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CYRILLE KENNE†, GÜNTER LEUGERING‡, AND GISÈLE MOPHOU§

Abstract. We consider a model of population dynamics with age dependence and spatial structure but unknown birth rate. Using the notion of Low-regret [9], we prove that we can bring the state of the system to a desired state by acting on the system via a localized distributed control. We provide the optimality systems that characterize the Low-regret control. Moreover, using an appropriate Hilbert space, we prove that the family of Low-regret controls tends to a so-called No-regret control, which we, in turn, characterize.

Key words. Population dynamics, incomplete data, optimal control, No-regret control, Low-regret control, Euler-Lagrange formula.

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1. Introduction.

We consider a population with age dependence and spatial structure, and we assume that the population lives in a bounded domain $\Omega \subset \mathbb{R}^3$, with boundary Γ of class C^2. We denote by $y = y(t, a, x)$, the distribution of individuals of age $a \geq 0$, at time $t \geq 0$ and location $x \in \Omega$. For the time $T > 0$ and the life expectancy of an individual $A > 0$, we set $U = (0, T) \times (0, A)$, $Q = U \times \Omega$, $\Sigma = U \times \Gamma$, $Q_A = (0, A) \times \Omega$, $Q_T = (0, T) \times \Omega$ and $Q_\omega = U \times \omega$, where ω is a non-empty open subset of Ω. We consider a model describing the dynamics of a population with age dependence and spatial structure:

$$\text{(1.1)} \qquad \left\{ \begin{array}{rcll} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - \Delta y + \mu y &=& f + v\chi_\omega & \text{in} \quad Q, \\ y &=& 0 & \text{on} \quad \Sigma, \\ y(0, \cdot, \cdot) &=& y^0 & \text{in} \quad Q_A, \\ y(\cdot, 0, \cdot) &=& \int_0^A g(a) y(t, a, x)\, da & \text{in} \quad Q_T, \end{array} \right.$$

where $y^0 \in L^2(Q_A)$, $f \in L^2(Q)$, the control $v \in L^2(Q)$ and χ_ω denote the characteristic function of the control set ω. The mortality rate $\mu = \mu(a) \geq 0$ is known and continuous on $[0, A]$, whereas the fertility rate $g = g(a) \in L^\infty(0, A)$ is unknown and positive. We assume, as in [1], that:

$$(H_1) \qquad \lim_{a \to A} \int_0^a \mu(s) ds = +\infty,$$

which means that each individual in the population dies before age A. For more literature on the population dynamics model and the signification of assumption (H_1), we refer to [1, 2, 3, 4, 5, 6, 7] and the reference therein.

Remark 1.1. Set

$$\text{(1.2)} \qquad W(T, A) = \left\{ \rho \in L^2(U; H_0^1(\Omega)); \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} \in L^2\left(U; H^{-1}(\Omega)\right) \right\}.$$

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Then we have (see [8]) that

$$W(T,A) \subset \mathcal{C}([0,T], L^2(Q_A)) \text{ and } W(T,A) \subset \mathcal{C}([0,A], L^2(Q_T)). \tag{1.3}$$

Under the assumption on the data, (1.1) has a solution $y(v,g) = y(t,a,x;v,g)$ in $W(T,A)$. We define the cost function

$$J(v,g) = \|y(v,g) - z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q_\omega)}, \tag{1.4}$$

where $z_d \in L^2(Q)$ and $N > 0$ are given. We want to solve the following optimization problem:

$$\inf_{v\in L^2(Q_\omega)} \sup_{g\in L^2(0,A)} J(v,g).$$

But observing that we could have $\sup\limits_{g\in L^2(0,A)} J(v,g) = +\infty$, we consider the optimization problem:

$$\inf_{v\in L^2(Q_\omega)} \sup_{g\in L^2(0,A)} (J(v,g) - J(0,0)). \tag{1.5}$$

Problem (1.5) is called No-regret control problem. The notions of No-regret control and Low-regret control were introduced by J. -L. Lions [9] in order to control a phenomenon described by a parabolic equation with missing initial condition. Let us recall that one obtains the Low-regret control problem by relaxing the No-regret control one. See (3.20),(3.21),(3.22) for the relaxation used in this article. The Low-regret control problem is a family of classical optimal control problems. The most difficult task is to prove that this family of controls (called Low-regret controls) converges towards the No-regret control. Also in [10], J. -L. Lions proved that these notion can be used in the framework of decomposition methods. In [11], O. Nakoulima et *al.* applied this notion to linear evolution equations with incomplete data and they proved that the Low-regret controls converges to the No-regret control for which they obtained a singular optimality system. B. Jacob et *al.* [12] generalized the notion of No-regret controls to population dynamics with incomplete initial data with a distributed control acting on the whole domain. They proved the existence and uniqueness of the No-regret control and gave a singular optimality system that characterizes this control. In the nonlinear case, this notion was considered by O. Nakoulima et *al.* [13] to control on the whole domain a nonlinear system with incomplete data. Observing on the one hand that the No-regret control is typically not easy to characterize and, on the other hand that the Low-regret cost function may not be convex, they proved by adapting this cost to a No-regret control that the adapted Low-regret controls converge towards this No-regret control characterized by a singular optimality system. In [14], J. Vélin studied systems governed by quasilinear equations with unknown boundary condition and a control acting on the whole domain. After established some regularity results for the control-to-state and control-perturbation applications and its derivatives, he proved by proceeding as in [13] that the adapted Low-regret controls converge towards a No-regret control characterized gain by a singular optimality. Note that in the above papers, the convergence of the Low-regret controls towards the No-regret control is obtained by controlling on the whole domain.

In this paper, we use the notion of No-regret and Low-regret to control a model describing the dynamics of population with age dependence and spatial structure with missing birth rate by acting on a part of the domain. Observing that with an unknown birth rate, the control problem considered is now non-linear, we start by proving some regularity results. Then we prove the existence of a No-regret control. We then regularize the No-regret control problem to a Low-regret control problem ((3.20),(3.21),(3.22)). We introduce an appropriate Hilbert space to obtain estimates on the states satisfying the optimality systems and by that characterize the Low-regret control. Then we prove that the adapted Low-regret controls converges towards a No-regret control and establish a singular optimality system that, in turn, characterizes this no-regret control.

The rest of this paper is structured as follows. In section 2, we give some regularity results. We study the Low-regret and no-regret control and their characterizations in section 3. A conclusion is given in section 4.

2. Preliminary results. In order to solve the optimization problem (3.2), we need some preliminary results.

In what follows, we adopt the following notation

$$\left\{ \begin{array}{lcl} L & = & \dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a} - \Delta + \mu I, \\ L^* & = & -\dfrac{\partial}{\partial t} - \dfrac{\partial}{\partial a} - \Delta + \mu I, \end{array} \right. \tag{2.1}$$

where I is the identity operator.

PROPOSITION 2.1. *Let $y = y(v, g)$ be a solution of (1.1). Then the application $(v, g) \mapsto y(v, g)$ is a continuous function from $L^2(Q_\omega) \times L^2(0, A)$ onto $L^2(U, H_0^1(\Omega))$.*

Proof. Let $(v_0, g_0) \in L^2(Q_\omega) \times L^2(0, A)$. We show that $(v, g) \mapsto y(v, g)$ is continuous at (v_0, g_0).

Set $\overline{y} = y(v, g) - y(v_0, g_0)$, then $\overline{y}$ is solution to the problem

$$\left\{ \begin{array}{rcll} L\overline{y} & = & v\chi_{Q_\omega} - v_0\chi_{Q_\omega} & \text{in} \quad Q, \\ \overline{y} & = & 0 & \text{on} \quad \Sigma, \\ \overline{y}(0, \cdot, \cdot) & = & 0 & \text{in} \quad Q_A, \\ \overline{y}(\cdot, 0, \cdot) & = & \eta & \text{in} \quad Q_T, \end{array} \right. \tag{2.2}$$

where for $(t, x) \in Q_T$,

$$\eta(t, x) = \int_0^A [g(a)y(t, a, x; v, g) - g_0(a)y(t, a, x; v_0, g_0)]\, da.$$

If we set $z = e^{-rt}\overline{y}$ with $r > 0$, then we obtain that z is solution to the problem

$$\left\{ \begin{array}{rcll} Lz + rz & = & v\chi_{Q_\omega} - v_0\chi_{Q_\omega} & \text{in} \quad Q, \\ z & = & 0 & \text{on} \quad \Sigma, \\ z(0, \cdot, \cdot) & = & 0 & \text{in} \quad Q_A, \\ z(\cdot, 0, \cdot) & = & e^{-rt}\eta & \text{in} \quad Q_T. \end{array} \right. \tag{2.3}$$

Multiplying the first equation of system (2.3) by z and integrating by parts over Q, we get

$$\begin{array}{rcl} \displaystyle\int_{Q_\omega} (v - v_0)z \; dxdtda & = & \dfrac{1}{2}\|z(T, \cdot, \cdot)\|^2_{L^2(Q_A)} - \dfrac{1}{2}\|z(0, \cdot, \cdot)\|^2_{L^2(Q_A)} \\ & + & \dfrac{1}{2}\|z(\cdot, A, \cdot)\|^2_{L^2(Q_T)} - \dfrac{1}{2}\|z(\cdot, 0, \cdot)\|^2_{L^2(Q_T)} \\ & + & \|\nabla z\|^2_{L^2(Q)} + \displaystyle\int_Q (r + \mu)z^2. \end{array}$$

From this we deduce that

$$\|\nabla z\|^2_{L^2(Q)} + r\|z\|^2_{L^2(Q)} \leq \frac{1}{2}\|z(\cdot,0,\cdot)\|^2_{L^2(Q_T)} + \frac{1}{2}\|v-v_0\|^2_{L^2(Q_\omega)} + \frac{1}{2}\|z\|^2_{L^2(Q)}, \tag{2.4}$$

because $\mu \geq 0$. On the other hand, observing that for $(t,x) \in Q_T$,

$$\begin{aligned} z(t,0,x) &= e^{-rt}\int_0^A [g(a)y(t,a,x,v,g) - g_0(a)y(t,a,x,v_0,g_0)]\ da \\ &= e^{-rt}\int_0^A [(g(a)-g_0(a))y(t,a,x,v,g)]\ da + \int_0^A g_0(a)z\ da, \end{aligned}$$

we obtain

$$\|z(\cdot,0,\cdot)\|^2_{L^2(Q_T)} \leq 2\|g-g_0\|^2_{L^\infty(0,A)}\|y\|^2_{L^2(Q)} + 2\|g_0\|^2_{L^\infty(0,A)}\|z\|^2_{L^2(Q)}.$$

Thus, (2.4) gives

$$\begin{aligned} \|\nabla z\|^2_{L^2(Q)} + \left(r - \|g_0\|^2_{L^2(0,A)} - \frac{1}{2}\right)\|z\|^2_{L^2(Q)} &\leq \|g-g_0\|^2_{L^2(0,A)}\|y\|^2_{L^2(Q)} \\ &+ \frac{1}{2}\|v-v_0\|^2_{L^2(Q_\omega)}. \end{aligned}$$

Choosing r such that $r > \|g_0\|^2_{L^2(0,A)} + \frac{1}{2}$, we have

$$\|z\|^2_{L^2(U;H^1_0(\Omega))} \leq \|g-g_0\|^2_{L^2(0,A)}\|y\|^2_{L^2(Q)} + \frac{1}{2}\|v-v_0\|^2_{L^2(Q_\omega)}.$$

From this we deduce

$$\|z\|_{L^2(U;H^1_0(\Omega))} \leq \|g-g_0\|_{L^2(0,A)}\|y\|_{L^2(Q)} + \frac{\sqrt{2}}{2}\|v-v_0\|_{L^2(Q_\omega)}.$$

Therefore,

$$\|\overline{y}\|_{L^2(U;H^1_0(\Omega))} \leq e^{rT}\|g-g_0\|_{L^2(0,A)}\|y\|_{L^2(Q)} + \frac{\sqrt{2}}{2}e^{rT}\|v-v_0\|_{L^2(Q_\omega)}.$$

As $(v,g) \to (v_0,g_0)$, we have $\overline{y} \to 0$ strongly in $L^2(U;H^1_0(\Omega))$. Hence $y(v,g) \to y(v_0,g_0)$ strongly in $L^2(U;H^1_0(\Omega))$ as $(v,g) \to (v_0,g_0)$. ∎

PROPOSITION 2.2. *Assume that the assumption of Proposition 2.1 holds true. Let $\lambda > 0$. Let $g,h \in L^2(0,A)$ and $v,w \in L^2(Q_\omega)$. Let also $y = y(v,g)$ be a solution (1.1). Set $\overline{y}_\lambda = \dfrac{y(v+\lambda w, g+\lambda h) - y(v,g)}{\lambda}$. Then $(\overline{y}_\lambda)$ converges strongly in $L^2(U;H^1_0(\Omega))$ as $\lambda \to 0$ to a function $\overline{y}$ solution of*

$$\left\{\begin{array}{rcllr} L\overline{y} &=& w\chi_{Q_\omega} & \text{in} & Q, \\ \overline{y} &=& 0 & \text{on} & \Sigma, \\ \overline{y}(0,\cdot,\cdot) &=& 0 & \text{in} & Q_A, \\ \overline{y}(\cdot,0,\cdot) &=& \int_0^A g(a)\overline{y}\ da + \int_0^A h(a)y(t,a,x;v,g)\ da & \text{in} & Q_T. \end{array}\right. \tag{2.5}$$

Proof. $\overline{y}_\lambda$ is a solution to the problem

$$\left\{\begin{array}{rcllr} L\overline{y}_\lambda &=& w\chi_{Q_\omega} & \text{in} & Q, \\ \overline{y}_\lambda &=& 0 & \text{on} & \Sigma, \\ \overline{y}_\lambda(0,\cdot,\cdot) &=& 0 & \text{in} & Q_A, \\ \overline{y}_\lambda(\cdot,0,\cdot) &=& \displaystyle\int_0^A g(a)\overline{y}_\lambda\ da + \int_0^A h(a)y(t,a,x;v+\lambda w, g+\lambda h)\ da & \text{in} & Q_T. \end{array}\right.$$

Define $y_\lambda = \overline{y}_\lambda - \overline{y}$, where $\overline{y}$ is a solution to (2.5). Then y_λ is a solution to

$$\text{(2.6)} \qquad \left\{ \begin{array}{rcll} Ly_\lambda &=& 0 & \text{in} \quad Q, \\ y_\lambda &=& 0 & \text{on} \quad \Sigma, \\ y_\lambda(0,\cdot,\cdot) &=& 0 & \text{in} \quad Q_A, \\ y_\lambda(\cdot,0,\cdot) &=& \displaystyle\int_0^A g(a) y_\lambda \, da + \eta_1 & \text{in} \quad Q_T, \end{array} \right.$$

where for $(t,x) \in Q_T$,

$$\eta_1(t,x) = \int_0^A h(a) \left[y(t,a,x; v+\lambda w, g+\lambda h) - y(t,a,x;v,g) \right] \, da.$$

We set $z_\lambda = e^{-rt} y_\lambda$ with $r > 0$. Then we obtain that z_λ is a solution to the problem

$$\text{(2.7)} \qquad \left\{ \begin{array}{rcll} Lz_\lambda + r z_\lambda &=& 0 & \text{in} \quad Q, \\ z_\lambda &=& 0 & \text{on} \quad \Sigma, \\ z_\lambda(0,\cdot,\cdot) &=& 0 & \text{in} \quad Q_A, \\ z_\lambda(\cdot,0,\cdot) &=& \int_0^A g(a) z_\lambda \, da + e^{-rt}\eta_1 & \text{in} \quad Q_T. \end{array} \right.$$

Multiplying the first equation of system (2.7) by z_λ and integrating by parts over Q, then using the fact that $\mu \geq 0$, we obtain

$$\text{(2.8)} \qquad \|\nabla z_\lambda\|^2_{L^2(Q)} + r\|z_\lambda\|^2_{L^2(Q)} \leq \frac{1}{2}\|z_\lambda(\cdot,0,\cdot)\|^2_{L^2(Q_T)}.$$

Since

$$\begin{array}{rcl} \|z_\lambda(\cdot,0,\cdot)\|^2_{L^2(Q_T)} &\leq& 2\|g\|^2_{L^2(0,A)}\|z_\lambda\|^2_{L^2(Q)} \\ &+& 2\|h\|^2_{L^2(0,A)}\|y(v+\lambda w, g+\lambda h) - y(v,g)\|^2_{L^2(U;H^1_0(\Omega))}, \end{array}$$

it follows from (2.8) that

$$\begin{array}{l} \|\nabla z_\lambda\|^2_{L^2(Q)} + (r - \|g\|^2_{L^2(0,A)})\|z_\lambda\|^2_{L^2(Q)} \leq \\ \|h\|^2_{L^2(0,A)}\|y(v+\lambda w, g+\lambda h) - y(v,g)\|^2_{L^2(U;H^1_0(\Omega))}. \end{array}$$

Choosing r such that $r > \|g\|^2_{L^2(0,A)}$, we deduce

$$\text{(2.9)} \qquad \|z_\lambda\|_{L^2(U;H^1_0(\Omega))} \leq \|h\|_{L^2(0,A)}\|y(v+\lambda w, g+\lambda h) - y(v,g)\|_{L^2(U;H^1_0(\Omega))}.$$

Hence,

$$\text{(2.10)} \qquad \|y_\lambda\|_{L^2(U;H^1_0(\Omega))} \leq e^{rT}\|h\|_{L^2(0,A)}\|y(v+\lambda w, g+\lambda h) - y(v,g)\|_{L^2(Q)}.$$

Passing to the limit in this latter identity when $\lambda \to 0$ and using the fact that the function $(v,g) \mapsto y(v,g)$ is continuous, it follows that $y_\lambda \to 0$ strongly in $L^2(U;H^1_0(\Omega))$. This means that $(\overline{y}_\lambda)$ converges to $\overline{y}$ strongly in $L^2(U;H^1_0(\Omega))$ as $\lambda \to 0$. ∎

Proposition 2.3. *The mapping*

$$\frac{\partial y}{\partial g}(\cdot, g) \colon L^2(Q_\omega) \to \mathcal{L}\left(L^2(0,A); L^2(U;H^1_0(\Omega)\right)$$

$$v \mapsto \frac{\partial y}{\partial g}(v,g),$$

is continuous.

Proof. From Proposition 2.2, we have that $\overline{y}(h) = \dfrac{\partial y}{\partial g}(v,g)(h)$ is a solution to

$$\left\{\begin{array}{rcll} L\overline{y}(h) &=& 0 & \text{in} \quad Q, \\ \overline{y}(h) &=& 0 & \text{on} \quad \Sigma, \\ \overline{y}(h)(0,\cdot,\cdot) &=& 0 & \text{in} \quad Q_A, \\ \overline{y}(h)(\cdot,0,\cdot) &=& \int_0^A g(a)\overline{y}(h)\, da + \int_0^A h(a) y(t,a,x;v,g)\, da & \text{in} \quad Q_T. \end{array}\right.$$

Let $v_1, v_2 \in L^2(Q_\omega)$. Set $\overline{y}_1(h) = \dfrac{\partial y}{\partial g}(v_1,g)(h)$, $\overline{y}_2(h) = \dfrac{\partial y}{\partial g}(v_2,g)(h)$ and take $\overline{z}(h) = e^{-rt}(\overline{y}_1(h) - \overline{y}_2(h))$, $r > 0$. It then follows that $\overline{z}(h)$ is a solution to problem

$$\left\{\begin{array}{rcll} L\overline{z}(h) + r\overline{z}(h) &=& 0 & \text{in} \quad Q, \\ \overline{z}(h) &=& 0 & \text{on} \quad \Sigma, \\ \overline{z}(h)(0,\cdot,\cdot) &=& 0 & \text{in} \quad Q_A, \\ \overline{z}(h)(\cdot,0,\cdot) &=& \displaystyle\int_0^A g(a)\overline{z}(h)\, da + e^{-rt}\eta_4 & \text{in} \quad Q_T, \end{array}\right. \tag{2.11}$$

where for $(t,x) \in Q_T$,

$$\eta_4(t,x) = \int_0^A h(a)(y(t,a,x;v_1,g) - y(t,a,x;v_2,g))\, da. \tag{2.12}$$

Multiplying the first equation of system (2.11) by $\overline{z}(h)$ and integrating by parts over Q, we obtain

$$\begin{array}{l} \dfrac{1}{2}\|\overline{z}(h)(T,\cdot,\cdot)\|^2_{L^2(Q_A)} + \dfrac{1}{2}\|\overline{z}(h)(\cdot,A,\cdot)\|^2_{L^2(Q_T)} - \dfrac{1}{2}\|\overline{z}(h)(\cdot,0,\cdot)\|^2_{L^2(Q_T)} \\ +\|\nabla\overline{z}(h)\|^2_{L^2(Q)} + \displaystyle\int_Q (r+\mu)\overline{z}(h) = 0. \end{array} \tag{2.13}$$

Observing

$$\begin{array}{rcl} \|\overline{z}(h)(\cdot,0,\cdot)\|^2_{L^2(Q_T)} &\leq& 2\|g\|^2_{L^2(0,A)}\|\overline{z}(h)\|^2_{L^2(Q)} \\ &+& \|h\|^2_{L^2(0,A)}\|y(v_1,g) - y(v_2,g)\|^2_{L^2(U;H_0^1(\Omega))} \end{array}$$

and choosing in (2.13) r such that $r > \|g\|^2_{L^2(0,A)}$, we deduce

$$\|\overline{z}(h)\|_{L^2(U;H_0^1(\Omega))} \leq \|h\|_{L^2(0,A)}\|y(v_1,g) - y(v_2,g)\|_{L^2(U;H_0^1(\Omega))}. \tag{2.14}$$

Therefore,

$$\|\overline{y}_1(h) - \overline{y}_2(h)\|_{L^2(U;H_0^1(\Omega))} \leq e^{rT}\|h\|_{L^2(0,A)}\|y(v_1,g) - y(v_2,g)\|_{L^2(U;H_0^1(\Omega))}, \tag{2.15}$$

from which we deduce

$$\begin{array}{rcl} |||\overline{y}_1 - \overline{y}_2||| &=& \displaystyle\sup_{h\in L^2(0,A), \|h\|\leq 1} \|\overline{y}_1(h) - \overline{y}_2(h)\|_{L^2(U;H_0^1(\Omega))} \\ &\leq& e^{rT}\|y(v_1,g) - y(v_2,g)\|_{L^2(U;H_0^1(\Omega))}, \end{array} \tag{2.16}$$

where $|||\cdot|||$ stands for the norm in $\mathcal{L}\left(L^2(0,A); L^2(U;H_0^1(\Omega))\right)$. This leads us to

$$\left|\left|\left|\frac{\partial y}{\partial g}(v_1,g) - \frac{\partial y}{\partial g}(v_2,g)\right|\right|\right| \leq e^{rT}\|y(v_1,g) - y(v_2,g)\|_{L^2(U;H_0^1(\Omega))}.$$

Passing to the limit in this latter inequality when $v_1 \to v_2$ while using Proposition 2.2, we obtain that $\frac{\partial y}{\partial g}(v_1,g)$ converges to $\frac{\partial y}{\partial g}(v_2,g)$ in $\mathcal{L}\left(L^2(0,A); L^2(U;H_0^1(\Omega))\right)$. ∎

3. Resolution of the optimization problem (1.5). In this section, we are concerned with the optimization problem (1.5). But because the map $g \mapsto y(v,g)$ from $L^2(0,A)$ to $L^2(U;H_0^1(\Omega)$ is non-linear, we replace the cost function defined in (1.4) by a new cost-function

$$J_1(v,g) = J(v,0) + \frac{\partial J}{\partial g}(v,0)(g). \tag{3.1}$$

Then, we consider the following new optimization problem:

$$\inf_{v\in L^2(Q_\omega)} \sup_{g\in L^2(0,A)} (J_1(v,g) - J_1(0,g)). \tag{3.2}$$

Let $y(v,0) \in L^2(U;H_0^1(\Omega))$ be the solution of

$$\left\{\begin{array}{rcll} Ly(v,0) & = & f + v\chi_\omega & \text{in} \quad Q, \\ y & = & 0 & \text{on} \quad \Sigma, \\ y(0,\cdot,\cdot) & = & y^0 & \text{in} \quad Q_A, \\ y(\cdot,0,\cdot) & = & 0 & \text{in} \quad Q_T. \end{array}\right. \tag{3.3}$$

Then we have the following result.

PROPOSITION 3.1. *For any $(v,g) \in L^2(Q_\omega) \times L^2(0,A)$, the following equality holds:*

$$J_1(v,g) = J(v,0) + 2\int_Q \left(\frac{\partial y}{\partial g}(v,0)(g)\right)(y(v,0) - z_d)\,dt\,da\,dx, \tag{3.4}$$

where J is the cost function defined in (1.4) and

$$\frac{\partial J}{\partial g}(v,0)(g) = \lim_{t\to 0} \frac{J(v,tg) - J(v,0)}{t}.$$

Proof. Observing on the one hand

$$\begin{array}{rcl} J(v,tg) & = & \|y(v,tg) - z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q_\omega)} \\ & = & J(v,0) + \|y(v,tg) - y(v,0)\|^2_{L^2(Q)} \\ & + & 2\displaystyle\int_Q (y(v,tg) - y(v,0))(y(v,0) - z_d)dt\,da\,dx, \end{array}$$

and on the other hand

$$\frac{\partial J}{\partial g}(v,0)(g) = \lim_{t\to 0} \frac{J(v,tg) - J(v,0)}{t},$$

using Proposition 2.2, we obtain that

$$\frac{\partial J}{\partial g}(v,0)(g) \quad = \quad 2\int_Q \left(\frac{\partial y}{\partial g}(v,0)(g)\right)(y(v,0) - z_d)dt\,da\,dx.$$

So

$$J_1(v,g) = J(v,0) + 2\int_Q \left(\frac{\partial y}{\partial g}(v,0)(g)\right)(y(v,0) - z_d)dt\,da\,dx.$$

PROPOSITION 3.2. *For any* $(v,g)\in L^2(Q_\omega)\times L^2(0,A)$, *we have*

$$J_1(v,g)-J_1(0,g)=J(v,0)-J(0,0)+2\int_0^A S(a;v)g(a)da, \tag{3.5}$$

where for any $a\in(0,A)$,

$$S(a;v)=\int_{Q_T}[y(t,a,x;v,0)\xi(v)(t,0,x)-y(t,a,x;0,0)\xi(0)(t,0,x)]\,dt\,dx \tag{3.6}$$

with $\xi(v)$, *a solution to*

$$\left\{\begin{array}{rcll} L^*\xi(v) &=& y(v,0)-z_d & \text{in} \quad Q,\\ \xi(v) &=& 0 & \text{on} \quad \Sigma,\\ \xi(v)(T,\cdot,\cdot) &=& 0 & \text{in} \quad Q_A,\\ \xi(v)(\cdot,A,\cdot) &=& 0 & \text{in} \quad Q_T. \end{array}\right. \tag{3.7}$$

Proof. In view of (3.4), we have

$$\begin{array}{rcl} J_1(v,g)-J_1(0,g) &=& J(v,0)-J(0,0)\\ &+& 2\displaystyle\int_Q\left(\frac{\partial y}{\partial g}(v,0)(g)\right)(y(v,0)-z_d)\,dt\,da\,dx\\ &-& \displaystyle\int_Q\left(\frac{\partial y}{\partial g}(0,0)(g)\right)(y(0,0)-z_d)\,dt\,da\,dx. \end{array} \tag{3.8}$$

From Proposition 2.2, we have that $\overline{y}(g)=\dfrac{\partial y}{\partial g}(v,0)(g)$ is the solution to

$$\left\{\begin{array}{lcll} L\overline{y}(g) &=& 0 & \text{in} \quad Q,\\ \overline{y}(g) &=& 0 & \text{on} \quad \Sigma,\\ \overline{y}(g)(0,\cdot,\cdot) &=& 0 & \text{in} \quad Q_A,\\ \overline{y}(g)(\cdot,0,\cdot) &=& \displaystyle\int_0^A g(a)y(t,a,x;v,0)\,da & \text{in} \quad Q_T. \end{array}\right. \tag{3.9}$$

So, if we multiply the first equation of (3.9) by $\xi(v)$ and integrate by parts over Q, we get

$$-\int_Q g(a)y(t,a,x;v,0)\xi(v)(t,0,x)\,dt\,da\,dx+\int_Q\overline{y}(g)\left(y(v,0)-z_d\right)\,dt\,da\,dx=0,$$

which can be rewritten as

$$\begin{array}{l}\displaystyle\int_Q\left(\frac{\partial y}{\partial g}(v,0)(g)\right)(y(v,0)-z_d)\,dt\,da\,dx=\\ \displaystyle\int_Q g(a)y(t,a,x;v,0)\xi(v)(t,0,x)\,dt\,da\,dx.\end{array} \tag{3.10}$$

We also have

$$\begin{array}{l}\displaystyle\int_Q\left(\frac{\partial y}{\partial g}(0,0)(g)\right)(y(0,0)-z_d)\,dt\,da\,dx=\\ \displaystyle\int_Q g(a)y(t,a,x;0,0)\xi(0)(t,0,x)\;dt\,da\,dx.\end{array} \tag{3.11}$$

□

Using (3.8), (3.10) and (3.11), it follows that

$$\begin{aligned}&J_1(v,g)-J_1(0,g)=J(v,0)-J(0,0)+\\&2\int_0^A g(a)\left[\int_{Q_T} y(t,a,x;v,0)\xi(v)(t,0,x)dadtdx - y(t,a,x;0,0)\xi(0)(t,0,x)\right]dadtdx\\&=J(v,0)-J(0,0)+2\int_0^A S(a;v)\,g(a)da.\end{aligned}$$

LEMMA 3.3. *Let $\xi(v)$ be the solution of problem (3.7). Then the application $v\mapsto \xi(v)$ is continuous from $L^2(Q_\omega)$ onto $L^2(U;H_0^1(\Omega))$.*

Proof. Let $v_1,v_2\in L^2(Q_\omega)$, and define $\bar\xi=\xi(v_1)-\xi(v_2)$. Then $\bar\xi$ is the solution to problem

$$\left\{\begin{array}{rcll} L^*\bar\xi &=& y(v_1,0)-y(v_2,0) & \text{in} \quad Q,\\ \bar\xi &=& 0 & \text{on} \quad \Sigma,\\ \bar\xi(T,\cdot,\cdot) &=& 0 & \text{in} \quad Q_A,\\ \bar\xi(\cdot,A,\cdot) &=& 0 & \text{in} \quad Q_T.\end{array}\right. \tag{3.12}$$

By setting $z=e^{-rt}\bar\xi$, it follows that z solves

$$\left\{\begin{array}{rcll} L^*z+rz &=& (y(v_1,0)-y(v_2,0))e^{-rt} & \text{in} \quad Q,\\ z &=& 0 & \text{on} \quad \Sigma,\\ z(T,\cdot,\cdot) &=& 0 & \text{in} \quad Q_A,\\ z(\cdot,A,\cdot) &=& 0 & \text{in} \quad Q_T.\end{array}\right. \tag{3.13}$$

If we multiply the first equation of system (3.13) by z and integrating by parts over Q, we get

$$\begin{aligned}&\frac{1}{2}\|z(0,\cdot,\cdot)\|^2_{L^2(Q_A)}+\frac{1}{2}\|z(\cdot,0,\cdot)\|^2_{L^2(Q_T)}+\|\nabla z\|^2_{L^2(Q)}+\int_Q (r+\mu)z^2\,dt\,da\,dx=\\&\int_Q (y(v_1,0)-y(v_2,0))ze^{-rt}\,dt\,da\,dx.\end{aligned}$$

It then follows

$$\frac{1}{2}\|z(\cdot,0,\cdot)\|^2_{L^2(Q_T)}+\|\nabla z\|^2_{L^2(Q)}+r\|z\|^2_{L^2(Q)}\le\frac{1}{2}\|y(v_1,0)-y(v_2,0)\|^2_{L^2(Q)}+\frac{1}{2}\|z\|^2_{L^2(Q)}.$$

Taking $r=\frac{1}{2}$ in this latter identity yields

$$\frac{1}{2}\|z(\cdot,0,\cdot)\|^2_{L^2(Q_T)}+\frac{1}{2}\|z\|^2_{L^2(U;H_0^1(\Omega))}\le\frac{1}{2}\|y(v_1,0)-y(v_2,0)\|^2_{L^2(Q)}.$$

Thus,

$$\|\bar\xi(\cdot,0,\cdot)\|^2_{L^2(Q_T)}+\|\bar\xi\|^2_{L^2(U;H_0^1(\Omega))}\le e^T\|y(v_1,0)-y(v_2,0)\|_{L^2(Q)}, \tag{3.14}$$

from which we deduce

$$\|\bar\xi\|_{L^2(U;H_0^1(\Omega))}\le e^{T/2}\|y(v_1,0)-y(v_2,0)\|_{L^2(Q)}.$$

Using Proposition 2.1, while passing to limit in this latter inequality when $v_1\to v_2$, we obtain that $\bar\xi\to 0$ strongly in $L^2(U;H_0^1(\Omega))$. This means that $\xi(v_1)\to\xi(v_2)$ strongly in $L^2(U;H_0^1(\Omega))$ as $v_1\to v_2$. ∎

PROPOSITION 3.4. *Let $S(\cdot;v)$ be the function defined in (3.6). Then the map $v \mapsto S(\cdot;v)$ is continuous form $L^2(Q_\omega)$ onto $L^2(0,A)$.*

Proof. Let v_1 and v_2. Then in view of (3.6),

$$\begin{aligned} S(a;v_1) - S(a;v_2) &= \int_{Q_T} (y(t,a,x;v_1,0) - y(t,a,x;v_2,0))\xi(v_1)(t,0,x)\,dt\,dx \\ &- \int_{Q_T} y(t,a,x;v_2,0)(\xi(v_2)(t,0,x) - \xi(v_1)(t,0,x))\,dt\,dx. \end{aligned}$$

Using the Cauchy Schwarz inequality, we have

$$\begin{aligned} &|S(a;v_1) - S(a;v_2)| \leq \|y(.,a,.;v_1,0) - y(.,a,.;v_2,0)\|_{L^2(Q_T)}\|\xi(v_1)(.,0,.)\|_{L^2(Q_T)} \\ &+\|y(.,a,.;v_2,0)\|_{L^2(Q_T)}\|\xi(v_2)(.,0,.) - \xi(v_1)(.,0,.)\|_{L^2(Q_T)}. \end{aligned}$$

Observing on the one hand that $\xi(v_2) - \xi(v_1)$ is solution of (3.12), and, on the other hand that, in view of (3.14),

$$\|\xi(v_2)(.,0,.) - \xi(v_1)(.,0,.)\|^2_{L^2(Q_T)} \leq e^{T/2}\|y(v_1,0) - y(v_2,0)\|_{L^2(Q)},$$

we have

$$\begin{aligned} &|S(a;v_1) - S(a;v_2)| \leq \|y(.,a,.;v_1,0) - y(.,a,.;v_2,0)\|_{L^2(Q_T)}\|\xi(v_1)(.,0,.)\|_{L^2(Q_T)} \\ &+e^{T/2}\|y(.,a,.;v_2,0)\|_{L^2(Q_T)}\|y(v_1,0) - y(v_2,0)\|_{L^2(Q)}. \end{aligned}$$

Hence,

$$\begin{aligned} &\int_0^A |S(a;v_1) - S(a;v_2)|^2 da \leq \\ &2\|y(.,a,.;v_1,0) - y(.,a,.;v_2,0)\|^2_{L^2(Q)}\|\xi(v_1)(.,0,.)\|^2_{L^2(Q_T)} + \\ &2e^T\|y(.,a,.;v_2,0)\|^2_{L^2(Q)}\|y(v_1,0) - y(v_2,0)\|^2_{L^2(Q)}. \end{aligned}$$

It then follows from Poincaré inequality,

$$\begin{aligned} &\|S(a;v_1) - S(a;v_2)\|_{L^2(0,A)} \leq \\ &C(\Omega)\|y(.,a,.;v_1,0) - y(.,a,.;v_2,0)\|_{L^2(U;H^1_0(\Omega))}\|\xi(v_1)(.,0,.)\|_{L^2(Q_T)} + \\ &C(\Omega)e^{T/2}\|y(.,a,.;v_2,0)\|_{L^2(Q)}\|y(v_1,0) - y(v_2,0)\|_{L^2(U;H^1_0(\Omega))}. \end{aligned}$$

In view of Proposition 2.1, it follows that $S(\cdot,v_1) \to S(\cdot,v_2)$ as $v_1 \to v_2$. ∎

LEMMA 3.5. *Let $S(.,v)$ be defined as in (3.6) for any $a \in L^2(0,A)$. For any $\gamma > 0$, we consider the sequences $y^\gamma = y(t,a,x;u^\gamma,0)$ and $\xi(u^\gamma)$, respectively, solutions of (3.3) and (3.7) with $v = u^\gamma$. Assume that there exists $C > 0$ independent of γ such that*

$$\|S(.,u^\gamma)\|_{L^2(0,A)} < C.$$

Assume also that $\hat{u} \in L^2(Q_\omega)$, $\hat{\xi}(.,0,.) \in L^2(Q_T)$ and $\hat{y} = y(t,a,x;\hat{u},0) \in L^2(U;H^1_0(\Omega))$ solution of (3.3) are such that

$$u^\gamma \rightharpoonup \hat{u} \text{ weakly in } L^2(U \times \omega), \tag{3.15a}$$

$$y^\gamma \rightharpoonup \hat{y} = y(t,a,x;\hat{u},0) \text{ weakly in } L^2(U,H^1_0(\Omega)), \tag{3.15b}$$

$$\xi(u^\gamma)(.,0,.) \rightharpoonup \hat{\xi}(.,0,.) \text{ weakly in } L^2(Q_T). \tag{3.15c}$$

Then we have

$$S(a;u^\gamma) \rightharpoonup S(a;\hat{u}) \text{ weakly in } \mathcal{D}'(0,A).$$

Proof. Let $\mathcal{D}((0,A))$ be the set of $\mathcal{C}^\infty$ function with compact support on $(0,A)$. Set for any $\phi \in \mathcal{D}((0,A))$

$$z^\gamma(t,x) = \int_0^A y(t,a,x;u^\gamma,0)\phi(a)da,\ (t,x) \in Q_T.$$

Then, in view of (3.15b), there exists a constant $C > 0$ independent of γ such that

$$\|z^\gamma\|_{L^2(Q_T)} \leq \|y^\gamma\|_{L^2(Q)}\|\phi\|_{L^2(0,A)} \leq C.$$

Consequently, there exists $z \in L^2(Q_T)$ such that

$$z^\gamma \rightharpoonup z \text{ weakly in } L^2(Q_T).$$

Moreover, we have

$$z(t,x) = \int_0^A y(t,a,x;\hat{u},0)\phi(a)da,\ (t,x) \in Q_T. \tag{3.16}$$

Because $y^\gamma = y(t,a,x;u^\gamma,0)$ solves (3.3) with $v = u^\gamma$, we have that z^γ solves

$$\begin{cases} \dfrac{\partial z^\gamma}{\partial t} - \Delta z^\gamma &= k^\gamma & \text{in} & Q_T, \\ z^\gamma &= 0 & \text{on} & (0,T)\times\Gamma, \\ z^\gamma(0) &= \displaystyle\int_0^A y^0(a,x)\phi(a)da & \text{in}\Omega, \end{cases}$$

where

$$k^\gamma(t,x) = \int_0^A (f + u^\gamma\chi_\omega)\phi da - \int_0^A \mu(a)y^\gamma\phi da - \int_0^A \frac{\partial y^\gamma}{\partial a}\phi da.$$

Consequently, in view of (3.15a) and (3.15b), we have there exists a positive constant C independent of γ such that

$$\|k^\gamma\|_{L^2(Q_T)} \leq \left(2\|f\|^2_{L^2(Q)} + 2\|u^\gamma\|^2_{L^2(Q_\omega)} + \|\mu\|^2_{L^\infty(0,A)}\|y^\gamma\|^2_{L^2(Q)}\right)^{1/2}\|\phi\|_{L^2(0,A)} + \|y^\gamma\|_{L^2(Q)}\left\|\frac{\partial\phi}{\partial a}\right\|_{L^2(0,A)} \leq C.$$

It then follows that there is $C > 0$, independent of γ, such that

$$\begin{cases} \|z^\gamma\|_{L^2((0,T);H^1_0(\Omega))} & \leq C, \\ \left\|\dfrac{\partial z^\gamma}{\partial t}\right\|_{L^2((0,T);H^{-1}(\Omega))} & \leq C. \end{cases}$$

Therefore, it follows from Aubin-Lions's Lemma that

$$z^\gamma \to z \text{ strongly in } L^2(Q_T), \tag{3.17}$$ ∎

where

$$z(t,x) = \int_0^A y(t,a,x;\hat{u},0)\phi(a)da,\ (t,x) \in Q_T$$

because of (3.16).

Now in view of (3.6)

$$S(a;u^\gamma)=\int_{Q_T}\left[y(t,a,x;u^\gamma,0)\xi(u^\gamma)(t,0,x)-y(t,a,x;0,0)\xi(0)(t,0,x)\right]\,dt\,dx.$$

Therefore for any $\phi\in\mathcal{D}(0,A)$,

$$\begin{aligned}\int_0^A S(a;u^\gamma)\phi(a)da &= \int_{Q_T}\int_0^A (y(t,a,x;u^\gamma,0)\phi(a)da)\xi(u^\gamma)(t,0,x)\,dt\,da\,dx\\ &- \int_Q y(t,a,x;0,0)\xi(0)(t,0,x)\phi(a)\,dt\,da\,dx\\ &= \int_{Q_T} z^\gamma(t,x)\xi(u^\gamma)(t,0,x)\,dt\,da\,dx\\ &- \int_Q y(t,a,x;0,0)\xi(0)(t,0,x)\phi(a)\,dt\,da\,dx\end{aligned}$$

Passing this latter identity to the limit while using (3.17), (3.16) and (3.15c), we obtain

$$\begin{aligned}\int_0^A S(a;u^\gamma)\phi(a)da &\to \int_{Q_T} z(t,x)\xi(\hat{u})(t,0,x)\,dt\,da\,dx\\ &- \int_Q y(t,a,x;0,0)\xi(0)(t,0,x)\phi(a)\,dt\,da\,dx\\ &= \int_{Q_T}\int_0^A (y(t,a,x;\hat{u},0)\phi(a)da)\xi(\hat{u})(t,0,x)\,dt\,da\,dx\\ &- \int_Q y(t,a,x;0,0)\xi(0)(t,0,x)\phi(a)\,dt\,da\,dx\quad \forall\phi\in\mathcal{D}(0,A),\end{aligned}$$

which in view of (3.6), proves

$$S(a;u^\gamma)\rightharpoonup S(a;\hat{u})\text{ weakly in }\mathcal{D}'(0,A).$$

From now on, we denote by $\mathcal{D}(\Theta)$ the set of $\mathcal{C}^\infty$ function with compact support on Θ.

3.1. Existence of an No-regret control and Low-regret control. In view of (3.5), the optimization problem (3.2) is equivalent to the following problem:

$$\inf_{v\in L^2(Q_\omega)}\sup_{g\in L^2(0,A)}[J(v,0)-J(0,0)+2\int_0^A S(a;v)g(a)da]. \tag{3.18}$$

As $\int_0^A S(a;v)g(a)da$ may be equal to 0 or not upper bounded, we consider the set,

$$\mathcal{M}=\left\{v\in L^2(Q_\omega);\int_0^A S(a;v)g(a)da=0,\quad\forall g\in L^2(0,A)\right\}. \tag{3.19}$$

Then $\mathcal{M}$ is strongly closed in $L^2(Q_\omega)$. We have on the one hand that the application $v\mapsto J(v,0)-J(0,0)$ is coercive on $L^2(Q_\omega)$, bounded below by $-J(0,0)$, and continuous because of (2.1), and on the other hand that the application $v\mapsto S(\cdot;v)$ is continuous on $L^2(Q_\omega)$. Consequently, using minimizing sequences and Lemma 3.5, we prove that there exists a No-regret control $\tilde{u}$ in $\mathcal{M}$ satisfying (3.18). We thus have proved the following result.

LEMMA 3.6. *There exists a solution* $\tilde{u}$ *of (3.18) in* $\mathcal{M}$.

As such a control $\tilde{u}$ is not easy to characterize, we consider for any $\gamma > 0$, the relaxed optimization problem, which we refer to as the *Low-regret-control problem*:

$$\inf_{v\in L^2(Q_\omega)} \sup_{g\in L^2(0,A)} \left[J(v,0) - J(0,0) + 2\int_0^A S(a;v)g(a)da - \gamma\|g\|^2_{L^2(0,A)} \right], \tag{3.20}$$

which by means of Fenchel-Legendre transform is equivalent to

$$\inf_{v\in L^2(Q_\omega)} \mathcal{J}^\gamma(v), \tag{3.21}$$

with

$$\mathcal{J}^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}\|S(\cdot;v)\|^2_{L^2(0,A)}. \tag{3.22}$$

PROPOSITION 3.7. *Let* $\gamma > 0$. *Then there exists at least in* $L^2(Q_\omega)$ *a Low-regret control* u_γ *solution of problem (3.21).*

Proof. We have $\mathcal{J}^\gamma(v) \geq -J(0,0)$ and $\mathcal{J}^\gamma(0) = 0$. Using using minimizing sequences, Proposition 2.1, Proposition 3.4 and Lemma 3.5, we prove as for Lemma 3.6 that problem (3.21) has at least one solution $u_\gamma \in L^2(Q_\omega)$. ☐

Remark 3.8. The uniqueness of $u_\gamma \in L^2(Q_\omega)$, solution of (3.21) is not guaranteed because the application $v \mapsto S(\cdot;v)$ from $L^2(Q_\omega)$ to $L^2(0,A)$ is not necessary strictly convex. Consequently, we are not sure that the Low-regret controls u_γ will converge towards the No-regret control $\widetilde{u} \in \mathcal{M}$. So, in order to obtain this convergence, we adapt the cost function $\mathcal{J}^\gamma$ to a given No-regret control.

3.2. Existence of the adapted low-regret control. Let $\widetilde{u}$ be a No-regret optimal control. For any $\gamma > 0$, we define the adapted cost function $\widetilde{\mathcal{J}}^\gamma$ by:

$$v \mapsto \widetilde{\mathcal{J}}^\gamma(v) = J(v,0) - J(0,0) + \|v - \widetilde{u}\|^2_{L^2(Q_\omega)} + \frac{1}{\gamma}\|S(\cdot;v)\|^2_{L^2(0,A)}. \tag{3.23}$$

Then, we consider the following optimal control problem:

$$\inf_{v\in L^2(Q_\omega)} \widetilde{\mathcal{J}}^\gamma(v). \tag{3.24}$$

PROPOSITION 3.9. *Let* $\gamma > 0$. *Then problem (3.24) has at least a solution* $\widetilde{u}_\gamma$ *in* $L^2(Q_\omega)$.

Proof. One proceeds as for the proof of Proposition 3.7 using the fact that $v \mapsto \widetilde{\mathcal{J}}^\gamma(v)$ is continuous on $L^2(Q_\omega)$(thanks to Proposition 2.1 and Proposition 3.4) and the fact that $\lim_{\|v\|_{L^2(Q_\omega)}\to+\infty} \mathcal{J}^\gamma(v) = +\infty$. ☐

PROPOSITION 3.10. *Let* $\widetilde{u}_\gamma \in L^2(Q_\omega)$ *be a solution of (3.24). Then there exist* $\widetilde{p}_\gamma = p(\widetilde{u}_\gamma) \in L^2(U;H^1_0(\Omega))$ *and* $\widetilde{q}_\gamma = q(\widetilde{u}_\gamma) \in L^2(U;H^1_0(\Omega))$ *such that* $\{\widetilde{y}_\gamma, \widetilde{\xi}_\gamma, \widetilde{p}_\gamma, \widetilde{q}_\gamma\}$ *is a solution of the systems:*

$$\left\{\begin{array}{rcll} L\widetilde{y}_\gamma &=& f + \widetilde{u}_\gamma\chi_{Q_\omega} & \text{in} \quad Q, \\ \widetilde{y}_\gamma &=& 0 & \text{on} \quad \Sigma, \\ \widetilde{y}_\gamma(0,\cdot,\cdot) &=& y^0 & \text{in} \quad Q_A, \\ \widetilde{y}_\gamma(\cdot,0,\cdot) &=& 0 & \text{in} \quad Q_T, \end{array}\right. \tag{3.25}$$

(3.26)
$$\begin{cases} L^*\widetilde{\xi}_\gamma &= \widetilde{y}_\gamma - z_d \quad in \quad Q,\\ \widetilde{\xi}_\gamma &= 0 \quad on \quad \Sigma,\\ \widetilde{\xi}_\gamma(T,\cdot,\cdot) &= 0 \quad in \quad Q_A,\\ \widetilde{\xi}_\gamma(\cdot,A,\cdot) &= 0 \quad in \quad Q_T,\end{cases}$$

(3.27)
$$\begin{cases} L\widetilde{p}_\gamma &= 0 \quad in \quad Q,\\ \widetilde{p}_\gamma &= 0 \quad on \quad \Sigma,\\ \widetilde{p}_\gamma(0,\cdot,\cdot) &= 0 \quad in \quad Q_A,\\ \widetilde{p}_\gamma(\cdot,0,\cdot) &= \dfrac{1}{\sqrt{\gamma}}\displaystyle\int_0^A y(t,a,x;\widetilde{u}_\gamma,0)S(a;\widetilde{u}_\gamma)\,da \quad in \quad Q_T,\end{cases}$$

(3.28)
$$\begin{cases} L^*\widetilde{q}_\gamma &= y(\widetilde{u}_\gamma,0) - z_d + \varrho^\gamma \quad in \quad Q,\\ \widetilde{q}_\gamma &= 0 \quad on \quad \Sigma,\\ \widetilde{q}_\gamma(T,\cdot,\cdot) &= 0 \quad in \quad Q_A,\\ \widetilde{q}_\gamma(\cdot,A,\cdot) &= 0 \quad in \quad Q_T,\end{cases}$$

and

(3.29)
$$(N+1)\widetilde{u}_\gamma - \widetilde{u} + \widetilde{q}_\gamma = 0 \qquad in \quad Q_\omega,$$

where $\widetilde{y}_\gamma = y(\widetilde{u}_\gamma,0)$, $\widetilde{\xi}_\gamma = \xi(\widetilde{u}_\gamma)$ *and*

(3.30)
$$\varrho^\gamma = \frac{1}{\sqrt{\gamma}}\widetilde{p}_\gamma + \frac{1}{\gamma}S(a;\widetilde{u}_\gamma)\xi(\widetilde{u}_\gamma)(t,0,x).$$

Proof. We write the Euler-Lagrange optimality condition that characterizes $\widetilde{u}_\gamma$:

$$\lim_{\lambda\to 0}\frac{\widetilde{\mathcal{J}}^\gamma(\widetilde{u}_\gamma+\lambda w) - \widetilde{\mathcal{J}}^\gamma(\widetilde{u}_\gamma)}{\lambda} = 0, \quad \forall w \in L^2(Q_\omega).$$

Using Proposition 2.2 and Proposition 3.2, we obtain after some calculations

(3.31)
$$\begin{aligned} 0 \;=\; & \int_Q \left(\frac{\partial y}{\partial v}(\widetilde{u}_\gamma,0)(w)\right)\left(y(\widetilde{u}_\gamma,0) - z_d + \frac{1}{\gamma}\xi(\widetilde{u}_\gamma)(\cdot,0,\cdot)S(\cdot;\widetilde{u}_\gamma)\right)dt\,da\,dx\\ +\; & \int_{Q_\omega}(\widetilde{u}_\gamma - \widetilde{u})w\,dt\,da\,dx + \int_{Q_\omega} N\widetilde{u}_\gamma\, w\,dt\,da\,dx\\ +\; & \frac{1}{\gamma}\int_Q \frac{\partial \xi}{\partial v}(\widetilde{u}_\gamma)(w)(\cdot,0,\cdot)\,y(\widetilde{u}_\gamma,0)S(\cdot;\widetilde{u}_\gamma)\,dt\,da\,dx, \quad \forall w\in L^2(Q_\omega),\end{aligned}$$

where $\overline{y}(w) = \dfrac{\partial y}{\partial v}(\widetilde{u}_\gamma,0)(w)$ and $\overline{\xi} = \dfrac{\partial \xi}{\partial v}(\widetilde{u}_\gamma)(w)$ are respectively solutions to

(3.32)
$$\begin{cases} L\overline{y}(w) &= w \quad \text{in} \quad Q,\\ \overline{y}(w) &= 0 \quad \text{on} \quad \Sigma,\\ \overline{y}(w)(0,\cdot,\cdot) &= 0 \quad \text{in} \quad Q_A,\\ \overline{y}(w)(\cdot,0,\cdot) &= 0 \quad \text{in} \quad Q_T\end{cases}$$

and

$$\left\{\begin{array}{rcll} L^*\overline{\xi} &=& \overline{y}(w) & \text{in} \quad Q,\\ \overline{\xi} &=& 0 & \text{on} \quad \Sigma,\\ \overline{\xi}(T,\cdot,\cdot) &=& 0 & \text{in} \quad Q_A,\\ \overline{\xi}(\cdot,A,\cdot) &=& 0 & \text{in} \quad Q_T. \end{array}\right. \tag{3.33}$$

To interpret (3.31), we use the adjoint state $\widetilde{q}_\gamma$ and $\widetilde{p}_\gamma$ solutions of (3.28) and (3.27) respectively.

So if we multiply the first term of (3.32) by a function $\widetilde{q}_\gamma$ and the first equation of (3.33) by a function $\dfrac{1}{\sqrt{\gamma}}p_\gamma$, then integrate by parts over Q, we, respectively, obtain

$$\int_Q \overline{y}(w)\left(y(\widetilde{u}_\gamma,0)-z_d+\frac{1}{\sqrt{\gamma}}p_\gamma+\frac{1}{\gamma}S(a;\widetilde{u}_\gamma)\xi(\widetilde{u}_\gamma)(t,0,x)\right)dt\,da\,dx = \int_{Q_\omega} w\,\widetilde{q}_\gamma\,dt\,da\,dx, \tag{3.34}$$

and

$$\frac{1}{\gamma}\int_Q \overline{\xi}(\cdot,0,\cdot)y(t,a,x;\widetilde{u}_\gamma,0)S(a;\widetilde{u}_\gamma)\,dt\,da\,dx = \frac{1}{\sqrt{\gamma}}\int_Q \overline{y}(w)\,p_\gamma\,dt\,da\,dx. \tag{3.35}$$

Combining (3.34), (3.35) and (3.31), we have

$$\int_{Q_\omega}((N+1)\widetilde{u}_\gamma-\widetilde{u}+\widetilde{q}_\gamma)w\,dt\,da\,dx=0, \quad \forall w\in L^2(Q_\omega),$$

which implies that

$$(N+1)\widetilde{u}_\gamma-\widetilde{u}+\widetilde{q}_\gamma = 0 \text{ in } Q_\omega. \tag{3.36}$$

□

Proposition 3.11. *Let $\widetilde{u}_\gamma\in L^2(Q_\omega)$ be a solution of (3.24). Let also $\widetilde{y}_\gamma$, $\widetilde{\xi}_\gamma$, $\widetilde{p}_\gamma$, $\widetilde{q}_\gamma$ and ϱ^γ be such that (3.25)-(3.30) hold true. Then we have following estimations:*

$$\|\widetilde{u}_\gamma\|_{L^2(Q_\omega)} \le C\left(N,\|\widetilde{u}\|_{L^2(Q_\omega)},\|y^0\|_{L^2(Q_A)},\|f\|_{L^2(Q)},\|z_d\|_{L^2(Q)}\right), \tag{3.37}$$

$$\frac{1}{\sqrt{\gamma}}\|S(\cdot;\widetilde{u}_\gamma)\|_{L^2(0,A)} \le C\left(\|\widetilde{u}\|_{L^2(Q_\omega)},\|y^0\|_{L^2(Q_A)},\|f\|_{L^2(Q)},\|z_d\|_{L^2(Q)}\right), \tag{3.38}$$

$$\|S(\cdot;\widetilde{u}_\gamma)\|_{L^2(0,A)} \le \sqrt{\gamma}C\left(\|\widetilde{u}\|_{L^2(Q_\omega)},\|y^0\|_{L^2(Q_A)},\|f\|_{L^2(Q)},\|z_d\|_{L^2(Q)}\right), \tag{3.39}$$

$$\|\widetilde{y}_\gamma\|_{L^2(U;H_0^1(\Omega))} \le C\left(N,T,\|\widetilde{u}\|_{L^2(Q_\omega)},\|y^0\|_{L^2(Q_A)},\|f\|_{L^2(Q)},\|z_d\|_{L^2(Q)}\right), \tag{3.40}$$

$$\|\widetilde{\xi}_\gamma\|_{L^2(U;H_0^1(\Omega))} \le C\left(T,\|\widetilde{u}\|_{L^2(Q_\omega)},\|y^0\|_{L^2(Q_A)},\|f\|_{L^2(Q)},\|z_d\|_{L^2(Q)}\right), \tag{3.41}$$

$$\|\widetilde{\xi}_\gamma(.,0,.)\|_{L^2(Q_T)} \le C\left(T,\|\widetilde{u}\|_{L^2(Q_\omega)},\|y^0\|_{L^2(Q_A)},\|f\|_{L^2(Q)},\|z_d\|_{L^2(Q)}\right), \tag{3.42}$$

$$\|\widetilde{p}_\gamma(.,0,.)\|_{L^2(Q_T)} \le C\left(\|\widetilde{u}\|_{L^2(Q_\omega)},\|y^0\|_{L^2(Q_A)},\|f\|_{L^2(Q)},\|z_d\|_{L^2(Q)}\right), \tag{3.43}$$

$$\|\widetilde{p}_\gamma\|_{L^2(U;H_0^1(\Omega))} \le C\left(T,\|\widetilde{u}\|_{L^2(Q_\omega)},\|y^0\|_{L^2(Q_A)},\|f\|_{L^2(Q)},\|z_d\|_{L^2(Q)}\right), \tag{3.44}$$

$$\|\varrho^\gamma\|_{\mathcal{D}'(Q)} \le C\left(N,T,\|\widetilde{u}\|_{L^2(Q_\omega)},\|y^0\|_{L^2(Q_A)},\|f\|_{L^2(Q)},\|z_d\|_{L^2(Q)}\right), \tag{3.45}$$

where from now on, $C(X)$ to denote a positive constant whose value varies from a line to another but depends on X.

Proof. We proceed in three steps.

Step 1. We prove the estimations (3.37)-(3.41).

As $\widetilde{u}_\gamma$ is solution of (3.24), we can write

$$\widetilde{\mathcal{J}}^\gamma(\widetilde{u}_\gamma) \leq \widetilde{\mathcal{J}}^\gamma(0) = \|\widetilde{u}\|_{L^2(Q_\omega)}. \tag{3.46}$$

It then follows from the definition of $\widetilde{\mathcal{J}}^\gamma$ and J given respectively by (3.23) and (1.4) that,

$$\begin{aligned}&\|\widetilde{y}_\gamma - z_d\|^2_{L^2(Q)} + N\|\widetilde{u}_\gamma\|^2_{L^2(Q_\omega)} + \|\widetilde{u}_\gamma - \widetilde{u}\|^2_{L^2(Q)} + \frac{1}{\gamma}\|S(\cdot;\widetilde{u}_\gamma)\|^2_{L^2(0,A)} \leq\\ &\|\widetilde{u}\|^2_{L^2(Q_\omega)} + \|y(0,0) - z_d\|^2_{L^2(Q)} = C\left(\|\widetilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}\right).\end{aligned}$$

Hence we deduce (3.37), (3.38), (3.39) and

$$\|\widetilde{y}_\gamma - z_d\|_{L^2(Q)} \leq C\left(\|\widetilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}\right). \tag{3.47}$$

Observing $\widetilde{y}_\gamma$ and $\widetilde{\xi}_\gamma$ are respectively solution of (3.25) and (3.26), proceeding as for $\overline{y}$ in pages 3-4, we obtain that

$$\|\widetilde{y}_\gamma\|_{L^2(U;H^1_0(\Omega))} \leq \frac{1}{\sqrt{2}}e^{2T}(\|y_0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|\widetilde{u}_\gamma\|_{L^2(Q_\omega)})$$

and

$$\|\widetilde{\xi}_\gamma(.,0,.)\|_{L^2(Q_T)} + \|\widetilde{\xi}_\gamma\|_{L^2(U;H^1_0(\Omega))} \leq \frac{\sqrt{2}}{2}e^{2T}\|\widetilde{y}_\gamma - z_d\|_{L^2(Q)},$$

from which we, respectively, deduce (3.40), (3.41) and (3.42) because of (3.37) and (3.47).

Step 2. We prove the estimations (3.43) and (3.44).

To prove (3.43), we observe that

$$\begin{aligned}&\left|\frac{1}{\sqrt{\gamma}}\int_0^A y(t,a,x;\widetilde{u}_\gamma,0)S(a;\widetilde{u}_\gamma)\ da\right| \leq\\ &\frac{1}{\sqrt{\gamma}}\|S(.;\widetilde{u}_\gamma)\|_{L^2(0,A)}\left(\int_0^A y(t,a,x;\widetilde{u}_\gamma,0)^2 da\right)^{1/2}.\end{aligned}$$

So using (3.38) and (3.47), we deduce

$$\begin{aligned}\int_{Q_T}\left|\frac{1}{\sqrt{\gamma}}\int_0^A y(t,a,x;\widetilde{u}_\gamma,0)S(a;\widetilde{u}_\gamma)\ da\right|^2 dt\,dx &\leq \frac{1}{\gamma}\|S(.;\widetilde{u}_\gamma)\|^2_{L^2(0,A)}\|\widetilde{y}_\gamma\|^2_{L^2(Q)}\\ &\leq C,\end{aligned}$$

where $C = C\left(\|\widetilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}\right) > 0$. This means

$$||p_\gamma(\cdot,0,\cdot)||_{L^2(Q)} \leq C\left(\|\widetilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}\right).$$

Since p_γ is solution of (3.27), proceeding as for $\overline{y}$ in pages 3-4 while using (3.43), we obtain

$$\|\widetilde{p}_\gamma\|_{L^2(U;H^1_0(\Omega))} \leq \frac{1}{\sqrt{2}}e^{2T}(\|y_0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|\widetilde{u}_\gamma\|_{L^2(Q_\omega)}).$$

Step 3. We prove (3.45).

We observe that $\widetilde{q}_\gamma$, the solution of (3.28), can be decomposed as $\widetilde{q}_\gamma = \widetilde{q}_\gamma^1 + \widetilde{q}_\gamma^2$, where $\widetilde{q}_\gamma^1$ is solution to

$$\left\{\begin{array}{rcll} L^*\widetilde{q}_\gamma^1 & = & \widetilde{y}_\gamma - z_d & \text{in} \quad Q, \\ \widetilde{q}_\gamma^1 & = & 0 & \text{on} \quad \Sigma, \\ \widetilde{q}_\gamma^1(T,\cdot,\cdot) & = & 0 & \text{in} \quad Q_A, \\ \widetilde{q}_\gamma^1(\cdot,A,\cdot) & = & 0 & \text{in} \quad Q_T, \end{array}\right. \tag{3.48}$$

and $\widetilde{q}_\gamma^2$ is solution to

$$\left\{\begin{array}{rcll} L^*\widetilde{q}_\gamma^2 & = & \dfrac{1}{\sqrt{\gamma}}p_\gamma + \dfrac{1}{\gamma}S(a;\widetilde{u}_\gamma)\xi(\widetilde{u}_\gamma)(t,0,x) & \text{in} \quad Q, \\ \widetilde{q}_\gamma^2 & = & 0 & \text{on} \quad \Sigma, \\ \widetilde{q}_\gamma^2(T,\cdot,\cdot) & = & 0 & \text{in} \quad Q_A, \\ \widetilde{q}_\gamma^2(\cdot,A,\cdot) & = & 0 & \text{in} \quad Q_T. \end{array}\right. \tag{3.49}$$

Proceeding as for $\overline{y}$ in pages 3-4, while using (3.47), we obtain

$$\|\widetilde{q}_\gamma^1\|_{L^2(U;H_0^1(\Omega))} \le C\left(T, \|\widetilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}\right). \tag{3.50}$$

Combining (3.31) and (3.35), we obtain

$$\begin{array}{rcl} 0 & = & \displaystyle\int_Q \overline{y}(w)(\widetilde{y}_\gamma - z_d)\,dt\,da\,dx \\ & + & \displaystyle\int_{Q_\omega} N\widetilde{u}_\gamma\, w\,dt\,da\,dx + \int_{Q_\omega}(\widetilde{u}_\gamma - \tilde{u})w\,dt\,da\,dx \\ & + & \displaystyle\int_Q \overline{y}(w)\left(\frac{1}{\sqrt{\gamma}}p_\gamma + \frac{1}{\gamma}\widetilde{\xi}_\gamma(0)S(a;\widetilde{u}_\gamma)\right)dt\,da\,dx, \forall w \in L^2(Q_\omega). \end{array} \tag{3.51}$$

Set

$$\mathcal{E} = \left\{\overline{y}(w), \quad w \in L^2(Q_\omega)\right\}. \tag{3.52}$$

Then $\mathcal{E} \subset L^2(Q)$. We define on $\mathcal{E}\times\mathcal{E}$ the inner product:

$$\langle \overline{y}(v), \overline{y}(w)\rangle_{\mathcal{E}} = \int_{Q_\omega} vw\ \ dt\,da\,dx + \int_Q \overline{y}(v)\overline{y}(w)\,dt\,da\,dx,\ \forall \overline{y}(v), \overline{y}(w) \in \mathcal{E}. \tag{3.53}$$

Then $\mathcal{E}$ endowed with the norm

$$\|\overline{y}(w)\|_{\mathcal{E}}^2 = \|w\|_{L^2(Q_\omega)}^2 + \|\overline{y}(w)\|_{L^2(Q)}^2,\ \forall \overline{y}(w) \in \mathcal{E} \tag{3.54}$$

is a Hilbert space. We set

$$T_\gamma(\widetilde{u}_\gamma) = \frac{1}{\sqrt{\gamma}}p_\gamma + \frac{1}{\gamma}\widetilde{\xi}_\gamma(0)S(a;\widetilde{u}_\gamma).$$

Then, in view of (3.51), we have for any $w \in L^2(Q_\omega)$,

$$\begin{array}{rcl} \displaystyle\int_Q T_\gamma(\widetilde{u}_\gamma)\overline{y}(w)dtdadx & = & -\displaystyle\int_Q \overline{y}(w)(\widetilde{y}_\gamma - z_d t)\,dt\,da\,dx \\ & - & \displaystyle\int_{Q_\omega} N\widetilde{u}_\gamma\, w\,dt\,da\,dx - \int_{Q_\omega}(\widetilde{u}_\gamma - \tilde{u})w\,dt\,da\,dx. \end{array} \tag{3.55}$$

Using the Cauchy Schwarz inequality, we have

$$\left| -\int_Q \overline{y}(w)(\widetilde{y}_\gamma - z_d t)\,dt\,da\,dx - \int_{Q_\omega} ((N+1)\widetilde{u}_\gamma - \widetilde{u})\,w\,dt\,da\,dx \right| \leq$$

$$\|\widetilde{y}_\gamma - z_d\|_{L^2(Q)} \|\overline{y}(w)\|_{L^2(Q)} + (N+1)\|\widetilde{u}_\gamma\|_{L^2(Q_\omega)} \|w\|_{L^2(Q_\omega)} +$$

$$\|\widetilde{u}\|_{L^2(Q_\omega)} \|w\|_{L^2(Q_\omega)}.$$

Therefore, using (3.47) and (3.37),

$$\left| -\int_Q \overline{y}(w)(\widetilde{y}_\gamma - z_d t)\,dt\,da\,dx - \int_{Q_\omega} ((N+1)\widetilde{u}_\gamma - \widetilde{u})\,w\,dt\,da\,dx \right| \leq$$

$$\left(\|\widetilde{y}_\gamma - z_d\|^2_{L^2(Q)} + [(N+1)\|\widetilde{u}_\gamma\|_{L^2(Q_\omega)} + \|\widetilde{u}\|_{L^2(Q_\omega)}]^2 \right)^{1/2} \|\overline{y}(w)\|_{\mathcal{E}} \leq C\|\overline{y}(w)\|_{\mathcal{E}},$$

where $C = C\left(N, T, \|\widetilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}\right) > 0$. It then follows from (3.55)

$$\left| \int_Q T_\gamma(\widetilde{u}_\gamma)\overline{y}(w) dtdadx \right| \leq C\|\overline{y}(w)\|_{\mathcal{E}}.$$

Consequently,

$$\|T_\gamma(\widetilde{u}_\gamma)\|_{\mathcal{E}'} = \left\| \frac{1}{\sqrt{\gamma}} p_\gamma + \frac{1}{\gamma} \widetilde{\xi}_\gamma(0) S(a; \widetilde{u}_\gamma) \right\|_{\mathcal{E}'} \leq C.$$

In particular,

$$\|\varrho^\gamma\|_{\mathcal{D}'(Q)} = \left\| \frac{1}{\sqrt{\gamma}} p_\gamma + \frac{1}{\gamma} \widetilde{\xi}_\gamma(0) S(a; \widetilde{u}_\gamma) \right\|_{\mathcal{D}'(Q)} \leq C, \tag{3.56}$$

where $C = C\left(N, T, \|\widetilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}\right) > 0$.

Now, proceeding as for $\overline{y}$ in pages 3-4, while using (3.56), we obtain that

$$\|\widetilde{q}^2_\gamma\|_{L^2(U; H^1_0(\Omega))} \leq C\left(N, T, \|\widetilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}\right). \tag{3.57}$$ □

Finally from (3.50) and (3.57) we deduce (3.45).

3.3. Characterization of the No-regret control.

PROPOSITION 3.12. *The adapted Low-regret optimal control $\widetilde{u}_\gamma$ converges $L^2(Q_\omega)$ to the No-regret control $\widetilde{u} \in \mathcal{M}$.*

Proof. In view of (3.37)-(3.42), there exists a subsequence of $(\widetilde{u}_\gamma, \widetilde{y}_\gamma, \widetilde{\xi}_\gamma, S(., \widetilde{u}_\gamma))$ still denoted by $(\widetilde{u}_\gamma, \widetilde{y}_\gamma, \widetilde{\xi}_\gamma, S(., \widetilde{u}_\gamma))$ and $\hat{u} \in L^2(Q_\omega)$, $\widetilde{y} \in L^2(U, H^1_0(\Omega))$, $\widetilde{\xi} \in L^2(U, H^1_0(\Omega))$, $\alpha \in L^2(0, A)$, $\tau \in L^2(QT)$ such that

$$\widetilde{u}_\gamma \rightharpoonup \hat{u} \text{ weakly in } L^2(Q_\omega), \tag{3.58}$$

$$\frac{1}{\sqrt{\gamma}} S(., \widetilde{u}_\gamma)) \rightharpoonup \alpha \text{ weakly in } L^2(0, A), \tag{3.59}$$

$$S(., \widetilde{u}_\gamma)) \to 0 \text{ strongly in } L^2(0, A), \tag{3.60}$$

$$\widetilde{y}_\gamma \rightharpoonup \widetilde{y} \text{ weakly in } L^2(U; H^1_0(\Omega)), \tag{3.61}$$

$$\widetilde{\xi}_\gamma \rightharpoonup \widetilde{\xi} \text{ weakly in } L^2(U; H^1_0(\Omega)), \tag{3.62}$$

$$\widetilde{\xi}_\gamma(., 0, .) \rightharpoonup \tau \text{ weakly in } L^2(Q_T). \tag{3.63}$$

If we multiply the first equation (3.25) by $\phi \in \mathcal{D}(Q)$ and the first equation in (3.26) by $\psi \in \mathcal{D}(Q)$ and integrate by parts over Q, we have

$$\int_Q \widetilde{y}_\gamma L^* \phi \, dt\, da\, dx = \int_Q (f + \widetilde{u}_\gamma \chi_\omega)\phi \, dt\, da\, dx$$

and

$$\int_Q \widetilde{\xi}_\gamma L\psi \, dt\, da\, dx = \int_Q (\widetilde{y}_\gamma - z_d)\psi \, dt\, da\, dx.$$

Passing in these two latter identities to the limit, while using (3.58), (3.61) and (3.62), we obtain

$$\int_Q \widetilde{y} L^* \phi \, dt\, da\, dx = \int_Q (f + \hat{u}\chi_\omega)\phi \, dt\, da\, dx$$

and

$$\int_Q \widetilde{\xi} L\psi \, dt\, da\, dx = \int_Q (\widetilde{y} - z_d)\psi \, dt\, da\, dx,$$

which after an integration by parts over Q give

$$\int_Q L\widetilde{y}\, \phi \, dt\, da\, dx = \int_Q (f + \hat{u}\chi_\omega)\phi \, dt\, da\, dx, \quad \forall \phi \in \mathcal{D}(Q)$$

and

$$\int_Q L^* \widetilde{\xi}\, \psi \, dt\, da\, dx = \int_Q (\widetilde{y} - z_d)\psi \, dt\, da\, dx, \quad \forall \psi \in \mathcal{D}(Q),$$

respectively. Hence, we can deduce

$$L\widetilde{y} = f + \hat{u}\chi_\omega \quad \text{in } Q \tag{3.64}$$

and

$$L^* \widetilde{\xi} = \widetilde{y} - z_d \quad \text{in } Q \tag{3.65}$$

Note that $\widetilde{y}, \widetilde{\xi} \in L^2(U, H_0^1(\Omega))$. This implies that $\widetilde{y}(t,a)|_\Gamma$ and $\widetilde{\xi}(t,a)|_\Gamma$ exist and belong to $L^2(\Gamma)$ for almost every $(t,a) \in U$. On the other hand from (3.64), (3.65) and the expression of the operator L and L^* given by (2.1), we have $\widetilde{y}, \widetilde{\xi} \in W(T,A)$. It follows from Remark 1.1 that $(\widetilde{y}(0,.,.), \widetilde{\xi}(T,.,.))$ exists and belongs $(L^2(Q_A))^2$ and $(\widetilde{y}(.,0,.), \widetilde{\xi}(.,A,.), \widetilde{\xi}(.,0,.))$ exists and belongs $(L^2(Q_T))^2$.

Now, if we multiply the first equation (3.25) by $\phi \in \mathcal{C}^\infty(\overline{Q})$ such that $\phi = 0$ on Σ, $\phi(\cdot, A, \cdot) = 0$ in Q_T and $\phi(T, \cdot, \cdot) = 0$ in Q_A and the first equation in (3.26) by $\psi \in \mathcal{C}^\infty(\overline{Q})$ such that $\psi = 0$ on Σ and $\psi(0, \cdot, \cdot) = 0$ in Q_A and integrate by parts over Q, we respectively have that

$$-\int_{Q_A} y^0\, \phi(0,a,x)\, da\, dx + \int_Q \widetilde{y}_\gamma\, L^* \phi \, da\, dx = \int_Q (f + \widetilde{u}_\gamma \chi_\omega)\phi \, dt\, da\, dx$$

and

$$\int_{Q_A} \widetilde{\xi}_\gamma(t,0,x)\, \psi(0,a,x)\, dt\, dx + \int_Q \widetilde{\xi}_\gamma\, L\psi \, da\, dx = \int_Q (\widetilde{y}_\gamma - z_d)\psi \, dt\, da\, dx$$

Passing these two latter identities to the limit while using (3.58), (3.61), (3.62) and (3.63), we obtain

$$\begin{array}{l} -\displaystyle\int_{Q_A} y^0\, \phi(0,a,x)\, da\, dx + \int_Q \widetilde{y} L^*\phi\, dt\, da\, dx = \int_Q (f+\hat{u}\chi_\omega)\phi\, dt\, da\, dx,\\ \forall \phi \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \phi|_\Gamma = 0,\ \phi(\cdot,A,\cdot)|_{Q_T} = 0,\ \phi(T,\cdot,\cdot)|_{Q_A} = 0, \end{array}$$

and

$$\begin{array}{l} \displaystyle\int_{Q_A} \tau\, \psi(0,a,x)\, dt\, dx + \int_Q \widetilde{\xi} L\psi\, dt\, da\, dx = \int_Q (\widetilde{y} - z_d)\psi\, dt\, da\, dx,\\ \forall \psi \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \psi|_\Gamma = 0,\ \psi(0,\cdot,\cdot)|_{Q_A} = 0, \end{array}$$

respectively, which after an integration by parts over Q give

$$\begin{array}{l} \displaystyle\int_Q (f+\hat{u}\chi_\omega)\phi\, dt\, da\, dx = -\int_{Q_A} (y^0 - \widetilde{y}(0,a,x))\, \phi(0,a,x)\, da\, dx +\\ \displaystyle\int_{Q_T} \widetilde{y}(t,0,x)\phi(t,0,x)\, dt\, dx - \int_\Sigma \widetilde{y}\frac{\partial \phi}{\partial \nu}\, dt\, da\, dx + \int_Q L\widetilde{y}\, \phi\, dt\, da\, dx,\\ \forall \phi \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \phi|_\Gamma = 0,\ \phi(\cdot,A,\cdot)|_{Q_T} = 0,\ \phi(T,\cdot,\cdot)|_{Q_A} = 0, \end{array}$$

and

$$\begin{array}{l} \displaystyle\int_Q (\widetilde{y} - z_d)\psi\, dt\, da\, dx = \int_{Q_A} (\tau - \widetilde{\xi}(t,0,x))\, \psi(0,a,x)\, dt\, dx +\\ \int_Q L^*\widetilde{\xi}\psi\, dt\, da\, dx + \int_{Q_A} \widetilde{\xi}(T,a,x)\psi(T,a,x)\, da\, dx +\\ \displaystyle\int_{Q_T} \widetilde{\xi}(t,A,x)\psi(t,A,x)\, dt\, dx + \int_\Sigma \widetilde{\xi}\frac{\partial \psi}{\partial \nu}\, dt\, da\, dx,\\ \forall \psi \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \psi|_\Gamma = 0,\ \psi(0,\cdot,\cdot)|_{Q_A} = 0. \end{array}$$

Using (3.64) and (3.65), we deduce from these latter identities that,

$$\begin{array}{l} 0 = -\displaystyle\int_{Q_A} (y^0 - \widetilde{y}(0,a,x))\, \phi(0,a,x)\, da\, dx +\\ \displaystyle\int_{Q_T} \widetilde{y}(t,0,x)\phi(t,0,x)\, dt\, dx - \int_\Sigma \widetilde{y}\frac{\partial \phi}{\partial \nu}\, dt\, da\, dx,\\ \forall \phi \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \phi|_\Gamma = 0,\ \phi(\cdot,A,\cdot)|_{Q_T} = 0,\ \phi(T,\cdot,\cdot)|_{Q_A} = 0, \end{array} \tag{3.66}$$

and

$$\begin{array}{l} 0 = \displaystyle\int_{Q_A} (\tau - \widetilde{\xi}(t,0,x))\, \psi(0,a,x)\, dt\, dx +\\ \displaystyle\int_{Q_A} \widetilde{\xi}(T,a,x)\psi(T,a,x)\, da\, dx +\\ \displaystyle\int_{Q_T} \widetilde{\xi}(t,A,x)\psi(t,A,x)\, dt\, dx + \int_\Sigma \widetilde{\xi}\frac{\partial \psi}{\partial \nu}\, dt\, da\, dx,\\ \forall \psi \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \psi|_\Gamma = 0,\ \psi(0,\cdot,\cdot)|_{Q_A} = 0. \end{array} \tag{3.67}$$

If we successively take in (3.66) and (3.67),

$$\begin{array}{l} \phi(\cdot,0,\cdot)|_{Q_T} = 0 \text{ and } \dfrac{\partial \phi}{\partial \nu}|_\Gamma = 0,\\ \psi(\cdot,0,\cdot)|_{Q_T} = \psi(\cdot,A,\cdot)|_{Q_T} = 0 \text{ and } \dfrac{\partial \psi}{\partial \nu}|_\Gamma = 0, \end{array}$$

$$\frac{\partial \phi}{\partial \nu}|_{\Gamma} = 0,$$
$$\psi(\cdot, A, \cdot)|_{Q_T} = 0 \text{ and } \frac{\partial \psi}{\partial \nu}|_{\Gamma} = 0,$$

then in (3.67),

$$\frac{\partial \psi}{\partial \nu}|_{\Gamma} = 0,$$

we successively obtain

$$\widetilde{y}(0, \cdot, \cdot) = y^0 \text{ in } Q_A, \tag{3.68}$$
$$\widetilde{\xi}(T, \cdot, \cdot) = 0 \text{ in } Q_A, , \tag{3.69}$$

$$\widetilde{y}(\cdot, 0, \cdot) = 0 \text{ in } Q_T, \tag{3.70}$$
$$\widetilde{\xi}(\cdot, A, \cdot) = 0 \text{ in } Q_T, \tag{3.71}$$

then

$$\widetilde{y} = 0 \text{ on } \Sigma, \tag{3.72}$$
$$\widetilde{\xi}(\cdot, 0, \cdot) = \tau \text{ in } Q_T, \tag{3.73}$$

and finally,

$$\widetilde{\xi} = 0 \text{ on } \Sigma. \tag{3.74}$$

Now, using (3.58),(3.61), (3.63), (3.73) and (3.39), we have from Lemma 3.5 that

$$S(., u^\gamma) \rightharpoonup S(., \hat{u}) \text{ weakly in } \mathcal{D}'(0, A).$$

Hence, using (3.60) and the uniqueness of the limit that

$$S(., \widetilde{u}_\gamma) \to S(., \hat{u}) = 0 \text{ strongly in } L^2(0, A).$$

Consequently,

$$\int_0^A S(a; \widetilde{u}_\gamma) g(a)\, da \to \int_0^A S(a; \hat{u}_\gamma) g(a)\, da = 0.$$

Thus $\hat{u} \in \mathcal{M}$ and we also have $\|S(.; \hat{u})\|_{L^2(0,A)} = 0$. Since $\widetilde{u}$ is a No-regret control and $\widehat{u} \in \mathcal{M}$, it follows from (3.18) that

$$J(\widetilde{u}, 0) - J(0, 0) \leq J(\widehat{u}, 0) - J(0, 0), \tag{3.75}$$

Observing that $\widetilde{u}_\gamma$ solves the problem $\inf\limits_{v \in L^2(Q_\omega)} \widetilde{\mathcal{J}}^\gamma(v)$, we have

$$\widetilde{\mathcal{J}}^\gamma(\widetilde{u}_\gamma) \leq \widetilde{\mathcal{J}}^\gamma(\widetilde{u}) = J(\widetilde{u}, 0) - J(0, 0), \tag{3.76}$$

which, in view of the definition of $\widetilde{\mathcal{J}}^\gamma$ given by (3.23), implies that

$$J(\widetilde{u}_\gamma, 0) - J(0, 0) + \|\widetilde{u}_\gamma - \widetilde{u}\|^2_{L^2(Q_\omega)} \leq \widetilde{\mathcal{J}}^\gamma(\widetilde{u}_\gamma) \leq \widetilde{\mathcal{J}}^\gamma(\widetilde{u}) = J(\widetilde{u}, 0) - J(0, 0).$$

Using the convexity and lower semi-continuity of J on $L^2(Q_\omega)$, (3.58) and (3.61), we obtain

$$J(\hat{u},0) - J(0,0) + \|\hat{u}-\widetilde{u}\|^2_{L^2(Q_\omega)} \leq \liminf_{\gamma\to 0} \widetilde{\mathcal{J}}^\gamma(\widetilde{u}_\gamma) \leq J(\widetilde{u},0) - J(0,0), \tag{3.77}$$

which combining with (3.75) gives

$$\|\hat{u}-\widetilde{u}\|^2_{L^2(Q_\omega)} \leq 0.$$

Hence,

$$\hat{u} = \widetilde{u} \text{ in } Q_\omega. \tag{3.78}$$

Thus the adapted Low-regret controls converge in $L^2(Q_\omega)$ to the No-regret control. Moreover from (3.78), (3.64), (3.68),(3.70) and (3.72), it follows that $\widetilde{y} = y(\widetilde{u},0) \in L^2(U; H^1_0(\Omega))$ unique solution of

$$\left\{\begin{array}{rcll} L\,\widetilde{y} &=& f + \widetilde{u}\chi_{Q_\omega} & \text{in} \quad Q, \\ \widetilde{y} &=& 0 & \text{on} \quad \Sigma, \\ \widetilde{y}(0,\cdot,\cdot) &=& y^0 & \text{in} \quad Q_A, \\ \widetilde{y}(\cdot,0,\cdot) &=& 0 & \text{in} \quad Q_T. \end{array}\right. \tag{3.79}$$

Similarly, from (3.65), (3.69),(3.71) and (3.74), we infer that $\widetilde{\xi} = \xi(\widetilde{u})$ is the unique solution of

$$\left\{\begin{array}{rcll} L^*\widetilde{\xi} &=& \widetilde{y} - z_d & \text{in} \quad Q, \\ \widetilde{\xi} &=& 0 & \text{on} \quad \Sigma, \\ \widetilde{\xi}(T,\cdot,\cdot) &=& 0 & \text{in} \quad Q_A, \\ \widetilde{\xi}(\cdot,A,\cdot) &=& 0 & \text{in} \quad Q_T. \end{array}\right. \tag{3.80}$$

∎

Proposition 3.13. *The No-regret control $\widetilde{u} \in \mathcal{M}$ is characterized by the functions $\widetilde{u}$, $\widetilde{y}$, $\widetilde{\xi}$, $\widetilde{p}$ and $\widetilde{q}$ unique solutions of the optimality system:*

$$\left\{\begin{array}{rcll} L\,\widetilde{y} &=& f + \widetilde{u}\chi_{Q_\omega} & \textit{in} \quad Q, \\ \widetilde{y} &=& 0 & \textit{on} \quad \Sigma, \\ \widetilde{y}(0,\cdot,\cdot) &=& y^0 & \textit{in} \quad Q_A, \\ \widetilde{y}(\cdot,0,\cdot) &=& 0 & \textit{in} \quad Q_T, \end{array}\right. \tag{3.81}$$

$$\left\{\begin{array}{rcll} L^*\widetilde{\xi} &=& \widetilde{y} - z_d & \textit{in} \quad Q, \\ \widetilde{\xi} &=& 0 & \textit{on} \quad \Sigma, \\ \widetilde{\xi}(T,\cdot,\cdot) &=& 0 & \textit{in} \quad Q_A, \\ \widetilde{\xi}(\cdot,A,\cdot) &=& 0 & \textit{in} \quad Q_T, \end{array}\right. \tag{3.82}$$

$$\left\{\begin{array}{rcll} L\,\widetilde{p} &=& 0 & \textit{in} \quad Q, \\ \widetilde{p} &=& 0 & \textit{on} \quad \Sigma, \\ \widetilde{p}(0,\cdot,\cdot) &=& 0 & \textit{in} \quad Q_A, \\ \widetilde{p}(\cdot,0,\cdot) &=& \lambda_1 & \textit{in} \quad Q_T, \end{array}\right. \tag{3.83}$$

$$\text{(3.84)} \qquad \begin{cases} L^*\widetilde{q} &= \widetilde{y} - z_d + \lambda_2 & in & Q, \\ \widetilde{q} &= 0 & on & \Sigma, \\ \widetilde{q}(T,\cdot,\cdot) &= 0 & in & Q_A, \\ \widetilde{q}(\cdot,A,\cdot) &= 0 & in & Q_T \end{cases}$$

and

$$\text{(3.85)} \qquad N\widetilde{u} + \widetilde{q} = 0 \ in \ Q_\omega,$$

where

$$\begin{aligned} \lambda_1 &= \lim_{\gamma\to 0} \frac{1}{\sqrt{\gamma}} \int_0^A y_\gamma(t,a,x,\widetilde{u}_\gamma) S(a;\widetilde{u}_\gamma)\ da, \\ \lambda_2 &= \lim_{\gamma\to 0} \left(\frac{1}{\sqrt{\gamma}} p_\gamma + \frac{1}{\gamma}\widetilde{\xi}_\gamma(0) S(a;\widetilde{u}_\gamma) \right). \end{aligned}$$

Proof. We have already proved (3.81) and (3.82) (see Page 22).

From (3.43), (3.56),(3.44) and (3.45), we have there exist $\lambda_1 \in L^2(Q_T)$, $\lambda_2 \in L^2(Q)$, $\widetilde{p} \in L^2(U; H_0^1(\Omega))$ and $\widetilde{q} \in \mathcal{D}'(Q)$ such that

$$\text{(3.86)} \qquad \widetilde{p}_\gamma(.,0,.) \rightharpoonup \lambda_1 \text{ in } L^2(Q_T),$$

$$\text{(3.87)} \qquad \frac{1}{\sqrt{\gamma}} p_\gamma + \frac{1}{\gamma}\widetilde{\xi}_\gamma(0) S(a;\widetilde{u}_\gamma) \rightharpoonup \lambda_2 \text{ in } L^2(Q),$$

$$\text{(3.88)} \qquad \widetilde{p}_\gamma \rightharpoonup \widetilde{p} \text{ in } L^2(U; H_0^1(\Omega)),$$

$$\text{(3.89)} \qquad \widetilde{q}_\gamma \rightharpoonup \widetilde{q} \text{ in } \mathcal{D}'(Q).$$

∎

Then, proceeding as for $\widetilde{y}_\gamma$ and $\widetilde{\xi}_\gamma$ in Pages 18-21 when passing to the limit in (3.27) and (3.28) and using (3.61), (3.87)-(3.89), we prove (3.83) and (3.84). To obtain (3.85), we pass to the limit in (3.36) while using (3.58), (3.78) and (3.89).

4. Conclusions.

We used the notion of No-regret and Low-regret to control a model describing the dynamics of population with age dependence and spatial structure with missing birth rate. In contrast to some works on the topic which need the control to acting on the whole domain so obtain the convergence optimality system that characterizes the Low-regret control towards the singular optimality characterizing the No-regret control, our control acts on a part of the domain. We then introduce an appropriate Hilbert space and apply the Aubin-Lions Lemma to an appropriate auxiliary problem to obtain the convergence of a adapted Low-regret control towards a No-regret control that we characterize.

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