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Proportional and maxmin fairness for the sensor location problem with chance constraints

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Abstract

In this paper we present a study on the Equitable Sensor Location Problem and we focus on the stochastic version of the problem where the surveying capacity of some sensors is measured as probability of intrusions detection. The Equitable Sensor Location Problem, which is an extension of the Equitable Facility Location Problem, considers installing surveying facilities as cameras/sensors in order to monitor and protect some important locations. Each location can be simultaneously protected by multiple facilities. Clearly this problem falls into the category of Maximal Coverage Location Problem and we focus on the equitable variant. The objective of the Equitable Sensor Location Problem is to provide equitable protection to all locations when the number of sensors that can be placed is limited. We study the resilient and ambiguous versions of this problem. The resilient sensor location problem considers the case when some sensors are assumed to fail partially

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or completely. The ambiguous version studies the case when the surveying probabilities are uncertain and represented by independent Bernoulli random variables with the corresponding ambiguity set containing the Bernoulli probability distributions. For each problem we consider two popular fairness measures which are the lexicographic optimal and proportionally fair solutions and provide an integer linear formulation together with the solution methodology. Numerical results for each studied problem are provided at the end of the paper.

Keywords: equity, equitable resource allocation, stochastic set covering, robust combinatorial optimization, \mathcal{NP} -hardness

1. Introduction

We consider the problem of installing facilities at strategic locations in order to monitor and protect numerous important locations. Each location can be simultaneously protected by multiple facilities. Concrete examples include airports where various locations, such as terminals, baggage areas, control towers, gates, runways, among others, must be protected. Other examples include shopping malls and entertainment parks (e.g., Disney World) where large numbers of people assemble at many locations, and strategic complexes such as hospitals, power plants, and military installations. Clearly this problem falls into the category of Maximal Coverage Location Problem. Covering problems are among the most studied combinatorial optimization problems. The Maximal Covering Location Problem (MCLP) was introduced in [10] and is \mathcal{NP} -hard [13]. Facility location problems are intensively studied in the literature, see for instance [3, 9, 14, 16, 17, 20] and the references therein.

In our study the objective is to provide (i) equitable protection to all locations when the number of sensors that can be placed is limited (ii) robust protection under possibly full or partial failures of sensors. The Equitable Sensor Location Problem [14, 17, 18] is an extension of the Equitable Facility Location Problem, see Ogryczak [25]. The latter considers placing facilities, such as police stations or emergency rooms, so as to provide equitable service to all neighborhoods, where people in each of the neighborhoods are served by the closest facility. Versions of this problem are proposed in Neidhardt, Luss, and Krishnan [24], and in Luss [[17] Section 7.2.2], while some other related works on fairness are given in [4, 5, 6, 32].

In this study we explore models with two different objective functions often used to model optimization problems with fairness criteria: (i) Lexicographic maximin (or minimax) optimization and (ii) Proportional fairness optimization. These criteria are among the most popular ones from the fairness studies and cover a large spectrum of applications. The contribution of this work stands in providing a full study on the different variants of the probabilistic equitable sensor location problem with the above two fairness criteria (other criteria like these using Gini coefficients and Lorenz curve orderings are beyond the scope of this study [1]). The contribution is twofold: (i) modeling and solving the probabilistic equitable sensor location problem and the resilient variant; (ii) modeling and solving the ambiguous equitable sensor location problem. Note that the resilient variant of the equitable sensor location stands for the case when sensors are subject to partial or complete failures while the ambiguous equitable sensor location problem considers the case with uncertain probabilities. Throughout this study we

use surveying capacity of sensors measured as probability of intrusions detection and assume that they are independent. This assumption enables us to obtain tractable computational methods as it will be shown in Sections 3 and 4. The paper is organized as follows. Section 2 gives preliminaries on equity and proportional fairness to be used in the rest of the paper. Section 3 studies the basic equitable sensor location problem and proposes an integer linear program. A similar integer linear program is proposed to compute the proportionally fair solution in linear time. Next, the section presents a similar model for the equitable resilient sensor location problem, assuming that sensors may fail. Section 4 is devoted to the ambiguous sensor location problem, where probabilities are assumed to vary in a finite set. We prove that the the proportional fairness version of the ambiguous problem is \mathcal{NP} -hard in the strong sense and present mixed-integer linear programming formulations. Section 5 presents numerical results. The numerical results show the value of the resilient and ambiguous solutions when compared to the deterministic ones.

2. Preliminaries

This section is devoted to preliminaries on equity and proportional fairness together with a result that will be useful in writing down the linear integer model for the basic sensor location problem. Next, we deduce the mathematical formulation of the problem and discuss solution methods for both equitable and proportionally fair variants.

Let us start by introducing formally the notion of equity as discussed in this paper. The equity notion is closely related to lexicographic optimization

as remarked in numerous studies such as [7, 15, 17, 23, 26, 30, 31]. We recall some definitions on lexicographic ordering, useful for a better understanding of the study. A vector γ is lexicographically greater (resp. lower) than γ' if there exists $s \in \{1, \dots, n\}$ such that $\gamma_p = \gamma'_p$, for all $p \in \{1, \dots, s - 1\}$ and $\gamma_s > \gamma'_s$ (resp. $\gamma_s < \gamma'_s$). A vector γ is lexicographically maximal (resp. minimal) in X if for every vector $\gamma' \in X$, γ is lexicographically greater (resp. lower) than or equal to γ' .

Let $\vec{\gamma}$ (resp. $\overleftarrow{\gamma}$) be the vector γ with its indices reordered so that the components are in non-decreasing (resp. non-increasing) order. A feasible vector is defined as leximin maximal [30] as follows: A vector $\gamma \in X$ is leximin maximal if for every vector $\gamma' \in X$, $\vec{\gamma}$ is lexicographically greater than or equal to $\vec{\gamma}'$. Similarly, one can define leximax minimality as follows: a vector $\gamma \in X$ is leximax minimal if for every vector $\gamma' \in X$, $\overleftarrow{\gamma}$ is lexicographically lower than or equal to $\overleftarrow{\gamma}'$.

Let us look now at the solution methodology. We define $\Gamma \subset \mathbb{R}^m$ as the set of vectors γ for which the following set is non-empty:

$$\{f_i(x) \geq \gamma_i; \quad i \in 1, \dots, m, x \geq 0, x \in \mathbb{R}^n\}. \quad (1)$$

We say that γ is feasible if $\gamma \in \Gamma$. Let us focus on the feasible vectors γ that are leximin maximal. Computing a leximin maximal vector for the system of inequalities (1) when $f_i(x)$ are linear is relatively easy as shown by the method in [22] or by the methods in [[17], Section 3.4] and in references therein. Then, one can compute a leximin maximal vector among the feasible vectors by solving a sequence of at most m linear programs. At iteration i one computes the highest value that can take the i^{th} smaller component of

the solution vector.

Similar results can be drawn for the following system of functions:

$$\{f_i(x) \leq \gamma_i; \quad i \in 1, \dots, m, x \geq 0, x \in \mathbb{R}^n\}, \quad (2)$$

where we look for a feasible vector γ which is lexicmax minimal.

Note that all the above is closely tied to the Pareto optimality concept and some work has been done to transpose these concepts to the stochastic, chance-constrained context under the name of p-efficient point (see [29, 32]).

The above results are shown for the systems where $x \in \mathbb{R}^n$. Nevertheless, the problem may be solved optimally for the discrete case (i.e., $x \in \mathbb{Z}_+^n$) by using the methods presented in [[17] Sections 7.2.3 and 7.3.2] and references therein. An effective iterative method based on OWA (Ordered Weighted Average) criteria initially proposed in [34] and further developed in [27] can also be used.

Let us consider some strictly increasing function ϕ and the system composed of functions $\phi \circ f_i$. Recall that the operator \circ stands for the function composition operator. It can be shown that the following result holds.

Proposition 1. *Let ϕ be a strictly increasing function in \mathbb{R} . A vector γ feasible for (1) is lexicmin maximal if and only if the vector $(\phi(\gamma_1), \dots, \phi(\gamma_m))$ is lexicmin maximal for the corresponding system composed of functions $\{\phi \circ f_i, i \in M\}$.*

Proof. The proof follows easily by contradiction. Namely, suppose that γ is lexicmin maximal for (1) and that $(\phi(\gamma_1), \dots, \phi(\gamma_m))$ (denoted $\phi(\gamma)$ for short) is not lexicmin maximal for the corresponding system. Hence, there exists a

vector $\eta \neq \phi(\gamma)$ that is leximin maximal for the system corresponding to $\phi \circ f$. Therefore, $\phi^{-1}(\eta) \neq \gamma$ is leximin maximal for (1), yielding the contradiction. The reverse is shown similarly. \square

The above result was first used in [22] and next in [21] when dealing with some specific routing problems ensuring load balancing in telecommunication networks. It is straightforward to see that this result can be extended to $x \in \mathbb{Z}_+^n$.

3. The stochastic equitable sensor location problem

3.1 Problem description

The Equitable Sensor Location Problem can be represented by a bipartite graph $G(N, M, A)$ with a set of nodes $N = \{1, \dots, n\}$ representing candidate sensor locations, a set of nodes $M = \{1, \dots, m\}$ representing locations that should be protected and a set A of directed links. A link from node $i \in N$ to node $j \in M$ implies that a sensor at i monitors node j . If there is no link from node i to node j , then a sensor at i does not monitor j . We assume that we have K sensors available to be placed in the candidate locations in order to protect the selective locations. Consider the bipartite graph in Figure 1a. Suppose that sensors are located at nodes 1 and 5, as represented in Figure 1b. Then, locations 2, 3 and 5 are monitored by both sensors and location 4 is monitored only by the sensor at node 5. If sensors are located at nodes 1 and 4, locations 3 and 5 are protected by both sensors, and locations 2 and 4 are protected by a single sensor.

We consider in this paper a probabilistic version of the problem, where the effective monitoring of node j by node i is represented by random vari-

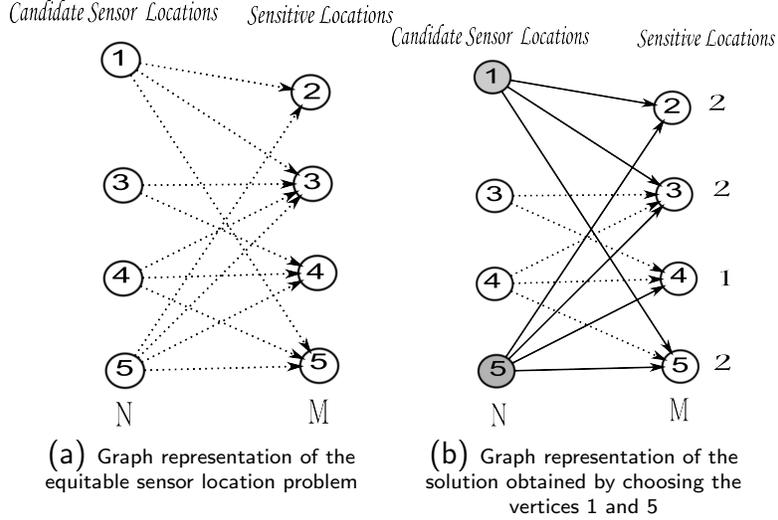


Figure 1: Graphical representation of the basic problem.

able a_{ij} . Specifically, a_{ij} is a Bernoulli random variable that takes value 1 with a given probability p_{ij} . Hence, p_{ij} represents the probability that a sensor at node i detects an intruder at node j under normal operating conditions. We further assume that the random variables $\{a_{ij}, i \in N, j \in M\}$ are independent and the models developed in this study require this condition. Assuming independence enables us to greatly simplify the calculations. While the assumption may not always hold, it can be satisfied by certain types of sensors in some particular cases. For example, sensors that use visualization (such as cameras) would fit such cases. A sensor in i may be partially or fully blocked from seeing some locations while others are unaffected. Notice that the case of $p_{ij} = 0$ is represented by the lack of link from i to j in the graph representation. Also, if all detection probabilities are equal to one, the problem reduces to the Equitable Facility Location Problem since each sensitive location is fully protected by a single monitoring sensor. In

contrast to the present study which assumes independence between the random variables, Beraldi and Ruszczynski [5, 6] and Saxena et al. [32] do not make any independence assumption.

3.2 Mathematical formulation

Let binary optimization variable x_i represent whether or not a sensor is placed in the candidate location i and $q_j(x)$ denote the probability that an intruder is not detected at node j . We obtain

$$\begin{aligned}
q_j(x) &= P\left(\sum_{i \in N} a_{ij}x_i < 1\right) \\
&= P(a_{ij}x_i < 1, \forall i \in N) && (a_{ij} \text{ is binary}) \\
&= \prod_{i \in N} P(a_{ij}x_i < 1) && (\text{independence}) \\
&= \prod_{i \in N} (1 - p_{ij}x_i) \\
&= \prod_{i \in N} (1 - p_{ij})^{x_i} && (x_i \text{ is binary.})
\end{aligned}$$

As said before, we consider two distinct objective functions in this section, the *lexicographic* one and the *proportional* one. Let us first focus on the lexicographic case. With respect to the $q_j(x)$ criterion, system (2) can be written as:

$$\left\{ q_j(x) \leq \gamma_j, j \in M, \sum_{i \in N} x_i = K, x \in \{0, 1\}^n \right\}, \quad (3)$$

where one looks for a feasible leximax minimal vector γ . The above problem seems hard at first sight since the criteria $q_j(x)$ is clearly non-linear. This is

where Proposition 1 comes into play. We can use the logarithmic function as function ϕ , which combined with the fact that x is a binary solution vector, allows to linearize the functions involved:

$$\log(q_j(x)) = \log\left(\prod_{i \in N} (1 - p_{ij})^{x_i}\right) = \sum_{i \in N} (\log(1 - p_{ij}))x_i,$$

and system (3) becomes

$$\left\{ \sum_{i \in N} (\log(1 - p_{ij}))x_i \leq \gamma_j, j \in M, \sum_{i \in N} x_i = K, x \in \{0, 1\}^n \right\}.$$

Therefore, computing the lexicmax minimal vector can be done using the approaches shown in [[17] Sections 7.2.3 and 7.3.2] and references therein, or [27].

Let us now turn to the problem of minimizing the proportional fairness, which is formally defined as

$$\sum_{j \in M} \log(q_j(x)). \tag{4}$$

In view of (4) above, solving the proportional fair sensor location problem amounts to solve

$$\begin{aligned} \min \quad & \sum_{j \in M} \sum_{i \in N} (\log(1 - p_{ij}))x_i \\ \text{s.t.} \quad & \sum_{i \in N} x_i = K \\ & x_i \in \{0, 1\}, \forall i \in N. \end{aligned}$$

Clearly, the above problem is tractable as it can be solved in $O(|N||M| + |N| \log |N|)$ by ordering the n coefficients $\{\sum_{j \in M} \log(1 - p_{ij}), i \in N\}$ in increasing order and choosing the K first elements.

3.3 The resilient sensor location problem

This subsection is devoted to the resilient sensor location problem. We consider the problem when some failures can occur and the system needs to be properly dimensioned in order to cover all possible failure's states.

Figure 1 presents the problem when all four sensors are operating and all links connecting nodes in N to nodes in M are operational. However some of these links may fail. For example all links emanating from node 1 in set N may fail (which is equivalent to a failure of the sensor in node 1). Figure 2 presents the scenario where the sensor at node 1 failed (the dashed links do not provide protection anymore). Under this scenario each of the locations 2, 3, 4, and 5 is now protected by a single sensor. If a partial failure of the sensor at node 1 occurs, some or all of the outgoing links from node 1 are not able to provide nominal protection. Their surveying capacity may be altered or lost. We deal with all these cases.

Let us introduce first some additional notation. Let \mathcal{S} be the set of possible failure states, each state s is represented by a matrix α_{ij}^s which gives the failure ratio for state s . We assume α_{ij}^s takes any value in $[0, 1]$ such that $\alpha_{i,j}^s = 0$ represents the total failure of covering location j from sensor i ; $\alpha_{ij}^s = 1$ means that this surveying capacity is not affected at all in state s while the remaining values represent partial degradation of the surveying from sensor i .

We denote with $q_j^s(x)$ the value of $q_j(x)$ over state $s \in \mathcal{S}$. Specifically,

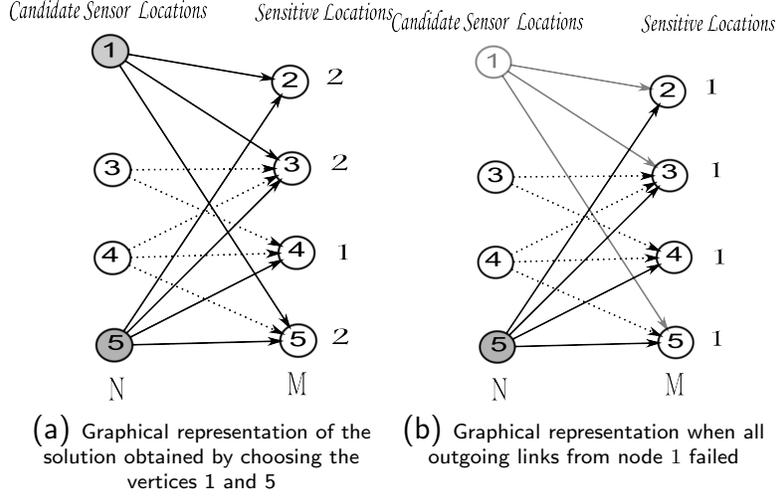


Figure 2: Graphical representation of the resilient sensor location problem.

we express $q_j^s(x)$ by using $\alpha_{ij}^s x_i$ instead of x_i . Let \mathcal{S} be the set of possible failure states, each of which is described by the table α^s . In this context, we need to find a solution (a placement of the sensors) such that each location ensures equitable protection level in all the possible states contained in \mathcal{S} . As in Section 3.2, we may deduce similar transformations:

$$q_j^s(x) = \prod_{i \in N} (1 - p_{ij} \alpha_{ij}^s x_i) = \prod_{i \in N} (1 - p_{ij} \alpha_{ij}^s)^{x_i} \quad (x_i \text{ is binary.})$$

Proceeding as before we obtain the following set which is similar to system (3):

$$\left\{ q_j^s(x) \leq \gamma_j, j \in M, s \in \mathcal{S}, \sum_{i \in N} x_i = K, x \in \{0, 1\}^n \right\}. \quad (5)$$

The above system can be handled similarly as system (3). Specifically, we can use Proposition 1 and the logarithmic function to linearize the sys-

tem. Next, computing the leximax minimal vector can be done using the approaches shown in [[17] Sections 7.2.3 and 7.3.2] and references therein, or [27]. The respective proportional fair problem is formulated below:

$$\begin{aligned} \min \quad & \sum_{j \in M} \sum_{s \in S} (\log(1 - p_{ij} \alpha_{i,j}^s)) x_i \\ \text{s.t.} \quad & \sum_{i \in N} x_i = K \\ & x \in \{0, 1\}^n. \end{aligned}$$

which can be solved in $O(|M||N||S| + |N| \log |N|)$ using a sort algorithm.

4. The ambiguous sensor location problem

In this section, we consider an ambiguous variant of the probabilistic sensor location problem, where probabilities p_{ij} are uncertain. This assumption makes sense in practice as the probabilities describe the normal operating conditions of the sensors. These are, however, likely to be affected by many sources of uncertainty, most of which are hard to predict accurately. Consider, for instance the location of surveillance cameras in an airport to secure points of interest. It may happen that some object is placed temporarily between the vision-field of the camera and the point of interest, thus reducing the probability of detecting an intruder in the point of interest.

This is modeled by introducing an ambiguity set that contains the possible probability distributions. Specifically, we are given nominal values and deviations for the probabilities, respectively denoted by \bar{p} and \hat{p} , and we

assume that p can be any discrete probability measure in the ambiguity set

$$\mathcal{P} := \{p \in \mathbb{R}_+^{n \times m} \mid p_{ij} = \bar{p}_{ij} - \hat{p}_{ij} \xi_i^j, \xi^j \in \Xi^j\},$$

where set $\Xi^j \subset \{0, 1\}^n$ is any 0–1 set. As often in robust optimization, one could equivalently consider 0–1 polytopes (polytopes having binary extreme points) because the functions involved in the robust constraints can be made linear in ξ . Notice that, since p is a probability measure, we must define \bar{p} and \hat{p} such that $0 \leq p_{ij} \leq 1$ for each $i \in N, j \in M$ and $p \in \mathcal{P}$, which amounts to impose that $0 \leq \hat{p}_{ij} \leq \bar{p}_{ij} \leq 1$ for each $i \in N$ and $j \in M$.

In the ambiguous setting, we replace the probability $q_j(x)$ that an intruder is not detected at node j by the *worst-case* probability that an intruder is not detected at node j , denoted $\mathbf{q}_j(x)$. Recalling from Section 3 that the effective monitoring of node j by node i is represented by the set of independent Bernoulli random variables a_{ij} , we can define $\mathbf{q}_j(x)$ formally as

$$\mathbf{q}_j(x) = \max_{p \in \mathcal{P}} P \left(\sum_{i \in N} a_{ij} x_i < 1 \right) = \max_{p \in \mathcal{P}} \prod_{i \in N} (1 - p_{ij} x_i).$$

Ambiguity sets have already been used in the context of ambiguous probabilistic constraints [12] and distributionally robust optimization [11], see also [33]. The main difference of our approach with these frameworks is that we stick here to ambiguity sets that contain only Bernoulli probability distributions, while the aforementioned works consider sets of continuous distributions that satisfy, for instance, moment-based constraints or statistical distance metrics (e.g., phi-divergences, Wasserstein distance) [19].

4.1 Linearizing the probability

We show in this section how the worst-case probability can be handled by using classical techniques of robust optimization. Given $\xi^j \in \Xi_{\Gamma}^j = \{\xi^j | \sum_{i \in N} \xi_i^j \leq \Gamma\}$ for some $\Gamma \leq n$ and each $j \in M$, we denote in the following $q_j^{\xi}(x)$ as the value of $q_j(x)$ associated to probability distribution given by $p_{ij} = \bar{p}_{ij} - \hat{p}_{ij}\xi_i^j$ for each $i \in N, j \in M$, namely,

$$q_j^{\xi}(x) = \prod_{i \in N} (1 - (\bar{p}_{ij} - \hat{p}_{ij}\xi_i^j)x_i).$$

Hence by definition, $q_j^{\xi}(x)$ and $\mathbf{q}_j(x)$ are linked through

$$\mathbf{q}_j(x) = \max_{\xi \in \Xi^j} q_j^{\xi}(x). \quad (6)$$

Using basic properties of logarithmic functions, we can rewrite $\log(q_j^{\xi}(x))$ as a linear function of ξ :

$$\log(q_j^{\xi}(x)) = \log \left(\prod_{i \in N} (1 - (\bar{p}_{ij} - \hat{p}_{ij}\xi_i^j)x_i) \right) \quad (7)$$

$$= \sum_{i \in N} \log(1 - (\bar{p}_{ij} - \hat{p}_{ij}\xi_i^j)x_i) \quad (8)$$

$$= \sum_{i \in N} \log(1 - \bar{p}_{ij}x_i) + \sum_{i \in N} \log \left(1 + \frac{\hat{p}_{ij}}{1 - \bar{p}_{ij}} \xi_i^j x_i \right) \quad (9)$$

$$= \sum_{i \in N} \log(1 - \bar{p}_{ij}x_i) + \sum_{i \in N} \log \left(1 + \frac{\hat{p}_{ij}}{1 - \bar{p}_{ij}} \right) \xi_i^j x_i, \quad (10)$$

where (8) comes from the fact that the logarithmic function of a product reduces to the sum of respective logarithmic functions, (9) is obtained by developing the logarithmic of a summation formula and since x_i is binary.

Finally, (10) follows from the fact that x_i and ξ_i^j are binary. To simplify notations, we introduce $\bar{\alpha}_{ij} = \log(1 - \bar{p}_{ij})$ and $\hat{\alpha}_{ij} = \log(1 + \frac{\hat{p}_{ij}}{1 - \bar{p}_{ij}})$, yielding

$$\log(q_j^\xi(x)) = \sum_{i \in N} \bar{\alpha}_{ij} x_i + \sum_{i \in N} \hat{\alpha}_{ij} \xi_i^j x_i. \quad (11)$$

4.2 Proportional fairness

Similarly to (4), the proportional fairness considers the logarithm of the worst-case probabilities $\mathbf{q}_j(x)$. We obtain for each $j \in M$ that

$$\log(\mathbf{q}_j(x)) = \log \left(\max_{\xi \in \Xi^j} q_j^\xi(x) \right) \quad (12)$$

$$= \max_{\xi \in \Xi^j} \log(q_j^\xi(x)) \quad (13)$$

$$= \max_{\xi^j \in \Xi^j} \left(\sum_{i \in N} \bar{\alpha}_{ij} x_i + \sum_{i \in N} \hat{\alpha}_{ij} \xi_i^j x_i \right), \quad (14)$$

where (12) follows from (6), (13) holds because the logarithm is a monotone increasing function, and (14) follows from (11). The resulting sensor location problem is a classical min max robust optimization problem:

$$\min_x \left(\sum_{j \in M} \max_{\xi^j \in \Xi^j} \left(\sum_{i \in N} \bar{\alpha}_{ij} x_i + \sum_{i \in N} \hat{\alpha}_{ij} \xi_i^j x_i \right) \right) \quad (15)$$

$$\text{s.t. } \sum_{i \in N} x_i = K \quad (16)$$

$$x_i \in \{0, 1\}, \forall i \in N. \quad (17)$$

We prove below that the above problem is \mathcal{NP} -Hard if the sets Ξ^j are arbitrary. Specifically, assuming that $|M| = 1$ we show that the proportional

ambiguous sensor location problem is \mathcal{NP} -hard in the weak sense when $|\Xi| = 2$ (where Ξ is the ambiguity set) while the problem is \mathcal{NP} -hard in the strong sense when the cardinality of Ξ is part of the input. These results are in line with the complexity results obtained for the robust counterparts of classical polynomially solvable combinatorial optimization problems, see the survey of [2].

To verify the first claim we present in AppendixA a polynomial reduction from the *partition problem*, defined as follows. Given a set $L = \{a_1, \dots, a_{|L|}\}$ of $|L|$ integers, one wants to find a subset S of L of cardinality $|L|/2$ such that $\sum_{l \in S} a_l = \sum_{l \in S \setminus L} a_l$.

Theorem 1. *The partition problem polynomially reduces to the decision version of the proportional ambiguous sensor location problem where $|\Xi| = 2$.*

The second claim is obtained through a reduction from the decision version of the *stable set problem*, presented in AppendixB. Given a simple graph $G = (V, E)$, where V is the set of vertices and E is the set of edges, a stable set $S \subseteq V$ is a set of vertices such that for all $u, v \in S$ we have that $(u, v) \notin E$. Hence, the decision version of the stable set problem can be stated as follows: given a graph G and an integer ℓ , one wants to determine if there is a stable set of cardinality at least ℓ .

Theorem 2. *The decision version of the stable set problem polynomially reduces to the decision version of the proportional ambiguous sensor location problem.*

Corollary 1. *The proportional ambiguous sensor location problem is \mathcal{NP} -hard in the weak sense when $|\Xi| = 2$ and in the strong when the cardinality*

of Ξ is part of the input.

In view of the above complexity results, we address the problem through mixed-integer linear programming. Hence, assume that each set Ξ^j corresponds to the set of extreme points of a polytope having a compact formulation. Said differently, $\Xi^j = \text{ext}(\{A^j\xi \leq b^j, \xi \geq 0\})$, where the matrix $A^j \in \mathbb{R}^{k \times n}$ and the vector $b^j \in \mathbb{R}^k$ characterize the polytope. A well-known example of such sets is the budgeted uncertainty set from [8]:

$$\text{conv}(\Xi_\Gamma^j) := \left\{ 0 \leq \xi_i^j \leq 1, i \in N, j \in M, \sum_{i \in N} \xi_i^j \leq \Gamma \right\}.$$

Then, we use classical techniques to reformulate problem (15)–(17) as a MILP by dualizing the inner maximization problems. Defining $\mathcal{K} = \{1, \dots, k\}$, we obtain

$$\begin{aligned} \max_{\xi^j \in \Xi^j} \left(\sum_{i \in N} \bar{\alpha}_{ij} x_i + \sum_{i \in N} \hat{\alpha}_{ij} \xi_i^j x_i \right) &= \sum_{i \in N} \bar{\alpha}_{ij} x_i + \max_{\xi^j \in \Xi^j} \sum_{i \in N} \hat{\alpha}_{ij} \xi_i^j x_i \\ &= \sum_{i \in N} \bar{\alpha}_{ij} x_i + \max_{\xi^j \in \text{conv}(\Xi^j)} \sum_{i \in N} \hat{\alpha}_{ij} \xi_i^j x_i \\ &= \sum_{i \in N} \bar{\alpha}_{ij} x_i + \begin{cases} \min_{u \geq 0} & \sum_{l \in \mathcal{K}} b_l^j u_l \\ \text{s.t.} & \sum_{l \in \mathcal{K}} A_{li}^j u_l \geq \hat{\alpha}_{ij} x_i, i \in N \end{cases} \\ &= \begin{cases} \min_{u \geq 0} & \sum_{i \in N} \bar{\alpha}_{ij} x_i + \sum_{l \in \mathcal{K}} b_l^j u_l \\ \text{s.t.} & \sum_{l \in \mathcal{K}} A_{li}^j u_l \geq \hat{\alpha}_{ij} x_i, i \in N \end{cases}. \end{aligned} \quad (18)$$

Therefore, the problem of minimizing the proportional fairness amounts to

solving the following MILP in optimization vectors x and u :

$$\begin{aligned}
& \min \sum_{j \in M} \sum_{i \in N} \bar{\alpha}_{ij} x_i + \sum_{j \in M} \sum_{l \in \mathcal{K}} b_l^j u_l \\
& \text{s.t.} \quad \sum_{i \in N} x_i = K \\
& \quad \sum_{l \in \mathcal{K}} A_{li}^j u_l \geq \hat{\alpha}_{ij} x_i, \quad i \in N \\
& \quad x_i \in \{0, 1\}, \forall i \in N \\
& \quad u \geq 0.
\end{aligned}$$

When A contains non-negative coefficients, an alternative approach proposed in [8] and extended in [28] relies on solving a sequence of deterministic problems. Specifically, let us denote by A' the submatrix obtained from A by not considering the upper bounds on ξ and let k' be the number of lines of A' (for instance, $k' = 1$ for the budgeted uncertainty set Ξ_{Γ}^j). The iterative algorithm proposed in [8, 28] solves the above min max robust optimization problem by solving $O((k'm)^{k'm}(nm)^{k'm})$ problems minimizing the proportional fairness with known probabilities. In particular, the min max robust problem is polynomially solvable if k' and m are constant.

4.3 Max-min fairness

In the ambiguous setting, the probability q_j is replaced by the worst-case probability \mathbf{q}_j . Hence, the system (3) becomes

$$\left\{ \mathbf{q}_j(x) \leq \gamma_j, \quad j \in M, \quad \sum_{i \in N} x_i = K, \quad x \in \{0, 1\}^n \right\},$$

where one looks for a feasible leximax minimal vector γ . Notice that, combining (14) with (18), we obtain immediately the following relation

$$\log(\mathbf{q}_j(x)) = \begin{cases} \min_{u \geq 0} & \sum_{i \in N} \bar{\alpha}_{ij} x_i + \sum_{l \in \mathcal{K}} b_l^j u_l \\ \text{s.t.} & \sum_{l \in \mathcal{K}} A_{li}^j u_l \geq \hat{\alpha}_{ij} x_i, \quad i \in N \end{cases}. \quad (19)$$

Using again Proposition 1, and replacing $\log(\mathbf{q}_j(x))$ by the rhs of (19), the problem amounts to finding the leximax minimal vector γ feasible for system

$$\left\{ \min_{\substack{\sum_{l \in \mathcal{K}} A_{li}^j u_l \geq \hat{\alpha}_{ij} x_i, \quad i \in N \\ u \geq 0}} \sum_{i \in N} \bar{\alpha}_{ij} x_i + \sum_{l \in \mathcal{K}} b_l^j u_l \leq \gamma_j, \quad j \in M, \quad \sum_{i \in N} x_i = K, \quad x \in \{0, 1\}^n \right\}, \quad (20)$$

Then, one readily verifies that, given γ , x is feasible for each constraint

$$\min_{\substack{\sum_{l \in \mathcal{K}} A_{li}^j u_l \geq \hat{\alpha}_{ij} x_i, \quad i \in N \\ u \geq 0}} \left(\sum_{i \in N} \bar{\alpha}_{ij} x_i + \sum_{l \in \mathcal{K}} b_l^j u_l \right) \leq \gamma_j$$

if and only if there exists a vector $u \geq 0$ that satisfies $\sum_{l \in \mathcal{K}} A_{li}^j u_l \geq \hat{\alpha}_{ij} x_i$ for each $i \in N$ such that x satisfies

$$\sum_{i \in N} \bar{\alpha}_{ij} x_i + \sum_{l \in \mathcal{K}} b_l^j u_l \leq \gamma_j$$

for each $j \in M$. Hence, we can replace system (20) with the following one

$$\left\{ \begin{array}{l} \sum_{i \in N} \bar{\alpha}_{ij} x_i + \sum_{l \in \mathcal{K}} b_l^j u_l \leq \gamma_j, \quad j \in M, \\ \sum_{i \in N} x_i = K, x \in \{0, 1\}^n, \sum_{l \in \mathcal{K}} A_{li}^j u_l \geq \hat{\alpha}_{ij} x_i, \quad i \in N, u \geq 0 \end{array} \right\}. \quad (21)$$

We can finally find a leximax minimal vector γ feasible for system (21) using the algorithm proposed in [[17], Sections 7.2.3 and 7.3.2] and in the references therein.

5. Numerical results

In this section, we report on the computational experiments obtained by applying the above models for different variants of the proportional equitable sensor location problems, namely Proportional Fair, Proportional Fair Resilient and Proportional Fair Ambiguous. All experiments have been carried out on an Intel(R) Core(TM) i7 CPU M60, 2.6Hz 4GB Ram machine and all formulations and algorithms were coded in C++, compiled with a GNU G++ 4.5 compiler and IBM CPLEX 12.3. In the rest of the section, we present the benchmark used in our computations as well as different numerical tests with respect to the proportionally fair sensor location problem and the resilient one towards the total and partial failure cases. We end this section with a few numerical results on the ambiguous equitable sensor location problem and conclude with some discussion of the obtained results.

5.1 Benchmark generation

To the best of our knowledge, there are no probabilistic instances defined for the sensor (facility) location problem. We therefore tested our algorithm on a set of instances generated randomly. We have built 10 instances per scenario (N, M) , where we consider 3 different values for N (the set of candidate locations), 10, 20 and 30; and three other values for the sensitive locations to be protected respectively 30, 40 and 50. We consider a quadratic grid of 100×100 as the space where both sensors and points of interest are placed randomly. We have generalized the problem studied and added the cost of sensors as part of parameters. Hence, we have considered two possible cost values (30 and 50) which are assigned randomly to each candidate location.

We assume that we have for each instance a number A of high quality sensors that are produced using a new and yet less mature technology. For the instances where $N = 10$ we have 3 of such sensors, for the instances where $N = 20$ we have 4 and in the instances where $N = 30$ we have 5. This will be important for analyzing the resilient solution and will be discussed in more details ahead. Finally, the surveillance probabilities assigned to a candidate sensor for a sensitive location are expressed as the function of its generation and the distance between both locations. Specifically, the probability that a sensor $i \geq A$ protects a location j is valued as $(1 - \frac{d(i,j)}{\sqrt{2} \cdot 100} \cdot \frac{1}{40})$ and for $i < A$ is evaluated as $(1 - \frac{d(i,j)}{\sqrt{2} \cdot 100} \cdot \frac{1}{20})$. Hence, the closer a sensor is to a location, the higher is the probability of protection for that location by that sensor. Similarly, the more expensive the sensor is, the better is the protection it provides.

Following above, we have generated 10 instances for each scenario (10, 30),

(10, 40), and (10, 50). Next, for scenarios (20, 30), (20, 40), (20, 50) we have taken the first set of instances and added 10 new locations to each of them. The same routine is used to generate (30, 30), (30, 40), (30, 50) using instances for (20, 30), (20, 40), (20, 50).

5.2 Proportionally fair sensor location problem

We analyze the quality regarding mean surveillance probability and standard deviation dispersion of the solutions obtained by the model presented for the proportionally fair sensor location problem. Indeed, a proportionally fair solution intends to reach a compromise between two objectives: the max-min fairness which seeks to reduce inequality among the protected locations versus the overall sum of protecting levels.

$ N $	$ M $	Protection	Variance	$ N $	$ M $	Protection	Variance
10	30	88.35%	4.69	10	30	93.14%	3.88
10	40	94.99%	4.22	10	40	97.57%	2.27
10	50	89.17%	5.25	10	50	94.81%	2.80
20	30	89.11%	3.70	20	30	95.10%	2.28
20	40	95.22%	3.84	20	40	97.67%	2.41
20	50	89.95%	3.85	20	50	95.30%	2.44
30	30	89.08%	4.84	30	30	95.17%	2.38
30	40	95.22%	3.84	30	40	97.78%	2.16
30	50	90.17%	3.53	30	50	95.56%	2.18

(a) Choosing 5 sensors
(b) choosing 7 sensors

Table 1: Results for the proportionally fair sensor location problem. We report the mean surveillance value as well as the variance.

Table 1 presents the results for the proportionally fair sensor location problem where M (resp. N) gives the set of locations to be protected (resp.

candidate sensor locations). We present the mean and the variance of the surveillance probability according to the number of installed sensors. Notice that, as expected, the surveillance probability is higher as the number of available sensors rises and the obtained solutions have very low variances. Notice that choosing more sensors allows us not only to improve the protection but also to decrease the variance, meaning that discrepancy between the protection levels in the different places are decreasing, turning the solution fairer.

5.3 Resilient sensor location problem

Concerning the resilient sensor location problem and scenarios of total failure we tested the same instances presented before under a set of scenarios that represents all the possibilities of failure (total, or partial) of 1, 2 or 3 sensors over the A sensors from the new technology. When a sensor fails totally, its surveillance in any location is zero. When a sensor fails partially, we assume that its surveillance in any location is reduced by 50%.

Intuitively, this represents the fact that these sensors provide a larger surveillance probability, but they are unstable. This is the case in many real instances, for example, we can imagine that the three first sensors are “new” or “untested” sensors, they provide better surveillance than the sensors we already have, but they are not fully reliable.

Tables 2-8, present the results for the resilient sensor location problem. We have tested two different sets of failing scenarios, in the first we consider a total failure of the high-tech sensors and in the second we consider a partial failure of the same sensors, (i.e., the surveying capacity of such sensors is reduced by 50%). This produces two sets of solutions, the first related to the

$ N $	$ M $	Prop. Fair		Prop. Fair Res	
		Prot	Var	Prot	Var
10	30	88.35%	4.69	80.72%	5.59
10	40	94.99%	4.22	94.70%	3.94
10	50	89.17%	5.25	85.45%	5.68
20	30	89.11%	3.70	87.19%	6.82
20	40	95.22%	3.84	94.93%	4.32
20	50	89.95%	3.85	88.10%	5.00
30	30	89.08%	4.85	88.10%	4.69
30	40	95.22%	3.84	94.55%	5.11
30	50	90.17%	3.53	88.86%	3.98

(a) Choosing 5 sensors

$ N $	$ M $	Prop. Fair		Prop. Fair Res	
		Prot	Var	Prot	Var
10	30	93.14%	3.88	90.32%	4.22
10	40	97.57%	2.27	97.05%	2.10
10	50	94.81%	2.80	93.25%	2.66
20	30	95.10%	2.28	93.92%	3.63
20	40	97.67%	2.41	97.64%	2.38
20	50	95.30%	2.44	93.98%	3.48
30	30	95.17%	2.38	94.37%	3.73
30	40	97.78%	2.16	97.63%	2.45
30	50	95.56%	2.18	94.84%	2.50

(b) Choosing 7 sensors

Table 2: Results for the resilient sensor location problem. We report the mean surveillance value as well as the variance for the scenario without failures.

		Total Failure				Partial Failure			
$ N $	$ M $	Prop. Fair		Prop. Fair Res		Prop. Fair		Prop. Fair Res	
		Prot	Var	Prot	Var	Prot	Var	Prot	Var
10	30	82.71%	5.38	80.72%	5.59	86.51%	5.25	82.72%	5.07
10	40	93.04%	4.73	94.70%	3.94	93.68%	4.37	94.70%	3.94
10	50	84.16%	5.47	85.45%	5.68	86.51%	5.36	85.95%	5.47
20	30	83.63%	4.84	87.19%	6.82	85.78%	4.27	87.19%	6.82
20	40	95.22%	3.84	94.93%	4.32	95.22%	3.84	94.93%	4.32
20	50	84.83%	4.53	88.10%	5.00	86.73%	4.16	88.10%	5.00
30	30	83.69%	6.13	88.10%	4.69	86.67%	5.43	88.10%	4.69
30	40	95.22%	3.84	94.55%	5.11	95.22%	3.84	94.55%	5.11
30	50	85.01%	4.60	88.86%	3.98	88.19%	4.11	88.86%	3.98

Table 3: Results for the resilient sensor location problem when choosing 5 sensors. We report the mean surveillance value as well as the variance for the worst-case among the three scenarios where one sensor in $\{1, 2, 3\}$ fails completely or partially.

scenarios with complete failure and the second related to the scenarios with partial failure.

In Table 2 we present the mean and the variance of the surveillance prob-

		Total Failure				Partial Failure			
N	M	Prop. Fair		Prop. Fair Res		Prop. Fair		Prop. Fair Res	
		Prot	Var	Prot	Var	Prot	Var	Prot	Var
10	30	89.95%	4.66	90.32%	4.22	91.98%	4.26	89.32%	3.97
10	40	96.63%	2.61	97.05%	2.10	97.43%	2.32	97.05%	2.10
10	50	92.46%	2.99	93.25%	2.66	93.48%	2.84	93.25%	2.66
20	30	92.41%	4.03	93.92%	3.63	93.31%	3.68	93.92%	3.63
20	40	96.87%	2.62	97.64%	2.38	96.97%	2.52	97.64%	2.38
20	50	93.04%	2.89	93.98%	3.48	93.95%	2.67	93.98%	3.48
30	40	97.78%	2.16	97.63%	2.45	97.78%	2.16	97.63%	2.45
30	30	92.79%	3.24	94.37%	3.73	93.87%	2.94	94.37%	3.73
30	50	93.32%	2.80	94.84%	2.50	94.22%	2.46	94.84%	2.50

Table 4: Results for the resilient sensor location problem when choosing 7 sensors. We report the mean surveillance value as well as the variance for the worst-case among the three scenarios where one sensor in $\{1, 2, 3\}$ fails completely or partially.

		Total Failure				Partial Failure			
N	M	Prop. Fair		Prop. Fair Res		Prop. Fair		Prop. Fair Res	
		Prot	Var	Prot	Var	Prot	Var	Prot	Var
10	30	71.33%	6.70	80.72%	5.59	81.65%	6.12	82.72%	5.07
10	40	88.92%	7.27	94.70%	3.94	91.12%	5.29	94.70%	3.94
10	50	84.16%	5.47	85.45%	5.68	84.16%	5.42	85.95%	5.47
20	30	72.64%	7.03	87.19%	6.82	78.14%	6.13	87.19%	6.82
20	40	92.31%	6.17	94.93%	4.32	93.73%	5.83	94.93%	4.32
20	50	84.83%	4.53	88.10%	5.00	85.87%	4.23	88.10%	5.00
30	30	72.96%	8.54	88.10%	4.69	81.15%	6.02	88.10%	4.69
30	40	92.31%	6.17	94.55%	5.11	93.86%	4.59	94.55%	5.11
30	50	85.01%	4.60	88.86%	3.98	86.81%	4.35	88.86%	3.98

Table 5: Results for the resilient sensor location problem when choosing 5 sensors. We report the mean surveillance value as well as the variance for the worst-case among the scenarios where two sensor in $\{1, 2, 3\}$ fails completely or partially.

ability according to the number of installed sensors for the scenario without any sensor failing. Notice that the solution obtained by solving the proportionally fair resilient model (called resilient solution) and the solution

		Total Failure				Partial Failure			
N	M	Prop. Fair		Prop. Fair Res		Prop. Fair		Prop. Fair Res	
		Prot	Var	Prot	Var	Prot	Var	Prot	Var
10	30	83.58%	6.04	84.14%	5.50	87.57%	4.94	89.32%	3.97
10	40	94.70%	3.94	97.05%	2.10	95.80%	3.64	97.05%	2.10
10	50	89.34%	3.91	93.25%	2.66	91.38%	3.21	93.25%	2.66
20	30	87.19%	6.82	93.92%	3.63	90.89%	3.82	93.92%	3.63
20	40	94.93%	4.32	97.64%	2.38	95.91%	3.12	97.64%	2.38
20	50	93.04%	2.89	93.98%	3.48	93.04%	2.96	93.98%	3.48
30	30	88.10%	4.69	94.37%	3.73	90.85%	3.89	94.37%	3.73
30	40	96.41%	3.57	97.63%	2.45	96.81%	3.18	97.63%	2.45
30	50	93.32%	2.80	94.84%	2.50	93.72%	2.75	94.84%	2.50

Table 6: Results for the resilient sensor location problem when choosing 7 sensors. We report the mean surveillance value as well as the variance for the worst-case among the scenarios where two sensor in $\{1, 2, 3\}$ fails completely or partially.

		Total Failure				Partial Failure			
N	M	Prop. Fair		Prop. Fair Res		Prop. Fair		Prop. Fair Res	
		Prot	Var	Prot	Var	Prot	Var	Prot	Var
10	30	57.30%	6.76	80.72%	5.59	67.26%	6.16	82.72%	5.07
10	40	61.82%	6.31	81.98%	5.03	81.12%	5.87	81.98%	5.03
10	50	72.36%	10.96	85.45%	5.68	76.16%	8.06	85.95%	5.47
20	30	72.64%	7.03	87.19%	6.82	74.18%	6.53	87.19%	6.82
20	40	75.44%	5.46	83.86%	4.35	85.08%	5.26	83.86%	4.35
20	50	74.14%	7.36	88.10%	5.00	81.94%	6.37	88.10%	5.00
30	30	72.96%	8.54	88.10%	4.69	76.46%	7.51	88.10%	4.69
30	40	75.44%	5.46	83.68%	5.61	86.44%	4.74	83.68%	5.61
30	50	74.86%	5.87	88.86%	3.98	81.46%	5.16	88.86%	3.98

Table 7: Results for the resilient sensor location problem when choosing 5 sensors. We report the mean surveillance value as well as the variance for the scenarios where the sensors numbered as 1,2 and 3 fail completely or partially.

obtained by solving the proportional fair model (called also equitable solution) are quite close to each other regarding the means and variances, the equitable solutions being slightly better.

		Total Failure				Partial Failure			
N	M	Prop. Fair		Prop. Fair Res		Prop. Fair		Prop. Fair Res	
		Prot	Var	Prot	Var	Prot	Var	Prot	Var
10	30	76.09%	5.59	84.14%	5.50	82.56%	5.25	89.32%	3.97
10	40	81.98%	5.03	89.52%	3.94	87.18%	4.83	93.52%	3.04
10	50	81.55%	7.39	88.48%	4.49	89.45%	6.87	91.71%	4.22
20	30	87.19%	6.82	93.92%	3.63	88.79%	6.13	93.92%	3.63
20	40	83.86%	4.35	92.58%	3.11	93.86%	3.85	95.57%	2.91
20	50	88.10%	5.00	93.98%	3.48	91.87%	4.65	93.98%	3.48
30	30	88.10%	4.69	94.37%	3.73	88.18%	4.29	94.37%	3.73
30	40	89.30%	4.06	92.90%	2.95	92.32%	3.15	94.45%	2.65
30	50	88.86%	3.98	94.84%	2.50	91.16%	3.75	94.84%	2.50

Table 8: Results for the resilient sensor location problem when choosing 7 sensors. We report the mean surveillance value as well as the variance for the scenarios where the sensors numbered as 1,2 and 3 fail completely or partially.

In all the remaining tables we compare the equitable solution with the resilient solution separately for total and partial failures. For example, in Table 3 we compare the behaviors of the equitable and resilient solutions for the scenarios where one “high-tech” sensor fails completely or partially, given that we must choose 5 sensors. In the first part of the table, labeled with *total failure*, we compare the mean surveillance and the variance of the equitable solution with the mean surveillance and variance of the resilient solution obtained under the total failure assumption. In the second part, we report similar comparisons as above but with the resilient solution obtained under the partial failure assumption. We run these solutions on scenarios in accordance with these assumptions, meaning that in the first part of the table we consider only scenarios with total failure and the second part, only scenarios with partial failure.

In Table 4 we compare the behaviors of the equitable and resilient so-

lutions for the scenarios where one “high-tech” sensor fails completely or partially, given that we must choose 7 sensors. Notice that the surveillance levels are very close, with some advantage to the resilient solution for both cases. In Tables 5 and 6 we present the comparison for the scenarios where two “high-tech” sensors fail totally or partially, again the resilient solution performs better. This is more perceivable when looking at the total failures case. Such behavior is found also in the case where we have three “high-tech” sensors failing represented in Tables 7 and 8.

To conclude, concerning the case of total failure of the sensors, notice that, as expected, the surveillance probability of the resilient solution is higher than the equitable solution in the scenarios with sensors failing completely. Notice also that, even in the scenario without failing sensors, the resilient solution provides a mean surveillance probability that is quite good, while in failure cases it gives systematically solutions notably better compared to the equitable solution.

5.4 *Ambiguous sensor location problem*

Here we adopt the budgeted uncertainty polytope described before. We consider for the sensors numbered from 1 to $|A|$ a deviation that is equal to the surveillance probability, meaning that these sensors can, in a way, become totally nonoperational; for the sensors numbered from $|A| + 1$ to N we consider a deviation of 25%. We set the value 5 for the parameter Γ_s (the maximum number of probabilistic deviations for each sensor s), for all the sensors, except for the first three sensors we have $\Gamma_{s_1} = \Gamma_{s_2} = \Gamma_{s_3} = 30$. We computed the Proportional Fair Ambiguous solution for the latter case and compared its behavior with that of the standard Proportional Fair solution

for two different scenarios. The first set of scenarios is built as follows: for each instance we set the probability values at their nominal value (i.e., no deviations are taken into account). For the second scenario we set the surveillance probabilities of each sensor s to their worst value for the lowest Γ_s of them while keep at their nominal value for the rest of probabilities. As we did before, we report the mean probability surveillance values and the standard deviations for both solutions, the basic proportionally fair solution and the proportionally fair ambiguous solution.

Notice that for both scenarios the proportionally fair solution and the ambiguous solution have mean surveillance probabilities close to each other, with a slight advantage to the ambiguous solution in the second scenario. An interesting result is that the variance is lower in the ambiguous solution for all tested cases of the second scenario, which shows the robustness of such solutions.

6. Concluding remarks

In this paper we have provided a study on different variants of the stochastic sensor location problem, namely the probabilistic equitable sensor location problem, the resilient variant and the ambiguous one. For each of them we have considered two popular fairness criteria that are lexicographic and proportional fairness and report solution methodology and a full complexity study. Obviously, one can examine these models with other objective functions that exhibit different notions of fairness. We show that the proportional fairness version of the ambiguous problem is \mathcal{NP} -hard in the strong sense while for the conventional and resilient cases it is polynomial.

$ N $	$ M $	Prop. Fair		Prop. Fair Amb	
		Prot	Var	Prot	Var
10	30	88.35%	4.69	80.72%	5.59
10	40	94.99%	4.22	95.45%	3.12
10	50	89.17%	5.25	89.59%	3.81
20	30	89.11%	3.70	87.19%	6.82
30	30	89.08%	4.85	88.10%	4.69
20	40	95.22%	3.84	94.19%	5.21
30	40	95.22%	3.84	93.69%	6.32
20	50	89.95%	3.85	87.73%	5.68
30	50	90.17%	3.53	89.97%	3.88

(a) Choosing 5 sensors

$ N $	$ M $	Prop. Fair		Prop. Fair Amb	
		Prot	Var	Prot	Var
10	30	93.14%	3.88	92.06%	3.72
10	40	97.57%	2.27	97.57%	2.27
10	50	94.81%	2.80	94.81%	2.80
20	30	95.10%	2.28	93.92%	3.63
20	40	97.67%	2.41	97.64%	2.38
20	50	95.30%	2.44	94.94%	2.30
30	30	95.17%	2.38	94.37%	3.73
30	40	97.78%	2.16	97.63%	2.45
30	50	95.56%	2.18	95.39%	2.31

(b) Choosing 7 sensors

Table 9: Results for the ambiguous sensor location problem, first scenario

Regarding the computational experiments, we must mention that the elapsed time to solve the instances goes in line with the complexity of the respective problems, that is from seconds in the proportional fairness applied to conventional and resilient cases, to minutes in the case of the ambiguous approach. Nevertheless, no instance took more than the time limit of 10 minutes to be solved. Although we are able to handle for the same amount

$ N $	$ M $	Prop. Fair		Prop. Fair Amb	
		Prot	Var	Prot	Var
10	30	77.78%	26.736	78.44%	5.93
10	40	92.05%	9.293	94.58%	3.56
10	50	86.36%	12.502	88.79%	4.71
20	30	79.59%	27.248	85.36%	7.59
20	40	91.91%	10.651	93.33%	5.76
20	50	87.95%	8.578	87.20%	6.05
30	30	79.78%	27.521	86.51%	5.67
30	40	92.05%	10.652	92.91%	6.62
30	50	87.64%	10.919	89.39%	4.82

(a) Choosing 5 sensors

$ N $	$ M $	Prop. Fair		Prop. Fair Amb	
		Prot	Var	Prot	Var
10	30	82.68%	28.28	90.12%	4.09
10	40	94.68%	9.16	97.01%	2.60
10	50	92.48%	8.87	94.02%	3.17
20	30	84.94%	28.90	92.58%	4.11
20	40	95.10%	8.51	97.22%	2.65
20	50	93.32%	8.81	94.62%	2.73
30	30	85.35%	29.04	92.99%	4.65
30	40	95.01%	9.29	97.25%	2.59
30	50	94.00%	6.58	95.05%	2.94

(b) Choosing 7 sensors

Table 10: Results for the ambiguous sensor location problem, second scenario

of time instances with N and M increased separately, increasing both has proven to lead to scalability issues quickly, for example, we might solve in 10 minutes a problem with $M = 10$ and $N = 100$ but a problem with $M = 55$ and $N = 55$ takes considerably more computation time. This may be expected as the size grows exponentially for such instances. A possible research direction is to analyze and propose methods that use parallelism

to solve the models. Notice that the models are highly decomposable hence parallelism should decrease the computation time and improve scalability.

Although the models in this paper focus on protection provided to multiple locations, variants of these models can be developed for other application areas. One example may consider the placement of base stations that provide service to mobile phones in cellular wireless networks, where the service experienced by a phone in a given location may depend on the location of multiple base stations. Another example may consider placement of generators that provide electrical power but are subject to failures. Moreover the links connecting a demand location to the supplying generators are also subject to failures. The service provided to each locations may then depend on the location of the generators and on the network topology that connects supply nodes to demand locations.

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AppendixA. Proof of Theorem 1

Consider an instance of the partition problem given by set L , and let us define for each $i \in N$

$$\underline{a}_i = \min \left(a_i, \frac{2}{|L|} \sum_{k \in N} a_k - a_i \right) \quad \text{and} \quad \bar{a}_i = \max \left(a_i, \frac{2}{|L|} \sum_{k \in N} a_k - a_i \right).$$

We first construct an instance of problem (15)–(17) that corresponds to the given instance of the partition problem, and show later how to construct the corresponding instance of the proportional ambiguous sensor location problem. Consider $|M| = 1$, $|N| = |L|$, $K = |L|/2$, $\bar{\alpha}_i = \underline{a}_i$ and $\hat{\alpha}_i = \bar{a}_i - \underline{a}_i$ for every $i \in N$. Finally, we define Ξ as $\{\xi^1, \xi^2\}$ where ξ^1 and ξ^2 are defined for each $i \in N$ by

$$\xi_i^1 = \frac{a_i - \underline{a}_i}{\bar{a}_i - \underline{a}_i} \quad \text{and} \quad \xi_i^2 = \frac{\frac{2}{|L|} \sum_k a_k - a_i - \underline{a}_i}{\bar{a}_i - \underline{a}_i}.$$

With these definitions, we see that $\bar{\alpha}_i + \hat{\alpha}_i \xi_i^1 = a_i$ and $\bar{\alpha}_i + \hat{\alpha}_i \xi_i^2 = \frac{2}{|L|} \sum_{k \in N} a_k - a_i$.

Let x be any vector feasible for problem (15)–(17) and let $S \subset N$ be the set of indices where $x_i = 1$. Then,

$$\max_{\xi \in \Xi} \sum_{i \in N} \bar{\alpha}_{ij} x_i + \sum_{i \in N} \hat{\alpha}_{ij} \xi_i^j x_i = \max \left(\sum_{i \in S} a_i, \sum_{k \in N} a_k - \sum_{i \in S} a_i \right) \quad (\text{A.1})$$

$$= \max \left(\sum_{i \in S} a_i, \sum_{i \in N \setminus S} a_i \right). \quad (\text{A.2})$$

Hence, the instance of the partition problem is a *yes* instance if and only if

the optimal solution cost of problem (15)–(17) is less than or equal to $\frac{\sum_{i \in N} a_i}{2}$.

We construct next an equivalent instance for the proportional ambiguous sensor location problem. First, notice that, due to constraints (16), we can add a constant M to all components of $\bar{\alpha}$ without affecting the optimal solution of the problem. Then, we define

$$\bar{p}_i = 1 - e^{\bar{\alpha}_i + M} \quad (\text{A.3})$$

$$\hat{p}_i = (e^{\hat{\alpha}_i} - 1)(1 - \bar{p}_i) \quad (\text{A.4})$$

for each $i \in N$, where

$$\bar{\alpha}_i = \log(1 - \bar{p}_i) \quad (\text{A.5})$$

$$\hat{\alpha}_i = \log\left(1 + \frac{\hat{p}_i}{1 - \bar{p}_i}\right). \quad (\text{A.6})$$

One readily verifies that choosing

$$M = -\max_{k \in N} \bar{\alpha}_k + \hat{\alpha}_k$$

yields values of \bar{p}_i and \hat{p}_i that satisfy $0 \leq \hat{p}_i \leq \bar{p}_i \leq 1$ for each $i \in N$. Moreover, the input \bar{p}_i and \hat{p}_i can be expressed by a number of digits that is polynomial in the number of digits of the original input.

AppendixB. Proof of Theorem 2

Consider an instance for the stable set problem, given by the graph $G = (V, E)$ and the integer ℓ . We construct an instance for the proportional ambiguous sensor location problem as follows: $|M| = 1$, $|N| = |V|$, $K = \ell$,

$\bar{p}_i = 1 - e^\theta$ and $\hat{p}_i = (e^\psi - 1)e^\theta$ for each $i \in N$ where θ and ψ are real numbers chosen such that $0 \leq \hat{p}_i \leq \bar{p}_i \leq 1$ for each $i \in N$. Using the previous reformulation and definitions (A.5) and (A.6), we obtain an instance of problem (15)–(17) defined by $\bar{\alpha}_i = \theta$ and $\hat{\alpha}_i = \psi$ for every $i \in N$. In order to define Ξ , let us consider $ch(e)$ the characteristic vector of an edge $e = (uv) \in E$, meaning $ch(e) \in \{0, 1\}^{|V|}$ and $ch_i(e) = 1$ if and only if $i = u$ or $i = v$. Then, we consider the finite uncertainty set $\Xi = \cup_{e \in E} ch(e)$.

Next, we show that there is a stable set of cardinality at least ℓ in the graph G if and only if the optimal solution cost of the proportional ambiguous sensor location problem associated is smaller than or equals to $\ell\theta + \psi$. Now we look at the two possible cases:

There exists a stable set of cardinality ℓ : Let $S \subseteq V$ be a stable set of cardinality at least ℓ . Consider the solution vector x for the proportional ambiguous sensor location problem defined as $x_i = 1$ if and only if $i \in S$. Notice that

$$\max_{\xi \in \Xi} \left(\sum_{i \in N} \theta x_i + \sum_{i \in N} \psi \xi_i x_i \right) = \sum_{i \in N} \theta x_i + \max_{e \in E} \sum_{i \in V} \psi ch_i(e) x_i \leq \ell\theta + \psi$$

There exists no stable set of cardinality ℓ : By contradiction, suppose that

$$\max_{\xi \in \Xi} \left(\sum_{i \in N} \theta x_i^* + \sum_{i \in N} \psi \xi_i x_i^* \right) \leq \ell\theta + \psi$$

for a solution x^* and we do not have a stable set of cardinality at least

ℓ . As x^* is a binary vector we have that

$$\begin{aligned} \max_{\xi \in \Xi} \left(\sum_{i \in N} \theta x_i^* + \sum_{i \in N} \psi \xi_i x_i^* \right) &= \sum_{i \in N} \theta x_i^* + \max_{\xi \in \Xi} \sum_{i \in N} \psi \xi_i x_i^* \\ &= \ell \theta + \psi \max_{\xi \in \Xi} \sum_{i \in N} \xi_i x_i^* \\ &\leq \ell \theta + \psi, \end{aligned}$$

which means that

$$\sum_{i \in V} c h_i(e) x_i^* \leq 1$$

for every $e \in E$. Hence $S^* = \{i \in V | x_i^* = 1\}$ is a stable set of G with cardinality at least ℓ .