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To cite this version: Besnard, Philippe *A Note on Directions for Cumulativity*. (2014) IfCoLog Journal of Logics and their Applications, 1 (2). 77-81. ISSN 2055-3714

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A Note on Directions for Cumulativity

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Abstract

Logical systems are often characterized as closure systems, by means of unary operators satisfying Reflexivity, Idempotence and Monotony. In order to capture non-monotone systems, Monotony can be replaced by Cumulativity, namely Restricted Cut and Cautious Monotony. This short note shows that in such a context, Restricted Cut is redundant.

Keywords: Non-Monotonic Consequence, Cumulativity.

1 Introduction

Tarski [7] introduced abtract logics, as consequence operations, later popularized by Scott [6] in connection with consequence relations à la Gentzen [3] (see also Gabbay [2]). The outcome is that abstract logics are often identified with closure operators over sets of formulas of a logical language. In symbols, given a logical language consisting of the set of formulas $\mathcal{F}$, a *consequence operator* is any C defined over $2^{\mathcal{F}}$ that satisfy all three axioms below:

$$\begin{array}{ll} X \subseteq C(X) & (Reflexivity) \\ C(C(X)) = C(X) & (Idempotence) \\ X \subseteq Y \Rightarrow C(X) \subseteq C(Y) & (Monotony) \end{array}$$

In an insightful attempt to also capture logical systems failing it, the last axiom was weakened by Makinson [5] into

$$X \subseteq Y \subseteq C(X) \Rightarrow C(X) = C(Y) \qquad (Cumulativity)$$

It was soon split into two "halves" dubbed Cautious Monotony and Restricted Cut.

$$X \subseteq Y \subseteq C(X) \Rightarrow C(X) \subseteq C(Y) \qquad (\textit{Cautious Monotony})$$
$$X \subseteq Y \subseteq C(X) \Rightarrow C(Y) \subseteq C(X) \qquad (\textit{Restricted Cut})$$

It is the purpose of this note to show that, in its more natural context (Reflexivity and Idempotence), Cumulativity is not the combination of the two "halves" but is actually equivalent to one of them, namely Cautious Monotony.

Actually, it is shown below that if all three of Reflexivity, Idempotence, and Cautious Monotony hold, then so does Restricted Cut. In semantical terms, it means that $f(A) \subseteq A$ together with $f(f(A)) = f(A)$ and $f(A) \subseteq B \subseteq A \Rightarrow f(B) \subseteq f(A)$ give $f(A) \subseteq B \subseteq A \Rightarrow f(A) \subseteq f(B)$. Indeed, from $f(B) \subseteq f(A) \subseteq B \subseteq A$, it follows that $f(B) \subseteq f(A) \subseteq B$, so $f(A) = f(f(A)) \subseteq f(B)$.

2 The One Direction for Cumulativity

We switch to a sequent presentation (freely drawn upon Kleene [4] and Dummett [1]) as the result is less obvious there, although no less striking. That is, we consider a generalization of sequents

$$X_1, \ldots, X_n \vdash Y$$

such that the antecedent $X_1, \ldots, X_n$ consists of countably many formulas given as a (finite) series of sets for the sake of brevity. Thus, the rules have the following general form where W_i, X_j, Y, Z denote countable sets of formulas and *proviso* is a condition:

$$\frac{X_1, \ldots, X_n \vdash Y}{W_1, \ldots, W_m \vdash Z} \quad \textit{proviso}$$

Such a rule means that if *proviso* is true then $W_1, \ldots, W_m \vdash Z$ can be derived from $X_1, \ldots, X_n \vdash Y$.

The axioms, written here as rules with no premises, are meant to encode Reflexivity and have the following form

$$\frac{}{X \vdash Y} \quad (X \cap Y \neq \emptyset)$$

As for the rules, first please observe that we cannot include Left Thinning due to the motivation for Cumulativity. For the sake of brevity, we do not provide all rules and we resort to a rule mixing Left Interchange and Left Contraction as follows

Merging:

$$\frac{W, X, Y \vdash Z}{W, Y \vdash Z} \quad (X \subseteq Y)$$

Idempotence:

$$\frac{\{x \mid X \vdash x\} \vdash Z}{X \vdash Z}$$

In a monotone compact logic, the effect of Idempotence is in fact obtained through Restricted Cut. For such a logic, Idempotence is indeed an admissible rule in a sequent system with Restricted Cut (even as an admissible rule). However, Cumulativity (i.e., Restricted Cut and Cautious Monotony) was originally motivated by non-monotone logics hence postulating Idempotence makes sense when considering these logics.

Cautious Monotony:

$$\frac{X \vdash Z}{X, Y \vdash Z} \quad (y \in Y \Rightarrow X \vdash y)$$

Restricted Cut:

$$\frac{X, Y \vdash Z}{X \vdash Z} \quad (y \in Y \Rightarrow X \vdash y)$$

This formulation is intended to exhibit the fact that, under the same proviso, Cautious Monotony and Restricted Cut trigger converse inferences.

Theorem: *Restricted Cut is an admissible rule in any system enjoying Idempotence, Merging and Cautious Monotony.*

Proof. 1. Applying Cautious Monotony ($X = \Gamma$ and $Y = \Theta$)

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Theta \vdash \Delta} \quad (\theta \in \Theta \Rightarrow \Gamma \vdash \theta)$$

Stated otherwise, if $(\theta \in \Theta \Rightarrow \Gamma \vdash \theta)$ then $(\gamma \in \{\gamma \mid \Gamma \vdash \gamma\} \Rightarrow \Gamma, \Theta \vdash \gamma)$.

2. Applying Cautious Monotony ($X = \Gamma, \Theta$ and $Y = \{\gamma \mid \Gamma \vdash \gamma\}$)

$$\frac{\Gamma, \Theta \vdash \Delta}{\Gamma, \Theta, \{\gamma \mid \Gamma \vdash \gamma\} \vdash \Delta} \quad (\gamma \in \{\gamma \mid \Gamma \vdash \gamma\} \Rightarrow \Gamma, \Theta \vdash \gamma)$$

3. In view of what we just proved in Step 1, we obtain

$$\frac{\Gamma, \Theta \vdash \Delta}{\Gamma, \Theta, \{\gamma \mid \Gamma \vdash \gamma\} \vdash \Delta} \quad (\theta \in \Theta \Rightarrow \Gamma \vdash \theta)$$

4. Trivially, if $(\theta \in \Theta \Rightarrow \Gamma \vdash \theta)$ then $\Theta \subseteq \{\gamma \mid \Gamma \vdash \gamma\}$ hence applying Merging gives

$$\frac{\Gamma, \Theta \vdash \Delta}{\Gamma, \{\gamma \mid \Gamma \vdash \gamma\} \vdash \Delta} \quad (\theta \in \Theta \Rightarrow \Gamma \vdash \theta)$$

5. According to the axioms, $\Gamma \subseteq \{\gamma \mid \Gamma \vdash \gamma\}$ hence applying Merging again gives

$$\frac{\Gamma, \Theta \vdash \Delta}{\{\gamma \mid \Gamma \vdash \gamma\} \vdash \Delta} \quad (\theta \in \Theta \Rightarrow \Gamma \vdash \theta)$$

6. Lastly, applying Idempotence yields

$$\frac{\Gamma, \Theta \vdash \Delta}{\Gamma \vdash \Delta} \quad (\theta \in \Theta \Rightarrow \Gamma \vdash \theta)$$

which is exactly Restricted Cut.

□

The author is aware of not being the first to figure all this out but after discussing with various colleagues working in the field, it appears that this was largely ignored, and unpublished to the best of his knowledge, hence it could justify a brief note such as the present one.

3 Conclusion

Reflexivity and Idempotence are the most desirable features of a logical system. If Monotony is to be weakened, Cumulativity conveys the attractive idea that intermediate conclusions could be, as premises, freely added or freely removed without changing the overall set of conclusions. However, we have shown that Cumulativity is, unexpectedly, captured by part of the idea, Cautious Monotony, although the latter only imposes that intermediate conclusions could be removed from the premises with no loss among conclusions. In other words, the other part of the idea, Restricted Cut, is actually otiose with respect to weakening Monotony, or Full Cut for that matter. Is there a context in which Restricted Cut would play some logical role? (Please observe that a formal role is possible, together with Reflexivity, as Idempotence ensues.)

References

[1] M. Dummett. *Elements of Intuitionism.* Oxford University Press, 2nd edition, 2000.

[2] D. Gabbay. *Semantical Investigations in Heyting's Intuitionistic Logic*, chapter Consequence Relations. Reidel, Dordrecht, 1981.

[3] G. Gentzen. *The Collected Works of Gerhard Gentzen (M. E. Szabo, editor)*, chapter Investigations into Logical Deduction. North-Holland, Amsterdam, 1969.

[4] S. Kleene. *Introduction to Metamathematics.* North-Holland, Amsterdam, 1952.

[5] D. Makinson. *Handbook of Logic in Artificial Intelligence and Logic Programming (D. M. Gabbay, C. J. Hogger, and J. A. Robinson (editors)*, volume III (D. Nute, volume co-ordinator), chapter General Patterns in Nonmonotonic Reasoning, Clarendon Press, Oxford, 1994.

[6] D. Scott. Completeness and Axiomatizability in Many-Valued Logic, *Proceedings of the Tarski Symposium (L. Henkin, editor)*, AMS, Providence, pp. 411-435, 1974.

[7] A. Tarski. *Logic, Semantics, Metamathematics (E. H. Woodger, editor)*, chapter On Some Fundamental Concepts of Metamathematics, Oxford University Press, 1956.