbb{E}[\mathbf{X}_0] = \frac{w_0}{D} \qquad \text{and}$$
$$\mathbb{E}[\mathbf{X}_1] = \frac{T}{D}\left(x_0 + \sum_{j=1}^{d-1} \frac{x_j}{1+\lambda^{j-1}}\right).$$

There remains to check that, up to a good choice of λ, it holds that

$$\frac{2^{k-3}+k}{k}\mathbb{E}[X_0] + \frac{2}{k}\mathbb{E}[X_1] \geq 1.$$

This translates to

$$2^{k-3}w_0 + \frac{kw_0}{\lambda} \geq T\left(k - 2\sum_{j=1}^{d-1}\frac{x_j}{1+\lambda^{j-1}}\right) \qquad \text{if } x_0 = 0, \text{ and to}$$
$$2^{k-3}w_0 \geq T\left(k - 2x_0 - 2\sum_{j=1}^{d-1}\frac{x_j}{1+\lambda^{j-1}}\right) \qquad \text{if } x_0 \neq 0.$$

We use the two following facts.

Fact 1: For every positive integer j, the function $\lambda \mapsto \left(1 + \frac{1}{\lambda^{j-1}}\right)^{1+\lambda^{j-1}}$ is non increasing on $(0, +\infty)$, and in particular always bounded from above by $\frac{3125}{1024}$ when $\lambda \geq 4$ and $j \geq 2$, and by $\left(1 + \frac{1}{\lambda^{j_0-1}}\right)^{1+\lambda^{j_0-1}}$ when $\lambda \geq 1$ and $j \geq j_0$.

Fact 2: For all real numbers y_0, A and B with $A > 1$ and $B > 0$, the maximum of the function $f \colon y \mapsto A^y(B - 2y)$ on the domain $[y_0, +\infty)$ is $f(y_0)$ when $B/2 - 1/\ln A \leq y_0$, and $\frac{2A^{B/2}}{e \ln A}$ otherwise.

Let us discriminate on the possible values for x_0, noting that $w_0 \geq \lambda^{1-x_0}$.

(i) When $x_0 = 0$, it suffices to show that

$$2^{k-3}\lambda + k \geq \prod_{j=1}^{d-1}\left(1 + \frac{1}{\lambda^{j-1}}\right)^{x_j}\left(k - 2\sum_{j=1}^{d-1}\frac{x_j}{1+\lambda^{j-1}}\right). \tag{6}$$

(ii) When $1 \le x_0 \le k/2$, it suffices to show that

$$2^{k-3}\lambda^{1-x_0} \ge \prod_{j=1}^{d-1}\left(1+\frac{1}{\lambda^{j-1}}\right)^{x_j}\left(k-2x_0-2\sum_{j=1}^{d-1}\frac{x_j}{1+\lambda^{j-1}}\right). \tag{7}$$

Recall that, according to the definition, each value x_j is an integer. Note that the right side of inequality (6) and that of inequality (7) are both at most 0 if $x_1 \ge k-2x_0$; so we may assume that $x_1 \in \{0,\ldots,k-2x_0-1\}$. Let us fix $\lambda = 4$, and prove the stronger statement that the right side of inequality (7), which we call $R7$, is always at most 2^{k-2x_0-1}. This implies both (6) and (7). We define $y_j \coloneqq \dfrac{x_j}{1+\lambda^{j-1}}$, for every $j \in \{1,\ldots,d-1\}$.

- If $x_1 = k-2x_0-1$, then

$$\begin{aligned}
R7 &= 2^{k-2x_0-1}\cdot\prod_{j=2}^{d-1}\left(1+\frac{1}{\lambda^{j-1}}\right)^{x_j}\left(1-2\sum_{j=2}^{d-1}\frac{x_j}{1+\lambda^{j-1}}\right)\\
&= 2^{k-2x_0-1}\cdot\prod_{j=2}^{d-1}\left(1+\frac{1}{\lambda^{j-1}}\right)^{(1+\lambda^{j-1})y_j}\left(1-2\sum_{j=2}^{d-1}y_j\right)\\
&\le 2^{k-2x_0-1}\cdot\prod_{j=2}^{d-1}\left(\frac{3125}{1024}\right)^{y_j}\left(1-2\sum_{j=2}^{d-1}y_j\right) && \text{by Fact 1}\\
&\le 2^{k-2x_0-1}\cdot\left(\frac{3125}{1024}\right)^{y}(1-2y) && \text{where } y \coloneqq \sum_{j=2}^{d-1}y_j\\
&\le 2^{k-2x_0-1}\cdot\max_{y\in\mathbb{R}^+}\left(\frac{3125}{1024}\right)^{y}(1-2y)\\
&\le 2^{k-2x_0-1} && \text{by Fact 2.}
\end{aligned}$$

- If $x_1 = k-2x_0-2$, then

$$R7 = 2^{k-2x_0-2}\cdot\prod_{j=2}^{d-1}\left(1+\frac{1}{\lambda^{j-1}}\right)^{x_j}\left(2-2\sum_{j=2}^{d-1}\frac{x_j}{1+\lambda^{j-1}}\right)$$

  If $x_j = 0$ for every $j \in \{2,\ldots,d-1\}$, then $R7 \le 2^{k-2x_0-1}$. Let us now assume otherwise, and set $j_0 \coloneqq \min\{j : x_j > 0\}$. In particular $x_{j_0} \ge 1$ and $y_{j_0} \ge \frac{1}{1+\lambda^{j_0-1}}$. Then

$$
\begin{aligned}
R7 &= 2^{k-2x_0-2} \cdot \prod_{j=j_0}^{d-1} \left(1+\frac{1}{\lambda^{j-1}}\right)^{x_j} \left(2 - 2\sum_{j=j_0}^{d-1} \frac{x_j}{1+\lambda^{j-1}}\right) \\
&= 2^{k-2x_0-2} \cdot \prod_{j=j_0}^{d-1} \left(1+\frac{1}{\lambda^{j-1}}\right)^{(1+\lambda^{j-1})y_j} \left(2 - 2\sum_{j=j_0}^{d-1} y_j\right) \\
&\leq 2^{k-2x_0-2} \cdot \prod_{j=j_0}^{d-1} \left(1+\frac{1}{\lambda^{j_0-1}}\right)^{(1+\lambda^{j_0-1})y_j} \left(2 - 2\sum_{j=j_0}^{d-1} y_j\right) && \text{by Fact 1} \\
&\leq 2^{k-2x_0-2} \cdot \left(1+\frac{1}{\lambda^{j_0-1}}\right)^{(1+\lambda^{j_0-1})y} (2-2y) && \text{where } y \coloneqq \sum_{j=j_0}^{d-1} y_j \geq \frac{1}{1+\lambda^{j_0-1}} \\
&\leq 2^{k-2x_0-2} \cdot \max_{y \geq \frac{1}{1+\lambda^{j_0-1}}} \left(1+\frac{1}{\lambda^{j_0-1}}\right)^{(1+\lambda^{j_0-1})y} (2-2y) \\
&\leq 2^{k-2x_0-1} && \text{by Fact 2.}
\end{aligned}
$$

- If $x_1 \leq k - 2x_0 - 3$, then

$$
\begin{aligned}
R7 &= 2^{x_1} \cdot \prod_{j=2}^{d-1} \left(1+\frac{1}{\lambda^{j-1}}\right)^{x_j} \left(k - 2x_0 - x_1 - 2\sum_{j=2}^{d-1} \frac{x_j}{1+\lambda^{j-1}}\right) \\
&= 2^{x_1} \cdot \prod_{j=2}^{d-1} \left(1+\frac{1}{\lambda^{j-1}}\right)^{(1+\lambda^{j-1})y_j} \left(k - 2x_0 - x_1 - 2\sum_{j=2}^{d-1} y_j\right) \\
&\leq 2^{x_1} \cdot \prod_{j=2}^{d-1} \left(\frac{3125}{1024}\right)^{y_j} \left(k - 2x_0 - x_1 - 2\sum_{j=2}^{d-1} y_j\right) && \text{by Fact 1} \\
&\leq 2^{x_1} \cdot \left(\frac{3125}{1024}\right)^{y} (k - 2x_0 - x_1 - 2y) && \text{where } y \coloneqq \sum_{j=2}^{d-1} y_j \\
&\leq 2^{x_1} \cdot \max_{y \in \mathbb{R}} \left(\frac{3125}{1024}\right)^{y} (k - 2x_0 - x_1 - 2y) \\
&\leq 2^{x_1} \frac{2\left(\frac{3125}{1024}\right)^{\frac{k-2x_0-x_1}{2}}}{e \ln\left(\frac{3125}{1024}\right)} && \text{by Fact 2} \\
&\leq 2^{k-2x_0-1} && \text{as } (k-2x_0-1)/2 > 1.
\end{aligned}
$$

We have shown that when $\lambda = 4$,

$$
\frac{2^{k-3}+k}{k} \mathbb{E}\left[\mathbf{X}_0\right] + \frac{2}{k} \mathbb{E}\left[\mathbf{X}_1\right] \geq 1.
$$

$\square$

We set $\lambda \coloneqq 4$, and apply Lemma 1 with $\left(\alpha_0(v), \alpha_1(v), \alpha_2(v)\right) = \left(\frac{2^{k(v)-3}+k(v)}{k(v)}, \frac{2}{k(v)}, 0\right)$ for every vertex $v \in V(G)$, where $k(v)$ is chosen such that $\frac{2\deg(v)+2^{k-3}+k}{k}$ is minimised in $k = k(v)$, and is always at least 4 since $\deg(v)$ is a non-negative integer. This ends the proof of Theorem 7.

4. Bounds on the inverse independence ratio

We focus on establishing upper bounds on the inverse independence ratios of graphs with bounded maximum degree and girth. These bounds are obtained by using the uniform distribution on $\mathcal{S}_\alpha(G)$, for G in the considered class of graphs, in Lemma 2 or Lemma 3.

4.1. Structural analysis of a neighbourhood.

We start by introducing some terminology.

Definition 1.

(1) A *pattern of depth* r is any graph G given with a root vertex v such that

$$\forall u \in V(G), \quad \mathrm{dist}_G(u,v) \leq r.$$

(2) A pattern P of depth r and root v is *d-regular* if it has maximum degree d and every vertex at distance at most $r-2$ from v in P has degree d.

Definition 2. For a given pattern P with root v, we let $W_i \coloneqq \{\, u \in V(P) \mid \mathrm{dist}_P(u,v) = i\}$. Let $\mathbf{S}$ be a maximum independent set chosen uniformly at random. We set $\mathbf{X}_i \coloneqq \mathbf{S} \cap W_i$ and $e_i(P) \coloneqq \mathbb{E}\left[|\mathbf{X}_i|\right]$ for each $i \in \{0,\dots,r\}$.

(1) The *constraint* associated to the pattern P of depth r is the vector

$$\mathbf{e}(P) \coloneqq (e_0(P),\dots,e_r(P)) \in \left(\mathbb{Q}^+\right)^{r+1}.$$

The *cardinality* $n_{\mathbf{e}(P)}$ of the constraint $\mathbf{e}(P)$ is the number of maximum independent sets of P.

(2) Given two constraints $\mathbf{e},\mathbf{e}' \in \left(\mathbb{Q}^+\right)^{r+1}$, we say that $\mathbf{e}$ is *weaker* than $\mathbf{e}'$ if, for any vector $\alpha \in \left(\mathbb{Q}^+\right)^{r+1}$ it holds that

$$\alpha^\top \mathbf{e}' \geq 1 \quad \Rightarrow \quad \alpha^\top \mathbf{e} \geq 1.$$

If the above condition holds only for all vectors $\alpha \in \left(\mathbb{Q}^+\right)^{r+1}$ with non-increasing coordinates, then we say that $\mathbf{e}$ is *relatively weaker* than $\mathbf{e}'$.

Note that $\mathbf{e}$ is weaker than $\mathbf{e}'$ if and only if

$$\forall i \in \{0,\dots,r\}, \quad e_i \geq e'_i,$$

and $\mathbf{e}$ is relatively weaker than $\mathbf{e}'$ if and only if

$$\forall i \in \{0,\dots,r\}, \quad \sum_{j=0}^{i} e_j \geq \sum_{j=0}^{i} e'_j.$$

Remark 2. Let P be a pattern such that one of its vertices u is adjacent with some leaves $u_1,\dots,u_k$ where $k \geq 2$. Then every maximum independent set of P contains $\{u_1,\dots u_k\}$ and not u. Consequently, $\mathbf{e}(P)$ is weaker than $\mathbf{e}(P \setminus \{u_3,\dots,u_k\})$ since, letting i be the distance of u_1 to the root of P, one has

$$e_j(P) = \begin{cases} e_j(P \setminus \{u_3,\dots,u_k\}) & \text{if } j \neq i, \text{ and} \\ e_i(P) = e_i(P \setminus \{u_3,\dots,u_k\}) + (k-2) & \text{if } j = i. \end{cases}$$

4.2. Tree-like patterns.

4.2.1. *Rooting at a vertex.* Fix a depth $r \geq 2$. Let G be a d-regular graph of girth at least $2r+2$, and let $\mathbf{S} \in \mathcal{S}_\alpha(G)$ be a maximum independent set drawn uniformly at random. For any fixed vertex v, we set $\mathbf{R} \coloneqq \mathbf{S} \setminus N^r[v]$, and $\mathbf{X_i}(v) \coloneqq \mathbf{S} \cap N^i(v)$, for each $i \in \{0, \dots, r\}$. Finally, we set $\mathbf{W} \coloneqq N^r[v] \setminus N(\mathbf{R})$. So $\mathbf{R}$ is the set of vertices in $\mathbf{S}$ at distance at least $r+1$ from v, and $\mathbf{W}$ is the set of vertices at distance at most r from v uncovered by $\mathbf{R}$. In particular, we know that $\mathbf{S} \cap N^r[v] \subseteq \mathbf{W}$.

Because $\mathbf{S}$ is a maximum independent set of G, it holds that $\mathbf{S} \cap N^r[v]$ is a maximum independent set of $G[\mathbf{W}]$. Conversely, if S_W is a maximum independent set of $G[\mathbf{W}]$, then $\mathbf{R} \cup S_W$ is a maximum independent set of S. Thus, for any independent set R of $G - N^r[v]$, if one conditions on the fact that $\mathbf{R} = R$, then $\mathbf{S} \cap N^r[v]$ is a uniform random independent set of $G[\mathbf{W}]$. The subgraph $G[\mathbf{W}]$ is a d-regular pattern of depth r with root vertex v, and since G has girth at least $2r+2$, it follows that $G[\mathbf{W}]$ is a tree. Let $\mathcal{T}_r(d)$ be the set of acyclic d-regular patterns of depth r.

We seek parameters $(\alpha_i)_{i \leq r}$ such that the inequality $\sum_{i=0}^{r} \alpha_i \mathbb{E}\left[|\mathbf{X}_i(v)|\right] \geq 1$ is satisfied regardless of the choice of v. To this end, it is enough to pick the rational numbers α_is in such a way that the inequality is satisfied in any tree $T \in \mathcal{T}_r(d)$, when v is the root vertex. In a more formal way, given any $T \in \mathcal{T}_r(d)$, the vector $\alpha = (\alpha_0, \dots, \alpha_r)$ must be compatible with the constraint $\mathbf{e}(T)$, that is, $\alpha^\top \mathbf{e}(T) \geq 1$ for each $T \in \mathcal{T}_r(d)$.

An application of Lemma 2 then lets us conclude that the desired bound is the solution to the following linear program.

$$\frac{|G|}{\alpha(G)} \leq \min \quad \alpha_0 + \sum_{i=1}^{r} \alpha_i d(d-1)^{i-1} \tag{8}$$
$$\text{such that} \begin{cases} \forall T \in \mathcal{T}_r(d), & \sum_{i=0}^{r} \alpha_i e_i(T) \geq 1, \\ \forall i \leq r, & \alpha_i \geq 0. \end{cases}$$

The end of the proof is made by computer generation of $\mathcal{T}_r(d)$, in order to generate the desired linear program, which is then solved again by computer computation. For the sake of illustration, we give a complete human proof of the case where $r = 2$ and $d = 3$. There are 10 trees in $\mathcal{T}_2(3)$. One can easily compute the constraint $(e_0(T), e_1(T), e_2(T))$ for each $T \in \mathcal{T}_2(3)$; they are depicted in Figure 5. Note that constraints $\mathbf{e}_8$, $\mathbf{e}_9$ and $\mathbf{e}_{10}$ are weaker than constraint $\mathbf{e}_7$, so we may disregard these constraints in the linear program to solve. Note also that constraint $\mathbf{e}_0$ is relatively weaker than constraint $\mathbf{e}_1$, and so may be disregarded as well, provided that the solution of the linear program is attained by a vector α with non-increasing coordinates, which will have to be checked. The linear program to solve is therefore the following.

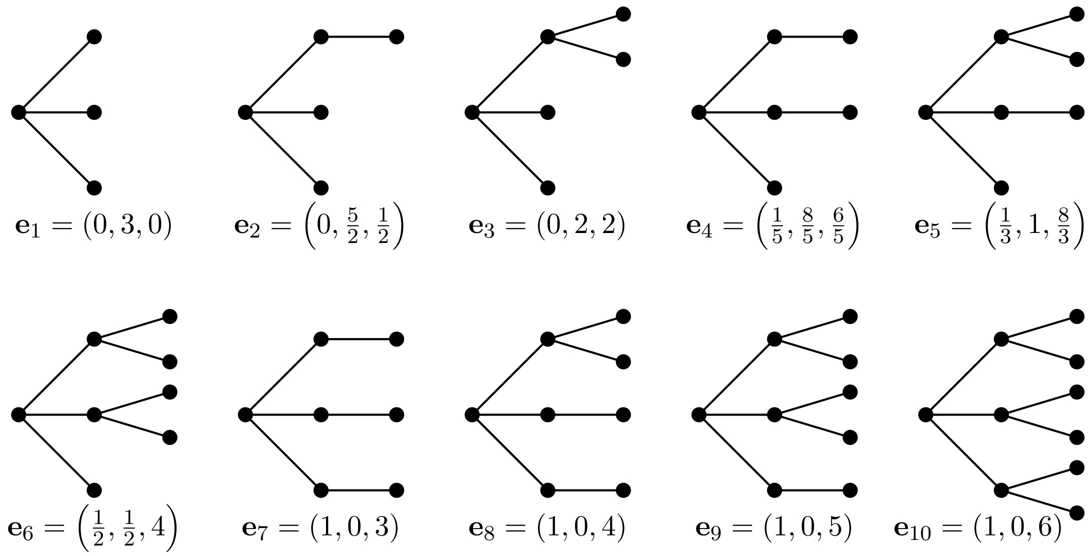

FIGURE 5. An enumeration of $\mathbf{e}(T)$ for all trees $T \in \mathcal{T}_2(3)$

$$
\begin{array}{ll}
\text{minimise} & \alpha_0 + 3\alpha_1 + 6\alpha_2 \\
\text{such that} & \begin{cases}
\frac{5}{2}\alpha_1 + \frac{1}{2}\alpha_2 & \geq \quad 1, \\
2\alpha_1 + 2\alpha_2 & \geq \quad 1, \\
\frac{1}{5}\alpha_0 + \frac{8}{5}\alpha_1 + \frac{6}{5}\alpha_2 & \geq \quad 1, \\
\frac{1}{3}\alpha_0 + \alpha_1 + \frac{8}{3}\alpha_2 & \geq \quad 1, \\
\frac{1}{2}\alpha_0 + \frac{1}{2}\alpha_1 + 4\alpha_2 & \geq \quad 1, \\
\alpha_0 + 3\alpha_2 & \geq \quad 1, \\
\forall i \in \{0,1,2\}, \quad \alpha_i & \geq \quad 0.
\end{cases}
\end{array}
$$

The solution of this linear program is $\frac{85}{31} \approx 2.741935$, attained by $\alpha = \left(\frac{19}{31}, \frac{14}{31}, \frac{4}{31}\right)$, which indeed has non-increasing coordinates. This is an upper bound on $i(3,6)^{-1}$, though we prove a stronger one through a more involved computation in Section 4.2.3.

4.2.2. *Inductive computation of the vectors* $\mathbf{e}(T)$. To compute $\mathbf{e}(T)$ for each $T \in \mathcal{T}_r(d)$, one can enumerate all the maximum independent sets of T and average the size of their intersection with each layer of T. For general graphs, there might be no better way of doing so, however the case of $\mathcal{T}_r(d)$ can be treated inductively by a standard approach: we distinguish between the maximum independent sets that contain the root and those that do not. We introduce the following notation.

Definition 3. Let $\mathbf{e}$ and $\mathbf{e}'$ be two vectors in $\left(\mathbb{Q}^+\right)^{r+1}$ where r is a positive integer. The *wedge* of $\mathbf{e}$ and $\mathbf{e}'$ is the vector $\mathbf{e} \vee \mathbf{e}' \in \left(\mathbb{Q}^+\right)^{r+1}$ given by

$$\mathbf{e} \vee \mathbf{e} \coloneqq \begin{cases} \dfrac{n_{\mathbf{e}}}{n_{\mathbf{e}} + n_{\mathbf{e}'}}\mathbf{e} + \dfrac{n_{\mathbf{e}'}}{n_{\mathbf{e}} + n_{\mathbf{e}'}}\mathbf{e}' & \text{if } \|\mathbf{e}\|_1 = \|\mathbf{e}'\|_1, \\ \mathbf{e} & \text{if } \|\mathbf{e}\|_1 > \|\mathbf{e}'\|_1, \\ \mathbf{e}' & \text{if } \|\mathbf{e}\|_1 < \|\mathbf{e}'\|_1. \end{cases}$$

The *cardinality* $n_{\mathbf{e}\vee\mathbf{e}'}$ of the wedge of $\mathbf{e}$ and $\mathbf{e}'$ is defined to be

$$n_{\mathbf{e}\vee\mathbf{e}'} \coloneqq \begin{cases} n_{\mathbf{e}} + n_{\mathbf{e}'} & \text{if } \|\mathbf{e}\|_1 = \|\mathbf{e}'\|_1, \\ n_{\mathbf{e}} & \text{if } \|\mathbf{e}\|_1 > \|\mathbf{e}'\|_1, \\ n_{\mathbf{e}'} & \text{if } \|\mathbf{e}\|_1 < \|\mathbf{e}'\|_1. \end{cases}$$

For a given tree $T \in \mathcal{T}_r(d)$ with root v, let $\mathbf{e}_0(T)$, (respectively $\mathbf{e}_1(T)$), be the vectors with values $\left(\mathbb{E}\left[|\mathbf{S}_T \cap W_i|\right]\right)_{i\le r}$ where $\mathbf{S}_T$ is a uniform random maximum independent set of T given that $v \notin \mathbf{S}_T$, (respectively $v \in \mathbf{S}_T$). It readily follows from Definition 3 that

$$\mathbf{e}(T) = \mathbf{e}_0(T) \vee \mathbf{e}_1(T).$$

Furthermore, the cardinality of $\mathbf{e}(T)$, that is the number of maximum independent sets in T, is exactly the cardinality of $n_{\mathbf{e}_0\vee\mathbf{e}_1}$.
We also need the following concept.

Definition 4. Let $\mathbf{e}$ and $\mathbf{e}'$ be two elements of $\left(\mathbb{Q}^+\right)^{r+1}$ where r is a positive integer. The *sum* of $\mathbf{e}$ and $\mathbf{e}'$ is the vector $\mathbf{e} \oplus \mathbf{e}' \in \left(\mathbb{Q}^+\right)^{r+1}$ given by

$$\mathbf{e} \oplus \mathbf{e}' \coloneqq \Big(e_0 + e'_0, \ldots, e_r + e'_r\Big).$$

The *cardinality* $n_{\mathbf{e}\oplus\mathbf{e}'}$ of the sum of $\mathbf{e}$ and $\mathbf{e}'$ is defined to be

$$n_{\mathbf{e}\oplus\mathbf{e}'} \coloneqq n_{\mathbf{e}} n_{\mathbf{e}'}.$$

If $T_1, \ldots, T_d$ are the subtrees of T rooted at the children of the root v (some of which might be empty), then it holds that

$$\mathbf{e}_0(T) = \left(0, \quad \bigoplus_{i=1}^{d} \mathbf{e}(T_i)\right) \quad \text{and} \quad \mathbf{e}_1(T) = \left(1, \quad \bigoplus_{i=1}^{d} \mathbf{e}_0(T_i)\right).$$

Furthermore, the cardinality of $\mathbf{e}_0(T)$ is indeed the product of the cardinalities $n_{\mathbf{e}(T_i)}$ for $i \in \{1, \ldots, d\}$, and the cardinality of $\mathbf{e}_1(T)$ is the product of the cardinalities $n_{\mathbf{e}_0(T_i)}$ for $i \in \{1, \ldots, d\}$.

We thus obtain an inductive way of computing $\mathbf{e}(T)$ by using the following initial values.

$$\begin{aligned} \mathbf{e}_0(\varnothing) &\coloneqq (0), & n_{\mathbf{e}_0(\varnothing)} &\coloneqq 1 \\ \mathbf{e}_1(\varnothing) &\coloneqq (0), & n_{\mathbf{e}_1(\varnothing)} &\coloneqq 0 \\ \mathbf{e}_0(\{v\}) &\coloneqq (0), & n_{\mathbf{e}_0(\{v\})} &\coloneqq 1 \\ \mathbf{e}_1(\{v\}) &\coloneqq (1), & n_{\mathbf{e}_1(\{v\})} &\coloneqq 1. \end{aligned}$$

Following the enumeration of the vectors $\mathbf{e}(T)$ for $T \in \mathcal{T}_r(d)$ described in Section 4.2.2, the following statement is obtained by computer calculus.

Lemma 6. *The solution to the linear program* (8) *is*

$$\mathcal{T}_3(3)\colon \quad \frac{5849}{2228} \approx 2.625224 \qquad \textit{with } \alpha = \left(\frac{953}{2228}, \frac{162}{557}, \frac{81}{557}, \frac{21}{557}\right),$$

$$\mathcal{T}_4(3)\colon \quad \frac{2098873192}{820777797} \approx 2.557176 \qquad \textit{with } \alpha = \Big(\frac{225822361}{820777797}, \frac{18575757}{91197533}, \frac{10597368}{91197533}, \frac{5054976}{91197533}, \frac{1172732}{91197533}\Big),$$

$$\mathcal{T}_5(3)\colon \quad \frac{29727802051155412}{11841961450578397} \approx 2.510378 \quad \textit{with } \alpha = \Big(\frac{3027359065168972}{11841961450578397}, \frac{2216425114872980}{11841961450578397}, \frac{2224040336719575}{23683922901156794}, \frac{2026654050681425}{47367845802313588}, \frac{403660478424775}{23683922901156794}, \frac{51149140376400}{11841961450578397}\Big),$$

$$\mathcal{T}_3(4)\colon \quad \frac{7083927}{2331392} \approx 3.038497 \qquad \textit{with } \alpha = \left(\frac{123345}{333056}, \frac{68295}{291424}, \frac{12283}{145712}, \frac{2911}{145712}\right),$$

$$\mathcal{T}_4(4)\colon \quad 3 \qquad \textit{with } \alpha = \left(\frac{7}{43}, \frac{6}{43}, \frac{19}{258}, \frac{7}{258}, \frac{1}{258}\right),$$

$$\mathcal{T}_2(5)\colon \quad \frac{69}{19} \approx 3.631579 \qquad \textit{with } \alpha = \left(\frac{37}{57}, \frac{6}{19}, \frac{4}{57}\right),$$

$$\mathcal{T}_3(5)\colon \quad \frac{7}{2} = 3.5 \qquad \textit{with } \alpha = \left(\frac{77}{282}, \frac{25}{141}, \frac{17}{282}, \frac{2}{141}\right).$$

4.2.3. *Rooting in an edge.* Definition 1 can be extended to a pattern with a root-edge instead of a root-vertex. The distance in a pattern P between a vertex w and an edge uv is defined to be $\min\{\mathrm{dist}_P(w,u), \mathrm{dist}_P(w,v)\}$. The depth of a pattern P rooted in an edge e is then the largest distance between e and a vertex in P. It is then possible to follow the same analysis as in Section 4.2.1 with edge-rooted patterns: in order for the edge-rooted pattern of depth r to always be a tree, the graph G must have girth at least $2r+3$. Let $\mathcal{T}'_r(d)$ be the set of acyclic edge-rooted d-regular patterns of depth r. By Lemma 3, the linear program to solve is now the following.

$$\frac{|G|}{\alpha(G)} \leq \min \quad 2\sum_{i=0}^{r} \alpha_i (d-1)^i \tag{9}$$

$$\text{such that} \begin{cases} \forall T \in \mathcal{T}'_r(d), & \displaystyle\sum_{i=0}^{r} \alpha_i e_i(T) \geq 1, \\ \forall i \leq r, & \alpha_i \geq 0. \end{cases}$$

For a given tree $T \in \mathcal{T}'_r(d)$ rooted in $e = uv$, it is possible to compute $\mathbf{e}(T)$ using the constraints associated to vertex-rooted trees. If T_u and T_v are the subtrees of T respectively rooted at u and at v, then it readily follows from Definitions 3 and 4 that

$$\mathbf{e}(T) = \Big(\mathbf{e}_0(T_u) \oplus \mathbf{e}_0(T_v)\Big) \vee \Big(\mathbf{e}_0(T_u) \oplus \mathbf{e}_1(T_v)\Big) \vee \Big(\mathbf{e}_1(T_u) \oplus \mathbf{e}_0(T_v)\Big). \tag{10}$$

Moreover, the cardinality of the constraint $\mathbf{e}(T)$, that is, the number of maximum independent sets of T, is precisely the cardinality of the vector obtained in the right side of (10).

Following the enumeration of the vectors $\mathbf{e}(T)$ for $T \in \mathcal{T}'_r(d)$ described earlier, the next statement is obtained by computer calculus.

Lemma 7. *The solution to the linear program* (9) *is*

$$\begin{array}{lll}
\mathcal{T}_2'(3): & \dfrac{30}{11} \approx 2.72727 & \text{with } \alpha = \left(\dfrac{1}{2}, \dfrac{13}{44}, \dfrac{3}{44}\right); \\
\mathcal{T}_3'(3): & \dfrac{125}{48} \approx 2.604167 & \text{with } \alpha = \left(\dfrac{11}{32}, \dfrac{5}{24}, \dfrac{3}{32}, \dfrac{1}{48}\right); \\
\mathcal{T}_4'(3): & \dfrac{14147193}{5571665} \approx 2.539132 & \text{with } \alpha = \left(\dfrac{98057}{506515}, \dfrac{159348}{1114333}, \dfrac{3688469}{44573320}, \dfrac{1752117}{44573320}, \dfrac{402569}{44573320}\right); \\
\mathcal{T}_2'(4): & \dfrac{41}{13} \approx 3.153846 & \text{with } \alpha = \left(\dfrac{11}{26}, \dfrac{3}{13}, \dfrac{2}{39}\right); \\
\mathcal{T}_3'(4): & \dfrac{127937}{42400} \approx 3.017382 & \text{with } \alpha = \left(\dfrac{5539}{16960}, \dfrac{1737}{10600}, \dfrac{257}{5300}, \dfrac{399}{42400}\right); \\
\mathcal{T}_2'(5): & \dfrac{18}{5} = 3.6 & \text{with } \alpha = \left(\dfrac{17}{45}, \dfrac{8}{45}, \dfrac{2}{45}\right).
\end{array}$$

The bounds obtained in Lemma 7 are valid for graphs of girth at least $2r+3$. It turns out that the same bounds, with the same α, remain valid for graphs of girth $2r+2=6$, when $r=2$ and $d \in \{3,4\}$. We were not able to check this for higher values of r or d, but we propose the following conjecture which would explain and generalise this phenomenon.

Conjecture 2. *Let P be a d-regular edge-rooted pattern of depth r and of girth $2r+2$. Then the constraint $\mathbf{e}(P)$ is weaker than some convex combination of constraints $\mathbf{e}(T)$ with $T \in \mathcal{T}_r'(d)$. More formally, there exist $T_1, \ldots, T_m \in \mathcal{T}_r'(d)$ and $\lambda_1, \ldots, \lambda_m \in [0,1]$ with $\sum_{i=1}^m \lambda_i = 1$ such that for any $\alpha \in (\mathbb{Q}^+)^{r+1}$,*

$$\alpha^\top \left(\sum_{i=1}^m \lambda_i \mathbf{e}(T_i)\right) \geq 1 \quad \Rightarrow \quad \alpha^\top \mathbf{e}(P) \geq 1.$$

4.3. **More complicated patterns.**

4.3.1. *Rooting at a vertex.* Let us fix a depth $r \geq 2$. Let G be a d-regular graph of girth $g \leq 2r+1$. We repeat the same analysis as in Section 4.2.1: we end up having to find a vector $\alpha \in \mathbb{Q}^{r+1}$ compatible with all the constraints generated by vertex-rooted d-regular patterns of depth r and girth g. Letting $\mathcal{P}_r(d,g)$ be the set of such patterns, we thus want that

$$\forall P \in \mathcal{P}_r(d,g), \quad \alpha^\top \mathbf{e}(P) \geq 1.$$

In this setting, we could do no better than performing an exhaustive enumeration of every possible pattern $P \in \mathcal{P}_r(d,g)$, and computing the associated constraint $\mathbf{e}(P)$ through an exhaustive enumeration of $\mathcal{S}_\alpha(P)$. The complexity of such a process grows fast, and we considered only depth $r \leq 2$ and degree $d \leq 4$. Since the largest value of the inverse independence ratio over the class of 3-regular graphs of girth 4 or 5 is known to be $\frac{14}{5} = 2.8$, and the one of 4-regular graphs of girth 4 is known to be $\frac{13}{4} = 3.25$, the only open value in these settings is for the class of 4-regular graphs of girth 5. Unfortunately, this method is not powerful enough to prove an upper bound lower than $\frac{13}{4}$, the obtained bound for $\mathcal{P}_2(4,5)$ being $\frac{82}{25} = 3.28$. It is more interesting to root the patterns in an edge.

4.3.2. *Rooting in an edge.* Similarly, we define $\mathcal{P}_r'(d,g)$ to be the set of edge-rooted d-regular patterns of girth g. For fixed r and g, we seek for the solution of the following linear program.

$$
\frac{|G|}{\alpha(G)} \leq \min \quad 2\sum_{i=0}^{r} \alpha_i (d-1)^i \tag{11}
$$

$$
\text{such that} \begin{cases} \forall P \in \mathcal{P}'_r(d,g), & \sum_{i=0}^{r} \alpha_i e_i(P) \geq 1, \\ \forall i \leq r, & \alpha_i \geq 0. \end{cases}
$$

Again, our computations were limited to the cases where $r \leq 2$ and $d \leq 4$. However, we managed to prove improved bounds for girth 6 when $d \in \{3, 4\}$, which seems to support Conjecture 2.

Lemma 8. *The solution to the linear program* (11) *is*

$$
\begin{aligned}
\mathcal{P}'_2(3,6): \qquad & \frac{30}{11} \approx 2.72727 \qquad && \textit{with } \alpha = \left(\frac{1}{2}, \frac{13}{44}, \frac{3}{44}\right), \\
\mathcal{P}'_2(4,6): \qquad & \frac{41}{13} \approx 3.153846 \qquad && \textit{with } \alpha = \left(\frac{11}{26}, \frac{3}{13}, \frac{2}{39}\right).
\end{aligned}
$$

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