Uniqueness result for the 3-D Navier-Stokes-Boussinesq equations with horizontal dissipation

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Abstract

In this paper, for the 3-D Navier-Stokes-Boussinesq system with horizontal dissipation, where there is no smoothing effect on the vertical derivatives, we prove a uniqueness result of solutions $(u, \rho) \in L^\infty_T\big(H^{0,s} \times H^{0,1-s}\big)$ with $(\nabla_h u, \nabla_h \rho) \in L^2_T\big(H^{0,s} \times H^{0,1-s}\big)$ and $s \in [1/2, 1]$. As a consequence, we improve the conditions stated in the paper [13] in order to obtain a global well-posedness result in the case of axisymmetric initial data.

keywords: Boussinesq sytem, horizontal dissipation, anisotropic inequalities, uniqueness, global well-posedness.
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1 Introduction and main results

The Navier-Stokes-Boussinesq system is obtained from the density dependent Navier-Stokes equations by using the Boussinesq approximation. It is widely used to model geophysical flows (for instance oceanical or atmospherical flows) whenever rotation and stratification play an important role (see [16]). We will consider the following so-called Navier-Stokes-Boussinesq equations with horizontal dissipation:

$$(NSB_h) \quad \begin{cases} \big(\partial_t + u\cdot\nabla\big)u - \Delta_h u + \nabla P = \rho e_3 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \big(\partial_t + u\cdot\nabla\big)\rho - \Delta_h \rho = 0 \\ div(u) = 0 \\ (u,\rho)_{|t=0} = (u_0, \rho_0), \end{cases}$$

where $\Delta_h := \partial_1^2 + \partial_2^2$ denotes the horizontal laplacian and $e_3 = (0,0,1)^T$ is the third vector of the canonical basis of $\mathbb{R}^3$.
The unknowns of the system are $u = (u^1, u^2, u^3)$, ρ and P which represent respectively: the velocity, the density and the pressure of the fluid.

In the following we will say that (u, ρ) is a solution of (NSB_h) if it is a weak solution in the classical sense (see for instance [3] pages 123,132 and 204). We recall also that from a solution (u, ρ) we may use a result of De Rham in order to recover a pressure P (which depends on u and ρ) and to obtain a distributional solution (u, ρ, P) of the system (NSB_h).

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Note that in (NSB_h) the diffusion only occur in the horizontal direction. This is a natural assumption for several cases of interest in geophysical fluids flows (see [16]). However $-\Delta_h$ is a less regularizing operator than the laplacian $-\Delta$ and we cannot expect a better theory than for the classical Navier-Stokes-Boussinesq equations:

$$(NSB) \quad \begin{cases} \big(\partial_t + u\cdot\nabla\big)u - \Delta u + \nabla P = \rho e_3 \quad \text{in } \mathbb{R}^+\times\mathbb{R}^3 \\ \big(\partial_t + u\cdot\nabla\big)\rho - \Delta\rho = 0 \\ div(u) = 0 \\ (u,\rho)_{|t=0} = (u_0,\rho_0), \end{cases}$$

In particular, the question of the global well-posedness of (NSB) and consequently of (NSB_h) remains largely open, but recently the system (NSB_h) has received a lot of attention from mathematicians (see for instance [13, 1, 17]) and significant progress in its analysis have been made.

Note again that (NSB_h) involves the operator $-\Delta_h$ which smooth only along the horizontal variables. Hence we need to estimate differently the horizontal and the vertical directions, and the natural functional setting for the analysis involves some anisotropic Sobolev and Besov spaces. The definitons of these spaces and some of their important properties are recalled in the next section.

In order to analyse (NSB_h) it is usefull to forget its second equation for a while, and to consider first the Navier-Stokes equations with horizontal laplacian:

$$(NS_h) \quad \begin{cases} \big(\partial_t + u\cdot\nabla\big)u - \Delta_h u + \nabla P = 0 \quad \text{in } \mathbb{R}^+\times\mathbb{R}^3 \\ div(u) = 0 \\ u_{|t=0} = u_0 \end{cases}$$

Several interesting studies for this last system were done. In [3], the authors proved the local existence and the global one for small data in $H^{0,s}$ for some $s > \frac{1}{2}$. The proof of the exsitence part in [3] uses deeply the structure of the equation and the fact that u is a divergence free vector field. The key point used in their estimates is related to the fact[1] that $H^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$ and that $H^s(\mathbb{R})$ is an algebra. Hence it is easy to deal with the term $u^h\cdot\nabla_h u$ by using some product rules in the well chosen spaces. Next, after using the divergence free condition together with some Littlewood-Paley stuffs in a clever way they were able to treat the term $u^3\partial_3 u$ with the same argument. Always in [3], the authors proved also a uniqueness result (but only for $s > \frac{3}{2}$, because of the term $w^3\partial_3 u$) by establishing a H^{0,s_0}-energy estimate for a difference between two solutions $w = u - v$, where $s_0 \in]\frac{1}{2}, s]$. Later in [12], D.Iftime had overcome the difficulty by remarking that it is sufficient to estimate w in $H^{-\frac{1}{2}}$ with respect to the vertical variable, and this only requires an $H^{\frac{1}{2}}$ regularity for u in the vertical direction. Then he proved a uniqueness result for any $s > \frac{1}{2}$, and the gap between existence and uniqueness was closed.

To do something similar with system (NSB_h) we begin by estimating the horizontal terms (terms which contain only horizontal derivatives) by using some product rules in the adequate Besov and Sobolev spaces. For the vertical terms (terms which contain only vertical derivatives) we follow in general the idea in [3] in order to transform them into terms similar to the horizontal ones by using the divergence free condition. Hence, for $s \in]\frac{1}{2}, 1]$, we first

[1]Recall that this argument permits to prove the uniqueness of weak-solution for the classical Navier-Stokes problem in dimension two.

propose to estimate the difference between two solutions $w = u - v$ in $H^{0,s-1}$ instead of $H^{0,-\frac{1}{2}}$ providing that the solution u already exists in the $H^{0,s}$ energy-space (see Appendix for a proof of an existence result). For the second equation, denoting the difference between two solutions $\theta = \rho_1 - \rho_2$ we remark that:

- The function ρ only appears in the third equation of u (the equation for the component u^3). Hence a priori, we only need to estimate ρ in the H^{s-1}-norm with respect to the vertical variable.
- In order to deal with the term $u^h \cdot \nabla_h \theta$, we must estimate θ with respect to the vertical variable in some space $H^{-\alpha}$, with $\alpha \geq 0$ and such that $H^s(\mathbb{R}) \times H^{-\alpha}(\mathbb{R})$ holds to be a subspace of $H^{-\alpha}(\mathbb{R})$. In fact, lemma 2 bellow says that the minimum index $-\alpha$ that can be chosen is $-\alpha = -s$.
- For the term $w^h \cdot \nabla_h \rho$, if we consider that ρ lies in some H^β-space, with respect to the vertical variable, then a direct application of the product rules shows that we need $\beta \geq 1 - s$. Moreover, because the system is hyperbolic in the vertical direction, we expect the loss of one derivative.

Hence we will estimate vertically ρ in H^{1-s} and θ in H^{-s}.

For the critical case where $s = \frac{1}{2}$, in [15] M.Paicu proved a uniqueness[2] result for (NS_h) in $L^\infty_T(H^{0,\frac{1}{2}}) \cap L^2_T(H^{1,\frac{1}{2}})$. It is clear that such a space falls to be embedded in L^∞ in the vertical direction which is the major problem that prevents using similar arguments to those in the case where $s > \frac{1}{2}$. In order to prove the uniqueness, the author in [15] established a double logarithm estimate (see (7)) and concluded by using the Osgood lemma. We will show that (NSB_h) can be treated in the same way.

Our main result is the following:

Theorem 1. ***Uniqueness***
Let $s \in [1/2, 1]$ and $(u, \rho), (v, \eta)$ be two solutions for system (NSB_h) in

$$L^\infty_{loc}(\mathbb{R}_+; H^{0,s}) \cap L^2_{loc}(\mathbb{R}_+; H^{1,s}) \times L^\infty_{loc}(\mathbb{R}_+; H^{0,1-s}) \cap L^2_{loc}(\mathbb{R}_+; H^{1,1-s})$$

Then $(u, \rho) = (v, \eta)$

As an interesting consequence, we can improve the results of global well-posedeness in the case of axisymmetric initial data established in [13].

Let us first recal some basic notions: We say that a vector field u is axisymmetric if it satisfies

$$\mathcal{R}_{-\alpha}(u(\mathcal{R}_\alpha(x))) = u(x), \quad \forall \alpha \in [0, 2\pi], \ \forall x \in \mathbb{R}^3,$$

where $\mathcal{R}_\alpha$ denotes the rotation of axis (Oz) and with angle α. Moreover, an axisymmetric vector field u is called without swirl if it has the form:

$$u(x) = u^r(r, z) e_r + u^z(r, z) e_z, \quad x = (x_1, x_2, x_3), r = \sqrt{x_1^2 + x_2^2} \text{ and } z = x_3.$$

We say that a scalar function f is axisymmetric, if the vector field $x \mapsto f(x) e_z$ is axisymmetric. We also denote by $\omega = \operatorname{curl} u$ the vorticity of u. Then we will prove:

[2]We should mention that the existence of solution in such scaling-invariant space is still an open problem even for the classical Navier-Stokes system.

Theorem 2. *Let $u_0 \in H^1(\mathbb{R}^3)$ be an axisymmetric divergence free vector field without swirl such that $\frac{\omega_0}{r} \in L^2$ and let $\rho_0 \in L^2$ be an axisymmetric function. Then there is a unique global solution (u, ρ) of the system (NSB_h). Moreover we have:*

$$u \in \mathcal{C}(\mathbb{R}_+; H^1) \cap L^2_{loc}(\mathbb{R}_+; H^{1,1} \cap H^{2,0}), \quad \frac{\omega}{r} \in L^\infty_{loc}(\mathbb{R}_+; L^2) \cap L^2_{loc}(\mathbb{R}_+; H^{1,0})$$
$$\rho \in \mathcal{C}(\mathbb{R}_+; L^2) \cap L^2_{loc}(\mathbb{R}_+; H^{1,0})$$

Note that in Theorem 2, we only assume that $(u_0, \rho_0) \in H^1 \times L^2$ whereas in [13] the authors consider a stronger condition. Namelly they assume that $(\nabla \times u_0, \rho_0)$ is in $H^{0,1} \times H^{0,1}$ or in $L^\infty \times H^{0,1}$. In both works the key point consists to establish an uniqueness result: it is the Theorem 1 for us, whereas in [13] the authors assume a strong initial condition in order to obtain some double exponential control in time for the gradient of u.

The paper is organized as follows: in section 2, for the reader's convenience we recall the required background concerning the functional spaces and some useful technical tools. In section 3, we establish several a priori estimates which are then used in section 4 to prove the two theorems above. Finally in Appendix we shall prove a result of well posedeness for (NSB_h) under some smallness conditions involving only T, the L^2-norm of ρ_0 and the $H^{0,s}$-norm of u_0. The uniqueness part of this result is a consequence of Theorem 1.

2 Notations and functional spaces

Throughout this paper we write $\mathbb{R}^3 = \mathbb{R}^2_h \times \mathbb{R}_v$ and for any vector $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, we will denote the two first componants by ξ_h and the last one by ξ_v, that is to say: $\xi = (\xi_1, \xi_2, \xi_3) := (\xi_h, \xi_v)$. Similarly, for any vector field $X = (X^1, X^2, X^3)$ we will write $X = (X^h, X^v)$ with the meaning that $X^h = (X^1, X^2)$ and $X^v = X^3$.

We will also use the notations:

$H^s_h = H^s(\mathbb{R}^2_h)$, $H^s_v = H^s(\mathbb{R}_v)$, $L^p_v(H^s_h) = L^p(\mathbb{R}_v; H^s_h)$ and $L^r_T L^p_h L^q_v = L^r(0, T; L^p(\mathbb{R}^2_h; L^q(\mathbb{R}_v)))$.

Recall that (NSB_h) involves the operator $-\Delta_h$ which only regularizes along the horizontal direction. Hence the regularity of the functions along the vertical variable must be mesured differently than the horizontal ones, and we then need some anisotropic function spaces. We now provide the definition of these spaces which are based on an anisotropic version of the Littlewood-Paley theory (see [2] for more details).

Let (ψ, φ) be a couple of smooth functions with value in $[0, 1]$ satisfying:

$$\text{Supp } \psi \subset \{\xi \in \mathbb{R} : |\xi| \le \frac{4}{3}\}, \qquad \text{Supp } \varphi \subset \{\xi \in \mathbb{R} : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$$
$$\psi(\xi) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi) = 1 \ \ \forall \xi \in \mathbb{R}, \quad \sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1 \ \ \forall \xi \in \mathbb{R}\backslash\{0\}.$$

Let a be a tempered distribution, $\hat{a} = \mathcal{F}(a)$ its Fourier transform and $\mathcal{F}^{-1}$ denotes the inverse of $\mathcal{F}$. We define the non-homogeneous dyadic blocks Δ_q and the homogeneous ones $\dot{\Delta}_q$ by setting:

$$\Delta_q^v a := \begin{cases} \mathcal{F}^{-1}\big(\varphi(2^{-q}|\xi_3|\hat{a})\big) \text{ for } q\in\mathbb{N} \\ \mathcal{F}^{-1}\big(\psi(|\xi_3|\hat{a})\big) \text{ for } q=-1 \\ 0 \text{ for } q\leq -2 \end{cases} \qquad \Delta_j^h a := \begin{cases} \mathcal{F}^{-1}\big(\varphi(2^{-j}|\xi_h|\hat{a})\big) \text{ for } j\in\mathbb{N} \\ \mathcal{F}^{-1}\big(\psi(|\xi_h|\hat{a})\big) \text{ for } j=-1 \\ 0 \text{ for } j\leq -2 \end{cases}$$

$$S_q^v := \Delta_q^v - \Delta_{q-1}^v \ \forall q\in\mathbb{Z}, \qquad S_j^h := \Delta_j^h - \Delta_{j-1}^h \ \forall j\in\mathbb{Z}$$

$$\dot{\Delta}_q^v a := \mathcal{F}^{-1}\big(\varphi(2^{-q}|\xi_3|\hat{a})\big)\ \forall\, q\in\mathbb{Z}, \qquad \dot{\Delta}_j^h a := \mathcal{F}^{-1}\big(\varphi(2^{-j}|\xi_h|\hat{a})\big)\ \forall\, j\in\mathbb{Z},$$

$$\dot{S}_q^v := \dot{\Delta}_q^v - \dot{\Delta}_{q-1}^v \ \forall q\in\mathbb{Z}, \qquad \dot{S}_j^h := \dot{\Delta}_j^h - \dot{\Delta}_{j-1}^h \ \forall j\in\mathbb{Z}.$$

We then have $a = \sum_{m\geq -1}\Delta_m a = \sum_{m\in\mathbb{Z}}\dot{\Delta}_m a$ for both horizontal and vertical decompositions. Moreover, in all the situations, i.e. for Δ, S with the same index of direction (horizontal or vertical) and in both homogeneous and nonhomogeneous cases it holds:

$$\Delta_m\Delta_{m'}a = 0 \text{ if } |m-m'|\geq 2$$
$$\Delta_m\big(S_{m'-1}a\Delta_{m'}a\big) = 0 \text{ if } |m-m'|\geq 5$$
$$\Delta_m \sum_{i\in\{0,1,-1\}}\sum_{m'\in\mathcal{Z}}\big(\Delta_{m'+i}a\Delta_{m'}a\big) = \Delta_m \sum_{i\in\{0,1,-1\}}\sum_{m'\geq m-N_0}\big(\Delta_{m'+i}a\Delta_{m'}a\big),$$

where $N_0\in\mathbb{N}$ can be chosen independently of a (we can in fact assume that $N_0=5$).

In what follows, we will use the so-called Bony decomposition (see [2]):

$$ab = T_a(b) + T_b(a) + R(a,b),$$
$$T_a(b) := \sum_{q\in Z} S_{q-1}a\Delta_q b, \qquad R(a,b) := \sum_{i\in\{0,1,-1\}}\sum_{q\in Z}\Delta_{q+i}a\Delta_q b.$$

Here again all the situations may be considered however particular cases must be precised by using the adequate notations. For instance if we consider the non-homogeneous version for the vertical variable, we have to add the exponent ^{v} in all the operators T_a, T_b, R, S_q and Δ_q.

Our analysis will be made in the context of the non-homogeneous and anisotropic Sobolev and Besov spaces:

Definition 1. *Let s, t be two real numbers and let p, q_1, q_2 be in $[1,+\infty]$, we define the space $(B^t_{p,q_1})_h(B^s_{p,q_2})_v$ as the space of tempered distributions u such that*

$$\|u\|_{(B^t_{p,q_1})_h(B^s_{p,q_2})_v} := \Big\|2^{kt}2^{js}\left\|\Delta_k^h\Delta_j^v u\right\|_{L^p}\Big\|_{l_k^{q_1}(\mathbb{Z};l_j^{q_2}(\mathbb{Z}))} < \infty$$

In the situation where $q_1=q_2=q$, we use the notation $B^{t,s}_{p,q} := (B^t_{p,q})_h(B^s_{p,q})_v$. If $p=q=2$ then this last space is denoted by $H^{t,s}$. If moreover $t=0$ then we have:

$$\|u\|_{H^{0,s}} \approx \big(\sum_{j\in\mathbb{Z}} 2^{2js}\left\|\Delta_j^v u\right\|_{L^2}^2\big)^{\frac{1}{2}}$$

Let $f\in B^{0,s}_{p,2}$, the following properties will be of constant use in the paper:

$$\forall s\in\mathbb{R},\ \exists c_q = c_q(f):\ \left\|\Delta_q^v f\right\|_{L^p} \leq c_q 2^{-sq}\|f\|_{B^{0,s}_{p,2}},\ \text{and} \sum_{q\geq -1} c_q^2 \leq 1 \tag{1}$$

$$\forall s<0,\ \exists \tilde{c}_q = \tilde{c}_q(f):\ \left\|S_q^v f\right\|_{L^p} \leq \tilde{c}_q 2^{-sq}\|f\|_{B^{0,s}_{p,2}},\ \text{and} \sum_{q\geq -1} \tilde{c}_q^2 \leq 1 \tag{2}$$

Other properties of these spaces can be found in [2] for the usual isotropic version, and in [4] for the anisotropic case. A very helpful tool related to the Bernstein lemma (see for instance lemma 2.1 in [6]) is:

Lemma 1. *Let $\mathcal{B}_h$ (resp. $\mathcal{B}_v$) be a ball of $\mathbb{R}^2_h$ (resp. $\mathbb{R}_v$) and $\mathcal{C}_h$ (resp. $\mathcal{C}_v$) a ring of $\mathbb{R}^2_h$ (resp. $\mathbb{R}_v$). Let also a be a tempered distribution and $\hat{a}$ its Fourier transform. Then for $1 \le p_2 \le p_1 \le \infty$ and $1 \le q_2 \le q_1 \le \infty$ we have:*

$$\begin{aligned}
&Supp\ \hat{a} \subset 2^k \mathcal{B}_h \implies \left\|\partial^\alpha_{x_h} a\right\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{k\left(|\alpha|+2\left(\frac{1}{p_2}-\frac{1}{p_1}\right)\right)} \|a\|_{L^{p_2}_h(L^{q_1}_v)} \\
&Supp\ \hat{a} \subset 2^l \mathcal{B}_v \implies \left\|\partial^\beta_{x_3} a\right\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{l\left(\beta+\left(\frac{1}{q_2}-\frac{1}{q_1}\right)\right)} \|a\|_{L^{p_1}_h(L^{q_2}_v)} \\
&Supp\ \hat{a} \subset 2^k \mathcal{C}_h \implies \|a\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{-kN} \sup_{|\alpha|=N} \left\|\partial^\alpha_{x_h} a\right\|_{L^{p_1}_h(L^{q_1}_v)} \\
&Supp\ \hat{a} \subset 2^l \mathcal{C}_v \implies \|a\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{-lN} \left\|\partial^N_{x_3} a\right\|_{L^{p_1}_h(L^{q_1}_v)}
\end{aligned}$$

We will also need some product rules in (non-homogeneous) Sobolev spaces:

Lemma 2. *Let $\sigma, \sigma', s, s_0 \in \mathbb{R}$ verifying $\sigma, \sigma' < 1,\ \sigma + \sigma' > 0,\ s_0 > 1/2, s \le s_0$ and $s + s_0 \ge 0$ then there exists a constant $C = C(\sigma, \sigma', s, s_0)$ such that:*

$$\|ab\|_{H^{\sigma+\sigma'-1,s}} \le C \|a\|_{H^{\sigma,s}} \|b\|_{H^{\sigma',s_0}}, \quad \forall a, b \in \mathcal{S}$$

Proof
Remark first that because $\|ab\|_{H^{\sigma+\sigma'-1,s}} = \left\| \|ab\|_{H^s_v} \right\|_{H^{\sigma+\sigma'-1}_h}$, we have only to prove that:

$$H^s(\mathbb{R}) \times H^{s_0}(\mathbb{R}) \subset H^s(\mathbb{R}). \tag{3}$$

Indeed, by using (3) together with the usual product rules with respect to the horizontal variables, the desired result follows (see for instance [2]). Note also that when $s_0 > s > \frac{1}{2}$, the inclusion (3) is trivial since in this case the space H^s is an algebra and clearly $H^{s_0} \hookrightarrow H^s$. It remains then only to prove (3) in the situation $s_0 > \frac{1}{2} \ge s$ and $s + s_0 \ge 0$. In order to do this, we use the Bony decomposition in the vertical variable: $ab = T^v_a b + T^v_b a + R^v(a, b)$.
For the first term, let us consider the two cases: $s < \frac{1}{2}$ and $s = \frac{1}{2}$.

The case $s < \frac{1}{2}$: By using the embedding $H^s(\mathbb{R}) \hookrightarrow B^{s-\frac{1}{2}}_{\infty,2}$, together with (2) we obtain for any $q \ge -1$:

$$\begin{aligned}
\left\|\Delta^v_q(T^v_a b)\right\|_{L^2(\mathbb{R})} &\lesssim \left\|S^v_{q-1} a\right\|_{L^\infty(\mathbb{R})} \left\|\Delta^v_q b\right\|_{(L^2\mathbb{R})} \lesssim c_q^2 2^{-q(s-\frac{1}{2})} 2^{-qs_0} \|a\|_{B^{s-\frac{1}{2}}_{\infty,2}(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})} \\
&\lesssim c_q^2 \max\{1, 2^{s_0-\frac{1}{2}}\} 2^{-qs} \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})} \lesssim c_q^2 2^{-qs} \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}.
\end{aligned}$$

It follows that

$$\|T^v_a b\|_{H^s(\mathbb{R})} \lesssim \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})} \tag{4}$$

The case $s = \frac{1}{2}$. We use the following estimate:

$$\left\|S^v_{q-1} a\right\|_{L^\infty(\mathbb{R})} \le \sum_{-1 \le j \le q} 2^{\frac{j}{2}} \left\|\Delta^v_j a\right\|_{L^2(\mathbb{R})} \lesssim \sqrt{q} \|a\|_{H^{\frac{1}{2}}(\mathbb{R})},$$

in order to obtain:

$$\left\|\Delta^v_q(T^v_a b)\right\|_{L^2(\mathbb{R})} \lesssim \left\|S^v_{q-1} a\right\|_{L^\infty(\mathbb{R})} \left\|\Delta^v_q b\right\|_{L^2(\mathbb{R})} \lesssim c_q \sqrt{q} 2^{-q(s_0-\frac{1}{2})} 2^{-\frac{q}{2}} \|a\|_{H^{\frac{1}{2}}(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}$$

Seen that $\forall\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all $q \in \mathbb{R}^+$: $\sqrt{q}2^{-q\varepsilon} \le C_\varepsilon$, we infer that:

$$\|T_a^v b\|_{H^{\frac{1}{2}}(\mathbb{R})} \lesssim \|a\|_{H^{\frac{1}{2}}(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}$$

and (4) follows for all $s \le \frac{1}{2} < s_0$. Moreover, by using the embedding $H^{s_0}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ together with the estimate:

$$\left\|\Delta_q^v(T_b^v a)\right\|_{L^2(\mathbb{R})} \lesssim \left\|S_{q-1}^v b\right\|_{L^\infty(\mathbb{R})} \left\|\Delta_q^v a\right\|_{L^2(\mathbb{R})} \lesssim \|b\|_{L^\infty(\mathbb{R})} c_q 2^{-qs} \|a\|_{H^s(\mathbb{R})},$$

we obtain:

$$\|T_b^v a\|_{H^s(\mathbb{R})} \lesssim \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}$$

For the reminder term, if $s + s_0 > 0$, then applying lemma 1 together with (2) gives:

$$\begin{aligned}\left\|\Delta_q^v(R(a,b))\right\|_{L^2(\mathbb{R})} &\lesssim 2^{\frac{q}{2}} \sum_{j\ge q-N_0} \left\|\Delta_j^v a\right\|_{L^2(\mathbb{R})} \left\|\tilde{\Delta}_j^v b\right\|_{L^2(\mathbb{R})}, \text{ where } \tilde{\Delta}_q^v := \sum_{i=\{-1,0,1\}} \Delta_{q+i}^v, \\ &\lesssim 2^{\frac{q}{2}} \sum_{j\ge q-N_0} \left(c_j^2 2^{-j(s+s_0)}\right) \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}.\end{aligned}$$

Consequently, for any $q \ge -1$ we get:

$$\begin{aligned}2^{qs}\left\|\Delta_q^v(R(a,b))\right\|_{L^2(\mathbb{R})} &\lesssim 2^{-q(s_0-\frac{1}{2})} \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})} \sum_{j\ge q-N_0} c_j^2 2^{-(j-q)(s+s_0)} \\ &\lesssim c_q 2^{-q(s_0-\frac{1}{2})} \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}.\end{aligned}$$

It is then easy to show that $\|R(a,b)\|_{H^s(\mathbb{R})} \lesssim \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}$.
If $s + s_0 = 0$, then along the same lines we can prove that:

$$\|R(a,b)\|_{B_{2,\infty}^{-\frac{1}{2}}(\mathbb{R})} \lesssim \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}$$

The last step consists to use the following inequality by taking $a = -\frac{1}{2}$ and $\varepsilon = s_0 - \frac{1}{2}$:

$$\|f\|_{B_{2,1}^{a-\varepsilon}} = \sum_{k\ge -1} 2^{k(a-\varepsilon)} \|\Delta_k f\|_{L^2} \le C(\varepsilon) \|f\|_{B_{2,\infty}^a}, \quad \forall a \in \mathbb{R}, \varepsilon > 0 \tag{5}$$

We get:

$$\|R(a,b)\|_{H^{-s_0}(\mathbb{R})} = \|R(a,b)\|_{H^s(\mathbb{R})} \lesssim \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}$$

which ends the proof. □

Another important result is the following commutator-type estimate:

Lemma 3. *Let u, f be regular where u is a divergence free vector field in $\mathbb{R}^3$. We have:*

$$\left\|\left[\Delta_q^v, S_{j-1}^v u^3(.,x_3)\right] f\right\|_{L_v^2 H_h^{-1/2}} \lesssim 2^{-q} \left\|S_{j-1}^v \nabla_h u(.,x_3)\right\|_{L_v^\infty L_h^2} \|f\|_{L_v^2 H_h^{1/2}}$$

Proof
The proof is essentialy based on the fact that $-\partial_3 u^3 = \nabla_h \cdot u^h$ and the following usual commutator estimate used with respect to the vertical variable:

$$\left\|\left[\Delta_j, a\right] b\right\|_{L^r} \lesssim 2^{-j} \|\nabla a\|_{L^p} \|b\|_{L^q}, \quad \text{with } \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \tag{6}$$

For the proof of estimates of type (6) one may see for example [2], and for a detailed proof of lemma 3, one may see [3]. □

Let us end this section by recalling the Osgood lemma (see for instance [2]):

Lemma 4. ***Osgood lemma***
Let g be a measurable function from $[t_0, T]$ to $[0, a]$, γ a locally integrable function from $[t_0, T]$ to $\mathbb{R}^+$ and μ a continuous and non-decreasing function from $[0, a]$ to $\mathbb{R}^+$. Assume that for some non-negative real number c, g satisfies:

$$g(t) \leq c + \int_{t_0}^{t} \gamma(\tau)\mu(g(\tau))d\tau, \quad a.e. \ t \in [t_0, T]$$

Then we have for a.a. $t \in [t_0, T]$:

$$c > 0 \implies -M(g(t)) + M(c) \leq \int_{t_0}^{t} \gamma(\tau)d\tau, \quad \textit{where } M(x) = \int_{x}^{a} \frac{d\tau}{\mu(\tau)}$$
$$c = 0 \textit{ and } \int_{0}^{a} \frac{d\tau}{\mu(\tau)} = \infty \implies g = 0$$

3 A priori estimates

In this section we establish the main a priori estimates required to prove the uniqueness in our theorems.
Let $\langle f, g\rangle := \langle f, g\rangle_{L^2(\mathbb{R}^3)}$ be the usual L^2-scalar product, and $\langle f, g\rangle_{\alpha,\beta}$ denotes the scalar product between f and g in $H^{\alpha,\beta}(\mathbb{R}^3)$. In order to simplify the redaction, we introduce the following notations:

$$L_1 := \sum_{q\geq -1} 2^{2q(s-1)}\langle \Delta_q^v(u^h \cdot \nabla_h w), \Delta_q^v w\rangle, \quad L_2 := \sum_{q\geq -1} 2^{2q(s-1)}\langle \Delta_q^v(u^3\partial_3 w), \Delta_q^v w\rangle$$
$$L_3 := \sum_{q\geq -1} 2^{2q(s-1)}\langle \Delta_q^v(w^h \cdot \nabla_h v), \Delta_q^v w\rangle, \quad L_4 := \sum_{q\geq -1} 2^{2q(s-1)}\langle \Delta_q^v(w^3\partial_3 v), \Delta_q^v w\rangle$$
$$L_5 := \sum_{q\geq -1} 2^{-2qs}\langle \Delta_q^v(u^h \cdot \nabla_h \theta), \Delta_q^v \theta\rangle, \quad L_6 := \sum_{q\geq -1} 2^{-2qs}\langle \Delta_q^v(u^3\partial_3 \theta), \Delta_q^v \theta\rangle$$
$$L_7 := \sum_{q\geq -1} 2^{-2qs}\langle \Delta_q^v(w^h \cdot \nabla_h \eta), \Delta_q^v \theta\rangle, \quad L_8 := \sum_{q\geq -1} 2^{-2qs}\langle \Delta_q^v(w^3\partial_3 \eta), \Delta_q^v \theta\rangle$$
$$L_9 := \sum_{q\geq -1} 2^{2q(s-1)}\langle \Delta_q^v \theta, \Delta_q^v(w^3)\rangle$$

We shall prove:

Proposition 1. *Let $s \in]1/2, 1]$. Then for $u, v, w, \rho, \eta, \theta$ verifying:*

$$\begin{aligned} &u, v, \nabla_h u, \nabla_h v \in H^{0,s}, && \rho, \eta, \nabla_h \rho, \nabla_h \eta \in H^{0,1-s} \\ &w, \nabla_h w \in H^{0,s-1}, && \theta, \nabla_h \theta \in H^{0,-s} \\ &\nabla \cdot u = \nabla \cdot v = \nabla \cdot w = 0, && \end{aligned}$$

we have:

$$L_1 \lesssim \|u\|_{1/2,s} \|\nabla_h w\|_{0,s-1} \|w\|_{1/2,s-1}, \quad L_2 \lesssim \|\nabla_h u\|_{0,s} \|w\|_{1/2,s-1}^2$$
$$L_3 \lesssim \|\nabla_h v\|_{0,s} \|w\|_{1/2,s-1}^2, \quad L_4 \lesssim \|v\|_{1/2,s} \left(\|w\|_{0,s-1} + \|\nabla_h w\|_{0,s-1}\right) \|w\|_{1/2,s-1}$$
$$L_5 \lesssim \|u\|_{1/2,s} \|\nabla_h \theta\|_{0,-s} \|\theta\|_{1/2,-s}, \quad L_6 \lesssim \|\nabla_h u\|_{0,s} \|\theta\|_{1/2,-s}^2$$
$$L_7 \lesssim \|\nabla_h \eta\|_{0,1-s} \|w\|_{1/2,s-1} \|\theta\|_{1/2,-s}, \quad L_8 \lesssim \|\eta\|_{1/2,1-s} \left(\|w\|_{0,s-1} + \|\nabla_h w\|_{0,s-1}\right) \|\theta\|_{1/2,-s}$$
$$L_9 \lesssim \|\theta\|_{0,-s} (\|\nabla_h w\|_{0,s-1} + \|w\|_{0,s-1})$$

Proof
in the following we denote by c_q some constant $c_q := c_q(u, v, w, \theta, \rho, t)$ with $\sum_{q\geq -1} c_q^2 \leq 1$ which comes from the fact (1) or (2). This constant is allowed to differ from one line to another.

• **L_1 estimate:**
Since $s + (s-1) > 0$, by using product lemma 2 between $H^{\frac{1}{2},s}$ and $H^{0,s-1}$, we obtain:

$$\begin{aligned} L_1 = \langle u^h \cdot \nabla_h w, w\rangle_{0,s-1} &\leq \left\| u^h \cdot \nabla_h w \right\|_{-1/2,s-1} \|w\|_{1/2,s-1} \\ &\lesssim \|u\|_{1/2,s} \|\nabla_h w\|_{0,s-1} \|w\|_{1/2,s-1} \end{aligned}$$

• **L_2 estimate:**
We write $L_2 = L_2^{(1)} + L_2^{(2)} + L_2^{(3)}$ where:

$$\begin{aligned} L_2^{(1)} &:= \sum_{q\geq -1} 2^{2q(s-1)} \sum_{|j-q|\leq N_0} \langle \Delta_q^v\big(\Delta_j^v(u^3) S_{j-1}^v(\partial_3 w)\big), \Delta_q^v w\rangle \\ L_2^{(2)} &:= \sum_{q\geq -1} 2^{2q(s-1)} \sum_{|j-q|\leq N_0} \langle \Delta_q^v\big(S_{j-1}^v(u^3) \Delta_j^v(\partial_3 w)\big), \Delta_q^v w\rangle \\ L_2^{(3)} &:= \sum_{q\geq -1} 2^{2q(s-1)} \sum_{i\in\{0,-1,1\}} \sum_{j\geq q-N_0} \langle \Delta_q^v\big(\Delta_{j+1}^v(u^3) \Delta_j^v(\partial_3 w)\big), \Delta_q^v w\rangle \end{aligned}$$

Then by using the embbeding of $H^{\frac{1}{2}}(\mathbb{R}_h^2)$ in $L^4(\mathbb{R}_h^2)$, Bernstein lemma for the vertical variable together with statement (1), we obtain:

$$\begin{aligned} L_2^{(1)} &\lesssim \|w\|_{1/2,s-1} \sum_{q\geq -1} c_q 2^{q(s-1)} 2^{q/2} \sum_{|j-q|\leq N_0} \left\| S_{j-1}^v w \right\|_{L_h^4 L_v^2} \left\| \Delta_j^v \nabla_h u \right\|_{L^2} \\ &\lesssim \|w\|_{1/2,s-1} \|w\|_{1/2,s-1} \|\nabla_h u\|_{0,s} \sum_{q\geq -1} c_q 2^{q(s-1/2)} \sum_{|j-q|\leq N_0} c_j^2 2^{j(1-s)} 2^{-sj} \\ &\lesssim \|w\|_{1/2,s-1} \|w\|_{1/2,s-1} \|\nabla_h u\|_{0,s} \sum_{q\geq -1} c_q 2^{q(s-1/2)} 2^{q(1-s)} 2^{-qs} \\ &\lesssim \|w\|_{1/2,s-1} \|w\|_{1/2,s-1} \|\nabla_h u\|_{0,s} \sum_{q\geq -1} c_q 2^{-q(s-1/2)} \\ &\lesssim \|w\|_{1/2,s-1} \|w\|_{1/2,s-1} \|\nabla_h u\|_{0,s} \end{aligned}$$

To estimate $L_2^{(2)}$, we consider the decomposition used in [3], by writing $L_2^{(2)} = A_1 + A_2 + A_3$ with:

$$\begin{aligned} A_1 &:= \sum_{q\geq -1} 2^{2q(s-1)} \langle S_q^v(u^3) \Delta_q^v(\partial_3 w), \Delta_q^v w\rangle \\ A_2 &:= \sum_{q\geq -1} 2^{2q(s-1)} \sum_{|j-q|\leq N_0} \langle \big(S_q^v - S_{j-1}^v\big)(u^3) \Delta_j^v(\partial_3 w), \Delta_q^v w\rangle \\ A_3 &:= \sum_{q\geq -1} 2^{2q(s-1)} \sum_{|j-q|\leq N_0} \langle \big[\Delta_q^v, S_{j-1}^v(u^3)\big] \Delta_j^v(\partial_3 w), \Delta_q^v w\rangle, \end{aligned}$$

where $\big[\Delta_q^v, S_{j-1}^v(u^3)\big]$ denotes the commutator between Δ_q^v and $S_{j-1}^v(u^3)$.

After integration by parts we obtain:

$$\begin{aligned}
A_1 &= -\frac{1}{2}\sum_{q\geq -1} 2^{2q(s-1)}\langle S_q^v(\partial_3 u^3)\Delta_q^v(w), \Delta_q^v w\rangle \\
&= \frac{1}{2}\sum_{q\geq -1} 2^{2q(s-1)}\langle S_q^v(\nabla_h\cdot u^h)\Delta_q^v(w), \Delta_q^v w\rangle \\
&\lesssim \|w\|_{1/2,s-1}\sum_{q\geq -1} c_q 2^{-q(s-1)}\left\|\Delta_q^v w\right\|_{L_h^4 L_v^2}\left\|S_q^v(\nabla_h u)\right\|_{L_h^2 L_v^\infty} \\
&\lesssim \|w\|_{1/2,s-1}^2\|\nabla_h u\|_{0,s}
\end{aligned}$$

In order to estimate A_2, we remark first that $S_q^v - S_{j-1}^v$ is spectraly supported away from 0, that is we can use lemma 1 to estimate A_2 just like $L_2^{(1)}$. Indeed:

$$\begin{aligned}
A_2 &\leq C\|w\|_{1/2,s-1}\sum_{q\geq -1} c_q 2^{q(s-1)}\sum_{|j-q|\leq N_0}\left\|(S_q^v - S_{j-1}^v)\partial_3 u^3\right\|_{L_h^2 L_v^\infty}\left\|\Delta_j^v w\right\|_{L_h^4 L_v^2} \\
&\leq C\|w\|_{1/2,s-1}\sum_{q\geq -1} c_q 2^{q(s-1)}\sum_{|j-q|\leq N_0}\left\|(S_q^v - S_{j-1}^v)\nabla u^h\right\|_{L_h^2 L_v^\infty}\left\|\Delta_j^v w\right\|_{L_h^4 L_v^2} \\
&\leq C\|w\|_{1/2,s-1}^2\|\nabla_h u\|_{0,s}\sum_{|i|\leq N_0}\sum_{q\geq -1} c_q 2^{q(s-1)} c_{j+i} 2^{(q+i)(1-s)} \\
&\leq C\|w\|_{1/2,s-1}^2\|\nabla_h u\|_{0,s}
\end{aligned}$$

Finally for A_3 we use the commutator estimate proved in lemma 3 to obtain:

$$\begin{aligned}
A_3 &\leq \|w\|_{1/2,s-1}\sum_{q\geq -1} c_q 2^{q(s-1)}\sum_{|j-q|\leq N_0}\left\|\left[\Delta_q^v, S_{j-1}^v(u^3)\right]\Delta_j^v(\partial_3 w)\right\|_{L^2\left(\mathbb{R}_{x_3};H^{-1/2}(\mathbb{R}^2)\right)} \\
&\leq \|w\|_{1/2,s-1}\sum_{q\geq -1} c_q 2^{q(s-1)}\sum_{|j-q|\leq N_0}\left\|S_{j-1}^v\nabla_h u(.,x_3)\right\|_{L_h^2 L_v^\infty}\left\|\Delta_j^v w\right\|_{1/2,0} \\
&\leq \|w\|_{1/2,s-1}^2\|\nabla_h u\|_{0,s}\sum_{i\in\{0,-1,1\}}\sum_{q\geq -1} c_q c_{q+i} \\
&\leq \|w\|_{1/2,s-1}^2\|\nabla_h u\|_{0,s}
\end{aligned}$$

Ditto for the the last term in this part, using the fact that $\partial_3 u^3 = -\nabla_h\cdot u^h$:

$$\begin{aligned}
L_2^{(3)} &\lesssim \|w\|_{1/2,s-1}\sum_{q\geq -1} c_q 2^{q(s-1)} 2^{q/2}\sum_{i\in\{0,-1,1\}}\sum_{j\geq q-N_0}\left\|\Delta_{j+i}^v w\right\|_{L_h^4 L_v^2}\left\|\Delta_j^v\nabla_h u\right\|_{L^2} \\
&\lesssim \|w\|_{1/2,s-1}\|w\|_{1/2,s-1}\|\nabla_h u\|_{0,s}\sum_{q\geq -1} c_q 2^{q(s-1/2)}\sum_{i\in\{0,-1,1\}}\sum_{j\geq q-N_0} c_j c_{j+i} 2^{j(1-s)} 2^{-sj} \\
&\lesssim \|w\|_{1/2,s-1}^2\|\nabla_h u\|_{0,s}\sum_{q\geq -1} c_q 2^{q(s-1/2)} 2^{q(1-2s)}\sum_{i\in\{0,-1,1\}}\sum_{j\geq q-N_0} c_j c_{j+i}
\end{aligned}$$

where we used the fact that $s\in]1/2,1]$ that is $1-2s<0$. We obtain finally:

$$\begin{aligned}
L_2^{(3)} &\lesssim \|w\|_{1/2,s-1}^2\|\nabla_h u\|_{0,s}\sum_{q\geq -1} c_q 2^{-q(s-1/2)} \\
&\lesssim \|w\|_{1/2,s-1}^2\|\nabla_h u\|_{0,s}
\end{aligned}$$

Remark: In the case where $s=1$ we don't have to deal with L_1+L_2 which is equal to 0 because of the identity $\langle u\cdot\nabla w, w\rangle = 0$

• L_3 **estimate:**
By using product lemma 2 between $H^{0,s}$ and $H^{1/2,s-1}$ we obtain:

$$\begin{aligned}L_3 = \langle w^h \nabla_h v, w\rangle_{0,s-1} &\leq \left\| w^h \nabla_h v \right\|_{-1/2,s-1} \|w\|_{1/2,s-1} \\ &\lesssim \|\nabla_h v\|_{0,s} \|w\|^2_{1/2,s-1}\end{aligned}$$

• L_4 **estimate:**
We write $L_4 = L_4^{(1)} + L_4^{(2)}$ where:

$$L_4^{(1)} := \sum_{q\geq -1} 2^{2q(s-1)} \sum_{j\geq q-N_0} \langle \Delta_q^v(\Delta_j^v w^3 S_{j+2}^v(\partial_3 v)), \Delta_q^v w\rangle$$

$$L_4^{(2)} := \sum_{q\geq -1} 2^{2q(s-1)} \sum_{|j-q|\leq N_0} \langle \Delta_q^v(S_{j-1}^v(w^3)\Delta_j^v \partial_3 v), \Delta_q^v w\rangle$$

Hense, by using again lemma 2, we infer that:

$$L_4^{(1)} \leq \|w\|_{1/2,s-1} \sum_{q\geq -1} c_q 2^{q(s-1/2)} \sum_{j\geq q-N_0} \left\|\Delta_j^v w^3\right\|_{L^2} \left\|S_{j+2}^v(\partial_3 v)\right\|_{L_h^4 L_v^2}$$

For $s \neq 1$, after certain calculations

$$\begin{aligned}\left\|S_{j+2}^v(\partial_3 v)\right\|_{L_h^4 L_v^2} &\leq \sum_{m\leq j+1} 2^{m(1-s)} 2^{sm} \|\Delta_m^v v\|_{L_h^4 L_v^2} \\ &\leq \left(\sum_{m\leq j+1} 2^{2m(1-s)}\right)^{1/2} \|v\|_{1/2,s} \\ &\lesssim 2^{j(1-s)} \|v\|_{1/2,s}\end{aligned}$$

Thus, by using lemma 1 together with the previous estimate and the divergence free condition on w, we find:

$$\begin{aligned}L_4^{(1)} &\lesssim \|w\|_{1/2,s-1} \|v\|_{1/2,s} \sum_{q\geq -1} c_q 2^{q(s-1/2)} \sum_{j\geq q-N_0} 2^{-js} \left\|\Delta_j^v \nabla_h w\right\|_{L^2} \\ &\lesssim \|w\|_{1/2,s-1} \|v\|_{1/2,s} \|\nabla_h w\|_{0,s-1} \sum_{q\geq -1} c_q 2^{q(s-1/2)} \sum_{j\geq q-N_0} c_j 2^{j(1-2s)} \\ &\lesssim \|w\|_{1/2,s-1} \|v\|_{1/2,s} \|\nabla_h w\|_{0,s-1} \sum_{q\geq -1} c_q 2^{q(s-1/2)} \left(\sum_{j\geq q-N_0} 2^{2j(1-2s)}\right)^{1/2} \|c_j\|_{l^2(\mathbb{N}\cup\{-1\})} \\ &\lesssim \|w\|_{1/2,s-1} \|v\|_{1/2,s} \|\nabla_h w\|_{0,s-1} \sum_{q\geq -1} c_q 2^{q(s-1/2)} 2^{q(1-2s)} \\ &\lesssim \|w\|_{1/2,s-1} \|v\|_{1/2,s} \|\nabla_h w\|_{0,s-1} \sum_{q\geq -1} c_q 2^{q(1/2-s)} \\ &\lesssim \|w\|_{1/2,s-1} \|v\|_{1/2,s} \|\nabla_h w\|_{0,s-1}\end{aligned}$$

For the second term we proceed as follows:

$$\begin{aligned}
L_4^{(2)} &\lesssim \|w\|_{1/2,s-1} \sum_{q\geq -1} c_q 2^{q(s-1)} 2^q \sum_{|j-q|\leq N_0} 2^{j-q} \left\|S_{j-1}^v w^3\right\|_{L_h^2 L_v^\infty} \left\|\Delta_j^v v\right\|_{L_h^4 L_v^2} \\
&\lesssim \|w\|_{1/2,s-1} \left\|w^3\right\|_{0,s} \sum_{|j|\leq N_0} \sum_{q\geq -1} c_q 2^{q(s-1)} 2^q \left\|\Delta_{j+q}^v v\right\|_{L_h^4 L_v^2} \\
&\lesssim \|v\|_{1/2,s} \|w\|_{1/2,s-1} \left\|w^3\right\|_{0,s} \sum_{|i|\leq N_0} \sum_{q\geq -1} c_q c_{q+i} \\
&\lesssim \|v\|_{1/2,s} \|w\|_{1/2,s-1} \left\|w^3\right\|_{0,s}
\end{aligned}$$

In order to close the estimates of $L_4^{(2)}$ we remark that, for any $s \in [\frac{1}{2}, 1]$, we have:

$$\begin{aligned}
\left\|w^3\right\|_{0,s} &\leq \left\|w^3\right\|_{0,s-1} + \left\|\partial_3 w^3\right\|_{0,s-1} \\
&\leq \left\|w^3\right\|_{0,s-1} + \|\nabla_h w\|_{0,s-1}
\end{aligned}$$

In the case where $s = 1$, note that the estimate can be obtained easly, by using product rules and the previous inequality, as the following:

$$\begin{aligned}
\langle w^3 \partial_3 v, w\rangle_{L^2} &\leq \left\|w^3 \partial_3 v\right\|_{-1/2,0} \|w\|_{1/2,0} \\
&\lesssim \left\|w^3\right\|_{0,1} \|\partial_3 v\|_{1/2,0} \|w\|_{1/2,0} \\
&\lesssim \|v\|_{1/2,1} \left(\|w\|_{L^2} + \|\nabla_h w\|_{L^2}\right) \|w\|_{1/2,0}
\end{aligned}$$

• L_5 **estimate:**

In order to estimate this term we proceed by duality by inferring firstly that:

$$L_5 \leq \left\|u^h \nabla_h \theta\right\|_{-\frac{1}{2},-s} \|\theta\|_{\frac{1}{2},-s}$$

Moreover, lemma 2 gives:

$$\left\|u^h \nabla_h \theta\right\|_{-\frac{1}{2},-s} \lesssim \left\|u^h\right\|_{\frac{1}{2},s} \|\nabla_h \theta\|_{0,-s}$$

It follows that:

$$L_5 \lesssim \left\|u^h\right\|_{\frac{1}{2},s} \|\nabla_h \theta\|_{0,-s} \|\theta\|_{\frac{1}{2},-s}$$

• L_6 **estimate:**

We use the Bony decomposition: $L_6 = L_6^{(1)} + L_6^{(2)} + L_6^{(3)}$, where:

$$\begin{aligned}
L_6^{(1)} &:= \sum_{q\geq -1} 2^{2q(-s)} \sum_{|j-q|\leq N_0} \langle \Delta_q^v\big(\Delta_j^v(u^3) S_{j-1}^v(\partial_3\theta)\big), \Delta_q^v \theta\rangle \\
L_6^{(2)} &:= \sum_{q\geq -1} 2^{2q(-s)} \sum_{|j-q|\leq N_0} \langle \Delta_q^v\big(S_{j-1}^v(u^3) \Delta_j^v(\partial_3\theta)\big), \Delta_q^v \theta\rangle \\
L_6^{(3)} &:= \sum_{q\geq -1} 2^{2q(-s)} \sum_{i\in\{0,-1,1\}} \sum_{j\geq q-N_0} \langle \Delta_q^v\big(\Delta_{j+1}^v(u^3) \Delta_j^v(\partial_3\theta)\big), \Delta_q^v \theta\rangle
\end{aligned}$$

For the first term, we use the Bernstein lemma together with usual sobolev embbeding and the free divergence condition to obtain:

$$\begin{aligned}
L_6^{(1)} &\leq C \|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{-qs} 2^{q/2} \sum_{|j-q|\leq N_0} \left\|S_{j-1}^v \theta\right\|_{L_h^4 L_v^2} \left\|\Delta_j^v \nabla_h \cdot u^h\right\|_{L^2} \\
&\lesssim \|\theta\|_{1/2,-s} \|\theta\|_{1/2,s-1} \|\nabla_h u\|_{0,s} \sum_{q\geq -1} c_q 2^{q(1/2-s)} \sum_{|j-q|\leq N_0} c_j^2 \\
&\lesssim \|\theta\|_{1/2,-s}^2 \|\nabla_h u\|_{0,s}
\end{aligned}$$

For $L_6^{(2)}$ we follow the same decomposition used for $L_2^{(2)}$, so we write: $L_6^{(2)} = B_1 + B_2 + B_3$, where:

$$B_1 := \sum_{q\geq -1} 2^{-2qs} \langle S_q^v(u^3)\Delta_q^v(\partial_3\theta), \Delta_q^v\theta\rangle$$

$$B_2 := \sum_{q\geq -1} 2^{-2qs} \sum_{|j-q|\leq N_0} \langle \big(S_q^v - S_{j-1}^v\big)(u^3)\Delta_j^v(\partial_3\theta), \Delta_q^v\theta\rangle$$

$$B_3 := \sum_{q\geq -1} 2^{-2qs} \sum_{|j-q|\leq N_0} \langle \big[\Delta_q^v, S_{j-1}^v(u^3)\big]\Delta_j^v(\partial_3\theta), \Delta_q^v\theta\rangle$$

After integration by parts we obtain:

$$\begin{aligned}
B_1 &= \frac{1}{2}\sum_{q\geq -1} 2^{-2qs} \langle S_q^v(\partial_3 u^3)\Delta_q^v(\theta), \Delta_q^v\theta\rangle \\
&= \frac{1}{2}\sum_{q\geq -1} 2^{-2qs} \langle S_q^v(\nabla_h u^h)\Delta_q^v(\theta), \Delta_q^v\theta\rangle \\
&\lesssim \|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{-qs} \left\|\Delta_q^v\theta\right\|_{L_h^4L_v^2} \left\|S_q^v(\nabla_h u)\right\|_{L_h^2L_v^\infty} \\
&\lesssim \|\theta\|_{1/2,s-1}^2 \|\nabla_h u\|_{0,s}
\end{aligned}$$

To estimate B_2 we remark first that $S_q^v - S_{j-1}^v$ is spectraly supported away from 0 that is we can use lemma 1 and estimate B_2 as A_2, indeed:

$$\begin{aligned}
B_2 &\leq C\|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{-qs} \sum_{|j-q|\leq N_0} \left\|\big(S_q^v - S_{j-1}^v\big)\partial_3 u^3\right\|_{L_h^2L_v^\infty} \left\|\Delta_j^v\theta\right\|_{L_h^4L_v^2} \\
&\leq C\|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{-qs} \sum_{|j-q|\leq N_0} \left\|\big(S_q^v - S_{j-1}^v\big)\nabla_h u^h\right\|_{L_h^2L_v^\infty} \left\|\Delta_j^v\theta\right\|_{L_h^4L_v^2} \\
&\leq C\|\theta\|_{1/2,-s}^2 \|\nabla_h u\|_{0,s} \sum_{|i|\leq N_0}\sum_{q\geq -1} c_q c_{q+i} \\
&\leq C\|\theta\|_{1/2,-s}^2 \|\nabla_h u\|_{0,s}
\end{aligned}$$

Finally for B_3 we use the commutator estimate proved in lemma 3 to obtain:

$$\begin{aligned}
B_3 &\lesssim \|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{-qs} \sum_{|j-q|\leq N_0} \left\|\big[\Delta_q^v, S_{j-1}^v(u^3)\big]\Delta_j^v(\partial_3\theta)\right\|_{L^2\left(\mathbb{R}_{x_3};H^{-1/2}(\mathbb{R}^2)\right)} \\
&\lesssim \|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{-qs} \sum_{|j-q|\leq N_0} \left\|S_{j-1}^v \nabla_h u(., x_3)\right\|_{L_h^2L_v^\infty} \left\|\Delta_j^v\theta\right\|_{1/2,0} \\
&\lesssim \|\theta\|_{1/2,-s}^2 \|\nabla_h u\|_{0,s} \sum_{i\in\{0,-1,1\}}\sum_{q\geq -1} c_q c_{q+i} \\
&\lesssim \|\theta\|_{1/2,-s}^2 \|\nabla_h u\|_{0,s}
\end{aligned}$$

For $L_6^{(3)}$, by using the same arguments we find:

$$\begin{aligned}
L_6^{(3)} &\lesssim \|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{-qs} 2^{q/2} \sum_{i\in\{0,-1,1\}}\sum_{j\geq q-N_0} \left\|\Delta_{j+i}^v\theta\right\|_{L_h^4L_v^2} \left\|\Delta_j^v \nabla_h \cdot u^h\right\|_{L^2} \\
&\lesssim \|\theta\|_{1/2,-s} \|\theta\|_{1/2,-s} \|\nabla_h u\|_{0,s} \sum_{q\geq -1} c_q 2^{-q(s-1/2)} \sum_{i\in\{0,-1,1\}}\sum_{j\geq q-N_0} c_j c_{j+i} \\
&\lesssim \|\theta\|_{1/2,-s}^2 \|\nabla_h u\|_{0,s}
\end{aligned}$$

• L_7 **estimate:**
This term can be estimated by using the following property based on product rules in dimension one together with inequality (5):

$$H^{s-1}(\mathbb{R}) \times H^{1-s}(\mathbb{R}) \subset B^{-\frac{1}{2}}_{2,\infty}(\mathbb{R}) \hookrightarrow H^{-s}$$

Indeed, based on the Bony decomposition with respect to the vertical variable we write $L_7 = L_7^{(1)} + L_7^{(2)} + L_7^{(3)}$, where:

$$L_7^{(1)} := \sum_{q\geq -1} 2^{-2qs} \sum_{|j-q|\leq N_0} \langle \Delta_q^v\big(S_{j-1}^v(w^h)\Delta_j^v(\nabla_h \eta)\big), \Delta_q^v\theta\rangle$$
$$L_7^{(2)} := \sum_{q\geq -1} 2^{-2qs} \sum_{|j-q|\leq N_0} \langle \Delta_q^v\big(\Delta_j^v(w^h)S_{j-1}^v(\nabla_h \eta)\big), \Delta_q^v\theta\rangle$$
$$L_7^{(3)} := \sum_{q\geq -1} 2^{-2qs} \sum_{i\in\{0,-1,1\}} \sum_{j\geq q-N_0} \langle \Delta_q^v\big(\Delta_j^v(w^h)\Delta_{j+i}^v(\nabla_h \eta)\big), \Delta_q^v\theta\rangle$$

By using inequality (2), similar arguments give then:

$$\begin{aligned} L_7^{(1)} &\leq \|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{-q(s-1/2)} \sum_{|j-q|\leq N_0} \left\|S_{j-1}^v w\right\|_{L_h^4 L_v^2} \left\|\Delta_j^v(\nabla_h \eta)\right\|_{L^2} \\ &\leq \|\nabla_h \eta\|_{0,1-s} \|w\|_{1/2,s-1} \|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{-q(s-1/2)} \sum_{|j-q|\leq N_0} c_j^2 \\ &\lesssim \|\nabla_h \eta\|_{0,1-s} \|w\|_{1/2,s-1} \|\theta\|_{1/2,-s} \end{aligned}$$

For the second term we proceed as follows:

$$\begin{aligned} L_7^{(2)} &\leq \|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{-q(s-1/2)} \sum_{|j-q|\leq N_0} 2^{1/2(j-q)} \big(\left\|\Delta_j^v w\right\|_{L_h^4 L_v^2} 2^{j(s-1)}\big)\big(2^{j(1-s-1/2)} \left\|S_{j-1}^v(\nabla_h \eta)\right\|_{L_h^2 L_v^\infty}\big) \\ &\leq \|\nabla_h \eta\|_{B^{0,1-s-1/2}_{\infty,2}} \|w\|_{1/2,s-1} \|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{-q(s-1/2)} \sum_{|j-q|\leq N_0} c_j^2 \\ &\lesssim \|\nabla_h \eta\|_{0,1-s} \|w\|_{1/2,s-1} \|\theta\|_{1/2,-s}, \end{aligned}$$

where we used the embedding $H^{0,1-s} \hookrightarrow B^{0,1-s-1/2}_{\infty,2}$ and the fact that $1-s-1/2<0$.
For the last term we proceed as follows:

$$\begin{aligned} L_7^{(3)} &\leq \|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{-q(s-1/2)} \sum_{i\in\{0,-1,1\}} \sum_{j\geq q-N_0} \left\|\Delta_j^v w\right\|_{L_h^4 L_v^2} \left\|\Delta_{j+i}^v(\nabla_h \eta)\right\|_{L^2} \\ &\leq \|\nabla_h \eta\|_{0,1-s} \|w\|_{1/2,s-1} \|\theta\|_{1/2,-s} \sum_{i\in\{0,-1,1\}} \sum_{q\geq -1} c_q 2^{-q(s-1/2)} \sum_{j\geq q-N_0} c_j c_{j+i} \\ &\lesssim \|\nabla_h \eta\|_{0,1-s} \|w\|_{1/2,s-1} \|\theta\|_{1/2,-s} \end{aligned}$$

• L_8 **estimate:**
We write $L_8 = L_8^{(1)} + L_8^{(2)} + L_8^{(3)}$ where:

$$L_8^{(1)} := \sum_{q\geq -1} 2^{-2qs} \sum_{|j-q|\leq N_0} \langle \Delta_q^v(\Delta_j^v w^3 S_{j-1}^v(\partial_3 \eta)), \Delta_q^v\theta\rangle$$

$$L_8^{(2)} := \sum_{q\geq -1} 2^{-2qs} \sum_{|j-q|\leq N_0} \langle \Delta_q^v(S_{j-1}^v(w^3)\Delta_j^v\partial_3 \eta), \Delta_q^v\theta\rangle$$

$$L_8^{(3)} := \sum_{q\geq -1} 2^{-2qs} \sum_{i\in\{0,-1,1\}} \sum_{j\geq q-N_0} \langle \Delta_q^v(\Delta_j^v w^3 \Delta_{j+i}^v(\partial_3\eta)), \Delta_q^v\theta\rangle$$

Then for the first term, we have:

$$L_8^{(1)} \leq \|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{-qs} \sum_{|j-q|\leq N_0} \left\|\Delta_j^v w^3\right\|_{L^2} \left\|S_{j-1}^v(\partial_3\eta)\right\|_{L_h^4 L_v^\infty}$$

Now we use:

$$\begin{aligned}\left\|S_{j-1}^v(\partial_3\eta)\right\|_{L_h^4 L_v^\infty} &\leq 2^j c_j 2^{-j(1-s-1/2)} \|\eta\|_{H_h^{1/2}(B_{\infty,2}^{1-s-1/2})_v} \\ &\lesssim 2^j c_j 2^{-j(1-s-1/2)} \|\eta\|_{H^{1/2,1-s}}\end{aligned}$$

and:

$$\left\|\Delta_j^v w^3\right\|_{L^2} \leq 2^{-j} c_j 2^{-j(s-1)} \|\nabla_h w\|_{H^{0,s-1}}$$

Therefore we find:

$$\begin{aligned}L_8^{(1)} &\leq \|\theta\|_{1/2,-s} \|\nabla_h w\|_{H^{0,s-1}} \|\eta\|_{H^{1/2,1-s}} \sum_{q\geq -1} c_q 2^{q(1/2-s)} \sum_{|j-q|\leq N_0} c_j^2 2^{1/2(j-q)} \\ &\lesssim \|\theta\|_{1/2,-s} \|\nabla_h w\|_{H^{0,s-1}} \|\eta\|_{H^{1/2,1-s}}\end{aligned}$$

Next, for the second term:

$$\begin{aligned}L_8^{(2)} &\leq \|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q \sum_{|j-q|\leq N_0} 2^{-(q-j)s} \left\|S_{j-1}^v w^3\right\|_{L_h^2 L_v^\infty} 2^{j(1-s)} \left\|\Delta_j^v(\eta)\right\|_{L_h^4 L_v^2} \\ &\leq \|\theta\|_{1/2,-s} \left\|w^3\right\|_{0,s} \|\eta\|_{1/2,1-s} \sum_{q\geq -1} c_q \sum_{|j-q|\leq N_0} c_j \\ &\leq \|\theta\|_{1/2,-s} \left\|w^3\right\|_{0,s} \|\eta\|_{1/2,1-s} \sum_{|j|\leq N_0} \sum_{q\geq -1} c_q c_{j+q} \\ &\lesssim \|\theta\|_{1/2,-s} \left(\left\|w^3\right\|_{0,s-1} + \|\nabla_h w\|_{0,s-1}\right) \|\eta\|_{1/2,1-s}\end{aligned}$$

For $L_8^{(3)}$ we have:

$$\begin{aligned}L_8^{(3)} &\leq \|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{-qs} 2^{q/2} \sum_{i\in 0,-1,1} \sum_{j\geq q-N_0} \left\|\Delta_j^v w^3\right\|_{L^2} \left\|\Delta_{j+i}^v(\partial_3\eta)\right\|_{L_h^4 L_v^2} \\ &\leq \|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{q(1/2-s)} \sum_{i\in 0,-1,1} \sum_{j\geq q-N_0} \left\|\Delta_j^v \nabla_h w\right\|_{L^2} \left\|\Delta_{j+i}^v \eta\right\|_{L_h^4 L_v^2} \\ &\leq \|\theta\|_{1/2,-s} \sum_{q\geq -1} c_q 2^{q(1/2-s)} \sum_{i\in 0,-1,1} \sum_{j\geq q-N_0} 2^{j(s-1)} \left\|\Delta_j^v \nabla_h w\right\|_{L^2} 2^{j(1-s)} \left\|\Delta_{j+i}^v \eta\right\|_{L_h^4 L_v^2} \\ &\lesssim \|\theta\|_{1/2,-s} \|\nabla_h w\|_{0,s-1} \|\eta\|_{1/2,1-s}\end{aligned}$$

Finally for the last term, we use the fact that $s \leq 1$ which implies that $2s-1 \leq 1$, and that w is a free divergence vector field to infer that:

$$\begin{aligned}L_9 &= 2^{-2(s-1)} \langle S_0^v\theta, S_0^v w^3\rangle + \sum_{q\geq 0} 2^{2q(s-1)} \langle \Delta_q^v\theta, \Delta_q^v w^3\rangle \\ &\lesssim \|S_0^v\theta\|_{L^2} \|S_0^v w\|_{L^2} + \sum_{q\geq 0} 2^{-qs} \left\|\Delta_q^v\theta\right\|_{L^2} 2^{q(s-1)} 2^{q(2s-1)} \left\|\Delta_q^v w^3\right\|_{L^2} \\ &\lesssim \|S_0^v\theta\|_{L^2} \|S_0^v w\|_{L^2} + \sum_{q\geq 0} 2^{-qs} \left\|\Delta_q^v\theta\right\|_{L^2} 2^{q(s-1)} \left\|\Delta_q^v \nabla_h \cdot w^h\right\|_{L^2} \\ &\lesssim \|\theta\|_{0,-s} \left(\|\nabla_h w\|_{0,s-1} + \|w\|_{0,s-1}\right)\end{aligned}$$

□

In the case where $s = \frac{1}{2}$ we can take up again the reasoning in [15], by estimating both equations in the same space $H^{0,-\frac{1}{2}}$. Note also that the terms $u \cdot \nabla u$ and $u \cdot \nabla \rho$ can be treated along the same way. More precisely one may prove the following proposition:

Proposition 2. *Let u, v, ρ and η be in space $L^\infty_T(H^{0,\frac{1}{2}})$ with $\nabla_h u, \nabla_h v, \nabla_h \rho, \nabla_h \eta$ in $L^2_T(H^{0,\frac{1}{2}})$ and u, v two divergence free vector fields. Let w, θ be in $L^\infty_T(H^{0,\frac{1}{2}})$ with $\nabla_h w$ and $\nabla_h \theta$ in $L^\infty_T(H^{0,\frac{1}{2}})$ solution of:*

$$\begin{cases} \partial_t w + u \cdot \nabla w - \Delta_h w + \nabla \varpi = \theta e_3 - w \cdot \nabla v \\ \partial_t \theta + u \cdot \nabla \theta - \Delta_h \theta = -w \cdot \nabla \eta \\ div(w) = 0 \end{cases},$$

and $\chi(t) := \|w(t)\|^2_{0,-\frac{1}{2}} + \|\theta(t)\|^2_{0,-\frac{1}{2}}$. For all $0 < t < T$, if $\chi(t) \leq e^{-2}$, then we have:

$$\frac{d}{dt}\chi(t) \leq Cf(t)\chi(t)\big(1 - ln\chi(t)\big)ln\big(1 - ln\chi(t)\big), \tag{7}$$

where f is a locally integrable function depending on the norms of $u, v, w, \rho, \eta, \theta$ in $H^{0,\frac{1}{2}} \cap H^{1,\frac{1}{2}}$

Proof

As mentioned before, the proof is essentialy based on [15], indeed by following lemma 4.2 from [15] we infer that:

$$\frac{d}{dt}\|w(t)\|^2_{0,-\frac{1}{2}} \leq Cf_1(t)\|w(t)\|^2_{0,-\frac{1}{2}}\big(1 - ln\|w(t)\|^2_{0,-\frac{1}{2}}\big)ln\big(1 - ln\|w(t)\|^2_{0,-\frac{1}{2}}\big) + C\chi(t)$$

where:

$$f_1 = \big(1 + \|u\|^2_{1,\frac{1}{2}} + \|v\|^2_{1,\frac{1}{2}} + \|w\|^2_{1,\frac{1}{2}}\big) \times \big(1 + \|\nabla_h u\|^2_{1,\frac{1}{2}} + \|\nabla_h v\|^2_{1,\frac{1}{2}} + \|\nabla_h w\|^2_{1,\frac{1}{2}}\big)$$

Note that, for $0 < x << 1$, the function $x \longmapsto x(1 - ln(x))ln(1 - ln(x))$ is non-decreasing. Then, for $\chi(t)$ small enough, and following the same approach in [15] we can prove that:

$$\frac{d}{dt}\|w(t)\|^2_{0,-\frac{1}{2}} \leq Cf(t)\chi(t)\big(1 - ln\chi(t)\big)ln\big(1 - ln\chi(t)\big) \tag{8}$$

$$\frac{d}{dt}\|\theta(t)\|^2_{0,-\frac{1}{2}} \leq Cf(t)\chi(t)\big(1 - ln\chi(t)\big)ln\big(1 - ln\chi(t)\big) \tag{9}$$

consequently by summing together (8) and (9) we obtain (7). □

4 Proof of the Theorems

4.1 Proof of Theorem 1

Let $(u, \rho, P), (v, \eta, \Pi)$ be two solutions for system (NSB_h), and $w := u - v, \theta := \rho - \eta, \varpi := P - \Pi$ denotes the difference functions. Then (w, θ, ϖ) satisfies

$$(\mathcal{Q}) \quad \begin{cases} \big(\partial_t + u \cdot \nabla\big)w - \Delta_h w + \nabla \varpi = \theta e_3 - w \cdot \nabla v \\ \big(\partial_t + u \cdot \nabla\big)\theta - \Delta_h \theta = -w \cdot \nabla \eta \\ \nabla \cdot w = 0 \\ w_{|t=0} = \theta_{|t=0} = 0 \end{cases}$$

• ***The case:*** $s \neq 1/2$
Recall first that by interpolation, we have:

$$\|w\|_{1/2,s-1} \lesssim \|w\|_{0,s-1}^{1/2} \|\nabla_h w\|_{0,s-1}^{1/2}$$

$$\|\theta\|_{1/2,-s} \lesssim \|\theta\|_{0,-s}^{1/2} \|\nabla_h \theta\|_{0,-s}^{1/2}$$

An easy consequence of the Young inequality tells that for any non negative real numbers a, b, A, B, D with $a + b = 2$, there exists $C > 0$ such that:

$$A^a B^b D \leq \frac{1}{100} B^2 + C A^2 D^{2/a}.$$

Hence by using these last inequalities in an appropriate way together with the estimates given in Proposition 1, it is not difficult to prove that:

$$\sum_{i=1}^{9} L_i \leq \frac{1}{2}\big(\|\nabla_h w\|_{0,s-1}^2 + \|\nabla_h \theta\|_{0,-s}^2 \big) + f(t)\big(\|w\|_{0,s-1}^2 + \|\theta\|_{0,-s}^2 \big), \tag{10}$$

where f is a locally in time integrable function given by:

$$f(t) = C\big(\|u\|_{1/2,s}^4 + \|u\|_{0,s}^2 + \|\nabla_h v\|_{0,s}^2 + \|v\|_{1/2,s}^{4/3} + \|\nabla_h \eta\|_{0,s}^2 + \|\eta\|_{1/2,s}^4 + 1\big)$$

On the other hand, by applying the operator Δ_q^v to $(\mathcal{Q})$, and by summing with respect to $q \in \mathbb{N} \cup \{-1\}$, then (10) leads to:

$$\|w(t)\|_{0,s-1}^2 + \|\theta(t)\|_{0,-s}^2 \leq \int_0^t f(\tau)\big(\|w(\tau)\|_{0,s-1}^2 + \|\theta(\tau)\|_{0,-s}^2 \big) d\tau$$

Finally, we can conclude by using Gronwall 's lemma.
• ***The case:*** $s = 1/2$
The uniqueness in this case can be deduced by applying the Osgood lemma to the estimate given in Proposition 2.

4.2 Proof of Theorem 2

The uniqueness result in Theorem 2 is a direct consequence of Theorem 1 when we take $s = 1$. We have then only to prove the existence of a global solution (u, ρ) for (NSB_h). The arguments given hereafter, based on the Friedrichs method, are very classical. (see for instance [2, 14, 15, 5] for more details)

For $n \in \mathbb{N}$, we consider the following approximate system:

$$(NSB_h)_n \quad \begin{cases} \partial_t u_n + \mathbb{E}_n(u_n \cdot \nabla u_n) - \Delta_h u_n + \nabla P_n = \rho_n e_3 \\ \partial_t \rho_n + \mathbb{E}_n(u_n \cdot \nabla \rho_n) - \Delta_h \rho_n = 0 \\ div(u_n) = 0 \\ P_n = \mathbb{E}_n \sum_{k,j} (-\Delta)^{-1} \partial_j \partial_k (u_n^j u_n^k) \\ (u_n, \rho_n)_{|t=0} = (\mathbb{E}_n u_0, \mathbb{E}_n \rho_0), \end{cases}$$

where $\mathbb{E}_n$ denotes the cut-off operator defined on $L^2(\mathbb{R}^3)$ by $\mathbb{E}_n u := \mathcal{F}^{-1}(\mathbb{1}_{B(0,n)} \hat{u})$.
It is then easy to see by using a fixed point argument that there exists some $T_n > 0$ for which $(NSB_h)_n$ admits a unique solution $(u_n, \rho_n) \in \mathcal{C}^\infty([0, T_n[, \mathcal{L}_n^{2,\sigma})$, where :

$$\mathcal{L}_n^{2,\sigma} := L_n^{2,\sigma} \times L_n^2(\mathbb{R}^3)$$

$$L_n^{2,\sigma} := \left\{ v \in \left(L^2(\mathbb{R}^3)\right)^3 : \nabla \cdot v = 0 \; and \; Supp\;(\hat{v}) \subset B(0,n) \right\}$$
$$L_n^2(\mathbb{R}^3) := \left\{ \rho \in L^2(\mathbb{R}^3) : Supp\;(\hat{\rho}) \subset B(0,n) \right\}$$

Moreover because u_n and ρ_n are regular, we can multiply the first equation in $(NSB_h)_n$ by u_n and the second one by ρ_n. Then, for $t \in [0, T_n[$, after integrating the corresponding terms over $[0,t[\times\mathbb{R}^3$, we obtain the classical uniform L^2-energy bounds:

$$\|u_n(t)\|_{L^2}^2 + 2\int_0^t \|\nabla_h u_n(\tau)\|_{L^2}^2 \, d\tau \leq 2(\|u_0\|_{L^2}^2 + t^2 \|\rho_0\|_{L^2}^2) \tag{11}$$
$$\|\rho_n(t)\|_{L^2}^2 + 2\int_0^t \|\nabla_h \rho_n(\tau)\|_{L^2}^2 \, d\tau \leq \|\rho_0\|_{L^2}^2 \tag{12}$$

Hence (u_n, ρ_n) is a global solution, that is for any $T > 0 : (u_n, \rho_n) \in \mathcal{C}^\infty([0,T[, \mathcal{L}_n^{2,\sigma})$ and it satisfies $(NSB_h)_n$ on $[0,T[\times\mathbb{R}^3$. Moreover, we may extract a subsequence, still denoted (u_n, ρ_n), such that:

$$(u_n, \rho_n) \overset{*}{\rightharpoonup} (u, \rho) \quad \text{in } L_T^\infty L^2 \cap L_T^2 H^{1,0}.$$

However, in order to pass to the limit in the non-linear terms, we will need some strong convergence property. To this end we can use the proposition 3.2 established in [13]. We obtain:

$$\|u_n(t)\|_{H^1}^2 + \int_0^t \|\nabla_h u_n(\tau)\|_{H^1}^2 \, d\tau \leq C_0 e^{C_0 t}$$

It follows that u_n is uniformly bounded in $L_T^\infty H^1 \cap L_T^2(H^{1,1} \cap H^{2,0})$.
Assume temporarily that:

$$\left(L_T^\infty L^2 \cap L_T^2 H^{1,0}\right) \cap \left(L_T^\infty H^{0,1} \cap L_T^2 H^{1,1}\right) \hookrightarrow L_T^4 L_v^2(L_h^4) \cap L_T^4 L_v^\infty(L_h^4) \tag{13}$$

we infer that:

- u_n is bounded in $L_T^4 L_v^2(L_h^4) \cap L_T^4 L_v^\infty(L_h^4)$
- ρ_n is bounded in $L_T^\infty L^2 \cap L_T^2 H^{1,0} \hookrightarrow L_T^4 L_v^2(L_h^4)$

Hence ∇P_n, $div(u_n \rho_n)$ and $div(u_n \otimes u_n)$ are bounded in $L_T^2 H^{-1}$, and this gives a bound for $\partial_t u_n$ and $\partial_t \rho_n$ in $L_T^2 H^{-1}$. We can then use Aubin's theorem in order to extract a new subsequence (still denoted (u_n, ρ_n)) that strongly converges to (u, ρ) in $L_T^2 L^2$. Now, we can pass to the limit $n \to \infty$ in all the terms in $(NSB_h)_n$, and we show that (u, ρ) is a solution of (NSB_h).

For the sake of completeness, we shall justify (13). Let $a \in \mathcal{S}(\mathbb{R}^3)$ and $s > \frac{1}{2}$ (note that in inequality (13) s is equal to 1). By using the lemma 1, we obtain for some non-negative number N to be fixed later:

$$\begin{aligned}\|a\|_{L_h^4(L_v^\infty)} &\leq \sum_{k\geq -1} \sum_{j \geq -1} \left\| \Delta_k^h \Delta_j^v a \right\|_{L_h^4(L_v^\infty)} \\ &\leq \sum_{N \geq k \geq -1} \sum_{j \geq -1} 2^{\frac{k}{2}} 2^{j(\frac{1}{2}-s)} 2^{js} \left\| \Delta_k^h \Delta_j^v a \right\|_{L^2} + \sum_{k \geq N+1} \sum_{j \geq -1} 2^{\frac{-k}{2}} 2^{j(\frac{1}{2}-s)} 2^{js} \left\| \nabla_h \Delta_k^h \Delta_j^v a \right\|_{L^2}\end{aligned}$$

Hence, because $s > \frac{1}{2}$ and $j \geq -1$, the Cauchy-Schwarz inequality gives:

$$\|a\|_{L_h^4(L_v^\infty)} \lesssim 2^{\frac{N}{2}} \|a\|_{L_h^2(H^s)_v} + 2^{-\frac{N}{2}} \|\nabla_h a\|_{L_h^2(H^s)_v}$$

Therefore, by choosing N such that $2^N = \frac{\|\nabla_h a\|_{L^2_h(H^s)_v}}{\|a\|_{L^2_h(H^s)_v}}$, we obtain:

$$\|a\|_{L^4_h(L^\infty_v)} \lesssim \|a\|^{\frac{1}{2}}_{L^2_h(H^s)_v} \|\nabla_h a\|^{\frac{1}{2}}_{L^2_h(H^s)_v}$$

On the other hand, the Minkowski inequality ensures that $L^4_h(L^\infty_v) \hookrightarrow L^\infty_v(L^4_h)$. This is sufficient to conclude the proof of (13).

5 Appendix

In this additional section we prove a result of well posedeness for (NSB_h) under some smallness conditions on: T, the norm of u_0 in $H^{0,s}(\mathbb{R}^3)$ and the norm of ρ in $L^2(\mathbb{R}^3)$. This result may not be optimal in this direction, but it gives the existence of a solution in a some new[3] situations where the Theorem 1 concerning the uniqueness can be applied:

Theorem 3. *Let $s \in]1/2, 1]$, $\delta \in [0, s]$ and $(u_0, \rho_0) \in H^{0,s} \times H^{0,\delta}$. We have:*

- *There exists $C_0 > 0$ such that if*

$$\|u_0\|^2_{0,s} + T\,\|\rho_0\|_{L^2}\left(\|u_0\|_{L^2} + \|\rho_0\|_{L^2}\left(1 + \frac{T}{2}\right)\right) < C_0^2$$

  then (NSB_h) has at least one solution (u, ρ) in $\mathcal{X}^{s,\delta}(T)$, where:

$$\mathcal{X}^{s,\delta}(T) := L^\infty_T H^{0,s} \cap L^2_T H^{1,s} \times L^\infty_T H^{0,\delta} \cap L^2_T H^{1,\delta}$$

- *The solution is unique if $\delta \geq 1 - s$*

Remark 1. *In the case where $s = \frac{1}{2}$, one may prove a similar result by taking the initial data u_0 in $L^2_h(B^{\frac{1}{2}}_{2,1})_v$. See for instance [14] where this is established for (NS_h)*

The proof of the existence part in theorem 3 requires the following lemma:

Lemma 5. *Let $s \in]1/2, 1]$, $\delta \in [0, s]$, then for all regulars vector fields a, b with $\nabla \cdot a = 0$, we have:*

$$|\langle a \cdot \nabla b, b\rangle_{0,\delta}| \lesssim \|b\|_{1/2,\delta}\left(\|a\|_{1,s}\|b\|_{1/2,\delta} + \|a\|_{1/2,s}\|b\|_{1,\delta}\right) \tag{14}$$

$$|\langle \rho, a^3\rangle_{0,s}| \leq \frac{1}{4}\|\rho\|_{L^2}\|a\|_{L^2} + \|\rho\|_{L^2}\|\nabla_h a\|_{0,s} \tag{15}$$

Proof
In order to prove (14), we follow the same approach as in [3], so we write:

$$\langle a \cdot \nabla b, b\rangle_{0,\delta} = \left\langle a^h \cdot \nabla_h b, b\right\rangle_{0,\delta} + \left\langle a^3 \partial_3 b, b\right\rangle_{0,\delta} \tag{16}$$

We remark next that lemma 2 gives $H^{1/2,s} \times H^{0,\delta} \longrightarrow H^{-1/2,\delta}$, which implies:

$$\begin{aligned} |\left\langle a^h \cdot \nabla_h b, b\right\rangle_{0,\delta}| &\lesssim \|b\|_{1/2,\delta}\left\|a^h \cdot \nabla_h b\right\|_{-1/2,\delta} \\ &\lesssim \|b\|_{1/2,\delta}\|a\|_{1/2,s}\|b\|_{1,\delta} \end{aligned}$$

[3]We have already treated in Theorem 2 some particular situations with axisymmetric initial data.

It remains now only to estimate the second term in the right hand side of (16). Indeed, Bony's decomposition tells that:

$$\Delta_q^v(a^3\partial_3 b) = \Delta_q^v\Big(\sum_{k\geq q-N_0} S_{k+2}(\partial_3 b)\Delta_k^v a^3 + \sum_{|q-k|\leq N_0} S_{k-1}a^3\Delta_k^v b\Big) \tag{17}$$

Note then firstly that:

$$\begin{aligned}\|S_{k+2}(\partial_3 b)\|_{L_h^4L_v^2} &\leq \sum_{m\leq k+1} 2^{m(1-\delta)}2^{\delta m}\,\|\Delta_m^v b\|_{L_h^4L_v^2} \leq \Big(\sum_{m\leq k+1} 2^{2m(1-\delta)}\Big)^{1/2}\|b\|_{1/2,\delta}\\ &\lesssim 2^{k(1-\delta)}\,\|b\|_{1/2,\delta}\end{aligned}$$

Moreover, by using the fact that $\partial_3 a^3 = -\nabla_h\cdot u^h$ together with the lemma 1, we obtain:

$$\left\|\Delta_k^v a^3\right\|_{L^2} \leq C2^{-k}\left\|\Delta_k^v\nabla_h\cdot u^h\right\|_{L^2}$$

This implies:

$$\begin{aligned}\left|\left\langle \Delta_q^v\sum_{k\geq q-N_0} S_{k+2}(\partial_3 b)\Delta_k^v a^3, \Delta_q^v b\right\rangle\right| &\lesssim \left\|\Delta_q^v b\right\|_{L_h^4L_v^2} 2^{q/2}\sum_{k\geq q-N_0}\|S_{k+2}(\partial_3 b)\|_{L_h^4L_v^2}\left\|\Delta_k^v a^3\right\|_{L^2}\\ &\lesssim \left\|\Delta_q^v b\right\|_{L_h^4L_v^2} 2^{q/2}\sum_{k\geq q-N_0} 2^{-k\delta}\,\|b\|_{1/2,\delta}\,\|\Delta_k^v\nabla_h u\|_{L^2}\\ &\lesssim \|b\|_{1/2,\delta}\,\|u\|_{1,s}\,2^{-q\delta}c_q\,\|b\|_{1/2,\delta}\,2^{q/2}\sum_{k\geq q-N_0} 2^{-k(\delta+s)}c_k\\ &\lesssim \|b\|_{1/2,\delta}^2\,\|u\|_{1,s}\,2^{-2q\delta}c_q 2^{q(1/2-s)}\end{aligned}$$

Hence, thanks to the assumption $s > 1/2$ we infer that:

$$\left|\left\langle \sum_{k\geq -1} S_{k+2}(\partial_3 b)\Delta_k^v a^3, b\right\rangle_{0,\delta}\right| \lesssim \|b\|_{1/2,\delta}^2\,\|u\|_{1,s}$$

In order to estimate the second term in the right hand side of (17), we use the decomposition proposed in [3]:

$$\begin{aligned}\langle\Delta_q^v\sum_{|q-k|\leq N_0} S_{k-1}a^3\Delta_k^v b, \Delta_q^v b\rangle &= \langle S_q(a^3)\partial_3\Delta_q^v b, \Delta_q^v b\rangle + \langle\sum_{|q-k|\leq N_0}\left[\Delta_q^v, S_{k-1}a^3\right]\partial_3\Delta_k^v b, \Delta_q^v b\rangle\\ &\quad+\langle\sum_{|q-k|\leq N_0}\big(S_q - S_{k-1}\big)a^3\Delta_k^v\partial_3 b, \Delta_q^v b\rangle := J_1^q + J_2^q + J_3^q\end{aligned}$$

By using an integration by parts, we obtain:

$$J_1^q = \left\langle S_q(a^3)\partial_3\Delta_q^v b, \Delta_q^v b\right\rangle = \frac{1}{2}\left\langle S_q(\partial_3 a^3)\Delta_q^v b, \Delta_q^v b\right\rangle$$

which shows that this term can be estimated in the same way as $\langle\nabla_h a\cdot b, b\rangle$ due to the divergence free condition. We get:

$$\Big|\sum_{q\geq -1} 2^{2q\delta}\left\langle S_q(a^3)\partial_3\Delta_q^v b, \Delta_q^v b\right\rangle\Big| \lesssim \|a\|_{1,s}\,\|b\|_{1/2,\delta}^2$$

Next from lemma 3, it easly follows:

$$\Big|\sum_{q\geq -1} 2^{2q\delta} J_2^q\Big| = \Big|\sum_{q\geq -1} 2^{2q\delta}\Big\langle \sum_{|q-k|\leq N_0} \big[\Delta_q^v, S_{k-1}a^3\big]\partial_3\Delta_k^v b, \Delta_q^v b\Big\rangle\Big| \lesssim \|a\|_{1,s}\sum_{q\geq -1} 2^{2q\delta}\left\|\Delta_q^v b\right\|_{1/2,0}$$
$$\lesssim \|a\|_{1,s}\|b\|_{1/2,\delta}^2$$

Finally the term J_3^q can be estimated just like $\langle S_{k-1}a^3\Delta_k^v b, \Delta_q^v b\rangle$ because the support of $S_q - S_{k-1}$ is included in an annulus in Fourier side. This ends the proof of (14).
In order to prove (15), we use the fact that $\nabla\cdot a = 0$ together with $s-1\leq 0$ to obtain:

$$\begin{aligned}
\left|\left\langle \rho, a^3\right\rangle_{0,s}\right| &= \left|2^{-2s}\langle S_0\rho, S_0 a^3\rangle + \sum_{q\geq 0} 2^{2qs}\langle \Delta_q^v\rho, \Delta_q^v a^3\rangle\right| \\
&\leq \frac{1}{4}\|S_0\rho\|_{L^2}\|S_0 a\|_{L^2} + \sum_{q\geq 0}\left\|\Delta_q^v\rho\right\|_{L^2} 2^{q(s-1)}2^{qs}\left\|\Delta_q^v\partial_3 a^3\right\|_{L^2} \\
&\leq \frac{1}{4}\|S_0\rho\|_{L^2}\|S_0 a\|_{L^2} + \sum_{q\geq 0}\left\|\Delta_q^v\rho\right\|_{L^2} 2^{qs}\left\|\Delta_q^v\nabla_h a\right\|_{L^2} \\
&\leq \frac{1}{4}\|\rho\|_{L^2}\|a\|_{L^2} + \|\rho\|_{L^2}\|\nabla_h a\|_{0,s},
\end{aligned}$$

This concludes the proof of lemma 5 □

5.1 Proof of Theorem 3

The uniqueness part of theorem 3 is clearly a direct consequence of Theorem 1. Hence it remains only to prove the existence part. This can be done in a similar way than explained in the proof of Theorem 2, and in particular the construction of the approximate sequence (u_n, ρ_n) does not add any difficulty. However, the uniform bounds must now be obtained in some new adequate norms. For simplicity, in the following we will drop the index of the approximate sequence.

We apply Δ_q^v in both equations for u and ρ from (NSB_h), then we multiply the first equation by $\Delta_q^v u$, the second one by $\Delta_q^v\rho$ and we sum over $q\geq -1$ to obtain:

$$\frac{d}{2dt}\|u(t)\|_{0,s}^2 + \|\nabla_h u(t)\|_{0,s}^2 \leq \left|\langle u\cdot\nabla u, u\rangle_{0,s}\right| + \left|\left\langle \rho, u^3\right\rangle_{0,s}\right| \tag{18}$$

$$\frac{d}{2dt}\|\rho(t)\|_{0,\delta}^2 + \|\nabla_h\rho(t)\|_{0,\delta}^2 \leq \left|\langle u\cdot\nabla\rho, \rho\rangle_{0,\delta}\right| \tag{19}$$

We prove firstly an uniform bound for u by using the L^2-energy estimate of ρ and u. Indeed, by taking $a = b = u$ in lemma 5, then (18) gives:

$$\frac{d}{dt}\|u\|_{0,s}^2 + 2\|\nabla_h u\|_{0,s}^2 \leq \|\rho\|_{L^2}\|u\|_{L^2} + \|\nabla_h u\|_{0,s}^2 + \|\rho\|_{L^2}^2 + C\|u\|_{0,s}\|\nabla_h u\|_{0,s}^2 \tag{20}$$

If we assume that:

$$\|u_0\|_{0,s}^2 + T\|\rho_0\|_{L^2}\left(\|u_0\|_{L^2} + \|\rho_0\|_{L^2}\left(1+\frac{T}{2}\right)\right) < C_0^2 \leq \frac{1}{4C^2} \tag{21}$$

then, we have: $\|u_0\|_{0,s} < C_0$.
Let us now asssume that there exists $T^{max}\in]0,T[$ satisfying:

$$T^{max} := \inf\{t\in[0,T] : \|u(t)\|_{0,s} = C_0\}$$

It follows that $\forall t \in [0, T^{max}[$: $\|u(t)\|_{0,s} < C_0 < \frac{1}{2C}$.
By using this last inequality in (20), and by integrating over $[0, t[$, we obtain for all $t \in [0, T^{max}[$:

$$\|u(t)\|_{0,s}^2 + \frac{1}{2}\int_0^t \|\nabla_h u(\tau)\|_{0,s}^2 \, d\tau \leq \|u_0\|_{0,s}^2 + \int_0^t \big(\|\rho(\tau)\|_{L^2}^2 + \|u(\tau)\|_{L^2} \|\rho(\tau)\|_{L^2} \, d\tau \big)$$

Hence, by using the L^2 energy estimate for ρ and u given by (11) and (12), we infer that:

$$\|u(t)\|_{0,s}^2 \leq \|u_0\|_{0,s}^2 + t \|\rho_0\|_{L^2} \left(\|u_0\|_{L^2} + \|\rho_0\|_{L^2} \big(1 + \frac{t}{2}\big) \right) \quad \forall t \in [0, T^{max}[,$$

then by using (21) and passing to the limit $t \longrightarrow T^{max}$, we obtain:

$$C_0^2 \leq \|u_0\|_{0,s}^2 + T \|\rho_0\|_{L^2} \left(\|u_0\|_{L^2} + \|\rho_0\|_{L^2} \big(1 + \frac{T}{2}\big) \right) < C_0^2,$$

which contradicts the existence of T^{max} and gives for all $t \in [0, T]$: $\|u(t)\|_{0,s} < C_0$. Therefore we have proved an uniform bound of u in $L_T^\infty H^{0,s}$. Plugging this bound into (20) gives a bound for u in $L_T^2 H^{1,s}$. Note that by using an argument of interpolation, this also gives a bound of u in $L_T^4 H^{1/2,s}$.

Next, we can estimate the right hand side of (19) by using lemma 5 with $a = u$ and $b = \rho$. We obtain after some calculations:

$$\frac{d}{dt} \|\rho(t)\|_{0,\delta}^2 + 2 \|\nabla_h \rho(\tau)\|_{0,\delta}^2 \, d\tau \leq \|\nabla_h \rho\|_{0,\delta}^2 + C A(t) \|\rho(t)\|_{0,\delta}^2 ,$$
$$\text{where } A(\cdot) = \|\nabla_h u(\cdot)\|_{0,s}^2 + \|u(\cdot)\|_{1/2,s}^4 \in L_T^1$$

Finally, by applying Gronwall 's lemma we obtain the adequate bound for ρ.

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