

GEOMETRIC REALIZATIONS OF TAMARI INTERVAL LATTICES VIA CUBIC COORDINATES

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GEOMETRIC REALIZATIONS OF TAMARI INTERVAL LATTICES VIA CUBIC COORDINATES

CAMILLE COMBE

Abstract. We introduce cubic coordinates, which are integer words encoding intervals in the Tamari lattices. Cubic coordinates are in bijection with interval-posets, themselves known to be in bijection with Tamari intervals. We show that in each degree the set of cubic coordinates forms a lattice, isomorphic to the lattice of Tamari intervals. Geometric realizations are naturally obtained by placing cubic coordinates in space, highlighting some of their properties. We consider the cellular structure of these realizations. Finally, we show that the poset of cubic coordinates is shellable.

Contents

Introduction 2
1. Preliminaries 3
1.1. Posets and lattices 3
1.2. Rooted trees and binary trees 4
1.3. Tamari intervals and interval-posets 6
2. Cubic coordinates and Tamari intervals 10
2.1. Tamari interval diagrams 10
2.2. Link with interval-posets 14
2.3. Cubic coordinates 16
2.4. Order structure 19
3. Geometric properties 22
3.1. Cubic realizations 22
3.2. Covering map 23
3.3. Cells and synchronized cubic coordinates 25
3.4. Volume of $\mathfrak{C}(CC)$ 30
3.5. EL-shellability 33
Acknowledgements 34
References 35

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Introduction

The Tamari lattices are partial orders having extremely rich combinatorial and algebraic properties. These partial orders are defined on the set of binary trees and rely on the rotation operation [Tam62]. We are interested in the intervals of these lattices, meaning the pairs of comparable binary trees. Tamari intervals of size n also form a lattice. The number of these objects is given by a formula that was proved by Chapoton [Cha06]:

$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}. \tag{0.0.1}$$

Strongly linked with associahedra, Tamari lattices have been recently generalized in many ways [BPR12, PRV17]. In this process, the number of intervals of these generalized lattices have also been enumerated through beautiful formulas [BMFPR12, FPR17]. Many bijections between Tamari intervals of size n and other combinatorial objects are known. For instance, a bijection with 3-connected planar triangulations is presented by Bernardi and Bonichon in [BB09] (see also [Fan18]). It has been proved by Châtel and Pons that Tamari intervals are in bijection with interval-posets of the same size [CP15].

We provide in this paper a new bijection with Tamari intervals, which is inspired by interval-posets. More precisely, we first build two words of size n from the Tamari diagrams [Pal86] of a binary tree. If they satisfy a certain property of compatibility, we build a Tamari interval diagram from these two words. We show that Tamari interval diagrams and interval-posets are in bijection. Then we propose a new encoding of Tamari intervals, by building $(n-1)$-tuples of numbers from Tamari interval diagrams. We call these tuples cubic coordinates. This new encoding has two obvious virtues: it is very compact and it gives a way of comparing in a simple manner two Tamari intervals, through a fast algorithm. On the other hand, some properties of Tamari intervals translate nicely in the setting of cubic coordinates. For instance, synchronized Tamari intervals [FPR17] become cubic coordinates with no zero entry. Besides, cubic coordinates provide naturally a geometric realization of the lattice of Tamari intervals, by seeing them as space coordinates. Indeed, all cubic coordinates of size n can be placed in the space $\mathbb{R}^{n-1}$. By drawing their cover relations, we obtain a directed graph. This gives us a realization of cubic coordinate lattices, which we call cubic realization. This realization leads us to many questions, in particular about the cells it contains. We characterize these cells in a combinatorial way, and we deduce a formula to compute the volume of the cubic realization in the geometrical sense. Another direction, more topological, involves the shellability of partial order. We show, drawing inspiration from the work of Björner and Wachs [BW96, BW97], that the cubic coordinates poset is EL-shellable, and as a consequence its associated complex is shellable.

This article is organized in three sections.

The first section is dedicated to reminders about some definitions, such as binary trees, Tamari intervals and interval-posets, and sets out the conventions used. Because of its key role in this work, the bijection between Tamari intervals and interval-posets is also recalled in this section.

In the second section, we define Tamari interval diagrams and show that they are in bijection, size by size, with interval-posets. We then define cubic coordinates and show that they are in bijection, size by size, with Tamari interval diagrams. Using this two bijections, and after having endowed the set of cubic coordinates with a partial order, we show that there is a poset isomorphism between the poset of cubic coordinates and the poset of Tamari intervals.

As pointed out above, the poset of cubic coordinates can then be realized geometrically. This cubic realization and the cells that compose it are the object of the third section. For each cell, we then associate a synchronized cubic coordinate, which is a cubic coordinate without letter 0. By relying upon this particular cubic coordinate, we give a formula to compute the volume of the cubic realization. Finally, we extend the result of Björner and Wachs on the Tamari posets to the Tamari interval posets, by showing that the cubic coordinate posets are EL-shellable.

This article is a complete version of [Com19]. All the proofs are given and several new results are presented, such as the EL-shellability of cubic coordinate posets.

General notations and conventions. Throughout this article, for all words u, we denote by u_i the i-th letter of u. For any integers i and j, $[i,j]$ denotes the set $\{i, i+1, \dots, j\}$. For any integer i, $[i]$ denotes the set $[1,i]$. All posets considered in this article are finite.

1. Preliminaries

In this first section we provide some basic notions of combinatorics and the conventions used afterwards. For this, we recall the definitions of lattices, binary trees, Tamari intervals, and interval-posets. Also, we recall the bijection given in [CP15].

1.1. Posets and lattices.

A *partially ordered set*, commonly called *poset*, is a pair $(\mathcal{P}, \preccurlyeq_{\mathcal{P}})$. When the context is clear, we simply denote this pair by $\mathcal{P}$.

When two elements x and y of $\mathcal{P}$ satisfy $x \preccurlyeq_{\mathcal{P}} y$, then we say that x and y are *comparable*. Otherwise, they are *incomparable*.

Let $x, y \in \mathcal{P}$ such that $x \preccurlyeq_{\mathcal{P}} y$ and $x \neq y$. The element y *covers* x, denoted by $x \lessdot_{\mathcal{P}} y$, for the partial order $\preccurlyeq_{\mathcal{P}}$ if, for all $z \in \mathcal{P}$ such that $x \preccurlyeq_{\mathcal{P}} z \preccurlyeq_{\mathcal{P}} y$, either $z = x$ or $z = y$. The binary relation $\lessdot_{\mathcal{P}}$ is called the *covering relation* of the poset $\mathcal{P}$. By a slight abuse of notation, the set of elements (x,y) such that $x \lessdot_{\mathcal{P}} y$ is also denoted by $\lessdot_{\mathcal{P}}$.

A *maximal element* of $\mathcal{P}$ is an element x such that if there is $y \in \mathcal{P}$ such that $x \preccurlyeq_{\mathcal{P}} y$, then $y = x$. Likewise, a *minimal element* of $\mathcal{P}$ is an element y such that if there is $x \in \mathcal{P}$ such that $x \preccurlyeq_{\mathcal{P}} y$, then $x = y$. A poset $\mathcal{P}$ is *bounded* if it has a unique maximal element and a unique minimal element for $\preccurlyeq_{\mathcal{P}}$.

Since a partial order is transitive, one can realize posets or lattices by knowing only covering relations. The natural way to realize posets is to draw their *Hasse diagrams*, by drawing a edge between all x and y in $\mathcal{P}$ such that $(x,y) \in \lessdot_{\mathcal{P}}$. For any $(x,y) \in \lessdot_{\mathcal{P}}$, we choose the convention to represent x at the top and y at the bottom in the Hasse diagrams. We will keep this convention for all realizations.

Let $x, y \in \mathcal{P}$, the *join* between x and y, denoted by $\vee_{\mathcal{P}}(x,y)$ (or $x \vee_{\mathcal{P}} y$), is defined by

$$\vee_{\mathcal{P}}(x,y) := \min_{\preccurlyeq_{\mathcal{P}}} \{z \in \mathcal{P} \ : \ x \preccurlyeq_{\mathcal{P}} z \text{ and } y \preccurlyeq_{\mathcal{P}} z\}. \tag{1.1.1}$$

The *meet* between x and y, denoted by $\wedge_{\mathcal{P}}(x,y)$ (or $x \wedge_{\mathcal{P}} y$), is defined by

$$\wedge_{\mathcal{P}}(x,y) := \max_{\preccurlyeq_{\mathcal{P}}} \{z \in \mathcal{P} \ : \ z \preccurlyeq_{\mathcal{P}} x \text{ and } z \preccurlyeq_{\mathcal{P}} y\}. \tag{1.1.2}$$

A poset $\mathcal{P}$ is a *join-semilattice* if for all $x, y \in \mathcal{P}$, $\vee_{\mathcal{P}}(x,y)$ exists. Likewise, a poset $\mathcal{P}$ is a *meet-semilattice* if for all $x, y \in \mathcal{P}$, $\wedge_{\mathcal{P}}(x,y)$ exists. A poset $(\mathcal{L}, \preccurlyeq_{\mathcal{L}})$ is a *lattice* if $\mathcal{L}$ is a join-semilattice and a meet-semilattice.

Let $\mathcal{P}$ be a poset and $u^{(1)}, u^{(2)} \in \mathcal{P}$ such that $u^{(1)} \preccurlyeq_{\mathcal{P}} u^{(2)}$. An *interval* $(u^{(1)}, u^{(2)})$ is the set of all elements between $u^{(1)}$ and $u^{(2)}$. The set of intervals of $\mathcal{P}$ is denoted by $\mathrm{int}(\mathcal{P})$. The *poset of intervals* of a poset $\mathcal{P}$ is the poset on the set $\mathrm{int}(\mathcal{P})$ endowed with the partial order $\preccurlyeq_{\mathrm{int}(\mathcal{P})}$ defined, for all $(u^{(1)}, u^{(2)}), (v^{(1)}, v^{(2)}) \in \mathrm{int}(\mathcal{P})$, by

$$(u^{(1)}, u^{(2)}) \preccurlyeq_{\mathrm{int}(\mathcal{P})} (v^{(1)}, v^{(2)}) \text{ if } u^{(1)} \preccurlyeq_{\mathcal{P}} v^{(1)} \text{ and } u^{(2)} \preccurlyeq_{\mathcal{P}} v^{(2)}. \tag{1.1.3}$$

In the same way, for $(u^{(1)}, u^{(2)}), (v^{(1)}, v^{(2)}) \in \mathrm{int}(\mathcal{L})$ such that $(u^{(1)}, u^{(2)}) \preccurlyeq_{\mathrm{int}(\mathcal{L})} (v^{(1)}, v^{(2)})$, a covering relation for the partial order $\preccurlyeq_{\mathrm{int}(\mathcal{L})}$ is defined.

The property of being a lattice is preserved under this construction.

Proposition 1.1.1. *If $(\mathcal{L}, \preccurlyeq_{\mathcal{L}})$ is a lattice, then $(\mathrm{int}(\mathcal{L}), \preccurlyeq_{\mathrm{int}(\mathcal{L})})$ is a lattice.*

Proof. Let $(u^{(1)}, u^{(2)}), (v^{(1)}, v^{(2)}) \in \mathrm{int}(\mathcal{L})$. First, we have to show that $\vee_{\mathcal{L}}(u^{(1)}, v^{(1)}) \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(2)}, v^{(2)})$. By the definition of the join, one has $u^{(2)} \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(2)}, v^{(2)})$ and $v^{(2)} \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(2)}, v^{(2)})$. Furthermore, since $u^{(1)} \preccurlyeq_{\mathcal{L}} u^{(2)}$ and $v^{(1)} \preccurlyeq_{\mathcal{L}} v^{(2)}$, one has $u^{(1)} \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(2)}, v^{(2)})$ and $v^{(1)} \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(2)}, v^{(2)})$. In addition, $\vee_{\mathcal{L}}(u^{(1)}, v^{(1)})$ is the minimal element of $\mathcal{L}$ satisfying $u^{(1)} \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(1)}, v^{(1)})$ and $v^{(1)} \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(1)}, v^{(1)})$. Thus, $\vee_{\mathcal{L}}(u^{(1)}, v^{(1)}) \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(2)}, v^{(2)})$.

From (1.1.3), one has

$$\begin{aligned}
&\vee_{\mathrm{int}(\mathcal{L})} \left((u^{(1)}, u^{(2)}), (v^{(1)}, v^{(2)})\right) \\
&= \min_{\preccurlyeq_{\mathrm{int}(\mathcal{L})}} \{(w^{(1)}, w^{(2)}) \in \mathrm{int}(\mathcal{L}) \ : \ (u^{(1)}, u^{(2)}) \preccurlyeq_{\mathrm{int}(\mathcal{L})} (w^{(1)}, w^{(2)}), (v^{(1)}, v^{(2)}) \preccurlyeq_{\mathrm{int}(\mathcal{L})} (w^{(1)}, w^{(2)})\} \\
&= \min_{\preccurlyeq_{\mathrm{int}(\mathcal{L})}} \{(w^{(1)}, w^{(2)}) \in \mathrm{int}(\mathcal{L}) \ : \ u^{(1)} \preccurlyeq_{\mathcal{L}} w^{(1)}, u^{(2)} \preccurlyeq_{\mathcal{L}} w^{(2)}, v^{(1)} \preccurlyeq_{\mathcal{L}} w^{(1)}, v^{(2)} \preccurlyeq_{\mathcal{L}} w^{(2)}\} \\
&= \left(\vee_{\mathcal{L}}(u^{(1)}, v^{(1)}), \vee_{\mathcal{L}}(u^{(2)}, v^{(2)})\right).
\end{aligned} \tag{1.1.4}$$

The case of the meet $\wedge_{\mathrm{int}(\mathcal{L})} \left((u^{(1)}, u^{(2)}), (v^{(1)}, v^{(2)})\right) = \left(\wedge_{\mathcal{L}}(u^{(1)}, v^{(1)}), \wedge_{\mathcal{L}}(u^{(2)}, v^{(2)})\right)$ is symmetrical. □

1.2. **Rooted trees and binary trees.** A *rooted tree*, or simply a *tree* in our context, is defined recursively as a node together with a (possibly empty) sequence of rooted trees. We shall use the standard terminology about trees like *root*, *edge*, *child*, *descendant*, *subtree*, *etc.* The *size* of a tree is its number of nodes. The nodes of the trees considered in this work are labeled by positive integers. We draw trees with the root at the top, where a node is depicted by ○ with its label inside the circle. A *forest* is a sequence of trees. From a forest $\mathfrak{f}$ of n trees, it is always possible to build a tree $\mathfrak{t}$ by taking the root of each element of $\mathfrak{f}$ and by linking all these roots to an artificial node, such that this artificial node

become the root of $\mathfrak{t}$. The size of the obtained tree is one plus the sum of all sizes of trees in $\mathfrak{f}$.

A *binary tree* (or *2-tree*) $\mathfrak{t}$ is either a leaf or a node attached through two edges to two binary trees, which are called respectively the *left subtree* and the *right subtree* of $\mathfrak{t}$. Recall that the *size* of a binary tree is its number of nodes. We denote by $\mathrm{T}_2(n)$ the set of binary trees of size n. The set of binary trees is enumerated by Catalan numbers. We draw binary trees with the root at the top and the leaves at the bottom, where a node is depicted by ○ and a leaf is depicted by ▫ (see for instance Figure 1).

Let $\mathfrak{t} \in \mathrm{T}_2(n)$. Each node of $\mathfrak{t}$ is numbered recursively, starting with the left subtree, then the root, and ending with the right subtree. An example is given in Figure 1. This numbering then establishes a total order on the nodes of a binary tree called the *infix order*. Afterwards, this numbering is used to refer to the nodes. The sequence of nodes numbered from 1 to n forms the *infix traversal*.

When the size n of $\mathfrak{t}$ satisfies $n \geqslant 1$, the *canopy* of $\mathfrak{t}$ is the word of size $n-1$ on the alphabet $\{0,1\}$ built by assigning to each leaf of $\mathfrak{t}$ a letter as follows. Any leaf oriented to the left (resp. right) is labeled by 0 (resp. 1). The canopy of $\mathfrak{t}$ is the word obtained by reading from left to right the labels thus established, forgetting the first and the last one (since there are always respectively 0 and 1). For instance, the binary tree in Figure 1 has for canopy the word 0110100. There is a link between infix order of a binary tree and its canopy. For a node of index i for the infix order in a tree $\mathfrak{t}$, the right subtree of i is a leaf oriented to the right if and only if the i-th letter of the canopy of $\mathfrak{t}$ is 1. The left subtree of i is a leaf oriented to the left if and only if the $(i-1)$-th letter of the canopy of $\mathfrak{t}$ is 0. The two direct implications can be proved by induction on the set of binary trees, for instance, see Lemma 4.3. of [Gir12]. The converses are simply given by the definition of the canopy.

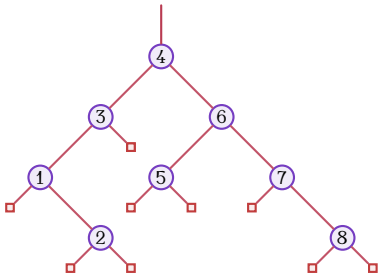


FIGURE 1. A binary tree of size 8 and the numbering of its nodes in the infix order.

A fundamental operation in binary trees is the *right rotation* [Tam62]. Let k and l be the indices for the infix order of two nodes of a binary tree $\mathfrak{t}$, such that the node k is the left child of the node l. Right rotation locally changes the tree $\mathfrak{t}$ so that l becomes the right child of k (see Figure 2). Equivalently, this means that the local configuration $((\mathfrak{a},\mathfrak{b}),\mathfrak{c})$ becomes $(\mathfrak{a},(\mathfrak{b},\mathfrak{c}))$, where $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ are the subtrees shown in Figure 2.

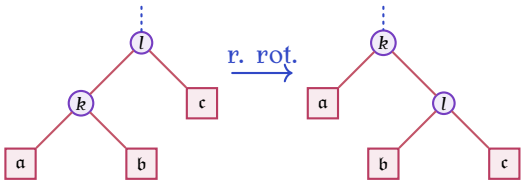


FIGURE 2. Right rotation of edge (k, l) in t (on the left), where a, b, and c are any subtrees.

1.3. **Tamari intervals and interval-posets.** For any $n \geqslant 0$, let $s, t \in T_2(n)$. We set $s \preccurlyeq_{ta} t$ if either $t = s$ or t is obtained by successively applying one or more right rotations in s. The set $T_2(n)$ endows with $\preccurlyeq_{ta}$ is the *Tamari lattice* of order n [HT72]. Moreover, s is covered by t, denoted by $s \lessdot_{ta} t$, if t is obtained from s by performing one right rotation.

In the literature, the Tamari lattice is closely related to the *associahedron*, or the *Stasheff polytope* after the work of Stasheff. More precisely, the Hasse diagram of the Tamari lattice is the 1-skeleton of the associahedron.

Let $s, t \in T_2(n)$. A *Tamari interval* of *size* n is an interval (s, t) for the Tamari order $\preccurlyeq_{ta}$. The set of Tamari intervals of size n is denoted by $\mathrm{int}(T_2(n))$.

The Tamari interval lattice is the set $\mathrm{int}(T_2(n))$ endowed with the partial order $\preccurlyeq_{\mathrm{int(ta)}}$. Let $n \geqslant 0$ and $(s, t), (s', t') \in \mathrm{int}(T_2(n))$, following (1.1.3), we have that $(s, t) \preccurlyeq_{\mathrm{int(ta)}} (s', t')$ if $s \preccurlyeq_{ta} s'$ and $t \preccurlyeq_{ta} t'$. According to Proposition 1.1.1, the poset so defined is a lattice. Moreover, it follows from the definition of $\preccurlyeq_{\mathrm{int(ta)}}$ that (s', t') covers (s, t) if

- ⋆ either s' is obtained by a single right rotation of an edge in s and $t' = t$,
- ⋆ or t' is obtained by a single right rotation of an edge in t and $s' = s$.

It is known from [Cha06] that Tamari intervals of size n are enumerated by

$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}. \tag{1.3.1}$$

The first numbers are

$$1, 1, 3, 13, 68, 399, 2530, 16965. \tag{1.3.2}$$

This sequence is Sequence **A000260** of [Slo].

Interval-posets are posets introduced by Châtel and Pons in [CP15] in order to study the Tamari lattice. Indeed, there is a poset isomorphism between the Tamari interval lattices and the set of interval-posets endowed with a certain partial order.

Let $n \geqslant 0$ and $\{\pi_1, \dots, \pi_n\}$ be a set of n symbols numbered from 1 to n. An *interval-poset* π is a partial order $\lhd$ on the set $\{\pi_1, \dots, \pi_n\}$ such that

(i) if $i < k$ and $\pi_k \lhd \pi_i$, then for all π_j such that $i < j < k$, one has $\pi_j \lhd \pi_i$,
(ii) if $i < k$ and $\pi_i \lhd \pi_k$, then for all π_j such that $i < j < k$, one has $\pi_j \lhd \pi_k$.

The *size* of an interval-poset is the cardinality of its underlying set. The set of interval-posets of size n is denoted by $\mathrm{IP}(n)$, and the elements of interval-poset are called *vertices*.

The two conditions (i) and (ii) of interval-posets are referred to as *interval-poset properties*. For any $i < j$, the relations $\pi_j \lhd \pi_i$ are known as *decreasing relations* and the relations $\pi_i \lhd \pi_j$ are known as *increasing relations*.

As it is shown in Figure 4b, the Hasse diagram of interval-posets can be drawn as directed graph where two vertices π_i and π_j are related by an arrow from π_i to π_j (resp. π_j to π_i) if $\pi_i \lhd \pi_j$ (resp. $\pi_j \lhd \pi_i$) where $i < j$.

Let $n \geqslant 0$ and $(\mathfrak{s}, \mathfrak{t}) \in \mathrm{int}(\mathrm{T}_2(n))$ and $\pi \in \mathrm{IP}(n)$. We will recall a bijection ρ relating on the one hand the restriction of π to its decreasing relations with the binary tree $\mathfrak{s}$, and on the other hand the restriction of π to its increasing relations with the binary tree $\mathfrak{t}$.

Thus, from the restriction of π to its decreasing (resp. increasing) relations we build a forest referred to as the *decreasing* (resp. *increasing*) *forest*, such that if $\pi_j \lhd \pi_i$ with $i < j$ (resp. $j < i$), then the node j is a descendant of the node i. Otherwise, if $\pi_j \not\lhd \pi_i$ with $i < j$ (resp. $j < i$) the node j is placed to the right (resp. left) of the node i.

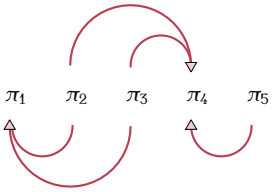
Note that we obtain a decreasing (resp. increasing) forest formed by trees labelled from the roots to the leaves in increasing (resp. decreasing) order. Moreover, the prefix (resp. suffix) traversal of the decreasing (resp. increasing) forest gives the sequence of labels $1, \ldots, n$. Let us add a virtual root node (without label) on the top of both decreasing and increasing forests to form two trees. We denote by $\mathfrak{s}'$ and $\mathfrak{t}'$ the trees respectively obtained from the decreasing and the increasing forests.

Let ρ be the map sending π to the pair $(\mathfrak{s}, \mathfrak{t})$ of binary trees defined such that the tree $\mathfrak{s}$ (resp. $\mathfrak{t}$) is the unique binary tree obtained by reading $\mathfrak{s}'$ (resp. $\mathfrak{t}'$) in the following way. For all label i, j in $\mathfrak{s}'$ (resp. $\mathfrak{t}'$), if a node j is a descendant of a node i in $\mathfrak{s}'$ (resp. $\mathfrak{t}'$), then j becomes a right (resp. left) descendant of the node i in $\mathfrak{s}$ (resp. $\mathfrak{t}$). If a node i is a left (resp. right) brother of a node j in $\mathfrak{s}'$ (resp. $\mathfrak{t}'$), then i becomes a left (resp. right) descendant of the node j in $\mathfrak{s}$ (resp. $\mathfrak{t}$).

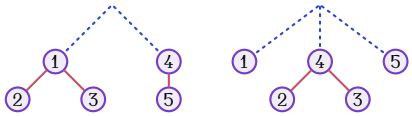
Figure 3 gives an example of construction by the bijection ρ of a Tamari interval from an interval-poset of size 5.

In this section, we shall draw interval-posets as follows. For any $i < j$, if $\pi_j \lhd \pi_i$ and there is no vertex π_k such that $\pi_k \lhd \pi_i$ and $j < k$, then we draw an arrow with source π_j and target π_i from below as shown in the example in Figure 4. Symmetrically, if $\pi_j \lhd \pi_k$ and $j < k$ and if there is no π_i such that $\pi_i \lhd \pi_k$ and $i < j$, then we draw an arrow with source π_j and target π_k from above. We refer to this directed graph with two types of arrows as the *minimalist representation* of π.

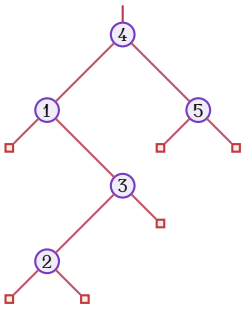
The closure for the interval-poset properties is given by adding the decreasing relations $\pi_j \lhd \pi_i$ for any relation $\pi_k \lhd \pi_i$ and by adding the increasing relations $\pi_j \lhd \pi_k$ for any relation $\pi_i \lhd \pi_k$, for any $i < j < k$. By taking the reflexive closure and the closure for the interval-poset properties, an interval-poset is obtained from the minimalist representation. The interest of the minimalist representation is justified later, in particular with Theorem 2.2.3. It is important to represent the decreasing relations and the increasing relations independently.



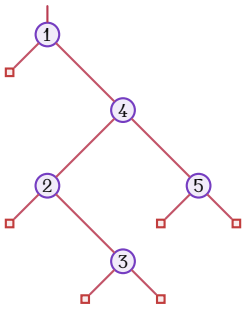
(A) Interval-poset of size 5.



(B) Decreasing (on the left) and increasing (on the right) forests

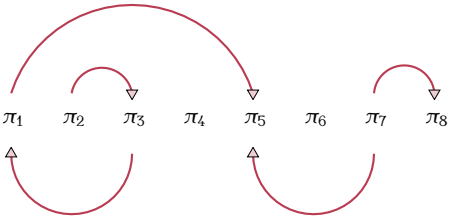


(C) Left binary tree.

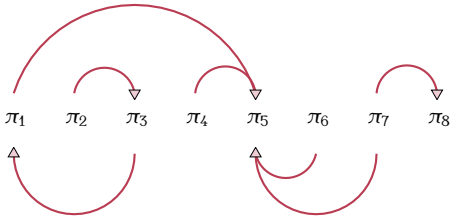


(D) Right binary tree.

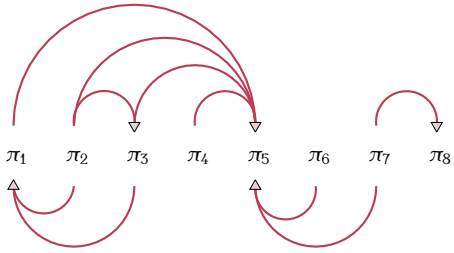
FIGURE 3. Construction of a Tamari interval from an interval-poset by ρ.



(A) Minimalist representation.



(B) Hasse diagram.



(C) Diagram with all apparent (except reflexive) relations.

FIGURE 4. Different representations of an interval-poset of size 8.

Let $n \geqslant 0$ and $\pi, \pi' \in \mathrm{IP}(n)$ and $(\mathfrak{s}, \mathfrak{t}) := \rho(\pi)$, $(\mathfrak{s}', \mathfrak{t}') := \rho(\pi')$. Let $(\star)$ (resp. $(\diamond)$) the following condition: π' is obtained from π by adding (resp. removing) only decreasing

(resp. increasing) relations of target a vertex π_k, such that if only one of these decreasing (resp. increasing) relations is removed (resp. added), then either π is obtained or the object obtained is not an interval-poset.

For the sequel, we need to recall that $(\mathfrak{s}', \mathfrak{t}')$ covers $(\mathfrak{s}, \mathfrak{t})$ if and only if π and π' satisfy either $(\star)$ or $(\diamond)$.

Lemma 1.3.1. *The interval-posets π and π' satisfy $(\star)$ (resp. $(\diamond)$) for the vertex π_i (resp. π_j) if and only if $\mathfrak{s}'$ (resp. $\mathfrak{t}'$) is obtained by a unique right rotation of the edge (i, j) in $\mathfrak{s}$ (resp. $\mathfrak{t}$) and $\mathfrak{t}' = \mathfrak{t}$ (resp. $\mathfrak{s}' = \mathfrak{s}$).*

Proof. Suppose π and π' satisfy $(\star)$ for the vertex π_i. Therefore, π' has more decreasing relations of target π'_i than the vertex π_i in π. Suppose that the vertices π_j and π_i are not related in π, and that π'_j and π'_i are related in π', with $i < j$. Then, by the interval-poset property (i), for any π'_k such that $i < k < j$, $\pi'_k \lhd \pi'_i$. Moreover, if we remove only one of these decreasing relations, we obtain either π or an object that is no longer an interval-poset. This means that the number of descending relations added in π' is minimal, or equivalently, that the vertex π_j is closest to the vertex π_i such that π_j and π_i are not related in π and $i < j$. This case is depicted in Figure 5. By the bijection ρ, add

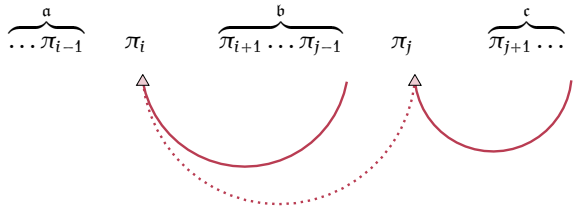
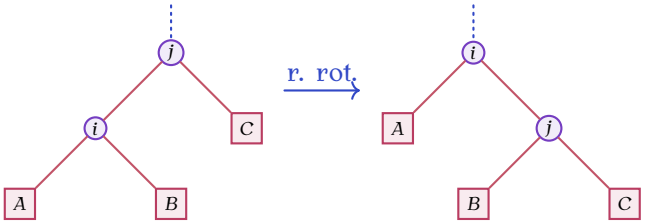


FIGURE 5. Interval-poset of the decreasing forest before (without dotted line) and after (with dotted line) the right rotation of the edge (i, j), where $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ may be empty.

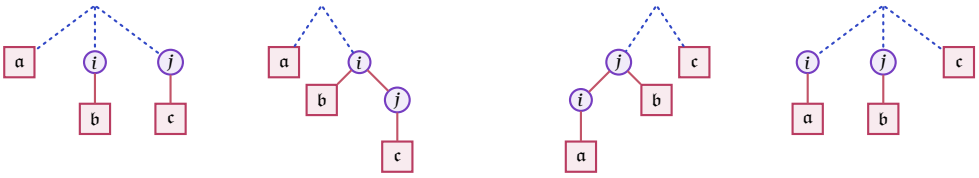
these decreasing relations of target π_i in π leads to the decreasing forest induced by $\mathfrak{s}'$ represented by Figure 6b. A unique right rotation is then made between the trees $\mathfrak{s}$ and $\mathfrak{s}'$ (see Figure 6a). Furthermore, since the increasing relations are unchanged between π and π', the increasing forests induced by $\mathfrak{t}$ and $\mathfrak{t}'$ are the same, and thus $\mathfrak{t}' = \mathfrak{t}$.

Reciprocally, suppose that $\mathfrak{s}'$ is obtained by a unique right rotation of the edge (i, j) in $\mathfrak{s}$ and that $\mathfrak{t}' = \mathfrak{t}$. The case is depicted by Figure 6a, and the two decreasing forests induced by $\mathfrak{s}$ and $\mathfrak{s}'$ are depicted by Figure 6b. By the bijection ρ, we then obtain the interval-poset whose restriction to decreasing relations is shown by Figure 5. Since $\mathfrak{t}' = \mathfrak{t}$, the increasing relations of the interval-posets associated with $(\mathfrak{s}, \mathfrak{t})$ and $(\mathfrak{s}', \mathfrak{t}')$ are the same. Finally, π' is obtained by adding only decreasing relations of target π_i in π. Furthermore, if only one of these relations is removed, then either π is obtained, or the object obtained is not an interval-poset. This means that π and π' satisfy $(\star)$.

Symmetrically, we show that π and π' satisfy $(\diamond)$ for π_j if and only if $\mathfrak{t}'$ is obtained by a unique right rotation of the edge (i, j) in $\mathfrak{t}$ and $\mathfrak{s}' = \mathfrak{s}$. Figure 6c and Figure 7 depicts this case. □



(A) Binary trees $\mathfrak{s}$ and $\mathfrak{s}'$ (resp. $\mathfrak{t}$ and $\mathfrak{t}'$).



(B) Decreasing forests induced by $\mathfrak{s}$ and $\mathfrak{s}'$.

(C) Increasing forests induced by $\mathfrak{t}$ and $\mathfrak{t}'$.

FIGURE 6. Right rotation of the edge (i, j) in the binary tree $\mathfrak{s}$ (resp. $\mathfrak{t}$), where $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are subtrees.

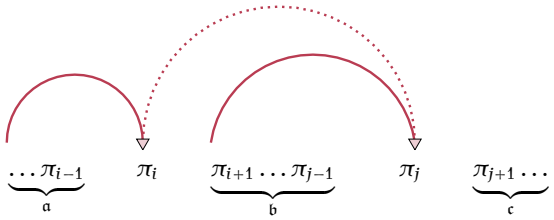


FIGURE 7. Interval-poset of the increasing forest before (with dotted lines) and after (without dotted lines) the right rotation of the edge (i, j), where $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ may be empty.

2. Cubic coordinates and Tamari intervals

The aim of this section is to build the poset of the cubic coordinates, then to establish the poset isomorphism between this poset and the poset of the Tamari intervals. To achieve this goal, we first define the Tamari interval diagrams based on the interval-posets. The cubic coordinates are then obtained from the Tamari interval diagrams.

2.1. Tamari interval diagrams.

Let us give the definition of a Tamari diagram, as formulated in [BW97]. For any $n \geqslant 0$, a *Tamari diagram* is a word u of length n on the alphabet $\mathbb{N}$ which satisfies the two following conditions:

(i) $0 \leqslant u_i \leqslant n - i$ for all $i \in [n]$,
(ii) $u_{i+j} \leqslant u_i - j$ for all $i \in [n]$ and $j \in [0, u_i]$.

The *size* of a Tamari diagram is its number of letters. For instance, the sets of Tamari diagrams of size 2, 3 and 4 are

$$\begin{gathered}\{00, 10\}, \qquad \{000, 100, 010, 200, 210\}, \\ \{0000, 0010, 0100, 0200, 0210, 1000, 1010, 2000, 2100, 3000, 3010, 3100, 3200, 3210\}.\end{gathered} \tag{2.1.1}$$

In the literature, Tamari diagrams are also known as bracket vectors, objects inspired by the right bracketing introduced in [HT72] by Huang and Tamari. Furthermore, Tamari diagrams are known to be enumerated by Catalan numbers

$$\mathrm{cat}(n) := \frac{1}{n+1}\binom{2n}{n}. \tag{2.1.2}$$

A dual version of Tamari diagrams can be defined by considering the opposite of Conditions (i) and (ii). For any $n \geqslant 0$, a *dual Tamari diagram* is a word v of length n on the alphabet $\mathbb{N}$ which satisfies the two following conditions:

(i) $0 \leqslant v_i \leqslant i-1$ for all $i \in [n]$,

(ii) $v_{i-j} \leqslant v_i - j$ for all $i \in [n]$ and $j \in [0, v_i]$.

The *size* of a dual Tamari diagram is its number of letters. In other words, $v = v_1 \dots v_n$ is a dual Tamari diagram if and only if $v_n \dots v_1$ is a Tamari diagram.

Note that the first condition of a Tamari diagram u and of a dual Tamari diagram v of size n implies that $u_n = 0$ and $v_1 = 0$.

A graphical representation of a Tamari diagram u of size n by needles and diagonals provides a simple way to check Condition (ii) of a Tamari diagram. For each position $i \in [n]$, we draw a needle from the point $(i-1, 0)$ to the point $(i-1, u_i)$ in the Cartesian plane. Condition (ii) says that one can draw lines of slope -1 passing through the x-axis and the top of each needle without crossing any other needle. For instance, the Tamari diagram 9021043100 is drawn by Figure 8. One can observe that none of its diagonals, drawn as dotted lines, crosses a needle.

Likewise, a graphical representation can be given for the dual Tamari diagram v of size n. One draws v in the same way as Tamari diagram, and Condition (ii) says that one can draw lines of slope 1 passing through the x-axis and the top of each needle without crossing any other needle. Figure 8 also depicts the dual Tamari diagram 0010040002.

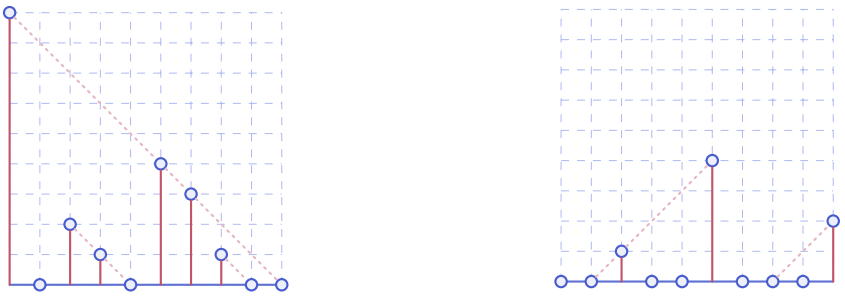


FIGURE 8. A Tamari diagram 9021043100 (on the left) and a dual Tamari diagram 0010040002 (on the right) of size 10.



FIGURE 9. A binary tree and the associated Tamari diagram of the same size.

For any $n \geqslant 0$, the set of Tamari diagrams of size n is in bijection with $T_2(n)$. Indeed, one builds from a Tamari diagram u of size n a binary tree $\mathfrak{s}$ recursively as follows. If $n = 0$, $\mathfrak{s}$ is defined as the leaf. Otherwise, let i be the smallest position in u such that u_i is the maximum allowed value, namely $n - i$. Then $\mathfrak{s}_1 := u_1 \dots u_{i-1}$ and $\mathfrak{s}_2 := u_{i+1} \dots u_n$ are also Tamari diagrams. One forms $\mathfrak{s}$ by grafting the binary trees obtained recursively by this process applied on $\mathfrak{s}_1$ and on $\mathfrak{s}_2$ to a new node. Reciprocally, for each node of index i of the tree $\mathfrak{s}$, labeled with an infix traversal, the value of the i-th letter of the corresponding Tamari diagram is given by the number of nodes in the right subtree of the node i. The complete demonstration is given in [Pal86].

In the case of dual Tamari diagrams, the construction of the binary tree $\mathfrak{t}$ is also recursive, except that it is the maximum position i in the dual Tamari diagram whose value is the highest allowed on that section of the word that should be chosen first. Similarly for the reciprocal, the procedure is identical, except that the value of the i-th letter in the dual Tamari diagram is given by the number of nodes in the left subtree of the node i in the tree $\mathfrak{t}$.

For instance, in Figure 1, the Tamari diagram is 10040210 and the dual Tamari diagram is 00230100. Figure 9 depicts the corresponding binary tree of the Tamari diagram 1003010.

Let $n \geqslant 0$ and u be a Tamari diagram, and v be a dual Tamari diagram, both of size n. The diagrams u and v are *compatible* if there are no i, j with $1 \leqslant i < j \leqslant n$ such that $u_i \geqslant j - i$ and $v_j \geqslant j - i$. If u and v are compatible, then the pair (u, v) is called *Tamari interval diagram*. The set of Tamari interval diagrams of size n is denoted by $\mathrm{TID}(n)$.

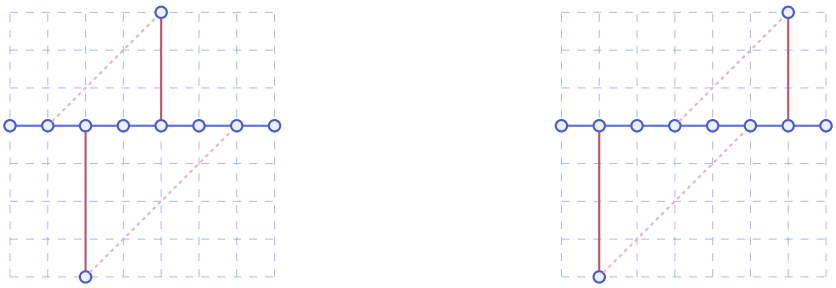


FIGURE 10. Two incompatible diagrams (on the left) and two compatible diagrams (on the right).

In other words, a Tamari diagram u of size n and a dual Tamari diagram v of size n are compatible if for any needle of position i and height $v_i \neq 0$ in v (resp. $u_i \neq 0$ in u), there is no needle of position j and height greater than or equal to $i - j$ in u (resp. $j - i$ in v) with $i - v_i \leqslant j \leqslant i - 1$ (resp. $i + 1 \leqslant j \leqslant i + u_i$) and $i \in [n]$.

For example, the two diagrams in Figure 8 are compatible. Figure 10 gives two other examples of two incompatible diagrams 00400000 and 00003000, and two compatible diagrams 04000000 and 00000030. Hereinafter, if u and v are compatible, we can also say that u and v satisfy the compatibility condition.

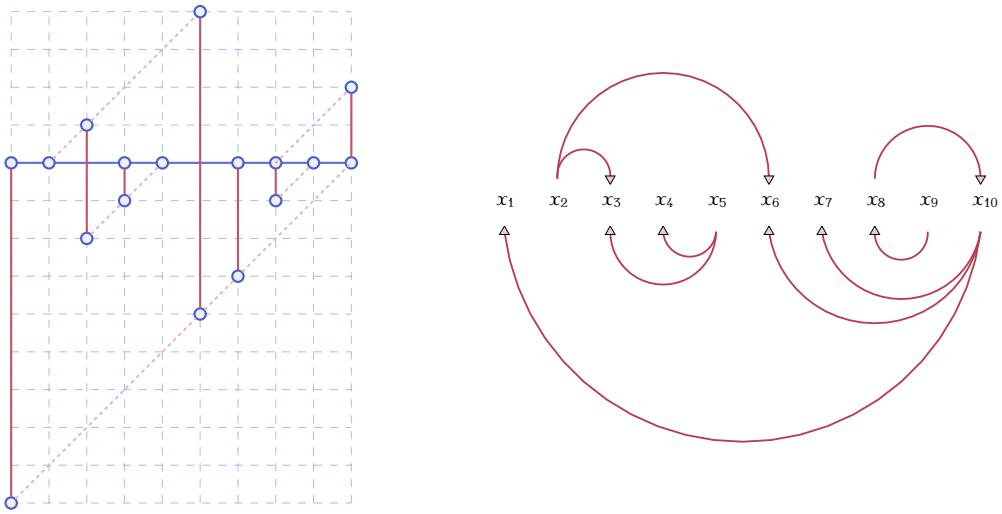


FIGURE 11. A Tamari interval diagram of size 10 (on the left) and its associated interval-poset (on the right).

As for Tamari diagrams and dual Tamari diagrams, a graphical representation of the Tamari interval diagram is also possible, as shown in Figure 10. Figure 11 gives the representation of the Tamari interval diagram (9021043100, 0010040002) formed by the two diagrams seen in Figure 8 which are compatible, where we have simply considered the symmetry relative to the abscissa axis of the Tamari diagram, and placed it under the dual Tamari diagram. Thus, the Tamari diagram u is drawn below and the dual Tamari diagram v is drawn above. With such a representation, it is then easy to verify that u and v are compatible. Indeed, any needle of u that is below the diagonal linking the top of the needle in position j in v to the abscissa point $j - v_j$, has a diagonal that intersects the x-axis strictly before the position j. Symmetrically, any needles of v that is above a diagonal linking the top of the needle in position i in u to the abscissa point $i + u_i$, has a diagonal that intersects the x-axis strictly after the position i.

One consequence of the compatibility condition is that each needle of non-zero height in the dual Tamari diagram v is always preceded by a needle of u of zero height. Symmetrically, each non-zero height needle in the Tamari diagram u is always followed by a

needle of v of zero height. In other words, for any $i \in [n]$, u_i and v_{i+1} can both be zero, but cannot both be non-zero.

2.2. **Link with interval-posets.** Let us show that there is a bijection between the set of Tamari interval diagrams and the set of interval-posets of the same size.

Let $n \geqslant 0$ and χ be the map sending a Tamari interval diagram (u, v) of size n to the relation

$$(\{\pi_1, \dots, \pi_n\}, \lhd) \tag{2.2.1}$$

where $\pi_{i+l} \lhd \pi_i$ for all $i \in [n]$ and $0 \leqslant l \leqslant u_i$, and $\pi_{i-k} \lhd \pi_i$ for all $i \in [n]$ and $0 \leqslant k \leqslant v_i$.

Proposition 2.2.1. *For any $n \geqslant 0$, the map χ has values in* $\mathrm{IP}(n)$.

Proof. Let $(u, v) \in \mathrm{TID}(n)$ and $\pi := \chi(u, v)$. First, we show that $\lhd$ is a partial order, then that interval-poset properties are satisfied.

(1) By the definition of χ one has $\pi_{i+l} \lhd \pi_i$ and $\pi_{i-k} \lhd \pi_i$ with $0 \leqslant l \leqslant u_i$ and $0 \leqslant k \leqslant v_i$ for all $\pi_i \in \pi$. Specifically, $\pi_i \lhd \pi_i$. This shows that π is reflexive.

(2) Let π_i, π_j and π_k be vertices of π with $i < j < k$.
   (a) Suppose that $\pi_j \lhd \pi_i$ and that $\pi_k \lhd \pi_j$. Then $\pi_j \lhd \pi_i$ implies that there is an integer $0 \leqslant i' \leqslant u_i$ such that $j = i + i'$. Therefore, by Condition (ii) of a Tamari diagram, $u_j = u_{i+i'} \leqslant u_i - i'$. Likewise, $\pi_k \lhd \pi_j$ implies that there is an integer $0 \leqslant j' \leqslant v_j$ such that $k = j + j'$. Still by the same condition, one has $u_k = u_{j+j'} \leqslant u_j - j'$. By using these two inequalities, we obtain that $u_i \geqslant u_k + i' + j'$. Since $i' + j' = k - i$, then we have $u_i \geqslant k - i$, which implies by the definition of χ that $\pi_k \lhd \pi_i$ in π.
   (b) Suppose that $\pi_j \lhd \pi_i$ and that $\pi_i \lhd \pi_k$. Therefore, $\pi_j \lhd \pi_k$ because $\pi_i \lhd \pi_k$ implies that each vertex between π_i and π_k is in relation with π_k.
   (c) Suppose that $\pi_i \lhd \pi_j$ and that $\pi_j \lhd \pi_k$. Then $\pi_i \lhd \pi_j$ implies that there is an integer $0 \leqslant i' \leqslant v_i$ such that $i = j - i'$. By Condition (ii) of a dual Tamari diagram, $v_i = v_{j-i'} \leqslant v_j - i'$. Likewise, $\pi_j \lhd \pi_k$ implies that there is an integer $0 \leqslant j' \leqslant v_j$ such that $j = k - j'$. By the same condition (ii), $v_j = v_{k-j'} \leqslant v_k - j'$. By these two inequalities, one has $v_k \geqslant v_i + i' + j'$. Since $i' + j' = k - i$, one has $v_k \geqslant k - i$, which implies by the definition of χ that $\pi_i \lhd \pi_k$ in π.
   (d) Suppose that $\pi_j \lhd \pi_k$ and that $\pi_k \lhd \pi_i$. Then $\pi_j \lhd \pi_i$ because $\pi_k \lhd \pi_i$ implies that all vertex between π_i and π_k is in relation with π_i.

   This shows that π is transitive. Notice that it is impossible to have the case $\pi_i \lhd \pi_k$ and $\pi_k \lhd \pi_j$ since π is the image of a Tamari interval diagram. Getting this case would contradict the fact that u and v are compatible. Similarly, the case $\pi_i \lhd \pi_j$ and $\pi_k \lhd \pi_i$ is impossible.

(3) Let $i < j$ and π_i, π_j be vertices of π. Suppose that $\pi_j \lhd \pi_i$ and that $\pi_i \lhd \pi_j$. By the definition of χ, $\pi_j \lhd \pi_i$ if and only if $u_i \geqslant j - i$. Likewise, $\pi_i \lhd \pi_j$ if and only if $v_j \geqslant j - i$. However, since u and v are compatible, this case is impossible. This shows that π is antisymmetric.

(4) The definition of χ implies directly that π satisfies the interval-poset properties, namely that for all π_i, π_j and π_k vertices of π with $i < j < k$, if $\pi_k \lhd \pi_i$, then $\pi_j \lhd \pi_i$, and if $\pi_i \lhd \pi_k$, then $\pi_j \lhd \pi_k$.

□

Let $n \geqslant 0$ and χ' be the map sending an interval-poset π of size n on a pair of words $(u, v) \in \mathbb{N}^n \times \mathbb{N}^n$, such that for all $i \in [n]$,

$$u_i := \#\{\pi_j \in \pi : \pi_j \lhd \pi_i \text{ and } i < j\}; \tag{2.2.2}$$

$$v_j := \#\{\pi_i \in \pi : \pi_i \lhd \pi_j \text{ and } i < j\}. \tag{2.2.3}$$

Lemma 2.2.2. *Let $n \geqslant 0$, $\pi \in \mathrm{IP}(n)$ and $(u, v) := \chi'(\pi)$. If $u_i \geqslant j - i$ (resp. $v_j \geqslant j - i$), then $\pi_j \lhd \pi_i$ (resp. $\pi_i \lhd \pi_j$), with $0 \leqslant i \leqslant j \leqslant n$.*

Proof. According to (2.2.2), the fact that $u_i \geqslant j - i$ means that there are at least $j - i$ vertices in decreasing relation to the vertex π_i. By the point (i) of interval-poset properties, this implies in particular that $\pi_j \lhd \pi_i$. Respectively, we show with the point (ii) of interval-poset properties that $v_j \geqslant j - i$ implies that $\pi_i \lhd \pi_j$. □

Theorem 2.2.3. *For any $n \geqslant 0$, the map $\chi : \mathrm{TID}(n) \to \mathrm{IP}(n)$ is bijective.*

Proof. Let us show that χ' is the inverse map of χ. Let $n \geqslant 0$, $\pi \in \mathrm{IP}(n)$ and $(u, v) := \chi'(\pi)$.

(1) Since π is an interval-poset, there are at most $n - i$ vertices of π in decreasing relation to π_i and at most $i - 1$ vertices of π in increasing relation to π_i for all $i \in [n]$. Therefore, the word u satisfies Condition (i) of a Tamari diagram and the word v satisfies Condition (i) of a dual Tamari diagram.

(2) Let π_i and π_{i+j} be vertices of π such that $i \in [n]$ and $j \in [0, u_i]$. By Lemma 2.2.2, the fact that $u_i \geqslant j$ means that $\pi_{i+j} \lhd \pi_i$. Thus, by transitivity of interval-posets, one has that for any $i + j \leqslant k \leqslant n$, if $\pi_k \lhd \pi_{i+j}$, then $\pi_k \lhd \pi_i$. Thus, $u_{i+j} + j \leqslant u_i$, which implies Condition (ii) of a Tamari diagram.
Symmetrically, Condition (ii) of a dual Tamari diagram is checked by considering π_i and π_{i-j} vertices of π such that $i \in [n]$ and $j \in [0, v_i]$.

(3) For all i, j such that $1 \leqslant i < j \leqslant n$ and $u_i \geqslant j - i$, suppose that $v_j \geqslant j - i$. By Lemma 2.2.2, the relation $u_i \geqslant j - i$ implies that $\pi_j \lhd \pi_i$. Likewise, the relation $v_j \geqslant j - i$ means that $\pi_i \lhd \pi_j$. Both of these implications lead to a contradiction with the antisymmetric nature of interval-posets. Necessarily, we have $v_j < j - i$, which implies that u and v are compatible.

The pair (u, v) is a Tamari interval diagram of size n. Finally, it is clear that $\chi(u, v) = \pi$ by construction. Therefore, the map χ is surjective.

Let (u, v) and (u', v') be two Tamari interval diagrams of size n, such that $(u, v) \neq (u', v')$ and such that $\chi(u, v) := \pi$ and $\chi(u', v') := \pi'$. So there is at least one letter of (u, v) and (u', v') such that $u_i \neq u'_i$ or $v_i \neq v'_i$, for $i \in [n]$. Therefore, the number of vertices of π in relation to the vertex π_i associated with the component u_i and v_i by χ is different from the number of vertices of π' in relation to the vertex π'_i associated with the component u'_i and v'_i by χ, we thus have $\pi \neq \pi'$. This shows that the map χ is injective. □

The minimalist representation of the interval-posets defined in Section 1 allows us to describe a direct construction of the corresponding Tamari interval diagram. Indeed, let us consider the minimalist representation of an interval-poset π of size n. For any relation $\pi_j \lhd \pi_i$ (resp. $\pi_i \lhd \pi_j$) drawn, with $1 \leqslant i < j \leqslant n$, we set $u_i := j - i$ (resp. $v_j := j - i$) and all other elements not involved in any relation to 0. This forms a pair of words (u, v) which is the inverse image of π by χ.

An example is given by Figure 11, where a Tamari interval diagram and its interval-poset which is its image by χ are shown.

2.3. **Cubic coordinates.** We describe in this part the set of cubic coordinates, and we show that there is a bijection between this set and the set of Tamari interval diagrams.

An $(n-1)$-tuple c on $\mathbb{Z}$ is a *cubic coordinate* if there is a Tamari interval diagram (u, v) of size n such that

$$c = (u_1 - v_2, u_2 - v_3, \dots, u_{n-1} - v_n). \tag{2.3.1}$$

The size of a cubic coordinate is its number of components plus one. The set of cubic coordinates of size n is denoted by $\mathrm{CC}(n)$. For instance, $(9, -1, 2, 1, -4, 4, 3, 1, -2)$ is a cubic coordinate of size 10 since there is the Tamari interval diagram $(9021043100, 0010040002)$ satisfying the conditions of the definition.

Besides, for any $n \geqslant 1$, let ϕ be the map sending an $(n-1)$-tuple c on $\mathbb{Z}$ to a pair (u, v) of words on $\mathbb{N}$, both of length n, such that u satisfies $u_n = 0$ and for any $i \in [n-1]$,

$$u_i = \max(c_i,\ 0), \tag{2.3.2}$$

and v satisfies $v_1 = 0$ and for any $2 \leqslant i \leqslant n$,

$$v_i = |\min(c_{i-1},\ 0)|. \tag{2.3.3}$$

Theorem 2.3.1. *For any $n \geqslant 0$, the map $\phi : \mathrm{CC}(n) \to \mathrm{TID}(n)$ is bijective.*

Proof. Let c and c' be two cubic coordinates of size n such that $c \neq c'$. Then there is a component c_i such that $c_i \neq c'_i$, with $i \in [n-1]$. By the map ϕ, one has then $u_i \neq u'_i$ or $v_{i+1} \neq v'_{i+1}$, namely $(u, v) \neq (u', v')$. Which shows that the map ϕ is injective.

Let $(u, v) \in \mathrm{TID}(n)$. Let $c := (u_1 - v_2, u_2 - v_3, \dots, u_{n-1} - v_n)$, the $(n-1)$-tuple whose components are given by the difference between u_i and v_{i+1} for any $i \in [n-1]$. Now if $u_i \neq 0$, then $v_{i+1} = 0$ for any $i \in [n-1]$. Therefore, $\phi(c) = (u, v)$, where (u, v) is indeed a Tamari interval diagram by hypothesis. By the definition of a cubic coordinate, one can conclude that $c \in \mathrm{CC}(n)$. This shows that the map ϕ is surjective. □

Therefore, by the map ϕ it is possible to build a cubic coordinate from a Tamari interval diagram and reciprocally. Graphically, we have to shift the upper part of a Tamari interval diagram (corresponding to the dual Tamari diagram) to the left by one position and collect the height of the needles from left to right. Then, we put a positive sign for the needles of the lower part of the Tamari interval diagram (corresponding to the Tamari diagram) and a negative sign for the upper part, and we forget the last needle of zero height. To reconstruct a Tamari interval diagram from a cubic coordinate, we reconstruct the needles of the Tamari diagram and the dual Tamari diagram from the components of the cubic

coordinate in the same way, and then we shift the dual Tamari diagram to the right by one position.

Using the map χ we can then directly give the cubic coordinate of an interval-poset π. In the same way that we shift the dual Tamari diagram one position to the left, we shift all the increasing relations of the interval-poset to the left by one vertex. Then, for each vertex π_i, we count the number of elements in increasing or decreasing relation of target π_i, out of reflexive relation, for all $i \in [n-1]$. These numbers become the components of positive sign if it is a decreasing relation, negative otherwise, of the cubic coordinate. As the increasing relations have been shifted, the number associated with the vertex π_n is always zero. Therefore, this vertex is forgotten for the cubic coordinate. In the same way, to construct an interval-poset from a cubic coordinate with each component of a cubic coordinate, we rebuild the increasing and decreasing relations on $n-1$ vertices, we add the vertex π_n, then we shift the increasing relations to the right.

Lemma 2.3.2. *Let $n \geqslant 0$ and $c \in \mathrm{CC}(n)$ such that there is a component $c_i \neq 0$, for $i \in [n-1]$. Let c' be the $(n-1)$-tuple such that $c'_i = 0$ and $c'_j = c_j$ for any $j \neq i$, with $j \in [n-1]$. Then c' is a cubic coordinate.*

Proof. Let $(u', v') := \phi(c')$ and (u'_j, v'_{j+1}) be the pair of letters corresponding to c'_j by the map ϕ, with $j \in [n-1]$. Since $c'_i = 0$, then $(u'_i, v'_{i+1}) = (0, 0)$. By hypothesis, all other pairs of letters are the same as those of $(u, v) := \phi(c)$. In order to show that c' is a cubic coordinate, we have to show that (u', v') is a Tamari interval diagram, namely that (u', v') satisfies the conditions of a Tamari diagram, of a dual Tamari diagram, and of compatibility. Clearly, with $(u'_i, v'_{i+1}) = (0, 0)$, all these conditions are satisfied for (u', v'). □

Depending on the case, either the definition of cubic coordinates or the definition of Tamari interval diagrams is used, as it is done for the proof of Lemma 2.3.2. For example, the following results are stated for Tamari interval diagrams.

Let $n \geqslant 0$. A Tamari interval diagram (u, v) of size n is *synchronized* if either $u_i \neq 0$ or $v_{i+1} \neq 0$ for any $i \in [n-1]$.

Likewise, a cubic coordinate c of size n is synchronized if $c_i \neq 0$ for any $i \in [n-1]$. The set of synchronized cubic coordinates of size n is denoted by $\mathrm{SCC}(n)$.

A Tamari interval $(\mathfrak{s}, \mathfrak{t})$ is synchronized if and only if the binary trees $\mathfrak{s}$ and $\mathfrak{t}$ have the same canopy [FPR17, PRV17]. The definition of the canopy is recalled in Section 1.

Proposition 2.3.3. *Let $n \geqslant 0$ and $(u, v) \in \mathrm{TID}(n)$. The Tamari interval diagram (u, v) is synchronized if and only if $\rho(\chi(u, v))$ is a synchronized Tamari interval.*

Proof. If (u, v) is not synchronized, then there is an index $i \in [n-1]$ such that $u_i = 0$ and $v_{i+1} = 0$. Let $\pi := \chi(u, v)$ be the interval-poset associated to (u, v), and $(\mathfrak{s}, \mathfrak{t}) := \rho(\chi(u, v))$. The two binary trees $\mathfrak{s}$ and $\mathfrak{t}$ are not synchronized if there is at least one letter of some index j in the canopy of the tree $\mathfrak{s}$ that is different from the letter of the same index j in the canopy of $\mathfrak{t}$. Let us show that (u, v) is not synchronized if and only if the binary trees $\mathfrak{s}$ and $\mathfrak{t}$ are not synchronized.

The letter u_i is equal to 0 if and only if there is no descending relation of target π_i in π, namely, if and only if the node i has no right child in the tree $\mathfrak{s}$ (see Section 1.3). To summarize, $u_i = 0$ if and only if the right subtree of the node i is a leaf oriented to the right. Now, as recall in Section 1.2, a leaf linked to the node i is oriented to the right if and only if the i-th letter in the canopy corresponding to $\mathfrak{s}$ is 1.

Symmetrically, $v_{i+1} = 0$ if and only if there is no increasing relation of target π_{i+1} in π, namely, if and only if the node $i+1$ has no left child in the tree $\mathfrak{t}$. Then, $v_{i+1} = 0$ if and only if the left subtree of the node $i+1$ is a leaf oriented to the left. As seen in Section 1, a leaf linked to the node $i+1$ is oriented to the left if and only if the i-th letter in the canopy corresponding to $\mathfrak{t}$ is 0.

To conclude, $u_i = 0$ and $v_{i+1} = 0$ if and only if the letter of index i in the canopy of the tree $\mathfrak{s}$ is different from the letter of index i in the canopy of the tree $\mathfrak{t}$. Therefore, (u, v) is not synchronized if and only if the binary trees $\mathfrak{s}$ and $\mathfrak{t}$ are not synchronized. □

An interval-poset π of size $n \geqslant 3$ is *new* if

(1) there is no decreasing relation of source π_n,
(2) there is no increasing relation of source π_1,
(3) there is no relation $\pi_{i+1} \lhd \pi_{j+1}$ and $\pi_j \lhd \pi_i$ with $i < j$.

The definition of a new interval-poset is given in [Rog20].

For any $n \geqslant 3$, a Tamari interval diagram (u, v) of size n is *new* if the following conditions are satisfied

(i) $0 \leqslant u_i \leqslant n - i - 1$ for all $i \in [n-1]$,
(ii) $0 \leqslant v_j \leqslant j - 2$ for all $j \in [2, n]$,
(iii) $u_k < l - k - 1$ or $v_l < l - k - 1$ for all $k, l \in [n]$ such that $k + 1 < l$.

Proposition 2.3.4. *Let $n \geqslant 3$ and $(u, v) \in \mathrm{TID}(n)$. The Tamari interval diagram (u, v) is new if and only if $\chi(u, v)$ is a new interval-poset.*

Proof. Let us show that $\pi := \chi(u, v)$ is not new if and only if (u, v) is not new. Theorem 2.2.3 leads to three cases.

- Let us consider the negation of (i) of a new Tamari interval diagram by assuming that $u_i = n - i$. By Lemma 2.2.2, this implies that $\pi_n \lhd \pi_i$ with $i \in [n-1]$. Reciprocally, if $\pi_n \lhd \pi_i$ with $i \in [n-1]$, then by the point (i) of interval-poset properties, all vertices between π_i and π_n are in decreasing relation to π_i. Since $u_i := \#\{\pi_j \in \pi : \pi_j \lhd \pi_i \text{ and } i < j\}$, it implies that $u_i = n - i$.
- Likewise, by Lemma 2.2.2, if $v_j = j - 1$, then $\pi_1 \lhd \pi_j$ with $j \in [2, n]$. By the point (ii) of interval-poset properties, we get the converse property.
- According to Lemma 2.2.2, if $u_i \geqslant j - i$, then $\pi_j \lhd \pi_i$, and if $v_{j+1} \geqslant j - i$, then $\pi_{i+1} \lhd \pi_{j+1}$ with $i < j$. We obtain the two converse properties with respectively the point (i) and the point (ii) of interval-poset properties. Specifically, by setting $l := j + 1$ and $k := i$, we find the formulation of the negation of (iii) of a new Tamari interval diagram, with $k + 1 < l$.

□

In [Rog20] it is shown that a Tamari interval is new if and only if the associated interval-poset is new. With Proposition 2.3.4 we get the following result.

Proposition 2.3.5. *Let $n \geqslant 3$ and $(u, v) \in \mathrm{TID}(n)$. The Tamari interval diagram (u, v) is new if and only if $\rho(\chi(u, v))$ is a new Tamari interval.*

Proposition 2.3.6. *Let $n \geqslant 3$ and $(u, v) \in \mathrm{TID}(n)$. If (u, v) is synchronized, then (u, v) is not new.*

Proof. Assume by contradiction that (u, v) is synchronized and new. Since (u, v) is new, one has $u_i < n - i$ for $i \in [n-1]$, and $v_j < j - 1$ for $j \in [2, n]$. In particular, $u_{n-1} = 0$ and $v_2 = 0$. This implies, since (u, v) is synchronized, that $u_1 \neq 0$ and $v_n \neq 0$. Furthermore, since (u, v) is new, Condition (iii) of a Tamari interval diagram is satisfied. Specifically, for any $k \in [n-2]$, either $u_k < 1$ or $v_{k+2} < 1$. Let us denote by $(*)$ this condition. Assuming that $u_1 \neq 0$, since (u, v) is synchronized, one has either $u_2 \neq 0$ or $v_3 \neq 0$. By $(*)$, the second choice is impossible, thus $u_2 \neq 0$. By the same reasoning, for every $k \in [n-2]$, $u_k \neq 0$. However, also by assumption one has $v_n \neq 0$. Therefore, $u_{n-2} \neq 0$ and $v_n \neq 0$ which is a contradiction with $(*)$. □

2.4. **Order structure.** Firstly, we endow the set of cubic coordinates with an order relation. Then we show that there is an isomorphism between this poset and the poset of Tamari intervals. The two bijections constructed in the first two parts of Section 2 allow us to establish this poset isomorphism.

Let $n \geqslant 0$ and $c, c' \in \mathrm{CC}(n)$. We set that $c \preccurlyeq c'$ if and only if $c_i \leqslant c'_i$ for all $i \in [n-1]$. Endowed with $\preccurlyeq$, the set $\mathrm{CC}(n)$ is a poset called the *cubic coordinate poset*.

Recall that the map ϕ is defined at the beginning of Section 2.3 and the map χ is defined at the beginning of Section 2.2. Let $(\mathfrak{s}, \mathfrak{t}), (\mathfrak{s}', \mathfrak{t}') \in \mathrm{int}(\mathrm{T}_2(n))$ and let $\psi := \phi^{-1} \circ \chi^{-1} \circ \rho^{-1}$ be the map from the Tamari interval poset to the cubic coordinate poset $\mathrm{CC}(n)$.

For the next results in all this section, let us denote by $c := \psi(\mathfrak{s}, \mathfrak{t})$, $c' := \psi(\mathfrak{s}', \mathfrak{t}')$ and $(u, v) := \phi(c)$, $(u', v') := \phi(c')$, and $\pi := \chi(u, v)$, $\pi' := \chi(u', v')$.

Lemma 2.4.1. *If $(\mathfrak{s}', \mathfrak{t}')$ covers $(\mathfrak{s}, \mathfrak{t})$, then there is a unique different component c_i between c and c' such that $c_i < c'_i$ and there is no cubic coordinate c'' different from c and c' such that $c \preccurlyeq c'' \preccurlyeq c'$.*

Proof. By Lemma 1.3.1 we know that $(\mathfrak{s}', \mathfrak{t}')$ covers $(\mathfrak{s}, \mathfrak{t})$ if and only if π and π' satisfy either $(\star)$ or $(\diamond)$. Let us assume that π and π' satisfy either $(\star)$ or $(\diamond)$ for the vertex π_i. By using (2.2.2) and (2.2.3), two cases are possible.

- Suppose that π and π' satisfy $(\star)$, then since only decreasing relations are added in π' relative to π, only u' is modified in (u', v') relative to (u, v). Furthermore, since π' is obtained by adding decreasing relations of target π_i in π, only the letter u'_i in u' is increased relative to u. Moreover, since the number of descending relations added in π is minimal, there cannot be any Tamari interval diagram between (u, v) and (u', v'), and thus no cubic coordinate between c and c'. In the end, the image by ϕ^{-1} of (u', v') is the cubic coordinate c' with $c'_i = u'_i$ and $c'_j = c_j$ for any $j \neq i$.

- ⋆ Suppose that π and π' satisfy $(\diamond)$, the arguments are roughly the same, with the difference that this time, only increasing relations are removed in π' relative to π. We obtain that only the component $c'_{i-1} = -v'_i$ of c' has increased relative to c.

In both cases, the implication is true. □

Note that if there is a unique different component c_i between c and c' such that $c_i < c'_i$ and there is no cubic coordinate c'' different from c and c' such that $c \preccurlyeq c'' \preccurlyeq c'$, then in particular c' covers c. Thus, Lemma 2.4.1 has the consequence that if $(\mathfrak{s}', \mathfrak{t}')$ covers $(\mathfrak{s}, \mathfrak{t})$, then c' covers c.

Lemma 2.4.2. *Let $n \geqslant 0$ and $c, c' \in \mathrm{CC}(n)$. If $c \preccurlyeq c'$, then there is a cubic coordinate c'' such that $u'' = u$ and $v'' = v'$, where $(u'', v'') := \phi(c'')$.*

Proof. The composition of bijections $\phi^{-1} \circ \chi^{-1}$ associates a pair of words (u, v) to a pair of comparable binary trees $(\mathfrak{s}, \mathfrak{t})$ such that u encodes the binary tree $\mathfrak{s}$ and v encodes the binary tree $\mathfrak{t}$. By this composition, u (resp. v) is obtained by counting in $\mathfrak{s}$ (resp. $\mathfrak{t}$) the number of left (resp. right) descendants of each node for the infix order. Additionally, we know that if $(\mathfrak{s}, \mathfrak{t}) \preccurlyeq_{\mathrm{int(ta)}} (\mathfrak{s}', \mathfrak{t}')$, then the interval $(\mathfrak{s}, \mathfrak{t}')$ is a Tamari interval because we always have $\mathfrak{s} \preccurlyeq_{\mathrm{ta}} \mathfrak{s}' \preccurlyeq_{\mathrm{ta}} \mathfrak{t}'$. The construction of $\phi^{-1} \circ \chi^{-1}$ and the fact that $(\mathfrak{s}, \mathfrak{t}')$ is a Tamari interval imply that the pair (u, v') is always a Tamari interval diagram. Therefore, c'' is a cubic coordinate. □

For any $c, c' \in \mathrm{CC}(n)$, let

$$\mathrm{D}^-\left(c, c'\right) := \{d \ : \ c_d \neq c'_d \text{ and } c'_d \leqslant 0\}, \tag{2.4.1}$$

and

$$\mathrm{D}^+\left(c, c'\right) := \{d \ : \ c_d \neq c'_d \text{ and } c_d \geqslant 0\}. \tag{2.4.2}$$

Now consider the case where c and c' share either their Tamari diagrams or their associated dual Tamari diagrams, then we have the two following lemmas.

Lemma 2.4.3. *Let $n \geqslant 0$ and $c, c' \in \mathrm{CC}(n)$. If $c \preccurlyeq c'$ such that $u = u'$ and $\#\mathrm{D}^-(c, c') = r$, then there is a chain*

$$\left(c = c^{(0)}, c^{(1)}, \ldots, c^{(r-1)}, c^{(r)} = c'\right), \tag{2.4.3}$$

such that $\#\mathrm{D}^-\left(c^{(i-1)}, c^{(i)}\right) = 1$ for all $i \in [r]$.

Proof. Let

$$\mathrm{D}^-\left(c, c'\right) = \{d_1, d_2, \ldots, d_r\} \tag{2.4.4}$$

with $d_{k-1} < d_k$ for all $k \in [2, r]$. For any $k \in [r]$, let $c^{(k)}$ be a tuple obtained by replacing in c all the components c_{d_i} by the components c'_{d_i} for $i \in [k]$. The tuple $c^{(k)}$ is a cubic coordinate. Indeed, by denoting $\phi(c^{(k)})$ by $(u^{(k)}, v^{(k)})$, one has that $u^{(k)} = u = u'$, so the compatibility with $v^{(k)}$ is always satisfied. Therefore, the only thing to check is that $v^{(k)}$ is a dual Tamari diagram. Condition (i) is naturally satisfied. Since $c \leqslant c'$, one has $v_i \geqslant v'_i$ for all $i \in [n]$. Therefore, Condition (ii) is satisfied because for $i \in [d_k]$ and $j \in [i+1, n]$,

$v_i^{(k)} = v'_i$ and $v_j^{(k)} = v_j$, and so $v_j^{(k)} - v_i^{(k)} = v_j - v'_i \geqslant v_j - v_i \geqslant j - i$. The word $v^{(k)}$ is then a dual Tamari diagram. Consider the chain

$$\left(c = c^{(0)}, c^{(1)}, \dots, c^{(r-1)}, c^{(r)} = c'\right). \tag{2.4.5}$$

For all $i \in [r]$, since we change only one component between $c^{(i-1)}$ and $c^{(i)}$, one has $\#\mathrm{D}^-\left(c^{(i-1)}, c^{(i)}\right) = 1$. □

Lemma 2.4.4. *Let $n \geqslant 0$ and $c, c' \in \mathrm{CC}(n)$. If $c \preccurlyeq c'$ such that $v = v'$ and $\mathrm{D}^+(c, c') = s$, then there is a chain*

$$\left(c = c^{(0)}, c^{(1)}, \dots, c^{(s-1)}, c^{(s)} = c'\right), \tag{2.4.6}$$

such that $\#\mathrm{D}^+\left(c^{(i-1)}, c^{(i)}\right) = 1$ for all $i \in [s]$.

Proof. The proof is similar to the demonstration of Lemma 2.4.3. Let

$$\mathrm{D}^+\left(c, c'\right) = \{d_1, d_2, \dots, d_s\} \tag{2.4.7}$$

with $d_{k-1} < d_k$ for all $k \in [2, s]$. For any $k \in [s]$, let $c^{(k)}$ be a tuple obtained by replacing in c all the components c_{d_i} by the components c'_{d_i} for $i \in [k]$. As we did in the proof of Lemma 2.4.3, we can check that, for any $k \in [s]$, the tuple $c^{(k)}$ is a cubic coordinate. Then, by consider the chain

$$\left(c = c^{(0)}, c^{(1)}, \dots, c^{(s-1)}, c^{(s)} = c'\right), \tag{2.4.8}$$

one has that $\#\mathrm{D}^+\left(c^{(i-1)}, c^{(i)}\right) = 1$ for all $i \in [s]$.

□

Theorem 2.4.5. *For any $n \geqslant 0$, the map ψ is a poset isomorphism.*

Proof. The map ψ is an isomorphism of posets if ψ and its inverse preserves the partial order. As these relations are transitive, Lemma 2.4.1 gives the direct implication. Suppose that $c \preccurlyeq c'$. According to Lemma 2.4.2, Lemma 2.4.3 and Lemma 2.4.4 there is always a chain between c and c' such that the components are independently increasing one by one. So we can see what happens when we change only one component c_i by c'_i at any step between c and c'.

Obviously, if $c_i = c'_i$, then $u_i = u'_i$ and $v_{i+1} = v'_{i+1}$ and no changes are made between the corresponding binary tree pairs. Suppose that $c_i < c'_i$, then three cases are possible.

- ⋆ Suppose that c'_i is positive and c_i is positive or null. The image by ϕ of c and c' differ for the letter u_i, namely $c'_i = u'_i$ and $c_i = u_i$, and $v_{i+1} = v'_{i+1} = 0$. The difference of a letter u_i between (u, v) and (u', v') is directly translated by the map χ: the interval-poset π' has more decreasing relations of target π_i than the vertex π_i in π. By the map ρ, it means that to go from the tree $\mathfrak{s}$ to the tree $\mathfrak{s}'$ at least one right rotation of the edge (i, j) is made, where j is the father of the node i in $\mathfrak{s}$.
- ⋆ Symmetrically, assume that c'_i is negative or null, then $c'_i = -v'_{i+1}$, $c_i = -v_{i+1}$ and $u_i = u'_i = 0$. By the map χ, the interval-poset π' has less decreasing relations of target π_{i+1} than the vertex π_{i+1} in π. This implies by ρ that to pass from the tree $\mathfrak{t}$ to the tree $\mathfrak{t}'$ at least one right rotation of the edge $(k, i+1)$ is made, where k is the right child of the node $i+1$ in $\mathfrak{t}$.

⋆ Finally, with Lemma 2.4.2, the case where c_i is negative and c'_i is positive falls into the conjunction of the two previous cases.

Therefore, $c \preccurlyeq c'$ implies that $(\mathfrak{s}, \mathfrak{t}) \preccurlyeq_{\text{int(ta)}} (\mathfrak{s}', \mathfrak{t}')$. Hence, the map ψ is an isomorphism of posets. □

Let us denote by $\lessdot$ the covering relation of the poset $\mathrm{CC}(n)$.

Proposition 2.4.6. *Let $n \geqslant 0$ and $c, c' \in \mathrm{CC}(n)$ such that $c \lessdot c'$. Then, there is a unique different component between c and c'.*

Proof. It is a consequence of Theorem 2.4.5 and Lemma 2.4.1. □

The following diagram provides a summary of the applications used in Section 2. Recall that $\psi = \phi^{-1} \circ \chi^{-1} \circ \rho^{-1}$, therefore this diagram of poset isomorphisms is commutative.

$$\begin{array}{ccc} \mathrm{TID}(n) & \xrightarrow{\chi} & \mathrm{IP}(n) \\ \uparrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \rho} \\ \mathrm{CC}(n) & \xleftarrow{\psi} & \mathrm{int}(\mathrm{T}_2(n)) \end{array} \tag{2.4.9}$$

A consequence of the poset isomorphism ψ is that the order dimension [MP90, Tro02] of the poset of Tamari intervals is at most $n - 1$.

3. Geometric properties

In this section, we give a very natural geometrical realization for the lattices of cubic coordinates. After defining the cells of this realization, we give some properties related to them. Finally, we show that the lattice of the cubic coordinates is EL-shellable.

3.1. Cubic realizations.

Theorem 2.4.5 provides a simpler translation of the order relation between two Tamari intervals. We provide the geometrical realization induced by this order relation, which is natural for cubic coordinates. In a combinatorial way we study the cells formed by this realization.

For any $n \geqslant 0$, the *cubic realization* of $\mathrm{CC}(n)$ is the geometric object $\mathfrak{C}(\mathrm{CC}(n))$ defined in the space $\mathbb{R}^{n-1}$ and obtained by placing for each $c \in \mathrm{CC}(n)$ a vertex of coordinates $(c_1, \ldots, c_{n-1})$, and by forming for each $c, c' \in \mathrm{CC}(n)$ such that $c \lessdot c'$ an edge between c and c'. Every edge of $\mathfrak{C}(\mathrm{CC}(n))$ is parallel to some vector in the canonical basis of $\mathbb{R}^{n-1}$.

Figure 12 shows the cubic realization of $\mathrm{CC}(3)$, where the elements are the vertices and the edges are the covering relations. Figure 13 shows the cubic realization of $\mathrm{CC}(4)$. In these drawings the negative sign components are denoted with a bar.

In algebraic topology, to define the tensor products of A_∞-algebras, one can use a cell complex called the *diagonal of the associahedron*. This complex has notably been studied by Loday [Lod11], by Saneblidze and Umble [SU04], and by Markl and Shnider [MS06]. More recently, there is a description of this object in [MTTV21]. The realization of this complex seems to be identical to the cubic realization, up to continuous deformation.

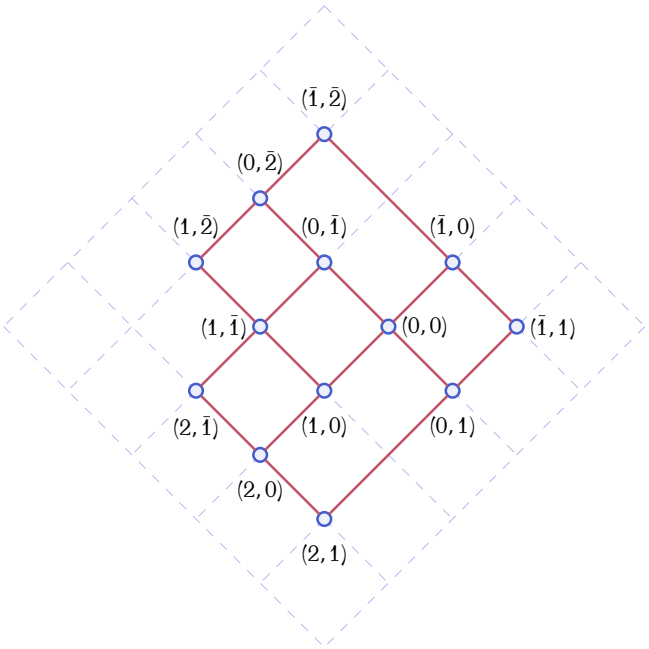


FIGURE 12. $\mathfrak{C}(\mathrm{CC}(3))$.

3.2. **Covering map.** Let $n \geqslant 0$. We define the set of

- ⋆ *input-wings* as the set $\mathcal{I}(\mathrm{CC}(n))$ containing any $c \in \mathrm{CC}(n)$ which covers exactly $n-1$ elements,
- ⋆ *output-wings* as the set $\mathcal{O}(\mathrm{CC}(n))$ containing any $c \in \mathrm{CC}(n)$ which is covered by exactly $n-1$ elements.

Let $n \geqslant 0$ and $c \in \mathrm{CC}(n)$. For $i \in [n-1]$, the *covering map* $\uparrow_i$ sends c to its covering differing only at index i, when such covering exists. We denote by $\uparrow c_i$ the letter which differs in $\uparrow_i(c)$.

In particular, for $n \geqslant 0$, a cubic coordinate c of size n is an output-wing if for any $i \in [n-1]$, $\uparrow_i(c)$ is well-defined.

Let $n \geqslant 0$ and $c \in \mathrm{CC}(n)$, and $(u, v) := \phi(c)$. If $\uparrow c_i$ is positive, then the letter u_i increases and becomes equal to $\uparrow c_i$ and v_{i+1} is equal to 0. Then, we define $\uparrow u_i :=\uparrow c_i$. If $\uparrow c_i$ is negative or null, then v_{i+1} decreases and becomes equal to $|\uparrow c_i|$ and u_i is equal to 0. Then, we set $\downarrow v_{i+1} := -\uparrow c_i$.

Lemma 3.2.1. *Let $n \geqslant 0$ and $c \in \mathrm{CC}(n)$, and $i \in [n-1]$ such that $\uparrow_i(c)$ is well-defined. Then,*

(i) if $c_i < 0$, then $\uparrow c_i \leqslant 0$,
(ii) if $c_i \geqslant 0$, then $\uparrow c_i > 0$.

Proof. Let us show the first implication, the second being obvious because the covering map always strictly increases a component. Let $c_i < 0$, and let c' be the $(n-1)$-tuple such that $c'_i = 0$ and $c'_j = c_j$ for any $j \neq i$, with $j \in [n-1]$. By Lemma 2.3.2, c' is a cubic

coordinate. As $c \leqslant c'$ and they differ only at the i-th component, by the definition of $\uparrow_i(c)$, we have $c \leqslant \uparrow_i(c) \leqslant c'$, thus $\uparrow c_i \leqslant c'_i = 0$. □

Let $c \in \mathrm{CC}(n)$. For all $i \in [n]$, let

$$\Uparrow_i(c) := \uparrow_i(\uparrow_{i+1} \dots (\uparrow_{n-1}(\uparrow_n(c)))), \tag{3.2.1}$$

with the convention that $\uparrow_n(c) := c$. For instance, for $c \in \mathrm{CC}(5)$, $\Uparrow_2(c) = \uparrow_2(\uparrow_3(\uparrow_4(\uparrow_5(c))))$.

Lemma 3.2.2. *Let $n \geqslant 0$ and $c \in \mathcal{O}(\mathrm{CC}(n))$. For all $i \in [n]$, $\Uparrow_i(c)$ is a cubic coordinate.*

Proof. For $i = n$, one has by convention that $\Uparrow_n(c)$ is a cubic coordinate. Let us suppose that for $i \in [n-1]$, $\Uparrow_{i+1}(c)$ is a cubic coordinate, and let us show that $\Uparrow_i(c)$ is also a cubic coordinate. Depending on the sign of $\Uparrow_{i+1}(c)_i$, two cases are possible.

Suppose that $\Uparrow_{i+1}(c)_i < 0$. In this case, consider c' the $(n-1)$-tuple obtained from $\Uparrow_{i+1}(c)$ by replacing the component $\Uparrow_{i+1}(c)_i$ by 0. By Lemma 2.3.2, c' is a cubic coordinate. Since $\Uparrow_{i+1}(c)_i < 0$ one has $\Uparrow_{i+1}(c) \preccurlyeq c'$. If c' covers $\Uparrow_{i+1}(c)$, then $c' = \Uparrow_i(c)$. Otherwise, it is always possible to find another cubic coordinate c'' between $\Uparrow_{i+1}(c)$ and c' such that $c'' = \Uparrow_i(c)$. In both cases, $\Uparrow_i(c)$ is a cubic coordinate.

Suppose that $\Uparrow_{i+1}(c)_i \geqslant 0$. Let us set $(u, v) := \phi(c)$, and $(x, y) := \phi(\Uparrow_{i+1}(c))$. Since u_i is not changed yet in x, one has $x_i = u_i$. Due to Condition (ii) of a Tamari diagram and the compatibility condition, there are two configurations, involving indices, which can make contradiction with the fact that (x, y) is still a Tamari interval diagram when x_i becomes $\uparrow x_i$.

(1) If there is an index j such that $1 \leqslant i < j \leqslant n$ and $y_j \geqslant j - i$ in y, then, since $y_j < v_j$, one has $v_j \geqslant j - i$ in v. By the compatibility condition, that implies $u_i < j - i$ in u. Moreover, since c is assumed to be an output-wing, $u_i < j - i - 1$ in u, so that u_i can be increased. This inequality remains true in x.

(2) If there is an index h such that $1 \leqslant i - h \leqslant u_h$, by Condition (ii) of a Tamari diagram, $u_i \leqslant u_h - i + h$ in u. This remains true in x because components with index smaller than i remain unchanged between c and $\Uparrow_{i+1}(c)$. Furthermore, since c is an output-wing, then $u_i < u_h - i + h$. This inequality remains true for $\Uparrow_{i+1}(c)$.

With these two configurations, let us build a cubic coordinate c' different from $\Uparrow_{i+1}(c)$ only for $\Uparrow_{i+1}(c)_i$, depending on which choices are available to increase u_i. Let us set $(u', v') := \phi(c')$.

(a) Suppose there is a j satisfying (1), and there is no h satisfying (2) in $\Uparrow_{i+1}(c)$. In this case, by choosing the minimal index j such that (1) holds, we set $u'_i := j - i - 1$ in c'. Thus, u'_i is also minimized, and since $u'_i < j - i$, the compatibility condition is satisfied in c'. Furthermore, since $\Uparrow_{i+1}(c)$ is assumed to be a cubic coordinate, all conditions in a Tamari diagram and a dual Tamari diagram are satisfied for c'. Therefore, our candidate c' is a cubic coordinate. Note that in the construction of c', other possible not minimal j satisfying (1) will not cause any problem.

(b) Suppose there is an h satisfying (2), and there is no j satisfying (1) in $\Uparrow_{i+1}(c)$. Then, by choosing the minimal index h such that (2) holds, we set $u'_i := u'_h -$

$i + h$. Therefore, Condition (ii) of a Tamari diagram is satisfied for u'. Also, by Condition (i) of a Tamari diagram, $u'_h \leqslant n - h$ which implies $u'_i \leqslant n - i$. Finally, the compatibility condition is also satisfied because it was assumed that there was no j satisfying (1). The tuple c' is thus a cubic coordinate. As for the previous case, other possible not minimal h satisfying (2) will not cause any problem.

(c) Suppose there is a j and an h satisfying (1) and (2) in $\Uparrow_{i+1}(c)$. In this case, we set $u'_i := \min\{u'_h - i + h,\ j - i - 1\}$. By the two previous cases, the tuple c' is a cubic coordinate.

(d) Otherwise, we set $u'_i := n - i$. The tuple c' is a cubic coordinate.

In any case, for u'_i fixed in c', either c' covers $\Uparrow_{i+1}(c)$, and so $c' = \Uparrow_i(c)$, or there is a cubic coordinate c'' between $\Uparrow_{i+1}(c)$ and c' such that $c'' = \Uparrow_i(c)$. In both cases, $\Uparrow_i(c)$ is a cubic coordinate, and differs by only one component from c'. □

Let $n \geqslant 0$ and $c \in \mathcal{O}(\mathrm{CC}(n))$. The cubic coordinate $\Uparrow_1(c)$ is the *corresponding input-wing* of c (the name comes from a corollary of Theorem 3.3.1). For instance $c = (0, -1, 1, -1, -5, 0, 1, -1, -3)$ is an output-wing, and its corresponding input-wing is $\Uparrow_1(c) = (1, 0, 2, 0, -4, 3, 2, 0, -2)$. By Lemma 3.2.2 such an element does exist. Note that performing the covering map on c in a different order than the one prescribed by (3.2.1) does not always result in the corresponding input-wing. This observation can already be made on the two pentagons of Figure 12.

3.3. **Cells and synchronized cubic coordinates.** In Figure 12 and Figure 13, we notice that a "cellular" organization appears. Thanks to the cubic coordinates, a combinatorial definition of these cells is provided. The aim is to have a better understanding of the realization of the cubic coordinate posets as a geometrical object.

For any $n \geqslant 0$, let $c, c' \in \mathrm{CC}(n)$ such that $c \preccurlyeq c'$. A *cell* is the set of points

$$\langle c, c' \rangle := \left\{ x \in \mathbb{R}^{n-1} : c_i \leqslant x_i \leqslant c'_i \text{ for all } i \in [n-1] \right\}. \tag{3.3.1}$$

By the definition, a cell is an orthotope, that is, a parallelotope whose edges are all mutually orthogonal or parallel. The *dimension* $\dim\langle c, c' \rangle$ of a cell $\langle c, c' \rangle$ is its dimension as an orthotope and it satisfies $\dim\langle c, c' \rangle = \#\mathrm{D}(c, c')$, where $\mathrm{D}(c, c') := \{d \ :\ c_d \neq c'_d\}$.

From now on, we denote by c^{out} any output-wing and by c^{in} its corresponding input-wing. Any particular cell $\langle c^{\mathrm{out}}, c^{\mathrm{in}} \rangle$ formed by an output-wing and by its corresponding input-wing is called a *cell-wing*.

A consequence of Lemma 3.2.1 is that for any cell-wing $\langle c^{\mathrm{out}}, c^{\mathrm{in}} \rangle$ of dimension $n - 1$, for all $i \in [n-1]$,

(i) if $c^{\mathrm{out}}_i < 0$, then $c^{\mathrm{in}}_i \leqslant 0$,

(ii) if $c^{\mathrm{out}}_i \geqslant 0$, then $c^{\mathrm{in}}_i > 0$.

Theorem 3.3.1. *Let $n \geqslant 1$ and $\langle c^{\mathrm{out}}, c^{\mathrm{in}} \rangle$ be a cell-wing of dimension $n - 1$, and c be a $(n-1)$-tuple such that for all $i \in [n-1]$, the component c_i is equal either to c^{out}_i or to c^{in}_i. Then c is a cubic coordinate.*

Proof. If all the components of c are equal to those of c^{out} (resp. to those of c^{in}), then c is a cubic coordinate. Suppose this is not the case, meaning that c has components of c^{out} and c^{in}.

Let us denote $(u_i^{\text{out}}, v_{i+1}^{\text{out}})$ (resp. $(u_i^{\text{in}}, v_{i+1}^{\text{in}})$) the pair of letters corresponding to c_i^{out} (resp. c_i^{in}) and (u_i, v_{i+1}) the one corresponding to c_i for any $i \in [n-1]$. By hypothesis on c^{out} and c^{in} the letter u_i which is equal to u_i^{out} or u_i^{in} satisfies $0 \leqslant u_i \leqslant n-i$ for any $i \in [n]$. Similarly, the letter v_i which is equal to v_i^{out} or v_i^{in} satisfies $0 \leqslant v_i \leqslant i-1$ for any $i \in [n]$. In order to show that c is a cubic coordinate, let us prove that u satisfies Condition (ii) of a Tamari diagram, v satisfies Condition (ii) of a dual Tamari diagram and (u, v) satisfies the compatibility condition.

(i) Let us show that for any choice of letters u_i and u_{i+j} with $i \in [n]$ and $j \in [0, u_i]$ one has $u_{i+j} \leqslant u_i - j$.
   - ⋆ If u_i and u_{i+j} are equal respectively to u_i^{out} and to u_{i+j}^{out} (resp. to u_i^{in} and to u_{i+j}^{in}), then Condition (ii) of a Tamari diagram is satisfies because c^{out} (resp. c^{in}) is a cubic coordinate.
   - ⋆ Suppose that $u_i = u_i^{\text{in}}$ and $u_{i+j} = u_{i+j}^{\text{out}}$. By the definition of c^{in} one has $u_{i+j}^{\text{out}} < u_{i+j}^{\text{in}}$. However $u_{i+j}^{\text{in}} \leqslant u_i^{\text{in}} - j$ because c^{in} is a cubic coordinate. Therefore, Condition (ii) of a Tamari diagram is satisfied.
   - ⋆ Suppose that $u_i = u_i^{\text{out}}$ and $u_{i+j} = u_{i+j}^{\text{in}}$. Let $c' := \Uparrow_{i+j}(c^{\text{out}})$. According to Lemma 3.2.2 c' is a cubic coordinate such that $c'_i = u_i^{\text{out}}$ and $c'_{i+j} = u_{i+j}^{\text{in}}$. Since Condition (ii) of a Tamari diagram is satisfied for c', it must also be satisfied for c.

(ii) Condition (ii) of a dual Tamari diagram is satisfied with similar arguments given for the previous case, applied to the dual Tamari diagram v.

(iii) Rather than showing the compatibility condition as it is stated, let us show the contrapositive. That is, for every $1 \leqslant i < j \leqslant n$ such that $v_j \geqslant j - i$, let us show that $u_i < j - i$.
   - ⋆ Clearly, if u_i and v_j are equal to u_i^{out} and v_j^{out} (resp. to u_i^{in} and v_j^{in}), then the compatibility condition is satisfied.
   - ⋆ Suppose that $u_i = u_i^{\text{out}}$ and $v_j = v_j^{\text{in}}$. If $v_j^{\text{in}} \geqslant j - i$, then for c^{out} one has $v_j^{\text{out}} \geqslant j - i$ because $v_j^{\text{in}} < v_j^{\text{out}}$. Since c^{out} is a cubic coordinate, this implies that $u_i^{\text{out}} < j - i$.
   - ⋆ Suppose that $u_i = u_i^{\text{in}}$ and $v_j = v_j^{\text{out}}$. If $v_j^{\text{out}} \geqslant j - i$, then for all $k \in [i, j-1]$, $u_k^{\text{out}} < j - k$ because c^{out} is a cubic coordinate and then satisfies the compatibility condition. Moreover, since $c^{\text{out}} \in \mathcal{O}(\mathrm{CC}(n))$ each component can be minimally increased independently of the others, thus $u_k^{\text{out}} < j - k - 1$ for all $k \in [i, j-1]$. For the same reason $u_{i+h} < u_i - h$ for all $h \in [0, u_i]$. These two reasons imply that if one builds the cubic coordinate $c' = \Uparrow_i (c^{\text{out}})$, then by the definition of the covering map one has $c'_i = u'_i < j - i$, because at worst, the covering map sends u_i^{out} to $j - i - 1$ (we have already seen this in the proof of Lemma 3.2.2). However, by the definition of c^{in} one has $u_i^{\text{in}} = u'_i$, that is $u_i^{\text{in}} < j - i$. Therefore, the compatibility condition between u^{in} and v_j^{out} is satisfied for c.

Thus, for all choices of letters of u and v one has that c is a cubic coordinate. □

One of the direct consequences of Theorem 3.3.1 is that for every cell-wing $\langle c^{\text{out}}, c^{\text{in}}\rangle$, at least 2^{n-1} cubic coordinates belong to this cell.

This theorem also implies that a corresponding input-wing covers $n-1$ cubic coordinates, and so is in particular an input-wing.

Moreover, due to the fact the Tamari interval lattice is self-dual, the number of output-wings is equal to the number of input-wings. Therefore, by Theorem 2.4.5, an input-wing is always a corresponding input-wing of some output-wing.

Let $n \geqslant 0$, and $\epsilon \in \{-1,1\}^{n-1}$, and $c \in \mathrm{CC}(n)$. The *ϵ-region* of c is the set

$$\mathcal{R}_\epsilon(c) := \{(x_1,\dots,x_{n-1}) \in \mathbb{R}^{n-1} \ : \ x_i < c_i \text{ if } \epsilon_i = -1,\ x_i > c_i \text{ otherwise}\}. \tag{3.3.2}$$

The cubic coordinate c is *external* if there is $\epsilon \in \{-1,1\}^{n-1}$ such that $\mathrm{CC}(n) \cap \mathcal{R}_\epsilon(c) = \emptyset$. The ϵ-region $\mathcal{R}_\epsilon(c)$ is then *empty*. Otherwise, c is *internal*.

Proposition 3.3.2. *Let $n \geqslant 0$ and $c \in \mathrm{CC}(n)$. If c is internal, then $\phi(c)$ is a new Tamari interval diagram.*

Proof. Instead, let us show that if $\phi(c)$ is not new, then c is external. Let us denote (u_i, v_{i+1}) the pair of letters corresponding to c_i by the map ϕ for $i \in [n-1]$.

Tamari interval diagram $\phi(c)$ is not new if there is

(1) either $i \in [n-1]$ such that $u_i = n - i$,
(2) or $j \in [2,n]$ such that $v_j = j - 1$,
(3) or $k, l \in [n]$ such that $u_k = l - k - 1$ and $v_l = l - k - 1$ with $k + 1 < l$.

Suppose there is some i satisfying (1), then there cannot be a cubic coordinate c' such that $c'_i > c_i$ because, by the definition of a Tamari diagram, $c'_i \leqslant n - i$. Similarly, if we assume that there is j satisfying (2), then there cannot be a cubic coordinate c' such that $c'_{j-1} < c_{j-1}$ because by the definition of a dual Tamari diagram, $c'_{j-1} \geqslant 1 - j$. If (3) is satisfied, then there cannot be a cubic coordinate c' such that $c'_k > c_k$ and $c'_{l-1} < c_{l-1}$. Indeed, if the letters u_k and v_l are increased in c, then the compatibility condition is contradicted, so the result cannot be a cubic coordinate. Since in each case at least one ϵ-region is empty, c is external. □

Proposition 3.3.3. *Let $n \geqslant 0$ and $c \in \mathrm{SCC}(n)$. Then c is external.*

Proof. By Proposition 2.3.6 we know that if c is synchronized, then $\phi(c)$ is not new. Now, we just saw from Proposition 3.3.2 that if $\phi(c)$ is not new, then c is external. □

We know that each cell-wing contains at least 2^{n-1} cubic coordinates on the edges. Now, let us show that it is possible to associate bijectively each cell-wing to a synchronized cubic coordinate.

Let $n \geqslant 1$ and $\langle c^{\text{out}}, c^{\text{in}}\rangle$ be a cell-wing of dimension $n-1$ and γ be the map defined by

$$\gamma(c_i^{\text{out}}, c_i^{\text{in}}) := \begin{cases} c_i^{\text{out}} & \text{if } c_i^{\text{out}} < 0, \\ c_i^{\text{in}} & \text{if } c_i^{\text{out}} \geqslant 0, \end{cases} \tag{3.3.3}$$

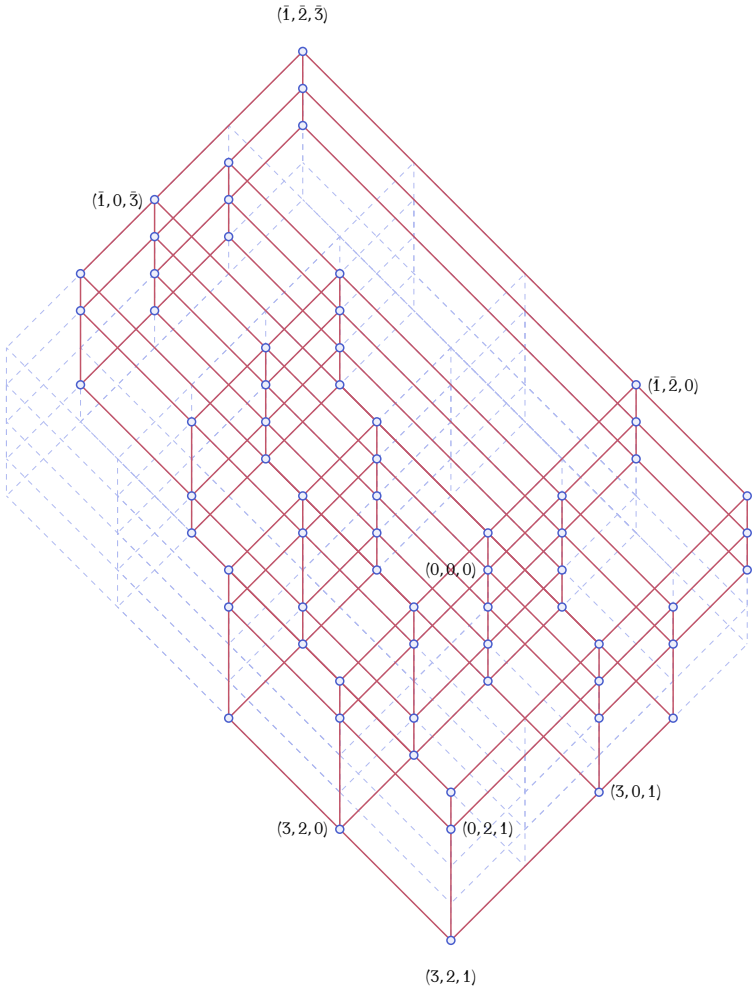


FIGURE 13. $\mathfrak{C}(\mathrm{CC}(4))$.

for all $i \in [n-1]$. Note that the components returned by the map γ are never zero. Let denote by $(u_i^{\mathrm{out}}, v_{i+1}^{\mathrm{out}})$ (resp. $(u_i^{\mathrm{in}}, v_{i+1}^{\mathrm{in}})$) the pair of letters corresponding to c_i^{out} (resp. c_i^{in}) by the map ϕ, for any $i \in [n-1]$. Thus, the map γ becomes

$$\gamma(c_i^{\mathrm{out}}, c_i^{\mathrm{in}}) := \begin{cases} -v_{i+1}^{\mathrm{out}} & \text{if } c_i^{\mathrm{out}} < 0, \\ u_i^{\mathrm{in}} & \text{if } c_i^{\mathrm{out}} \geqslant 0. \end{cases} \tag{3.3.4}$$

Let Γ be the map defined by

$$\Gamma\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle := (\gamma(c_1^{\mathrm{out}}, c_1^{\mathrm{in}}), \gamma(c_2^{\mathrm{out}}, c_2^{\mathrm{in}}), \dots, \gamma(c_{n-1}^{\mathrm{out}}, c_{n-1}^{\mathrm{in}})). \tag{3.3.5}$$

For instance, the cell-wing $\langle(0,-1,1,-1,-5,0,1,-1,-3),(1,0,2,0,-4,3,2,0,-2)\rangle$ is sent by Γ to $(1,-1,2,-1,-5,3,2,-1,-3)$.

Theorem 3.3.4. *For any $n \geqslant 1$, the map Γ is a bijection from the set of cell-wings of dimension $n-1$ to* $\mathrm{SCC}(n)$.

Proof. The components of $\Gamma\langle c^{\text{out}}, c^{\text{in}}\rangle$ belong to either c^{out} or c^{in}. In both cases, it is a non-zero component. According to Theorem 3.3.1, $\Gamma\langle c^{\text{out}}, c^{\text{in}}\rangle$ is therefore a cubic coordinate of size n. Moreover, this cubic coordinate is synchronized because none of its components is null.

Let $\langle c^{\text{out}}, c^{\text{in}}\rangle$ and $\langle e^{\text{out}}, e^{\text{in}}\rangle$ be two cell-wings of dimension $n-1$ such that $\Gamma\langle c^{\text{out}}, c^{\text{in}}\rangle = \Gamma\langle e^{\text{out}}, e^{\text{in}}\rangle$. Let us denote $(u_i^{\text{out}}, v_{i+1}^{\text{out}})$ (resp. $(u_i^{\text{in}}, v_{i+1}^{\text{in}})$) the pair of letters corresponding to c_i^{out} (resp. c_i^{in}) and $(x_i^{\text{out}}, y_{i+1}^{\text{out}})$ (resp. $(x_i^{\text{in}}, y_{i+1}^{\text{in}})$) the pair of letters corresponding to e_i^{out} (resp. e_i^{in}) by the map ϕ, for all $i \in [n-1]$.

To suppose that $\Gamma\langle c^{\text{out}}, c^{\text{in}}\rangle = \Gamma\langle e^{\text{out}}, e^{\text{in}}\rangle$ is equivalent to suppose that for all $i \in [n-1]$, $\gamma(c_i^{\text{out}}, c_i^{\text{in}}) = \gamma(e_i^{\text{out}}, e_i^{\text{in}})$. The map Γ is injective if, for every $i \in [n-1]$, $c_i^{\text{out}} = e_i^{\text{out}}$ and $c_i^{\text{in}} = e_i^{\text{in}}$. Suppose that there is some index i such that $c_i^{\text{out}} \neq e_i^{\text{out}}$ or $c_i^{\text{in}} \neq e_i^{\text{in}}$, and we take the smallest such index. Then, two cases have to be considered: either $\gamma(c_i^{\text{out}}, c_i^{\text{in}}) = u_i^{\text{in}}$ or $\gamma(c_i^{\text{out}}, c_i^{\text{in}}) = -v_{i+1}^{\text{out}}$.

(1) Suppose that $\gamma(c_i^{\text{out}}, c_i^{\text{in}}) = u_i^{\text{in}}$.
   - ⋆ In this case, $\gamma(e_i^{\text{out}}, e_i^{\text{in}}) = x_i^{\text{in}}$ and $u_i^{\text{in}} = x_i^{\text{in}}$. Moreover, since $u_i^{\text{in}} \neq 0$ (resp. $x_i^{\text{in}} \neq 0$), then necessarily $v_{i+1}^{\text{in}} = 0$ (resp. $y_{i+1}^{\text{in}} = 0$). Therefore, $c_i^{\text{in}} = e_i^{\text{in}}$.
   - ⋆ On the other hand , the fact that $u_i^{\text{in}} > 0$ (resp. $x_i^{\text{in}} > 0$) implies by Lemma 2.3.2 that $0 \leqslant u_i^{\text{out}} < u_i^{\text{in}}$ and $v_{i+1}^{\text{out}} = 0$ (resp. $0 \leqslant x_i^{\text{out}} < x_i^{\text{in}}$ and $y_{i+1}^{\text{out}} = 0$). Thus, one has $v_{i+1}^{\text{out}} = y_{i+1}^{\text{out}}$. Therefore, the only way for the hypothesis to be true is that $u_i^{\text{out}} \neq x_i^{\text{out}}$.

     Without loss of generality, suppose that $u_i^{\text{out}} < x_i^{\text{out}}$. By the definition of the covering map, one has $x_i^{\text{out}} < x_i^{\text{in}}$. This implies, in addition to the hypothesis that $x_i^{\text{in}} = u_i^{\text{in}}$, that $u_i^{\text{out}} < x_i^{\text{out}} < u_i^{\text{in}}$.

     Let $c := \Uparrow_{i+1}(c^{\text{out}})$ and $e := \Uparrow_{i+1}(e^{\text{out}})$, both cubic coordinates by Lemma 3.2.2. By construction, $c_j = c_j^{\text{out}}$ (resp. $e_j = e_j^{\text{out}}$) for all $j \in [i]$ and $c_k = c_k^{\text{in}}$ (resp. $e_k = e_k^{\text{in}}$) for all $k \in [i+1, n-1]$.

     By minimality of i, we have that $c_j = e_j$ for all $j \in [i]$. Moreover, by the hypothesis that $\Gamma\langle c^{\text{out}}, c^{\text{in}}\rangle = \Gamma\langle e^{\text{out}}, e^{\text{in}}\rangle$, we have that $u_k^{\text{in}} = x_k^{\text{in}}$ for $k \in [i+1, n-1]$. Indeed, if $u_k^{\text{in}} > 0$ (resp. $x_k^{\text{in}} > 0$) then necessarily $\gamma(c_k^{\text{out}}, c_k^{\text{in}}) = u_k^{\text{in}}$ (resp. $\gamma(e_k^{\text{out}}, e_k^{\text{in}}) = x_k^{\text{in}}$) and so $u_k^{\text{in}} = x_k^{\text{in}}$. Otherwise, $u_k^{\text{in}} = x_k^{\text{in}} = 0$. Note that because we know nothing about v_k^{in} and y_k^{in} for $k \in [i+2, n]$, we cannot say that $\uparrow_i(c)$ and $\uparrow_i(e)$ are equal.

     Now, let c' be a tuple such that $c'_i = x_i^{\text{out}}$ and $c'_j = c_j$ for all $j \neq i$ and let (u', v') the pair of words corresponding to c' by the map ϕ. Let us show that c' is a cubic coordinate.

     By construction, since the word v' is the dual Tamari diagram of c, v' is a dual Tamari diagram. Likewise, since the word u' is the Tamari diagram of $\uparrow_i(e)$, u' is a Tamari diagram.

     Moreover, we know that between c, c' and $\uparrow_i(c)$, only one positive letter changes, with $c_i = u_i^{\text{out}}$, $c'_i = x_i^{\text{out}}$ and $\uparrow c_i = u_i^{\text{in}}$, and we have established that $u_i^{\text{out}} < x_i^{\text{out}} < u_i^{\text{in}}$. Since the letter u_i^{in} satisfies the compatibility condition with the letters of v^{in} in $\uparrow_i(c)$, then all letter lower in position i satisfies this condition

as well. Therefore, u' and v' are compatible and c' is a cubic coordinate distinct from c and $\uparrow_i(c)$ such that $c \preccurlyeq c' \preccurlyeq \uparrow_i(c)$.
However, if c' is a cubic coordinate, then by the definition of the covering map $\uparrow c_i := u_i^{\text{in}} = x_i^{\text{out}}$, and so $\uparrow_i(c) := \Uparrow_i(c^{\text{out}}) = c'$. This is not possible with the assumption that $u_i^{\text{in}} = x_i^{\text{in}}$, and so that $\gamma(c_i^{\text{out}}, c_i^{\text{in}}) = \gamma(e_i^{\text{out}}, e_i^{\text{in}})$.

(2) Suppose that $\gamma(c_i^{\text{out}}, c_i^{\text{in}}) = -v_{i+1}^{\text{out}}$. In this case $\gamma(e_i^{\text{out}}, e_i^{\text{in}}) = -y_{i+1}^{\text{out}}$ and $v_{i+1}^{\text{out}} = y_{i+1}^{\text{out}}$. By rephrasing the arguments of the case (1) for the dual, we show that $c_i^{\text{out}} = e_i^{\text{out}}$ and $c_i^{\text{in}} = e_i^{\text{in}}$.

This shows that the map Γ is injective.

Now let us show that the cardinal of the set of cell-wings of dimension $n-1$ is equal to the cardinal of $\mathrm{SCC}(n)$. Recall that the set of cells of size n is exactly $\mathcal{O}(\mathrm{CC}(n))$. Furthermore, by the poset isomorphism ψ we know that these elements are the Tamari intervals having $n-1$ elements covering in the Tamari interval lattices. In [Cha18] Chapoton shows that the set of these Tamari intervals has the same cardinal as the set of synchronized Tamari intervals (see Theorem 2.1 and Theorem 2.3 from [Cha18]). Finally, Proposition 2.3.3 allows us to conclude that the cardinal of $\mathrm{SCC}(n)$ and the cardinal of the set of cell-wings of dimension $n-1$ are equal. Thus, the map Γ is bijective. □

Let us also defined the map $\bar{\gamma}$ by

$$\bar{\gamma}(c_i^{\text{out}}, c_i^{\text{in}}) := \begin{cases} c_i^{\text{in}} & \text{if } c_i^{\text{out}} < 0, \\ c_i^{\text{out}} & \text{if } c_i^{\text{out}} \geqslant 0, \end{cases} \tag{3.3.6}$$

for all $i \in [n-1]$. Then $\bar{\Gamma}$ is defined by

$$\bar{\Gamma}\langle c^{\text{out}}, c^{\text{in}}\rangle := (\bar{\gamma}(c_1^{\text{out}}, c_1^{\text{in}}), \bar{\gamma}(c_2^{\text{out}}, c_2^{\text{in}}), \dots, \bar{\gamma}(c_{n-1}^{\text{out}}, c_{n-1}^{\text{in}})). \tag{3.3.7}$$

By Theorem 3.3.1, $\bar{\Gamma}\langle c^{\text{out}}, c^{\text{in}}\rangle$ is a cubic coordinate belonging to $\langle c^{\text{out}}, c^{\text{in}}\rangle$, called *opposite cubic coordinate*. For the synchronized cubic coordinate c associated with $\langle c^{\text{out}}, c^{\text{in}}\rangle$ by Γ, denote c^{op} the opposite cubic coordinate. All the components of c^{op} are different from those of c, and these differences are the greatest possible. For any synchronized cubic coordinate c, such a cubic coordinate c^{op} always exists and is unique.

Note that the map Γ only returns the positive components of c^{in} and the negative components of c^{out}. Conversely, the map $\bar{\Gamma}$ returns the positive components of c^{out} and the negative components of c^{in}. We already know that the latter combination is always possible for any comparable cubic coordinates according to Lemma 2.4.2. On the other hand, this is not the case for the first mentioned combination.

3.4. **Volume of $\mathfrak{C}(\mathrm{CC})$.** Now let us take a closer look at the geometry of the cubic realization. We already know that there are at least 2^{n-1} cubic coordinates forming an outline of each cell-wing. The following notions will allow us to say more.

A point x of $\mathbb{R}^{n-1}$ is *inside* a cell $\langle c, c'\rangle$ if, for any $i \in [n-1]$, $c_i \neq c'_i$ implies $c_i < x_i < c'_i$. A cell $\langle c, c'\rangle$ is *pure* if there is no cubic coordinate inside $\langle c, c'\rangle$. The *volume* $\mathrm{vol}\langle c, c'\rangle$ of

$\langle c, c' \rangle$ is its volume as an orthotope and it satisfies

$$\operatorname{vol} \langle c, c' \rangle = \prod_{i \in \mathrm{D}(c,c')} (c'_i - c_i). \tag{3.4.1}$$

Lemma 3.4.1. *Let $n \geqslant 1$ and $\langle c^{\mathrm{out}}, c^{\mathrm{in}} \rangle$ be a cell-wing of dimension $n-1$. The cell $\langle c^{\mathrm{out}}, c^{\mathrm{in}} \rangle$ is pure.*

Proof. Suppose there is a cubic coordinate c such that $c_i^{\mathrm{out}} < c_i < c_i^{\mathrm{in}}$ for all $i \in [n-1]$. By Lemma 3.2.1 we know that if $c_i^{\mathrm{out}} < 0$, then $c_i^{\mathrm{in}} \leqslant 0$ and if $c_i^{\mathrm{out}} \geqslant 0$, then $c_i^{\mathrm{in}} > 0$. However, since $c_i^{\mathrm{out}} < c_i < c_i^{\mathrm{in}}$, then c_i is different from 0. In the end, if such a cubic coordinate c exists, it would be synchronized. But then, there would be a cubic coordinate both synchronized and internal by hypothesis. This is impossible according to Proposition 3.3.3. □

We showed with Theorem 3.3.1 that each cell-wing contains at least 2^{n-1} cubic coordinates. By Lemma 3.4.1, we know that each cell-wing $\langle c^{\mathrm{out}}, c^{\mathrm{in}} \rangle$ is pure, and then has only cubic coordinates on its border.

Let $n \geqslant 1$ and $\langle c^{\mathrm{out}}, c^{\mathrm{in}} \rangle$ be a cell-wing of dimension $n-1$. Since between c^{out} and c^{in} all components are different, one has $\mathrm{D}(c^{\mathrm{out}}, c^{\mathrm{in}}) = n-1$, and so the volume of $\langle c^{\mathrm{out}}, c^{\mathrm{in}} \rangle$ satisfies

$$\operatorname{vol} \langle c^{\mathrm{out}}, c^{\mathrm{in}} \rangle = \prod_{i=1}^{n-1} (c_i^{\mathrm{in}} - c_i^{\mathrm{out}}). \tag{3.4.2}$$

Let us denote by c^0 the cubic coordinate such that $c_i^0 = 0$ for any $i \in [n-1]$. To compute $\operatorname{vol} \langle c^{\mathrm{out}}, c^{\mathrm{in}} \rangle$ from the synchronized cubic coordinate c associated by Γ, we must first compute the volume of the cell formed by c^0 and c.

By Lemma 3.2.1, any cell-wing is included in an ϵ-region of the c^0 cubic coordinate. This means that no cell-wing can be cut by a line passing by the origin c^0 and a cubic coordinate of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$ or $(0, \ldots, 0, -1, 0, \ldots, 0)$.

According to Lemma 2.3.2, for any cubic coordinate, replacing any component by 0 gives a cubic coordinate. In other words, for any cubic coordinate c, there are $n-1$ cubic coordinates related to c which are its projections on the lines passing by c^0 and a cubic coordinate of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$ or $(0, \ldots, 0, -1, 0, \ldots, 0)$. Therefore, even if c^0 and c are not comparable, we consider the cell, denoted by $\langle c \rangle$, between c^0 and c, such that the volume of this cell satisfies

$$\operatorname{vol} \langle c \rangle = \prod_{i \in \mathrm{D}(c,c^0)} |c_i|. \tag{3.4.3}$$

Note that the dimension of a cell is less than or equal to $n-1$. Moreover, $\langle c \rangle$ can be no-pure, and may even contain other cells of the same dimension.

By the map Γ, the components of the synchronized cubic coordinate c of the cell-wing $\langle c^{\mathrm{out}}, c^{\mathrm{in}} \rangle$ are the greatest in absolute value between c^{out} and c^{in}. Therefore, in the cell-wing $\langle c^{\mathrm{out}}, c^{\mathrm{in}} \rangle$, c is the furthest cubic coordinate from c^0. In particular, $\langle c \rangle$ contains the cell-wing $\langle c^{\mathrm{out}}, c^{\mathrm{in}} \rangle$ and the dimension of $\langle c \rangle$ is $n-1$.

Let $n \geqslant 0$ and $c \in \mathrm{SCC}(n)$. Since by the definition, all components of c are different from 0, one has $\mathrm{D}(c, c^0) = n - 1$. Therefore,

$$\mathrm{vol}\,\langle c\rangle = \prod_{i=1}^{n-1} |c_i|. \tag{3.4.4}$$

Let us endow the set $\mathrm{SCC}(n)$ with the partial order $\preccurlyeq_{\mathrm{s}}$ such that for $c, c' \in \mathrm{SCC}(n)$ one has $c' \preccurlyeq_{\mathrm{s}} c$ if c'_i and c_i have the same sign and $|c'_i| \leqslant |c_i|$ for any $i \in [n-1]$.

Lemma 3.4.2. *For any $n \geqslant 1$, let $\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$ be a cell-wing of dimension $n-1$, and $c := \Gamma\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$. For any $x \in \mathbb{R}^{n-1}$ such that $x \in \langle c\rangle$, if $x \notin \langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$, then there is $c' \in \mathrm{SCC}(n)$ different from c such that $c' \preccurlyeq_{\mathrm{s}} c$ and $x \in \langle c'\rangle$.*

Proof. Let c^{op} be the opposite cubic coordinate of c. Since $x \notin \langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$ and $x \in \langle c\rangle$, then necessarily $c^{op} \neq c^0$. For the same reasons, there is an index i such that $|x_i| < |c_i^{op}|$ where $c_i^{op} \neq 0$. Let us build from such index i the $(n-1)$-tuple ∇c such that $\nabla c_i = c_i^{op}$ and $\nabla c_j = c_j$ for all $j \neq i$. According to Theorem 3.3.1, ∇c is a cubic coordinate and belongs to the cell-wing $\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$. Also, ∇c is a synchronized cubic coordinate which satisfies $\nabla c \preccurlyeq_{\mathrm{s}} c$ and which is different from c. We can then associate to ∇c a cell, which is strictly included in $\langle c\rangle$. Then $x \in \langle \nabla c\rangle$. □

Since by Lemma 3.4.1 all cell-wings are pure, Lemma 3.4.2 implies that $\langle c\rangle \subseteq \coprod_{c' \preccurlyeq_{\mathrm{s}} c} \Gamma^{-1}(c')$, and since the reciprocal inclusion is obvious, one has the following result.

Lemma 3.4.3. *Let $n \geqslant 0$ and $c \in \mathrm{SCC}(n)$. Then*

$$\langle c\rangle = \coprod_{c' \preccurlyeq_{\mathrm{s}} c} \Gamma^{-1}(c'). \tag{3.4.5}$$

Let $n \geqslant 0$ and $c \in \mathrm{SCC}(n)$. The *synchronized volume* of c is defined by

$$\mathrm{sv}(c) := \mathrm{vol}\,\langle c\rangle - \sum_{\substack{c' \preccurlyeq_{\mathrm{s}} c \\ c' \neq c}} \mathrm{sv}(c'). \tag{3.4.6}$$

Note that (3.4.6) is a Möbius inversion [Sta12].

Proposition 3.4.4. *Let $n \geqslant 1$ and $\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$ be a cell-wing of dimension $n-1$. By setting $c := \Gamma\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$, we have*

$$\mathrm{vol}\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle = \mathrm{sv}(c). \tag{3.4.7}$$

Proof. This is a consequence of Lemma 3.4.3 and of (3.4.6). □

With Proposition 3.4.4 we are able to compute, for any $n \geqslant 0$, the volume of $\mathfrak{C}(\mathrm{CC}(n))$ depending on synchronized cubic coordinates,

$$\mathrm{vol}(\mathfrak{C}(\mathrm{CC}(n))) = \sum_{c \in \mathrm{SCC}(n)} \mathrm{sv}(c). \tag{3.4.8}$$

3.5. **EL-shellability.** In [BW96] and [BW97], Björner and Wachs generalized the method of labellings of the cover relations of graded posets to the case of non-graded posets. In particular, they showed the EL-shellability of the Tamari poset [BW97].

Let $\mathcal{P}$ be a bounded poset and Λ be a poset, and $\lambda : \lessdot_{\mathcal{P}} \to \Lambda$ be a map. For any saturated chain $\left(x^{(1)}, \dots, x^{(k)}\right)$ of $\mathcal{P}$, we set

$$\lambda\left(x^{(1)}, \dots, x^{(k)}\right) := \left(\lambda\left(x^{(1)}, x^{(2)}\right), \dots, \lambda\left(x^{(k-1)}, x^{(k)}\right)\right). \tag{3.5.1}$$

We say that a saturated chain of $\mathcal{P}$ is *λ-increasing* (resp. *λ-decreasing*) if its image by λ is an increasing (resp. decreasing) word for the order relation $\preccurlyeq_\Lambda$. We say also that a saturated chain $\left(x^{(1)}, \dots, x^{(k)}\right)$ of $\mathcal{P}$ is *λ-smaller* than a saturated chain $\left(y^{(1)}, \dots, y^{(k)}\right)$ of $\mathcal{P}$ if $\lambda\left(x^{(1)}, \dots, x^{(k)}\right)$ is smaller than $\lambda\left(y^{(1)}, \dots, y^{(k)}\right)$ for the lexicographic order induced by $\preccurlyeq_\Lambda$. The map λ is called *EL-labeling* (edge lexicographic labeling) of $\mathcal{P}$ if for any $x, y \in \mathcal{P}$ satisfying $x \preccurlyeq_{\mathcal{P}} y$, there is exactly one λ-increasing saturated chain from x to y, and this chain is λ-minimal among all saturated chains from x to y. Any bounded poset that admits an EL-labeling is *EL-shellable* [BW96, BW97].

The EL-shellability of a poset $\mathcal{P}$ implies several topological and order theoretical properties of the associated order complex $\Delta(\mathcal{P})$ built from $\mathcal{P}$. Recall that the faces of this simplicial complex are all the chains of $\mathcal{P}$. Moreover, if $\mathcal{P}$ has at most one λ-decreasing chain between any pair of elements, then the Möbius function of $\mathcal{P}$ takes values in $\{-1, 0, 1\}$. In this case, the simplicial complex associated with each open interval of $\mathcal{P}$ is either contractible or has the homotopy type of a sphere [BW97].

For the sequel, we set Λ as the poset $\mathbb{Z}^3$ wherein elements are ordered lexicographically. Let $(c, c') \in \lessdot$ such that, for $i \in [n-1]$, $c_i < c'_i$, and let $\lambda : \lessdot \to \mathbb{Z}^3$ be the map defined by

$$\lambda(c, c') := (\varepsilon, i, c_i), \tag{3.5.2}$$

where $\varepsilon := \begin{cases} -1 & \text{if } c_i < 0, \\ 1 & \text{else.} \end{cases}$

Note that by Proposition 2.4.6, the index i such that $c_i < c'_i$ is unique.

Theorem 3.5.1. *For any $n \geqslant 0$, the map λ is an EL-labeling of $\mathrm{CC}(n)$. Moreover, there is at most one λ-decreasing chain between any pair of elements of $\mathrm{CC}(n)$.*

Proof. Let $c, c' \in \mathrm{CC}(n)$ such that $c \preccurlyeq c'$. By Lemma 2.4.2, there is a cubic coordinate c'' such that $u'' = u$ and $v'' = v'$ with $(u'', v'') := \phi(c'')$. Let

$$\mathrm{D}^-\left(c, c''\right) = \{d_1, d_2, \dots, d_r\} \tag{3.5.3}$$

with $d_{k-1} < d_k$ for all $k \in [2, r]$, and

$$\mathrm{D}^+\left(c'', c'\right) = \{d'_1, d'_2, \dots, d'_s\}, \tag{3.5.4}$$

with $d'_{k-1} < d'_k$ for all $k \in [2, s]$.

By Lemma 2.4.3, there is a chain between c and c''

$$\left(c, c^{(1)}, \dots, c^{(r-1)}, c^{(r)} = c''\right), \tag{3.5.5}$$

where, for $k \in [r]$, $c^{(k)}$ be a cubic coordinate obtained by replacing in c all the components c_{d_i} by the components c''_{d_i} for $i \in [k]$.

By Lemma 2.4.4, there is a chain between c'' and c'

$$\left(c'', c'^{(1)}, \dots, c'^{(s-1)}, c'^{(s)} = c'\right), \tag{3.5.6}$$

where, for $k \in [s]$, $c'^{(k)}$ be a cubic coordinate obtained by replacing in c'' all the components c''_{d_i} by the components c'_{d_i} for $i \in [k]$.

Let us consider the chain obtained by concatenating the two chains (3.5.5) and (3.5.6). Since in this chain only one component differs between two consecutive cubic coordinates, a saturated chain μ can be constructed by considering all the cubic coordinates between them. For both chains (3.5.5) and (3.5.6), the components are independently increasing one by one from the left to the right. By construction, it implies that μ is λ-increasing for the lexicographic order induced by (3.5.2).

Moreover, any other choice of saturated chain between c and c' implies choosing, at a certain step k, a greater label for the lexicographical order than the label (ε, k, c_k) of μ, and then having to choose the label (ε, k, c_k) afterwards. Thus, in addition to being λ-increasing, the saturated chain μ is unique and is λ-minimal among all saturated chains from c to c'.

If a saturated chain λ-decreasing exists between c and c', it is built by first changing the different and negative components between c and c'' from right to left, and then changing the different and positive components between c'' and c' from right to left. For the same reason that any saturated λ-increasing chain is unique for any interval, if it exists, the λ-decreasing chain is also unique. □

For instance, in Figure 12, the λ-increasing saturated chain between $(-1, -2)$ and $(2, 1)$ is the chain

$$((-1, -2), (0, -2), (0, -1), (0, 0), (1, 0), (2, 0), (2, 1)), \tag{3.5.7}$$

and

$$\lambda((-1, -2), \dots, (2, 1)) = ((-1, 1, -1), (-1, 2, -2), (-1, 2, -1), (1, 1, 0), (1, 1, 1), (1, 2, 0)). \tag{3.5.8}$$

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